Lindenmayer Systems

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- Designed to explore organic growth
- Creates complex shapes from simple rules
- For example, with the rules:

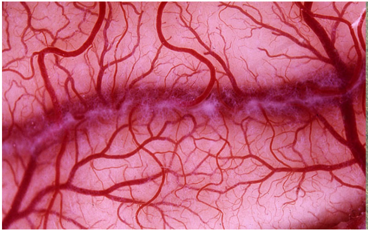
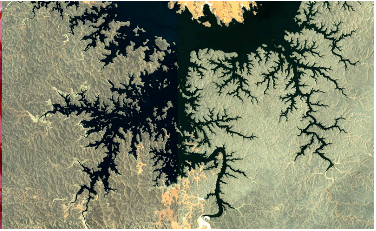
$$0em3ex1.3ex0 \to 1, \qquad 1 \to 01$$

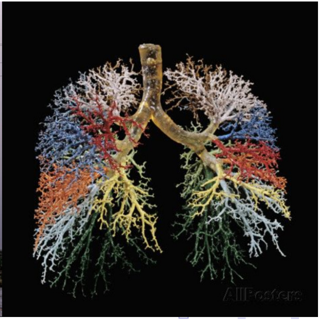
we have this string transformation:

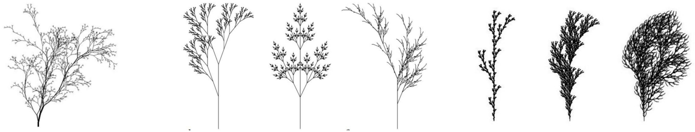
$$0em2.7ex1ex01101 = 0\ 1\ 1\ 0\ 1 \longrightarrow 1\ 01\ 01\ 1\ 01 = 10101101$$

- All operations take place *simultaneously*
- L-systems are *term-rewriting systems*

All these lovely fractal trees in the city of Melbourne, Australia, in the winter.
Once you start looking, you can't stop seeing them



These are all examples from *The Algorithmic Beauty of Plants* by Aristid Lindenmayer and Przemysław Prusinkiewicz, available at
`http://algorithmicbotany.org/papers/abop/abop.pdf`

- Sets of rules describe how one string of symbols will be expanded to a new string
- Each symbol corresponds to a *turtle graphics* instruction:
  - `F`: Move forward
  - `-`: Turn left
  - `+`: Turn right
  - `[`: Memorize current position and heading
  - `]`: Move to most recently memorized position and heading

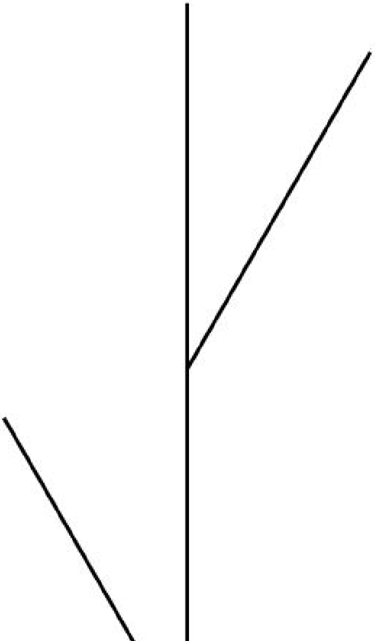
For example, this sequence of symbols:

`F[+F]F[-F]F`

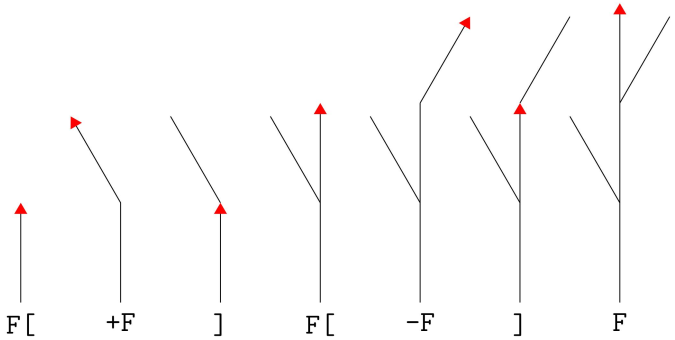
has this output:

We can clearly alter the output by changing the angle of the turns, and the length of the move forward.

In this example, the angle is 26°



This shows how the turtle draws a path with branches:



It's all done from the point of view of the turtle. A side of *Koch's snowflake* can be computed by the rules:

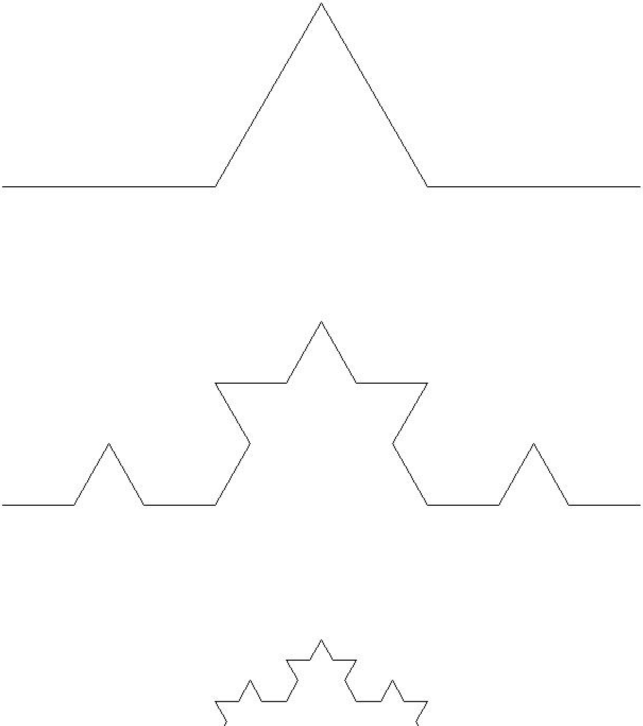
- Start: F
- Modify: F $\rightarrow$ F+F--F+F (with turns of 60°)
- At every further step, each F is replaced by the string F+F--F+F

- The second iteration produces
  `F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F`

- The third iteration produces:
  `F+F--F+F+F+F--F+F--F+F--F+F+F+F--F+F+F+F--F+F+F+F--`

- and so on...

First iteration:
Second iteration:
Third iteration:
Fourth iteration:



Remember the `F` $\to$ `F+F--F+F` iteration? How many symbols are in the n^{th} string?

Let f_n be the number of `F`'s, and k_n be the number of other symbols in the n^{th} string. We have:

$$\begin{aligned} f_{n+1} &= 4f_n, \quad f_1 = 4 \\ k_{n+1} &= 4f_n + k_{n-1}, \quad k_1 = 4 \end{aligned}$$

It follows immediately that

$$f_n = 4^n \text{ and } k_n = 4 + 4^2 + 4^3 + \cdots + 4^n = \frac{4}{3}(4^n - 1).$$

The total length is thus

$$f_n + k_n = 4^n + \frac{4}{3}(4^n - 1) = \frac{1}{3}(7(4^n) - 4).$$

```
import turtle as t  # "turtle" is a turtle graphics mod
```

```
# Lindenmayer system (a) from ABOP figure 1.24(a), p 25
def edgetree(level, size, angle):
    if (level==0):
t.fd(size)
    else:
edgetree(level-1, size/3, angle)
t.lt(angle)
edgetree(level-1, size/3, angle)
t.bk(size/3)
t.rt(angle)
edgetree(level-1, size/3, angle)
t.rt(angle)
edgetree(level-1, size/3, angle)
t.bk(size/3)
t.lt(angle)
edgetree(level-1, size/3, angle)
```

Racket is a modern lisp; descended from Scheme.

```
;;   F -> F[+F]F[-F]F

(require furtle) ;; furtle is a simple but fast turtle
(: ltree_b (-> Real Real Real TurtleF))  ;; typed Racke
(define (ltree level size angle)
  (if (= level 0)
      (turtles (forward size))
      (turtles (ltree (- level 1) (/ size 3) angle)
       (save)
       (left angle) (ltree (- level 1) (/ size 3) angle
       (restore)
       (ltree (- level 1) (/ size 3) angle)
       (save)
       (right angle) (ltree (- level 1) (/ size 3) angl
       (restore)
       (ltree (- level 1) (/ size 3) angle))))
```

Fractal dimension can be defined by the "box-counting measure":

Suppose our picture is subdivided into boxes of size b, and $N(b)$ boxes are needed to cover the shape. Its dimension can be defined as

$$\lim_{b \to 0} \frac{\log(N(b))}{\log(1/b)}.$$

For example, take a curve of length k. As $b \to 0$, we would find that

$$N(b) \to \frac{k}{b}.$$

Thus

$$\lim_{b \to 0} \frac{\log(N(b))}{\log(1/b)} = \lim_{b \to 0} \frac{\log(k/b)}{\log(1/b)} = \lim_{b \to 0} 1 - \frac{\log(k)}{\log(b)} = 1.$$

In general a fractal will have a non-integer dimension between 1 and 2.

Thanks, folks!