Week 2: Problem Set

Summations Mini-course

ALP

July 9, 2021

This set of problems is meant to be done independently with no outside help besides a four-function calculator if necessary. Tools like Wolfram Alpha and Desmos are not allowed. You may not ask for help from anyone besides an ALP instructor. Problems are meant to be a challenge, and we do not intend for everyone to complete all the problems. However, we encourage students to give each problem at least a solid attempt.

1 Problems

Minimum is [0 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "Equality when equal"
>
> *Katelyn Zhou*

[1 ♟] Problem 1 (AHSME). Let $F = .48181\cdots$ be an infinite repeating decimal with the digits 8 and 1 repeating. When F is written as a fraction in lowest terms, what is the absolute difference between the numerator and denominator?

[2 ♟] Problem 2. Find the units digit of $\sum_{k=1}^{2021} k^k$.

[4 ♟] Problem 3 (AHSME). The sum of an infinite geometric series with common ratio r such that $|r| < 1$, is 15, and the sum of the squares of the terms of this series is 45. Find the first term of the series.

[5 ♟] Problem 4 (Aops Mock AMC 10). Let G denote the sum of an infinite geometric series and the first term of the series is not 0. Suppose the sum of the squares of the terms is $2G$, and that of the cubes is $\frac{64G}{13}$. The sum of the first three terms of the original series can be represented as a fraction $\frac{m}{n}$ in lowest terms. Find $m+n$.

[7 ♟] Problem 5 (OMO). Let a_n denote the remainder when $(n+1)^3$ is divided by n^3; in particular, $a_1 = 0$. Compute the remainder when $a_1 + a_2 + \cdots + a_{2013}$ is divided by 1000.

[7 ♟] Problem 6 (HMMT). The value of

$$\sum_{1 \le a < b < c} \frac{1}{2^a 3^b 5^c}$$

(i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$) can be represented as $\frac{m}{n}$ in lowest terms. Find $m+n$.

[8 ♟] Problem 7 (OMO). Compute $\left\lceil \sum_{k=2018}^{\infty} \frac{2019! - 2018!}{k!} \right\rceil$. (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

[8 ♟] Problem 8 (AMC). Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)?$$

[9 ♟] Problem 9 (AIME). For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \le 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \le 2005$ with $S(n)$ even. Find $|a-b|$.

[9 ♟] Problem 10 (OMO). Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to the sum of the digits of n. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.

[16 ♟] Problem 11. Define

$$\omega_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Also, define $\Gamma_n = \sum_{k=1}^{n} \omega_n$. Then, let

$$\Omega_n = \sum_{k=1}^{n} (-1)^{k-1} \Gamma_k.$$

Lastly, define φ_{2n} to be the sum of the reciprocals of the even numbers until $2n$ and φ_{2n+1} be the be the sum of the reciprocals of the odd numbers until $2n+1$. Show for any n that

$$\Omega_{2n+1} < 2n + 2 - \varphi_{2n+1}$$
$$\Omega_{2n} < \varphi_{2n} - 2n.$$

Note: Problem 11 is a writing problem! This means that for this problem, you must write a solution. You might get partial credit if you submit an incomplete solution depending on how far you progressed, so make sure to submit even if you didn't fully solve the problem. You may use LaTeX or scan a hand-written solution. Submit via upload in the Google Form.