Week 3: Problem Set

Summations Mini-course

ALP

July 17, 2021

This set of problems is meant to be done independently with no outside help besides a four-function calculator if necessary. Tools like Wolfram Alpha and Desmos are not allowed. You may not ask for help from anyone besides an ALP instructor. Problems are meant to be a challenge, and we do not intend for everyone to complete all the problems. However, we encourage students to give each problem at least a solid attempt.

1 Problems

Minimum is [0 ♟]. Problems denoted with ♜ are required. (They still count towards the point total.)

> "There is a figure that links the personal and the collective shadow; it is the Trickster. Jung describes the Trickster as 'the summation of all the inferior traits of character in individuals.'"
>
> *Christopher Perry*

[1 ♟] Problem 1. Let
$$S = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{99}{100!}.$$
S can be written as $\frac{m}{n}$ for relatively prime positive integers m, n. Find the last three digits of $m + n$.

[2 ♟] Problem 2 (AIME). Consider the sequence defined by $a_k = \dfrac{1}{k^2 + k}$ for $k \geq 1$. Given that
$$a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29}$$
for positive integers m and n with $m < n$, find $m + n$.

[2 ♟] Problem 3. The value of
$$\frac{1}{3^2 + 1} + \frac{1}{4^2 + 2} + \frac{1}{5^2 + 3} + \cdots$$
can be represented as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

[4 ♟] Problem 4 (Intermediate Algebra). Let $a_1, a_2, \ldots, a_{2018}$ be the roots of the polynomial
$$x^{2018} + x^{2017} + \cdots + x^2 + x - 1345 = 0.$$
Compute
$$\sum_{n=1}^{2018} \frac{1}{1 - a_n}.$$

[5 ♟] Problem 5. If
$$\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$$
then determine the value of N.

[5 ♟] Problem 6. The value of
$$\sum_{k=1}^{\infty} \frac{k+2}{k(k+1)2^k}$$
can be represented as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

[6 ♟] Problem 7. The value of

$$\sum_{k=1}^{99} k!(k^2+k+1)$$

can be represented as $j \times k! - l$ for positive integers j, k, l where l is minimized, and k is maximized. Find $j+k+l$.

[7 ♟] Problem 8. The infinite sum

$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}$$

can be represented as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m+n$.

[8 ♟] Problem 9. Let

$$S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{99}} + \frac{1}{\sqrt{100}}.$$

Find S rounded to the nearest integer.

[10 ♟] Problem 10 (OMO). The sequence $\{a_n\}$ satisfies $a_0 = 1, a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$

What is $\frac{1}{S}$?

[14 ♟] Problem 11 (Intermediate Algebra). Prove that there is only one real solution to the equation

$$(x^{2006}+1)(x^{2004}+x^{2002}+x^{2000}+\cdots+x^2+1) = 2006x^{2005}.$$

Note: Problem 11 is a writing problem! This means that for this problem, you must write a solution. You might get partial credit if you submit an incomplete solution depending on how far you progressed, so make sure to submit even if you didn't fully solve the problem. You may use LaTeX or scan a hand-written solution. Submit via upload in the Google Form.