Basic Counting Techniques

bissue

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1 Setting the Stage

1.1 Bomb Defusal

Nearly every problem in competitive math (and in non-competitive math, too!) can be categorized into one of four subjects: Algebra, Geometry, Number Theory, and Counting, the first two of which are relatively standard school subjects and the third of which is a natural variant of the first. But the subject of "Counting", first taught to kindergarteners who still struggle to speak in complete sentences, sticks out. How could counting present a challenge when it is as straightforward as this?

Exercise 1.1. How many letters are in this question?

But counting in competitive math is not the same as the monotonous counting exercises your teachers drilled you on back in elementary school. The counting exercises discussed in this handout are nearly impossible to complete by means of finger raising; instead, various and ideas and shortcuts are used to quicken the time spent from a couple minutes to a couple seconds.

The trouble is that counting presents a much greater emphasis on **conceptualizing** ideas than memorizing ideas. Sure, the ideas required are not at all as *advanced* as those required in, say, Calculus, but the applications of those ideas often end up being so twisted and distorted that it is not too uncommon to have a single misconception break down an entire solution, leaving just the residue of a wrong answer behind.

Fortunately, as long as you have the basics strong and grounded within your mind, Counting should follow naturally. Be careful, be clever, be confident, and victory will be yours.

> *Remark 1.2.* Instead of "Counting", people often describe the subject with the word "Combinatorics". The two words technically have different meanings, but in most contexts, they're interchangeable. Combinatorics tends to refer more to competitive math and not kindergarten math, however.

1.2 From A to B

The following problems should not present too much of a challenge:

Example 1.3
1. How many positive integers are between 1 and 10, including both 1 and 10?
2. How many positive integers are between 1 and 20, **not** including 1 or 20?
3. How many odd integers are between -4 and 5, **not** including 5?

Proof. Any method will do, as long as you don't do this:

⚠ The answers to the first and second question, respectively, are $10 - 1 = \boxed{9}$ and $20 - 1 = \boxed{19.}$ (Why does this not work?)

Instead, the answers should be, in order, 10, 18, and 4. □

When you encounter problems like these in actual competitions, you will often see them written using words such as "inclusive" and "exclusive" in place of "including" or "not including". This is done simply for the sake of being concise. The words **inclusive** and **exclusive** are defined as follows:

Fact 1.4. The set of all numbers between A and B, **inclusive,** includes both A and B. The set of all numbers between A and B, **exclusive,** does not include A or B (meaning it excludes both of them).

Exercise 1.5.

1. How many positive integers are between 24 and 54, inclusive?
2. How many integers are between -20 and 16, exclusive?
3. In terms of integers A and B, how many integers are between A and B, inclusive?
4. In terms of integers A and B, how many integers are between A and B, exclusive?

⚠ The answers to parts 3 and 4 in the above exercise are **not** $|A - B|$. If you have that as an answer, look back at Example 1.3 to see why this logic is incorrect.

Example 1.6
How many integer multiples of 3 are between 1 and 100, inclusive?

Proof. Observe that, in this case, the word "inclusive" signifies that 1 and 100 should be **considered** to be counted. However, they are still not counted, regardless, because they are not integer multiples of 3. (As a result, an equivalent answer would arise had the problem author written "exclusive" instead.)

Watch out not to make the following mistake:

⚠ There are $100 - 1 - 1 = 98$ numbers between 1 and 100. Additionally, one-third of all numbers are multiples of 3, so we should take $\frac{1}{3}$ of 98 to get the answer of $\boxed{\frac{98}{3}}$.

The reason why this is incorrect is because multiples of three within the range of 1 and 100 are **not** spread out to form an exact one-third distribution. (Check this with the set $\{1, 2, 3, 4, 5\}$.) However, we can expect that the distribution will be *about* one-third, so if our final answer is extremely far away from $\frac{98}{3}$, we know we likely made a mistake.

We know that the set of all multiples of three between 1 and 100 looks something like this:

$$\{3, 6, 9, 12, \cdots, 93, 96, 99\}.$$

How many numbers are in this set? If we write the numbers like so, it becomes a little clearer:

$$\{3 \times 1,\ 3 \times 2,\ 3 \times 3,\ 3 \times 4,\ \cdots,\ 3 \times 31,\ 3 \times 32,\ 3 \times 33\}.$$

Now we can observe that the number of number of elements in this set is the same as the number of numbers in the set of positive integers from 1 to 33, inclusive, which is $\boxed{33.}$ We can verify that this is indeed very close to $\frac{98}{3}$ to check our answer.

Remark 1.7. What we used in this solution is called a **bijection.** A bijection is just a fancy way of saying that every number or thing we are counting may be mapped to a unique number or thing in a different set. In this case, we found a bijection from

$$\{3, 6, 9, 12, \cdots, 93, 96, 99\} \leftrightarrow \{1, 2, 3, 4, \cdots, 31, 32, 33\}.$$

Bijections are helpful because the new set of things we are counting may often be counted in a much easier manner than the original set. They typically appear in the form of "perspective shifts"; be on the lookout for them, especially on particularly tough problems!

□

Exercise 1.8. How many integer multiples of 4 are between -29 and 48, inclusive?

Exercise 1.9. How many positive integers between 11 and $(2021 \times 14 + 2)$ are one more than a multiple of 7?

You may be inclined to classify problems into different categories, as you might classify equations as quadratics or cubics. In the case of Combinatorics, however, this kind of thinking often does not yield much.

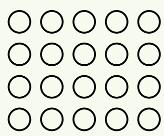
No two counting problems are identical, just as no two snowflakes are copies of one another; just because a problem doesn't ask precisely for the number of multiples of A between B and C, does not mean that the ideas in this section will be useless for that particular problem.

It could be argued that the heart of Combinatorics is not in the problems themselves, but rather, in the connections between the problems...

1.3 From A to B, and to C, and to D

As typical of Combinatorics, we begin with a more straightforward example and build up from there:

Example 1.10
How many circles are in the array below?



Proof. Instead of counting the circles one-by-one, we note that there are 4 rows and 5 columns, so there are a total of 4×5 circles. The reason why we multiply the numbers is because, for each of the 5 columns, there are 4 circles in that column. □

Exercise 1.11. How many spheres would there be, packed within a prism with 4 spheres lengthwise, 5 spheres widthwise, and 7 spheres heightwise?

Example 1.12
How many ordered pairs of positive integers (a, b) are there with $a \leq 5$ and $b \leq 4$?

Proof. We could list all such ordered pairs as such:

$$(5,4),\ (4,3),\ (3,3),\ (2,4),\ (5,1),\ \cdots$$

in a disorganized manner, but we would be bound to make mistakes. Instead, we try to organize our counting. What are all such ordered pairs with $a = 1$?

$$(1,1),\ (1,2),\ (1,3),\ (1,4)$$

What about $a = 2$?

$$(2,1),\ (2,2),\ (2,3),\ (2,4)$$

What about $a = 3$?

$$(3,1),\ (3,2),\ (3,3),\ (3,4)$$

We see that for every choice of a, there are exactly 4 choices of b that yield a valid ordered pair. There are 5 choices for a, and each one gives 4 choices, so the answer is $5 \times 4 = \boxed{20.}$ □

Did you catch the relation between this example and the previous? In the first example, every choice of a column yielded four circles. In the second example, every choice of a yielded four ordered pairs. Therefore, in both examples, we had $5 \times 4 = 20$ as our final answer.

Exercise 1.13. Can you describe a **bijection** between ordered pairs and circles in the array? In other words, how can you assign each ordered pair a unique circle in the array?

We could have skipped a lot of that proof by simply stating the following: There are 5 choices for a, and there are 4 choices for b, so there are $5 \times 4 = 20$ choices in total. (Why is it not $5 + 4 = 9$?) Try to test your understanding of this concept with the following problem:

Example 1.14
Student ID Numbers at ALP High School consist of three digits. The first digit must be even, the second digit must be greater than 3, and the third digit must be nonzero. How many possible ID numbers are there?

Remark 1.15. We always encourage you to try out the examples before reading the solution! Can you figure out how to extend the ideas from the previous example onto this one?

Proof. Again, listing the set of all possible ID numbers, as such, would be futile:

$$241, 242, 257, 888, 444, \cdots$$

Just as in the previous example, it would be nice for us to be able to categorize these ID numbers into groups so we may count them more easily.

Exercise 1.16. In the previous example, the groups were made based on the value of a. How might we construct the groups for this example?

A natural way to proceed would be to construct groups based on the value of the first digit. We have five of these groups:

$$0XY,\ 2XY,\ 4XY,\ 6XY,\ \text{and } 8XY$$

Observe that each of these groups contains the same number of ID numbers! Thus, if we simply count the number of ID numbers for one group, we may multiply by 5 to get the answer.

In the previous example, we were immediately done, since it was obvious that each group contained 4 ordered pairs. In this case, it is not so obvious.

Exercise 1.17. How many ID numbers are in each group? (Hint: Pretend the first digit is 0; the answer should be the same regardless. We now wish to count the number of possible ordered pairs (X, Y). How do we do this?)

To count the number of ID numbers in each group, we simply do the same thing again! We can categorize each possible XY into groups based on the value of X. This leaves the following groupings:

$$4Y,\ 5Y,\ 6Y,\ 7Y,\ 8Y,\ 9Y$$

It is clear that each one of these groups gives 9 choices for XY.[1] Thus, the number of possible XYs equals $6 \times 9 = 54$, since each of the six groups contains 9 XY values. Now that we know there are 54 possible XY values, we see that each of the groups

$$0XY,\ 2XY,\ 4XY,\ 6XY,\ 8XY$$

provides 54 different ID numbers. Since there are 5 groups, there must be a total of $5 \times 54 = \boxed{270}$ ID numbers. □

(Check your understanding!)
Note that an equivalent problem would have been the following:

> How many ordered pairs of positive integers (a, b, c) exist, where $a \le 5$, $b \le 6$, and $c \le 9$?

1. Why is this **bijection** beteween problems valid?
2. Observe that the answer to this question is simply $5 \times 6 \times 9$. Why is this the case?

What we just saw in the previous example and in the exercise above is the following:

Fact 1.18. (Fundamental Principle of Counting) [a] The number of ordered tuples

$$(a_1, a_2, \cdots, a_k)$$

in which each a_i may take on b_i different values, is equal to

$$b_1 \times b_2 \times \cdots \times b_k.$$

[1] In case you are uncertain on why this is true, note that $4Y$, for example, could be 41, 42, 43, and so on up to 49, giving 9 choices in total.

[a] This isn't called the Fundamental Counting Principle for nothing! Make sure you fully understand it before moving on.

Again, note that most problems will not tell you to count ordered tuples, but rather, they will ask you to count something which may be translated into ordered tuples through a **bijection.**

Exercise 1.19. Gamma wishes to create a lock that has as many combinations as possible, in order to prevent people from guessing her passcode. Which of these lock set-ups has the most number of combinations?

1. 200 digits, each one being either 0 or 1.
2. 50 digits, each one being an integer from 1 to 10, inclusive.
3. A string of 25 consecutive letters.

Exercise 1.20. How many four-digit, even positive integers have a hundreds digit of 0 and a tens digit between 3 and 7, inclusive?

One last problem before we move on:

Example 1.21
How many strings of five digits have only distinct digits? (Valid strings would be 01234 or 18769, while invalid strings would be 19942 or 81104.)

Proof. If we were simply told to count the number of strings of five digits, we would have 10^5 as our answer. In this problem, however, we need to deal with the added condition that no two digits are the same.

In spite of this added restriction, let us carry on with our grouping method anyway:

$$0WXYZ,\ 1WXYZ,\ 2WXYZ,\ \cdots,\ 9WXYZ$$

The difference is that in the third group $2WXYZ$, for example, none of W, X, Y, or Z may equal 2. However, the key aspect of this strategy is that *every group is identical.* It turns out that the same is true for this problem as well! (Why is this the case?) Thus, we might as well count the number of strings in the group $0WXYZ$ and multiply that number by 10 to get the final answer.

Our new problem is to find the number of strings of four digits that have only distinct digits and are not equal to 0. Do you see the resemblance to our original problem? We may once again create the following nine groups, all of which are identical:

$$1XYZ, 2XYZ, 3XYZ, \cdots, 9XYZ$$

We now wish to find the total number of strings XYZ, with distinct digits not equal to zero or one, and multiply the resulting answer by 9, and *then* multiply the resulting answer by 10.

Rather than continuing with this, we can spot a pattern. The first digit offers 10 choices. The second digit offers 9 choices. The third has 8, the fourth has 7, and the fifth has

6. Thus, the total number of strings must be $10 \times 9 \times 8 \times 7 \times 6 = \boxed{30240}$, by Fact 1.19. Notice that we don't know exactly what, say, the seven choices for the fourth digit *are*, but we do know that there are seven possible choices, so we multiply by seven nonetheless.

If you aren't so sure about the validity of this logic, you can continue to work out the grouping idea and see that the resulting answer is the same. □

> *Remark 1.22.* Let's say we instead grouped our counting by starting with the fifth digit, then the fourth, and so on until we reached the first. Then the fourth digit would have nine options, not seven! Why does this not change the final answer?

Exercise 1.23. How many strings of four digits have distinct digits? Six digits? Twenty digits?

Exercise 1.24. How many five-digit positive integers (not beginning with zero) have distinct digits?

Exercise 1.25. In how many ways may a set of six distinct blocks be ordered from left to right?

2 Lights, Camera, . . .

Before we move on, let's recap what we have discovered so far:

> **(Fact 1.19, Restated.)** Suppose we want to count the number of possible sequences of k things. Suppose that we may **construct** the sequence by choosing one of a_1 things to be the first element of the sequence, one of a_2 things to be the second element of the sequence, and so on, so that we may choose one of a_n things to be the nth element of the sequence for all n. Then the total number of sequences is equal to
>
> $$a_1 \times a_2 \times \cdots \times a_k$$

Thus, if we are able to translate what we are counting into sequences, in which we may assign a **fixed** number a_n to each thing in the sequence signifying the number of ways we may choose the value of that particular thing, then we have solved the problem.

Exercise 2.1. What three-syllable word starting with **b** describes this so-called "translation"?

Unfortunately, many problems are not so simple as to be obliterated by Fact 1.19 within a few seconds, since these fixed values of a_i may not even exist. In these cases, we may need to apply *additional strategies* to reduce the problem to something more familiar.

2.1 Action?

For this section, instead of showing you the tricks and then instructing you on how to solve the problems, we will show you the problems first so you may think about them before reading the solutions. Even though most of these problems require ideas you may have not seen before, none of these ideas are *advanced*, so it may be possible for you to come up with them on your own. (Just to keep you thinking critically, one of the problems in this bunch doesn't even require any new ideas!)

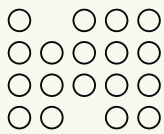
As you read further, we encourage you to pause whenever a new idea is introduced so you can try it out on some of the problems you were stuck on before. Section 2.4 will provide solutions to all four problems so you can see what you missed on some of the trickier ones.

1. How many three-digit positive integers do not have its first two digits as 4 and 2, in some order?
2. How many three-digit positive integers are multiples of either 2 or 3, but not 6?
3. How many even, five-digit positive integers have distinct digits?
4. How many five-digit positive integers do not have an odd digit sum?
5. Call a string of four letters *wordish* if it contains at least one vowel (not including Y). How many strings of four letters are *wordish*?

2.2 Defying the Script

As in before, we introduce our first new concept through a more basic example:

Example 2.2
How many circles are in the array below?



Proof. Instead of simply counting the circles one-by-one, we note that the circles form a 4×5 rectangle, excluding two circles. Previously, we just multiplied 4×5 to get 20 as the answer. This missing circle doesn't prevent us from doing that; we just need to be careful to subtract the missing circles to get $20 - 2 = \boxed{18}$ as the actual count. □

The technique used in the previous solution, although simplistic, is essential enough that it deserves its own name: **Complementary Counting.** When might you think Complementary Counting would be useful?

Fact 2.3. How many things satisfy property A, but do **not** satisfy property B?

Solution: Suppose there are m things that satisfy property A, and n things that satisfy properties A and B. Then the answer to the original problem equals

$$\boxed{m - n.}$$

Although complementary counting seems like it would only double our workload, since it requires us to find to values in order to indirectly calculate one, it is often beneficial whenever m and n are much easier to find than "the number of things satisfying A but not B".

In the example at the start of this section, property A would be "being inside the array", while property B would be "being one of the missing circles". In this case, m and n are not hard to find, while counting the total number of circles directly is a bit harder.

Remember that the only reason we bother with these "techniques" is because it helps lessen the workload. If complementary counting doesn't seem to make the problem any easier, don't use it!

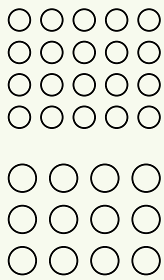
Exercise 2.4. Why might complementary counting **not** help for Exercise 3?

Exercise 2.5. How many three-digit numbers (not beginning with zero) are **not** made up of the digits 1, 7, and 8, in any order?

2.3 Break it Down

One more circle example before we're ready to perform:

Example 2.6
How many circles are there?



Proof. This time, there is no direct way to count the circles. However, based on the positioning of the circles, there is a natural way to organize our counting to be as efficient as possible. The upper rectangular array has $4 \times 5 = 20$ circles, while the lower rectangular array has $3 \times 4 = 12$ circles. Thus, the total number of circles equals $20 + 12 = \boxed{32.}$ □

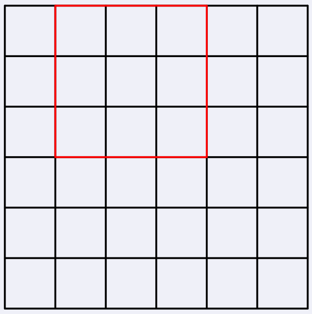
This method of dividing our counting into sections is called **casework**, since it literally is working with cases. Technically, we have done casework already when solving Example 1.14, for example; our cases were divided based on the first digit of the ID number, and each case after that was divided by the second digit of the ID number. However, in that form of casework, each case was identical to every other case, so multiplication would suffice. The term "casework" often refers to when these cases are not identical, such as in Example 2.6.

Casework, when performed properly, can demolish problems towering thousands of feet in the air like a wrecking ball. However, it is essential to plan out the casework before starting any calculations. If any of the following ends up happening, the attempt at casework will most likely be futile:

- There are too many cases (10+, typically) that cannot be completed within a reasonable amount of time.
- The cases are not well defined or cannot be identified precisely.
- Each individual case is not any easier to solve than the original problem.

One last remark: in general, casework is not considered as elegant a solution as other methods. If the casework is going to be quick, then go for it; if it seems a bit lengthy, then try to spend a bit more time looking for a slicker solution.

Exercise 2.7. How many squares can be found in the 6×6 grid below? (Count a square only if its edges coincide with those of the grid. An example of such a square is outlined in red.)



2.4 Action!

Our final rehearsal is over; it's time to perform!

Example 2.8
How many three-digit positive integers do not have its first two digits as 4 and 2, in some order?

Proof. First, observe that we can't simply apply the same multiplication idea from Fact 1.18 for this problem. For example, how are we supposed to know how many different values the second digit can take on? If the first digit is neither 2 nor 4, then the second digit can take on any of the ten digits from 0 to 9. However, if the first digit is 2 or 4, then the second digit only has nine possible choices.

Observe that the number of choices of the second digit is dependent on the choice for the first digit. This suggests two cases:

> **Case 1:** The first digit is neither 2 nor 4. Then the first digit can take on 7 values (numbers one through nine, excluding 2 and 4), the second digit can take on 10 values, and the third digit can also take on 10 values. This gives a total of $7 \times 10 \times 10 = 700$ three-digit numbers.
>
> **Case 2:** The first digit is either 2 or 4. Then the first digit can take on 2 values, and the second digit can take on 9 values.[2] Finally, the third digit can take on 10 values, as always. This gives a total of $2 \times 9 \times 10 = 180$ three-digit numbers.

[2] We don't know what these nine values are, but we do know that there are nine of them.

Overall, the number of three-digit numbers is simply the sum of both cases, or $700 + 180 = \boxed{880}$. □

Remark 2.9. Observe that any three-digit number is in either Case 1 or Case 2, and no three-digit number is in both cases 1 and 2. If we instead made these cases:

> **Case 1:** The first digit is neither 2 nor 4.
>
> **Case 2:** The second digit is neither 2 nor 4.

then we would both be missing some three-digit numbers and counting some twice, which give the wrong answer.

A complementary counting solution to this problem exists as well, outlined by the following exercises.

1. How many three-digit positive integers are there when not considering the added condition?
2. Intuitively, the number of positive integers violating the given condition will be little, so it may be better to count the number of excluded numbers and employ complementary counting. How many three-digit positive integers **do** have its first two digits as 4 and 2, in some order?
3. How do we finish the problem given the numerical answers to the previous two questions?

Notice that this example can be solved with both complementary counting and casework. If you ever have extra time on a single problem, a good way to check your answer is to try a different approach and see if you get the same answer.

Example 2.10
How many three-digit positive integers are multiples of either 2 or 3, but not 6?

Proof. If we tried to list all such positive integers, the list would look a bit irregular:

$$100, 104, 105, 106, 110, 111, 112, 116, 117, \cdots$$

It might be possible to extract some sort of pattern from this list of numbers, but a much quicker approach exists as well.

The question essentially asks, "How many three-digit numbers satisfy the property that they are a multiple of either 2 or 3, but do **not** satisfy the property that they are a multiple of 6?" If you look at Fact 2.3, this description fits the requirements for Complementary Counting perfectly, so let's give it a shot!

First, how many three-digit numbers are multiples of either 2 or 3? It might be intuitive to say the following:

⚠ There are 450 even three-digit numbers and 300 three-digit numbers that are multiples of three. Thus, there are $450 + 300 = 750$ three-digit numbers that are

⚠ either a multiple of 2 or 3.

To see why this logic does not hold, let's break down what we are really doing. This is secretly an exercise in casework, in which our two cases are the following:

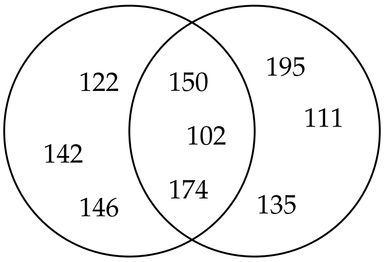
> **Case 1:** The three-digit number is a multiple of 2. There are 450 such three-digit numbers.
>
> **Case 2:** The three-digit number is a multiple of 3. There are 300 such three-digit numbers.

However, if we look at Remark 2.9, we observe that both cases must cover each three-digit number we wish to count exactly once. Thus, if a number is a multiple of either 2 or 3, then it should be counted under one case **and only one case.** However, some numbers that are multiples of either 2 or 3 are listed under both cases and are thus counted twice.

These numbers that have been counted twice are the multiples of both 2 and 3, meaning they are multiples of 6. There are 150 such numbers, so subtracting these 150 three-digit numbers that have been counted twice gives 600 such numbers satisfying "property A" of our complementary counting.

Luckily, finding the number of numbers that violate property B is simple. This is just the number of three-digit numbers that are multiples of 6, which we have found to be equal to 150. Therefore, we simply take $600 - 150$ to find $\boxed{450}$ as the answer. □

If the "double-counting" aspect of this proof was confusing, the following Venn Diagram might be helpful:



The left circle contains all three-digit multiples of 2, and the right circle contains all three-digit multiples of 3. The number of three-digit numbers that are either a multiple of 2 or 3 is the number of numbers in at least one of the two circles.

Ideally, the two circles would be non-intersecting, so the sum of the sizes of both circles would give the desired answer. However, when the circles do intersect, the numbers in both circles are counted twice in the sum of the sizes, so we have to subtract the intersection of both circles. In set theory terms,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This problem provides a sneak-peak at PIE, or the Principle of Inclusion and Exclusion, which will be explained more thoroughly in a future handout.

One final remark:

Remark 2.11. If you are familiar with modular arithmetic, you might recognize that a number N is a multiple of either 2 or 3, but not 6, if and only if

$$N \equiv 2, 3, \text{ or } 4 \pmod 6$$

Thus, another valid approach would be casework on the residue of N modulo 6. Alternatively, one could note that each "grouping" of residues modulo 6 adds three numbers to the count. We have the following 149 natural groupings:

$$\{102, 103, 104, 105, 106, 107\}$$

$$\{108, 109, 110, 111, 112, 113\}$$

$$\cdots$$

$$\{990, 991, 992, 993, 994, 995\}$$

Each one of these groupings has exactly 3 three-digit numbers which satisfy the requirements, and the remaining set of numbers

$$\{100, 101, 996, 997, 998, 999\}$$

contains 3 more numbers. Thus, by simply analyzing the set of numbers we wish to count, we can achieve the same answer of $149 \times 3 + 3 = 450$ in an entirely different manner.

Example 2.12
How many even, five-digit positive integers have distinct digits?

Proof. We already know how to count the number of five-digit positive integers with distinct digits, but now we also need to keep track of the "even" restriction.

Looking back at Fact 1.18, we might be encouraged to try to determine the number of ways each place value may be chosen. However, we soon reach a problem; the number of choices for the last digit is dependent on the number of even digits used in the previous four! How might we get around this?

We can fix this slightly by defining the value of the leading digit first, and then defining the value of the last digit. The remaining three digits will then always have 8, 7, and 6 choices, so we just have to multiply $8 \times 7 \times 6$ onto the number of ways to choose the first and last digit.

Despite our efforts, we still have to acknowledge the fact that the choice of the leading digit changes the number of possibilities for the last digit. In particular, if the leading digit is odd, the last digit has 5 options, whereas if it is even, the last digit only has 4. This suggests a casework approach:

> **Case 1.** The leading digit is odd. Then the leading digit has 5 choices, and the last digit has 5 as well, giving $5 \times 5 = 25$ options total.
>
> **Case 2.** The leading digit is even. Then the leading digit only has 4 choices (2, 4, 6, and 8), and the last digit only as 4 as well, giving $4 \times 4 = 16$ options total.

Summing these two cases together, we achieve a total of 41 outcomes for the first and last digits. Multiplying $41 \times 8 \times 7 \times 6$ gives $\boxed{13776}$ as the answer. □

Remark 2.13. We may "sanity-check" our answer, just to make sure we're not completely off, by noting that without the even condition, there are 27216 possibilities. We can estimate that around half of our five-digit numbers will be even, so the answer should be approximately 13608, which is very close to 13776. Note that it is not exactly 13608, however!

Exercise 2.14. How many odd, five-digit positive integers have distinct digits? (Hint: You should be able to solve this problem almost instantly.)

Example 2.15
How many five-digit positive integers do not have an odd digit sum?

Proof. This problem, surprisingly, is the problem that doesn't involve anything related to casework or complementary counting!

First, note that without the "odd digit sum" condition, the answer is simply $9 \times 10 \times 10 \times 10 \times 10$. However, it seems nearly impossible to take the "odd digit sum" into account when computing the number of possibilities.

In the past, we have had to count things made up of multiple components, where each component only had a fixed number of possibilities. It turns out that we can actually do the same here! First, we choose the first four digits however we want (of course, ensuring that hte leading digit is not zero). This gives:

$$9 \times 10 \times 10 \times 10$$

choices for the first four digits. Now, the key observation is this:

Claim — Regardless of how the first four digits are chosen, the fifth digit has exactly five possibilities that result in an odd digit sum.

Exercise 2.16. Convince yourself that this is true. (You might need to play around with some examples.)

Thus, the answer is just $9 \times 10 \times 10 \times 10 \times 5 = \boxed{45000.}$ □

Example 2.17
Call a string of four letters *wordish* if it contains at least one vowel (not including Y). How many strings of four letters are *wordish*?

Proof. One possible initial reaction to this problem might look like the following:

- Okay, so under normal conditions, there would simply be 26^4 choices.
- Now, we need to make sure at least one of these letters is a vowel. So which one is a vowel?
- Let's say that the first one is a vowel. Then there are 5×26^3 possibilities.

- And if the second one is a vowel, we also have 5×26^3. Continuing this, it seems that there should be $5 \times 26^3 \times 4$.
- However, some of these outcomes were counted multiple times. For example, EEEE would be counted in the case of having a vowel in the first position, second position, third position, and four position, making it counted three more times than necessary.
- So, now we need to consider the cases of having two vowels, which there are... six ways to choose the locations of. And then we have to compute that, do some weird subtraction to account for double counting that will eventually get confusing, and only then will we be done. Sigh...

This approach, which is essentially a very complex example of casework, should *theoretically* work, but with the number of steps necessary, it is quite likely that some mistake will be made in logic or computation that ruins the solution entirely, making it worth zero points.

Instead, the trick to solving this problem is a clever application of Complementary Counting! If you think about it, counting the number of four-letter words that do **not** contain at least one vowel is much easier than counting the number of words that do. This is because, without any vowels, each letter has 21 options, making a total of:

$$21^4 \text{ invalid words.}$$

Thus, the answer is simply the number of four-letter words without the vowel constraint, minus the number of words that fail the vowel constraint, which is $\boxed{26^4 - 21^4.}$ □

3 Summary

The key, driving force behind all Combinatorics problems is the following fact:

Fact 3.1 (Fundamental Principle of Counting). The number of ordered tuples

$$(a_1, a_2, \cdots, a_k)$$

in which each a_i may take on b_i different values, is equal to

$$b_1 \times b_2 \times \cdots \times b_k.$$

However, it is best to try your hardest to conceptualize this fact, not memorize it. If you're unsure of when this fact should be used, don't be afraid to ask us for help!

Additionally, Combinatorics problems may also require certain "tricks". The ones we have covered were Complementary Counting and Casework, described below:

Fact 3.2 (Complementary Counting). How many things satisfy property A, but do **not** satisfy property B?

Solution: Suppose there are m things that satisfy property A, and n things that satisfy properties A and B. Then the answer to the original problem equals

$$\boxed{m - n.}$$

Fact 3.3 (Casework). To count the number of things satisfying a certain property, we might be able to categorize these things into groups, where the number of things in each group is easy to compute.

⚠ When doing casework, make sure that none of your cases have any items in common! (See Remark 2.9.)

If the cases do have something in common, then the process demonstrated in Example 2.10 would be necessary.

Again, these two "tricks" are relatively intuitive; trying to remember the exact statements described below would most likely be pointless.

For almost all of the Combinatorics you will see, make sure to focus on only two things: **conceptualization** and **connections** – not memorization!

4 Problems

Minimum is [35 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "geo good combo bad"
>
> *me*

[2 ♟] Problem 1. How many (not necessarily distinct) letters can be found in the array of words shown below?

absolute abstract counting elephant redundant
partying eeeeeeee sprinkle bacteria creation
everysin glewordh asexactl yeightle tterssss

[3 ♟] Problem 2. A certain school assigns each student a unique ID number, made up of a sequence of five three-digit numbers. If each three-digit number is only allowed to be either, 123, 234, 345, or 199, how many valid ID numbers are there?

[3 ♟] Problem 3 (AoPS). The Gnollish language consists of 3 words, "splargh," "glumph," and "amr." In a sentence, "splargh" cannot come directly before "glumph"; all other sentences are grammatically correct (including sentences with repeated words). How many valid 3-word sentences are there in Gnollish?

[5 ♟] Problem 4. How many strings of five digits are there, such that no two consecutive digits sum to ten? (A valid example would be 01104.)

[5 ♟] Problem 5 (AoPS). How many sequences of 6 digits $x_1, x_2, \ldots, x_6$ can we form, given the condition that no two adjacent x_i have the same parity? Leading zeroes are allowed. (Parity means 'odd' or 'even'; so, for example, x_2 and x_3 cannot both be odd or both be even.)

[5 ♟] Problem 6 (AoPS). How many three-digit numbers are multiples of neither 5 nor 7?

[5 ♟] Problem 7 (CEMC). Ten circles are all the same size. Each pair of these circles overlap but no circle is exactly on top of another circle. What is the greatest possible total number of intersection points of these ten circles?

[6 ♜] Problem 8 (AoPS). Suppose that I have an unlimited supply of identical math books, history books, and physics books. All are the same size, and I have room on a shelf for 8 books. In how many ways can I arrange eight books on the shelf if no two books of the same type can be adjacent?

[5 ♟] Problem 9 (MathCounts). How many numbers can be expressed as the sum of two or more distinct elements of the set $\{0, 1, 2, 4, 8, 16\}$?

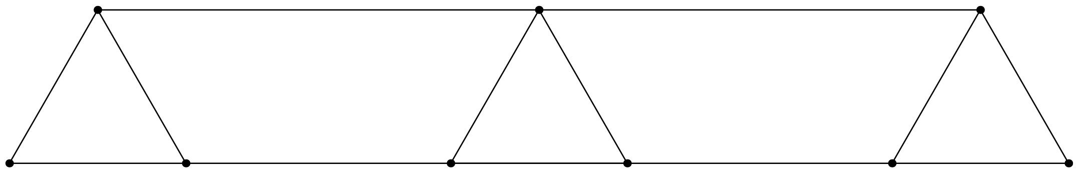
[6 ♟] Problem 10 (2020 AMC 8). Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

[5 ♟] Problem 11. A certain ALP Problem Set contains ten questions, each worth a distinct number of points between 1 and 10, inclusive. In how many ways can someone choose some subset of problems to solve in order to earn at least 50 points?

[5 ♟] Problem 12 (AoPS). A zealous geologist is sponsoring a contest in which entrants have to guess the age of a shiny rock. He offers these clues: the age of the rock is formed from the six digits 2, 2, 2, 3, 7, and 9, and the rock's age begins with an odd digit. How many possible values are there for the rock's age?

[5 ♟] Problem 13 (MathCounts). How many positive, three-digit integers contain at least one 3 as a digit but do not contain a 5 as a digit?

[7 ♟] Problem 14 (AoPS). Each of the nine dots in this figure is to be colored red, white or blue. No two dots connected by a segment (with no other dots between) may be the same color. How many ways are there to color the dots of this figure?



[7 ♟] Problem 15 (MathCounts). How many even, three-digit positive integers have the property that exactly two of the integer's digits are equal?

[16 ♜] Problem 16. The following set of problems are all related in some way. Try to figure out what they have in common!

1. **[1]** A certain bin has 512 marbles. The number of spotted marbles is the same as the number of non-spotted marbles. How many spotted marbles are there?
2. **[7]** How many sequences of nine integers, each between 0 and 9, inclusive, exist so that their sum is at least 41?
3. **[8]** How many strings of ten digits, each either a zero or a one, exist such that there are more zeroes than ones?