More Counting

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1 Combinatorics Basics

In the previous counting handout, we covered a few strategies to cover some counting problems. But naturally, counting can get much more complicated than that. In this handout we go over some more advanced strategies for counting, such as casework, restrictions, stars and bars, and the principle of inclusion-exclusion, all while building on the concepts of the previous handout. As with many topics in competition math, it's very important to understand the previous topics before proceeding, so don't be afraid to go back and review the previous handout!

1.1 Factorials

Example 1.1
How many rearrangements of the word "MATH" are there?

Solution. We can apply the concept of constructive counting here, by writing any given rearrangement one letter at a time. For example, for the first letter, we have 4 options (namely "M", "A", "T", and "H"). No matter what we place for the first letter, we will have 3 options for the second letter, because we can choose any of the 4 letters *except* the letter we already used (for example, if we had "T" as the first letter, we could use "M", "A", or "H" as the second letter). Similarly, we have 2 options for the third letter, as we can use any of the 4 letters except the 2 we already used. Finally, for the last letter we only have one option left, so our final answer is $4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$. □

Factorial
The factorial of n for some non-negative integer n, denoted as $n!$, is equal to $n(n-1)(n-2)\dots 2 \cdot 1$. By convention we define $0! = 1$. In particular, note that $n!$ is the number of ways to rearrange n distinct letters (proof left as an exercise).

Exercise 1.2. Use similar reasoning as the example to show that $n!$ is equal to the number of ways to rearrange n distinct letters.

Exercise 1.3. Compute $1!$, $2!$, $3!$, $4!$, and $5!$.

Exercise 1.4. How many different standings are there in a 5 person race, assuming there are no ties?

So now we know how to rearrange letters (or really, any objects - they do not have to be letters) in a row, as long as they are distinct. But what if we have duplicate letters?

Example 1.5
How many ways are there to rearrange the letters of "LETTER"?

Solution. Note that if all 6 letters were distinct, then the answer would just be $6! = 720$. However, the issue is that we have 2 "T"s and 2 "E"s, so that would be overcounting. For example, let us relabel the letters as L, E_1, T_1, T_2, E_2, R. Then if we said $6!$ as our answer, we would be counting $LE_1T_1T_2E_2R$ and $LE_2T_1T_2E_1R$ as 2 different rearrangements, when in reality they're the same.

So how much are we overcounting by? Well, note that for any given rearrangement, we can switch T_1 and T_2 around and get the same arrangement. Thus, that is overcounting by 2 times. However, the exact same logic holds for E_1 and E_2, so we are overcounting by a total of $2 \cdot 2 = 4$ times. In other words, for any given arrangement, we are counting it 4 times when we should be counting it once. Thus, our answer would be $\frac{6!}{4} = \boxed{180}$. □

Note that we didn't actually need any new formulas for this special problem - instead, we just fell back on principles we already knew, such as factorials. This is a common theme in competition math, and particularly in combinatorics - at the end of the day, remember that contests like the AMC are designed to test your problem solving skills, not how well you can memorize formulas. Although you will need to memorize certain things to save time on contests, it's still very important to *understand* the reasoning behind the formulas for things like factorials, instead of blindly memorizing, so that way we can make modifications as needed and apply the concepts to other problems.

Exercise 1.6. How many ways are there to rearrange the letters of "MISSISSIPPI"?

Exercise 1.7. Suppose we have 3 soccer balls, 3 basketballs, and 2 volleyballs that we want to line up in a row. How many ways are there to do so?

Exercise 1.8 (2002 AMC 10B). Using the letters A, M, O, S, U, we can form 5! five-letter "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies what position?

1.2 Permutations

Example 1.9
Suppose 8 (distinct) people are running in a race. Assuming there are no ties, how many ways are there to decide who gets first, second, and third place?

Solution. Notice the similarity to Exercise 1.1.8. In particular, if we wanted to decide the rankings of all 8 runners, the answer would just be 8!. This is because we could simply arrange the runners in a row, and have the first one finish first, the second one finish second, and so on. However, since we only want to order the 3 fastest runners, 8! is not the right answer.

Instead we resort to constructive counting again. Note that there are 8 possibilities for first place. Then once we decide first place, we can have any of the other 7 people finish second, and any of the remaining 6 people finish third. Thus, our answer is $8 \cdot 7 \cdot 6 = \boxed{336}$.

Another way to think about this is to determine the order of all 8 people, which gets us an answer of 8!. Then, we can figure out how much we are overcounting by - in particular, for any order of the top 3, we are counting it 5! times, for each of the 5! orderings of the remaining 5 people. This means we are overcounting by a factor of 5!, so the answer will be $\frac{8!}{5!} = \boxed{336}$ as well. As an extra step, try writing out the factorials explicitly and simplifying the resulting fraction, then comparing it to the $8 \cdot 7 \cdot 6$ we got from constructive counting. Do you see why the two answers should be the same? □

Exercise 1.10. Show that in general, if n people are running a race and we want to decide who the top r runners are ($0 \leq r < n$), then there are $n(n-1)(n-2)\ldots(n-r+3)(n-r+2)(n-r+1) = \frac{n!}{(n-r)!}$ ways to do so.

Permutations

For $0 \leq r \leq n$, ${}_nP_r$ is defined as $\frac{n!}{(n-r)!}$, and from the Exercise 1.2.2 is the number of ways to pick the top r runners in a race. More generally, it is the number of ways to pick r things from a set of n things, where the order that we pick in *does* matter. It is also denoted as $P(n,r)$ or P_r^n and is read as "n permute r."

Exercise 1.11. When we defined factorials, we stated that $0! = 1$. Use the formula for permutations when $n = r$ and compare the result to what the answer should be, using Exercise 1.1.8, to determine why $0!$ should equal 1.

Exercise 1.12. Compute ${}_6P_2$, ${}_5P_5$, and ${}_4P_3$

Exercise 1.13. If you did Exercise 1.2.5 correctly, you may have noticed ${}_4P_3 = {}_4P_4 = 4!$. In fact, in general ${}_nP_{n-1} = n!$. Figure out why this is true, using both the formula for permutations, and by reasoning through using constructive counting.

Exercise 1.14. How many ways are there to pick 4 distinct objects from a set of 8, and then line them up in a row?

1.3 Combinations

So now we know how many ways there are to choose r things from a set of n, where the order that we choose things in *does* matter. But what if order didn't matter?

Example 1.15

How many ways are there to choose 3 students from a class of 8 to form the student council?

Solution. Note that, if order did matter, then the answer would simply be ${}_8P_3 = 336$ (because this problem would be identical to Example 1.2.1). However, since order does not matter, we are overcounting. But just how much are we overcounting by? Note that for any set of 3 people, we are counting it $3! = 6$ times because there are 6 ways to rearrange the 3 people. Thus we are counting each combination as 6, so we should divide 336 by 6 to get $\frac{336}{6} = \boxed{56}$. □

Exercise 1.16. Show that in general, we want to pick a set of r things from a set of n ($0 \leq r \leq n$), then there are $\frac{{}_nP_r}{r!}$ ways to do so.

Combinations

For $0 \leq r \leq n$, ${}_nC_r$ is defined as $\frac{{}_nP_r}{r!}$, and from the Exercise 1.3.2 is the number of ways to pick r people from a set of n when order *does not matter*. By plugging in the formula for ${}_nP_r$, ${}_nC_r = \frac{n!}{r!(n-r)!}$ as well. It is also denoted as $C(n,r)$, C_r^n, or $\binom{n}{r}$ and is read as "n choose r."

Example 1.17 (1992 MAΘ)
We are given 5 lines and 2 circles in a plane. What is the maximum number of possible intersection points along these 7 figures?

Solution. First we will deal with the most complicated figures. The 2 circles can intersect each other at 2 points, at most. Each of the 5 lines can intersect each of the 2 circles at 2 points, at most, for $5 \cdot 2 = 10$ more intersections. Now for any pair of lines, they can form at most one more intersection, and there are $\binom{5}{2} = 10$ ways to pick a pair of lines, so we get $2 + 10 + 10 = \boxed{22}$ as our final answer. Note that this only shows our answer cannot be larger than 22, and we have not shown 22 is actually possible yet. Try to draw a configuration that achieves this answer, to prove the answer is indeed 22. □

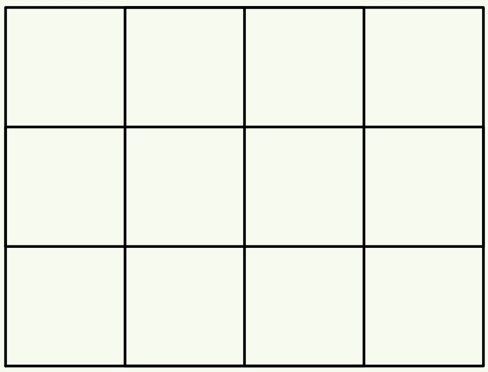
Example 1.18 (2006 AIME II Problem 4)
Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$$

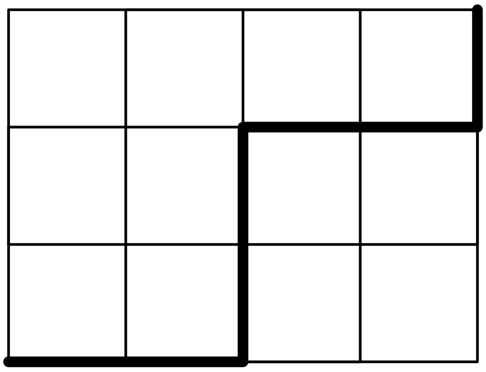
An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

Solution. Note that a_6 is the smallest variable, so $a_6 = 1$. Now note that for any set $\{a_1, a_2, a_3, a_4, a_5\}$ (disregarding order) there is exactly one way to rearrange them so that $a_1 > a_2 > a_3 > a_4 > a_5$. Similarly, for any set $\{a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ there is exactly one arrangement so that $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$ holds. Thus, we can simply decide what the 2 sets will be. To do this we can choose 5 of the numbers $\{2, 3, \ldots, 12\}$ to form $\{a_1, a_2, a_3, a_4, a_5\}$ and have the other 6 form the other set. There are a total of $\binom{11}{5} = \boxed{462}$ ways to do this. □

Example 1.19 (Pathwalking)
How many ways are there to get from the bottom left corner of the following diagram to the top right, only moving up and right along the grid lines?



Solution. This problem belongs to a common type called path-walking (you can probably see why!) The idea is that for any given path, we can write down the moves that we make (either R for moving right one segment, or U for moving up one segment). For example, the following path would correspond to “RRUURRU."



Then any valid path will consist of exactly 3 “U"s and 4 “R"s, because we cannot move down or left. As such, we just need to find how many 7 letter sequences have 3 “U"s and 4 “R"s. But this is much easier to count! We can either view this as the number of ways to rearrange the letters of “UUURRRR," which we covered in Section 2.1.1, or we can directly choose 3 of the letters to be “U" and fill in the rest with “R" to get a final answer of $\binom{7}{3} = \boxed{35}$. □

Exercise 1.20. Prove that $\binom{n}{r} = \binom{n}{n-r}$ using both the formula and the “picking objects from a group" definition.

Exercise 1.21. Compute $\binom{7}{2}$ and $\binom{7}{5}$ and verify they are equal. Also compute $\binom{7}{0}$ and $\binom{7}{7}$ and confirm they are what you would expect them to be, using the “choosing from a set" definition.

Exercise 1.22 (2020 AMC 10B). Ms. Carr asks her students to select 5 of the 10 books to read on her classroom reading list. Harold randomly selects 5 books from this list, and Betty does the same. How many ways can they select the books such that there are exactly 2 books that they both select?

Exercise 1.23 (2005 AIME II). A game uses a deck of n different cards, where $n \geq 6$ is an integer. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.

2 Counting with Restrictions

In the last section we covered permutations and combinations, which are the number of ways to pick a certain amount of things from a group. But problems will not always be this direct. For example, what if we added certain restrictions when picking the things?

Example 2.1
Suppose we want to pick 3 people to play on a basketball team, out of a set of 9. However, Albert hates basketball and refuses to play. How many ways are there to pick the 3 people without picking Albert?

Solution. This is a fairly straightforward restriction, because we can just ignore Albert when picking the team. As such, we have $9 - 1 = 8$ people to choose from, so the answer is simply $\binom{8}{3} = \boxed{56}$. □

We now consider some more tricky restrictions, where we can no longer directly apply a formula.

Example 2.2
Suppose we want to pick 4 people to play on a soccer ball team now, out of a set of 8. However, Brooke and Chloe are enemies and refuse to play on the same team. How many ways are there to pick the team?

Solution. We can no longer resort to directly applying our combination or permutation formulas now. Instead, we will have to do some thinking. We can consider the various cases:

1. **Case 1: Only Brooke is on the team.** Here, we can pick the remaining $4 - 1 = 3$ players (since we already know Brooke is playing) from the set of $8 - 2 = 6$ people (since we know Brooke is playing and Chloe is not, we do not need to account for them). This gives us $\binom{6}{3} = 20$ ways to pick the team for this case.

2. **Case 2: Only Chloe is on the team.** Note that this is identical to case 1, except Brooke and Chloe, are switched. If you aren't convinced, try working through the reasoning to get 20 for this case as well.

3. **Case 3: Neither Chloe nor Brooke are on the team.** This case is very similar to Example 1.4.1. Note that we only need to choose four people out of the $8 - 2 = 6$ that are not Chloe and Brooke, so we get $\binom{6}{4} = 15$ possibilities for this case.

Adding up all the cases, we get $20 + 20 + 15 = \boxed{55}$ as our final answer. □

Example 2.3
Suppose we have 5 people that we want to line up in a row. However, John and James refused to sit next to each other. How many ways are there to arrange the 5 people?

Solution. We could use casework here, and it will eventually work - however, a big aspect of competition math is the time factor, and casework is slow and error-prone, especially if the number gets larger than 5. Instead, it is easier to use complementary counting, so we will go over that solution (if you have time, try to do it with casework too!). If we ignore the extra restriction with James and John, we have a total of $5! = 120$ ways to seat them. However, this is clearly overcounting, since we are including the cases where John and James are next to each other, so we need to subtract those out. But how many ways are there for John and James to sit next to each other?

The key idea is to treat John and James as a single being, so we have 4 beings. Then there are $4! = 24$ ways to arrange them. However, we have to multiply this by 2, because there are 2 ways for John and James to sit together - either John could be first, or James could be first. Thus there are a total of $2 \cdot 24 = 48$ arrangements where John and James are adjacent, so our final answer is $120 - 48 = \boxed{76}$. □

Example 2.4
How many ways are there to arrange 6 people in a circle, where rotations count as the same arrangement?

Solution. If we were arranging the people in a line, the answer would simply be $6! = 720$. However, this would be overcounting since the people are in a circle and rotations are the same. Instead, note that for any particularly rearrangement, there are 6 different rotations that are all counted as the same, because there are 6 people. Thus we are overcounting by 6 times, so we need to divide by 6 to get $\frac{6!}{6} = \boxed{120}$. □

Exercise 2.5. How many ways are there to place 5 keys on a keychain, where two configurations are the same if we can somehow get from one to the other without taking keys off (i.e. we can rotate the keychain or flip it).

Exercise 2.6. How many ways are there for a teacher to divide a class of 9 into a group of 4 and a group of 5, if Alice and Bob refuse to be in the same group?

Exercise 2.7 (2019 AMC 10A). A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

3 Recursion

The idea behind recursion is that sometimes, we have problems that are too complicated to do with casework, or even other strategies such as complementary counting. When this happens, we can often create a “smaller" and “simpler" version of the problem, then use that to help solve our big problem (if our simpler version is still too complicated, we can we can make it even smaller, and keep repeating until we arrive at a problem that is easy to solve). Recursion is especially common on some harder AIME problems, although it also does come up on the AMCs and is definitely worth learning. Don’t worry if this doesn’t totally make sense - recursion is best learned through examples.

Example 3.1 (AoPS Intermediate Counting & Probability)
Find the number of 10-digit ternary sequences (that is, sequences using only 0, 1, or 2) such that the sequence does not contain two consecutive zeros (leading 0s are allowed, so something like 0120 is a valid sequence).

Solution. You could try casework here, but you’ll likely find that with 10 digits, there are simply too many cases to deal with all of them, and there’s a high chance that you’ll make some computational error anyway. If you try complementary counting or PIE, you’ll still get a very complicated expression. Instead, we can think about how we could make the problem simpler: we could try removing the restricting condition of “no consecutive zeros", which would just give us 3 options for each digit and an answer of 3^{10}. However, this modified version doesn’t help us solve our current problem.

We can also make the problem simpler by reducing the length of the sequence. If we consider a 9-digit ternary sequence, that’s still too difficult. But what if we instead just did a 1-digit sequence? In that case, we simply have 3 sequences (namely 0, 1, and 2). Let's try something slightly more difficult: 2-digits. That gives us 8 sequences because all of the $3^2 = 9$ sequences aside from 00 work. So what about 3-digits? Here the problem becomes fairly complex, so we should come up with a smarter strategy. Let us consider casework on the first digit of the 3-digit sequence.

1. **Case 1: The first digit is not 0 (so it is either 1 or 2)**. Then for the remaining 2 digits, we can take any sequence that does not contain 2 consecutive 0s. But we already know how many such 2-digit sequences there are - we just calculated it and got 8! So in this case, we have 2 choices for the first digit and 8 for the remaining digits, for a total of $2 \cdot 8 = 16$.

2. **Case 2: The first digit is 0**. Then the second digit will have to be a 1 or a 2. For the last digit, we can take any of the 3 digits - in other words, we can take any combination of $3 - 2 = 1$ digit without consecutive 0s. So in this case, we have 2 options for the second digit and 3 for the remaining digits (which is just the third), for a total of $2 \cdot 3 = 6$.

Thus, for 3-digit sequences we have 22 sequences. But wait a second - we can repeat this process for 4 digits too! In fact, if we let a_n be the number of n digit sequences without consecutive zeros, we can show that $a_n = 2a_{n-1} + 2a_{n-2}$ for all $n > 2$, using the same logic as we did above. From here, we can calculate a_n one at a time to get the sequence is $a_1, a_2, \ldots$ is $3, 8, 22, 60, 162, 448, 1224, 3344, 9136, 24960, \ldots$, so the tenth number, or $\boxed{24960}$ is the answer we want. □

Example 1.5.1 is a fairly common use of recursion - if we want to construct a sequence or string of length n that satisfies some restrictive condition, we can often let a_n denote the answer and write a formula for a_n in terms of smaller a_i, so we can then "build up" to the final answer. However, as we'll see in the next example this doesn't only apply to sequences.

Example 3.2 (2006 AIME I Problem 11)
A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

- Any cube may be the bottom cube in the tower.
- The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers than can be constructed. Find T.

Solution. Again we see that we a constructing something with a certain length (or in this case, height), following some complicated condition, so we turn to recursion. Let a_n be the answer when 8 is replaced with n. By casework we see $a_1 = 1$ and $a_2 = 2$. So how to we get from a_2 to a_3? Well note that we can place the cube with side length 3 in 3 spots - either on top of the 2, on top of the 1, or right at the bottom. Thus $a_3 = 3a_2 = 6$. In general, when going from a tower of height n to $n + 1$, we can place the $n + 1$ cube right after the n cube, the $n - 1$ cube, or at the bottom, so we have 3 choices. Thus, $a_{n+1} = 3a_n$, so by solving this recursion we see $a_8 = 2 \cdot 3^6 = \boxed{1458}$.

Sidenote: Notice that our recursion breaks down when going from 1 to 2, as $a_2 = 2 \neq 3 \cdot a_1$. This is because in our recursive step, we placed the cube of length $n+1$ after $n-1$ in one of the 3 cases, and if $n = 1$ this clearly makes no sense. □

Exercise 3.3 (2007 AMC 12A). Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

Exercise 3.4 (2019 AMC 10B). How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

3.1 Multiple Sequence Recursions

In the previous examples, we only had one sequence - namely a_n was the answer for length n. However, sometimes you might find it more natural to have 2 or more sequences accounting for two different cases. This is generally only necessary for AIME level problems, so if you are preparing for the AMC this section won't be necessary, although it is still good practice.

Example 3.5 (2008 AIME I Problem 11)
Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

Solution. You could try letting a_n be the number of such sequences with length n, as we did in the previous examples. However, you will find it very difficult to relate a_n to smaller a_i. Instead, we can consider 2 cases: if the sequence ends in A or B. Let a_n be the number of sequences of length n that end in A, and define b_n similarly with B. We can now try to relate a_n and b_n to smaller values of a_i and b_i.

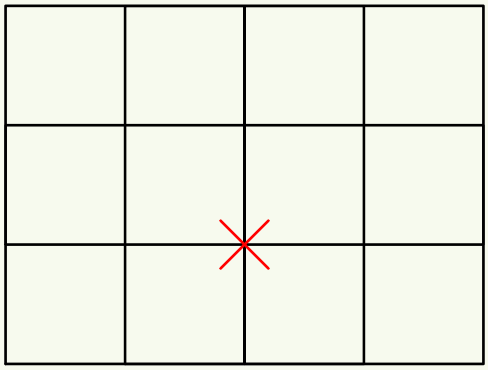
Note that if we have a sequence of length n ending with A, then the last 2 letters must be A, so we added 2 As to any string of length $n-2$. Thus, $a_n = a_{n-2} + b_{n-2}$. Similarly, if a sequence ends in B, then if the second to last letter is an A, we simply added a B to a sequence of length $n-1$ ending with A. If the second to last letter is B, then we added 2 Bs to a sequence of length $n-2$ ending with B (because if the sequence of length $n-2$ ended with A, we would have 2 Bs at the end of our sequence, which is not an odd number). Thus $b_n = a_{n-1} + b_{n-2}$.

Obviously these recursions only make sense for $n > 2$, so we have to do $n \leq 2$ by hand. We can check $a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0$, so from here we can solve the recursion to get $a_1, a_2, \ldots$ is $0, 1, 1, 1, 3, 2, 6, 6, 11, 16, 22, 37, 49, 80, \ldots$ and $b_1, b_2, \ldots$ is $1, 0, 2, 1, 3, 4, 5, 10, 11, 21, 27, 43, 64, 92, \ldots$. Thus the total number of sequences of length 14 is $a_{14} + b_{14} = 80 + 192 = \boxed{172}$. □

Exercise 3.6 (2001 AIME I). A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

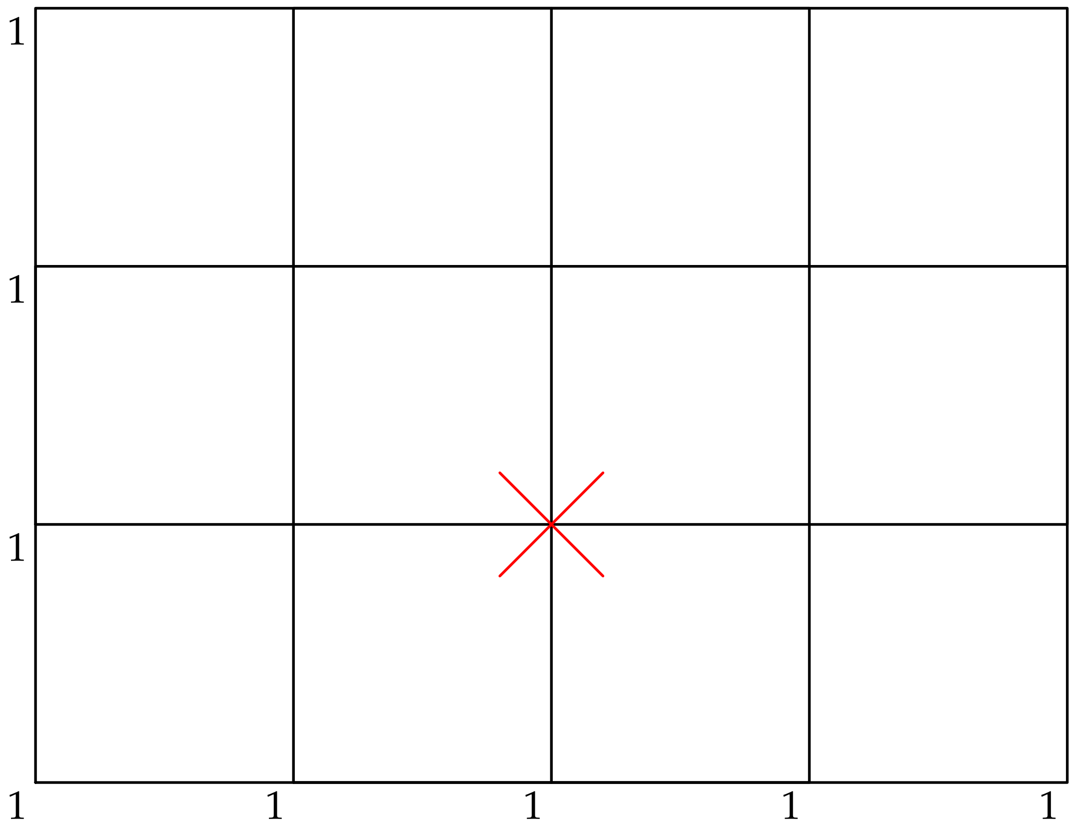
3.2 Pathwalking

Example 3.7 (Pathwalking Revisited)
How many ways are there to get from the bottom left corner of the following diagram to the top right, only moving up and right along the grid lines, *without going through the crossed out point*?

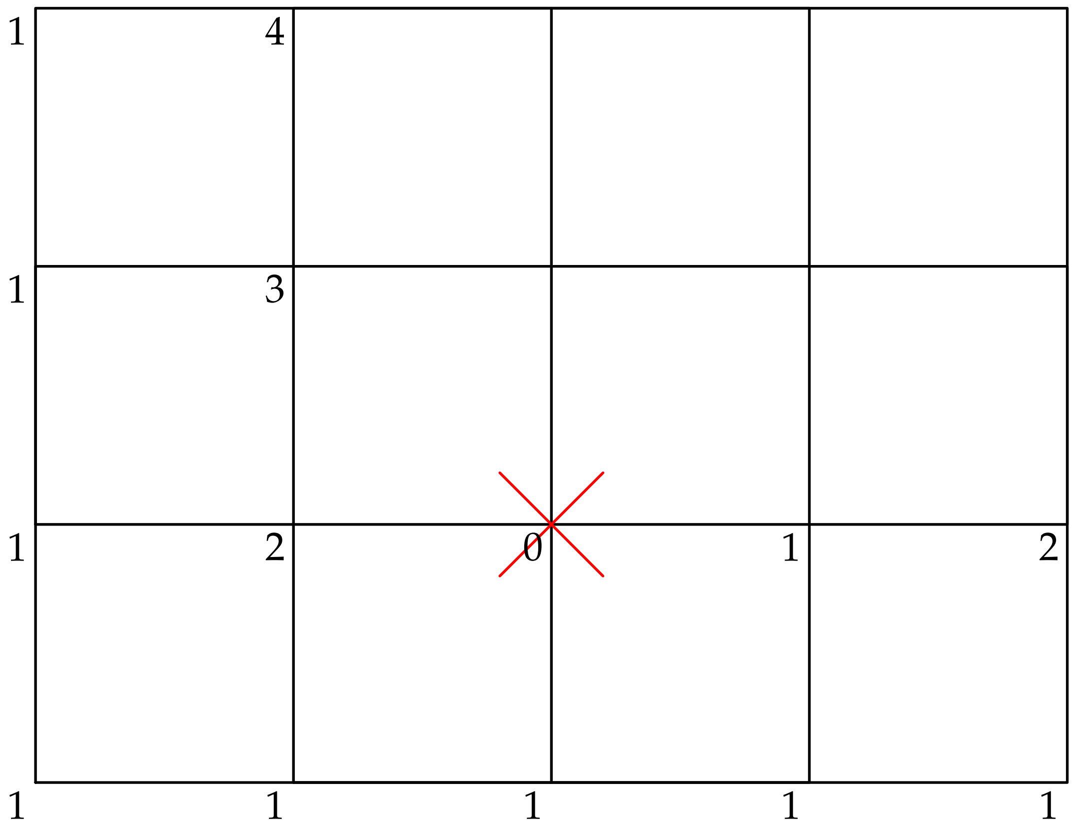


Solution. We will present two solutions, one using recursion and one not.

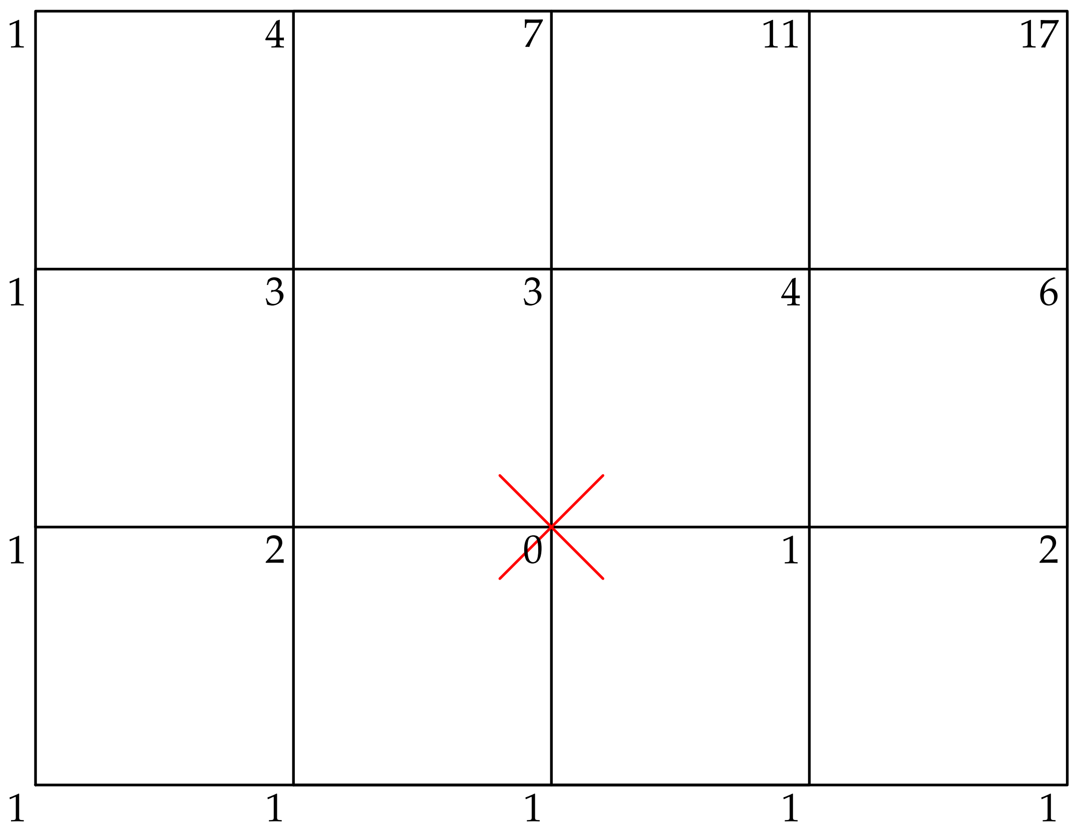
Solution 1. To use recursion on this problem, we can reduce the size of the grid until we get a more manageable casework (ignoring the condition for now). If the grid were just a single point, then the answer would be easy - it would just be 1, since we don't move at all. Furthermore, if either of the dimensions of the grid were 0, then we would just have a line, and the answer would still be 1 (seeing as we can only move in one direction). The corresponding points below have been labeled with the number of ways to get there starting from the bottom left.



Now notice that for any other point P that is not on the left or bottom edge, in order to reach P we must either reach the point directly to the left of P and move left, or reach the point directly below P and move up. Thus we can recursively fill in the rest of the grid, where every point is the sum of the number below and to the left of it. However, since we cannot pass through the crossed out point, there are 0 ways to reach that crossed out point, so we will label it with a 0. For example, we can fill in the second row and column as shown:



We can then continue filling in the grid in this fashion, labeling each point with the number of ways to get there and arriving at the following:



Since we want the number of ways to reach the top-right corner, we simply take the number at that position, which is $\boxed{17}$.

Solution 2. We can use a similar strategy as we did for Example 1.3.5, by using a clever way of counting the number of paths. However, this time we will have to deal with the restriction. You could try using casework, but it is better to use complementary counting. We already know the total number of ways to reach the top right corner without the restriction is simply $\binom{7}{3} = 35$ by Example 1.3.5.

However, we have to subtract out the number of paths that go through the crossed out point. Note that to get to the crossed out point, we will need to go right 2 times and go up once, so using the concept from Example 1.3.5, there are $\binom{3}{1} = 3$ ways to get to the crossed out point. But we want to find the number of ways to get to the top right through the crossed out point, not just the number of ways to get to the crossed out point. As such we need to multiply by the number of ways to get from the crossed out point to the top right, which involves going right twice and up twice. There are $\binom{4}{2} = 6$ ways to do this, so there are a total of $3 \cdot 6$ paths through the crossed out point. Thus, our answer is $35 - 3 \cdot 6 = \boxed{17}$. □

Both of the strategies for pathwalking are very important, and you should have a good understanding of both. Notice the differences between the two - the first one was much slower, but also much more reliable. If we added 2 crossed out points, or even more, then the second strategy would be very difficult to use. However, when there are few to little restrictions, the second strategy is much quicker.

Exercise 3.8 (2010 AMC 12A). A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ at each step?

Exercise 3.9 (2020 COMC). An ant walks from the bottom left corner of a 10×10 square grid to the diagonally-opposite corner, always walking along grid lines and taking as short a route as possible. Let $N(k)$ be the number of different paths that ant could follow if it makes exactly k turns. Find $N(6) - N(5)$.

4 Stars and Bars

We begin this section with a motivating example.

Example 4.1
John has 10 \$1 bills that he wants to distribute between his 3 children. How many ways are there to do this?

Solution. We could try doing casework on this problem, but with 10 bills and 3 people, this would be extremely tedious and error-prone, and it would only worsen as the numbers got larger. So instead, we interpret this another way. We can draw the 10 bills (denoted by $*$s) as follows:

$$* * * * * * * * * *$$

Then note that dividing these among 3 people is the same as placing 2 dividers (represented by $|$) between the bills. It's because we can give the bills before the first divider to the first child, give the bills after the first divider but before the second one to the second child, and give the bills after the second divider to the third child. For example, the following configuration corresponds to giving \$3 to the first child, \$2 to the second, and \$5 to the third.

$$***|**|*****$$

Thus we have a total of 12 "things" (namely, 10 $*$s and 2 $|$s), and we just need to place the 2 $|$s in a row. But we already know how to do this! By section 2.1, the answer is simply $\binom{12}{2} = \boxed{66}$. ☐

This sort of problem is abundant in competition math, so make sure you understand the solution very well. This strategy is called stars and bars because the $*$ are the stars and the $|$ are the bars. You may also hear it called sticks and stones or balls and urns.

Exercise 4.2. Prove that in general, if we want to distribute n (identical!) objects between r people, then the number of ways to do so is simply $\binom{n+r-1}{r-1}$. Note that the objects must be identical - if all objects are distinct, simply use constructive counting on each one.

Exercise 4.3. Find the number ways of to place 20 students in 4 classes.

Exercise 4.4. Find the number of non-negative integer solutions to $a + b + c + d = 18$.

4.1 Stars and Bars with Restrictions

So now we know how to deal with typical distribution problems. But competition problems will rarely be a direct application of such a formula. So what do we do if the problem has more restrictions?

Example 4.5
Find the number of ways for John to distribute his \$10 to his 3 children again, if the eldest, Mary, insists on having at least \$4.

Solution. For this sort of restriction, we can simply give Mary \$4 immediately, then distribute the remaining \$6 as we normally would. Using the stars and bars formula gives us $\binom{6+3-1}{3-1} = \boxed{28}$. ☐

Example 4.6
Find the number of ways to distribute five \$1 bills *and* three \$5 bills between 3 kids.

Solution. Note that unlike with our previous examples, here, the "stars" (bills) are not all identical, so we cannot directly apply the formula. So instead, let just consider the simpler problem, with just the five \$1 bills. We already know the answer to that is $\binom{5+3-1}{3-1} = 21$. Similarly, the answer to just the \$5 version is $\binom{3+3-1}{3-1} = 10$. But by constructive counting, the answer to our original problem is simply the answer to \$1 version, multiplied by the \$5 version! This is because we can first distribute the \$1 bills, then distribute the \$5 bills. Thus, our final answer is just $21 \cdot 10 = \boxed{210}$. ☐

Example 4.7
How many solutions are there to $a+b+c \leq 15$, if a, b, and c are positive integers?

Solution. First, note that if the $\leq$ were an equal sign, then the problem would straightforwardly be a distribution problem. It's is because we can interpret the problem as distributing 15 bills between 3 children and letting a, b, and c be the amounts that the 3 children get. But our problem here deals with positive integers, unlike most of our distribution methods so far which deal with nonnegative integers (for example, in Example 2.4.5, a child can receive 0 bills, however, if our variables are positive here, they cannot).

So we will start by converting this to nonnegative integers, to use a method we are familiar with. We can do this by replacing a with $a' = a-1$, b with $b' = b-1$, and c with $c' = c-1$. It is clear that since a, b and c are positive, a', b', and c' will be nonnegative. Then our inequality will become $(a'+1)+(b'+1)+(c'+1) \leq 15$, or $a'+b'+c' \leq 12$. Now if the $\leq$ were an equal sign, we would know how to solve this - namely, the answer would be $\binom{12+3-1}{3-1} = 91$.

But since we have inequality, the problem still requires a bit of work. We *could* just test every value of $a'+b'+c'$ from 0 to 12, but that would require quite a bit of work. Ideally, we want to change inequality to equality. So how do we do that? Well, we can add another variable - in particular, let $d' = 12-a'-b'-c'$. Then note that since $a'+b'+c' \leq 12$, d' will be nonnegative. Then we have $a'+b'+c'+d' = 12$, where all 4 variables are nonnegative. And we know how to solve this! By stars and bars, the answer is simply $\binom{12+4-1}{4-1} = \boxed{455}$. □

Exercise 4.8 (Brilliant). How many ways are there to choose a 5-letter "word" (combination of letters) from the 26-letter English alphabet with replacement, where words that are anagrams are considered the same (for example, ABBCD is the same as BBCAD)?

5 Principle of Inclusion-Exclusion

We'll begin this section with a type of problem that's common on competitions, and shows a simple example of the so-called Principle of Inclusion-Exclusion (PIE).

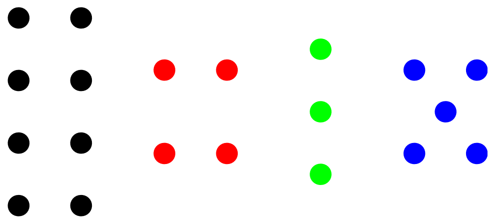
Example 5.1
In a class of 20 people, 7 of them play basketball and 8 of them play soccer. If 8 people don't play either sport, how many people play both sports?

Solution. Let's consider which combinations of sports any given person can play. They can either play both sports, play neither sport, play basketball but not soccer, or play soccer but not basketball. If we let the number of people in each of these 4 groups be a, b, c, and d respectively, then we have $a+b+c+d = 20$, because those 4 groups have no overlap by definition, so each of the 20 people must be in exactly one group.

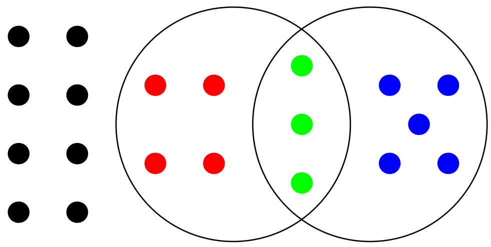
Now b is the number of people that play neither sport, so $b = 8$ by the problem statement. 7 people play basketball, but we don't know which of these 7 people play soccer and which don't; this means that we have $a+c = 7$, because a represents the group of people that play basketball *and* soccer, while c represents the people that play basketball but not soccer, so combined they form all the basketball players. Using similar logic, we get that the number of soccer players is $a + d$, which must be equal to 8. Now we have 4 equations and 4 variables, so we can solve for all the variables using your favorite method to get $a = 3$, $c = 4$, and $d = 5$ (we already knew $b = 8$). Since a is the number of people who play both sports, this is actually what we want, so our answer is $\boxed{3}$.

Now the above method works fine, but it's not very clear what exactly we did, aside from setting up a bunch of algebraic equations and solving them. In particular, this method is quite slow, requires a lot of setup, and isn't very intuitive. As an example of this, try solving this problem again, except this time with *3* sports. How many variables would we need, and how many equations would we need? Well, any specific person could either play the first sport or not play the first sport, play the second sport or not play it, and play the third sport or not play it, so we need $2 \cdot 2 \cdot 2$ groups, denoting the 2 options for each sport. This means we would need 8 variables and 8 equations - you can see how things would quickly get out of hand, with 4, 5, or even more sports.

So how would we find a quicker way to solve this? Well, let's take a look at a diagram to visualize things better. In the diagram below, we use dots to represent people - black represents the people that like neither sport, red represents those that only like basketball, blue represents those that only like soccer, and green represents those that like both.



But to draw this diagram, we needed to know how many people were in each group, which doesn't help us solve the problem. So how does this help? The key idea here is to draw use a Venn Diagram, with each circle representing a sport.



Now let use count the people, in terms of each circle. In particular, the total number of people that do a sport is represented by the two combined circles. But note that the two combined circles is not just adding the two circles separately - the overlap would get counted twice in that case, so we need to subtract out the overlap. This then gives us the formula that

# of people doing a sport = # of people playing basketball + # of people playing soccer
- # of people doing both

We can verify this with the numbers from this problem. Again let a people play both sports. Then this formula gives $20 - 8 = 7 + 8 - a$, or $a = 3$, which is the same answer we got above (the $20 - 8$ comes from the fact that 8 of the 20 people play no sport, so $20 - 8$ play at least one sport). $\square$

5.1 Set notation

We now introduce some notation, to make our solution from above more concise (that way, we don't have to write "number of people doing a sport" every time).

> **Sets**
> A set is a group of objects. Sets can contain anything, most commonly numbers, but in the context of PIE we will be using sets to represent groups of objects/things. Sets are usually denoted by capital English letters, such as S.

> **Cardinality of a set**
> The *cardinality* of a set S is just the size of the set, or more formally, the number of objects in the set. The cardinality of S is denoted $|S|$.

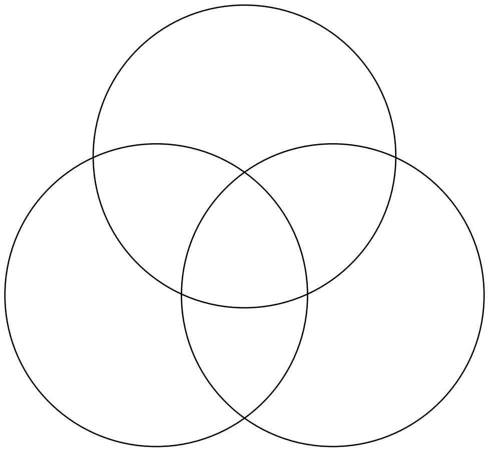
To give an example of sets and cardinality, consider our previous problem. We can let B denote the set of people who play basketball, and then the cardinality of B is 7, since there are 7 people that play basketball. More generally, you can just think of a set as a container that has a bunch of items - in this case the 7 basketball players.

> **Union and Intersection**
> The union of two sets is another set. It is the set of all things that are in either one of the sets. The union of A and B is $A \cap B$. The intersection of two sets is also another set, and it is the set of all things that are in *both* sets. The intersection of A and B is denoted $A \cap B$.

Still using our original example, we can let B denote the set of people who play basketball, and let S denote the people that play soccer. Then the set of people that play both sports is $S \cap B$, and those that play either one is $S \cup B$. Then going back to the formula we had in the last section, we get that $|S \cup B| = |S| + |B| - |S \cap B|$, because $|S \cup B|$ is the number of people playing either sport, $|S \cap B|$ is the number playing both, and $|S|$ and $|B|$ are the numbers playing soccer and basketball respectively. Plugging in the values given the problem, we once again get $20 - 8 = 8 + 7 - a$, or $a = 3$.

5.2 General PIE

So, what was the point of the last section? After all, we just introduced a lot of complicated notation, and got the exact same answer. Well, the reason we introduce this set notation is because it allows us to easily generalize the Principle of Inclusion-Exclusion, to when there are more than 2 sports (or really any types of groups). For example, let's consider the case with 3 sets A, B and C, again using a Venn Diagram.



To find the combined size of these 3 circles (in other words, $|A \cup B \cup C|$, let's start by just adding the areas of all 3 circles. Then this is just $|A| + |B| + |C|$, but just like with the 2 set case, this overcounts a lot. In particular, the sections that are the overlap of 2 circles get counted 2 times, and we only want to count them once. So we will subtract each of these sections, which are represented by $|A \cap B|, |B \cap C|$, and $|C \cap A|$. Then this gives us the expression $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$. However, this *still* isn't the correct expression, because if we look at the intersection of all 3 circles (i.e. the central section), we have counted that 3 times, but then we also subtracted it out 3 times (in each of $|A \cap B|, |B \cap C|$, and $|C \cap A|$). Since we also want to count that section once, we need to add it back once, to give us the final, correct equation:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

And with that, we now present the generalized PIE formula.

Principle of Inclusion-Exclusion

Given sets $A_1, A_2, \ldots, A_n$, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

We won't provide a full rigorous proof for the generalized statement, but if you understood the proofs for $n = 2, 3$, the idea is very similar. Essentially the idea is just to make sure you count each section of the Venn Diagram once (note an actual Venn Diagram isn't actually possible for more than 3 sets), and exactly once, making sure you are not overcounting/undercounting any sections. However, it is important to emphasize that *you should not memorize PIE!!!*. This is true of many math formulas, but especially true of PIE - oftentimes, the problems you encounter will have some small modification that make the formula useless. Instead, you will need to reason through the logic, and make sure that you count each object once, and only once.

In general, PIE is most useful when we want to count objects that satisfy at least one of many properties. Keep in mind that unlike the previous examples, oftentimes the sizes of the individual sets and their intersections will not be given to you - you will need to figure those out yourself, and *then* apply PIE! We'll now go over some examples of such a problem.

Example 5.2
How many 5 digit numbers have the property that the product of the first two digits is even?

Solution. First, let's consider when the product of two digits is even. If either (or both!) of the digits is even, then the product is even. The only way for the product to be odd is for both digits to odd! Thus, we want to first choose the first two digits such that at least one is even, and then we can choose the other 3 digits however we like. But this is very reminiscent of PIE!

In particular, we can find how many 5 digit numbers have the first digit as even, then add the number of 5 digit numbers with an even second digit, then subtract the overlap of these two sets (which is when both digits are even). For the first digit to be even, we can have it be $2, 4, 6$, or 8 (since the first digit can't be 0), and the other 4 digits can each take on any of the 10 values, so we have $4 \cdot 10^4$ possibilities by constructive counting.

Similarly, if the second digit is even, it can be $0, 2, 4, 6$, or 8, for a total of 5 possibilities. The first digit can be any of 9 possibilities (again, it can't be 0!), and the other 3 digits each have 10 possibilities, for a total of $5 \cdot 9 \cdot 10^3$ in this case. Now we need to add these two values up, then subtract the numbers with *both* the first 2 digits even. There are again 4 possibilities for the first digit, 5 for the second, and 10 for the other 3 digits, for a total of $4 \cdot 5 \cdot 10^3$. Thus, our final answer is $4 \cdot 10^4 + 5 \cdot 9 \cdot 10^3 - 4 \cdot 5 \cdot 10^3 = \boxed{65000}$. We can do a quick sanity check and note that this seems reasonable - there are 90000 5 digit numbers, and most of the numbers should have at least one of the first two digits being even, so 65000 seems reasonable.

Note that this problem can also be done with complementary counting, and is perhaps even easier that way. We won't go over that here, but feel free to try it on your own! □

Example 5.3 (AoPS Intermediate Counting & Probability)
A school offers 3 foreign language classes: Arabic, Japanese, and Russian. There are 50 students in at least one class. 18 students take both Arabic and Japanese, 15 take both Arabic and Russian, 13 take both Japanese and Russian, and 7 take all 3 languages. How many students take at least 2 languages?

Solution. If we blindly just apply PIE, we might get an expression like $18 + 15 + 13 - 7$. This is *not* the right answer. Let us consider why. When we take the expression $18 + 15 + 13$, we count each student who is taking exactly 2 languages once. However, for the students that take 3 languages, we are counting them a total of 3 times - once for Arabic and Japanese (in the "18" term), once for Arabic and Russian (in the "15" term), and once for Japanese and Russian (in the "13" term). Since we only want to count these students once, we need to subtract each of these students twice. Thus, our final expression will be $18 + 15 + 13 - 2 \cdot 7 = \boxed{32}$, where the $2 \cdot 7$ comes from the fact that we subtract each student taking all 3 classes twice. □

The previous problem is a great example of why you shouldn't just blindly apply PIE, because there's a good chance the problem isn't just about memorizing a formula. Instead, it's much better to carefully reason through the problem, ensuring that you're counting each relevant object/person once, and exactly once. In fact, notice that we didn't actually use the PIE formula in either of the examples above - instead we just used logic and arithmetic! Again, we want to emphasize that *you should not be blindly memorizing PIE* - this is true of most competition formulas/theorems, but especially true for PIE.

Example 5.4 (2014 AIME II Problem 9)
Ten chairs are arranged in a circle. Find the number of subsets of this set of chairs that contain at least three adjacent chairs.

Solution. This is a rather tricky problem, and as such don't be scared to reread this section multiple times, or ask for help. As is often the case with harder examples, the PIE is not simple, so you could likely also do casework or other methods as well - if you'd like an extra challenge, try to find other solutions on your own!

So we start by just naively counting how many ways there are to have a subset of 3. On initially inspection, there are 10 ways to choose a consecutive set of 3 chairs (this is because, say, there are 10 ways to choose the location of the rightmost chair). For the other 7 chairs, we can either include them or not, for a total of $10 \cdot 2^7$ possibilities.

However, this is massively overcounting many subsets. For example, if we take the subset with all 10 chairs, then we are counting this 10 times - once for each choice of 3 chairs. So we will need to subtract out the overcount, but unfortunately this is extremely difficult, even with PIE.

We'll start by breaking the subsets up into two cases - one case where there is only one consecutive block of chairs with at least 3 chairs, and one case where there are multiple such blocks (for example, taking consecutive 3 chairs, skipping the next 2, then taking the next 3 would fall into the second case, whereas taking 7 consecutive chairs and none of the rest would fall into the first).

For the first case, let the length of the longest consecutive block be $k \geq 3$. By definition, this is the only block that has length of at least 3. Then note that we are counting each such case $k-2$ times, once for each consecutive set of 3 chairs in that block of k. We only want to count each case once, so we need to subtract this case $k-3$ times - this can be done by counting the number of subsets of the 10 chairs with a consecutive block of 4 chairs. This is because when we count a consecutive block of 4 chairs, then for any subset with a block of k chairs, we will be counting it a total of $k-3$ times - similar to how with blocks of 3 chairs, we count it a total of $k-2$ times. Similar to when we counted the number of subsets with a block of 3, the number of subsets with a block of 4 will be $10 \cdot 2^6$ (without adjusting for overcount). However, there is one more thing to note - if $k = 10$, then we are counting the subset 10 times in each case, so by subtracting out the blocks of 4 case, we are counting it $10 - 10$ times. Since we want to count it once, we need to add it back, to get a total of $10 \cdot 2^7 - 10 \cdot 2^6 + 1 = 641$ (think about why this issue only occurs for $k = 10$!). Note that this is a very subtle point, but on the AIME just this one mistake would cost you the entire problem!

Now we need to consider the case where there are multiple disjoint (i.e. separate) blocks of length at least 3. This can be done with casework. If the blocks are both of length 3, there are 25 such cases (this is just a relatively simple casework exercise, so we will leave the details for you to work out). If one block is length 3 and the other is length 4, there are 20 cases. If there is a block of 3 and a block of 5, there are 10 cases. If there are 2 blocks of 4, there are 5 cases. Anything else is impossible - we cannot have more than 2 such blocks, and any other combination of lengths will result in the blocks not being disjoint. Thus, we have a total of $25 + 20 + 10 + 5 = 60$ cases that we are overcounting, so our answer $641 - 60 = \boxed{581}$. □

6 Problems

Minimum is [19 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

[2 ♟] Problem 1 (2019 AMC8). Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

[2 ♟] Problem 2. How many 5 digit numbers start with 2 "1"s or end with 2 "1"s?

[2 ♟] Problem 3 (AoPS Intermediate Counting & Probability). How many positive integers less than $100,000$ are a square or a cube?

[3 ♟] Problem 4. How many 5-digit integers have their digits in non-decreasing order (in other words, a digit can't be smaller than a previous one, although it can be equal)?

[3 ♟] Problem 5 (2017 AMC 10B). Call a positive integer **monotonous** if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

[3 ♟] Problem 6. How many 5 digit numbers contain only the digits 0, 1, and 2, and have at most one "1"?

[4 ♟] Problem 7 (2009 AMC 12B). Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

[4 ♟] Problem 8. Find the number of (not necessarily positive) integer solutions to $x + y + z = 10$, if x, y, and z are less than 20.

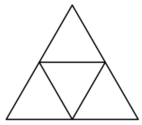
[4 ♟] Problem 9 (1983 AIME). The numbers 1447, 1005 and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

[4 ♟] Problem 10 (2001 AIME I). A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

[5 ♟] Problem 11 (2000 AIME II). Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

[5 ♟] Problem 12 (2006 AIME II). There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color.

Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles may be formed?



[6 ♟] Problem 13 (AoPS Intermediate Counting & Probability). How many 6-digit numbers have at least a 1, a 2, and a 3 in their digits?

[6 ♟] Problem 14 (2013 AIME II). A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.