3 Splitting the Summand

"When solving a summation problem, telescoping is either the most fun you'll have or the least fun you'll have."

- Mahith Gottipati

3.1 An Introduction

One good strategy when you have a summation with a complicated term is to split the term into the sum of several simpler ones. This is mainly used in a technique known as *Telescoping Sums*. This is a type of manipulation, but since the technique is especially important, we deemed that it needed its own week.

3.2 How it works

We observe that

Fact 41.

$$\sum_{i=k}^{n}(f(i)+g(i)) = \sum_{i=k}^{n} f(i) + \sum_{i=k}^{n} g(i)$$

for integers $n \geq k$ and functions $f(i)$ and $g(i)$ on the integers between k and n.

This is a consequence of the associative property of addition.

Example 42
Determine

$$\sum_{i=1}^{n} i(i+1).$$

Solution. We have

$$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n}(i^2+i) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}.$$

If we combine the two fractions we end up with $\boxed{\dfrac{n(n+1)(n+2)}{3}}$. $\square$

3.2.1 Exercises

Exercise 43 (AHSME). Find the sum of the first n terms of the sequence

$$1,\ (1+2),\ (1+2+2^2),\ \ldots\ (1+2+2^2+\cdots+2^{n-1})$$

in terms of n.

Exercise 44 (2019 AIME I). Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{321 \text{ digits}}.$$

Find the sum of the digits of N.

3.3 Telescoping Sums

Often you want to transform the main summand into the difference of two terms, so that cancellations will occur. This series of cancellations is called telescoping; in many examples of splitting the summand telescoping occurs. One straightforward example of such cancellation is as follows:

Example 45
Determine

$$\sum_{i=0}^{5} x^{i+1} - x^{i}$$

Solution. This simplifies to

$$(x^6 - x^5) + (x^5 - x^4) + (x^4 - x^3) + (x^3 - x^2) + (x^2 - x^1) + (x^1 - x^0) = \boxed{x^6 - 1}.$$

□

(It can also be factored as $(x-1)\sum_{i=0}^{5} x^i$, which is the formula for difference of nth powers, but that is a different topic.)

The technique that you have just seen in the previous example is known as telescoping sums. This technique involves splitting each individual summand into multiple pieces. Most of the pieces will cancel out as we add these summands together to get our desired sum. I like to think of it like a big sum "collapsing" into just a few terms that can be dealt with easily. In more rigorous terms, we can say the following.

Fact 46.

$$\sum_{i=k}^{n} (f(i) - f(i+1)) = f(k) - f(n+1)$$

for a function f defined on the integers between k and $n+1$ and integers $n > k$.

Telescoping sums are found everywhere, and I mean *literally* everywhere. They're like the Wilhelm Scream of summation problems, and a lot of them follow a formulaic, predictable format (although this is not always the case). Whenever you get stuck on a summation, consider giving telescoping sums a try.

In example 25, the summand was already split for you (into x^{i+1} and x^i) and you only had to figure out how the terms cancelled. Usually, it's not that simple, and you'll usually have to figure out what you need to split the summand into in order for the terms to cancel, or telescope. In a sense, you're "reverse engineering" the sum, trying to figure out how the sum was contrived.

3.4 Reverse Engineering

Here is a classic example:

Example 47
Determine
$$\sum_{i=1}^{2021} \frac{1}{i(i+1)}.$$

Solution. Notice that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Our sum is
$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2021} - \frac{1}{2022}\right),$$
Notice how all the terms are going to cancel out except for $\frac{1}{1}$ and $-\frac{1}{2022}$. So our sum is just $1 - \frac{1}{2022} = \boxed{\frac{2021}{2022}}$. □

We can also use telescoping to evaluate infinite sums. This can be done by taking larger and larger partial sums, and showing that they come arbitrarily close to some constant. The infinite sum will evaluate to that constant. Let's extend the previous example and make it an infinite sum.

Example 48
Determine
$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}.$$

Solution. First, we derive a formula for
$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$
Like before, we have that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Our sum is
$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right),$$
Again, notice how all the terms are going to cancel out except for $\frac{1}{1}$ and $-\frac{1}{n+1}$. So our sum is actually just $1 - \frac{1}{n+1} = \frac{n}{n+1}$. As n becomes arbitrarily large, $\frac{n}{n+1}$ will come arbitrarily close to 1, so the infinite sum we have here converges to $\boxed{1}$. □

Now let's go onto a slightly harder example, which involves breaking up a large sum into smaller sums (technique from last week's class).

Example 49
Compute
$$\sum_{i=1}^{63} \frac{6-\lfloor \log_2 i \rfloor}{i(i+1)}.$$

Solution. First of all, the $i(i+1)$ in the denominator looks a bit reminiscent of our previous example. Using the common substitution

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

we get the slightly-less-scary-looking

$$\sum_{i=1}^{63} \frac{6-\lfloor \log_2 i \rfloor}{i} - \frac{6-\lfloor \log_2 i \rfloor}{i+1}$$

Now what should we do about the logs? Some cancellation would be nice. In order to get a bunch of consecutive terms to cancel out, we can split the terms into smaller sums based on their values of $6-\lfloor \log_2 i \rfloor$:

$$\sum_{i=1}^{1}\left(\frac{6}{i}-\frac{6}{i+1}\right)+\sum_{i=2}^{3}\left(\frac{5}{i}-\frac{5}{i+1}\right)+\sum_{i=4}^{7}\left(\frac{4}{i}-\frac{4}{i+1}\right)$$
$$+\sum_{i=8}^{15}\left(\frac{3}{i}-\frac{3}{i+1}\right)+\sum_{i=16}^{i=31}\left(\frac{2}{i}-\frac{2}{i+1}\right)+\sum_{i=32}^{63}\left(\frac{1}{i}-\frac{1}{i+1}\right)$$

And now, we can use the power of Telescoping Sums to simplify all of these smaller summations.

$$= \left(\frac{6}{1}-\frac{6}{2}\right)+\left(\frac{5}{2}-\frac{5}{4}\right)+\left(\frac{4}{4}-\frac{4}{8}\right)+\left(\frac{3}{8}-\frac{3}{16}\right)+\left(\frac{2}{16}-\frac{2}{32}\right)+\left(\frac{1}{32}-\frac{1}{64}\right)$$
$$= \frac{6}{1}+\left(-\frac{6}{2}+\frac{5}{2}\right)+\left(-\frac{5}{4}+\frac{4}{4}\right)+\left(-\frac{4}{8}+\frac{3}{8}\right)+\left(-\frac{3}{16}+\frac{2}{16}\right)+\left(-\frac{2}{32}+\frac{1}{32}\right)-\frac{1}{64}$$
$$= 6-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}-\frac{1}{64} = 6-\frac{63}{64} = \boxed{\frac{321}{64}}.$$

□

While this solution still required some computation, notice how by using techniques such as grouping terms and telescoping sums, we were able to reduce the computation greatly. Notice how we also used the geometric sequence sum formula in the last line to hastily find the sum of $-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}-\frac{1}{64}$.

3.5 A Harder Example, and Partial Fraction Decomposition

Here's a difficult looking problem that will introduce us to our big idea of using Partial Fraction Decomposition as a way to engineer a way a given sum can telescope.

Example 50 (Dennis Chen)
Determine

$$\sum_{a=1}^{\infty} \frac{32a}{16a^4 + 24a^2 + 25}.$$

Solution. This example is reminiscent of example 47 in that we have a polynomial in terms of the summation index (this time it's a, not i) in the denominator, and we have to find a way to split it in such a way that the sum telescopes. The only difference is this one is much more scary and it's not factored.

So you can see it's probably best to practice to factor the denominator. Since there is no a^3 or a term, it's probably going to be factored in the form $(xa^2 + ya + z)(xa^2 - ya + z)$, which is a factorization that appears decently frequently on math competitions. In this form, it's clear that $x = 4$ and $z = 5$, and expanding it, we get $2xz - y^2 = 24 \implies y = 4$, so $16a^4 + 24a + 25 = (4a^2 + 4a + 5)(4a^2 - 4a + 5)$.

By this point, we might guess that the telescoping sum would probably look something like

$$\frac{1}{4a^2 - 4a + 5} - \frac{1}{4a^2 + 4a + 5}.$$

Computing the first few terms, we get $\frac{1}{5} - \frac{1}{13} + \frac{1}{13} - \cdots$, so it definitely *looks* like a telescoping sum. However, to confirm this, we could find that this nicely works out because

$$4(a+1)^2 - 4(a+1) + 5 = 4a^2 + 8a + 4 - 4a - 4 + 5 = 4a^2 + 4a + 5,$$

which is exactly the previous term. Due to the two terms in the denominator being basically the same, we could easily guess this. Either way, computing $\frac{1}{4a^2-4a+5} - \frac{1}{4a^2+4a+5}$, we get it's simply $\frac{8a}{16a^4+24a+25}$, or a quarter of the value of the summation we want to compute. The value of this quarter-sum is simply $\frac{1}{5}$, meaning the value of the original sum is $4 \cdot \frac{1}{5} = \boxed{\frac{4}{5}}$.

□

Remark 51. This problem demonstrates the power of factoring and telescoping. An intimidating sum fell to simply factoring and noticing that $4(a+1)^2 - 4(a+1) + 5 = 4a^2 + 8a + 4 - 4a - 4 + 5 = 4a^2 + 4a + 5$. By all means, figuring this out is not a simple task, but computing and noticing that the terms telescope is really all you need to figure it out.

Guessing that the telescoping sum would look something like $\frac{1}{4a^2-4a+5} - \frac{1}{4a^2+4a+5}$ in the above problem doesn't always work for telescoping sum problems. More often then not, you will have to "solve" for the numerators in a process that is usually known as ***Partial Fraction Decomposition***. We'll explore this idea more later on.

In algebra, the ***partial fraction decomposition*** of a rational fraction is the process by which some fractional term is "split open" into two or more simpler fractions. Usually the stuff in the numerator and denominator are polynomials of some kind, but you can also see partial fraction decomposition used to break apart fractions involving factorials and other strange functions. These fractions do not have to be positive - in fact, you need both positive and negative terms for things to cancel out (which is your ultimate goal in many telescoping sum problems). So a lot of times you'll be adding fractions with negative numerators and denominators.

You've probably learned before how multiple fractions can be added to yield one fraction. In partial fraction decomposition, you're essentially starting with that one fraction, then trying to figure out how it can be best expressed as the sum of fractions in order to optimize cancellation in whatever problem you're doing. Here's an example of how one would use partial fraction decomposition to solve the problem presented above.

Solution. We want to simplify the summand by splitting it into two simpler fractions. We see $16a^4 + 24a^2 + 25 = (4a^2 - 4a + 5)(4a^2 + 4a + 5)$. Let

$$\frac{M}{4a^2 - 4a + 5} + \frac{N}{4a^2 + 4a + 5} = \frac{32a}{(4a^2 - 4a + 5)(4a^2 + 4a + 5)}$$

for terms M and N. We want to find M, N such that the left hand side evaluates to the right hand side under fraction addition.

We want $M(4a^2 + 4a + 5) + N(4a^2 - 4a + 5) = 32a$ to hold for infinitely many a, so clearly it must hold for all a and thus both sides are the same polynomial in terms of a.

So the left hand side is equivalent to the polynomial $32a$, which lets us solve for the values of M and N easily. It's not that hard to see that $M = -N$ in order for the a^2 and constant terms to cancel out. Plus $4M - 4N = 32$ because the a terms must also come out equal on both ends. It's not hard to see that $M = 4$ and $N = -4$.

So, our Partial Fraction Decomposition turns out to be

$$\frac{4}{4a^2 - 4a + 5} - \frac{4}{4a^2 + 4a + 5}.$$

Since $4a^2 + 4a + 5 = 4(a+1)^2 - 4(a+1) + 5$, the infinite sum telescopes. The only remaining term is the first term which is $\boxed{\frac{4}{5}}$. □

Partial fraction decomposition was blatantly made the main point of this late AMC 10 problem that also shows that it's really not just a novelty trick for summation problems. Just like the previous problem, this problem looks very intimidating which is probably one of the reasons why it was placed so late. Partial fraction decomposition is often used in problems with scary polynomial fractions like these.

Example 52 (2019 AMC 10A)
Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

Solution. Note that clearing the denominators yields

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q).$$

Just like before, this equation must hold for s so the right side is the same polynomial as the left side, which is just the constant 1.

The key insight is that now, the equation still must hold if $s \in \{p, q, r\}$. (Misleading problem statement!) Letting $s = p$ yields $1 = A(p-q)(p-r)$, so $\frac{1}{A} = (p-q)(p-r)$. Similarly, $\frac{1}{B} = (q-p)(q-r)$ and $\frac{1}{C} = (r-p)(r-q)$. Hence

$$\begin{aligned}\frac{1}{A} + \frac{1}{B} + \frac{1}{C} &= (p-q)(p-r) + (q-p)(q-r) + (r-p)(r-q) \\ &= p^2 + q^2 + r^2 - (pq + qr + rp) = (p+q+r)^2 - 3(pq + qr + rp),\end{aligned}$$

which can be easily be evaluated by Vieta's formulae to be $\boxed{244}$. □

Remark 53. In most classes, partial fraction decomposition refers specifically and only to the breakdown of fractions with polynomial numerators and denominators. In this course we'll be dealing with a lot of fractions without polynomial numerators and denominators that still get split up using similar tactics/ideas in order to simplify our sums. For the sake of simplicity, I've just decided to lump all of these ideas into one vague umbrella of "partial fraction decomposition."

Exercise 54. Find

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}.$$

3.6 Not just for evaluating sums

Example 55 (Brilliant)
Let F_n denote the n-th Fibonacci number, satisfying $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$. Prove that $\sum_{k=1}^{n} \frac{1}{F_{k-1}F_{k+1}} < 1$ for all $n \geq 1$.

In telescoping problems like these, we can show that a sum is bounded above by some constant c if it can always be represented as $c - f(n)$ where $f(n)$ is some function of the number of terms in the sum, n. In this problem, we'll see how this can be done with a telescoping sum approach, keeping in mind that we want all terms to cancel except for a term equal to 1 minus a term that is a function of n.

Let $F_{k-1} = x$ and $F_k = y$. Then, we have $\frac{1}{F_{k-1}F_{k+1}} = \frac{1}{x(x+y)}$, as the Fibonacci numbers satisfy $F_k + F_{k-1} = F_{k+1}$. This factoring of the denominator once again seems reminiscent of previous problems. We're going to try to split this term into two fractions, one with a x going into the denominator, and one with a $(x+y)$ going into the denominator. We make the critical observation that

$$\frac{1}{x(x+y)} = \frac{1/y}{x} - \frac{1/y}{x+y} = \frac{1}{xy} - \frac{1}{y(x+y)}.$$

Plugging back $x = F_{k-1}$ and $y = F_k$, we have the seemingly miraculous identity

$$\frac{1}{F_{k-1}F_{k+1}} = \frac{1}{F_{k-1}F_k} - \frac{1}{F_kF_{k+1}}.$$

So for an arbitrary choice of n, the original sum is going to evaluate to

$$\frac{1}{F_0F_1} - \frac{1}{F_1F_2} + \frac{1}{F_1F_2} - \frac{1}{F_2F_3} + \cdots + \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}} = \frac{1}{F_0F_1} - \frac{1}{F_nF_{n+1}} = 1 - \frac{1}{F_nF_{n+1}}.$$

However, for all $n \geq 1$ we have $F_nF_{n+1} \geq 1$, thus $\frac{1}{F_nF_{n+1}} > 0$ so the given sum is always less than 1.

In the following solution, we'll see a slick trick to easily find the telescoping form without having to experiment!

Solution. Note, for all $k \geq 1$,

$$\begin{aligned}\frac{1}{F_{k-1}F_{k+1}} &= \frac{F_k}{F_{k-1}F_kF_{k+1}} \\ &= \frac{F_{k+1} - F_{k-1}}{F_{k-1}F_kF_{k+1}}.\end{aligned}$$

Factoring out $\frac{1}{F_k}$, we see

$$\begin{aligned}\frac{F_{k+1}-F_{k-1}}{F_{k-1}F_kF_{k+1}} &= \frac{1}{F_k}\cdot\frac{F_{k+1}-F_{k-1}}{F_{k+1}F_{k-1}}\\ &= \frac{1}{F_k}\cdot\left(\frac{1}{F_{k-1}}-\frac{1}{F_{k+1}}\right)\\ &= \frac{1}{F_{k-1}F_k}-\frac{1}{F_kF_{k+1}}.\end{aligned}$$

Plugging this into the original, we see it telescopes and the only remaining terms are $\frac{1}{F_0F_1}$ and $-\frac{1}{F_nF_{n+1}}$, thus the value of the sum is

$$\frac{1}{F_0F_1}-\frac{1}{F_nF_{n+1}} = 1-\frac{1}{F_nF_{n+1}} < 1,$$

and we are done. □

Onto another telescoping problem.

Example 56 (Brilliant)
Find the sum of all k such that

$$\sum_{n=1}^{k}\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}} \le \frac{11}{12}.$$

Solution. In this problem, it seems that this sum will start off less than $\frac{11}{12}$, then grow larger than $\frac{11}{12}$ as k grows. Let's use some wishful thinking. Perhaps we can telescope the sum so that for an arbitrary k, the sum can be written as $c - f(k)$, where $f(k)$ is a function of k. Then we would just need to be looking for k such that $f(k) \ge \frac{1}{12}$.

Before we go further, let's rationalize the denominator. Multiplying the fraction by $\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)\sqrt{n}-n\sqrt{n+1}}$, we get

$$\begin{aligned}\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}\cdot\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)\sqrt{n}-n\sqrt{n+1}} &= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)^2-n^2(n+1)}\\ &= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)\,(n+1-n)}\\ &= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)}.\end{aligned}$$

Now, we rewrite $\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)}$ as $\frac{(n+1)\sqrt{n}}{n(n+1)} - \frac{n\sqrt{n+1}}{n(n+1)}$, so we get

$$\begin{aligned}\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)} &= \frac{(n+1)\sqrt{n}}{n(n+1)} - \frac{n\sqrt{n+1}}{n(n+1)}\\ &= \frac{\sqrt{n}}{n} - \frac{\sqrt{n+1}}{n+1}\\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.\end{aligned}$$

This sum now telescopes! We have

$$\sum_{n=1}^{k} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}-1}{\sqrt{k+1}},$$

and we must calculate when this is less than or equal to $\frac{11}{12}$. Cross multiplying, we get

$$12\sqrt{k+1} - 12 \leq 11\sqrt{k+1} \implies \sqrt{k+1} \leq 12 \implies k \leq 143,$$

so the values of k such that the inequality is satisfied is all $k \leq 143$. Using our knowledge of the sum of the integers from 1 to n, we get the sum of all values of k is simply $\frac{143 \cdot 144}{2} = \boxed{10296}$.

$\square$

Exercise 57. Prove that $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \frac{1}{\sqrt{9}+\sqrt{11}} + \cdots + \frac{1}{\sqrt{9997}+\sqrt{9999}} \geq 24.$

3.7 Splitting into more than two parts

Sometimes we need to split a summand into more than 2 parts in order for it to telescope. Not all sums telescope using the tried-and-true "positive and negative cancel out" trick. Sometimes we're going to need a little bit more firepower. These problems are hard, and it's hard to see them. The best way to get better at these summations is by building intuition through practice.

Example 58 (OMMC 2021)
The sum

$$\frac{1^2-2}{1!}+\frac{2^2-2}{2!}+\frac{3^2-2}{3!}+\cdots+\frac{2021^2-2}{2021!}$$

can be expressed as a rational number N. Find the last 3 digits of $2021!\cdot N$.

You can try to do this problem with the techniques given previously in the section, but to my knowledge they will not work. The summand we're working with is

$$\frac{n^2-2}{n!}$$

for some positive integer n. How can we possibly get a degree 2 term up there in the numerator? We want to split this summand into multiple fractions, and it's pretty clear that they're going to have factorials in their denominators.

If you were a contestant at OMMC 2021 this year, this is where your struggles with the problem might have ended. But perhaps you used a bit of wishful thinking. How can get n^2-2 in the numerator? What if we could split it... into more than 2 fractions...

With a stroke of genius or just trial and error, you may have come up with the following "magical" telescoping solution:

Solution. See that for any positive integer n, we can express $\frac{n^2-2}{n!}$ as $\frac{1}{(n-2)!}+\frac{1}{(n-1)!}-\frac{2}{n!}$ when $n\geq 2$. From there, most of the terms in the sequence telescope and we are left with

$$\frac{1^2-2}{1!}+\frac{1}{0!}+\frac{1}{1!}+\frac{1}{1!}-\frac{1}{2020!}-\frac{1}{2021!}-\frac{1}{2021!}.$$

Multiplying by 2021! yields some large multiple of 1000 minus 2021 and 2 so our answer is $1000-21-2=\boxed{977}$. □

One may be interested in how one could find the magical identity $\frac{n^2-2}{n!}=\frac{1}{(n-2)!}+\frac{1}{(n-1)!}-\frac{2}{n!}$. You really can't solve for the numbers in the numerators like you can with a typical partial fraction decomposition, since you don't know what denominators are involved. (This is in contrast with when a polynomial is on the bottom, in which case we can just factor that polynommial).

For starters, we definitely know that we need fractions in the form of $\frac{a}{(n-1)!}$, $\frac{b}{n!}$ and maybe even some with $(n-2)!$ in the denominator. We also know we want some fractional terms. Through some experimentation, we can find that $\frac{1}{(n-1)!} - \frac{2}{n!} = \frac{n-2}{n!}$. Now, all we need is to turn this $n-2$ into a n^2-2, but how would we do that?

This is where the fraction $\frac{c}{(n-2)!}$ comes into play. We have

$$\frac{c}{(n-2)!} + \frac{n-2}{n!} = \frac{n^2-2}{n!},$$

so

$$\frac{cn! + (n-2)(n-2)!}{(n-2)!n!} = \frac{(n^2-2)(n-2)!}{(n-2)!n!}.$$

Multiplying by $(n-2)!n!$ on both sides and bringing the $(n-2)(n-2)!$ to the RHS, we have $cn! = (n^2-2-n+2)(n-2)! = (n(n-1))(n-2)!$. However, note that $n(n-1)(n-2)! = n!$, hence $c=1$ and our partial fraction decomposition is simply $\frac{1}{(n-2)!} + \frac{1}{(n-1)!} - \frac{2}{n!}$, and we can finish the problem as above.

Remark 59. The partial fraction decomposition above is quite magical which is why despite its short solution, it was considered a difficult question. To see such partial fraction decompositions, you must have a ton of intuition and can't be afraid to mess around a bit. If you didn't mess around in this problem, then there is a very slim chance you would've seen the partial fraction decomposition, making it very hard to solve the problem.

Telescoping sums involving factorials are often found on computational competitions. This is because factorials of numbers that are close to one another often share lots of terms with each other. So if you see a sum involving factorials, consider splitting the summand into separate parts involving factorials of various integers.

Exercise 60. Find

$$\sum_{k=2}^{\infty} \frac{1}{k(k^2-1)}.$$

3.8 A Hard Problem

Yeah, this problem is so much harder than any of the previous problems, it gets its very own section! Really, this problem is very hard until you figure out the 'trick' to solving it, and from there it's pretty much straightforward. Without further ado, here's the problem.

Example 61 (AoPS Intermediate Algebra)
For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty}\frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n}.$$

Now let's take a moment to process this really quick. So, we got some wacky notation $\langle n\rangle$ which is the closest integer to $\sqrt{n}$, and we have to do 2 to the power of that then 2 to the power of... I think you can guess why this problem gets its own section now. So how should we start? Here's an idea: let's evaluate $\langle n\rangle$ for the first few values of n. Here they are in a table:

$\langle n\rangle$	n
$\langle 1\rangle$	1
$\langle 2\rangle$	1
$\langle 3\rangle$	2
$\langle 4\rangle$	2
$\langle 5\rangle$	2
$\langle 6\rangle$	2
$\langle 7\rangle$	3
$\langle 8\rangle$	3
$\langle 9\rangle$	3
$\langle 10\rangle$	3
$\langle 11\rangle$	3
$\langle 12\rangle$	3

Though it looks like $\langle n\rangle$ follows a pattern, this pattern would prove very hard to evaluate, as we don't have $\langle n\rangle$ in terms of n, or anything even related to n, really. Now comes the crux move of this problem; bounding!

Let $\langle n\rangle = m$. Note, we have $m-\frac{1}{2} < \sqrt{n} < m+\frac{1}{2}$, since either we rounded down in which case we have $m+\frac{1}{2} > \sqrt{n}$ as $\{\sqrt{n}\} < \frac{1}{2}$. In the case that we round up, then we have $\sqrt{n} > m-\frac{1}{2}$, for similar reasoning as the rounding down case, Thus, we've established the bound on $\sqrt{n}$, in terms of m.

Squaring this bound, we get $m^2 - m + \frac{1}{4} < n < m^2 + m + \frac{1}{4} \implies m^2 - m + 1 \le n \le m^2 + m$, as n and m are both integers. Thus, the possible values of n are $m^2 - m + 1, m^2 - m, \ldots, m^2 + m$, as m ranges from 1 to infinity. Hence, we can rewrite our original sum as

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} = \sum_{m=1}^{\infty} \sum_{n=m^2-m+1}^{m^2+m} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

However, remember we define $\langle n \rangle = m$, so we can $\langle n \rangle = m$ and $-\langle n \rangle = -m$ and factor that out. Then, after factoring that out all that is left is the sum of $\frac{1}{2^n}$ for $m^2 - m + 1 \le n \le m^2 + m$. Note that

$$\begin{aligned}\sum_{n=m^2-m+1}^{m^2+m} \frac{1}{2^n} &= \frac{1}{2^{m^2-m+1}} + \frac{1}{2^{m^2-m}} + \cdots + \frac{1}{2^{m^2+m}} \\ &= \frac{2^{2m-1} + 2^{2m-2} + \cdots + 1}{2^{m^2+m}} \\ &= \frac{2^{2m} - 1}{2^{m^2+m}}.\end{aligned}$$

Then, multiplying this by $2^m + 2^{-m}$ (which was the part of the sum that we factored out), we get the original sum is simply equal to

$$\sum_{m=1}^{\infty} (2^m + 2^{-m}) \cdot \frac{2^{2m} - 1}{2^{m^2+m}} = \sum_{m=1}^{\infty} (2^{-m^2+2m} - 2^{-m^2-2m}).$$

This sum telescopes, hopefully. There are a ton of similarities between $-m^2 + 2m$ and $-m^2 - 2m$. If we list out the first few terms in our sum, then we get

$$2^1 - 2^{-3} + 2^0 - 2^{-8} + 2^{-3} - 2^{-15} + 2^{-8} - 2^{-24} + \cdots.$$

Hey, the 2^{-3} and -2^{-3} cancel, and the same case for the 2^{-8} and -2^{-8}! This happens because $-(m+2)^2 + 2(m+2) = -m^2 - 4m - 4 + 2m + 4 = -m^2 + 2m$, so the 2^{-m^2+2m} and $-2^{-(m+2)^2-2(m+2)}$ cancel. Now we are sure that this sum telescopes! The only terms that remain once we telescope are just 2^1 and 2^0, so our answer is $\boxed{3}$.

Just like that, through the power of bounding and cleverly manipulating our sum to make it telescope, we've solved this problem. This particular $\langle n \rangle$ doesn't show up very often, but this idea of bounding and creating a double sum which we then manipulate is common, so make sure to remember this idea! Though this solution wasn't insanely long, you probably see why this problem got it's own section! Now, onto the summary.

3.9 Summary

Splitting the summand allows for the simplification of complicated looking sums. The process involves splitting each of the summands in the sum into parts. Oftentimes these parts "telescope" or cancel out with each other and make the sum easier to compute. Telescoping sums become arguably the most ubiquitous summation technique at high level competitions such as HMMT. With practice, it can become easy to identify sums might be made to telescope.

Telescoping sums are usually what I call a "you get it or you don't question." That means that you either get the solution, or you don't. There is no in between. The problem is just a trivial computation once you know how the sum telescopes, but without noticing the telescope, doing the problem is impossible as there is often no alternative solution.

And indeed, actually finding the way the sum telescopes is a bit of an art. Smoothly contrived telescoping sums can prove difficult even to the most experienced competitors. They have tons of dead ends, and the solution is a bit like finding a needle in a haystack of possible choices. Practice, intuition, and wishful thinking are your best friends when tackling these problems.

3.10 Important Terms

1. Telescoping Sums
2. Partial Fraction Decomposition

3.11 Practice Problems

Problem 15. Simplify

$$\sum_{i=1}^{100}(2^i - 1) = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{100} - 1).$$

Problem 16. Prove that the sum of the first n squares is

$$\frac{n(n+1)(2n+1)}{6}$$

via a Telescoping Sums strategy.

Problem 17. Determine

$$\sum_{n=0}^{\infty} \frac{n^2 - 2^n}{3^{n+1}}.$$

Problem 18 (Holiday Problems). Determine the infinite sum

$$\sum_{n=4}^{\infty} \sqrt{\left(\log_{n-1} n\right)^2 + \left(\log_{n-2}(n-1)\right)^2 - \log_{n-2}(n^2)}.$$

Problem 19 (IMC 2015). Evaluate

$$\sum_{n=0}^{\infty} \frac{2^{2^n}}{4^{2^n} - 1} = \frac{2^1}{4^1 - 1} + \frac{2^2}{4^2 - 1} + \frac{2^4}{4^4 - 1} + \frac{2^8}{4^8 - 1} + \ldots$$

Problem 20. Simplify

$$S = \sum_{k=30}^{89} \frac{1}{\sin k^\circ \sin(k+1)^\circ}.$$

Problem 21 (Intermediate Algebra). Determine the exact value of the series

$$\frac{1}{5+1} + \frac{2}{5^2+1} + \frac{4}{5^4+1} + \frac{8}{5^8+1} + \frac{16}{5^{16}+1} + \cdots.$$