MATH 137 Assignment 3 Due by: **11 a.m.**, Friday, October 9

Submit your assignment in the drop box corresponding to the class section in which you are registered. Clearly put your name and I.D. number at the top of the first page and **underline your last name.** Submit your solutions in the same order as that of the questions appearing herein. Feel free to copy and use the assignment templates that appear on UW-ACE.

Attempt the questions on your own, but if you get help or collaborate with someone, then acknowledge the names of those who helped you. However, any outright copying of assignments will be reported.

Your solutions must have legible handwriting, and must be presented in clear, concise and logical steps that fully reveal what you are doing. Use of good English sentences to explain your reasoning is highly encouraged.

Questions such as those to be handed in, as well as those that are recommended, may appear on exams. To avoid frustration and disappointment, **get started on your assignment early**.

The following problems from Stewart's book are **recommended**. Make sure you know how to solve problems such as these.

- Section 2.2, pp. 96-99 # 23

  Note. If you know how to factor, this limit can be determined precisely.

- Section 2.3, pp. 106-108 # 31(c), 32(c), 33, 35, 38, 39, 43, 44, 47, 61

- Section 2.5, pp. 128-130 # 41, 45, 46, 47, 63

- Problems Plus, pp. 170-171 # 1, 7

  Hint for # 1. Multiply the expression by $\frac{\sqrt[3]{x}^2+\sqrt[3]{x}+1}{\sqrt{x}+1}$, and then to get back to where you started, multiply by the reciprocal of the above.

Hand in your solutions to the following 11 problems including MAPLE Lab # 1.

1. It can be shown that $\frac{\sin x}{x} \to 1$ as $x \to 0$. If you have not yet seen a proof of this fact, you will see it soon. This fact is one of the more important applications of the squeeze principle. Use this fact to show that $\dfrac{1-\cos x}{x} \to 0$

as $x \to 0$.
Hint. Obviously $\dfrac{1-\cos x}{x} = \dfrac{1-\cos x}{x}\dfrac{1+\cos x}{1+\cos x}$.
Now do some multiplying, use a bit of trigonometry and then some basic limit rules in conjunction with the continuity of the trigonometric functions.

2. Use the limit laws along with the basic limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to evaluate $\lim_{x\to 0} \dfrac{x^2(3+\sin x)}{(x+\sin x)^2}$.

3. With a bit of thinking, evaluate $\displaystyle\lim_{x\to 0} \frac{x}{|2x-1|-|2x+1|}$.

   Hint. When x is close to zero, what is the sign of $2x-1$ and of $2x+1$?

4. Find $\displaystyle\lim_{x\to 0} \arctan x \cos(1/x)$, and justify your method.

5. Let a be any real number and let

$$f(x) = \begin{cases} 1-x^2 & \text{when } x < a \\ x^2 & \text{when } x \geq a \end{cases} .$$

   Find the values of a that make f a continuous function on all of $\mathbb{R}$. Then, for each suitable value of a, sketch the resulting continuous function.

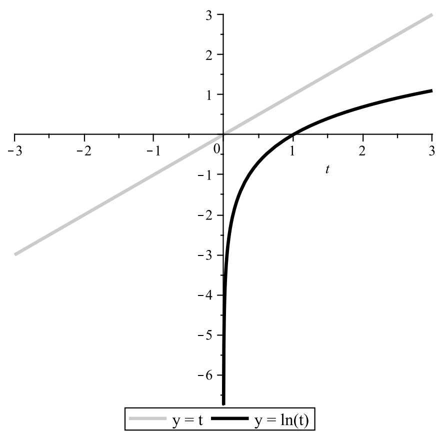
6. Let f be a function such that

$$\sin(\pi x) \leq f(x) \leq \frac{1}{4x(1-x)} \text{ for all } x \text{ in the interval } (0,1)$$

   Find $\lim_{x\to 1/2} f(x)$ and explain your reasoning.
   Decide if f continuous at $a = 1/2$ and explain your reasoning.

7. Here's a little tip. If you want to prove that some $f(x) \to 0$ as $x \to a$, just check that the absolute value $|f(x)| \to 0$ as $x \to a$. This can come in very handy when you want to use the squeeze principle, because you automatically have the lower squeeze, namely $0 \leq |f(x|$. Keeping this in mind, take the following steps (a) – (d) to prove that $x \ln x \to 0$ as $x \to 0^+$.

   (a) We can see from the graphs below and take it as given that $0 \leq \ln t < t$ when $t \geq 1$.



If $0 < x \leq 1$, prove that $|x \ln x| \leq 1$.
Hint. First explain why $|x \ln x| = x \ln\left(\frac{1}{x}\right)$, then observe that $\frac{1}{x} \geq 1$, then use the inequality given above.

(b) If $0 < x \leq 1$, use part (a) to explain why $|\sqrt{x} \ln(\sqrt{x}\,)| \leq 1$.

(c) Using the fact $x \ln x = \sqrt{x}^2 \ln\left(\sqrt{x}^2\right)$ along with part (b), show that $|x \ln x| \leq 2\sqrt{x}$ when $0 < x \leq 1$.

(d) Prove that $x \ln x \to 0$ as $x \to 0^+$.

(e) After observing that $x^x = e^{\ln(x^x)}$ use part (d) to find $\lim_{x \to 0^+} x^x$.
Explain the role that continuity plays in the solution to part (e).

8. Let $f(x) = x^3 + bx^2 + cx + d$ be a general cubic polynomial with the coefficient in front of x^3 adjusted to be a 1.

   (a) Explain why $f(x) > 0$ when $x > 0$ and very large, and why $f(x) < 0$ when $x < 0$ and very large.
   Hint. Rewrite f as $f(x) = x^3(1+b/x+c/x^2+d/x^3)$ and then inspect the sign of the part in brackets when x is very large.

   (b) Use the above information to show that every cubic must cut the x-axis in at least one place, i.e. prove every cubic has a real root.

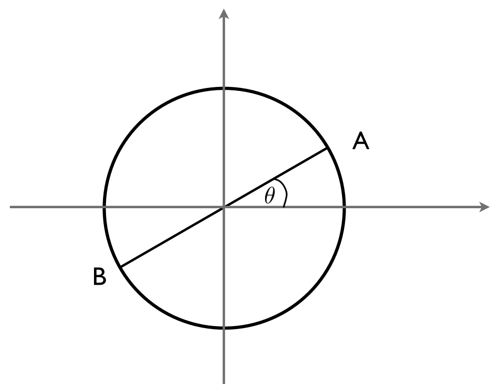
   (c) Does every degree 4 polynomial cut the x-axis? Explain your answer.

9. First put you calculator in radian mode.

   (a) Prove that the equation $x^2 = \cos x$ has a positive root.

(b) Use the bisection method, along with your calculator, to estimate the positive root of the above equation with error that is at most $1/16$. Be sure to clearly show the steps of your bisection method.

10. Two points on the earth's equator are called *antipodal* if they are diametrically across from each other, as shown.



A and B are antipodal

Assuming continuity of temperature variations as we travel around the equator, prove that at there are two antipodal points on the equator having the same temperature.
Hint. Let $t(\theta)$ be the temperature at a general point A on the equator as a function of the angle θ that the point makes with respect to some axis as shown above. Now study the function $f(\theta) = t(\theta) - t(\theta + \pi)$.

11. Complete MAPLE Lab #1, and attach your printed output, along with your answers to the related questions, to the end of this assignment.