Week 4: Problem Set

Summations Mini-course

ALP

July 24, 2021

This set of problems is meant to be done independently with no outside help besides a four-function calculator if necessary. Tools like Wolfram Alpha and Desmos are not allowed. You may not ask for help from anyone besides an ALP instructor. Problems are meant to be a challenge, and we do not intend for everyone to complete all the problems. However, we encourage students to give each problem at least a solid attempt.

1 Problems

Minimum is [0 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "It seems to me that no one picture can ever be a final summation of a personality. There are so many facets in every human being that it is impossible to present them all in one photograph."
>
> *Arnold Newman*

[3 ♟] Problem 1 (AMC). The product

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})$$

can be simplified as $a^n - b^n$ for positive integers a, b, n. where a and b are minimized and n is maximized. Find $a+b+n$.

[3 ♟] Problem 2 (HMMT). Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. The product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \ldots$$

can be written as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m+n$.

[4 ♟] Problem 3 (HMMT). Let $f(n)$ be the number of distinct prime divisors of n less than 6. Compute

$$\sum_{n=1}^{2020} f(n)^2.$$

[4 ♟] Problem 4 (HMMT). Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the remainder when

$$\sum_{\substack{2 \le n \le 50 \\ \gcd(n,50)=1}} \phi^!(n)$$

is divided by 50.

[4 ♟] Problem 5. For each positive integer n let d_n denote the greatest common divisor of n and $(2019-n)$. Find the value of

$$d_1 + d_2 + \cdots d_{2018} + d_{2019}.$$

[5 ♟] Problem 6 (HMMT). Daniel wrote all the positive integers from 1 to n inclusive on a piece of paper. After careful observation, he realized that the sum of all the digits that he wrote was exactly 10,000. Find n.

[6 ♟] Problem 7 (HrishiP). Let $S = (2 \cdot 3 \cdot 5)^5$. Find the number of divisors that

$$\sum_{nm|S} \gcd(n, m)$$

has.

[8 ♟] Problem 8 (HMMT). Given that

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}}$$

can be represented as $\frac{m}{n}$ for coprime positive integers m and n, find $m + n$.

[9 ♟] Problem 9 (HMMT). Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c \in \mathbb{N}}} (a + b + c),$$

where $\mathbb{N}$ denotes the positive integers.

[9 ♟] Problem 10. Find the number of positive divisors of

$$\sum_{i=0}^{2021} \sum_{j=0}^{2021} \frac{(2021 + i + j)!}{2021! \times i! \times j! \times 3^{i+j}}$$

[10 ♟] Problem 11 (Bulgaria). Let $\omega(n)$ denote the number of not necessarily distinct prime divisors of n. For example, $12 = 3 \cdot 2 \cdot 2 \Rightarrow \omega(12) = 3$, since there are 3 (not needed to be distinct) primes that divide 12. Compute

$$\sum_{n=1}^{1989} (-1)^{\omega(n)} \left\lfloor \frac{1989}{n} \right\rfloor .$$

[10 ♟] Problem 12 (HMMT). For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$. (The operation $\oplus$ is called the exclusive or.) Simplify

$$\sum_{k=0}^{2^n - 1} \left(k \oplus \left\lfloor \frac{k}{2} \right\rfloor \right)$$

Note: Problem 12 is a writing problem! This means that for this problem, you must write a solution. You might get partial credit if you submit an incomplete solution depending on how far you progressed, so make sure to submit even if you didn't fully solve the problem. You may use $\LaTeX$ or scan a hand-written solution. Submit via upload in the Google Form.