Optimal Policy in the Sequence Space

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Let X be an $n_X T$ vector of endogenous variables. Let ε be an $n_\varepsilon T$ vector of exogenous variables. Our model will be represented as $f(X, \varepsilon) = 0$. We partition X into $[Y' \quad Z']'$ where Z are policy instruments. Our objective function is $U(Y, \varepsilon)$. It is w.l.o.g. to omit Z from U because we can define an auxiliary variable $Y_k = Z$. The Lagrangian is

$$\mathcal{L} = U(Y, \varepsilon) - \lambda' f(Y, Z, \varepsilon)$$

and the FOCs are

$$\begin{aligned} U_Y' &= \lambda' f_Y \\ 0 &= \lambda' f_Z \end{aligned}$$

Combining the two FOCs we have

$$0 = U_Y' f_Y^{-1} f_Z.$$

Define $U_X' = [U_Y' \quad 0]'$ as U does not depend on Z, define $f_X = [f_Y \quad f_Z]$, and define $\Theta = -f_X^{-1} f_Z$ as the casual effects of the policy instruments on X. Transposing the FOCs we arrive at the policy criterion

$$\Theta' U_X = 0.$$

We assume we can compute U_X. Computing Θ can require some care (see below). But suppose we have Θ, then we have form the system

$$\begin{pmatrix} f(X,\varepsilon) \\ \Theta(X,\varepsilon)'U_X(X,\varepsilon) \end{pmatrix} = 0.$$

We solve the system above using Newton's method. For computational tractability we do not differentiate Θ in the Newton step. So the update is

$$X^{(j+1)} = X^{(j)} - \begin{pmatrix} f_X^{(j)} \\ \Theta^{(j)\prime}U_{XX}^{(j)} \end{pmatrix}^{-1} \begin{pmatrix} f^{(j)} \\ \Theta^{(j)\prime}U_X^{(j)} \end{pmatrix}.$$

Suppose you specify the model such that policy instruments are simply exogenous. It will likely then turn out that f_X is not invertible and one cannot form Θ using f_X^{-1}. In practice, we specify the model so that the policy instrument is set according to a rule subject to shocks. So the model is now

$$F(X,\varepsilon) \equiv \begin{pmatrix} f(X,\varepsilon) \\ g(X,\nu) \end{pmatrix} = 0$$

where g is a policy rule and ν are policy shocks. We then have the total derivative (assuming $d\varepsilon = 0$)

$$F_X dX + F_\nu d\nu = 0$$

leading to

$$\frac{dX}{d\nu} = F_X^{-1}F_\nu.$$

Define $\tilde{\Theta} = \frac{dX}{d\nu}$, which are the causal effects of the policy *shocks*. Define $M = \frac{dZ}{d\nu}$ which are the effects of the shocks on the policy instruments (M is the lower $T \times T$ block of $\tilde{\Theta}$). Note $\tilde{\Theta} = \frac{dX}{dZ}\frac{dZ}{d\nu} = \Theta M$.

We will impose the policy criterion $\tilde{\Theta}'U_X = M'\Theta'U_X = 0$ instead of $\Theta'U_X = 0$. If M is full rank, then the two are the same. If M is not full rank, it could be the case that we are in the null space of M' and we have not imposed the first-order conditions with respect to the choices of the policy instruments.

In practice, M can be slightly rank deficient without causing a problem. The main reason that M is rank deficient in monetary models is the forward guidance puzzle. We can devise a path of policy shocks that perturbs i_T by some tiny ε that endogenously pushes down inflation by much larger amounts at earlier dates. We then can offset these endogenous policy changes with positive policy shocks. It ends up looking like a path of policy shocks that has no effect on the policy instrument and this dimension of the column space of M maps to zero. If $\Theta' U_X$ is that we have a very particular failure of the FOCs in the null space of M'.