Polynomials

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Polynomials are a class of functions.

Definition 0.1 (Polynomial)
For constants $a_0, a_1, \ldots a_n, a_n \neq 0$, and non-negative integer n,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a **polynomial**.

For example, $x^3 + 12, 4xy^2 + 3x - 5$, etc. are polynomials.

- The **degree** of polynomial $P(x)$ is the largest exponent of x in the polynomial.
- $a_1, a_2, \ldots, a_n$ are the **coefficients** of polynomial $P(x)$ with degree n.
- a_n is the **leading coefficient** of polynomial $P(x)$ with degree n, and we call $P(x)$ **monic** if $a_n = 1$.
- A **root** of polynomial $P(x)$ is a value of x for $P(x) = 0$.

1 Polynomial Arithmetic

As with numbers, polynomials can also be added, subtracted, multiplied, and divided. We call combining polynomials combining like terms. Let's take a look.

1.1 Addition

Adding polynomials is simple; just combine them like terms.

Example 1.1
Add
$$(x^2 + 4x + 4) + (2x^2 + 5x + 9).$$

Solution. We have $x^2 + 2x^2 = 3x^2$. Next, $4x + 5x = 9x$. Lastly, $4 + 9 = 13$. So, this is just $\boxed{3x^2 + 9x + 13}$. □

In general, if we define two polynomials

$$P(X) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 \text{ and } Q(X) = q_m x^m + p_{m-1} x^{m-1} + \cdots + q_1 x + q_0,$$

where $n > m$, then

$$P(X) + Q(X) = p_n x^n + \cdots + (p_m + q_m) x^m + (p_{m-1} + q_{m-1}) x^{m-1} + \cdots + (p_1 + q_1) x + (p_0 + q_0).$$

1.2 Subtraction

Subtracting is the same thing; think of subtraction as adding negatives.

Example 1.2

Let $P(x) = x^4 + 2x^2 + 1$ and $Q(x) = x^5 + 3x^4 + x^2 + 4x + 1$. Find $Q(x) - P(x)$.

Solution. We need to find

$$Q(x) - P(x) = (x^5 + 3x^4 + x^2 + 4x + 1) - (x^4 + 2x^2 + 1).$$

This is just

$$(x^5 + 3x^4 + x^2 + 4x + 1) - x^4 - 2x^2 - 1 = \boxed{x^5 + 2x^4 - x^2 + 4x}.$$

$\square$

In general, given our previous definition of $P(X)$ and $Q(X)$,

$$P(X) - Q(X) = p_n x^n + \cdots + (p_m - q_m)x^m + (p_{m-1} - q_{m-1})x^{m-1} + \cdots + (p_1 - q_1)x + (p_0 - q_0).$$

1.3 Multiplication

When multiplying polynomials together, we use the distributive property, which multiplies each term in one polynomial to each term in the other polynomial. Then, we combine like terms and arrange them so that the power of x is in decreasing order.

Example 1.3

If $F(x) = x^2 + 2x + 4$ and $S(x) = x^2 + 5x + 9$, find $F(x) \times S(x)$.

Solution. We multiply each term of $F(x)$ by each term of $S(x)$ and simplifying (by combining):

$$x^2 \cdot x^2 + x^2 \cdot 5x + x^2 \cdot 9 + 2x \cdot x^2 + 2x \cdot 5x + 2x \cdot 9 + 4 \cdot x^2 + 4 \cdot 5x + 4 \cdot 9$$
$$x^4 + 5x^3 + 9x^2 + 2x^3 + 10x^2 + 18x + 4x^2 + 20x + 36$$
$$\boxed{x^4 + 7x^3 + 23x^2 + 38x + 36}.$$

$\square$

In general, given our previous definition of $P(X)$ and $Q(X)$,

$$P(X) \cdot Q(X) = q_n p_m x^{m+n} + \cdots + (p_2 q_0 + p_1 q_1 + p_0 q_2)x^2 + (p_1 q_0 + p_0 q_1)x + p_0 q_0.$$

1.4 Division

Theorem 1.4

For every polynomial $p(x)$, $q(x)$ and $r(x)$ can be uniquely chosen such that $f(x)$ can be expressed as

$$f(x) = p(x)q(x) + r(x).$$

It is required that $\deg r < \deg p$.

Exercise 1.5. Prove this theorem with induction!

When $r(x) = 0$, we say that $p(x), q(x)$, are **factors** of $f(x)$, just like we do in for integers.

> **Remark 1.6.** This is the same idea as the **division theorem** of number theory.

But how do we divide polynomials? Recall the method of long division, shown here.

$$
\begin{array}{r|r}
 & 16 \\ \hline
7) & 112 \\
 & -112 \\ \hline
 & 0
\end{array}
$$

We can do similarly for polynomials.

$$
\begin{array}{r|l}
 & x^2 + 0x + 1 \\ \hline
x+1) & x^3 + x^2 + x + 1 \\
 & -x^3 - x^2 \\ \hline
 & 0x^2 + x \\
 & 0x^2 + 0x \\ \hline
 & x + 1 \\
 & -x - 1 \\ \hline
 & 0
\end{array}
$$

In general, this is the process: We first take the polynomial that will be the divisor ($p(x)$ from the division theorem). If the degree of the polynomial getting divided is n more than that of the polynomial divisor, multiply $p(x)$ by ax^m, where a is a constant such that a times the leading coefficient of $p(x)$ equals the leading coefficient of the polynomial being divided. Note that this works on any 2 polynomials. Here some examples:

$$
\begin{array}{r|l}
 & x^3 + 2x^2 - x + 0 \\ \hline
x^2+x+1) & x^5 + 3x^4 + 2x^3 + x^2 + x + 1 \\
 & -x^5 - x^4 - x^3 \\ \hline
 & 2x^4 + x^3 + x^2 \\
 & -2x^4 - 2x^3 - 2x^2 \\ \hline
 & -x^3 - x^2 + x \\
 & x^3 + x^2 + x \\ \hline
 & 0x^2 + 2x + 1 \\
 & 0x^2 + 0x + 0 \\ \hline
 & 2x + 1
\end{array}
$$

Another one:

$$
\begin{array}{r|l}
 & x^3 - 18x^2 + 96x - 120 \\ \hline
x^2-3x+2) & x^5 - 21x^4 + 152x^3 - 444x^2 + 432x \\
 & -x^5 + 3x^4 - 2x^3 \\ \hline
 & -18x^4 + 150x^3 - 444x^2 \\
 & 18x^4 - 54x^3 + 36x^2 \\ \hline
 & 96x^3 - 408x^2 + 432x \\
 & -96x^3 + 288x^2 - 192x \\ \hline
 & -120x^2 + 240x \\
 & 120x^2 - 360x + 240 \\ \hline
 & -120x + 240
\end{array}
$$

Notice the first example has a $2x+1$ at the bottom, and the second one has a $-120x+240$. These are the **remainders** of the polynomial. They cannot get divided anymore.

What if you wanted to find this remainder quickly?

Theorem 1.7 (Remainder Theorem)
The remainder when $f(x)$ is divided by $x-k$ is $f(k)$.

Proof. Theorem 1.4 is helpful here. Our divisor is $p(x) = (x-k)$. In other words, we have

$$f(x) = (x-k)q(x) + r(x).$$

Setting $x = k$, we we find that $f(k) = r(k)$, the remainder. □

This leads to an important result.

Corollary 1.8 (Factor Theorem)
If r is a root of $f(x)$ (specifically, $f(r) = 0$), then $x - r$ is a factor of $f(x)$.

Proof. A simple application of the remainder theorem: take $x = r$. □

Example 1.9 (BMC)
Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

Solution. Since we are dividing by $x^3 - x$, the $\deg r$ is at most 2. Therefore, let $r(x)$, the remainder, be $ax^2 + bx + c$, assuming a, b, c is real.

When plugging in any arbitrary value of x, we know that $x^{81} + x^{49} + x^{25} + x^9 + x$ is a huge number, plus $p(x)$, the quotient, is unknown.

To tackle the latter problem, we set $x^3 - x = 0$. The values of x that satisfy this equation is $-1, 0, 1$. Indeed, these values of x end the former problem, being the number isn't huge anymore.

Plugging it in to $r(x)$,

$$r(-1) = a - b + c = -5,$$

$$r(0) = c = 0,$$

$$r(1) = a + b + c = 5.$$

Solving the system of equations, we have $(a, b, c) = (0, 5, 0)$, and so our remainder is $\boxed{5x}$. □

> **Remark 1.10.** Using long division here would take a while and would be a sub-optimal strategy during a timed competition

Example 1.11 (2017 AMC 12A Problem 23)
For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

Solution. By the Factor theorem, we can let $g(x) = (x-a)(x-b)(x-c)$. Also, we know that

$$(x-a)(x-b)(x-c) \mid f(x),$$

since a, b, c are roots of $f(x)$. Recall the division theorem. It tells us that $f(x) = p(x)g(x) + r(x)$, for some polynomials $p(x)$ and $r(x)$. However, since $g(x)$ divides $f(x)$, we know that $r(x) = 0$. We know that $f(x)$ has degree 4, and $g(x)$ has degree 3, so that implies $p(x)$ has degree 1, or it is linear.

So let $p(x) = x + q$. Then,

$$f(x) = p(x)g(x)$$
$$x^4 + x^3 + bx^2 + 100x + c = (x^3 + ax^2 + x + 10)(x + q).$$

The coefficients of both sides of that equality is equal. So, looking at the x^3 term. We have that

$$1 = a + q.$$

Next, the coefficient of x gives

$$100 = 10 + q \implies q = 90.$$

Thus, $a = -89$. You may want to compute roots or find b, but that is not needed. Note that $f(1) = p(1)g(1)$, and we already know what $p(x), g(x)$ is. Performing computation, we find that

$$p(1) \times g(1) = 91 \times -77 = \boxed{-7007}.$$

□

2 Roots

Definition 2.1 (Root)
A **root** of a polynomials $P(x)$ is a value r such that $P(r) = 0$. Polynomials can have multiple roots.

You know how to find the roots of a linear or quadratic polynomial, but how do we find roots of higher degree polynomials? We discuss some methods here.

Firstly, how many roots does a polynomial have? Here is the answer:

Theorem 2.2 (Fundamental Theorem of Algebra)
A polynomial with complex coefficients $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has at least one (distinct) complex root.

The proof is too advanced and unimportant (not AMC level at all). Notice that this implies that a polynomial of degree n has n roots, of which we include multiplicity. This fact leads to many corollaries.

We can use the Factor Theorem from the last section to quickly find roots. Here is an example.

Example 2.3
Find the roots of
$$f(x) = x^3 - 6x^2 + 11x - 6.$$

Solution. It is a good idea to look for some easily accessible roots. We can see that $f(1) = 0$. By the Factor theorem, we can factor this out of $f(x)$. It will be of the form

$$f(x) = (x-1)(x^2 + ax + b).$$

We must have $b = 6$, so that the constant term is -6. Similarly, we must have $-ax^2 - x^2 = -6x^2$, so $a = 5$. Now, we have

$$f(x) = (x-1)(x^2 - 5x + 6.)$$

By the usual methods of solving quadratics, we can find the roots of $x^2 - 5x + 6$ as $2, 3$. So, the roots are $x \in \{1, 2, 3\}$. These are the only solutions by the Fundamental Theorem of Algebra. $\square$

That's great, but how can we know what roots to try guessing? Here, the rational root theorem is useful.

Theorem 2.4 (Rational Root Theorem)
Consider a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ such that all the a_i are integers, and $a_n \neq 0$. Then if $f(x)$ has a rational root $\frac{p}{q}$, such that p, q are relatively prime, then p is a divisor of a_0 and q is a divisor of a_n.

Proof. Given $\frac{p}{q}$ is a rational root of a polynomial $f(x) = a_n x^n + x_{n-1}x^{n-1} + \cdots + a_0$, where the a_n's are integers, we wish to show that $p|a_0$ and $q|a_n$. Since $\frac{p}{q}$ is a root,

$$0 = a_n \left(\frac{p}{q}\right)^n + \cdots + a_0$$

Multiplying by q^n, we have:

$$0 = a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_0 q^n$$

Examining this in modulo p, we have $a_0 q^n \equiv 0 \pmod{p}$. As q and p are relatively prime, $p|a_0$. With the same logic, but with modulo q, we have $q|a_n$, which completes the proof. $\square$

This is a very powerful theorem that narrows the list of possible solutions we need to test. Note that this implies that when $a_n = 0$ for such an $f(x)$, all the roots are integers.

Also, this theorem gives a list of **possible roots**, which may or may not work.

Example 2.5
Find all rational roots of
$$3x^3 - 5x^2 + 5x - 2.$$

Solution. By the Rational Roots theorem, we have the list of possible roots is
$$\pm 1, \pm 2, \pm \tfrac{1}{3}, \pm \tfrac{2}{3}.$$
If we test all of these, only $\frac{2}{3}$ works as a root. □

3 Vieta's Formulas and Newton's Sums

3.1 Vieta's Formulas

Theorem 3.1 (Vieta's Formulas for Quadratics)
Given a quadratic equation $P(x) = ax^2 + bx + c$ with roots r and s, then
$$r + s = -\frac{b}{a} \text{ and } rs = \frac{c}{a}.$$

Proof. If r and s are the roots of the quadratic $P(x)$, then by the Factor Theorem, we have:
$$P(x) = ax^2 + bx + c = a(x - r)(x - s).$$
Expanding out the RHS yields:
$$P(x) = ax^2 + bx + c = ax^2 - a(r + s) + ars.$$
Since, by matching-up coefficients of like-terms, $-a(r + s) = b$ and $ars = c$, it follows that
$$r + s = -\frac{b}{a} \text{ and } rs = \frac{c}{a}.$$
□

Example 3.2 (2003 AMC 10A Problem 5)
Let d and e denote the solutions of $2x^2+3x-5 = 0$. What is the value of $(d-1)(e-1)$?

Solution. Notice
$$(d - 1)(e - 1) = de - (d + e) + 1.$$
By Vieta's Formulas, we know $d + e = -\frac{3}{2}$ and $de = -\frac{5}{2}$, and
$$de - (d + e) + 1 = -\frac{5}{2} + \frac{3}{2} + 1 = \boxed{0}.$$
□

Example 3.3 (2003 AMC 10A Problem 18)
What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

Solution. Multiplying both sides by x,

$$\frac{2003}{2004}x^2 + x + 1 = 0.$$

We want to find the value of $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$. By Vieta's Formulas,

$$a + b = -\frac{1}{\frac{2003}{2004}} = -\frac{2004}{2003} \text{ and } ab = \frac{1}{\frac{2003}{2004}} = \frac{2004}{2003}.$$

Therefore, $\frac{a+b}{ab} = \frac{-\frac{2004}{2003}}{\frac{2004}{2003}} = \boxed{-1}$. □

Exercise 3.4 (Brilliant)**.** If the sum of the reciprocals of the roots of the quadratic

$$3x^2 + 7x + k = 0$$

is $\frac{7}{3}$, what is k?

We're able this Vieta's Formulas technique on quadratics, but could we generalize it for higher-degree polynomials as well? The answer to that is yes!

Before generalizing Vieta's Formulas for any polynomial, we need to define what elementary symmetric sums are.

Definition 3.5 (Elementary Symmetric Sum)
A multiset is similar to a set but can contain duplicate numbers. Let the internal product of a multiset be the product of all the elements in the multiset. Consider a multiset with n numbers. Then, consider all the multisets containing k of the numbers in the original multiset, where $1 \leq k \leq n$, and without repetition of elements. The k-th elementary symmetric sum of the multiset of n numbers is the sum of all internal products of the multisets with k numbers.

For example, if $n = 3$, and the set of numbers is $\{x, y, z\}$, then

- First Elementary Symmetric Sum: $S_1 = x + y + z$
- Second Elementary Symmetric Sum: $S_2 = xy + xz + yz$
- Third Elementary Symmetric Sum: $S_3 = xyz$

Exercise 3.6. What are the elementary symmetric sums of the six variables $\{q, r, s, t, u, v\}$.

Now that we have a good enough understanding of elementary symmetric sums, let's dive into generalizing Vieta's Formulas.

Theorem 3.7 (Generalization of Vieta's Formulas to Higher Degree Polynomials)

If $r_0, r_1, \ldots, r_n$ are the roots of polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and we denote s_k as the k-th elementary symmetric sum of the set of n variables $\{r_1, r_2, \ldots, r_{n-1}, r_n\}$, then

$$s_k = (-1)^k \frac{a_{n-k}}{a_n}.$$

The proof of the generalization of Vieta's Formulas for higher-degree polynomials is rather bashy and complicated, so it remains unimportant for the reader (and AMC-level, for that matter). Let's go through a few examples.

Example 3.8 (2008 AIME II Problem 7)

Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

Solution. By Vieta's Formulas, $r + s + t = 0$, and so

$$\begin{aligned} r+s &= -t, \\ s+t &= -r, \\ t+r &= -s. \end{aligned}$$

Substituting back into the problem statement, we want to find the value of $-(r^3+s^3+t^3)$. We know

$$r^3 + s^3 + t^3 - 3rst = (r+s+t)(r^2+s^2+t^2-rs-st-tr) = 0.$$

Therefore, $-(r^3 + s^3 + t^3) = -3rst$, and by Vieta's Formula, $-3rst = -3(\frac{-2008}{8}) = -3(-251) = \boxed{753}$. □

Example 3.9 (2020 USMCA)

Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^3}{d^2+2}.$$

Solution. Because a is a root of the quartic polynomial,

$$\begin{aligned} a^4 + 2a + 4 &= 0 \\ \implies a^4 + 2a &= -4 \\ \implies a^3 + 2 &= -\frac{4}{a}. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{a^2}{a^3+2} &= -\frac{a^3}{4} \\ &= -\frac{1}{4}\left(-\frac{4}{a}-2\right) \\ &= \frac{1}{a}+\frac{1}{2},\end{aligned}$$

and likewise for b, c, and d. Therefore,

$$\frac{a^2}{a^3+2}+\frac{b^2}{b^3+2}+\frac{c^2}{c^3+2}+\frac{d^3}{d^2+2} = 2+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

□

Example 3.10

There exists real a, b, c such that

$$\begin{aligned}a+b+c&=5,\\ a^2+b^2+c^2&=7, \text{ and}\\ a^3+b^3+c^3&=11.\end{aligned}$$

Find the value of

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution. Note that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}.$$

We have that $(a+b+c)^2$ is

$$a^2+b^2+c^2+2(ab+bc+ca)=25.$$

Substituting known values,

$$ab+bc+ca=9.$$

Next, we have that

$$\begin{aligned}(a^2+b^2+c^2)(a+b+c)&=7\cdot 5=35.\\ a^3+b^3+c^3+2(a^2b+b^2c+c^2a+ab^2+ac^2+bc^2)&=35\\ a^2b+b^2c+c^2a+ab^2+ac^2+bc^2&=12.\end{aligned}$$

Lastly,

$$\begin{aligned}(a+b+c)^3&=125\\ a^3+b^3+c^3+3(a^2b+b^2c+c^2a+ab^2+ac^2+bc^2)+6abc&=125\\ 11+3(12)+6abc&=125\\ abc&=13.\end{aligned}$$

So the desired value is $\boxed{\frac{9}{13}}$. □

Example 3.11 (1984 USAMO Problem 1)
In the polynomial $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$, the product of 2 of its roots is -32. Find k.

Solution. Utilizing Vieta's formulas, we have:

$$\begin{aligned} a+b+c+d &= 18, \\ ab+ac+ad+bc+bd+cd &= k, \\ abc+abd+acd+bcd &= -200, \\ abcd &= -1984. \end{aligned}$$

From the last of these conditions, we see that $cd = \frac{abcd}{ab} = \frac{-1984}{-32} = 62$. In this way, the later condition becomes $-32+ac+ad+bc+bd+62 = k$, thus $ac+ad+bc+bd = k-30$. The key incentive is to factor the left-hand side as a result of two binomials: $(a+b)(c+d) = k-30$, so we just need to know $a+b$ and $c+d$. Let $p = a+b$ and $q = c+d$. Substituting our known qualities for ab and cd into the third condition, $-200 = abc+abd+acd+bcd = ab(c+d)+cd(a+b)$, we have $-200 = -32(c+d)+62(a+b) = 62p-32q$. Plus, the primary condition, $a+b+c+d = 18$, gives $p+q = 18$. In this way we have two equations in p and q, which we solve to get $p = 4$ and $q = 14$. Subsequently, we have $\underbrace{(a+b)}_{4}\underbrace{(c+d)}_{14} = k-30$, yielding $k = 4\cdot 14+30 = \boxed{86}$. $\square$

For our last example, we look at a recent high-level AIME problem.

Example 3.12 (2019 AIME I Problem 10)
For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\le j<k\le 673} z_j z_k\right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let

$$\begin{aligned} P(x) &= (x-z_1)(x-z_2)\cdots(x-z_{673}). \\ &= x^{673} + Ax^{672} + Bx^{671} + \cdots. \end{aligned}$$

Then, we have that

$$P(x)^3 = x^{2019} + 3Ax^{2018} + \left(3A^2+3B\right)x^{2017} + \cdots.$$

Equating coefficients with the original polynomial of the problem statement, we have that $3A = 20$ and $3A^2 + 3B = 19$. Solving we get that

$$(a, b) = \left(\frac{20}{3}, -\frac{343}{9}\right).$$

So by Vieta's, the expression needed is $\frac{343}{9}$, and the answer is $\boxed{352}$. □

Exercise 3.13 (1999 JBMO Problem 1)**.** Let a, b, c, x, y be five real numbers such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$. If a, b, c are all distinct numbers prove that their sum is zero.

Exercise 3.14 (2014 AIME II Problem 5)**.** Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

3.2 Newton's Sums

Newton's Sums efficiently calculates sums of the roots of a polynomial raised to a power.

Theorem 3.15 (Newton's Sums for Quadratics)

Let r, s be the roots of a quadratic polynomial, s_k denote the kth elementary symmetric sum of the set of roots, and $p_k = r^k + s^k$. Then

$$p_k = p_{k-1}s_1 - p_{k-2}s_2,$$

for $k \geq 3$ with $p_1 = s_1$ and $p_2 = p_1s_1 - 2s_2$.

Example 3.16 (Brilliant)

Let a polynomial $P(x)$ be defined as $P(x) = x^2 - 2x + 6$ with complex roots a and b. Then what is the value of $a^{10} + b^{10}$?

Solution. By Vieta's Formulas, $s_1 = 2$ and $s_2 = 6$. Therefore,

$$p_k = 2p_{k-1} - 6p_{k-2}.$$

Given this, $p_1 = 2$ and $p_2 = -8$, and applying recurrence relations repeatedly gives

$$p_{10} = 2(-3808) - 6(-2528) = \boxed{7552}.$$

□

Before generalizing, let's check another small case.

Theorem 3.17 (Newton's Sums for Cubics)

Let r, s, t be the roots of a cubic polynomial, s_k denote the kth elementary symmetric sum of the set of roots, and $p_k = r^k + s^k + t^k$. Then

$$p_k = p_{k-1}s_1 - p_{k-2}s_2 + p_{k-3}s_3,$$

for $k \geq 4$ with $p_1 = s_1, p_2 = p_1s_1 - 2s_2$, and $p_3 = p_2s_1 - p_1s_2 + 3s_3$.

Let's revisit a problem we covered in Vieta's Formulas sub-section.

Example 3.18 (2008 AIME II Problem 7)
Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

Solution. By Vieta's Formulas, $r + s + t = 0$, and so

$$\begin{aligned} r + s &= -t, \\ s + t &= -r, \\ t + r &= -s. \end{aligned}$$

Substituting back into the problem statement, we wish to find the value of $-(r^3+s^3+t^3)$. By Newton's Sums for Cubics,

$$p_3 = p_2 s_1 - p_1 s_2 + 3s_3.$$

By Vieta's Formulas, $p_1 = s_1 = 0$, and therefore our equation evaluates to

$$p_3 = 3s_3.$$

Also by Vieta's Formulas, $s_3 = -\frac{2008}{8} = -251$. So, $p_3 = 3(-251) = -753$. Because we want to find $-p_3$, our answer is $\boxed{753}$. □

Finally, we generalize Newton's Sums for All Polynomials.

Theorem 3.19 (Newton's Sums for All Polynomials)
Let $x_1, x_2, \ldots, x_n$ be the roots of a n-th degree polynomial, s_k denote the kth elementary symmetric sum of the set of roots, and $p_k = x_1^k + x_2^k + \cdots + x_n^k$. If $i \leq k-1$, then

$$p_{k-1} = p_{k-2}s_1 - p_{k-3}s_2 + \cdots + (-1)^{k-3}p_1 s_{k-2} + (-1)^{k-2}s_{k-1}(k-1),$$

and for $i \geq k$, then

$$p_i = \sum_{j=1}^{k}(-1)^{j+1}e_j p_{i-j} = p_{i-1}s_1 - p_{i-2}s_2 + \cdots + (-1)^{k+1}p_{i-k}s_k.$$

Let's try a few more examples.

Example 3.20 (2019 AMC 12A Problem 17)
Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3-5x^2+8x-13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\, s_k + b\, s_{k-1} + c\, s_{k-2}$ for $k = 2,\ 3,\ \ldots$ What is $a + b + c$?

Solution. Applying Newton's Sums, we have

$$s_{k+1} + (-5)s_k + (8)s_{k-1} + (-13)s_{k-2} = 0,$$

so

$$s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2},$$

we get the answer as $5 + (-8) + 13 = \boxed{10}$. □

Recall example 4.3.9. That was pretty bashy and required many algebraic manipulations. However, with the power of Newton's Sums, the problem is more intuitive – build a polynomial or apply another tactic. Let's look at a similar problem!

Example 3.21 (AoPS)

There exist three positive integers such that the sum of the integers is 14, the sum of the squares of the integers is 78, and the sum of the cubes of the integers is 476. The sum of the reciprocals of the integers can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Solution. Let r, s, and t be the three positive integers, and suppose they are the roots of the cubic polynomial $x^3 + ax^2 + bx + c$. Denote $p_k = r^k + s^k + t^k$. Applying Newton's Sums,

$$\begin{aligned} p_1 + a &= 0, \\ p_2 + ap_1 + 2b &= 0, \\ p_3 + ap_2 + bp_1 + 3c &= 0. \end{aligned}$$

By the given conditions, $p_1 = 14, p_2 = 78, p_3 = 476$. Working our way through these equations, we get $a = -14, b = 59, c = -70$. Thus, our cubic polynomial is

$$x^3 - 14x^2 + 59x - 70.$$

Nicely enough, the cubic polynomial can be factored into $(x-2)(x-5)(x-7)$, and so the three positive integers are $2, 5$, and 7. Finally, we get that

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{7} = \frac{59}{70},$$

so our answer is $m + n = \boxed{129}$. □

Remark 3.22. Alternatively, one could use Guess-and-Check.

Exercise 3.23 (Brilliant)**.** If the roots of $p(x) = x^3 + 3x^2 + 4x - 8$ are a, b, and c, what is the value of

$$a^2(1 + a^2) + b^2(1 + b^2) + c^2(1 + c^2)?$$

Exercise 3.24 (2012 OMO)**.** Let S denote the sum of the 2011th powers of the roots of the polynomial $(x - 2^0)(x - 2^1)\cdots(x - 2^{2010}) - 1$. How many ones are in the binary expansion of S?

Before moving onto Graphic Transformations, we'll leave off with a very significant polynomial identity, which you'll possibly use in the problem set.

Theorem 3.25 (Difference of Polynomials)

If $P(x)$ has integer coefficients than $a - b$ divides $P(a) - P(b)$.

4 Graphic Transformations

This section was heavily inspired by CNCM's lecture on Polynomial Transformations.

Definition 4.1 (Vertical Stretch and Compression)
For $y = af(x)$, if $|a| > 1$, then the graph of $f(x)$ will be vertically stretched by a factor of a, and if $0 < |a| < 1$, then the graph of $f(x)$ will be vertically compressed by a factor of a.

To find the coordinates of a transformed function of this form, you multiply the $y-$coordinates by a.

Definition 4.2 (Horizontal Stretch and Compression)
For $y = f(bx)$, if $|b| > 1$, the graph of $f(x)$ compresses horizontally, and if $0 < |b| < 1$, then the graph of $f(x)$ stretches horizontally.

To find the coordinates of a transformed function of this form, you multiply the $x-$coordinates by $\frac{1}{a}$.

Definition 4.3 (Reflections)
Take the function $y = f(x)$.

- $y = -f(x)$ reflects $f(x)$ across the $x-$axis.
- $y = f(-x)$ reflects $f(x)$ across the $y-$axis.

Definition 4.4 (Horizontal Shift)
Take the function $y = f(x)$.

- $y = f(x + c)$ will shift $f(x)$ left c units.
- $y = f(x - c)$ will shift $f(x)$ right c units.

Definition 4.5 (Vertical Shift)
Take the function $y = f(x)$.

- $y = f(x) + d$ will shift $f(x)$ up c units.
- $y = f(x) - d$ will shift $f(x)$ down c units.

These definitions and properties of transformations come through intuition.

Exercise 4.6. Take an arbitrary function f. Play-around with function f and transform it – what is the graphical effect from this, why does that occur, and does it match our definitions?

Example 4.7 (Varsity Tutors)
List the transformations that have been enacted upon the following equation:

$$f(x) = 4[6(x-3)^4] - 7.$$

Solution. This equation is based off the parent function $y = x^4$. The general form for transformations like this is

$$f(x) = a[b(x-c)^4] + d.$$

Because $a = 4$, the function has been vertically stretched by a factor of 4, $b = 6$, the function has been horizontally stretched by a factor of 6, $c = 3$, the function has been translated 3 units right, and $d = -7$, the function has been translated 7 units down. $\square$

Exercise 4.8 (Varsity Tutors)**.** What transformations have been enacted upon $g(x) = 4(2x-6)^5$ when compared to its parent function, $f(x) = x^5$?

Let's look at an application of Graphic Transformations to manipulate the roots of a polynomial.

Example 4.9 (CNCM)
Given that a, b, c are the roots of polynomial $x^3 - 21x^2 + 3x + 2 = 0$, compute $(a+1)(b+1)(c+1)$.

Solution. Let's try to transform the polynomial such that the roots of the transformed polynomial are $a+1, b+1$, and $c+1$; from there, we can use Vieta's Formulas. Shifting the polynomial to the right by 1 unit increases our roots by 1, exactly what we need. So, shifting the function gets $(x-1)^3 - 21(x-1)^2 + 3(x-1) + 2$, which expands to $x^3 - 24x^2 + 48x - 23$. Thus, using Vieta's Formulas, our answer is $\boxed{23}$. $\square$

Remark 4.10. Vieta's Formulas trivially solves this example and other examples like it. In that case, why do we even use Graphic Transformations for examples like this? Well, transformations like these are helpful during Analytic Geometry and Calculus and are purely used to humor you.

Example 4.11 (CNCM)
Given that α and β are the roots of $x^2 - 21x + 3 = 0$, compute $\alpha^2 + \beta^2$.

Solution. We're not able to transform it with $\pm\sqrt{x}$ off the bat because then it wouldn't be a polynomial anymore! Becoming a bit more creative, we see that the polynomial

$$(-x)^2 - 21(-x) + 3 = 0$$

has roots $-\alpha, -\beta$, however when we multiply it with the original polynomial, the signs cancel, and our roots are squared. So, we multiply those two polynomials together to get

$$x^4 - 435x^2 + 9 = 0,$$

and applying our transformation and Vieta's Formulas, our desired result is $\boxed{435}$. $\square$

5 Problem Set

Problem 5.1 (2020 HMMT)**.** Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

Problem 5.2 (1973 USAMO)**.** Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Problem 5.3 (1982 ISL)**.** Let $p(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1 and with one of its roots equal to the product of the other two. Show that $2p(-1)$ is a multiple of $p(1) + p(-1) - 2(1 + p(0))$.

Problem 5.4 (2001 AIME I)**.** Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

Problem 5.5 (2005 AIME I)**.** Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$.

Problem 5.6 (2003 AIME II)**.** Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Problem 5.7 (2007 HMMT)**.** The complex numbers α_1, α_2, α_3, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Problem 5.8 (2005 AIME I)**.** The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n where m and n are relatively prime positive integers, find $m + n$.

Problem 5.9 (2019 CMIMC)**.** Let a, b and c be the distinct solutions to the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of

$$\frac{1}{a(b^2 + c^2 - a^2)} + \frac{1}{b(c^2 + a^2 - b^2)} + \frac{1}{c(a^2 + b^2 - c^2)}.$$

Problem 5.10 (David's Problem Stash)**.** Let a, b, and c be nonzero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

Problem 5.11 (2020 AIME II)**.** Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 5.12 (1984 AIME)**.** Determine $w^2 + x^2 + y^2 + z^2$ if

$$\frac{x^2}{2^2-1} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} = 1$$
$$\frac{x^2}{4^2-1} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} = 1$$
$$\frac{x^2}{6^2-1} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} = 1$$
$$\frac{x^2}{8^2-1} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} = 1$$

Problem 5.13 (2014 AIME I)**.** Let $x_1 < x_2 < x_3$ be three real roots of equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

Problem 5.14 (CNCM)**.** Given that α and β are the roots of $x^2 - 21x + 3 = 0$, compute $\frac{1}{\alpha+1} + \frac{1}{\beta+1}$.

Problem 5.15 (2019 AMC 10A)**.** Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

Problem 5.16 (2015 AIME II)**.** Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Problem 5.17 (CNCM)**.** Compute the following expression:

$$\sum_{n=0}^{4} \frac{1}{e^{\frac{2ni\pi}{5}} + 1}.$$

Problem 5.18 (2018 HMMT)**.** Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \leq x_1, x_2, x_3, x_4 \leq 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$.

Problem 5.19 (2017 CMIMC)**.** The polynomial $P(x) = x^3 - 6x - 2$ has three real roots, α, β, and γ. Depending on the assignment of the roots, there exist two different quadratics Q such that the graph of $y = Q(x)$ pass through the points (α, β), (β, γ), and (γ, α). What is the larger of the two values of $Q(1)$?

Problem 5.20 (HMMT February 2020)**.** Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.