1 Introduction and Closed Forms

"Summations represent the true beauty of math, yet present its annoying nuances."

- Chirag Samantaroy, probably

1.1 Down the rabbit hole

In this handout, you will be introduced to basic summation notation, and formulas for simple sums that show up often. We'll sometimes call a summation a "sum" for short. So the word sum will be used a verb (to *sum* things up) or a noun (the *sum* of several things). Since sequences and summations often appear together, you will also be exposed to some basic types of sequences (and summations of these sequences, of course).

What is a summation? Definitions vary from person to person, but I tend to think of a *summation* as a certain amount of terms added together. These terms are called *summands*, and we can manipulate them in a variety of ways to make summing them easier.

You can sum real numbers, complex numbers, and even expressions involving variables. They don't have to be distinct. We can take the summation of all the roots, real and complex, of the polynomial $x^{2021} - 77x^{2020} + 42$, and we'll get 77. We can take the summation of the monic quadratic polynomials dividing $x^3 + 7x^2 + 14x + 8$ and get

$$(x^2 + 3x + 2) + (x^2 + 6x + 8) + (x^2 + 5x + 4) = 3x^2 + 14x + 14.$$

You get the point: while we'll quickly see that summations of real numbers are far more common in contests, keep it in the back of your mind that we are not restricted to them.

You can have any amount of terms in a summation. You have just 1 term in a summation (That would be a very boring summation). You can even have an infinite number of terms in a summation.

As you can see, what constitutes a summation is vague and not agreed upon. That doesn't mean that they are not important in contests. Typically, a "summation problem" on the AMC 10/12 or other math contests will ask you to compute, or simplify, a decently large sum of summands. There are techniques that are commonly used to solve these types of questions, and my hope is to teach you about these techniques through these 4 handouts.

1.2 Sum of the first n positive integers

We'll start by exploring a classic idea that you probably will be familiar with.

Example 1
Determine the sum of the first 10 positive integers.

Solution. Brute force it out and get $1+2+3+4+5+6+7+8+9+10 = \boxed{55}$. □

How could we have computed this if we wanted the sum of the first 100 integers, or the first 1000 integers? Let's see.

Example 2
Determine the sum of the first 1000 positive integers.

Solution. Clearly, our brute force tactic isn't going to work anymore. However, recount the properties of addition that you learned in elementary school, which allow us to move terms in our sum around. You might be able to see a cleaner method we can use, and this method involves pairing numbers off in the following manner:

$$\begin{aligned} 1+1000 &= 1001, \\ 2+999 &= 1001, \\ 3+998 &= 1001, \\ \underbrace{\cdots}_{\text{496 pairs}} &= 1001, \\ 500+501 &= 1001, \end{aligned}$$

Note that since the sum of each of the 500 pairs will be equal to 1001, the sum of all 500 pairs (and thus, the sum of the first 1000 positive integers) is equal to $500 \cdot 1001 = \boxed{500500}$. □

> *Remark 3 (Carl Friedrich Gauss).* Gauss was a famous mathematician born in 1777. When Gauss was in elementary school, his teacher assigned a problem similar to the example above as 'busy work' for the students. However, Gauss figured out the correct answer within few minutes (using the method of pairing shown above), surprising the teacher. Gauss would then move on to prove several major theorems, such as the fundamental theorem of algebra and Descartes's rule of signs (just to name a few).

We may extend the logic above to show a general closed formula for the sum of the first n positive integers.

Example 4
Determine the sum of the first n positive integers.

Solution. If n is even, as seen above, we can split the n positive integers into $\frac{n}{2}$ pairs of numbers, each with sum $n+1$.

$$\begin{aligned} 1+n &= n+1, \\ 2+(n-1) &= n+1, \\ 3+(n-2) &= n+1, \\ \underbrace{\cdots}_{\text{some number of pairs}} &= n+1, \\ \frac{n}{2}+(\frac{n}{2}+1) &= n+1, \end{aligned}$$

The sum of the $\frac{n}{2}$ pairs will be equal to $\frac{n}{2}(n+1)$. If n is odd, we start by using the same strategy as before. That is, we split our first n positive integers into $\frac{n-1}{2}$ pairs of numbers, each with sum $n+1$.

$$\begin{aligned} 1+n &= n+1, \\ 2+(n-1) &= n+1, \\ 3+(n-2) &= n+1, \\ \underbrace{\cdots}_{\text{some number of pairs}} &= n+1, \\ \frac{n-1}{2}+\frac{n+3}{2} &= n+1, \end{aligned}$$

But now we have a number in the middle, which is equal to $\frac{n+1}{2}$. We add that to the sum of the rest of the pairs. Our final sum evaluates to

$$\frac{(n-1)}{2}\cdot(n+1)+\frac{n+1}{2}=\frac{n^2+n}{2}=\frac{n}{2}(n+1).$$

Thus, regardless of parity, the sum of the first n positive integers is equal to $\boxed{\frac{n(n+1)}{2}}$. $\square$

Fact 5. For all positive integers n, the sum of the first n positive integers is equal to

$$\boxed{\frac{n(n+1)}{2}}.$$

1.3 Two types of sequences

Here, we'll introduce two common types of sequences. A *sequence* is a collection of elements (we call them terms as well), where repetitions of an element are allowed, and the order in which the elements appear matters. Elements don't have to be numbers, they can be anything. So

$$3, 7, 3, \pi, \text{an apple}, \text{a pear}, 56i + 8, \text{squareman}, \sqrt{1829419281}, \text{2008 Honda Accord}$$

is one example of a sequence. At the same time, this sequence is different from

$$3, 7, 3, \pi, \text{a pear}, \text{an apple}, 56i + 8, \text{squareman}, \sqrt{1829419281}, \text{2008 Honda Accord}$$

because two of the elements are switched. The number of elements in the sequence is called the *length* of the sequence. Both of these sequences have length 10. Sometimes, we will be working with sequences that have an infinite number of terms. These sequences have a defined first term, but no last term.

Clearly sequences of numbers are going to be more useful than sequences of fruits, AoPS users, or Japanese family cars, so we'll concentrate on them first. If a sequence of numbers $a_1, a_2, \ldots a_n$ (with length n) satisfies the property that

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = d$$

for some number d, it is called an *arithmetic sequence* or arithmetic progression. The number d is called the *common difference* of this sequence. The nth term of an arithmetic sequence may be written as $a + (n-1)d$, where $a = a_1$ is the initial term and d is the common difference.

The sequence of the first n positive integers is an example of an arithmetic sequence, with common difference 1. Another example of an arithmetic sequence is $7, 7 + \pi, 7 + 2\pi, 7 + 3\pi$. (What's the common difference of this sequence?)

If a sequence of numbers $a_1, a_2, \ldots a_n$ (with length n) has the property that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}} = r$$

for some number r, it is called a *geometric sequence* or geometric progression. The number r is called the *common ratio* of this sequence. In a geometric sequence, each term a_k may be written as $a \cdot r^{k-1}$, such that $a = a_1$ is the initial term, r is the common ratio, for all integers $1 \leq k \leq n$. The sequence of say, the first n powers of 2 is an example of a geometric sequence. (What's the common ratio and initial term of this sequence?)

The terms in an arithmetic sequence or a geometric sequence do not have to be integers, rational numbers, or even real numbers. Constant sequences like $187, 187, 187, 187$, are both arithmetic and geometric. Inexperienced competitors might forget that arithmetic sequences and geometric sequences can have negative common differences or common ratios. Sequences like $1, -2, 4, -8$ are still geometric and sequences like $7, 2, -3, -8$ are still arithmetic.

We may define arithmetic sequences and geometric sequences that have infinite length. In general, if an infinite sequence $a_1, a_2, a_3 \ldots$ satisfies the property that $a_{k+1} - a_k$ is constant over all choices of k, then that sequence is called an infinite arithmetic sequence. If an infinite sequence $a_1, a_2, a_3 \ldots$ satisfies the property that $\frac{a_{k+1}}{a_k}$ is constant over all choices of k, then that sequence is called an infinite geometric sequence.

Problems using the most elementary properties of arithmetic sequences and geometric sequences are ubiquitous throughout the AMC tests.

Example 6 (2008 AMC 12A)
The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

Solution. Using logarithm properties, we have $a_1 = 3\log a + 7\log b$, $a_2 = 5\log a + 12\log b$, and $a_3 = 8\log a + 15\log b$. Since $a_2 - a_1 = a_3 - a_2$, we have

$$2\log a + 5\log b = 3\log a + 3\log b,$$

so

$$2\log b = \log a$$

and $b^2 = a$.

Thus the sequence has initial term $13\log b$ with a common difference of $9\log b$. The 12th term is $(13 + 11(9)) \times \log b = 112 \times \log b = \log b^{112}$ so $n = \boxed{112}$. $\square$

Example 7 (2009 AMC 12B)
The fifth and eighth terms of a geometric sequence of real numbers are $7!$ and $8!$ respectively. What is the first term?

Solution. Call the nth term of the series be ar^{n-1}, where $a = a_1$ is the initial term and r is the common ratio. Because

$$\frac{8!}{7!} = \frac{ar^7}{ar^4} = r^3 = 8,$$

it follows that $r = 2$ and the first term of the sequence is $a = \frac{7!}{2^4} = \frac{7!}{16} = \boxed{315}$. $\square$

In both problems, the only thing we had to know about geometric sequences and arithmetic sequences were the bare-bones basic definitions. The AMCs like to bring in other unrelated concepts (like logarithms and factorials in this case) in order to "spice" their problems up. These problems are a token example of how you will need to connect ideas from several different areas of math in order to succeed on the AMCs. Even if these ideas are elementary, connecting them together proves challenging for a lot of competitors.

1.3.1 Exercises

Exercise 8 (1959 AHSME). By adding the same constant to $20, 50, 100$ a geometric progression results. What is the common ratio?

Exercise 9 (2010 AMC 12A). The first four terms of an arithmetic sequence are p, 9, $3p-q$, and $3p+q$. What is the 2010^{th} term of this sequence?

1.4 Summing Arithmetic and Geometric Sequences

Oftentimes in math contests you will be expected to quickly find sums of some arithmetic and geometric sequences. We will present examples and formulae pertaining to both types of sums. Do not think that these strategies are solely used in the latter half of the AMC 10/12 test. Problem 4 on a recent AMC 10 test involved summing up an arithmetic sequence.

Example 10
Find the sum

$$9 + 4 + (-1) + (-6) + (-11) + \cdots + (-101).$$

Solution. We're working with the arithmetic sequence $9, 4, -1, -6, \ldots$ See that the common difference of this sequence is -5, and not 5. Arithmetic sequences that are decreasing have negative common differences, and arithmetic sequences that are increasing have positive common differences. Our initial term is 9. If we have n terms in this sequence, then we know that $9 + (n-1)(-5) = -101$. (Putting an n instead of an $n-1$ is another mistake that I commonly see: indexes matter!) Solving for n gives $n = 23$, so there are 23 summands.

Pairing up opposite terms of integers worked when we were summing the first n positive integers, so let's try it here. We have $\frac{23-1}{2} = 11$ pairs of terms each summing to -92 that can be formed in the following manner:

$$\begin{aligned} 9 + (-101) &= -92, \\ 4 + (-96) &= -92, \\ (-1) + (-91) &= -92, \\ \underbrace{\cdots}_{\text{some number of pairs}} &= -92, \\ (-41) + (-51) &= -92. \end{aligned}$$

If you don't understand how this pairing strategy works, I suggest you write the numbers of this sequence in a list and pair them up "visually," confirming that each of them adds up to -92.

But now we have a number in the middle, which is equal to $\frac{9+(-101)}{2} = -46$. We add that to the sum of the rest of the pairs. Our final sum evaluates to

$$11 \cdot (-92) + (-46) = -46 \times 23 = \boxed{-1058}.$$

$\square$

This is an overexplained solution, but on the AMC 10/12 you're going to need to know this technique like the back of your hand. As we'll see later, this pairing strategy is not limited to arithmetic sequences, and can be found as a critical step in many summations of various difficulty levels.

Now, we will take a look at an example of a summation problem involving terms of a geometric sequence.

Example 11
Simplify the sum

$$4+4\cdot 3+4\cdot 3^2+\cdots+4\cdot 3^{2020}.$$

Solution. $4, 4\cdot 3, 4\cdot 3^2, \ldots$ is a geometric sequence with initial term $a = a_1 = 4$ and common ratio $r = 3$. In this case, the nth term can be written as $4\cdot 3^{n-1}$. So there's 2021 terms. From there though, things get a lot less clearer to proceed. We can't pair up opposite terms in the same way we did to an arithmetic sequences. So how can we possibly evaluate this sum without evaluating some gigantic power of 3?

Let's start "playing around" with the problem. If we multiply each term in the sequence, we'll get the very next term in the sequence. That's a very neat "defining" property of the summands, and it's clear that we have to use this in order to compute our sum properly. What happens if instead of multiplying each individual term in the sum by 3, we multiply the whole sum by 3?

Suppose the value of our sum is S. It turns out, we get

$$3S = 4\cdot 3+4\cdot 3+4\cdot 3^2+\cdots+4\cdot 3^{2021}.$$

Compare this sum with our original sum S side by side. A lot of terms are the same. This brings us to our second big idea. We subtract the original equation

$$S = 4+4\cdot 3+4\cdot 3^2+\cdots+4\cdot 3^{2020}$$

from our new one. The idea behind this subtraction is to cancel all the shared terms between the two equations: a common theme in summation problems. We ultimately get

$$\begin{aligned} 3S - S &= \left(4\cdot 3+4\cdot 3+4\cdot 3^2+\cdots+4\cdot 3^{2021}\right) - \left(4+4\cdot 3+4\cdot 3^2+\cdots+4\cdot 3^{2020}\right) \\ &= 4\cdot 3^{2021} - 4. \end{aligned}$$

Ding ding ding! We can now see that $2S = 4\cdot 3^{2021} - 4$, and solving the equation gives

$$S = \boxed{2\cdot 3^{2021} - 2}.$$

□

Just like how we did in this well-trodden example, many times in summations problems you're going to be trying to exploit interesting properties of the summands you're summing, and using these interesting properties in novel and clever ways. Math competition resources will always tell you to "play around with the problem" or "use some wishful thinking," and in summation problems these techniques will prove especially critical.

A good way to stay organized when doing summation problems is by setting variables. Set a variable like S to be the sum we're looking for. Oftentimes, through some clever manipulations, you might be able to find an equation relating to S, helping you solve the problem.

It turns out, we can generalize the ideas posed in the previous examples. For any positive integer n, there exists closed forms for the sum of the first n terms of an arithmetic sequence, and the first n terms of a geometric sequence. Don't try to memorize these formulas. If you have a good understanding about how arithmetic and geometric sequences work, rederiving these formulas will become routine. We present a proof of the geometric sequence formula here and leave the proof of the sum of an arithmetic sequence formula as an exercise.

Fact 12. For all positive integers n, the sum of an arithmetic sequence with n terms $a_1, a_2, a_3, \ldots a_n$ equals

$$n \cdot \frac{(a_1 + a_n)}{2}.$$

An intuitive way I use to make sense of this formula is to think of $\frac{(a_1+a_n)}{2}$ as the "average value" of the terms in the sequence. Then, we can just multiply that average value by the number of terms in the sequence to get our final summation value.

Fact 13. For all positive integers n, the sum of a geometric sequence with n terms, an initial term of $a = a_1$, and a common ratio of r equals

$$\frac{a(r^n - 1)}{r - 1}.$$

Solution. We're looking at finding the sum

$$S = a_1 + a_1 r + a_1 r^2 + \ldots a_1 r^{n-1}.$$

Notice that

$$Sr = a_1 r + a_1 r^2 + a_1 r^3 + \ldots a_1 r^n.$$

Subtracting the two equations gives

$$S \times (r-1) = \left(a_1 r + a_1 r^2 + a_1 r^3 + \dots a_1 r^n\right) - \left(a_1 + a_1 r + a_1 r^2 + \dots a_1 r^{n-1}\right).$$

But now notice all the terms are going to cancel out on the right hand side besides $a_1 r^n$ and $-a_1$. So we have

$$S(r-1) = a_1(r^n - 1) = a(r^n - 1),$$

and dividing both sides by $r - 1$ gives the result. □

Don't focus on all the scary looking letters and variables. Focus on the concepts instead and how they're shared with the earlier Example 9. In this proof, we generated another sum in terms of S (by multiplying it by the common ratio) that had all of its terms "shifted over" by one index. By subtracting the two sums, we were able to force cancellation. If you're easily intimidated by this notation (as I still am to this day), try applying the logic in the proof to geometric sequences with initial terms and ratios that are easy to work with. That way, you get an idea of how the proof works without sifting through mindnumbing algebraic notation.

1.4.1 Exercises

Exercise 14. Show that the sum of the first n powers of 2,

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1}$$

is equal to $2^n - 1$.

Exercise 15. Show the general Arithmetic Sequence Formula: for all positive integers n, the sum of an arithmetic sequence with n terms $a_1, a_2, a_3, \dots a_n$ equals

$$n \cdot \frac{(a_1 + a_n)}{2}.$$

1.5 Arithmetico-Geometric Sums

The powerful idea we used to evaluate a geometric sequence sum can be also used to evaluate the sum of the following scary sequence:

Example 16
Simplify

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 2^2 + \cdots + 100 \cdot 2^{99}.$$

Note that the nth term in the sum is the product of n and 2^{n-1}.

Solution. Let's call our sum S. The sequence $1, 2, 2^2 \ldots 2^{99}$ is an example of a geometric sequence, and $1, 2, 3 \ldots 100$ is an example of an arithmetic sequence. Put them together... and we have something that is very strange indeed.

Pairing opposite terms doesn't look viable (you can verify this on your own), so let's see if the technique we tried before still holds. That is, we multiply S by the common ratio of the geometric sequence that's involved.

$$2S = 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^4 + \ldots 100 \cdot 2^{100}.$$

That's a bit tricky. When we subtract S and $2S$, we don't get our terms to completely cancel, like how we did with a geometric sequence. However, you might see that we actually get something familiar that we do know how to evaluate! See that

$$S = 2S - S = 100 \cdot 2^{100} - 2^{99} - 2^{98} - \cdots - 1 = 100 \cdot 2^{100} - 2^{100} + 1 = \boxed{99 \cdot 2^{100} + 1}.$$

By subtracting $2S$ and S, we miraculously got a sum of a geometric sequence. Even if we're unable to snap all our terms away, we often find that we can reduce an unknown sum into something that's familiar. □

The sequence $1 \cdot 2, 2 \cdot 2^2, 3 \cdot 2^3, \ldots 100 \cdot 2^{100}$ is an example of an arithmetico-geometric sequence, because the nth term of the sequence can be written as the product of the nth term of a geometric sequence (powers of 2) and the nth term of a arithmetic sequence (positive integers). You will evaluate a general formula for the sum of such a sequence as an exercise. They show up occasionally in contest problems.

1.5.1 Exercises

Exercise 17. Define a *arithmetico-geometric sequence* $x_1, x_2, x_3, \ldots x_n$ such that $x_k = a_k g_k$ for all integers $1 \le k \le n$, given an arithmetic sequence $a_1, a_2, \ldots a_n$ and a geometric sequence $g_1, g_2, \ldots g_n$. Determine a general formula for the sum of an *arithmetico-geometric sequence*.

1.6 Hey, what's that weird looking E?

We use the ***sigma*** ($\sum$) notation to express a particular sum in a concise (albeit scary) way.

Fact 18. For integer k, n such that $n \geq k$ and $f(i)$ denotes some function defined for integers i that are greater than or equal to k and less than or equal to n,

$$\sum_{i=k}^{n} f(i) = f(k) + f(k+1) + \cdots + f(n).$$

For instance,

$$\sum_{i=1}^{100} (2^i - 1) = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{100} - 1).$$

Furthermore, we have that

$$\sum_{i=1}^{n} c = \overbrace{c + c + c + \cdots + c}^{n \text{ summands}} = nc.$$

for any constant c and any positive integer n.

Using what we know about the sum of the first n positive integers, see that the sum

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

for any positive integer n.

In particular, notice that if the function $f(i)$ is linear (that is, if it can be written in the form of $mx + b$ for some constants m and b), we are summing an arithmetic sequence. If the function $f(i)$ is exponential (can be written as $a \times r^i$ for constants a and b), we are summing a geometric sequence. Try convincing yourself why this is true.

We can write our new arithmetic and geometric sequence summations formulas using this new summation notation. We have that

$$\sum_{i=1}^{n} a + (i-1)d = a + (a+d) + \cdots + (a + (n-1)d) = n \cdot \frac{(2a + (n-1)d)}{2}$$

and

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

for arbitrarily defined constants $a, r,$ and d.

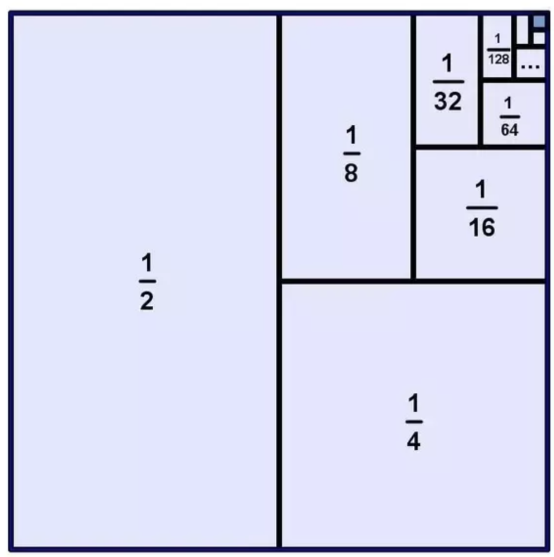
1.7 Summing Infinity

We can also use the sigma notation to express infinite sums, which are sums of an infinite number of terms. These infinite sums may either converge or diverge. We often call these sums infinite *series* (that is, the infinite series is the summation of the infinite sequence, and the terms infinite sum and infinite series are used interchangeably to my knowledge). If the sum converges, this means that the infinite series has partial sums (we'll get into this concept later) that come arbitrarily close to some real number r. The infinite sum is considered equal to this real number r.

For instance, a famous example of an infinite series that converges is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots ,$$

which converges to 1. This is an example of a sum of an infinite geometric sequence (I've heard it referred to as an infinite geometric series as well). There are "Proofs Without Words" of this result that you might be able to find online. Here's one, can you see how it illustrates this result?



Here's a hint: look at the square as having an area of one. Then what do the areas of the rectangles represent?

If the infinite series does not converge, we say that it diverges. Then, we consider it to evaluate to an undefined value. Forgoing all the advanced vocabulary, we may think of a sum that converges as one that will eventually be equal to a certain number, and a sum that diverges as one that will keep growing forever (either in the positive or negative direction). If a contest asks you to determine the value of an infinite sum, it has to converge. Otherwise, they would effectively be asking for the answer of "undefined."

We use the following notation to express infinite sums:

Fact 19. For a positive integer k and a function $f(i)$ defined for all integers $i \geq k$,

$$\sum_{i=k}^{\infty} f(i)$$

Is defined to be the evaluation of the infinite sum

$$f(k) + f(k+1) + f(k+2) + \cdots ,$$

that is if this sum converges.

As seen above,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

On the other hand,

$$\sum_{i=1}^{\infty} 2^i = 2 + 4 + 8 + \cdots$$

does not converge, and we say that this sum is undefined. Notice that this is also the sum of an infinite geometric sequence, but it doesn't converge in the same way that the first sum does.

Infinite sums are a challenging concept that will not be further explored in this section (bummer!), but we decided to give you a glimpse of things to come. We will explain how to actually compute these strange sums in a section of Week 2's handout. In the meantime, explore the behavior of these sums on your own. If you were a mathematician, how would you decide how to evaluate sums like $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$ or $2 + 4 + 8 + \cdots$? Which sums should be given a fixed value, and which ones shouldn't?

1.8 Summary

Summations are defined as a relatively large amount terms added together. These terms can be real numbers, complex numbers, and expressions involving one or more variables. In competitions, you will be required to evaluate summations, but also to apply them to problems that don't scream "BREAK OUT THE SIGMAAAAAA" right away. You were introduced to basic summations of geometric and arithmetic sequences, which pop up frequently. Finally, you learned how the elusive and scary sigma notation works. Getting used to it is hard, but we'll give you lots of examples throughout this course.

Along the way, you might have noticed a couple ideas that we used to make solving our summation problems easier. For instance, when we were evaluating the sum of the first n integers, we were able to pair together summands that were "opposite each other" in our sum, and this often allowed us to reduce our *summation problem* into a *multiplication problem*. When we were evaluating the sum of a finite geometric sequence, we were able to force some *cancellation* by manipulating our sum in a clever way. These are two ideas that we will emphasize again and again.

Next week, we will build on what we have learned this week to some more ubiquitous summation strategies.

Competitions love to use summations to make their problems look fancy or scary. Don't get fooled and don't get scared. Oftentimes when you plunge into the mess of algebraic notation, things clear up quite quickly. Until next time!

1.9 Important Terms

1. Summation
2. Summand
3. Arithmetic Sequence
4. Geometric Sequence
5. Common Difference
6. Common Ratio
7. Sigma Notation
8. Infinite Series

1.10 Practice Problems

Problem 1 (2021 AMC 10A). A cart rolls down a hill, travelling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

Problem 2 (1991 AHSME). The measures (in degrees) of the interior angles of a convex hexagon form an arithmetic sequence of integers. Let m be the measure of the largest interior angle of the hexagon. What is the largest possible value of m, in degrees?

Problem 3 (2011 AIME II). The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

Problem 4 (1987 AIME). Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive positive integers.

Problem 5 (2002 AMC 10B). Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \text{ and } a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

Problem 6. If

$$1 + x + x^2 + x^3 + \ldots + x^9 = 9,$$

find $x^{2000} + \frac{1}{x^{2000}}$.

Problem 7 (2011 AIME II). The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.