The use of cubic splines in the solution of two-point boundary value problems

By D. J. Fyfe*

A method of obtaining an approximate solution to two-point boundary value problems by use of of cubic splines, which was suggested by Bickley (1968) is developed. Error analysis is carried out and from the results a method of deferred correction is obtained. The use of unequal intervals is considered. An algorithm for computing the solution to prescribed accuracy in the case of equal intervals is described and a numerical example given.

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1. Bickley (1968) suggested that cubic splines may be used to obtain an approximate solution to the two-point boundary value problem:

$$y'' + p(x)y' + q(x)y = r(x), \quad a \leqslant x \leqslant b \tag{1}$$

with boundary conditions

$$\alpha_0 y + \beta_0 y' = \gamma_0 \quad \text{at} \quad x = a \tag{2}$$

and
$$\alpha_n y - \beta_2 y' = \gamma_n \quad \text{at} \quad x = b. \tag{3}$$

This paper examines this method, and error estimates are obtained which enable a deferred correction to be made. The possibility of using unequal intervals is also considered.

In Section 2, the relevant results of Curtis and Powell (1967), are quoted for the convenience of the reader. In Section 3, the method for the solution of equation (1) is developed whilst in Section 4 the unequal interval case is considered. In Section 5 a procedure is described which enables results to be obtained to a pre-assigned accuracy, with the minimum of computation, and in Section 6 some numerical results are given and discussed.

2. Let $f(x)$ be a function with continuous derivatives in the range $a \leqslant x \leqslant b$. Divide the range into n intervals by inserting knots at the points $x_0, x_1, \ldots, x_n$ where $a = x_0 < x_1 < \ldots . < x_n = b$, then $s(x)$ is a cubic spline interpolating function for $f(x)$ if

(i) $s(x)$ is a cubic polynomial in each interval $[x_i, x_{i+1}]$,

(ii) $s(x_i) = f(x_i)$, $i = 0(1)n$,

and (iii) $s'(x)$ and $s''(x)$ are continuous.

The third derivative $s'''(x)$ will be discontinuous at the knots x_i, $i = 1(1)n - 1$. It can be shown that if $s(x)$ is a cubic spline it must have the form

$$s(x) = a_0 + b_0(x - x_0) + \tfrac{1}{2}c_0(x - x_0)^2 + \tfrac{1}{6}\sum_{k=0}^{n-1} d_k(x - x_k)^3_+ + , a \leqslant x \leqslant 6, \tag{4}$$

where

$$z_+ \begin{cases} = z \ , \ z \geqslant 0 \\ = 0 \ , \ z < 0. \end{cases}$$

Let us now assume that the knots x_i are equally spaced in $[a, b]$, at interval h so that $x_i = a + ih$. In this case the following relationships can be obtained:

$$h[s'(x_{i-1}) + 4s'(x_i) + s'(x_{i+1})] = 3[f(x_{i+1}) - f(x_{i-1})], \tag{5}$$

$$h^2 s''(x_i) = 6[s(x_{i+1}) - s(x_i)] - 2h[2s'(x_i) + s'(x_{i+1})], \tag{6}$$

and
$$h^3 s'''(x_{i+}) = 12[s'(x_i) - s(x_{i+1})] + 6h[s'(x_i) + s'(x_{i+1})], \tag{7}$$

where $s'''(x_{i+})$ denotes the value of $s'''(x)$ in (x_i, x_{i+1}).

Using operator notation (5) may be written in the form

$$(E^{-1} + 4 + E)hs'(x_i) = 3(E - E^{-1})f(x_i)$$

and hence

$$hs'(x_i) = \left\{\frac{3(E - E^{-1})}{E^{-1} + 4 + E}\right\} f(x_i).$$

If we now put $E = e^{hD}$ and expand in powers of hD, we obtain

$$s'(x_i) = f'(x_i) - \frac{1}{180}h^4 f^{v}(x_i) + O(h^6). \tag{8}$$

Similarly (6) and (7) give

$$s''(x_i) = f''(x_i) - \frac{1}{12}h^2 f^{iv}(x_i) + \frac{1}{360}h^4 f^{vi}(x_i) + O(h^6) \tag{9}$$

and

$$s'''(x_{i+}) = f'''(x_i) + \frac{1}{2}hf^{iv}(x_i) + \frac{1}{12}h^2 f(x_i) - \frac{1}{360}h^4 f^{vii}(x_i) - \frac{1}{1440}h^5 f^{viii}(x_i) + O(h^6). \tag{10}$$

From (10) we now obtain

$$\frac{1}{2}[s'''(x_{i+}) + s'''(x_{i-})] = f'''(x_i) + \frac{1}{12}h^2 f^{v}(x_i) + O(h^3) \tag{11}$$

and
$$s'''(x_{i+}) - s'''(x_{i-}) = hf^{iv}(x_i) - \frac{1}{720}h^5 f^{viii}(x_i) + O(h^7) \tag{12}$$

(12) gives a very good estimate of $hf^{iv}(x_i)$ and from (4) we see that

$$s'''(x_{i+}) - s'''(x_{i-}) = d_i = hf^{iv}(x_i) + O(h^5). \tag{13}$$

* *Department of Mathematics, Woolwich Polytechnic, London, SE*18

We now define $e(x) = f(x) - s(x)$ and substitute (8), (9), (11) and (12) in the Taylor series expansion of $e(x_i + \theta h)$ obtaining

$$e(x_i + \theta h) = \frac{\theta^2(\theta - 1)^2}{24} h^4 f^{iv}(x_i) + \frac{\theta(\theta^2 - 1)(3\theta^2 - 2)}{360} h^5 f^{v}(x_i) + O(h^6) \quad (14)$$

for $0 \leqslant \theta \leqslant 1$.

3. The number of coefficients in (4) is $n + 3$ and Bickley (1968) has shown that satisfying (1) by colocation at the $n + 1$ knots together with equations (2) and (3) gives precisely the requisite number of equations.

Differentiating (4) twice we obtain

$$s'(x) = b_0 + c_0(x - x_0) + \tfrac{1}{2}\sum_{k=0}^{n-1} d_k(x - x_k)_+^2 \quad (15)$$

and $$s''(x) = c_0 + \sum_{k=0}^{n-1} d_k(x - x_k)_+. \quad (16)$$

Substituting $s(x)$, $s'(x)$ and $s''(x)$ into (1) gives, for satisfaction at the ith knot,

$$c_0 + \sum_{k=0}^{n-1} d_k(x_i - x_k)_+ + p_i\Big[b_0 + c_0(x_i - x_0) + \tfrac{1}{2}\sum_{k=0}^{n-1} d_k(x_i - x_k)_+^2\Big] + q_i\Big[a_0 + b_0(x_i - x_0) + \tfrac{1}{2}c_0(x_i - x_0)^2 + \tfrac{1}{6}\sum_{k=0}^{n-1} d_k(x_i - x_k)_+^3\Big] = r_i \quad (17)$$

where $p_i = p(x_i)$ etc.

Satisfying the boundary conditions leads to

$$\alpha_0 a_0 + \beta_0 b_0 = \gamma_0, \quad (18)$$

and $$\alpha_n\Big[a_0 + b_0(x_n - x_0) + \tfrac{1}{2}c_0(x_n - x_0)^2 + \tfrac{1}{6}\sum_{k=0}^{n-1} d_k(x_n - x_k)^3\Big] - \beta_n\Big[b_0 + c_0(x_n - x_0) + \tfrac{1}{2}\sum_{k=0}^{n-1} d_k(x_n - x_k)^2\Big] = \gamma_n. \quad (19)$$

(19), (17) with $i = n(-1)0$ and (18) produce a system of $n + 3$ equations in the unknowns $a_0, b_0, c_0, d_0, d_1, \ldots d_{n-1}$, which is upper triangular with a single sub-diagonal. The forward elimination has only one multiplier at each step and the solution is thus simple.

Let the spline obtained by solving (17), (18) and (19) be $s^{(0)}(x)$. We wish to find the spline which satisfies

$$(s_i'' + e_i'') + p_i(s_i' + e_i') + q_i s_i = r_i, \quad (20)$$

where e'_i and e''_i are given by (8) and (9). We have obtained $s^{(0)}(x)$ which satisfies

$$s_i^{(0)''} + p_i s_i^{(0)'} + q_i s_i^{(0)} = r_i. \quad (21)$$

If we now let $\epsilon(x) = s(x) - s^{(0)}(x)$ be a correction spline, we have, by subtracting (21) from (20)

$$\epsilon_i'' + p_i\epsilon_i' + q_i\epsilon_i = -e_i'' - p_i e_i'. \quad (22)$$

Substituting for e_i' and e_i'', we obtain

$$\epsilon_i'' + p_i\epsilon_i' + q_i\epsilon_i = -\frac{1}{12}h^2 y_i^{iv} + \frac{1}{360}h^4 y_i^{vi} - \frac{1}{180}p_i h^4 y_i^{v} + O(h^6)$$

in which the major term in the right-hand side is the first.

However, using equation (13) we have a good estimate for $-\frac{1}{12}h^2 y_i^{iv}$ at each of the internal knots, namely $-\frac{1}{12}hd_i$. Using linear extrapolation, estimates of $-\frac{1}{12}h^2 y^{iv}$ and $-\frac{1}{12}h^2 y^{iv}$ can be obtained to sufficient accuracy, for example

$$-\frac{1}{12}h^2 y_0^{iv} = -\frac{1}{12}h^2(2y_1^{iv} - y_2^{iv}) + \frac{1}{12}h^4 y_0^{vi} = -\frac{1}{12}h(2d_1 - d_2) + O(h^4).$$

The two boundary conditions become

$$\alpha_0\epsilon_0 + \beta_0\epsilon_0' = -\frac{1}{180}p_0 h^4 y_0^{v} + O(h^6)$$

and $$\alpha_n\epsilon_n - \beta_n\epsilon_n' = -\frac{1}{180}p_n h^4 y_n^{v} + O(h^6).$$

We thus obtain the correction spline $\epsilon(x)$ by solving the equations

$$\epsilon_i'' + p_i\epsilon_i' + q_i\epsilon_i = \left\{\begin{array}{ll} -\frac{1}{12}h(2d_1 - d_0), & i = 0 \\ -\frac{1}{12}hd_i, & i = 1(1)n-1 \\ -\frac{1}{12}h(2d_{n-1} - d_{n-2}) & i = n, \end{array}\right\} \quad (23)$$

with boundary conditions

$$\alpha_0\epsilon_0 + \beta_0\epsilon_0' = 0. \quad (24)$$

and $$\alpha_n\epsilon_n - \beta_n\epsilon_n' = 0. \quad (25)$$

Since $\epsilon(x)$ is a cubic spline it will have the form of (4) and so the solution of (23), (24), (25) is just the same as the solution of (17), (18), (19) with a new right-hand side. This will involve very little additional computation. A better approximation to $s(x)$ is then easily obtained from $s^{(0)}(x) + \epsilon(x)$.

4. The use of unequal intervals is now considered in a manner similar to that of Curtis and Powell (1967). We assume that for $a \leqslant x \leqslant x_k$ an interval $2h$ is used and for $x_k \leqslant x \leqslant b$ an interval h is used. Let $s_+(x)$ and $s_-(x)$ be the splines that would have been obtained if intervals h and $2h$ had been used throughout the range.

For $x \geqslant x_k$, $s(x)$ differs from $s_+(x)$ by a spline $\sigma_+(x)$ which is zero at each knot and whose effect diminishes as x increases. Curtis and Powell show that for $x \geqslant x_k$, $s(x) = s_+(x) + \sigma_+(x)$, where

$$\sigma_+(x) = \lambda_+[h^2(x-x_k) - \sqrt{3}h(x-x_k)^2 + (\sqrt{3}-1)(x-x_k)^3 + 2\sqrt{3}\sum_{j=1}^{\infty}(-2+\sqrt{3})^j(x-x_k-jh)_+^3] \quad (26)$$

and for $x \leqslant x_k$, $s(x) = s_-(x) + \sigma_-(x)$, where

$$\sigma_-(x) = \lambda_-[4h^2(x-x_k) - 2\sqrt{3}(x-x_k)^2 + (\sqrt{3}-1)(x-x_k)^3 - 2\sqrt{3}\sum_{j=1}^{\infty}(-2+\sqrt{3})^j(-x+x_k-2jh)_+^3]. \quad (27)$$

λ_+ and λ_- are determined from the continuity conditions and are given by

$$\lambda_+ = \frac{1}{12\sqrt{3}}hf^{iv}(x_k) - \frac{1}{36}h^2f^{v}(x_k) + O(h^3) \quad (28)$$

and $$\lambda_- = \frac{1}{48\sqrt{3}}hf^{iv}(x_k) + \frac{1}{72}h^2f^{v}(x_k) + O(h^3). \quad (29)$$

We have for $x \geqslant x_k$

$$e(x) = f(x) - s(x) = f(x) - s_+(x) - \sigma_+(x) = e_+(x) - \sigma_+(x).$$

Thus $e'(x) = e'_+(x) - \sigma'_+(x)$ and substituting from equations (8), (26) and (28) we obtain

$$e'(x_k) = -\frac{1}{12\sqrt{3}}h^3f^{iv}(x_k) + \frac{1}{30}h^4f^{v}(x_k) + O(h^5). \quad (30)$$

Similarly

$$e''(x_k) = \frac{1}{4}h^2f^{iv}(x_k) - \frac{\sqrt{3}}{18}h^3f^{v}(x_k) + O(h^4) \quad (31)$$

and $$d_k = \frac{15-\sqrt{3}}{8}hf^{iv}(x_k) - \frac{\sqrt{3}}{4}h^2f^{v}(x_k) + O(h^3). \quad (32)$$

In general for $j \geqslant 1$

$$e'(x_{k+j}) = \frac{1}{180}h^4f^{v}(x_{k+j}) - \frac{(\sqrt{3}-2)^j}{12\sqrt{3}}h^3f^{iv}(x_k) + \frac{(\sqrt{3}-2)^j}{36}h^4f^{v}(x_k) + O(h^5), \quad (33)$$

$$e''(x_{k+j}) = \frac{1}{12}h^2f^{iv}(x_{k+j}) + \frac{(\sqrt{3}-2)^j}{6}h^2f^{iv}(x_k) - \frac{\sqrt{3}(\sqrt{3}-2)^j}{18}h^3f^{v}(x_k) + O(h^4), \quad (34)$$

and $$d_{k+j} = hf^{iv}(x_{k+j}) + (\sqrt{3}-2)^jhf^{iv}(x_n) - \frac{(\sqrt{3}-2)^j}{\sqrt{3}}h^2f^{v}(x_k) + O(h^3). \quad (35)$$

Also

$$e'(x_{k-j}) = \frac{4}{45}h^4f^{v}(x_{k-j}) - \frac{(\sqrt{3}-2)^j}{12\sqrt{3}}h^3f^{iv}(x_k) - \frac{(\sqrt{3}-2)^j}{18}h^4f^{v}(x_k) + O(h^5), \quad (36)$$

$$e''(x_{k-j}) = \frac{1}{3}h^2f^{iv}(x_{k-j}) - \frac{(\sqrt{3}-2)^j}{12}h^2f^{iv}(x_k) - \frac{\sqrt{3}(\sqrt{3}-2)^j}{6}h^3f^{v}(x_k) + O(h^4), \quad (37)$$

and $$d_{k-j} = 2hf^{iv}(x_{k-j}) - \frac{(\sqrt{3}-2)^j}{4}hf^{iv}(x_k) - \frac{(\sqrt{3}-2)^j}{2\sqrt{3}}h^2f^{v}(x_k) + O(h^3). \quad (38)$$

As a consequence of these results, we see that the error in the first derivatives near a change of interval size is of order h^3, whereas for the equal interval case it is of order h^4. Secondly, there is an h^3 term in $e''(x_j)$ which is not present in the equal interval case and thirdly close to an interval change $d_i = hf^{iv}(x_i) + O(h^2)$, whereas away from interval changes $d_i = hf^{iv}(x_i) + O(h^5)$. These facts mean that the deferred correction described in Section 3 will not be as effective near an interval change and at best the values obtained for $s(x_i)$ will have errors of order h^3.

5. In this section a procedure is described which will produce an equal interval spline for use as an interpolating spline over the whole range $a \leqslant x \leqslant b$. The spline will give results to a prescribed accuracy at any point in the range and will involve the minimum convenient number of knots consistent with such accuracy.

The choice of interval is determined by two separate considerations. First, we must ensure that the values at the knots are determined to sufficient accuracy, and secondly, assuming that the values at the knots are correct, that the interpolation error at an interval point of any interval is sufficiently small. These two requirements are not necessarily related although it is often found that when the second condition is satisfied the first will also hold.

Let us consider the second requirement. We assume therefore that we have obtained a spline $s(x)$ which satisfies the conditions in Section 2. Then the error $e(x_i + \theta h)$ is given by (14). Thus we have

$$|e(x_i+\theta h)| \leqslant \max_{\substack{x_i \leqslant x \leqslant x_{i+1} \\ 0 \leqslant \theta \leqslant 1}} \left\{\frac{\theta^2(\theta-1)^2}{24}h^4|f^{iv}(x)|\right\} = \frac{1}{384}h^4 \max_{x_i \leqslant x \leqslant x_{i+1}} |f^{iv}(x)|. \quad (39)$$

At the internal knots we have from (13) $f^{iv}(x_i) \doteqdot \frac{d_i}{h}$.

Thus we may take as an estimate of the maximum error in $x_i \leqslant x \leqslant x_{i+1}$,

$$\phi_i = \frac{1}{384}h^3 \max\{|d_i|, |d_{i+1}|\} \quad i = 1(1)n-2, \quad (40)$$

and let $\phi_0 = \frac{1}{384}h^3|d_1|$ and $\phi_{n-1} = \frac{1}{384}h^3|d_{n-1}|$.

Therefore if we let $\phi = \max_{i=1(1)n-1}\{\phi_i\}$, then if the maximum error is to be less than ϵ, the interval control test will be

$$\phi = \frac{h^3}{384}\max_{i=1(1)n-1}\{|d_i|\} < \epsilon. \quad (41)$$

We now consider the problem of ensuring that the values at the knots have been determined to sufficient accuracy. Since we are solving a boundary value problem and so have information about the behaviour of the solution at the ends of the range, we expect the errors in the values at the knots to be greater away from the boundaries. Equation (41) necessitates the use of an iterative scheme in which a value for h is guessed, the values d_i and hence the error estimate calculated, and from this estimate a new value of h is determined and the process repeated. In order to estimate the accuracy of the values at the knots we make use of the values obtained for two successive values of h.

Let h_1 and h_2 be two intervals such that certain x_i are knots in each partition of the range. Let the values obtained from the two splines at one of these knots be $y\{h_1\}$ and $y\{h_2\}$. Since, from Section 3, the error is of order h^4 we have, letting y_i be the exact solution,

$$y_i = y_i\{h_1\} + B_i h_1^4 + O(h_1^6)$$

and

$$y_i = y_i\{h_2\} + B_i h_2^4 + O(h_2^6).$$

Subtracting we obtain

$$B_i \doteqdot -\frac{y_i\{h_1\} - y_i\{h_2\}}{h_1^4 - h_2^4}$$

and hence let

$$\eta_i = |e_i\{h_2\}| \doteqdot \left|\frac{h_2^4}{h_1^4 - h_2^4}[y_i\{h_1\} - y_i\{h_2\}]\right|. \tag{42}$$

Thus if we require that the maximum error at the knots is ϵ_k (not necessarily the same as ϵ) we use as interval control test

$$\eta = \max\{\eta_i\} < \epsilon_k. \tag{43}$$

We apply this test at the quarter points of the range which requires that h be chosen so that the number of knots is a multiple of 4. If in any example it is considered desirable to apply test (43) at additional points this can be done by ensuring that the required knots occur in two successive approximations.

Test (43) implies that even if the guessed value of h used initially satisfies test (41), a second value of h must be used in order that (43) can be applied.

The computational procedure is therefore as follows: guess a value of h and evaluate $s(x)$ as described in Section 3. Evaluate ϕ from (41) and if $\phi > \epsilon$, evaluate a new value of h from:

$$h_{r+1} = (\epsilon/\phi)^{1/4} h_r. \tag{44}$$

This new value of h is now modified until $\dfrac{b-a}{h}$ is a multiple of 4. Then the coefficients of a new spline are found and the test (41) reapplied. If this is still not satisfied a new h is found as above.

When an h satisfying (41) is found test (43) is applied, using current and previous values of h. If test (43) fails a new value of h is determined from

$$h_{r+1} = (\epsilon_k/\eta)^{1/4} h_r. \tag{45}$$

This interval is also modified so that $\dfrac{b-a}{h}$ is a multiple of 4. This test is then reapplied as necessary.

When an interval satisfying both tests is obtained the spline then calculated can be used to produce the solution $y = s(x)$ for any x in $[a, b]$.

6. The following numerical example (Fox, 1957) is used to illustrate the method

$$y'' + \frac{4x}{1+x^2}y' + \frac{2}{1+x^2}y = 0 \tag{46}$$

with boundary conditions $y(0) = 1$, $y(2) = 0\cdot2$. The exact solution is $y = \dfrac{1}{1+x^2}$.

The problem was first solved using the procedure described in the previous section with ϵ and ϵ_k both set equal to $\frac{1}{2} \times 10^{-4}$. The initial value of n was 4 and the tests chose $n = 8, 12, 16$ before both were satisfied. The results at the knots and the mid-points of each interval, are given in **Table 1**. The dramatic improvement obtained by using deferred correction can be seen from the fact that at $x = 1$ the errors in $s^{(0)}(x)$ and $s(x)$ are $-0\cdot00150618$ and $-0\cdot00000254$ respectively. The maximum error occurs at $x = 0\cdot4375$ and is less than one unit in the fourth decimal place.

Table 1

x	$s^{(0)}(x)$	$s(x)$	$e(x) \times 10^4$
0·0000	1·000 000 00	1·000 000 00	0·0000
0·0625		0·996 112 84	−0·0489
0·1250	0·984 893 16	0·984 647 51	−0·3212
0·1875		0·966 073 10	−0·3536
0·2500	0·942 052 03	0·941 233 10	−0·5663
0·3125		0·911 098 53	−0·6650
0·3750	0·878 230 64	0·875 784 26	−0·7193
0·4375		0·839 422 10	−0·7783
0·5000	0·801 969 78	0·800 067 84	−0·6784
0·5625		0·759 710 92	−0·6700
0·6250	0·721 238 96	0·719 151 14	−0·5002
0·6875		0·679 090 50	−0·4541
0·7500	0·642 056 85	0·640 029 29	−0·2929
0·8125		0·602 377 35	−0·2441
0·8750	0·568 188 45	0·566 384 48	−0·1280
0·9375		0·532 233 90	−0·0937
1·0000	0·501 506 18	0·500 002 54	−0·0254
1·0625		0·469 725 50	−0·0073
1·1250	0·442 567 25	0·441 376 79	0·0252
1·1875		0·414 907 69	0·0317
1·2500	0·391 141 35	0·390 239 69	0·0422
1·3125		0·367 284 15	0·0423
1·3750	0·346 595 75	0·345 941 83	0·0412
1·4375		0·326 110 78	0·0387
1·5000	0·308 140 10	0·307 689 04	0·0327
1·5625		0·290 575 92	0·0297
1·6250	0·274 966 06	0·274 675 89	0·0222
1·6875		0·259 896 50	0·0197
1·7500	0·246 318 11	0·246 152 57	0·0127
1·8125		0·233 362 62	0·0110
1·8750	0·221 523 56	0·221 452 78	0·0051
1·9375		0·210 352 91	0·0042
2·0000	0·200 000 00	0·200 000 00	0·0000

The form of the solution suggests that this example would benefit from the use of unequal intervals (indeed Fox (1957) uses this example to illustrate the use of unequal intervals in the finite difference method). We therefore again solve equation (46) with $n = 16$ but this time with knots at the points $0(\frac{1}{12})\frac{2}{3}(\frac{1}{6})2$. Modified deferred correction terms for use on the right-hand side of equation (23) close to $x = \frac{2}{3}$, are obtained from equations (31) to (38). The results are given in **Table 2.** The maximum error occurs at the mid-point of the interval including the knot where the interval size changed and is slightly smaller than the maximum error in the equal interval case.

It can be seen that very little advantage is gained by using unequal intervals. This is because the deferred correction is less effective. The interval control tests are easier to implement if equal intervals are used and so the use of unequal intervals is, in general, not recommended.

The results obtained using this method are better than using the usual finite difference method with the same number of knots. Also this method produces a spline function which may be used to obtain the solution at any point in the range, whereas the finite difference method only obtains the solution at the chosen knots.

Finally, the author would echo the comment of Curtis and Powell that the more one uses splines the more one likes them.

Table 2

x	$s^{(0)}(x)$	$s(x)$	$e(x) \times 10^4$
0·0000	1·000 000 00	1·000 000 00	0·0000
0·0416		0·998 270 42	−0·0352
0·0833	0·993 500 95	0·993 115 31	−0·1186
0·1250		0·984 630 11	−0·1473
0·1666	0·973 834 52	0·972 995 63	−0·2265
0·2083		0·958 429 28	−0·2662
0·2500	0·942 524 70	0·941 209 57	−0·3310
0·2916		0·921 637 30	−0·3730
0·3333	0·901 808 93	0·900 041 48	−0·4148
0·3750		0·876 757 47	−0·4514
0·4166	0·854 274 33	0·852 117 91	−0·4691
0·4583		0·826 448 50	−0·4965
0·5000	0·802 507 21	0·800 049 81	−0·4981
0·5416		0·773 206 38	−0·5202
0·5833	0·748 827 65	0·746 163 68	−0·4969
0·6250		0·719 149 00	−0·4788
0·6666	0·695 137 04	0·692 356 03	−0·4834
0·7500		0·640 060 41	−0·6041
0·8333	0·592 843 37	0·590 179 80	−0·1587
0·9166		0·543 397 18	−0·0010
1·0000	0·502 164 20	0·499 978 68	0·2131
1·0833		0·460 038 22	0·2567
1·1666	0·425 101 56	0·423 497 72	0·3169
1·2500		0·390 213 52	0·3039
1·3333	0·361 048 66	0·359 971 73	0·2827
1·4166		0·332 538 75	0·2476
1·5000	0·308 331 92	0·307 672 33	0·1998
1·5833		0·285 132 13	0·1638
1·6666	0·265 048 11	0·264 694 35	0·1153
1·7500		0·246 145 20	0·0864
1·8333	0·229 435 85	0·229 294 73	0·0463
1·9166		0·213 964 69	0·0262
2·0000	0·200 000 00	0·200 000 00	0·0000

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