2 Manipulations

"Wait. Ok, I actually can't stop drooling over my summation."

- Evan Chang

2.1 Hey, this title is vague

In this handout, I'll walk you through some basic manipulations that we use when computing sums. You've probably seen some of these manipulations before, but going over the strategies in a more detailed fashion is always worthwhile.

Some manipulations will involve manipulating individual terms to turn them into something that we can work with. Other manipulations involve grouping terms together. A powerful technique called induction will also prove useful in evaluating summations. There's a multitude of different types manipulations, and we can't possibly cover them all. But remember that when you come across a scary looking summation, odds are that one or two clever tricks will make the whole thing fall apart.

2.2 Pairing up terms

We'll start with an idea brought up last week when we evaluated the sum of certain arithmetic sequences. Let's take a moment to review this strategy.

Example 20
Determine the sum of the first n positive integers.

Solution. If n is even, then, as seen above, we can split the n positive integers into $\frac{n}{2}$ pairs of numbers, each with sum $n+1$, so the sum of the $\frac{n}{2}$ pairs will be equal to $\frac{n}{2}(n+1)$.

Otherwise, if n is odd, then we can split the n positive integers into $\frac{n-1}{2}$ pairs of numbers, each with sum $n+1$.This leaves a number in the middle, which is equal to $\frac{n+1}{2}$. Our final sum is

$$\frac{(n-1)}{2} \cdot (n+1) + \frac{n+1}{2} = \frac{n^2+n}{2} = \frac{n}{2}(n+1).$$

Thus, for all n, the sum of the first n positive integers is equal to $\boxed{\frac{n(n+1)}{2}}$. □

In this sum, we were able to greatly simplify the sum by *pairing up terms* that each had a constant sum. In this case, the pairs were the first and last, second and second last, third and third last, and so on. Creating these "opposite pairs" is a ubiquitous strategy in all types of summation problems, not just arithmetic sequences. It can reduce relatively complicated sums to just a few basic operations. Here is a more complicated example.

Example 21
Evaluate the sum

$$\sum_{n=1}^{1000} \frac{3}{9^{n/1001}+3}.$$

Solution. We create 500 pairs of summands, such that the nth term in the sum gets paired with the 1001-nth term in the sum.

$$\frac{3}{9^{\frac{n}{1001}}+3} + \frac{3}{9^{\frac{1001-n}{1001}}+3} = \frac{18+3\cdot 9^{\frac{1001-n}{1001}}+3\cdot 9^{\frac{n}{1001}}}{18+3\cdot 9^{\frac{1001-n}{1001}}+3\cdot 9^{\frac{n}{1001}}} = 1,$$

so all the opposite pairs add up to 1, and the answer is $\frac{1000}{2} = \boxed{500}$. □

We can get a clue that pairing is going to work, by looking at the *indexes* in this problem. We're summing from 1 to 1000, The fact that we have also 1001 in this sum suggests that it's a good idea to compare n and $1001-n$. After making the critical observation that $9^{\frac{n}{1001}} \cdot 9^{\frac{1001-n}{1001}} = 9$, the problem starts to quickly unravel.

2.3 Grouping up terms

Pairing up the numbers is a simple and effective strategy, but sometimes we need to *group terms* in the sum in other ways. Finding the slickest ways to group terms is a bit of a skill. The most obvious strategies might not be the best. Case in point:

Example 22 (Purple Comet)
Evaluate the sum $1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2 + \cdots - 1000^2 + 1001^2$.

In this sum, notice we have a pattern of addition, addition, subtraction, subtraction that repeats every 4 terms. The inexperienced competitor might just group the terms like this:

$$(1^2 + 2^2 - 3^2 - 4^2) + (5^2 + 6^2 - 7^2 - 8^2) + \cdots + (997^2 + 998^2 - 999^2 - 1000^2) + 1001^2.$$

This actually works perfectly fine: you would get that for all integers n, $n^2 + (n+1)^2 - (n+2)^2 - (n+3)^2 = -8n - 12$. You would then have to do some decent calculation using the arithmetic sequence formula to arrive at an answer. We encourage you to confirm on your own that the answer is 1001. However, if you thought about it for a little longer, you may have found the following alternate idea:

Solution. Notice $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$ for all integers n. See that we may group our terms as follows:

$$1^2 + (2^2 - 3^2 - 4^2 + 5^2) + (6^2 - 7^2 - 8^2 + 9^2) + \cdots + (997^2 - 998^2 - 1000^2 + 1001^2).$$

This is just equal to $1 + 250 \cdot 4 = \boxed{1001}$. □

This solution is clearly preferable to the first one. Groups that all have some constant sum are easier to deal with then groups that are equal to nonconstant values.

Realize that we do not have to group terms in our sum into groups of the same size.

Example 23 (Purple Comet)
Let d_k be the greatest odd divisor of k for $k = 1, 2, 3, \ldots$. Find $d_1 + d_2 + d_3 + \ldots + d_{255}$.

This number theory and algebra problem can seem a little bit intimidating at first glance. The crucial idea for this problem is to notice that for a positive integer k, if n is the exponent of 2 in k's prime factorization, then $d_k = \frac{k}{2^n}$. From there, you might see the following solution, based on the formula for the sum of the first n odd numbers (this was an exercise in Week 1's handout).

Solution. We group the terms of the sum into groups based on the exponent of 2 in their prime factorization. This exponent can be from 0 to 7.

In general, see that the sum of d_k for all integers $1 \le k \le 255$ divisible by 2^n but not by 2^{n+1} is just

$$\frac{2^n}{2^n} + \frac{3 \cdot 2^n}{2^n} + \frac{5 \cdot 2^n}{2^n} + \cdots + \frac{2^n(2^{8-n} - 1)}{2^n} = 1 + 3 + 5 + \cdots + (2^{8-n} - 1) = (2^{7-n})^2 = 4^{7-n}.$$

So we are summing the geometric series $1 + 4 + 4^2 + \cdots + 4^7 = \frac{4^8 - 1}{3} = \boxed{21845}$. $\square$

Exercise 24 (2017 AIME I). For a positive integer n, let d_n be the units digit of $1 + 2 + \cdots + n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

2.4 Induction

Induction is a powerful strategy that can be used to prove a multitude of different theorems. There are already a multitude of great handouts covering induction, and the technique is more common in AIME-style problems and Olympiad problems. This is why our explanation of this topic here will be a bit brief.

Let $P(n)$ be a statement that is true or false depending on the value of some integer n. For example, $P(n)$ might be "$1+3=|n|$." Then, $P(n)$ is true if $n=\pm 4$. It could also be "$\sqrt{n^2}=n$." Then, $P(n)$ is true for all nonnegative integers n.

The most basic form of the *Principle of Mathematical Induction*, or just *Induction*, states that if $P(1)$ is true, and if the validity $P(k)$ implies the validity of $P(k+1)$ for any positive integer k, then we know that $P(n)$ is valid for all positive integers n (It doesn't say anything about $P(n)$ for values of $n<1$). Here is some useful terminology that is commonly used in induction proofs.

1. Showing that $P(1)$ is true is known as showing the "*base case*."
2. Showing that the validity $P(k)$ implies the validity of $P(k+1)$ for any positive integer k is known as showing the "*inductive step*."

Note that all induction proofs must show the base case AND the inductive step. A common beginner mistake when writing up induction proofs is to neglect either the base case or the inductive step. Now, let's see this principle in action.

Example 25

For all positive integers n, the sum of the first n positive squares is $\frac{n(n+1)(2n+1)}{6}$. In other words:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. In this case, our statement $P(n)$ is going to be:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We want to show that $P(n)$ holds, or is true, for all positive integers n. We use induction.

Our base case is trivial to show. The sum of the first 1 positive squares is just $1^2=1=\frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$, so $P(1)$ is true.

For our inductive step, we have to show that for a positive integer k, that if

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6},$$

then

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

To show this, note that

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Which shows our inductive step. By induction, we have shown that the formula holds for all positive integers n. $\square$

Induction is often compared to toppling over a row of dominoes: If we topple the first domino, and if the dominoes are placed such that if one domino falls then the next domino will also falls, then all the dominoes in the row will fall. Similarly, if we prove that $P(1)$ is true, and we have shown that IF $P(k)$ is true THEN $P(k+1)$ is also true as well for all positive integers k, then we have shown that $P(2), P(3), \ldots$ will all be true.

Induction can prove to be a powerful tool, but its main drawback in a computational setting is that we can only proceed with inductive logic in a summation problem if we already have an idea of what the sum evaluates to. So the technique of induction can justify our logic, but we first have to get an idea of what a sum evaluates to, which ultimately takes quite a bit of intuition, wishful thinking and more often then not, plain old brute force. This is why induction is not a common technique in most computational contests, but is commonly seen at the olympiad level (where you are asked to give reasoning for a mathematical statement). We also often use induction when writing solutions to contest problems, as in solutions we have to rigorously prove any assumptions we made.

When solving a summation problem in a test, competitors often evaluate by hand versions of the given sum that use smaller values in place of the given ones. Based on what these smaller versions evaluate too, one might be able to make an educated guess on what the original sum evaluates to. For example, we might guess that the sum of the first 1001 odd numbers is 1001^2, if we tried finding the sum of the first 3 odd numbers or first 4 odd numbers. Induction is often used to verify whatever guess we find.

Remark 26. Induction is a simple concept that unfortunately has to be explained in confusing mathematical terminology for the sake of rigor. Before you read the next section, think deeply about how the logic used in an Induction proof is able to show the desired conclusion.

2.4.1 Exercises

Exercise 27. Prove that the sum of the first n positive cubes equals

$$\left(\frac{(n)(n+1)}{2}\right)^2.$$

Exercise 28. The Fibonacci numbers are an infinitely long sequence of integers $F_0, F_1, F_2 \ldots$ defined so that $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 1$. Prove that for all positive integers n,

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

2.5 Simplifying the Summand

Oftentimes rewriting or simplifying a scary looking summand can reduce the problem to a trivial exercise. A special case of this is splitting the summand, which gets its own week (I can't stop hyping it up, can't I?)

Example 29 (2021 DMC 10)

$$S = \frac{(3^0+1)^3+1}{(3^0)^2+3^0+1} + \frac{(3^1+1)^3+1}{(3^1)^2+3^1+1} + \cdots + \frac{(3^{15}+1)^3+1}{(3^{15})^2+3^{15}+1}.$$

If S can be expressed as $\frac{3^a+b}{c}$ for 3 and c, and b and c relatively prime, and what is $a+b+c$?

Solution. Note, each term can be expressed in the form $\frac{(x+1)^3+1}{x^2+x+1}$ where x is some power of 3. The numerator expands to x^3+3x^2+3x+2. Let's try factoring this. By Rational Root Theorem, the possible rational roots are $2, 1, -1$ and -2, and testing each of these, we see only $x = -2$ works.

Using polynomial division, we see $\frac{x^3+3x^2+3x+2}{x+2} = x^2+x+1$. This means $\frac{(x+1)^3+1}{x^2+x+1} = x+2$. The sum is

$$3^0+3^1+3^2+\cdots 3^{15}+2\cdot 16,$$

and using $3^0+3^1+3^2+\cdots+3^n = \frac{3^{n+1}-1}{2}$, we get S is

$$\frac{3^{16}-1}{2}+32 = \frac{3^{16}+63}{2},$$

so $a+b+c = 16+63+2 = \boxed{84}$. □

In this mock test, I bet a lot of people were intimidated by this scary and bizarre looking sum. However, once we saw the idea of polynomials and polynomial division in the summand, the problem quickly became a quick computation.

This isn't as much of a type of manipulation as it is a general strategy and mentality that you should be having coming into a summation question. As we saw here, sometimes we'll have to zoom in on the individual summands and consider other ways in which we can think about them.

2.5.1 Exercises

Exercise 30 (Purple Comet). Let $S = 2^4+3^4+5^4+7^4+\cdots+17497^4$ be the sum of the fourth powers of the first 2014 prime numbers. Find the remainder when S is divided by 240.

2.6 Evaluating Infinite Sums

In this section, we return to the idea of *infinite sums*. Earlier, we said that an infinite sum *converges* if *partial sums* of the infinite sum come arbitrarily close to some constant. Let's refine this crude idea. In general, infinite sums share the following rule:

Fact 31. An infinite sum
$$\sum_{i=k}^{\infty} f(i)$$
converges to a real number r if and only if for arbitrarily large integers $n > k$, the sum
$$\sum_{i=k}^{n} f(i)$$
becomes arbitrarily close to r (we say it "approaches" r).

Let's see how we can use this to prove the following crucial formula:

Fact 32. Let r be some real number with absolute value less than one. Then the sum of an infinite geometric sequence with initial term a_1,
$$\sum_{i=1}^{\infty} a_1 r^{i-1} = a_1 + a_1 r + a_1 r^2 + \ldots$$
is
$$\frac{a_1}{1-r}.$$

Proof. Notice that for arbitrarily large positive integers n,
$$\sum_{i=1}^{n} a_1 r^{i-1} = a_1 + a_1 r + a_1 r^2 + \ldots a_1 r^{n-1} = \frac{a_1(r^n - 1)}{r-1}.$$
But since the absolute value of r is less than 1, r^n becomes arbitrarily small and so $r^n - 1$ gets arbitrarily close to -1. Thus, our sum approaches approaches $\frac{a_1}{1-r}$. □

To evaluate an infinite sum S that we are given converges, sometimes we might express our sum in terms of itself, creating an equation that we can then solve for in terms of S. Consider this alternate "proof" of the sum of an infinite geometric sequence:

Proof. We're trying to find
$$S = \sum_{i=1}^{n} a_1 r^{i-1} = a_1 + a_1 r + a_1 r^2 + \ldots.$$

Notice that

$$S \cdot r = r \cdot \left(\sum_{i=1}^{n} a_1 r^{i-1} \right) = a_1 r + a_1 r^2 + a_1 r^3 + \ldots$$

Subtracting the two equations yields $S(1-r) = a_1 \implies S = \frac{a_1}{1-r}$. □

But then, why does this reasoning work for only values of r with an absolute value less than one? In the first proof this is explicitly addressed, but in this one it's nowhere to be found. The problem with this proof is that it assumes that S converges, when this might not be the case. Of course, all infinite sums you're going to see in competitions are going to be convergent (cause how else would there be an answer?) so you don't have to be too careful. Once we know that our infinite sum converges to a number, we are free to manipulate it to express that number in terms of itself and solve for its value.

Some infinite summations can be calculated as the product or the sum of other infinite summations that we already know converge. Just like we can manipulate real numbers to get a real result, we can manipulate converging infinite summations to form a new infinite summation that we also know converges! Here's an example:

Example 33 (2018 AMC 12A)

Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. Determine the infinite sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \cdots$$

of the reciprocals of the elements of A.

Solution. Notice that this infinite sum can actually be represented as the product of three geometric sequences. This follows from the fact that each element of A can be expressed exactly one way as the product of a power of 2, a power of 3, and a power of 5.

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots\right)$$

Then, just use the formula for infinite geometric series to get $2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4}$. □

In this example, we were able to "factor" this infinite sum just like we would a polynomial, or a positive integer, and express it as the product of several infinite sums that we do know how to calculate. We don't even have to show that the sum is convergent as long as we can express it as a manipulation of sequences that we do know converge. This is a common idea in infinite summation problems.

Let's look at an example from the challenging HMMT competition. Once we confirm that the sum in question converges, we will smoothly manipulate the sum to create an equation involving the value of the sum.

Example 34 (HMMT 2006)
Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find
$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

Solution. We have Fibonacci numbers (see exercise 28) in the numerator and powers of 4 in the denominator. Tough! Thankfully, at least it's not hard to verify via induction that $a_n < 2^n$ for all positive integers n, and
$$\sum_{n=1}^{\infty} \frac{2^n}{4^{n+1}}$$
is a geometric sequence with ratio $\frac{2}{4} = \frac{1}{2} < 1$ so it converges. So our desired sum is bounded by above and clearly it is bounded below as well, so it must converge (Since this was a computational contest you could've just assumed this from the start).

The fact that $a_{n+2} = a_{n+1} + a_n$ gives us an idea. If we're able to manipulate our sum X to get $a_{n+1} + a_n$ somewhere, then we'll be able to write this as a_{n+2} by definition. So maybe we can use our recursive formula for our sequence a_1, a_2 in order to express X in terms of itself. And indeed you played around for a bit, you may have found the following:

Because the sum converges, we may observe the following manipulation. If X is our desired sum, then see that $4X = \sum_{n=1}^{\infty} \frac{a_n}{4^n} = \sum_{n=0}^{\infty} \frac{a_{n+1}}{4^{n+1}}$. Furthermore,

$$\begin{aligned}
4X + X &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{4^{n+1}} + \sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}} \\
&= \frac{a_1}{4} + \sum_{n=1}^{\infty} \left(\frac{a_n}{4^{n+1}} + \frac{a_{n+1}}{4^{n+1}} \right) \\
&= \frac{a_1}{4} + \sum_{n=1}^{\infty} \left(\frac{a_{n+2}}{4^{n+1}} \right) \\
&= \frac{a_1}{4} - \frac{a_2}{4^1} - \frac{a_1}{4^0} + 16 \sum_{n=1}^{\infty} \left(\frac{a_n}{4^{n+1}} \right) \\
&= 1 + 16X.
\end{aligned}$$

So $4X + X = 1 + 16X$ and solving this gives $X = \boxed{\frac{1}{11}}$. □

In this problem, we had a sum X that we knew converged to some real number. We first considered the sum of $4X$ and X. Then, we were able to use the recursive formula for the sequence $a_1, a_2, \ldots$ in a slick way to express this result in terms of our original sum X, ending up with a single variable equation that we solved to finish the problem swiftly.

There's another way to solve this problem that is arguably slicker, involving subtracting equations instead of adding them. In other words, we use $a_{n+2} - a_{n+1} = a_n$ this time around. Using our knowledge that it converges from above, let S denote the given sum. We again multiply this by 4, but this time, instead of adding another S to $4S$, we subtract an S from it! Then, we solve for S as follows:

$$\begin{aligned}
S &= \frac{1}{4^2} + \frac{1}{4^3} + \frac{2}{4^4} + \cdots \\
4S &= \frac{1}{4^2} + \frac{2}{4^3} + \frac{3}{4^4} + \cdots + \frac{1}{4} \\
4S - S &= \frac{0}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots + \frac{1}{4} \\
3S &= \frac{1}{4} + \frac{S}{4} \\
12S &= 1 + S \\
S &= \boxed{\frac{1}{11}}.
\end{aligned}$$

A lot of competitors are keen on adding equations and expressions to each other, but are reluctant to subtract equations and expressions from each other. The successful competitor should be familiar with both ideas.

Onto our last example for this handout, it's a pretty hard one!

Example 35 (AoPS Intermediate Algebra)
Let x be a complex number such that $x^{2011} = 1$ and $x \neq 1$. Compute the sum

$$\frac{x^2}{x-1} + \frac{x^4}{x^2-1} + \frac{x^6}{x^3-1} + \cdots + \frac{x^{4020}}{x^{2010}-1}.$$

Solution. Let S denote the given sum, so

$$S = \sum_{n=1}^{2010} \frac{x^{2n}}{x^n - 1}. \tag{2.1}$$

Reversing the order of the terms, we get S also equals

$$S = \sum_{n=1}^{2010} \frac{x^{4022-2n}}{x^{2011-n} - 1}.$$

We are given $x^{2011} = 1$, thus

$$\frac{x^{4022-2n}}{x^{2011-n}-1} = \frac{x^{-2n}}{x^{-n}-1} = \frac{1}{x^n - x^{2n}} = \frac{1}{x^n(1-x^n)},$$

meaning

$$S = \sum_{n=1}^{2010} \frac{1}{x^n(1-x^n)}. \tag{2.2}$$

Now, the key step is to add equations (2.1) and (2.2). Adding them, we get

$$2S = \sum_{n=1}^{2010} \frac{x^{3n}-1}{x^n(x^n-1)},$$

and factoring $x^{3n}-1$ as $(x^n-1)(x^{2n}+x^n+1)$, we get

$$2S = \sum_{n=1}^{2010} \frac{x^{2n}+x^n+1}{x^n}$$

$$2S = \sum_{n=1}^{2010} x^n + 1 + \frac{1}{x^n}.$$

Then, this sum becomes

$$\begin{aligned} 2S &= x + x^2 + \cdots + x^{2010} + 2010 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{2010}} \\ &= x + x^2 + \cdots + x^{2010} + 2010 + \frac{x^{2010} + x^{2009} + \cdots + x}{x^{2011}}. \end{aligned}$$

However, since $x^{2011} - 1 = (x-1)(x^{2010} + x^{2009} + \cdots + x + 1) = 0$, we have $x^{2010} + x^{2009} + \cdots + x = -1$, and the value of $2S$ is simply $-1 + 2010 - 1 = 2008$, implying the answer is $\boxed{1004}$. □

Remark 36. Seeing a slick solution like this is not an easy task! First, you must see the 'reversing trick' and substituting $x^{2011} = 1$. Then, you also must see the 'adding trick', which is a fairly common thing in manipulations problems such as these.

Some words of wisdom:

1. Whenever given equations like $x^n = 1$, take advantage and substitute wherever you can.
2. Be warned, not all summations are as clean as these, remember the polar form of the Roots of Unity! (Note: The solutions to $x^n = 1$ are called the n-th Roots of Unity. You will learn about these very briefly later in the handout.)
3. Simplify, simplify, simplify! Working with nasty expressions is no simple task, working with simpler expressions is much easier.

Remark 37. Why would we reverse the series? The motivation was to rewrite the sum in a different form, which might be easier to work with. The main reason is that $4022 = 2 \cdot 2011$, so we would get $x^{4022} = 1$, and we could rewrite the terms as $\frac{x^{4022}-2n}{x^{2011-n}-1}$. We know that $x^{2011} = x^{4022} = 1$, so this is not very hard to work with.

Remark 38. Why would we add the two together? The two are screaming to be added together. They both share $(x^n - 1)$ (one has the negation of this, but still they are basically the same) in the denominator, so it would be pretty clean.

2.6.1 Exercises

Exercise 39. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

Exercise 40. Show that if the sum of an infinite arithmetic sequence converges, then it must converge to the number 0.

2.7 Summary

In this handout, we took a detailed look at some critical strategies that we use in summation problems. First, we learned that we can pair up terms (usually in a first-last, second-secondlast, third-thirdlast, etc. strategy). We also learned how we can pair or group terms in other ways to simplify the sum. We saw how induction is a powerful technique, and one of its applications is proving that summations evaluate to a value, or some closed-form formula involving summations holds. We took a first look at summation strategies that involve transforming the summand in some particular way, which we'll take a deep dive into in later weeks of this course. After, we looked at infinite summations, which mainly center around proving that an infinite summation converges to some real number, or showing that the infinite summation is equivalent to the manipulation of several other convergent infinite summations. Finally, we looked at certain "special" techniques to evaluate infinite sums, such as expressing the infinite sum as the sum or product of other infinite sums or subtracting the sum from a multiple of itself (dubbed the "subtracting trick").

The vast diversity of summation techniques makes choosing the right one a bit of a guessing game (and things will only get worse once we get to Telescoping Sums). Oftentimes the ability to quickly navigate an ocean of possible strategies is what separates the great competitor from the exceptional.

2.8 Important Terms

1. Pairing up terms
2. Index
3. Grouping terms
4. Principle of Mathematical Induction or Induction
5. Base case
6. Inductive step
7. Infinite sum
8. Converge
9. Partial sums (Of an infinite sum)

2.9 Practice Problems

Problem 8. Show that the sum of the first n positive integers is $\frac{n(n+1)}{2}$ through induction.

Problem 9 (AMC 10B 2016). The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

Problem 10 (2014 AMC 12A). A five-digit palindrome is a positive integer with respective digits $abcba$, where a is non-zero. Let S be the sum of all five-digit palindromes. What is the sum of the digits of S?

Problem 11 (MathLeague). Evaluate

$$\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)+\cdots.$$

Problem 12. Find the value of

$$\sum_{r=0}^{2017}\frac{1}{3^r+\sqrt{3^{2017}}}.$$

Problem 13 (2020 AMC 10A Problem 21). There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1}=2^{a_1}+2^{a_2}+\ldots+2^{a_k}.$$

What is k?

Problem 14 (Intermediate Algebra). Let $\tau=\frac{1+\sqrt{5}}{2}$. Find

$$\sum_{n=0}^{\infty}\frac{\lfloor \tau^n \rceil}{2^n}.$$

Note: For a real number x, $\lfloor x \rceil$ denotes the integer closest to x.