Week 1: Problem Set

Summations Mini-course

ALP

July 1, 2021

This set of problems is meant to be done independently with no outside help besides a four-function calculator if necessary. Tools like Wolfram Alpha and Desmos are not allowed. You may not ask for help from anyone besides an ALP instructor. Problems are meant to be a challenge, and we do not intend for everyone to complete all the problems. However, we encourage students to give each problem at least a solid attempt.

1 Problems

Minimum is [0 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "Mathematicians just love sigma notation for two reasons. First, it provides a convenient way to express a long or even infinite series. But even more important, it looks really cool and scary, which frightens nonmathematicians into revering mathematicians and paying them more money."
>
> *Calculus II for Dummies*

[3 ♟] Problem 1 (AMC). How many sets of two or more consecutive positive integers have a sum of 15?

[3 ♟] Problem 2 (AMC). How many terms are there in the arithmetic sequence $13, 16, 19, \ldots, 70, 73$?

[3 ♟] Problem 3 (AMC). The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

[4 ♟] Problem 4. Evaluate $|1+2-3-4+5+6-7-8+\cdots+2018-2019-2020|$.

[5 ♟] Problem 5. If

$$(-2)+(-2)^2+(-2)^3+\cdots+(-2)^{100}$$

can be expressed as $\frac{2^a-b}{c}$ in simplest form, find $a+b+c$.

[5 ♟] Problem 6 (AHSME). The sum of n terms of an arithmetic progression is 153, and the common difference is 2. If the first term is an integer, and $n > 1$, then what is the number of possible values for n?

[5 ♟] Problem 7 (AIME). Initially Alex, Betty, and Charlie had a total of 444 peanuts. Charlie had the most peanuts, and Alex had the least. The three numbers of peanuts that each person had formed a geometric progression. Alex eats 5 of his peanuts, Betty eats 9 of her peanuts, and Charlie eats 25 of his peanuts. Now the three numbers of peanuts each person has forms an arithmetic progression. Find the number of peanuts Alex had initially.

[5 ♟] Problem 8 (AHSME). Let $a_1, a_2, \cdots, a_k$ be a finite arithmetic sequence with

$$a_4+a_7+a_{10}=17$$

and

$$a_4+a_5+\cdots+a_{13}+a_{14}=77.$$

If $a_k=13$, then what is k?

[6 ♟] Problem 9 (AIME). The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

[7 ♟] Problem 10 (AIME). A strictly increasing sequence of positive integers $a_1, a_2, a_3, \cdots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1.

[8 ♟] Problem 11 (AMC). The first four terms in an arithmetic sequence are $x+y$, $x-y$, xy, and x/y, in that order. The fifth term can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

[9 ♟] Problem 12. Let

$$a = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \cdots + \frac{1001^2}{2001}$$

and

$$b = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \cdots + \frac{1001^2}{2003}.$$

What is the integer closest to $a-b$?

[14 ♟] Problem 13 (Intermediate Algebra). Let $n \geq 2$ be a positive integer. The equation

$$nx^{n-1} + (n-1)x^{n-2} + \cdots + 3x^2 + 2x + 1 = n^2$$

has a rational root between 1 and 2. Find this rational root in terms of n.

Note: Problem 13 is a writing problem! This means that for this problem, you must write a solution. You might get partial credit if you submit an incomplete solution depending on how far you progressed, so make sure to submit even if you didn't fully solve the problem. You may use LaTeX or scan a hand-written solution. Submit via upload in the Google Form.