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SPLINE FUNCTION APPROXIMATIONS FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. A procedure for obtaining spline function approximations for solutions of the initial value problem in ordinary differential equations is presented. The proposed method with quadratic and cubic splines is shown to be related to the well-known trapezoidal rule and Milne-Simpson method, respectively. The method is shown to be divergent, however, when higher degree spline functions are used.

1. Introduction and description of the method. This paper contains proofs for the results announced under the same title in [4]. We recall that discrete variable methods for numerically solving differential equations (see [3]) provide approximate solutions as discrete tabular values at usually equidistant values of the independent variable. Some global methods, such as repeated Taylor series expansions (made computationally practical by Moore [5]), have arbitrarily high order convergence, and yet produce approximations which are not, in general, differentiable over the entire range of integration. The object of this paper is to search for global approximate solutions having several continuous higher derivatives. Following the suggestion of I. J. Schoenberg [6], we obtain solutions in the form [7] of a spline function $S(x)$ of degree $m \geqq 2$ and continuity class C^{m-1}.

The method to be described produces convergent quadratic and cubic spline approximations. When applied with higher degree splines, the method is divergent.

Let the differential equation be

$$y' = f(x, y), \qquad 0 \leqq x \leqq b, \tag{1}$$

about which we make the following assumptions. If $T = \{(x, y) \mid 0 \leqq x \leqq b\}$, then we assume that $f(x, y) \in C^{m-2}$ in T and that it satisfies the Lipschitz condition

$$|f(x, y) - f(x, y^*)| \leqq L\, |y - y^*| \quad \text{if} \quad 0 \leqq x \leqq b. \tag{2}$$

If $m \geqq 3$ then (2) is equivalent to the boundedness of $\partial f/\partial y$ in T. These conditions on $f(x, y)$ guarantee the existence of a unique solution to (1) for any initial condition.

Our construction of the approximate solution $S(x) = S_m(x)$ is as fol-

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lows. Let $y(x)$ be the solution of (1) determined by the initial value $y(0) = y_0$. Let $n > m$ be an integer, $h = b/n$ and let $S(x)$, $0 \leqq x \leqq b$, be a spline function of degree m, class C^{m-1} and having its knots at the points $x = h, 2h, \cdots, (n-1)h$.

We define the first component of $S(x) = S_m(x)$ by

$$S(x) = y(0) + y'(0)x + \cdots \tag{3}$$
$$+ \frac{1}{(m-1)!} y^{(m-1)}(0)x^{m-1} + \frac{1}{m!} a_0 x^m, \qquad 0 \leqq x \leqq h,$$

with the last coefficient a_0 as yet undetermined. We now determine a_0 by requiring that $S(x)$ should satisfy (1) for $x = h$. This gives the equation

$$S'(h) = f(h, S(h)) \tag{4}$$

to be solved for a_0. In terms of $\xi = a_0 h^{m-1}/(m-1)!$ it is seen that (4) is an equation in ξ which is conveniently solved by iteration.

Having determined the polynomial (3), repeat the same steps in the interval $[h, 2h]$: we define

$$S(x) = \sum_{k=0}^{m-1} \frac{1}{k!} S^{(k)}(h)(x-h)^k + \frac{1}{m!} a_1(x-h)^m, \qquad h \leqq x \leqq 2h, \tag{5}$$

and determine a_1 so as to satisfy the equation

$$S'(2h) = f(2h, S(2h)).$$

Continuing in this manner we obtain a spline function $S_m(x)$ satisfying the equation

$$S_m'(\nu h) = f(\nu h, S_m(\nu h)), \qquad \nu = 0, 1, \cdots, n. \tag{6}$$

THEOREM 1. *If $h < m/L$, then the spline function $S_m(x)$ exists and is uniquely defined by the above construction.*

Proof. Over the interval $[\nu h, (\nu+1)h]$ we define

$$S(x) = \sum_{k=0}^{m-1} \frac{1}{k!} S^{(k)}(\nu h)(x-\nu h)^k + \frac{1}{m!} a_\nu(x-\nu h)^m$$
$$= A_\nu(x) + \frac{1}{m!} a_\nu(x-\nu h)^m, \qquad \nu = 0, 1, \cdots, n-1.$$

Thus $A_\nu(x)$ is uniquely determined by the spline continuity conditions, and a_ν is to be found from relation (6) replacing ν by $\nu + 1$. Relation (6) will be satisfied if and only if

$$a_\nu = \frac{(m-1)!}{h^{m-1}} \left\{ f\left((\nu+1)h, A_\nu((\nu+1)h) + \frac{1}{m!} a_\nu h^m\right) - A_\nu'((\nu+1)h) \right\} = g_h(a_\nu). \tag{7}$$

One Lipschitz constant for $g_h(t)$ is Lh/m independently of ν, where L is the Lipschitz constant for $f(x, y)$. Hence for $h < m/L$ we have that $g_h(t)$ is a strong contraction mapping, and (7) has a unique fixed point a_ν which may be found by iteration.

2. The consistency relation for a spline function. Let $\mathcal{S}$ denote the class of spline functions on $[0, b]$ of degree m, class C^{m-1} with knots $x_\nu = \nu h$, $\nu = 1, \cdots, n-1$. Let $s(x) \in \mathcal{S}$. If restricted to the interval $[0, (m-1)h]$, $s(x)$ depends on $(m+1) + (m-2) = 2m - 1$ linear parameters. It follows that the $2m$ quantities,

$$s(\nu h), \quad s'(\nu h), \qquad \nu = 0, 1, \cdots, m-1, \tag{8}$$

cannot be linearly independent. In fact we have the next theorem, the proof of which is due to Schoenberg [6].

THEOREM 2. *For any spline function $s(x) \in \mathcal{S}$, there is a unique linear relation between the quantities* (8) *given by*

$$\sum_{\nu=0}^{m-1} a_\nu^{(m)} s(\nu h) = h \sum_{\nu=0}^{m-1} b_\nu^{(m)} s'(\nu h), \tag{9}$$

whose coefficients may be written as

$$\begin{aligned} a_\nu^{(m)} &= (m-1)!(Q_m(\nu) - Q_m(\nu+1)), \\ b_\nu^{(m)} &= (m-1)!\,Q_{m+1}(\nu+1), \end{aligned} \tag{10}$$

where

$$Q_{m+1}(x) = \frac{1}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} (x-i)_+^m, \tag{11}$$

which is a B-spline.

Proof. Consider the convolution of two infinite sequences defined by

$$\{a_n\} * \{b_n\} = \{c_n\},$$

where

$$c_n = \sum_k a_k b_{n-k}, \qquad -\infty < n < \infty.$$

The following two properties hold for convolution:

$$\begin{aligned} \{a_n\} * \{b_n\} &= \{b_n\} * \{a_n\}, \\ \{a_n\} * \{b_n\} &= \{a_{n-\nu}\} * \{b_{n+\nu}\}. \end{aligned} \tag{12}$$

By a suitable change of scale we let $h = 1$ in (9) without loss of generality. We now consider the B-spline (see [7] and [8]) $Q(x) = Q_{m+1}(x)$ defined by (11) and examine the convolution of $\{Q'(n)\}$ with $\{Q(n-\nu)\}$. Applying relations (12), we have

$$\{Q'(n)\} * \{Q(n-\nu)\} = \{Q(n-\nu)\} * \{Q'(n)\} \tag{13}$$
$$= \{Q(n)\} * \{Q'(n-\nu)\}.$$

If we apply the representation theorem [1, p. 80] we may write the arbitrary spline function $s(x)$ as

$$s(x) = \sum_{\nu} c_\nu Q(x - \nu).$$

In particular, we have

$$s(n) = \sum_{\nu} c_\nu Q(n - \nu),$$
$$s'(n) = \sum_{\nu} c_\nu Q'(n - \nu). \tag{14}$$

Combining relations (13) and (14) finally yields

$$\{Q'(n)\} * \{s(n)\} = \{Q(n)\} * \{s'(n)\}. \tag{15}$$

We observe from (11) that $Q(x) = Q_{m+1}(x)$ vanishes outside the interval $(0, m + 1)$. Taking the element corresponding to $n = m$ of the convolution on each side of (15) now yields a result in the form of (9) with coefficients

$$a_\nu^{(m)} = Q'_{m+1}(m - \nu), \qquad b_\nu^{(m)} = Q_{m+1}(m - \nu). \tag{16}$$

But the coefficients defined in (10) differ from these only by a constant factor of $(m - 1)!$ because B-splines have the symmetry properties

$$Q_{m+1}(m - x) = Q_{m+1}(x + 1),$$
$$Q'_{m+1}(m - x) = -Q'_{m+1}(x + 1),$$

and the differentiation property

$$Q'_{m+1}(x + 1) = Q_m(x + 1) - Q_m(x).$$

TABLE 1

Coefficients of the consistency relations

	$a_\nu^{(m)}$					$b_\nu^{(m)}$				
m \ ν	0	1	2	3	4	0	1	2	3	4
2	−1	1				1/2	1/2			
3	−1	0	1			1/3	4/3	1/3		
4	−1	−3	3	1		1/4	11/4	11/4	1/4	
5	−1	−10	0	10	1	1/5	26/5	66/5	26/5	1/5

These relations can be verified directly from (11), and this completes the proof.

From (10) and (11) we easily obtain the numerical values of Table 1.

THEOREM 3. *The values* $S_m(\nu h)$, $\nu = 0, 1, \cdots, n$, *obtained in* §1 *are precisely the values furnished by the discrete multistep method described by the recurrence relation*

$$\sum_{\nu=0}^{m-1} a_\nu^{(m)} y_{k-m+1+\nu} = h \sum_{\nu=0}^{m-1} b_\nu^{(m)} y'_{k-m+1+\nu}, \quad k = m - 1, \cdots, n, \tag{17}$$

if the starting values

$$y_0 = S_m(0), \quad y_1 = S_m(h), \cdots, \quad y_{m-2} = S_m((m - 2)h) \tag{18}$$

are used.

Proof. For $h < L/m$ only one sequence $\{y_\nu\}$, $\nu = m - 1, \cdots, n$, satisfies (17) with starting values (18). By the consistency relation, however, the sequence $\{S_m(\nu h)\}$ satisfies (17) and obviously has starting values (18). Thus the values $S_m(\nu h)$, $\nu = m - 1, \cdots, n$, must coincide with the points y_ν generated by the corresponding multistep method.

Note that our spline integration procedure appears in §1 as a one-step method. Theorem 3 now shows that it also furnishes some of the solutions of the $(m - 1)$-step method (17). We next consider several important examples of this result.

3. Quadratic spline functions and the trapezoidal rule.

If we consult Table 1 and apply Theorem 3 for $m = 2$, we find the corresponding 1-step method to be

$$y_k - y_{k-1} = \tfrac{1}{2}h(y'_{k-1} + y_k') = \tfrac{1}{2}h(f(x_{k-1}, y_{k-1}) + f(x_k, y_k)),$$

which is the trapezoidal rule. Thus, by Theorem 3, the trapezoidal rule will give the same discrete solution $S_2(kh)$ as the quadratic spline method described in §1 for $m = 2$. In this case, y_0 is trivially the only starting value needed.

We now define the step function $S^{(m)}(x) = S_m^{(m)}(x)$ at the knots $x_k = kh$, $k = 1, \cdots, n - 1$, by the usual arithmetic mean:

$$S^{(m)}(x_k) = \tfrac{1}{2}[S^{(m)}(x_k - \tfrac{1}{2}h) + S^{(m)}(x_k + \tfrac{1}{2}h)], \quad k = 1, \cdots, n - 1. \tag{19}$$

THEOREM 4. *If* $f(x, y) \in C^2$ *in* T, *then there exists a constant* K *such that, for all* $h < 2/L$,

$$| S_2(x) - y(x)| < Kh^2, \; | S_2'(x) - y'(x)| < Kh^2, \; | S_2''(x) - y''(x)| < Kh,$$

if $x \in [0, b]$, *provided* $S_2''(x_k)$ *is given by* (19) *with* $m = 2$.

The proof of Theorem 4 requires the following lemmas. Let $y(x)$ be the unique solution to (1) with values $y_k = y(x_k)$ at the ponts $x_k = kh$, $k = 0, 1, \cdots, n$. Similar notation is used for $y'(x_k)$, $S(x_k)$ and $S'(x_k)$.

LEMMA 1. *If* $| S(x_k) - y(x_k)| < Kh^p$ *and* $S'(x_k) = f(x_k, S(x_k))$, *then there exists a constant* K^* *such that*

$$| S(x_k) - y(x_k)| < K^*h^p \quad \textit{and} \quad | S'(x_k) - y'(x_k)| < K^*h^p.$$

Proof. This is an immediate consequence of the Lipschitz condition (2). We have explicitly

$$\begin{aligned} | S'(x_k) - y'(x_k)| &= |f(x_k, S(x_k)) - f(x_k, y(x_k))| \\ &\leqq L | S(x_k) - y(x_k)| < LKh^p. \end{aligned}$$

Simply let $K^* = \max\{K, LK\}$.

LEMMA 2. *Let* $y(x) \in C^{m+1}[0, b]$ *and let* $S(x)$ *be a spline function of degree* m *having its knots at the points* x_k, $k = 1, 2, \cdots, n - 1$, *and such that the conditions*

$$\begin{aligned} &| S^{(r)}(x_k) - y^{(r)}(x_k)| = O(h^{p_r}), \\ &\qquad r = 0, 1, \cdots, m-1, \quad k = 0, 1, \cdots, n-1, \end{aligned} \tag{20}$$

$$\begin{aligned} &| S^{(m)}(x) - y^{(m)}(x)| = O(h), \\ &\qquad x_k < x < x_{k+1}, \quad k = 0, 1, \cdots, n-1, \end{aligned} \tag{21}$$

are satisfied. Then,

$$| S(x) - y(x)| = O(h^p), \qquad x \in [0, b], \tag{22}$$

where

$$p = \min_{r=0,1,\cdots,m} (r + p_r), \tag{23}$$

where $p_m = 1$, *and furthermore,*

$$| S^{(m)}(x) - y^{(m)}(x)| = O(h), \qquad x \in [0, b]. \tag{24}$$

Proof. Let $x_k < x \leqq x_{k+1}$. Expanding by Taylor's theorem and writing $\omega = x - x_k \leqq h$, we obtain

$$y(x) = \sum_{r=0}^{m-1} \frac{1}{r!} \omega^r y^{(r)}(x_k) + \frac{1}{m!} \omega^m y^{(m)}(\xi), \qquad x_k < \xi < x, \tag{25}$$

$$S(x) = \sum_{r=0}^{m-1} \frac{1}{r!} \omega^r S^{(r)}(x_k) + \frac{1}{m!} \omega^m S^{(m)}(\xi). \tag{26}$$

Note that $S^{(m)}(x)$ is constant for $x_k < x < x_{k+1}$. Subtraction of (25) from (26) gives

$$| S(x) - y(x) | \leqq \sum_{r=0}^{m-1} \frac{1}{r!} h^r | S^{(r)}(x_k) - y^{(r)}(x_k) |$$
$$+ \frac{1}{m!} h^m | S^{(m)}(\xi) - y^{(m)}(\xi) | = O(h^p),$$

in view of (20), (21) and (23). This establishes (22).

To prove (24) it is sufficient, in view of (21), to consider the knots x_k, $k = 1, 2, \cdots, n-1$. By (19) and (21),

$$S^{(m)}(x_k) = \tfrac{1}{2}[S^{(m)}(x_k - \tfrac{1}{2}h) + S^{(m)}(x_k + \tfrac{1}{2}h)]$$
$$= \tfrac{1}{2}[y^{(m)}(x_k - \tfrac{1}{2}h) + y^{(m)}(x_k + \tfrac{1}{2}h)] + O(h).$$

But, since $y(x) \in C^{m+1}[0, b]$,

$$y^{(m)}(x_k - \tfrac{1}{2}h) = y^{(m)}(x_k) - \tfrac{1}{2}hy^{(m+1)}(\xi_1), \qquad x_k - \tfrac{1}{2}h < \xi_1 < x_k,$$
$$y^{(m)}(x_k + \tfrac{1}{2}h) = y^{(m)}(x_k) + \tfrac{1}{2}hy^{(m+1)}(\xi_2), \qquad x_k < \xi_2 < x_k + \tfrac{1}{2}h.$$

Thus we finally obtain

$$S^{(m)}(x_k) = y^{(m)}(x_k) + O(h), \qquad k = 1, 2, \cdots, n-1.$$

This completes the proof.

Proof of Theorem 4. By Theorem 3, the quadratic spline values $S_k = S(x_k)$ are the same as the values generated by the trapezoidal rule, which is known [3, p. 199 and p. 248] to be a second order method. Therefore there exists a constant K_0 such that

$$| S_k - y_k | < K_0 h^2, \qquad k = 0\ 1, \cdots, n.$$

It follows immediately from Lemma 1 that (20) is satisfied, taking $m = p_0 = p_1 = 2$.

Expanding $S'_{k+1} = S'(x_{k+1})$ and $y'_{k+1} = y'(x_{k+1})$ by Taylor's theorem gives

$$S'_{k+1} = S_k' + hS''(x),$$
$$y'_{k+1} = y_k' + hy''(\xi), \qquad x_k < \xi < x_{k+1},$$

for any x in (x_k, x_{k+1}). Therefore,

$$h | S''(x) - y''(\xi)| \leqq | S_k' - y_k' | + | S'_{k+1} - y'_{k+1} |,$$

and by Lemma 1,

$$S''(x) = y''(\xi) + O(h),$$

which, because $| \xi - x | < h$, we may write as

$$S''(x) = y''(x) + O(h).$$

Thus the hypothesis (21) of Lemma 2 is satisfied.

Applying the lemma twice, allowing $S(x)$ and $S'(x)$, successively, to assume the role of $S(x)$ in the lemma, establishes the first two inequalities of Theorem 4. The third statement follows from (24), noting that $f \in C^2$ in T implies $y(x) \in C^3[0, b]$ as required by the hypothesis of Lemma 2.

4. Cubic spline functions and the Milne-Simpson method. Again from Table 1 we find for $m = 3$ the recurrence relation

$$\begin{aligned} y_k - y_{k-2} &= \tfrac{1}{3}h(y_{k-2}' + 4y_{k-1}' + y_k') \\ &= \tfrac{1}{3}h(f(x_{k-2}, y_{k-2}) + 4f(x_{k-1}, y_{k-1}) + f(x_k, y_k)), \end{aligned}$$

which is one way of expressing Simpson's rule. Hence, by Theorem 3, the Milne-Simpson method will furnish the discrete solution $S_3(kh)$ provided that y_0 and $y_1 = S_3(h)$ are used as starting values.

THEOREM 5. *If $f(x, y) \in C^3$ in T, then there exists a constant K such that, for all $h < 3/L$,*

$$| S_3(x) - y(x)| < Kh^4, \qquad | S_3'(x) - y'(x)| < Kh^3,$$
$$| S_3''(x) - y''(x)| < Kh^2, \qquad | S_3'''(x) - y'''(x)| < Kh,$$

if $x \in [0, b]$, provided $S_3'''(x_k)$ is given by (19) *with $m = 3$.*

The proof of this convergence theorem also depends upon some lemmas. The notation of §3 is employed. The Milne-Simpson method with infinite correction is known [3, p. 248] to be of the fourth order provided the starting values have fourth order accuracy. We therefore begin by considering the error in the starting value $S(h) = S_3(h)$.

LEMMA 3. *Let $m = 3$. There exists a constant K such that*

$$| S(h) - y(h)| < Kh^4.$$

Proof. Consider the expressions

$$\begin{aligned} S(h) &= y_0 + hy_0' + \tfrac{1}{2}h^2y_0'' + \tfrac{1}{6}h^3a_0, \\ y(h) &= y_0 + hy_0' + \tfrac{1}{2}h^2y_0'' + \tfrac{1}{6}h^3y_0''' + \tfrac{1}{24}h^4y^{(4)}(\xi), \quad 0 < \xi < h. \end{aligned}$$

Thus we have

$$| S(h) - y(h)| = \tfrac{1}{6}h^3 \,|(a_0 - y_0''') - \tfrac{1}{4}hy^{(4)}(\xi)|, \tag{27}$$

and the lemma will follow if we show that $| a_0 - y_0''' | = O(h)$. We begin by demonstrating that a_0 is uniformly bounded as a function of h. Referring to (7) with $m = 3$, we have

$$g_h(a_0) = \frac{2}{h^2}\{f(h, y_0 + hy_0' + \tfrac{1}{2}h^2y_0'' + \tfrac{1}{6}h^3a_0) - y_0' - hy_0''\}. \tag{28}$$

Recall that $g_h(t)$ is a strong contraction mapping for all $h < 3/L$. In particular, for $h \leqq 1/L$,

$$| g_h(t_1) - g_h(t_2)| \leqq \tfrac{1}{3} | t_1 - t_2 | .$$

Thus we obtain

$$| g_h(a_0)| - | g_h(0)| \leqq | g_h(a_0) - g_h(0)| \leqq \tfrac{1}{3} | a_0 | .$$

Since $g_h(a_0) = a_0$,

$$| a_0 | - | g_h(0)| \leqq \tfrac{1}{3} | a_0 | ,$$

which implies that

$$| a_0 | \leqq \tfrac{3}{2} | g_h(0)| . \tag{29}$$

Clearly we have

$$y(h) = y_0 + hy_0' + \tfrac{1}{2}h^2y_0'' + O(h^3),$$

so (28) now gives us

$$\begin{aligned} | g_h(0) | &= \frac{2}{h^2} | f(h, y_0 + hy_0' + \tfrac{1}{2}h^2y_0'') - y_0' - hy_0'' | \\ &= \frac{2}{h^2} | f(h, y(h) + O(h^3)) - y_0' - hy_0'' | \\ &= \frac{2}{h^2} | y'(h) + O(h^3) - y_0' - hy_0'' | \\ &= \frac{2}{h^2} | y_0' + hy_0'' + O(h^2) - y_0' - hy_0'' | \leqq M \end{aligned}$$

for some constant M. From (29) our uniform bound becomes

$$| a_0 | \leqq \tfrac{3}{2}M$$

for all $h \leqq 1/L$. Since uniform spacing is required over the interval $[0, b]$, there is only a finite number of possible values of h between $1/L$ and $3/L$. Thus $| a_0 |$ is uniformly bounded for all $h < 3/L$.

Equation (27) now becomes

$$| S(h) - y(h)| = O(h^3),$$

and Lemma 1 with $p = 3$ yields

$$\begin{aligned} S'(h) &= y'(h) + O(h^3) \\ &= y_0' + hy_0'' + \tfrac{1}{2}h^2y_0''' + O(h^3). \end{aligned}$$

From the definition of $S(x)$, we also have

$$S'(h) = y_0' + hy_0'' + \tfrac{1}{2}h^2a_0 .$$

Combining the last two equations and solving for a_0, we obtain

$$a_0 = y_0''' + O(h).$$

Equation (27) finally becomes

$$| S(h) - y(h)| = O(h^4).$$

This completes the proof.

Remark. Since Lemma 3 shows that the starting value $S(h)$ has error $O(h^4)$, we may conclude that

$$\begin{aligned} S(x_k) &= y(x_k) + O(h^4), \\ S'(x_k) &= y'(x_k) + O(h^4), \qquad k = 1, 2, \cdots, n, \end{aligned} \tag{30}$$

by Lemma 1 ($p = 4$) and the comments about the Milne-Simpson method preceding Lemma 3.

The next result, needed for proving Theorem 5, is stated in a general setting, and its applications are not restricted to spline functions.

LEMMA 4. *Let $y(x) \in C^4[0, b]$ and let x_k and $x_{k+1} = x_k + h$ be in* $[0, b]$. *Suppose $P(x)$ is the unique cubic polynomial that satisfies the Hermite interpolating conditions*

$$\begin{aligned} P(x_k) &= y(x_k), \qquad & P'(x_k) &= y'(x_k), \\ P(x_{k+1}) &= y(x_{k+1}), \qquad & P'(x_{k+1}) &= y'(x_{k+1}). \end{aligned} \tag{31}$$

Then there exists a constant K such that

$$| P'''(x_k) - y'''(x_k)| < Kh.$$

Proof. Let the cubic polynomial be

$$P(x) = \alpha_k + \beta_k(x - x_k) + \gamma_k(x - x_k)^2 + \delta_k(x - x_k)^3. \tag{32}$$

The coefficient δ_k of x^3 is characterized by

$$P'''(x) = 6\delta_k = 6P(x_k, x_k, x_{k+1}, x_{k+1}) = 6y(x_k, x_k, x_{k+1}, x_{k+1}),$$

using Steffensen's notation for divided differences. From a general property of divided differences (see [2, p. 66]), we have

$$P'''(x) = 6y(x_k, x_k, x_{k+1}, x_{k+1}) = y'''(\xi), \quad x_k < \xi < x_{k+1}, \tag{33}$$

while

$$| y'''(\xi) - y'''(x_k)| = | \xi - x_k |\cdot| y^{(4)}(\eta)| < Kh, \tag{34}$$

where $x_k < \eta < \xi$. Relations (33) and (34) clearly imply

$$| P'''(x_k) - y'''(x_k)| < Kh,$$

so the lemma is established.

Proof of Theorem 5. Denote the cubic spline component over $[x_k, x_{k+1}]$ by

$$S(x) = \alpha_k + \beta_k(x - x_k) + \gamma_k(x - x_k)^2 + \delta_k(x - x_k)^3, \quad x_k \leqq x \leqq x_{k+1}.$$

Solving the system analogous to (31) of Lemma 4 for δ_k, we obtain

$$\delta_k = \frac{1}{h^3}(2S_k + hS_k' - 2S_{k+1} + hS_{k+1}').$$

We may express δ_k using relations (30) as follows:

$$\begin{aligned}\delta_k &= \frac{1}{h^3}(2y_k + hy_k' - 2y_{k+1} + hy_{k+1}') + O(h)\\ &= \tfrac{1}{6}P_3'''(x_k) + O(h),\end{aligned}$$

where $P_3(x)$ is the unique cubic that interpolates the data y_k, y_k', y_{k+1} and y_{k+1}' taken from $y(x)$.

Let $x_k < x < x_{k+1}$. Now $S'''(x) \equiv 6\delta_k$, and Lemma 4 implies

$$\begin{aligned}S'''(x) &= P_3'''(x_k) + O(h)\\ &= y'''(x_k) + O(h)\\ &= y'''(x) + (x_k - x)y^{(4)}(\xi) + O(h).\end{aligned}$$

By assumption, $|x_k - x| < h$, so we obtain

(35) $$S'''(x) = y'''(x) + O(h), \quad x_k < x < x_{k+1}, \quad k = 0, 1, \cdots, n-1.$$

Thus (21) is satisfied with $m = 3$.

The step function $S'''(x)$ is constant over (x_k, x_{k+1}) so we may write

$$\begin{aligned}y(x_{k+1}) &= y_k + hy_k' + \tfrac{1}{2}h^2y_k'' + \tfrac{1}{6}h^3y'''(\xi), \qquad x_k < \xi < x_{k+1},\\ S(x_{k+1}) &= S_k + hS_k' + \tfrac{1}{2}h^2S_k'' + \tfrac{1}{6}h^3S'''(\xi).\end{aligned}$$

Subtracting and taking absolute values, we obtain using the first relation of (30),

$$\begin{aligned}|S(x_{k+1}) - y(x_{k+1})| &= |S_k - y_k + h(S_k' - y_k') + \tfrac{1}{2}h^2(S_k'' - y_k'')\\ &\quad + \tfrac{1}{6}h^3(S'''(\xi) - y'''(\xi))|\\ &= O(h^4).\end{aligned}$$

Relation (35) plus both fourth order convergence properties (30) at the knots now show that

(36) $$S_k'' = y_k'' + O(h^2).$$

It follows from (30) and (36) that (20) is satisfied with $m = 3$, $p_0 = p_1 = 4$, $p_2 = 2$. We note also that $f \in C^3$ in T implies $y(x) \in C^4[0, b]$.

The first three inequalities in the statement of Theorem 5 are established by applying Lemma 2 three times, with $S(x)$, $S'(x)$, and $S''(x)$, successively, assuming the role of $S(x)$ in the lemma. The fourth inequality follows from (24).

The quadratic and cubic spline methods considered above present several advantages over the standard trapezoidal and Milne-Simpson multistep methods. Our procedure of §1 produces smooth, accurate, global approximations to the solution of (1)and its first few derivatives. Furthermore, this method requires essentially the same computational effort as the corresponding discrete variable method, which does not supply such information. Another advantage of the spline method over the multistep method (17) is that the step-size h can be changed at any step, if necessary, without added complications.

Another interesting observation should be made. The spline function $S_m(x)$ constructed in §1 is identical to the spline of same degree obtained by Hermite interpolating the points and slopes generated by the respective multistep method. In particular, for $m = 2$, the unique cubic polynomial which interpolates any two consecutive values and slopes from the trapezoidal rule automatically reduces to a parabola. In the Milne-Simpson case, any two consecutive cubic polynomial interpolants have the same second derivative at their common knot and hence form part of the spline $S_3(x) \in C^2[0, b]$. These comments enable us easily to construct a convergent global solution over a given subset of $[0, b]$ and a convergent discrete solution over the remainder of the integration range.

5. Higher degree spline functions and unstable multistep methods. From Table 1 we observe that the multistep methods associated with spline functions of degree greater than three are unfamiliar. In fact we have the following negative result.

THEOREM 6. *The solutions $S_m(x)$ are divergent as $h \to 0$ for $m \geqq 4$.*

Proof. We will show that the multistep methods given by Theorem 3 are unstable and hence divergent for $m \geqq 4$. By a theorem due to Dahlquist [3, Theorem 5.5, p. 218], the multistep method (17) is stable only if the zeros of the "associated polynomial" $P(z)$ have modulus not exceeding unity. From (10) and (17) we see that

$$P(z) = \sum_{\nu=0}^{m-1} a_\nu^{(m)} z^\nu = (z - 1)(m - 1)! \sum_{\nu=1}^{m-1} Q_m(\nu) z^{\nu-1}.$$

Using the definition (11) of Q_m, we readily obtain

$$\begin{aligned} P(z) &= (z - 1)[z^{m-2} + (2^{m-1} - m)z^{m-3} + \cdots + 1] \\ &= (z - 1)(z - r_2) \cdots (z - r_{m-1}) = (z - 1)\bar{P}(z). \end{aligned}$$

The sum of the zeros of $\bar{P}(z)$ is given by

$$\sum_{\nu=2}^{m-1} r_\nu = m - 2^{m-1}. \tag{37}$$

Taking the moduli of both sides of (37) we have

$$\sum_{\nu=2}^{m-1} | r_\nu | \geqq | \sum_{\nu=2}^{m-1} r_\nu | = 2^{m-1} - m > m - 1 \tag{38}$$

for $m \geqq 4$. Let $| r_{\max} | = \max_\nu | r_\nu |$. Then (38) becomes

$$(m - 2)| r_{\max} | > m - 1,$$

so we see that $| r_{\max} | > 1$ for $m \geqq 4$. This fact proves that the multistep method, and hence the corresponding spline solution, is divergent.

Numerical results, obtained on the CDC 1604 at the University of Wisconsin Computing Center, for the equation $y' = y$, $y(0) = 1$, exhibit marked divergence for $m = 4, 5, 6, 7$. The positive conclusions of §3 and §4 have also been tested numerically.

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