4 Advanced Strategies

4.1 Entering the Twilight Zone

In the past few weeks, we've taught you ideas that will allow you to solve almost all summation problems on the AMCs. But summation problems will not always be your typical telescoping or pairing problem, and part of the reason why they have remained so popular is that there's just so much to do with the topic. In this handout, we'll be covering various advanced summation techniques that are not often seen at the AMC level, but come up frequently in higher level competitions.

These concepts are hard to understand but for the sake of brevity we will only briefly touch upon each of them. They will also use material you might not have seen previously. If your curiosity is aroused by anything covered here, we recommend you ask one of our teachers for further information or do a bit of self study.

4.1.1 "Generalized" Summation Notation

As we'll see later, the sigma notation can be used like this:

Fact 62. Let P be some condition involving an arbitrary integer i that can be satisfied or not satisfied depending on the value of i. Let $f(i)$ denotes some function defined for integers i satisfying the condition. Then

$$\sum_{P} f(i)$$

equals the sum of the quantity $f(i)$ over all integers i satisfying the condition (there can be infinitely many of them).

Of course, i can be replaced with any variable. Let's look at a few examples.

$$\sum_{d|N} d$$

represents the sum of the positive divisors of a positive integer N.

$$\sum_{a\in S} \frac{1}{a}$$

represents the sum of the reciprocals of the numbers in a set of nonzero integers. You can think of our typical summation notation as a special way of writing the property "the integer i lies between some integer k and some integer n."

4.2 Using Summations Techniques in Products

Oftentimes when evaluating a *product* of terms, the same ideas of forcing cancellation and manipulation are going to carry over. Sometimes a problem will ask you to directly evaluate a product of terms. This type of problem comes up less frequently than summation problems but is still good to become familiar with.

Fact 63. In general, we denote

$$\prod_{i=k}^{n} f(i)$$

as the product

$$f(k) \times f(k+1) \times \cdots \times f(n),$$

for integers $k \leq n$ and a function $f(i)$ on the integers between k and n inclusive.

As you'll see in one of the exercises, products can also be infinite, and these products of infinite numbers of terms are said to converge if "partial products" (if that's a term??) of the infinite product approach a particular value.

Fact 64. In general, we denote

$$\prod_{i=k}^{\infty} f(i)$$

as the infinite product

$$f(k) \times f(k+1) \times f(k+2) \times \cdots,$$

for an integer k and a function $f(i)$ on the integers greater than or equal to k.

Fact 65. An infinite product

$$\prod_{i=k}^{\infty} f(i)$$

converges to a real number r if and only if the product

$$\prod_{i=k}^{n} f(i)$$

gets arbitrarily close to r when n gets arbitrarily large, for an integer k and a function $f(i)$ on the integers greater than or equal to k. If an infinite product does not converge, we say it diverges (evaluates to an undefined value).

Sometimes, a problem will make one of the terms 0, forcing the entire sum to evaluate to 0.

Example 66 (Purple Comet)
Determine $\prod_{n=3}^{33}(2n^4 - 25n^3 + 33n^2)$.

Solution. Notice that this can be expressed as

$$\prod_{n=3}^{33} n^2(2n-3)(n-11).$$

When $n = 11$, when expression is 0. Hence, the product is $\boxed{0}$. □

Such a boring problem. Let's take a look at something more interesting.

Example 67 (HMMT)
Simplify

$$\frac{2^3-1}{2^3+1} \times \frac{3^3-1}{3^3+1} \times \cdots \times \frac{n^3-1}{n^3+1}.$$

Solution. In this problem, multiple summation ideas are going to carry over. First, simplify each term in the product using the difference/sum of cubes formula:

$$\frac{(2-1)(2^2+2+1)}{(2+1)(2^2-2+1)} \times \frac{(3-1)(3^2+3+1)}{(3+1)(3^2-3+1)} \times \cdots \times \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}.$$

Let's split our product into two products that we're going to multiply together. These are

$$\frac{1}{3} \times \frac{2}{4} \times \cdots \times \frac{n-1}{n+1}$$

and

$$\frac{7}{3} \times \frac{13}{7} \times \cdots \times \frac{n^2+n+1}{n^2-n+1}$$

It's pretty clear how the terms in the first product cancel out. The first product will leave us with $\frac{2}{(n+1)n}$. For the second product, observe that $(n-1)^2 + (n-1) + 1 = n^2 - n + 1$. It's clear that our final answer is

$$\frac{2}{(n+1)n} \times \frac{n^2+n+1}{3} = \boxed{\frac{2(n^2+n+1)}{3(n+1)n}}.$$

□

Exercise 68 (CMC). Let $a_0, a_1, \ldots, a_{2019}$ be real numbers such that $a_0 = 0$ and $a_n + a_{n+1}^{-1} = 2$ for every integer $n \geq 0$. What is the value of the product $a_1 a_2 \cdots a_{2019}$?

Exercise 69 (HMMT). Compute

$$\prod_{n=0}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{3^n} + \left(\frac{1}{4}\right)^{3^n}\right).$$

4.3 Nested sums

We'll first give an example.

Example 70 (2018 AMC 12B)
What is

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)?$$

Solution. An inexperienced competitor would've been very flustered to see this so early on in the AMC 12 test, especially if they had never seen such a problem before. There's not just one weird E, but two weird E's in the problem. So how should we make sense of it?

Recount our earlier definition of sums, and you might see that the way to think about this is to just treat $\sum_{j=1}^{100}(i+j)$ as the function of i that we're going to be summing up over all i from 1 to 100 inclusive. What does this simplify to? By using the formula for the sum of an arithmetic sequence, you'll be able to see that $\sum_{j=1}^{100}(i+j) = 100i + 5050$. Summing this quantity over all i gives us the answer of $\boxed{1,010,000}$. $\square$

Notice another way to think about this problem: we're summing the quantity $(i+j)$ over all distinct ordered pairs of integers (i,j) such that i and j are between 1 to 100 inclusive. Verify on your own that we can solve this problem by considering for an arbitrary integer k from 1 to 100, how many times it appears as a variable in this ordered pair.

These sums involving multiple sums are called *nested sums* because they "fit" inside one another to create a new sum. There can be any number of them. While they might seem strange, if you think of each of nested sum as a function in terms of a variable, you'll see that it's really nothing new. However, as you'll see, we'll be able to do interesting things with nested sums that make them very powerful tools.

Exercise 71. Simplify

$$\sum_{i=1}^{100}\sum_{j=1}^{100}\sum_{k=1}^{100} ijk.$$

Exercise 72. Simplify

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i^2-j^2).$$

4.4 Switching order of summation

Take a look at the last summation we looked at, which was

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j).$$

You might have noticed that by swapping the "placement" of the two sigmas, the value of the sum wouldn't change. In other words,

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)=\sum_{j=1}^{100}\sum_{i=1}^{100}(i+j).$$

What if we made the sum a little bit less symmetrical? Does something like

$$\sum_{i=1}^{100}\sum_{j=1}^{200}(i+7j)=\sum_{j=1}^{200}\sum_{i=1}^{100}(i+7j)$$

hold? You might be able to confirm that both sums represent the sum of the quantity $(i+7j)$ over all ordered pairs of integers (i,j) where i and j are integers between 1 and 100, and 1 and 200, respectively.

This is an example of a critical property of nested summations that comes up frequently in higher level competitions. The proof is just a extension of logic we have presented already.

Fact 73 (Switching order of summation). Let P and Q be conditions involving arbitrary integers i and j. Let $f(i,j)$ be some quantity dependent on integers i,j satisfying the conditions. Then

$$\sum_{P}\sum_{Q}f(i,j)=\sum_{Q}\sum_{P}f(i,j).$$

I like to think of it not so much as a theorem or fact, but as a strategy or manipulation of its own. The notation looks scary but the concept is simple. You will see the concept of switching order of summation as a crucial step in summation problems, often used in conjunction with other insights.

Example 74 (2008 HMMT Guts)
Find

$$\sum_{k=2}^{\infty}\sum_{j=2}^{2008}\frac{1}{j^k}.$$

Solution. See that

$$\sum_{k=2}^{\infty}\sum_{j=2}^{2008}\frac{1}{j^k} \implies \sum_{j=2}^{2008}\sum_{k=2}^{\infty}\frac{1}{j^k} \implies \sum_{j=2}^{2008}\frac{\frac{1}{j^2}}{1-\frac{1}{j}} = \sum_{j=2}^{2008}\frac{1}{j(j-1)}.$$

Now see that

$$\sum_{j=2}^{2008}\frac{1}{j(j-1)} \implies \sum_{j=2}^{2008}\left(\frac{1}{j-1}-\frac{1}{j}\right) \implies 1-\frac{1}{2008} \implies \boxed{\frac{2007}{2008}}.$$

□

In this problem, we saw how switching the order of summation changed a strange looking problem into a simple telescoping exercise. Usually problems will be a bit more discreet about a switching the order of summation application. The original statement of this problem in contest went like this.

Example 75
Let $f(r) = \sum_{j=2}^{2008}\frac{1}{j^r} = \frac{1}{2^r}+\frac{1}{3^r}+\cdots+\frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

Don't be fooled: this is the same exact problem. However, until one writes the sum in nested form, the overt switching order of summation application is cleverly hidden from view.

Exercise 76 (HMMT). Compute

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{k}{2^{n+k}}.$$

4.5 Averages and Linearity of Expectation

This section introduces the idea of using *expected values* of terms to solve summation problems. What if we could determine the average value of all the terms in our sum? Then, we would just be able to multiply that average value by the number of terms in our sum to determine the final value. Finding this average value will sometimes involve tricks from combinatorics, such as Linearity of Expectation.

This idea is used as a way to deal with problems involving sums of integers of a certain type. Because of Linearity of Expectation, we can determine what is the "average value" of each digit, then add all these averages up to find the expected value of the integer.

Example 77 (2014 AMC 12A)
A five-digit palindrome is a positive integer with respective digits $abcba$, where a is non-zero. Find the sum of all five-digit palindromes.

Solution. This problem is one that can be reasoned with intuitively. If we choose a random palindrome, what is its expected value? We look at each digit individually.

Clearly the average value of each of the "a" digits is going to be 5, (each integer from 1 to 9 shows up equally often) and the average value of the "b" and "c" digits is going to be 4.5. (each integer from 0 to 9 shows up equally often). So our average value is going to be $5 \cdot 10^4 + 4.5 \cdot 10^3 + 4.5 \cdot 10^2 + 4.5 \cdot 10^1 + 5 = 55000$. Since there are $9 \cdot 10 \cdot 10 = 900$ palindromes, we have $900 \cdot 55000 = \boxed{49500000}$ as our final answer. □

As you can see, the Linearity of Expectation is a concept similar to Splitting the Summand. We've learned earlier that the value of a sum of a particular summand over a certain set of values is equal to the sum of the *sums of the components making up that summand* over that same set of values. In the same way, the average value of a particular term is equal to the sum of the average values of the components that make it up.

Example 78 (SADGIME)
Define the function $\omega(n) = \sum_{i=1}^{k} (a_i - a_i a_{i+1})$ for all positive integers n, given that n can be written as $a_1 a_2 a_3 \dots a_k$ in base 2 where $a_1 \neq 0$, and a_{k+1} is defined as a_1. For example,

$$\omega(26) = \omega(11010_2) = (1-1\cdot 1) + (1-1\cdot 0) + (0-0\cdot 1) + (1-1\cdot 0) + (0-0\cdot 1) = 2.$$

Simplify

$$\sum_{i=0}^{2^N-1} \omega(i).$$

Solution. We're summing the quantity $\omega(n)$ over all n with N or less digits in base 2. The first thing we should be aware of is that a_{k+1} will always evaluate to 1, so $a_k - a_k a_{k+1}$will always be 0. Now, if we choose a random number with N or less digits in base 2, what is the expected value of $\omega(n)$ we get?

Consider the following idea: when is $a_i - a_i a_{i+1}$ not equal to 0? You might have seen that this only holds when $a_i = 1$ and $a_{i+1} = 0$, in which case the expression evaluates to 1. So we're actually trying to compute the expected number of times the consecutive sequence of digits 10 appears in n's binary representation! Now we can just think about this quantity over all N-digit binary integers where leading zeros are allowed. Make sure you see why we can think about the numbers as having leading zeroes in this new definition of $\omega(n)$.

The key insight now is to once again, use a Linearity of Expectation strategy. There are $n-1$ locations in the binary representation (in the n total digits, ignoring leading zeroes) where this 10 can show up, and at each location, the 10 shows up with $\frac{1}{4}$ probability. So the expected value is $\frac{n-1}{4}$.

So our final answer is $\frac{n-1}{4} \cdot 2^n = \boxed{2^{n-2}(n-1)}$. □

Perhaps a more intuitive solution would probably be guessing the closed form through evaluating small cases by hand, then proving it using an induction strategy, by writing the sum of $\omega(n)$ up to 2^N in terms of previous sums. We'll leave that solution to you.

Most summation problems that can be done with a "Linearity of Expectation" approach can also be done with a Splitting the Summand approach, which is also more general and versatile. However, it's often a bit quicker to use your intuition to reason out what the average value of a term is in a sum, then go through the trouble of splitting the summand. This sometimes makes the strategy a useful one to know when you're strapped for time.

Exercise 79. Find the sum of all 3-digit integers that do not have a digit of 3 in their decimal representation (so 187 is in our sum, but 137 is not).

4.6 Combinatorial Sums

In this section we'll briefly discuss sums involving combinations which can often involve some sort of counting insight. These show up decently often on the AIME and come up occasionally at the AMC 10/12 level. ALP will be covering combinatorics topics like these in more detail in other lectures. Often combinatorial sums will involve *combination notation*, which looks like this:

Fact 80. For nonnegative integers $k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

denotes the number of ways to choose k objects out of n total objects in such a way that the order in which the objects are chosen doesn't matter.

This notation is used all over math and we cannot possibly cover everything you need to know about it in a single section of this handout. However, there are a few elementary facts that you should be familiar with:

Fact 81. For nonnegative integers $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Fact 82 (Pascal's identity). For nonnegative integers $k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We can easily prove Pascal's identity algebraically by representing both sides in terms of factorials. But a more interesting way is as follows. We propose two methods to count how to choose how many ways we may choose k objects out of n total objects without order. The first one is obvious, $\binom{n}{k}$ gives us what we want. For our second method, choose an object A out of the n total objects. We do some casework. If we must include A, there are $\binom{n-1}{k-1}$ ways to choose k out of n objects. If we must not include A, there are $\binom{n-1}{k}$ ways to choose k out of n objects. So our answer is $\binom{n-1}{k-1} + \binom{n-1}{k}$ when counting in this way.

Now the key is to realize that both methods are counting the same thing, so they produce the same result. Therefore, the identity holds.

Example 83
Prove that
$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

Solution. We count the number of ways to choose an unordered subset of any size (possibly empty) of n total elements. One method is to just realize that each of the n elements can either be or not be in a given subset, yielding two choices. So our answer is just 2^n.

The other method is to perform casework on the number of elements in the subset. The number of elements in the subset can be anywhere from 0 to n. The number of subsets with size k for all integers $0 \le k \le n$ can be written as $\binom{n}{k}$. So our answer is $\sum_{k=0}^{n}\binom{n}{k}$.

Both methods count the same thing, so they produce the same result. Therefore, the identity holds. □

We can extend this strategy to look at more complicated sums, which are sometimes called "combinatorial identities."

Example 84 (Vandermonde's Identity)
Prove that
$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

Solution. We count the number of ways to choose an unordered subset of size n out of $a+b$ total elements. One method is to just find $\binom{a+b}{n}$.
Split the $a+b$ total elements into a subset A of size a and a disjoint subset B of size b. The other method is to perform casework on the number of elements in the subset that are also in A. The number of such elements can be anywhere from 0 to n. If the number of such elements is k, then clearly there are $\binom{a}{k}\binom{b}{n-k}$ possibilities. Summing this over all possible values of k gives the result of

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

Both methods count the same thing, so they produce the same result. Therefore, the identity holds.

□

These combinatorial identities are a bit scary to look at, but are actually quite intuitive to grasp once you have a good foundation in combinatorics and counting. The key is to interpret the multiplications and additions not as algebra, but as counting techniques.

Exercise 85. Simplify the sum

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 \cdots + \binom{n}{n}^2.$$

Exercise 86 (2020 AIME I). A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

4.7 Number Theory Sums

Using number theory to evaluate a sum's remainder modulo some integer is a setup that is commonly seen throughout all math competitions. Furthermore, summations involving functions related to number theory are commonly seen in AIME-level competitions and even at the Olympiad level. Here are a few examples of such functions:

Fact 87. The *divisor-counting function*, sometimes expressed as $d(n), v(n), \tau(n), \sigma_0(n)$, denotes the number of positive divisors of an integer n.

Fact 88. The *phi function*, usually written as $\varphi(n)$, denotes the number of positive integers less than or equal to n that are also relatively prime to n.

Fact 89. The *sum of divisors function*, usually written as $S(n)$ or $\sigma_1(n)$, denotes the sum of all the positive divisors of an integer n.

These three functions are all known as *multiplicative functions*. A function $f(n)$ over the positive integers is multiplicative if $f(m)f(n) = f(mn)$ for all relatively prime pairs of positive integers m, n. For example, since 69420 and 187 are relatively prime, we're going to have that $d(69420) \times d(187) = d(69420 \times 187)$ and similarly for other functions like $\varphi(n)$ and $S(n)$.

The divisor-counting, phi, and sum of divisors functions are just a few of the many number theory functions that are found in summation problems. They are interesting in that they can be computed given a number's *prime factorization*. We have the following critical results:

Fact 90. Suppose that a number N's prime factorization is $p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$ for distinct primes $p_1, p_2, \ldots p_k$ and positive integers $e_1, e_2, \ldots e_k$. Then, we have the following:

1. $d(n) = (e_1 + 1) \times (e_2 + 1) \times \cdots \times (e_k + 1)$.
2. $\varphi(n) = N \times \prod_{i=1}^{k} (1 - \frac{1}{p_i})$.
3. $S(n) = \prod_{i=1}^{k} (1 + p_i^1 + p_i^2 + \cdots + p_i^{e_i})$.

These facts are not trivial to prove, but for the sake of brevity we'll have to exclude them in this handout. The good news is that for almost all problems, this is really all you're going to need to know about these scary functions. However, using them in conjunction with other summation techniques is what makes number theory sums challenging.

This following idea relating to the sum of divisors formula is one that I've seen multiple times on math competitions.

Example 91
Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

Solution. Prime factorizing N, we see $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. The sum of N's odd divisors are the sum of the factors of N without 2, and the sum of the even divisors is the sum of the odds subtracted by the total sum of divisors.
The sum of odd divisors is given by

$$a = (1 + 3 + 3^2 + 3^3 + 3^4 + 3^5)(1 + 5)(1 + 7)(1 + 17 + 17^2)$$

and the total sum of divisors is

$$(1 + 2 + 4 + 8)(1 + 3 + 3^2 + 3^3 + 3^4 + 3^5)(1 + 5)(1 + 7)(1 + 17 + 17^2) = 15a.$$

Thus, our ratio is

$$\frac{a}{15a - a} = \frac{a}{14a} = \boxed{\frac{1}{14}}.$$

□

What we're doing in this solution is comparing the ratio between 1 and the $2 + 2^2 + 2^3$ parts of the $(1 + 2 + 2^2 + 2^3)$ term in the sum of divisors formula.
Sometimes sums will ask you to sum up a certain quantity over all positive integers that divide a certain integer n. The following is a classic result:

Example 92
If n is a positive integer, prove that

$$\sum_{d|n} \varphi(d) = n.$$

Solution. The key insight to proving this result is using the fact that for a divisor d of n, the number of positive integers k less than or equal to n such that $\gcd(k, n) = \frac{n}{d}$ is just $\varphi(d)$. This is because each valid k can be written as the product of $\frac{n}{d}$ and some integer between 1 and d that shares no prime factors with d, and any such product is a valid k.

To finish this problem, we note that for every integer k between 1 and n, there's always going to exist exactly one divisor d of n such that $\gcd(k, n) = \frac{n}{d}$. Since $\gcd(k, n)$ is a divisor of n, we can just let $d = \frac{n}{\gcd(k,n)}$.

By summing up the number of integers k between 1 and n such that $\gcd(k, n) = \frac{n}{d}$ over all d, we end up covering all integers between 1 and n so our result is n, as desired. □

This concept is a bit tricky: essentially what we are doing is splitting all the numbers between 1 and n into groups based on what their greatest common divisor with n is. Then we are matching the number of elements in each of these groups with terms of the sum $\sum_{d|n} \varphi(d) = n$. If you are having trouble understanding the idea, try following our logic for smaller choices of n.

Example 93 (2021 HMMT Algebra/Number Theory Round)
Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x, and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n. Compute $\frac{S}{n}$.

Solution. The key idea is to interpret our sum over ordered pairs of integers (x, y) of positive integers for which xy divides n as a *double sum*. The first sum is over all divisors y of n. the second sum is over all x dividing $\frac{n}{y}$. Make sure to convince yourself that our given sum and this new nested sum are one and the same.

We have that

$$\begin{aligned}\sum_{xy|n} \varphi(x) \cdot y &= \sum_{y|n} y \sum_{x|\frac{n}{y}} \varphi(x) \\ &= \sum_{y|n} y \left(\frac{n}{y}\right) \\ &= \sum_{y|n} n.\end{aligned}$$

So, for every divisor of y of n, we add n once. Since n has $2^{10} = 1024$ divisors, $S = 1024n$ and the requested value is $\boxed{1024}$. $\square$

This idea is often seen in simplifying sums over ordered tuples of positive integers satisfying a particular constraint. It is usually easier to work with nested single variable sums than a single multiple-variable sum.

Exercise 94 (PUMAC). Ggiven that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, simplify

$$p(\pi) = \sum_{n=1}^{\infty} \frac{d(n)}{n^2}.$$

4.8 Exploiting Symmetry

Well, we've saved the best for last, cause this is one of our favorite summation ideas. It is best illustrated with an example.

Example 95 (2013 HMMT Algebra Round)
Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

Whoa. Is that **seven** nested infinite sums?

Solution. If you were a typical competitor, this is where your struggles with the problem probably would have ended. However, suppose that you were unfazed by the scary notation and tried to make sense of the mess. You may see that the problem becomes easier once one realizes how to make use of the fact that the sum is very symmetric with respect to the 7 variables $a_1, a_2, \ldots, a_7$. Look at the numerator, each variable appears exactly once. Look at the denominator, each variable appears in the same way there. Furthermore, each of the variables is summed from 0 to infinity.

Let's split up the numerator of the summand into seven different fractions;

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\left(\frac{a_1}{3^{a_1+a_2+\cdots+a_7}}+\frac{a_2}{3^{a_1+a_2+\cdots+a_7}}+\cdots+\frac{a_7}{3^{a_1+a_2+\cdots+a_7}}\right).$$

Now you see that we can split this large sum into 7 smaller sums, each of them being the sum of the quantity

$$\frac{a_k}{3^{a_1+a_2+\cdots+a_7}}$$

over all nonnegative integer values of $a_1, a_2, \ldots a_7$ and one of the indices k between 1 and 7. Now, you might see that something very cool happens when we do this. *By symmetry, each of these sums evaluates to the same amount.* This is a tricky idea. Make sure you see why this is true before we go on. Our sum is equivalent to

$$7\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1}{3^{a_1+a_2+\cdots+a_7}}.$$

Now, you might see that our sum is still quite symmetric when it comes to the variables $a_2, a_3, \ldots a_7$. Each of them appears in the same way in the denominator of the summand fraction. This motivates representing our summand as the product of several simpler terms:

$$7\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\left(\frac{a_1}{3^{a_1}}\cdot\frac{1}{3^{a_2}}\cdots\frac{1}{3^{a_7}}\right).$$

The last critical step is the idea of *factoring* a large sum back into smaller sums. In this case, we can factor our sum into 7 sums multiplied together with a constant, 7. We'll leave it to you to verify that our sum factors as

$$7\left(\sum_{a=0}^{\infty}\frac{a}{3^a}\right)\left(\sum_{a=0}^{\infty}\frac{1}{3^a}\right)^6.$$

The rest is just computation. The 6 geometric series sums all evaluate to $\frac{1}{1-1/3}=\frac{3}{2}$ by the infinite geometric series formula; the first sum requires the formula for the sum of an infinite arithmetico-geometric series. Multiplying these together yields our final answer, which is

$$7\cdot\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^6=\boxed{\frac{15309}{256}}.$$

□

A key takeaway from this problem is to first of all, never be afraid of scary-looking summation problems. Problem writers intentionally use fancy notation and lots of sigmas and everything to make the problem look artificially hard.

In this problem, we were given a large and difficult sum that we split into small sums that we knew were going to evaluate to the same amount by symmetry. So, we only had to find the value of *one* of these smaller sums, which ended up being easily factorable as the product of several well known infinite series.

In the next example, we're going to be doing the opposite. We'll be given a "smaller" sum that we don't know how to evaluate, *construct* other sums that evaluate to the same value *by symmetry*, and add them all together to magically form a sum that we do know how to calculate.

Example 96 (2019 HMMT Algebra Round)
Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

Solution. In the desired sum, swapping variables won't change anything due to symmetry. Thus, we can permute the variable $3!=6$ ways to create 6 sums all summing to the same thing. Summing these permutations, we have

$$\begin{aligned}
&\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)} \\
&= \frac{\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{\sum_{sym} ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.}{6} \\
&= \frac{\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{3(a+b)(b+c)(c+a)}{4^{a+b+c}(a+b)(b+c)(c+a)}}{6} \\
&= \frac{\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{3}{4^{a+b+c}}}{6} \\
&= 6^{-1}\cdot 3\left(\sum_{a=1}\frac{1}{4^a}\right)\left(\sum_{b=1}\frac{1}{4^b}\right)\left(\sum_{c=1}\frac{1}{4^c}\right) \\
&= 6^{-1}\cdot 3\cdot\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)^3 \\
&= 6^{-1}\cdot 3\cdot\tfrac{1}{27} = \boxed{\tfrac{1}{54}}.
\end{aligned}$$

□

Symmetry is an idea that comes up frequently across all areas of math, and contests will often test contestants on how well they can spot it, implement it, and use it. The reason why we can only explain it in a very vague way is that there's really no solid definition of what symmetry is, and it's really something that you have to dig deep and understand at an intuitive level by doing exercises. Oftentimes the symmetry involved with a particular problem will be cleverly hidden by lots of notation or lots of words. A general rule you should try to follow is to ask yourself, "What is symmetrical about this setup? What variables/things/elements of this problem essentially play the same role? What can I do to this element of the problem to achieve the same result?"

Exercise 97. Evaluate

$$\sum_{a=1}^{9}\sum_{b=1}^{9}\sum_{c=1}^{9}(abc+ab+bc+ca+a+b+c).$$

Exercise 98. If the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{ab(a+b)}{5^{a+b+c}(b-c)(c-a)}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, what is $m+n$?

4.9 Summary

In this handout, you were exposed to generalized summation notation, products, and some advanced strategies that are used in summation problems. Ultimately, there are only so many things we can cover in a series of 4 handouts, and there are many more wonderful summation concepts that are found throughout all of mathematics. Generating functions, Roots of Unity Filter, Dirichlet Convolutions, are just a few important summation ideas in higher level competitions that we ultimately had to leave out in this handout.

Of course, just like any other topic, many summation problems on contests use never-seen-before concepts. You might find yourself throwing everything you've learned in this course at these problems, but a lot of them will not be solved until you take a step back, take a deep breath, and look at it from an alternate angle. One of the downsides of knowing too much theory is that you can often be led down the wrong path, while you really should be thinking about how to tackle this problem in an outside-the-box manner.

This is the last handout in our 4-part Minicourse. We hope that you've learned a bit more about summations and just how much can be done with the idea of adding lots of weird things together. Until next time!

4.10 Important Terms

1. Generalized Summation Notation
2. Products
3. Nested Sums
4. Switching Order of Summation
5. Linearity of Expectation
6. Combination Notation
7. Sum of divisors function
8. Number of divisors function
9. Phi function
10. Symmetry

4.11 Practice Problems

Problem 22 (2008 AMC 10A). Evaluate

$$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdot\dots\cdot\frac{4n+4}{4n}\cdot\dots\cdot\frac{2008}{2004}.$$

Problem 23. Simplify

$$\sum_{j=0}^{10}\sum_{i=0}^{j}\binom{10}{j}\times\binom{j}{i}.$$

Problem 24 (AHSME). If $T_n = 1+2+3+\cdots+n$ and

$$P_n = \frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdot\dots\cdot\frac{T_n}{T_n-1}$$

for $n = 2,3,4,\cdots$, then simplify P_{1991}.

Problem 25 (OMO). For all positive integers k, define $f(k) = k^2+k+1$. Compute the largest positive integer n such that

$$2015f(1^2)f(2^2)\cdots f(n^2) \geq \Big(f(1)f(2)\cdots f(n)\Big)^2.$$

Problem 26. Let $d(n)$ denote the number of positive divisors of a positive integer n. Determine

$$\sum_{n=1}^{999}\left(\left\lfloor\frac{1000}{n}\right\rfloor - d(n)\right).$$

Problem 27. Determine

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(l+m+n)!}{l!\cdot m!\cdot n!\cdot 4^{n+m+l-1}}\right).$$

Problem 28 (2006 AIME I). Given that a sequence satisfies $x_0 = 0$ and $|x_k| = |x_{k-1}+3|$ for all integers $k \geq 1$, find the minimum possible value of $|x_1+x_2+\cdots+x_{2006}|$.