Stability Analysis of Heat and Wave Equations using Lyapunov Functionals

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1. Objective

This research paper investigates the use of Lyapunov functionals for assessing the stability of two classes of partial differential equations (PDEs): the parabolic heat equation and the hyperbolic wave equation. The analysis spans 1D and multidimensional domains with Neumann and Robin boundary conditions. Key distinctions between finite- and infinite-dimensional dynamical systems are also addressed [Lessard(2022)].

2. Background

2.1. Lyapunov Stability Concepts

2.1.1 Finite-Dimensional Systems

For finite-dimensional systems defined by ordinary differential equations (ODEs), Lyapunov's method provides a robust systematic approach to verify stability. Given a system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, a scalar function $V(x)$ satisfying $V(x) > 0$ and $\dot{V}(x) < 0$ in a neighborhood of the origin implies Lyapunov stability.

2.1.2 Lyapunov Functionals in Infinite-Dimensional Systems

In the analysis of PDEs, the state of the system is described by a function $u(x,t)$ evolving over a spatial domain Ω. PDEs evolve in infinite-dimensional spaces, requiring generalizations of Lyapunov theory and a slightly modified notion of norm [Baker and Bergen(1969)].

A Lyapunov *functional*, $V[u(t)]$, assigns a scalar value to $u(x,t)$, often representing energy or norms. It must satisfy:

- $V[u(t)] \geq 0$,
- $V[u(t)] = 0$ only at equilibrium,
- $\frac{dV}{dt} \leq 0$.

For the heat equation we utilize the L2 norm to construct the Lyapunov functional:

$$V[u] = \frac{1}{2} \int_\Omega u^2 \, dx,$$

and for the damped wave equation the total mechanical energy to construct the Lyapunov functional:

$$V[u] = \frac{1}{2} \int_\Omega \left(u_t^2 + c^2 |\nabla u|^2 \right) dx.$$

2.2. Boundary Conditions

Boundary conditions significantly affect solution behavior. Dirichlet conditions fix function values at the boundary (e.g., $u = 0$), Neumann conditions fix the derivative (e.g., $u_x = 0$), Robin conditions combine the two (e.g., $u_x - \alpha u = 0$), and periodic conditions match function and derivative values across opposite boundaries [Strauss(2022)].

3. Heat Equation Analysis

3.1. 1D with Neumann Boundary Conditions

Consider the 1D heat equation:

$$u_t = \kappa u_{xx}, \quad x \in (0, L), \tag{1}$$

$$u_x(0, t) = 0, \quad u_x(L, t) = 0. \tag{2}$$

The Lyapunov functional is defined as:

$$V(t) = \frac{1}{2} \int_0^L u(x, t)^2 \, dx.$$

Differentiating with respect to time:

$$\dot{V}(t) = \kappa \int_0^L u u_{xx} \, dx = -\kappa \int_0^L u_x^2 \, dx \leq 0.$$

By Poincaré's inequality:

$$V(t) \leq V(0) e^{-\frac{2\kappa}{L^2} t},$$

implying exponential decay and asymptotic stability.

3.2. 1D with Robin Boundary Conditions

Now assume Robin boundary conditions:

$$u_x(0, t) - \alpha u(0, t) = 0, \quad u_x(L, t) + \alpha u(L, t) = 0, \quad \alpha > 0.$$

The time derivative of the same Lyapunov functional becomes:

$$\dot{V}(t) = -\kappa \left(\int_0^L u_x^2 \, dx + \alpha u(0)^2 + \alpha u(L)^2 \right) \leq 0.$$

Robin conditions introduce boundary dissipation, accelerating energy decay.

3.3. Multidimensional Generalization

In $\Omega \subset \mathbb{R}^n$, the heat equation is:

$$u_t = \kappa \nabla^2 u.$$

We define the Lyapunov functional:

$$V(t) = \frac{1}{2} \int_\Omega u^2 \, dx,$$

which captures the total internal energy of the domain.

Differentiating with respect to time:

$$\dot{V}(t) = \int_\Omega u u_t \, dx = \kappa \int_\Omega u \nabla^2 u \, dx.$$

To simplify the second term, we apply Green's first identity, which states:

$$\int_\Omega u \nabla^2 u \, dx = -\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, dS.$$

Thus, the derivative becomes:

$$\dot{V}(t) = -\kappa \int_\Omega |\nabla u|^2 \, dx + \kappa \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, dS.$$

The boundary integral depends on the boundary condition:

- Neumann BCs: $\frac{\partial u}{\partial n} = 0 \Rightarrow \dot{V}(t) = -\kappa \int_\Omega |\nabla u|^2 \, dx \leq 0$,
- Robin BCs: $\frac{\partial u}{\partial n} + \beta u = 0 \Rightarrow \dot{V}(t) = -\kappa \left(\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, dS \right) \leq 0$.

This confirms energy decay in both cases, with stronger damping under Robin conditions.

4. Wave Equation Analysis

4.1. 1D Damped Case

The 1D damped wave equation is:

$$u_{tt} = c^2 u_{xx} - \gamma u_t, \quad \gamma > 0.$$

Define the energy functional:

$$V(t) = \frac{1}{2} \int_0^L \left(u_t^2 + c^2 u_x^2 \right) dx.$$

Differentiating:

$$\dot{V}(t) = -\gamma \int_0^L u_t^2 \, dx \leq 0.$$

This confirms decay of energy due to damping and hence, asymptotic Lyapunov stability.

4.2. Multidimensional Damped Case

For $\Omega \subset \mathbb{R}^n$, the damped wave equation is given by:

$$u_{tt} = c^2 \nabla^2 u - \gamma u_t,$$

where $\gamma > 0$ introduces damping into the system.
We define the total mechanical energy functional as:

$$V(t) = \frac{1}{2} \int_\Omega \left(u_t^2 + c^2 |\nabla u|^2 \right) dx,$$

which captures both kinetic and potential energy of the system.
Taking the time derivative:

$$\dot{V}(t) = \int_\Omega u_t u_{tt} \, dx + c^2 \int_\Omega \nabla u \cdot \nabla u_t \, dx. \tag{3}$$

Substitute the PDE into the first term:

$$\int_\Omega u_t u_{tt}\, dx = \int_\Omega u_t(c^2\nabla^2 u - \gamma u_t)\, dx = c^2 \int_\Omega u_t \nabla^2 u\, dx - \gamma \int_\Omega u_t^2\, dx.$$

To evaluate $\int_\Omega u_t \nabla^2 u\, dx$, we use Green's first identity again:

$$\int_\Omega u_t \nabla^2 u\, dx = -\int_\Omega \nabla u_t \cdot \nabla u\, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, dS.$$

Therefore, combining terms:

$$\dot{V}(t) = -\gamma \int_\Omega u_t^2\, dx + c^2 \left(-\int_\Omega \nabla u_t \cdot \nabla u\, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, dS \right) \tag{4}$$

$$+ c^2 \int_\Omega \nabla u \cdot \nabla u_t\, dx. \tag{5}$$

The interior gradient terms cancel:

$$-\int_\Omega \nabla u_t \cdot \nabla u\, dx + \int_\Omega \nabla u \cdot \nabla u_t\, dx = 0.$$

Thus, we arrive at:

$$\dot{V}(t) = -\gamma \int_\Omega u_t^2\, dx + c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, dS.$$

Under homogeneous Neumann ($\frac{\partial u}{\partial n} = 0$) or Dirichlet ($u = 0 \Rightarrow u_t = 0$ on $\partial\Omega$) boundary conditions, the boundary term vanishes. Therefore,

$$\dot{V}(t) = -\gamma \int_\Omega u_t^2\, dx \leq 0,$$

which confirms energy dissipation and asymptotic Lyapunov stability.

5. Numerical Implementation in Two Dimensions

In order to test the stability of the Wave and Heat equations, a backward Euler numerical method was implemented to model the time-varying behavior of both systems using methods from [Taylor(2022)]. Both systems were modeled with Neumann boundary conditions such that there is no flux along the boundaries (heat / energy stays trapped within the system).

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0,$$

5.1. 2D Heat Equation

The 2D heat equation governs diffusion phenomena and is expressed as:

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where κ is the thermal diffusivity constant. Using backward Euler discretization in time and second-order central differences in space, the scheme becomes:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \kappa \left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} \right).$$

This can be rearranged into a linear system:

$$(\mathbf{I} - r\mathbf{L})\,\mathbf{u}^{n+1} = \mathbf{u}^n, \quad \text{where} \quad r = \frac{\kappa \Delta t}{\Delta x^2},$$

and $\mathbf{L}$ represents the discrete Laplace operator over the grid.

5.2. 2D Damped Wave Equation

The 2D damped wave equation models wave propagation with viscous damping:

$$\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where $\gamma > 0$ controls damping.
Using a central difference method in time and space, the update rule is:

$$u_{i,j}^{n+1} = (2 - \gamma \Delta t) u_{i,j}^n - (1 - \gamma \Delta t) u_{i,j}^{n-1} + \left(\frac{c \Delta t}{\Delta x} \right)^2 \left(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n \right).$$

The transient behavior of both the 2D wave and heat equations was modeled using an initial Gaussian disturbance at the center of the x, y-plane. The initial and final states for the heat equation are shown in Figures 1 and 2, and for the wave equation in Figures 3 and 4. These results visually confirm the Lyapunov stability of these PDEs. In addition, GIFs were created which show the transient behavior of both of these systems. `https://drive.google.com/drive/folders/1doLs9_SG7jFNHVB14FjTXqlm0r6ZKJ6w?usp=drive_link`

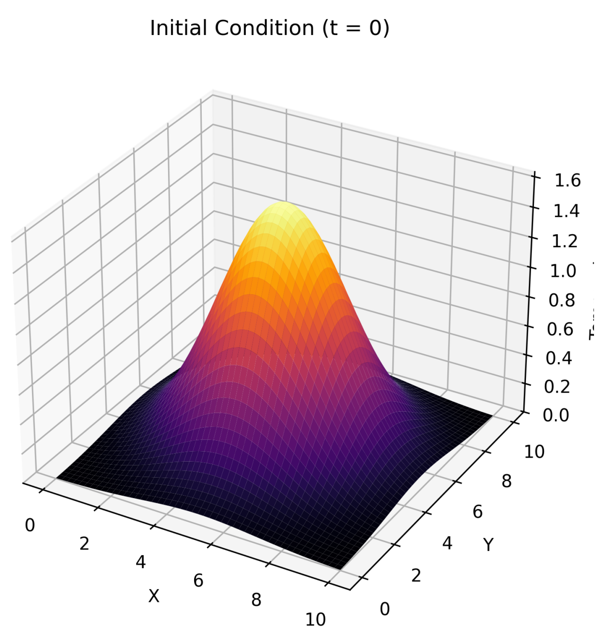

Figure 1: Initial state of 2D heat equation

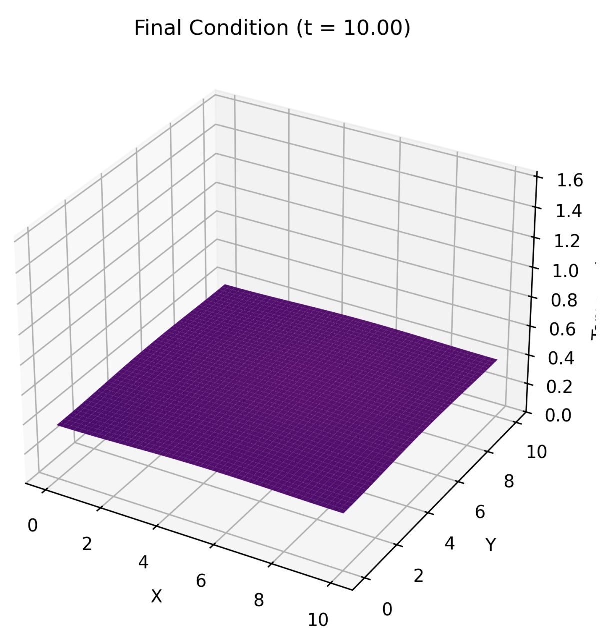

Figure 2: Final state of 2D heat equation

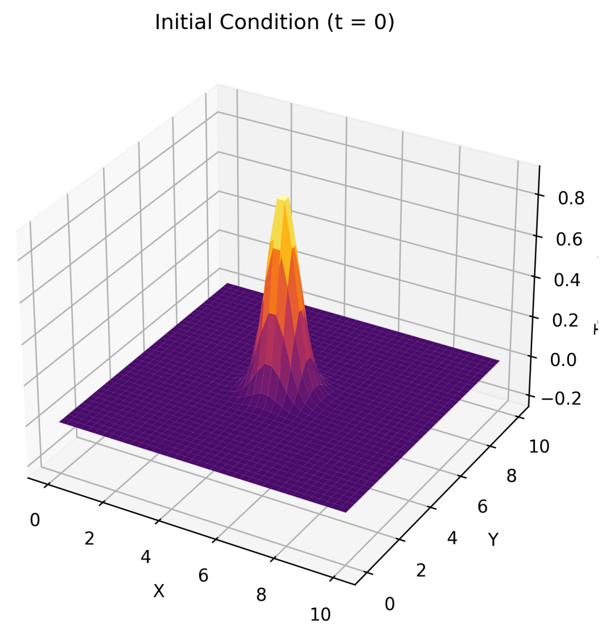

Figure 3: Initial state of 2D damped wave equation

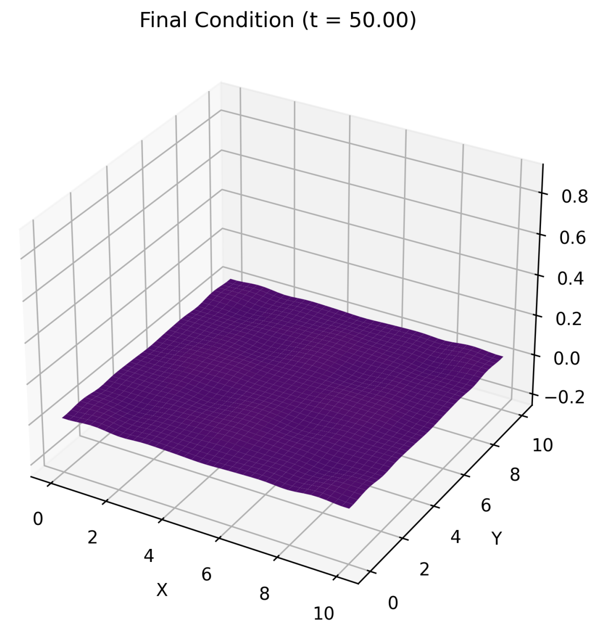

Figure 4: Final state of 2D damped wave equation

6. Conclusion

This report demonstrates that Lyapunov functionals provide a framework for analyzing the stability of PDEs. For the heat equation, the L^2 norm captures diffusion-driven energy decay. In contrast, the wave equation requires energy-based functionals reflecting both kinetic and potential energy. Boundary conditions influence stability, and functional analysis remains crucial in infinite-dimensional systems.

References

[Lessard(2022)] Lessard, L., "Lecture 13: Norms and Input-Output Stability," ECE 717: Advanced Linear Systems, 2022. Available at `https://laurentlessard.com/teaching/ece717/lectures/13.%20norms%20and%20BIBO%20stability.pdf`.

[Baker and Bergen(1969)] Baker, R. A., and Bergen, A. R., "Lyapunov Stability and Lyapunov Functions of Infinite Dimensional Systems," *IEEE Transactions on Automatic Control*, Vol. 14, No. 4, 1969, pp. 325–327. https://doi.org/10.1109/TAC.1969.1099292.

[Strauss(2022)] Strauss, W. A., "Heat Equation Notes," https://example.edu/heat-eqn-notes, 2022. Based on lecture derivations of the heat equation and separation of variables.

[Taylor(2022)] Taylor, S., "Numerical Methods for the Heat Equation," , 2022. URL https://www.math.lsa.umich.edu/~krasny/REU2022/taylor-report.pdf, university of Michigan REU Final Report.