Sequences and Series

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Contents

1 Introduction

Sequences and series are common throughout competition problems. In this handout, we'll discuss different tips and techniques for many types of sequences and series.

1.1 What is a sequence and series?

Definition 1.1 (Sequence)
A **sequence** is an ordered list of numbers that follow a pattern.

Definition 1.2 (Series)
A **series** is the sum of the elements in a given sequence.

2 Summations and Products

Remark 2.1. In this section, we use $\times$ instead of $\cdot$ to signify multiplication. This is intentional for easier viewing, but note that we, for the most part, use $\cdot$, and this notation will be applied later on.

2.1 Sigma (Summation) Notation

Definition 2.2 (Summation Notation)
We have that

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n)$$

where $f(x)$ is a function and $n \geq m$.

Here, we say i is a **dummy variable**. It cycles through all the integer numbers from m to n, inclusive.

Example 2.3
What is the value of

$$\sum_{i=3}^{10}(i+5) = (3+5) + (4+5) + (5+5) + \cdots + (10+5)?$$

Solution. Expanding,

$$(3+5) + (4+5) + (5+5) + \cdots + (10+5) = 8 + 9 + 10 + \cdots + 15.$$

Notice something: by pairing up the 8 and 15, 9 and 14, and so on and so forth, we end up with a bunch of 23's. Since this is much easier to add, we have

$$23 + 23 + 23 + 23 = 23 \cdot 4 = \boxed{92}.$$

□

Exercise 2.4. Compute

$$\sum_{i=1}^{50} 2i + 1 = 3 + 5 + 7 + \cdots + 101.$$

2.2 Pi (Product) Notation

Definition 2.5 (Product Notation)

We have that

$$\prod_{i=m}^{n} f(i) = f(m) \times f(m+1) \times f(m+2) \times \cdots \times f(n)$$

where $f(x)$ is a function and $n \geq m$.

Product notation works exactly like summation notation except that it multiplies the terms instead of adding them.

Example 2.6

If p is a positive integer, find the greatest possible value of p such that 2^p divides

$$\prod_{i=1}^{4} 2i.$$

Solution. Since the numbers are so small, we can just write the whole product out and prime-factorize it:

$$\begin{aligned} 2 \times 4 \times 6 \times 8 &= 2 \times 2^2 \times (2 \times 3) \times 2^3 \\ &= 2^7 \times 3. \end{aligned}$$

Therefore, the greatest possible value of p is $\boxed{7}$. □

Exercise 2.7. Express 6! using product notation.

Exercise 2.8. Find the value of

$$\frac{\prod_{i=1}^{50} 2i \cdot \prod_{i=0}^{50} (2i+1)}{\prod_{i=1}^{102} i}.$$

2.3 Double Summations and Double Products

Definition 2.9 (Double Summations)

We have that

$$\sum_{i=p}^{k}\sum_{j=q}^{l} f(i,j) = f(p,q) + f(p,q+1) + \cdots$$
$$+f(p+1,q) + f(p+1,q+1) + \cdots + f(k,l-1) + f(k,l)$$

where $f(i,j)$ is a function, $k \geq p$, and $l \geq q$. This double summation can also be written as:

$$\sum_{i=p}^{k}\sum_{j=q}^{l} f(i,j) = \sum_{j=q}^{l} f(p,q) + \sum_{j=q}^{l} f(p+1,q) + \sum_{j=q}^{l} f(p+2,q) + \cdots + \sum_{j=q}^{l} f(k,q).$$

As you can see, a **double summation** is essentially a summation inside another summation.

Example 2.10

Find

$$\sum_{i=1}^{10}\sum_{j=2}^{6} (2i+j) = (2\cdot 1+2) + (2\cdot 1+3) + \cdots + (2\cdot 10+5) + (2\cdot 10+6).$$

Solution. We get rid of the outer summation:

$$\sum_{j=2}^{6}(2+j) + \sum_{j=2}^{6}(4+j) + \sum_{j=2}^{6}(6+j) + \cdots + \sum_{j=2}^{6}(20+j).$$

Now, we **evaluate the inner sum**. We compute each of these terms:

$$\begin{aligned}
\sum_{j=2}^{6}(2+j) &= 2\cdot 5 + \sum_{j=2}^{6} j \\
&= 2\cdot 5 + 20 \\
\sum_{j=2}^{6}(4+j) &= 4\cdot 5 + \sum_{j=2}^{6} j \\
&= 4\cdot 5 + 20 \\
&\vdots \\
\sum_{j=2}^{6}(20+j) &= 20\cdot 5 + \sum_{j=2}^{6} j \\
&= 20\cdot 5 + 20.
\end{aligned}$$

Then, we sum all of them up:

$$\begin{aligned}
(2+4+6+\cdots+20) + 20\cdot 10 &= 5(1+2+3+\cdots+10) + 20\cdot 10 \\
&= 550 + 200 \\
&= \boxed{750}.
\end{aligned}$$

□

Likewise, a **double product** is a product inside another product.

Definition 2.11 (Double Product)

$$\prod_{i=p}^{k}\prod_{j=q}^{l} f(i,j) = f(p,q) \times f(p,q+1) \times \cdots$$
$$\times f(p+1,q) \times f(p+1,q+1) \times \cdots \times f(k,l-1) \times f(k,l),$$

where $f(i,j)$ is a function, $k \geq p$, and $l \geq q$. This double product can also be written as:

$$\prod_{i=p}^{k}\prod_{j=q}^{l} f(i,j) = \prod_{j=q}^{l} f(p,q) \times \prod_{j=q}^{l} f(p+1,q) \times \prod_{j=q}^{l} f(p+2,q) \times \cdots \times \prod_{j=q}^{l} f(k,q).$$

Remark 2.12. We can still evaluate the inner product; we just multiply the terms instead of adding them.

Example 2.13

The double product

$$\prod_{i=8}^{12}\prod_{j=6}^{15} 2^i 3^j = 2^8 3^6 \times 2^8 3^7 \times \cdots \times 2^9 3^6 \times 2^9 3^7 \times \cdots \times 2^{12} 3^{14} \times 2^{12} 3^{15}$$

can be written as $2^a 3^b$ for positive integers a and b. Find the value of $a+b$.

Solution. Let's get rid of the outer product:

$$\prod_{j=6}^{15} 2^8 3^j \times \prod_{j=6}^{15} 2^9 3^j \times \prod_{j=6}^{15} 2^{10} 3^j \times \cdots \times \prod_{j=6}^{15} 2^{15} 3^j.$$

Then, we **evaluate the inner product**. We compute each of these terms:

$$\begin{aligned}
\prod_{j=6}^{15} 2^8 3^j &= (2^8)^{10} 3^{105} \\
&= 2^{80} 3^{105} \\
\prod_{j=6}^{15} 2^9 3^j &= (2^9)^{10} 3^{105} \\
&= 2^{90} 3^{105} \\
&\vdots \\
\prod_{j=6}^{15} 2^{15} 3^j &= (2^{15})^{10} 3^{105} \\
&= 2^{150} 3^{105}.
\end{aligned}$$

Now, we multiply all of them together:

$$2^{80}3^{105} \times 2^{90}3^{105} \times 2^{100}3^{105} \times \cdots \times 2^{150}3^{105} = 2^{920}3^{840}.$$

Thus, $a = 920$ and $b = 840$, so $a + b = \boxed{1760}$. □

Exercise 2.14. Evaluate

$$\sum_{i=1}^{10} \sum_{j=1}^{i} 2^j.$$

Exercise 2.15. The value of

$$\prod_{i=1}^{100} \prod_{j=1}^{100} i\sqrt{j}$$

is equal to $(100!)^a$ for a positive integer value of a. Find a.

> **Remark 2.16.** Note that triple, quadruple, etc. summations/products exist too. They work the same way as double summations/products. To compute them, check the innermost summation/product and then check the outer ones in order.

3 Arithmetic and Geometric Sequences and Series

3.1 Arithmetic Sequences

Definition 3.1 (Arithmetic Sequence)
An **arithmetic sequence** is a sequence of numbers where each term is given by adding a fixed value to the earlier term.

For example, $-5, -1, 3, 7, 11, \cdots$ is an arithmetic sequence because each term is four more than the earlier term (in this case, 4 is the **common difference** of the sequence).

Theorem 3.2 (n^{th} Term of an Arithmetic Sequence)
The n^{th} term of an arithmetic sequence is

$$a_n = a_1 + d(n - 1),$$

where a_1 is the first term and d is the common difference.

Let's look at an example:

Example 3.3
Given an arithmetic sequence with first term 12 and common difference 4, what is the eighth term?

Solution. By the formula for the n^{th} term of an arithmetic sequence,

$$a_8 = a_1 + d(8 - 1).$$

We are given $a_1 = 12$ and $d = 4$. So,

$$\begin{aligned} a_8 &= 12 + 4(8-1) \\ \implies a_8 &= 12 + 4(7) \\ \implies a_8 &= 12 + 28 \\ \implies a_8 &= \boxed{40}. \end{aligned}$$

□

3.2 Arithmetic Series

Example 3.4
What is the sum of the first 100 whole numbers?

Solution. Manually adding all of those numbers would be tedious. However, notice $1, 2, \ldots, 100$ is an arithmetic sequence. We pair up to 1 and 100, 2 and 99, and so forth. Each pair of terms sum to 101, and there are 50 pairs in total. Therefore, the total sum is $50 \cdot 101 = \boxed{5050}$. □

All arithmetic series can be summed using this way, leading us to derive a generalization.

Theorem 3.5 (Generalized Form for Arithmetic Series)
The sum of the arithmetic sequence $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_n}{2} \cdot \left(\frac{a_n - a_1}{d} + 1\right),$$

where d is the common difference.

Proof. Note that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2}$, etc. Thus, the sum of the terms is $\frac{a_1+a_n}{2}$ times the number of terms. Say $a_1 = 0 \cdot d + a_1$, then $a_2 = 1 \cdot d + a_1$, and so on till $a_n = k \cdot d + a_1$. Thus, the number of terms are $\frac{a_n - a_1}{d} + 1$ (basically 0 to k). Therefore, the sum is

$$\frac{a_1 + a_n}{2} \cdot \left(\frac{a_n - a_1}{d} + 1\right).$$

□

3.3 Geometric Sequences

Definition 3.6 (Geometric Sequence)
A **geometric sequence** is a sequence of numbers in which each term is given by multiplying the earlier term by a fixed value.

For example, $1, 2, 4, 8, 16, \cdots$ is a geometric sequences because each term is 2 times the earlier term (in this case, 2 is the **common ratio** of the sequence).

Theorem 3.7 (n^{th} Term of a Geometric Sequence)
The n^{th} term of a geometric sequence is

$$a_n = a_1 \cdot r^{(n-1)},$$

where a_1 is the first term and r is the common ratio.

We look at an example:

Example 3.8
Given a geometric sequence with first term 3 and common ratio 2, what is the fifth term?

Solution. By the n^{th} term of a geometric sequence theorem,

$$a_n = a_1 \cdot r^{(n-1)}.$$

We are given $a_1 = 3$ and $r = 2$. So,

$$\begin{aligned} & a_5 = 3 \cdot 2^{(5-1)} \\ \implies & a_5 = 3 \cdot 2^4 \\ \implies & a_5 = 3 \cdot 16 \\ \implies & a_5 = \boxed{48}. \end{aligned}$$

□

Exercise 3.9. If the second term of a geometric sequence is 21 and the fourth term is 84, find the first term and the common ratio.

3.4 Geometric Series

3.4.1 Finite Geometric Series

A finite geometric series is a geometric series with a specific number of terms.

Example 3.10
Given a geometric sequence with first term 3 and common ratio 5, what is the sum of the first eight terms?

Solution. Let

$$S_8 = a_1 + a_2 + \cdots + a_8.$$

The equation can be re-written as

$$S_8 = 3 + 3 \cdot 5 + \cdots + 3 \cdot 5^7.$$

Multiplying both sides by 5,

$$5S_8 = 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^8.$$

Subtracting the second equation from the first yields

$$4S_8 = 3 \cdot 5^8 - 3.$$

Then,

$$\begin{aligned} 4S_8 &= 3(5^8 - 1) \\ \implies S_8 &= \frac{3(5^8 - 1)}{4}. \end{aligned}$$

Simplifying gives $\boxed{292968}$. □

We're capable of summing all finite geometric series using the same method, implying for us to acquire a general statement.

Theorem 3.11 (Generalized Form for a Finite Geometric Series)
The sum of the first n terms of a geometric sequence is given by

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1},$$

where a_1 is the first term in the sequence, and r is the common ratio.

Proof. Let

$$S_n = a_1 + a_2 + \cdots + a_n.$$

Thus, the equation can be re-written as

$$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1}.$$

Multiplying both sides by r,

$$S_n r = a_1 r + a_1 r^2 + \cdots + a_1 r^n.$$

Subtracting the original equation from this equation,

$$S_n r - S_n = a_1 r^n - a_1.$$

Factoring S_n out of the expression on the LHS and a_1 out of the expression on the RHS,

$$S_n(r - 1) = a_1(r^n - 1).$$

Isolating S_n,

$$S_n = a_1 \cdot \frac{r^n - 1}{r - 1}.$$

□

3.4.2 Infinite Geometric Series

An infinite geometric series is the sum of an infinite geometric sequence.

Example 3.12
Evaluate
$$1+\frac{1}{2}+\frac{1}{4}+\cdots.$$

Solution. Let
$$1+\frac{1}{2}+\frac{1}{4}+\cdots=S.$$
To utilize symmetry, we multiply $\frac{1}{2}$ to the equation:
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\frac{S}{2}.$$
Subtracting the second equation from the first, we get
$$1=\frac{S}{2}.$$
Thus, $S=\boxed{2}$. □

We're able to sum all infinite geometric series using this way, driving us to get an abstraction.

Theorem 3.13 (Generalized Form for Infinite Geometric Series)
The sum of the infinite geometric sequence $a_1, a_2, \cdots$ is
$$\frac{a_1}{1-r},$$
where a_1 is the first term and r is the common ratio, and $|r|<1$.

Proof. Let the sequence be
$$S=a_1+a_1r+a_1r^2+a_1r^3+\cdots.$$
Multiplying by r yields,
$$S\cdot r=a_1r+a_1r^2+a_1r^3+\cdots.$$
We subtract these two equations to get
$$S-Sr=a_1.$$
Finally, we can factor and divide to get
$$\begin{aligned} S(1-r)&=a_1\\ \implies S&=\frac{a_1}{1-r}. \end{aligned}$$
□

We're able to prove this, sure, but why is $|r| < 1$ necessary?

Suppose we had the infinite geometric series

$$2 + 6 + 18 + \cdots.$$

Now, applying the formula, the sum evaluates to $\frac{2}{1-3} = -1$. All terms are positive, so why is the sum negative?

Essentially, when $|r| \geq 1$, the sum doesn't reach a definite limit, rather it increases indefinitely as more terms are added (the formal term being it **diverges**). Therefore, $|r| < 1$, for the sum to reach a definite limit (the formal term being we want it to **converge**).

Exercise 3.14. Find $\displaystyle\sum_{i=1}^{\infty} 2^{1-2i} - 2^{2i} = \frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} \cdots$.

One common instance of summing infinite geometric sequences is the decimal expansion of most rational numbers.

Example 3.15

Prove $0.99999\ldots = 1$.

Solution. We're able to rewrite $0.99999\ldots$ as

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots.$$

We recognize this as an infinite geometric series, with first term $\frac{9}{10}$ and common ratio $\frac{1}{10}$. Therefore, it has a value of

$$\begin{aligned}\frac{\frac{9}{10}}{1 - \frac{1}{10}} &= \frac{\frac{9}{10}}{\frac{9}{10}} \\ &= \boxed{1}.\end{aligned}$$

□

3.5 A Few Examples

Now that we've gone through the theory of Arithmetic, and Geometric, Sequences + Series, let's now apply those concepts to competition problems!

Example 3.16 (1984 AIME Problem 1)

Find the value of $a_2+a_4+a_6+a_8+\ldots+a_{98}$ if a_1, a_2, $a_3\ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$.

Solution. $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$ can be rewritten as

$$98a_1 + 1 + 2 + \ldots + 97 = 137.$$

By Arithmetic Series Formula,

$$1 + 2 + \ldots + 97 = \frac{97 \cdot 98}{2}.$$

We want to find the value of $a_2 + a_4 + a_6 + a_8 + \ldots + a_{98}$, which is

$$49a_1 + 1 + 3 + \ldots + 97.$$

By Arithmetic Series Formula,

$$1 + 3 + \ldots + 97 = 49^2.$$

From our first equation, $49a_1 = \frac{137-97\cdot 49}{2}$. So, our answer is $\frac{137-97\cdot 49+2\cdot 49^2}{2} = \boxed{093}$. □

Example 3.17 (2004 AMC 10A Problem 18)

A sequence of three real numbers forms an arithmetic progression with the first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression?

Solution. Let the 3 real numbers be $9, 9+a, 9+2a$, a is our common difference. The text states that adding 2 and 20 to the second and third term (respectively) gives us a geometric progression, so the three terms in the geometric progression are $9, 9+a+2, 9+2a+20 \implies 9, 11+a, 29+2a$. Let's set its common ratio as r. Since those terms are in a geometric progression, we have $9r = 11 + a \implies a = 9r - 11$ and $9r^2 = 29 + 2a$. Let's substitute for a the question asks to find the third term in the geometric progression. When we do that, we get $9r^2 = 7 + 18r \implies 9r^2 - 18r - 7$. Solving for the roots, we get $-\frac{1}{3}, \frac{7}{3}$. Note that the question never said we were restricted to positive ratios/differences, so we are free to try both values of r and see which one has the smallest possible value for the third term. When the ratio is $-\frac{1}{3}$, the 3rd term is 1. When we try $\frac{7}{3}$, the 3rd term is 49. Therefore, our answer is $\boxed{1}$. □

Example 3.18 (2006 AMC 10A Problem 19)

How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

Solution. We first realize that the number of distinct non-similar triangles equals the number of unordered triples

$$(a_1, a_2, a_3)$$

where a_1, a_2, a_3 are the measures of the three angles in degrees. Since a_1, a_2, a_3 are in arithmetic progression, we can rewrite them as

$$a_2 - d, a_2, a_2 + d$$

for common difference d. Now, since the sum of the angles in a triangle is 180,

$$180 = 3 \cdot a_2 \implies a_2 = 60$$

$a_2 - d$ and $a_2 + d$ have to be positive integers, so d is an integer, and we have the inequality $0 < d < 60$. Therefore, $d \in \{1, 2, \ldots 59\}$ which is 59 distinct possible values of d, so there are $\boxed{59}$ possible combinations of the angles in the triangle. □

Example 3.19 (2011 AIME II Problem 5)

The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

Solution. Because the sum of the first 2011 terms is bigger than the sum of the second 2011 terms, the common ratio must be less than one. The sum of the first 2011 terms is 200, the second 2011 is 180, so the proportion of the second 2011 terms to the first 2011 terms is $\frac{9}{10}$. Following a similar example, the amount of the third 2011 terms is $\frac{9}{10} * 180 = 162$. Thus, the amount of the first 6033 terms is $200 + 180 + 162 = \boxed{542}$. □

Example 3.20 (2012 AIME I Problem 2)

The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Solution. If the sum of the original sequence is $\displaystyle\sum_{i=1}^{n} a_i$ then the sum of the new sequence can be expressed as $\displaystyle\sum_{i=1}^{n} a_i + (2i-1) = n^2 + \sum_{i=1}^{n} a_i$. Therefore, $836 = n^2 + 715 \implies n = 11$. Now the middle term of the original sequence is simply the average of all the terms, or $\frac{715}{11} = 65$, and the first and last terms average to this middle term, so the desired sum is simply three times the middle term, or $\boxed{195}$. □

Example 3.21 (2016 AIME I Problem 1)

For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Solution. By infinite geometric series, $S(a) = \frac{12}{1-a}$ and $S(-a) = \frac{12}{1+a}$. So,

$$S(a)S(-a) = \left(\frac{12}{1-a}\right)\left(\frac{12}{1+a}\right) = \frac{144}{1-a^2} = 2016.$$

Notice

$$S(a) + S(-a) = \frac{12}{1-a} + \frac{12}{1+a} = \frac{24}{1-a^2},$$

and so the answer is $\frac{2016}{6} = \boxed{336}$. □

Exercise 3.22 (2012 OMO)**.** If

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{8x^4} + \frac{1}{16x^5} + \cdots = \frac{1}{64},$$

and x can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, find $m + n$.

Exercise 3.23 (2012 AIME II)**.** Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

4 Arithmetico-Geometric Series

Definition 4.1 (Arithmetico-Geometric Series)
An **arithmetico-geometric series** is the sum of consecutive terms in an arithmetico-geometric sequence defined as $t_n = a_n g_n$, where a_n and g_n are the nth terms of arithmetic and geometric sequences, respectively.

Let's go over an example.

Example 4.2
Evaluate

$$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

Let the sum be S. Then

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots,$$

and dividing by 2 yields,

$$\frac{S}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots.$$

Note that if we subtract the second equation from the first, we get

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1,$$

so $S = \boxed{2}$.

In this kind of problem, we divide the equation by the common ratio portrayed in the denominator, subtract the original equation by the new equation, and then check the sum. This transforms it into a standard geometric series with a possible extra term, which we can then check. Note this method of multiplying then subtracting, it will be used a lot when evaluating sums.

Exercise 4.3. Evaluate the infinite sum

$$\frac{6}{7} + \frac{13}{49} + \frac{20}{343} + \cdots$$

Exercise 4.4. A set S is defined as follows: if n is in S, then $\frac{n}{3^{n-1}}$ is also in S. Given that 1 is in S, find the sum of all elements in S.

5 Telescoping

Telescoping is a powerful method to compute sums with a large or infinite number of terms. Just like a telescope, which can collapse into itself, telescoping can make computation much easier. By rewriting each term in a favorable way that allows cancellation of most of the terms, the remaining expression is often very neat and simple.

5.1 Partial Fraction Decomposition

The method of splitting the fraction is known as partial fraction decomposition. This is often very useful in telescoping problems.

Using this technique, you factor the denominator and decompose it according to the right form. The following are the main forms of partial fraction decomposition:

- Linear Factors: $$\frac{ax+b}{(x+a_1)(x+a_2)} = \frac{A}{x+a_1} + \frac{B}{x+a_2}$$

- Repeated Linear Factors: $$\frac{ax+b}{(x+a_1)(x+a_2)^2} = \frac{A}{x+a_1} + \frac{B}{x+a_2} + \frac{C}{(x+a_2)^2}$$

- Irreducible Quadratic Factors: $$\frac{ax^2+bx+c}{(x-a_1)(x^2+a_2)} = \frac{A}{x-a_1} + \frac{Bx+C}{x^2+a_2}$$

To solve for the variables, we usually multiply both sides by the denominator of the original fraction, then plug-in values of the variable that cancel out all but one of the constants.

Example 5.1

Use partial fraction decomposition to rewrite $\dfrac{6x+2}{(x-5)(x+3)}$.

Solution. We write
$$\frac{6x+2}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}.$$
Multiplying both sides by $(x-5)(x+3)$ gives
$$6x+2 = A(x+3) + B(x-5).$$

As stated above, we wish to plug-in two values of x–once to cancel A, and once to cancel B. To do this, we must make $x+3=0$ to cancel A, and $x-5=0$ to cancel B. Plugging in $x=-3$ gives $B\cdot(-8) = -16$ so $B=2$. Plugging in $x=5$ gives $A\cdot 8 = 32$, so $A=4$. Thus, our result is
$$\boxed{\frac{4}{x-5} + \frac{2}{x+3}}.$$

□

5.2 Additive Cancellation

Additive cancellation in telescoping often uses partial fraction decomposition. The main goal in these cases is to rewrite each term as a difference of two fractions so that most fractions appear twice (once as a positive term, once as a negative term) and cancel with each other.

Let's take a look at a classic example.

Example 5.2 (Well-Known)

Evaluate
$$\sum_{n=2}^{100} \frac{1}{(n-1)\cdot n} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100}.$$

Solution. First, we utilize "shifting the index." That is, instead of have
$$\frac{1}{(n-1)\cdot n},$$
we can make it
$$\frac{1}{n\cdot (n+1)}.$$
However, this also means that n now starts at 2 and ends at 99 in the summation. Though, this doesn't affect much for our purposes.

Now, suppose we let
$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1},$$
for constants A and B. Multiplying both sides by $n(n+1)$,
$$1 = A(n+1) + Bn \implies 1 = (A+B)n + A.$$
Therefore, $A = 1$ and $B = -1$. So,
$$\frac{1}{n\cdot (n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$
Hence, the original expression becomes
$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right).$$
Note that every fraction starting from $\frac{1}{2}$ to $\frac{1}{99}$ appears twice in the expression – once as the $\frac{1}{n}$, and once as the $-\frac{1}{n+1}$. Thus, they cancel each other, and we are left with $\frac{1}{1} - \frac{1}{100}$, which is $\boxed{\frac{99}{100}}$. □

That was a rather *simple* example; let's look at a harder one.

Example 5.3 (2018 MPFG)

Consider the sum
$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}}.$$

Determine $\lfloor S_{4901} \rfloor$. Recall that if x is a real number, then $\lfloor x \rfloor$ (the *floor* of x) is the greatest integer that is less than or equal to x.

Solution. First, we note that

$$\sqrt{k+2}-\sqrt{k}=(\sqrt{k+2}-\sqrt{k})\left(\frac{\sqrt{k+2}+\sqrt{k}}{\sqrt{k+2}+\sqrt{k}}\right)=\frac{2}{\sqrt{k+2}+\sqrt{k}}.$$

We can also observe that

$$\frac{1}{\sqrt{k+2}}=\frac{2}{2\sqrt{k+2}}<\frac{2}{\sqrt{k+2}+\sqrt{k}}<\frac{2}{\sqrt{k}+\sqrt{k}}=\frac{1}{\sqrt{k}}.$$

So, we have that

$$\frac{1}{\sqrt{k+2}}<\frac{1}{\sqrt{k+2}+\sqrt{k}}<\frac{1}{\sqrt{k}}.$$

We sum $\frac{1}{\sqrt{k+2}}<\frac{1}{\sqrt{k+2}+\sqrt{k}}$ over odd k from 1 to $2n-3$ to get $S_n-1<\sqrt{2n-1}-1$. We then sum $\frac{1}{\sqrt{k+2}+\sqrt{k}}<\frac{1}{\sqrt{k}}$ over odd k from 1 to $2n-1$ to get $\sqrt{2n+1}-1<S_n$, giving

$$\sqrt{2n+1}-1<S_n<\sqrt{2n-1}.$$

For $n=4901$, we get $\sqrt{2(4901)-1}=\sqrt{9801}=99$, giving $98<S_{4901}<99$, and an answer of $\boxed{98}$. □

Exercise 5.4. Compute $\displaystyle\sum_{n=2}^{\infty}\frac{n}{n^2-1}=\frac{2}{3}+\frac{3}{8}+\frac{4}{15}\cdots$.

5.3 Multiplicative Cancellation

The second technique that we'll introduce is multiplicative cancellation, in which the factors are multiplied, and a lot of them get canceled through multiplication and division. For this technique, the goal is to rewrite the numerators and denominators such that they have common cancellated factors.

Let's look at an example.

Example 5.5 (2006 HMMT November)

Find

$$\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{4^2}{4^2-1}\cdots\frac{2006^2}{2006^2-1}.$$

Solution. We first use Difference of Squares to expand the denominators:

$$\begin{aligned}\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{4^2}{4^2-1}\cdots\frac{2006^2}{2006^2-1}&=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2006\cdot 2006}{2005\cdot 2007}\\&=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2005}{2006}\cdot\frac{2006}{2005}\cdot\frac{2006}{2007}.\end{aligned}$$

After performing some cancellations, we get:

$$\frac{2}{1}\cdot\frac{2006}{2007}=\boxed{\frac{4012}{2007}}.$$

□

6 Recursion

Recursion, seemingly trivial, actually has a variety of uses in all forms of math, ranging from computational math to even calculus. For now, we will tackle some common recursion techniques. But first, what is recursion? There are many forms of recursion, but we will mainly focus on the computational side. Furthermore, we'll tackle the algebraic side recursion; combinatorial recursion will appear in later handouts.

The most basic recursion example is the Fibonacci sequence.

Example 6.1 (Fibonacci Sequence)

The Fibonacci sequences goes as follows: $1, 1, 2, 3, 5, 8, 13, ...$, where each next term is the sum of the two previous terms in the sequence. In other words, for F_n, we have $F_n = F_{n-1} + F_{n-2}$, and this is the **recurrence relation**.

As given from arithmetic and geometric sequences and series, one might wonder, is there a closed-form (another way to express the recursion that doesn't involve previous terms) for this sequence? There is! But before we delve into how to find it, let us go through some basic methods to find closed forms.

Example 6.2

Find the closed form of the recurrence relation $a_n = a_{n-1} + n$ where $a_0 = 2$ and $n \geq 1$.

Solution. Let us first generate a few equations.

$$\begin{aligned}
a_1 &= a_0 + 1 && (1)\\
a_2 &= a_1 + 2 && (2)\\
a_3 &= a_2 + 3 && (3)\\
a_4 &= a_3 + 4 && (4)\\
&\vdots\\
a_n &= a_{n-1} + n && (n)
\end{aligned}$$

Although it may seem concealed, but when we move terms around, we get

$$\begin{aligned}
a_1 - a_0 &= 1 && (1)\\
a_2 - a_1 &= 2 && (2)\\
a_3 - a_2 &= 3 && (3)\\
a_4 - a_3 &= 4 && (4)\\
&\vdots\\
a_n - a_{n-1} &= n && (n)
\end{aligned}$$

We see the common terms of a_1 in (1) and (2), the common a_2 terms in (2) and (3), and so on. This motivates us to add the equations. Note that $(1) + (2) = a_2 - a_0 = 3$. How convenient, the a_1 terms cancelled out! What if we repeatedly added them? We would get

$$a_1 - a_0 + a_2 - a_1 + ... + a_n - a_{n-1} = a_n - a_0 = 1 + 2 + 3 + 4 + ... + n.$$

Wait a minute, the right side looks familiar! This is the sum of an arithmetic sequence, with common difference 1. Therefore, we have

$$a_n - a_0 = \frac{n(n+1)}{2}.$$

But since $a_0 = 2$, we have $a_n = \boxed{\frac{n(n+1)}{2} + 2}$ as our closed form. Now let's move onto the bulk part of closed forms. □

Example 6.3

Find the closed form of the Fibonacci sequence, which has the recurrence relation of $F_n = F_{n-1} + F_{n-2}$.

Solution. Well this seems quite difficult! We have the sequence of $1, 1, 2, 3, 5, 8, 13, \ldots$. This clearly cannot be arithmetic, as the first two terms are the same, yet the third term is not. Similarly, this could not be geometric. We introduce a new concept called **characteristic equations**.

After graphing out a couple of terms of $y = F_n$, we suspect it may seem like a geometric sequence, so we might let $F_n = r^n$ for any constant r. Similarly, this gives

$$F_{n-1} = r^{n-1}, F_{n-2} = r^{n-2}.$$

Therefore, we have

$$r^n = r^{n-1} + r^{n-2}.$$

Note that $r \neq 0$, so dividing by r^{n-2} yields $r^2 = r + 1 \implies r^2 - r - 1 = 0$. Note that this has roots of $\frac{1\pm\sqrt{5}}{2}$. Uh oh, there are two roots. Which do we pick? Well, how about we let

$$F_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

How can we solve for a and b? Luckily, we know a few conditions, namely $F_0 = 1, F_1 = 1$. Therefore, we get $F_0 = a + b = 1$. Similarly, $F_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1$. We arrive at the system of equations

$$a + b = 1$$
$$a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Multiplying the first equation by $\frac{1-\sqrt{5}}{2}$ and subtracting the two yields

$$a\left(\frac{1+\sqrt{5}}{2}\right) - a\left(\frac{1-\sqrt{5}}{2}\right) = \sqrt{5}a = 1 - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

or $a = \frac{1+\sqrt{5}}{2\sqrt{5}}$. Therefore, $b = 1 - a = \frac{1-\sqrt{5}}{2\sqrt{5}}$. We have found both our unknown values, so plugging them in yields

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}.$$

□

Theorem 6.4 (Characteristic Equation Generalization)

For any linear recurrence $a_n = \alpha a_{n-1} + \beta a_{n-2}$, we obtain the characteristic equation as $\lambda^2 - \alpha\lambda - \beta = 0$. If we denote the roots of this equation as r_1, r_2, we get $a_n = \lambda_1(r_1)^n + \lambda_2(r_2)^n$. To find the constants λ_1, λ_2, we need two known pieces of information about the recurrence relation.

> **Remark 6.5.** If you were wondering about recurrence relations with more terms or is non-linear, it is beyond the scope of the topic, but note that clever substitutions can help reduce the number of terms. If substitutions do not work, another elementary method would be telescoping.

Exercise 6.6. n people sit in a circle. Each round, each person randomly looks to their left or right. If two adjacent people are facing each other, both of them exit the circle. Find a formula, in terms of n, for the expected several rounds until no one is left in the circle.

For more material on recursion, view Euclid's Orchard's Recursion in the AMC and AIME Handout and or Primeri's Recursion Handout.

7 Periodic Sequences

Definition 7.1 (Periodic Sequence)

A **periodic sequence** is a sequence for which similar terms are repeated.

For example, the sequence

$$a_1, a_2, \cdots, a_n, a_1, \cdots$$

is a periodic sequence, and we say it has a period of n. Usually, in problems involving periodic sequences, you're asked to find the nth term of a sequence, where n is large. But, since the sequence is periodic, that term equals a term that shows up earlier. Yet, to exploit this, or even gain the knowledge it is periodic in the first place, we look at the first few terms of this sequence and see if we can find a pattern.

Example 7.2 (2001 AMC 12 Problem 25)

Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$ in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

Solution. Denote the sequence to be $\{a_i\}$. We can compute the first few terms as:

$$\begin{aligned} a_3 &= \frac{2001}{x}, \\ a_4 &= \frac{a_3+1}{a_2} = \frac{2001+x}{2000x}, \\ a_5 &= \frac{a_4+1}{a_3} = \frac{1+x}{2000}, \\ a_6 &= \frac{a_5+1}{a_4} = x, \\ a_7 &= \frac{a_6+1}{a_5} = 2000. \end{aligned}$$

From here, it's obvious that the sequence repeats itself, as two consecutive terms uniquely find the next; we can check that $x = 2001$, $\frac{2001}{x} = 2001$, $\frac{2001+x}{2000x} = 2001$ and $\frac{1+x}{2000} = 2001$ each has one solution, yielding us a grand total of $\boxed{4}$. □

Example 7.3 (2020 AIME II Problem 6)
Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1}+1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution. We are obviously given $t_1 = 20$ and $t_2 = 21$, so now we are able to find the numerical value of t_3 using this information:

$$\begin{aligned} t_3 &= \frac{5t_{3-1}+1}{25t_{3-2}} \\ \implies t_3 &= \frac{5t_2+1}{25t_1} \\ \implies t_3 &= \frac{5(21)+1}{25(20)} \\ \implies t_3 &= \frac{105+1}{500} \\ \implies t_3 &= \frac{106}{500} \\ \implies t_3 &= \frac{53}{250}. \end{aligned}$$

Now using this information, as well as the earlier information, we are able to decide the value of t_4:

$$\begin{aligned} t_4 &= \frac{5t_{4-1}+1}{25t_{4-2}} \\ \implies t_4 &= \frac{5t_3+1}{25t_2} \\ \implies t_4 &= \frac{5(\frac{53}{250})+1}{25(21)} \\ \implies t_4 &= \frac{\frac{53}{50}+1}{525} \\ \implies t_4 &= \frac{\frac{103}{50}}{525} \\ \implies t_4 &= \frac{103}{26250}. \end{aligned}$$

Now using this information, as well as the earlier information, we are able to find the value of t_5:

$$\begin{aligned} t_5 &= \frac{5t_{5-1}+1}{25t_{5-2}} \\ \implies t_5 &= \frac{5t_4+1}{25t_3} \\ \implies t_5 &= \frac{5(\frac{103}{26250})+1}{25(\frac{53}{250})} \\ \implies t_5 &= \frac{\frac{53}{5250}+1}{\frac{53}{10}} \\ \implies t_5 &= \frac{\frac{5353}{5250}}{\frac{53}{10}} \\ \implies t_5 &= \frac{101}{525} \end{aligned}$$

Now using this information, as well as the earlier information, we are able to find the value of t_6:

$$\begin{aligned} t_6 &= \frac{5t_{6-1}+1}{25t_{6-2}} \\ \implies t_6 &= \frac{5t_5+1}{25t_4} \\ \implies t_6 &= \frac{5(\frac{101}{525})+1}{25(\frac{103}{26250})} \\ \implies t_6 &= \frac{\frac{101}{105}+1}{\frac{103}{1050}} \\ \implies t_6 &= \frac{\frac{206}{105}}{\frac{103}{1050}} \\ \implies t_6 &= 21. \end{aligned}$$

Alas, we have figured this sequence is period 5! Thus, $t_{2020} = t_5 = \frac{101}{525}$, and so our answer is $\boxed{626}$. □

8 Problem Set

Problem 8.1 (2015 AMC 8)**.** An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

Problem 8.2 (AoPS Staff)**.** The geometric series $a + ar + ar^2 + \cdots$ has a sum of 25, and the terms involving odd powers of r have a sum of 15. What is r?

Problem 8.3 (1999 USAMTS)**.** Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$$

Problem 8.4 (2003 AIME I)**.** In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

Problem 8.5 (1989 AIME)**.** If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

Problem 8.6 (2011 Stanford)**.** Evaluate the sum

$$\sum_{n \geq 1} \frac{7n + 32}{n(n+2)} \cdot \left(\frac{3}{4}\right)^n .$$

Problem 8.7. Let $T(n) = \dfrac{n \cdot (n+1)}{2}$ be the nth triangular number, and $U(n) = \dfrac{1}{T(n)}$. Find the value of

$$\prod_{n=1}^{50} (T(2n-1) \cdot U(2n)) = \frac{1 \cdot 2}{2} \cdot \frac{2}{2 \cdot 3} \cdots \frac{2}{100 \cdot 101}$$

Problem 8.8. Compute the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \cdots$$

which includes all terms of the form $\dfrac{1}{2^n 5^m}$ for non-negative integers n, m.

Problem 8.9. Let $\tau(n)$ be the number of distinct positive divisors of n. Compute

$$\sum_{d \mid 1001} \tau(d)$$

that is, the sum of $\tau(d)$ for all divisors of 1001.

Problem 8.10. Evaluate the sum

$$\sum_{n=2}^{98} \frac{1}{n(n+2)} = \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \cdots + \frac{1}{98 \cdot 100}.$$

Problem 8.11 (2012 SMT)**.** Express

$$\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdot \cdots \cdot \frac{16^3-1}{16^3+1}$$

as a fraction in lowest terms.

Problem 8.12. Find a closed form for the recurrence relation $b_{n+1} = 4b_n - 3$.

Problem 8.13. How many ways are there to tile a 10×2 grid with 1×2 dominoes?

Problem 8.14. Five numbers form an arithmetic sequence with a mean of 18. If the mean of the squares of the five numbers is 374, what is the greatest of the five original numbers?

Problem 8.15 (2002 AIME II)**.** Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

Problem 8.16. Unit square $ABCD$ is colored black. The diamond formed by the midpoints of its sides is colored white. Then the square formed by the midpoints of the sides of that diamond is colored black, and so on. What is the quantity which the fraction of $ABCD$ which is colored black approaches?

Problem 8.17 (2019 AMC 10A)**.** A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

Problem 8.18 (2020 AMC 10A)**.** There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is k?

Problem 8.19 (2004 AIME II)**.** A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms $a_{2n-1}, a_{2n}, a_{2n+1}$ are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

Problem 8.20 (AoPS)**.** If a, b, and c form an arithmetic progression, and

$$\begin{aligned} a &= x^2 + xy + y^2, \\ b &= x^2 + xz + z^2, \\ c &= y^2 + yz + z^2, \end{aligned}$$

where $x + y + z \neq 0$, prove that x, y, and z also form an arithmetic progression.