Exponents, Logarithms, and Radicals

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Contents

1 Introduction

This handout will cover the topics of exponents and logarithms. These topics often appear together and are extremely useful in problems that include repeated multiplication and logarithm notation. Exponents and Logarithms also appear in real-life situations. However, in this handout, we will focus on how to solve competition problems. Logarithms only appear on the AMC 12 and AIME, but not on the AMC 10. However, exponents appear consistently in all these contests.

2 Exponents

2.1 Introduction to Exponents

In this section, we will give some definitions and properties of exponential operations. Then, we'll go through some examples demonstrating how to use techniques to solve these problems in competitions.

We will begin with a simple definition of exponent notation. Exponent notation significantly simplifies writing when there is intense repeated multiplication.

Exponent

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n}$$

We call the a in the term the ***base*** and the n the ***exponent***. Some simple rules used for exponent operation are shown here.

Fact 2.1 (Exponent Rules). For all $a \neq 0$:

1. $a^b \cdot a^c = a^{b+c}$
2. $a^b \div a^c = a^{b-c}$
3. $a^c \cdot b^c = a \cdot b^c$
4. $(a^b)^c = a^{bc}$
5. $a^{\frac{b}{c}} = \sqrt[c]{a^b}$
6. $a^{-b} = \frac{1}{a^b}$
7. $a^0 = 1$

These properties should not be memorized but should be used enough times, so they are as familiar as adding and subtracting. These exponent properties will be helpful and handy in later problems. Readers are encouraged to try and prove these properties by themselves.

2.2 Some Problems

Example 2.2
Find x if $5^{x^2-7x} = 25^5$.

Solution. The terms on the left and right-hand sides have different bases, making them hard to deal with. We can change the right-hand side to $(5^2)^5 = 5^{10}$. Now our problem becomes straightforward. We know that the base is the same, so the exponent must also be the same: $x^2 - 7x = 10$. Moving the terms to one side and factoring gives us solutions $\boxed{2, 5}$. □

We see a standard problem-solving strategy in this problem. We know how to handle terms with the same base much easier, so we manipulated the terms to get what we wanted.

Example 2.3
Find all values of n such that $6^2 \cdot (6^n)^n = 6^n \cdot 6^n \cdot 6^n$.

Solution. The RHS is easily simplifiable by using (1) from Fact 2.1 to see that

$$6^n \cdot 6^n \cdot 6^n = 6^{3n}.$$

Now we turn to the LHS, and we see that $(6^n)^n$ can be simplified to 6^{n^2} using (3) from Fact 2.1. The LHS can be further simplified

$$6^2 \cdot (6^n)^n = 6^{n^2+2}.$$

Using the same logic as the last problem, we can now see that $n^2 + 2 = 3n$. After moving the terms to one side, and factoring we can see that the solutions for n are $\boxed{1, 2}$. □

Example 2.4
If $9^{x-1} = y$, find 3^{2x+3}.

Solution. We can start by noting that our answer will include y. We note that our desired term has a base of 3. So, to get closer to the answer, we changed our given term to base three.

$$9^{x-1} = 3^{2(x-1)} = 3^{2x-2} = y.$$

We can now compare the 3^{2x-2} to the desired term 3^{2x+3}. We notice that the exponents have a difference of 5. Therefore, $3^{2x-2} \cdot 3^5 = 3^{2x+3}$. So we know that $y \cdot 243 = 3^{2x+3}$. So the answer is $\boxed{243y}$. □

Example 2.5
If $8^x = 27$, find 4^{2x-3}.

Solution. We look at our desired expression and we can take the -3 out to get

$$\frac{1}{64} \cdot 4^{2x}.$$

So we've simplified the problem so that we only need to find 4^{2x}. To do that, we can notice that 4 and 8 are both powers of 2 so we can change our given and desired to

$$8^x = 2^{3x} = 27, 4^{2x} = 2^{4x}.$$

We can manipulate our first equation to get

$$(2^x)^3 = 27 \implies 2^x = 3.$$

Now, we can obtain our desired expression:

$$2^{4x} = (2^x)^4 = 3^4 = 81.$$

Therefore, we plug this back in the the first expression to get the answer $\boxed{\frac{81}{64}}$. $\square$

We've now seen multiple times how computing in the same base can be very convenient and is always a useful tactic to try because of how much it simplifies the problem.

Example 2.6 (2005 AIME I)
The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum.

This problem is relatively difficult to the ones shown above and requires more ingenious manipulations. The reader is encouraged to try this problem before looking at the solution.

Solution. Having multiple terms in the exponent is hard to deal with so from our exponent rules, we can change the equation to

$$\frac{1}{4} \cdot 2^{333x} + 4 \cdot 2^{111x} = 2 \cdot 2^{111x} + 1.$$

We can observe that there are multiple 2^{111x}'s in the equation, so to make the problem simpler, we let

$$y = 2^{111x}.$$

Now the equation becomes

$$\frac{1}{4}y^3 + 4y = 2y^2 + 1.$$

Moving everything to one side and multiplying everything by four, we get

$$y^3 - 8y^2 + 16y - 4 = 0.$$

We could continue the problem by bashing and finding the three roots to these equations. However, we can observe that the question tells us to look for the sum of the three real roots. But, when we solve this equation, we are solving for y. By Vieta's formulas, we know that if the roots of this equation are y_1, y_2, y_3 then

$$\begin{aligned} y_1 + y_2 + y_3 &= 8, \\ y_1y_2 + y_1y_3 + y_2y_3 &= -16, \\ y_1y_2y_3 &= 4. \end{aligned}$$

Each of y_1, y_2, y_3 correspond to a solution of x, so we can set

$$\begin{aligned} y_1 &= 2^{111x_1}, \\ y_2 &= 2^{111x_2}, \\ y_3 &= 2^{111x_3}. \end{aligned}$$

x_1, x_2, x_3 are the three roots of this equation so our goal is to find their sum. Now we can substitute these back into the equations above to get

$$\begin{aligned} 2^{111x_1} + 2^{111x_2} + 2^{111x_3} &= 8, \\ 2^{111x_1+111x_2} + 2^{111x_1+111x_3} + 2^{111x_2+111x_3} &= -16, \\ 2^{111x_1+111x_2+111x_3} &= 4. \end{aligned}$$

We see something similar to the sum of x_1, x_2, x_3 in the third equation and we can change it to

$$2^{111(x_1+x_2+x_3)} = 2^2.$$

Then we get the equation $111(x_1 + x_2 + x_3) = 2$, and solving we get the sum of the roots to be $\boxed{\dfrac{2}{111}}$. □

Our main steps, in the beginning, were taking down the constants from the exponent, and we did that because then we could get our y which made the problem look much simpler. Setting another variable to simplify the problem into a polynomial is often useful. Then, we used Vieta's Formulas to help get the desired answer.

> *Remark 2.7.* Always be aware of what the problem is asking. Then, you can find simpler ways to solve the problem.

2.3 Summary

Exponents are a useful tool to simplify problems. Some many techniques and tricks can be used to solve these problems. Some of these tricks:

1. Changing to the same base when possible.
2. Taking constants down from exponent.
3. Remembering the end goal, so you don't sidetrack.

2.4 Exercises

Exercise 2.8. Solve for x in the equation $\log_3 x^2 - 6x + 32 = 3$

Exercise 2.9 (HMMT). Find the product of all real x for which

$$2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3}.$$

Exercise 2.10 (HMMT). Compute the positive real number x satisfying

$$x^{2x^6} = 3.$$

3 Logarithms

In this section, we will first explore some basic definitions and properties of logarithms. Then, we'll incorporate them into solving problems in the later sections. There will be exercises for practice at the end.

3.1 Definitions and Basic Examples

Logarithms are closely related to exponents and can be considered in the same "family" of operations. Simply put, logarithms are the reverse of exponents. Logarithms are to exponents what subtraction is to add and what division is to multiplication.

Logarithmic Notation
$\log_a b = x \iff a^x = b.$

In this example the a is the ***base*** and will be verbally said: "the log base a of b is x."

Restrictions of Logarithms
For $\log_a b$, we define $a > 0$, $b > 0$, and $a \neq 1$.

We define that the base of a logarithm can only be positive because when it is negative, non-integer values will involve imaginary numbers. Readers are encouraged to explore why this is.

Logarithmic form and exponential form are often interchangeable and are both useful in solving problems. Before diving into problems, let's first show some properties when operating with logarithms.

3.2 Logarithmic Properties

Fact 3.1 (Logarithm Properties).

1. $\log_a b + \log_a c = \log_a bc$
2. $\log_a b - \log_a c = \log_a \frac{b}{c}$
3. $\log_{a^m} b^n = \frac{n}{m} \cdot \log_a b$
4. $\log_a b = \frac{\log_c b}{\log_c a}$
5. $\log_a b \cdot \log_b c = \log_a c$

These are the basic logarithmic properties and are EXTREMELY useful in solving problems. The proofs of these properties are not shown here. However, readers are encouraged to prove it themselves to help gain some familiarity with operating with logarithms.

3.3 Quick Exercises

Exercise 3.2.

1. Find $\log_5 25$.
2. Find $\log_2 128$.
3. Find $\log_4 8$.

Exercise 3.3.

1. Find $\log_2 8 + \log_2 8$.
2. Find $\log_4 9 \cdot \log_3 8$.
3. Find $\log_4 12 + 2\log_4 3 - 3\log_4 6$.

3.4 Let's Solve Some Problems

Example 3.4 (NYSML)
Compute the numerical value of $5^{\log_{10} 2} \cdot 2^{\log_{10} 3} \cdot 5^{\log_{10} 9} \cdot 2^{\log_{10} 6}$.

Solution. We start by observing that we can simplify the problem by combine terms with the same base using our exponent properties.

$$(5^{\log_{10} 2} \cdot 5^{\log_{10} 9}) \cdot (2^{\log_{10} 3} \cdot 2^{\log_{10} 6}) = 5^{\log_{10} 2 + \log_{10} 9} \cdot 2^{\log_{10} 3 + \log_{10} 6}.$$

We recognize that we can use number 1. from our properties to simply to

$$5^{\log_{10} 18} \cdot 2^{\log_{10} 18}.$$

Again we can observe that these terms have the same exponent, so we can combine them to

$$10^{\log_{10} 18}.$$

Now this goes back to the basic definition of a logarithm. The exponent is $\log_{10} 18$ which by definition is "the number we raise 10 to, to get 18". So when we raise 10 to this number, naturally, we get 18. Therefore, $\boxed{18}$ is our answer. □

Notice how in this problem, we used many of our exponent properties. This shows how connected they are and how logarithms and exponents are dependent on each other. Our main tactic was to simplify as much as possible by combining what was possible to get a simpler expression. This should be our objective for most problems.

Example 3.5 (1984 AIME)
Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

Solution. At first this problem seems all over the place. However, we notice that we can significantly simplify the problem by applying some logarithmic properties. We can change the two equations to

$$\frac{1}{3} \cdot \log_2 a + \log_2 b = 5$$
$$\log_2 a + \frac{1}{3} \cdot \log_2 b = 7.$$

Noticed how we changed everything to base 2. Similar to exponents, working with the same base is much easier. However, we notice that the we cannot add the two logarithms together because there are $\frac{1}{3}$ at the beginning of one term in each of the equations. We could bring the $\frac{1}{3}$ in to make it

$$\log_2 \sqrt[3]{a} + \log_2 b = 5$$
$$\log_2 a + \log_2 \sqrt[3]{b} = 7,$$

but with the cube roots everything seems a little messy. We notice that if we add the equations before we brought the $\frac{1}{3}$ in, then we will have the same coefficients. After adding them, we get

$$\frac{4}{3} \cdot (\log_2 a + \log_2 b) = 12.$$

We can multiply both sides by $\frac{3}{4}$ and simplify to get

$$\log_2 ab = 9.$$

Changing this to exponent for we get $2^9 = ab$, therefore, ab is $\boxed{512}$. □

In this problem, we see that similar to exponents, working with the same base is much more convenient, so we should try to work with them as much as possible. We also see how working with different coefficients can be rather annoying, and they require some manipulations to solve.

Example 3.6 (2019 AMC 12B)
Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b}$$

and all four terms on the left are positive integers, where log denotes the base-10 logarithm. What is ab?

Solution. We see that having logarithms nested in square roots is much more annoying compared to having square roots in logarithms. However, we observe that the problem tells us that all the terms are positive integers. Integers. Note that a must be a power of 10 for $\log a$ to be an integer. But, the entire $\sqrt{\log a}$ is an integer, so that means a must be a square power of 10. The same logic applies to b, so we assume:

$$a = 10^{m^2}, b = 10^{n^2}.$$

We also note that with this new assigning of variables, our goal of ab becomes

$$10^{m^2 n^2}.$$

We can now plug in to our equation to get

$$\sqrt{\log 10^{m^2}} + \sqrt{\log 10^{n^2}} + \log \sqrt{10^{m^2}} + \log \sqrt{10^{n^2}}.$$

Using our properties we an simplify this to

$$m + n + \frac{1}{2} \cdot m^2 + \frac{1}{2} \cdot n^2.$$

Multiplying by two and moving terms to one side we get

$$m^2 + n^2 + 2m + 2n - 200 = 0.$$

We see the $m^2, 2m$ and $n^2, 2n$ which motivates us to complete the square and simplify to

$$(m+1)^2 + (n+1)^2 = 202$$

We sense that we cannot simplify further. But, we observe that there exist few squares below 202, so we can start plugging and checking which cases are correct. After some quick checking, we see that the only case that works is when $m+1 = 11$ and $n+1 = 9$ or vice versa.

$$11^2 + 9^2 = 202.$$

Therefore, we get $m = 10, n = 8$ so

$$ab = 10^{100+64} = \boxed{10^{164}}.$$

□

One of the main ideas in this problem was utilizing the fact that the problem told us that the terms were positive integers, and we used that to create new variables that would be easier to work with. We also started testing cases when we got to $(m+1)^2 + (n+1)^2 = 202$. Don't be afraid to do some bashing when cases are small. It is often a useful technique at the end of problems.

Example 3.7 (2020 AMC 12B)
Find the value of $\sqrt{\log_2 6 + \log_3 6}$.

Solution. Off the bat, it doesn't seem like we can do anything to simplify the expression further. We could try changing the base to 6 but that doesn't seem to lead us anywhere. In these situations, we create terms to work with. We can do this by changing the expression to

$$\sqrt{1 + \log_2 3 + 1 + \log_3 2} = \sqrt{2 + \log_2 3 + \log_3 2}.$$

This might seem useless, but if we let $x = \log_2 3$, we see that we can express everything with x because $\log_2 3$ and $\log_3 2$ are just reciprocals of each other. So we get

$$\sqrt{2 + x + \frac{1}{x}}.$$

This next step requires some insight into algebraic manipulations. We can see that the term inside the square root can actually be expressed as

$$(\sqrt{x} + \frac{1}{\sqrt{x}})^2.$$

Then, the square and square root cancel out and we're left with $\sqrt{x} + \frac{1}{\sqrt{x}}$. Plugging in $\log_2 3$ as x, we get the answer $\boxed{\log_2 3 + \log_3 2}$. □

One of the key components of this problem was that it required us to see that the term inside the square root was a square. This comes from the intuition of doing problem after problem, and soon you familiarize yourself with these techniques. This shows why doing many problems is essential to spot these tricky situations and manage them properly.

Example 3.8 (2007 AIME I)
Let
$$N = \sum_{k=1}^{1000} k \left(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor \right).$$
Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

Solution. In the beginning, the problem might seem complicated with lots of symbols, but after looking into some cases, we see that it is rather simple. We notice that there is a ceiling minus a floor inside the summation. We can ignore the summation and look at the term inside first
$$k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor).$$
We note that if $\log_{\sqrt{2}} k$ is not an integer, then the expression inside the parentheses equals one because the ceiling goes to integer right above and the floor goes to the integer right below. Therefore, our inside term simplifies to just
$$k.$$
However, this is only true when our $\log_{\sqrt{2}} k$ is not an integer. So for that to be an integer, k has to be a power of $\sqrt{2}$. But, since k is also an integer, we can say that k has to be a power of 2. In that case, our expression will equal zero because the left term equals the right term. To solve the equation now, we can first forget about the powers of 2 and solve
$$N = \sum_{k=1}^{1000} k.$$
We solve this to be $\frac{1001*1000}{2} = 500500$. However, we've overcounted the powers of 2, so we now sum all the powers of 2 less than 1000 and subtract it from our sum.
$$1 + 2 + 4 + \cdots + 512 = 1024 - 1 = 1023.$$
And $500500 - 1023 = 499477 \equiv \boxed{477} \pmod{1000}$. □

The key to this problem was realizing that the ceiling and floor functions were reduced to 1 whenever the term inside was not an integer. This can be thought of intuitively or can also be gotten from testing some small cases. After that, the problem was just simple calculating.

3.5 Summary

Some of these problems require tricks that might not be easy to think of the first time around. You will only be able to build this intuition through working on more problems and noticing the patterns that emerge when solving them. However, there are some popular tricks and ideas that you should build intuition to solve problems more easily.

1. Most of the time, simplifying by combining terms is your friend.
2. Working with the same base and same coefficients are much easier.
3. Adding or subtracting logarithms are much easier than logarithms that are multiplied or divided.

4. Be aware of the end goal of the problem and try to create something similar.
5. Creating new variables and solving a new polynomial can simplify problems.
6. We often take the logarithm of both sides when there is a logarithm in the exponent

3.6 Exercises

Exercise 3.9 (AHSME). If $x, y > 0$,$\log_y x + \log_x y = \frac{10}{3}$, and $xy = 144$, then find $\frac{x+y}{2}$.

Exercise 3.10 (AIME). Let x, y, z exceed 1 and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.

Exercise 3.11 (AIME). The lengths of the sides of a triangle with positive are are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.

Exercise 3.12. Let $x = \log_b 3$ and $y = \log_b 2$. Find $x - y$.

4 Radicals

4.1 Introduction

Problems involving radicals are essential computing expressions that might seem complicated and messy with roots nested in roots and other ugly terms. In this section, we'll learn how to deal with terms ugly terms and how to simplify the expressions to something we can solve.

4.2 Common Concepts

One of the major ideas in this chapter is ***rationalizing the denominator***. For example, we can look at the term

$$\frac{1}{\sqrt{3}-\sqrt{2}}.$$

We don't like our denominator being irrational partly because it is extremely difficult to estimate the value of a term like that. To rationalize the denominator, we multiply top and bottom by its ***radical conjugate***.

Radical Conjugate
The radical conjugate for an expression $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$.

The logic behind this is that we want to rationalize the denominator, so we have to find a way to rid the square roots. We multiply by $\sqrt{a}-\sqrt{b}$ because this creates an $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ which results in an difference of squares type expression and we result with

$$a - b.$$

Note: you should not forcefully memorize an idea like this but should think about why we are multiplying what we are and how it is simplifying our problem. There will be other instances where rationalizing the denominator is not as simple as this.

Going back to our original example we can multiply top and bottom by the radical conjugate to get

$$\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \sqrt{3}+\sqrt{2}.$$

Now, this expression is a lot prettier.

Another common concept is a square root nested in a square root.

Example 4.1
Simplify $\sqrt{6+2\sqrt{5}}$

Solution. This configuration shows up commonly in problems involving radicals, so we should get used to it. Our main goal is to get rid of the double square root. To do that, we try to change the expression inside the larger square root into a perfect square. Because then we have

$$\sqrt{(A)^2} = |A|.$$

So we can set the equation

$$\sqrt{6+2\sqrt{5}} = a + b\sqrt{5}$$

for some a and b. Then we can square both sides to get

$$6+2\sqrt{5} = a^2 + 5b^2 + 2ab\sqrt{5}.$$

Because our irrational terms are $2\sqrt{5}$ on the left and $2ab\sqrt{5}$ on the right, we know that they are equal. Setting the rational parts equal to each other and irrational parts equal, we get the system of equations

$$\begin{aligned} 2ab &= 2 \\ a^2 + 5b^2 &= 6. \end{aligned}$$

So we get $ab = 1$ so we try $a = 1$ and $b = 1$ and they both work. Usually, this step requires so trial and error. So we can plug this back into our original equation to get

$$\sqrt{6+2\sqrt{5}} = 1 + \sqrt{5}.$$

Now, we've gotten rid of the double square roots, which were our goal. □

Getting familiar and simplifying these square roots is important so here are some exercises.

4.3 Quick Exercises

Exercise 4.2.

1. Simplify $\sqrt{3+2\sqrt{2}}$
2. Simplify $\sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}$
3. Simplify $\sqrt{49+28\sqrt{3}}$

Exercise 4.3 (ARML). Find the ordered pair of positive integers (a, b), with a<b, for which

$$\sqrt{1+\sqrt{21+12\sqrt{3}}} = \sqrt{a}+\sqrt{b}$$

4.4 Problems

Example 4.4
Rationalize the denominator for

$$\frac{1}{\sqrt[3]{7}-1}$$

Solution. We can blindly use the formula we used to rationalize this denominator. However, we see that this is a cube root, so that method will not work. Our logic behind rationalizing the denominator was to multiply top and bottom by an expression such that the denominator becomes rational. Because there is a $\sqrt[3]{7}$, we know we'll have to get that term to the third power somehow. So we remember the factorization

$$a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

We let $a = \sqrt[3]{7}$ and $b = 1$ in this scenario. Therefore to rationalize the denominator we multiply top and bottom by (a^2+ab+b^2) which is also

$$\sqrt[3]{49}+\sqrt[3]{7}+1.$$

Then we get

$$\frac{\sqrt[3]{49}+\sqrt[3]{7}+1}{(\sqrt[3]{7}-1)(\sqrt[3]{49}+\sqrt[3]{7}+1)}.$$

And from our factorization we know that our denominator simplifies as well as the entire fraction.

$$\boxed{\frac{\sqrt[3]{49}+\sqrt[3]{7}+1}{6}}.$$

□

We realize that there are multiple different ways to rationalize the denominator, and we can improvise to what the problem gives us to rationalize. Remembering factorizations is often helpful, so you know how to get rid of the roots.

Example 4.5 (AHSME)
Simplify

$$\frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}}.$$

Solution. When we look at the term on the right, we can use our trick of simplifying a square root inside a square root to get

$$\sqrt{3-2\sqrt{2}} = \sqrt{2}-1.$$

Now, we look at our fraction. If you try to use our trick from above, you will see that it does not work for this problem. So we need a different approach this time. Squaring the fraction would get rid of a lot of radicals, so we let

$$A = \frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}$$
$$A^2 = \frac{2\sqrt{5}+2\sqrt{(\sqrt{5}+2)(\sqrt{5}-2)}}{\sqrt{5}+1}.$$

Through further simplifying, we see that

$$A^2 = \frac{2+2\sqrt{5}}{\sqrt{5}+1}.$$

This is obviously 2. So we know that

$$A = \sqrt{2}.$$

So going back to our original equation, we get

$$\sqrt{2}-(\sqrt{2}-1) = \boxed{1}.$$

□

In this problem, we see that our trick with double radicals does not always work. No one-trick can solve all radical problems, so we have to be innovative and think about how to deal with the roots. In this problem, we saw that squaring would create a difference of squares inside the radical, which would simplify significantly.

Example 4.6 (AIME)
Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

Solution. The square roots with large numbers seem terrifying, but we observe that the degree of the roots is all powers of 2. This reminds us of the factorization

$$a^n - 1 = (a-1)(a^{n-1}+a^{n-2}\cdots a+1).$$

The denominator of x has a great resemblance to this factorization. So we will have

$$a = \sqrt[16]{5}.$$

If we multiply the top and bottom by $\sqrt[16]{5}-1$ the term will become

$$\frac{4(\sqrt[16]{5}-1)}{4} = \sqrt[16]{5}-1 = x.$$

So

$$(\sqrt[16]{5}-1+1)^{48} = \boxed{125}.$$

□

Example 4.7 (Sweden)
Solve the equation
$$(\sqrt{2}+1)^x + (\sqrt{2}-1)^x = 6.$$

Solution. We are not used to having a variable in the exponent. So we try to simplify by assigning new variables.
$$a = (\sqrt{x}+2)^x$$
$$b = (\sqrt{2}-1)^x.$$
From the given condition we know that $a+b=6$ and additionally we see that
$$ab = ((\sqrt{2}+1)^x)((\sqrt{2}-1)^x) = 1.$$
So, we can make a polynomial with a and b as its roots.
$$x^2 - 6x + 1 = 0.$$
Using the quadratic formula to solve this, we get
$$\frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}.$$
So we know that a and b are $3+2\sqrt{2}, 3-2\sqrt{2}$, not necessarily in order. So we want to find all x such that $(\sqrt{2}+1)^x = 3 \pm 2\sqrt{2}$ From some quick checking we see that
$$(\sqrt{2}+1)^2 = 3+2\sqrt{2}$$
works when $x=2$. Now we have to find x such that $(\sqrt{2}+1)^x = 3-2\sqrt{2}$. We see that $3+2\sqrt{2}$ and $3-2\sqrt{2}$ are reciprocals (check this yourself). So, we know that
$$(\sqrt{2}+1)^{-2} = 3-2\sqrt{2}.$$
So we get our solutions $\boxed{x = 2, -2}$. □

In this problem, we again see the power of creating new variables that help simplify the problem and how making a polynomial with roots of the numbers you want to find can be very helpful. These are some of the major strategies when solving these problems.

4.5 Summary

There is not a one-trick pony when solving radical problems. However, there are general concepts that we strive to achieve when simplifying a problem,m and we use the main direction to decide what to do specifically. We can logically think about how we can manipulate our equations to get what we want to achieve. This can be done through gaining more experience in problem-solving. Here are some general concepts

1. When rationalizing, try to remember some factorization
2. Don't mindlessly simplify. Always have a main goal.
3. Creating polynomials to help solve ugly terms is very helpful.
4. Usually, the fewer radicals there are, the easier it is to solve the problem.

4.6 Exercises

Exercise 4.8 (AoPS). Simplify the sum $\sqrt[3]{18+5\sqrt{13}}+\sqrt[3]{18-5\sqrt{13}}$.

Exercise 4.9 (AMC 10). Evaluate

$$\sqrt{9+6\sqrt{2}}+\sqrt{9-6\sqrt{2}}.$$

Exercise 4.10 (AIME). The number

$$\sqrt{104\sqrt{6}+468\sqrt{10}+\sqrt{14415}+2006}$$

can be written as$a\sqrt{2}+b\sqrt{3}+c\sqrt{5}$, where a,b,c are positive integers. Find abc.

5 Problems

Minimum is [26 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "Everyone hated Calculus. Quadratic equations, parabolas, logarithms, trigonometry - you name it. It was like floating in an endless, frictionless void traveling at x miles per hour. Solve for x."
>
> *Andrew Sturm*

[3 ♟] Problem 1 (AMC 12). What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

[4 ♟] Problem 2 (AIME). Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

[4 ♟] Problem 3 (AIME). Find the last three digits of the product of the positive roots of $\sqrt{1995}x^{\log_{1995} x} = x^2$.

[5 ♟] Problem 4 (AMC 12). The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

[5 ♟] Problem 5 (AHSME). If

$$x + \sqrt{x^2 - 1} + \frac{1}{x - \sqrt{x^2 - 1}} = 20,$$

evaluate

$$x^2 + \sqrt{x^4 - 1} + \frac{1}{x^2 + \sqrt{x^4 - 1}}.$$

[5 ♟] Problem 6 (NYSML). Find the ordered triple of positive inters (a, b, c) for which

$$(\sqrt{5} + \sqrt{2} - \sqrt{3})(4\sqrt{a} + \sqrt{b} - 2\sqrt{c}) = 12.$$

[6 ♟] Problem 7 (AIME). The system of equations

$$\begin{aligned}
\log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\
\log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\
\log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0
\end{aligned}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

[6 ♟] Problem 8 (AIME). The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a + b + c$.

[7 ♟] Problem 9 (AIME). Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[7 ♟] Problem 10 (AMC 12). Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

[8 ♟] Problem 11 (AIME). There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60$$
$$\log_{10} y + 2\log_{10}(\operatorname{lcm}(x,y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

[8 ♟] Problem 12 (AoPS). Show that

$$\sqrt{x - 4\sqrt{x-4}} + 2 = \sqrt{x + 4\sqrt{x-4}} - 2$$

for all $x \geq 8$.