Conics

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1 Introduction

If you stack two cones tip to tip and pass planes through them, you will be able to create parabolas, circles, ellipses, and hyperbolas, which are known as the four main **conics**. (Can you see how they get their name?)

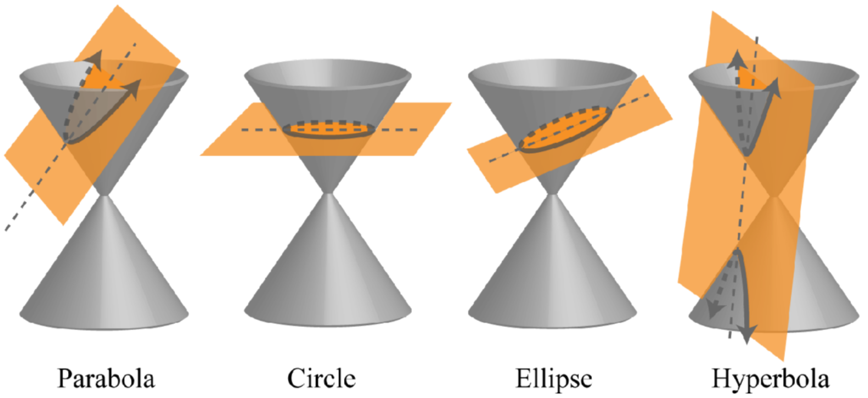


Figure 1: Creating conics out of cones

Here is a useful definition:

Locus
A **locus** is the set of all points that satisfy a certain property.

For example, the locus of all points P such that P has an x-coordinate of 1 is the line $x = 1$.

Conics are all loci.

Conics have historically not been popular on competitions, although in recent years, they have made a return. However, they are helpful to know and are also covered extensively in school math, though not on a competitive level.

The structure of this handout is simple. For each of the four main conics (that is, parabolas, circles, ellipses, and hyperbolas) we will give:

1. The definition of the conic, and important definitions relating to the conic, like focus, axis, etc.
2. Properties of the conic. This can be things like symmetry, shape, area, perimeter and more.
3. The general equation of the conic.

The second to last section will be example problems, intended to demonstrate how to use the theory given, and the last section will be a problem set.

By the end of this handout, you should have the necessary theory to solve all conic questions on the AMC/AIME.

2 Parabolas

2.1 What are Parabolas?

Let's visualize a line.

Figure 2: Literally, a line

Now, let's visualize a point not on the line. (Preferably above the line, for the sake of this demonstration).

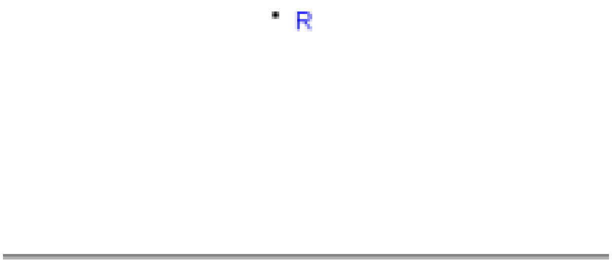


Figure 3: A point above our literal line

Great! A parabola is the locus of points which are equidistant from our literal line and our point. Before we look at one, try to imagine one. The line is known as the **directrix** while the point is known as the **focus**. Here is the formal definition:

> Parabola
> A **parabola** is the locus of all points P such that P is equidistant from another point Q and a line ℓ.

If you remember the class on polynomials, this may look familiar. Indeed, the graph of a quadratic (which has the highest degree 2), is a parabola.

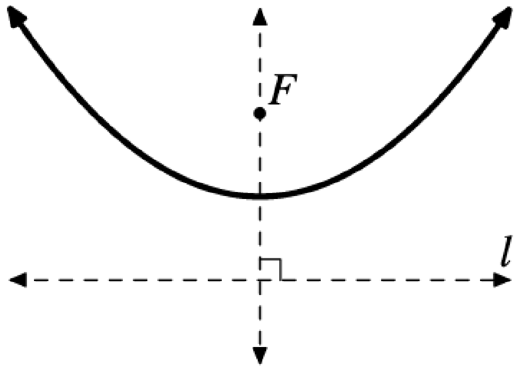


Figure 4: A parabola!

2.2 Properties of Parabolas

If we look at Figure 4 again, we may notice a line that was not our 'literal line' (that is to say, it was not our directrix). This line passes through the focus and intercepts our 'literal line' (this time, it is our directrix) perpendicularly. What do you notice about this line?

You may notice that the parabola is symmetric across the line. Indeed, you would be correct. This line is known as the **axis of symmetry** for a parabola. If you have time, take a second and prove how it is actually symmetric.

Parabolas can open both vertically or horizontally. The ones that open vertically (either upward or downward) are functions of x, and the ones that open horizontally (either left or right) are functions of y. Generally, functions of x are more commonly seen.

2.3 Equation for a Parabola

The standard equation for a parabola which opens either upwards or downwards (ie, if it is a function of x), is given by

$$f(x) = a(x-h)^2 + k,$$

where a is a nonzero constant and h and k are constants as well (but they are able to be 0.

What do h and k represent? The coordinate (h, k) is actually the **vertex** of the parabola. It follows that if you have a parabola with the formula $f(x) = ax^2$ and we move it h units to the right and k units upwards, then you have a parabola of the formula $f(x) = a(x-h)^2 + k$.

3 Circles

3.1 What are Circles?

Circle

A circle is the set of all points that are some distance r from a point. In particular, all the points are equidistant from that point, call it O, and at a distance of r. We call r the **radius** of the circle, and O its **center**.

3.2 Properties of Circles

Note that every circle is similar to every other circle. As a result, there is a homothety taking one circle from every other circle. Circles are actually a subset of ellipses, see the next section.

3.3 Equation of a circle

The general form of a circle is

$$(x-h)^2 + (y-k)^2 = r^2.$$

The point (h, k) is the center, and r is the radius.

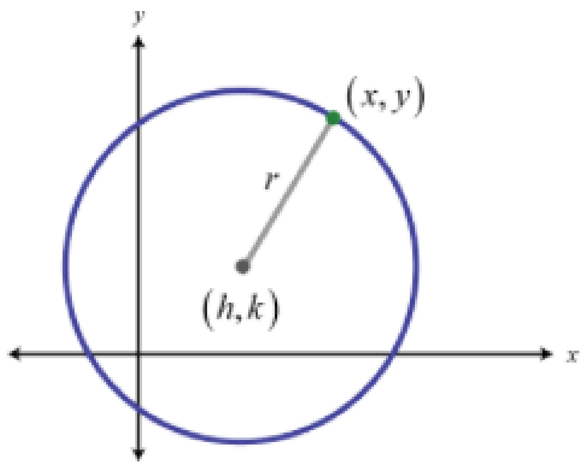


Figure 5: The circle with center (h, k)

4 Ellipses

4.1 What are Ellipses?

Formally, an ellipse is defined as:

Ellipse

An **ellipse** is the locus of all points P on the plane such that for two fixed points A and B also on the plane, the distance $PA + PB$ is constant

Note that in the image, the ellipse looks like a "stretched" circle. Indeed, circles are actually ellipses. They occur at the case $A = B$, and the value of $PA + PB$ is just the diameter.

The **major axis** is the longer diameter of the ellipse. The **minor axis** is the smaller diameter. The **foci** (plural for **focus**) are the two points A, B.

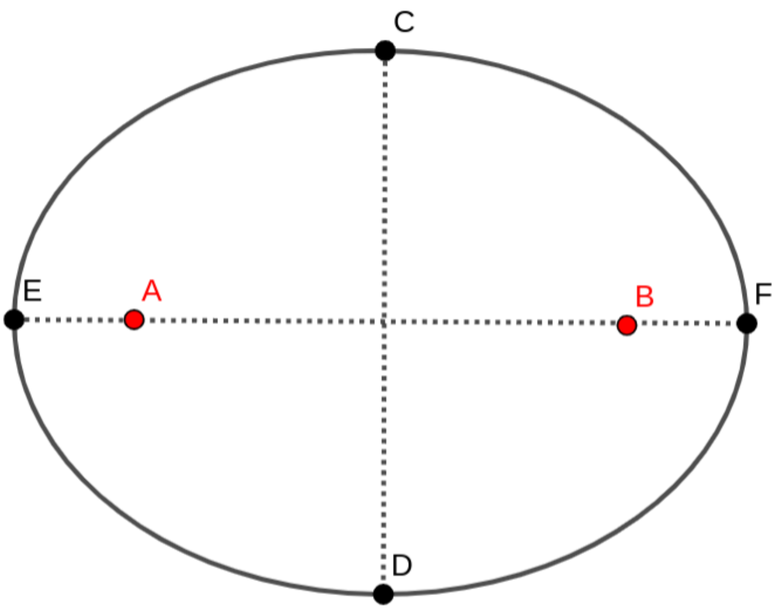


Figure 6: An ellipse with foci A and B

In Figure 6, the foci are A and B, the major axis is EF, and the minor axis is CD. The **semi-major axis** and **semi-minor axis** are half of the major and minor axes, respectively.

4.2 Properties of Ellipses

Let the semi-major axis be equal to a, and let the semi-minor axis be b.

Theorem 4.1
The area of an ellipse is $ab\pi$.

The actual proof involves calculus, so we will skip it. This formula is quite intuitive if we think about our characterization of ellipses as stretched circles.

Remark 4.2. We would expect that the circumference have a quite simple formula as well, however, this is not the case. See here.

Here is a sometimes useful definition:

Eccentricity
The **eccentricity** of a an ellipse is $\frac{c}{a}$ where $2c$ is the distance between the foci of the ellipse. It measures how "elongated" a conic is.

Remark 4.3. Let e be the eccentricity. We have:

- A circle if $e = 0$
- An ellipse if $0 < e < 1$
- A parabola if $e = 1$
- A hyperbola if $e > 1$
- a line if $e = \infty$.

Ellipses are symmetric over both axes (major and minor). Although rare, this is worth mentioning as well:

Theorem 4.4
A ray of light shot from one focus that bounces of the ellipse will go through the other ellipse.

One last property: In an ellipse, $(2b)^2 + (2c)^2 = (2a)^2$.

4.3 Equation of an Ellipse

As before, let the semi-major axis be equal to a, and let the semi-minor axis be b. The equation of an ellipse centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Observe that $a = b$ gives the circle equation.

5 Hyperbolas

5.1 What are hyperbolas?

Hyperbola
A hyperbola is defined as the locus of all points P such that for fixed points A, B the value of $|PA - PB|$ is constant.

Looking at 7, we see that the graph is a "bow." We call the halves this bow **branches**.

Similar to ellipses, we have two **foci**: the points A and B. The vertices are

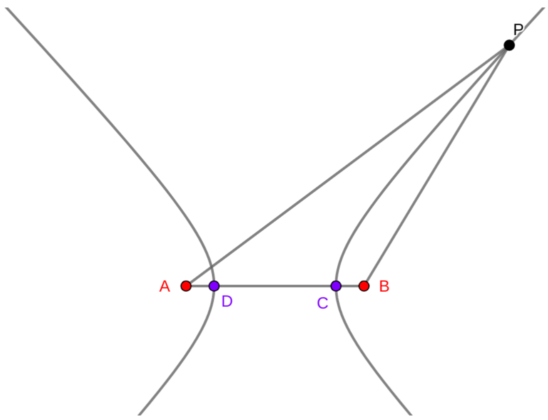


Figure 7: A Hyperbola with foci A and B

We say D and C are **vertices** (plural for **vertex**) of this hyperbola. They are the "closest" to the foci, and the hyperbola takes it's sharpest turn at the vertices. Notice that the distance between the vertices is the constant distance.

Not labeled on the diagram, hyperbolas also have two **asymptotes**.

Asymptotes
We call the asymptotes of the graph of an equation the lines that the graph will never cross, although get arbitrarily close to.

5.2 Properties of Hyperbolas

We have already discussed the graph of a hyperbola, however there is one more thing to add. The graph has two axes of symmetry. One is the line AB, and the other is the line ℓ passing through the midpoint of AB such that $\ell \perp AB$.

There isn't much else to say here.

5.3 Equation of a Hyperbola

The equation of a hyperbola that opens left and right (as in figure 7) is

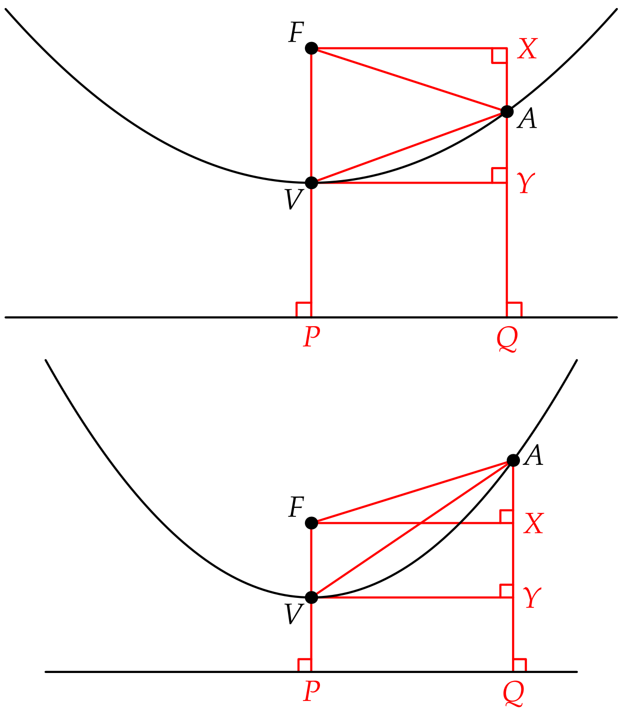
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

with a, b, h, k having the same definitions as in the ellipse section.

6 Worked Through Problems

Example 6.1 (2021 AMC 12A/20)
Suppose that on a parabola with vertex V and a focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

Solution. Let the directrix of the hyperbola be ℓ. (We are motivated to do so by the properties it can give). Observe that there are two configurations here (we can check this because the problem statement asks for the sum of all possible values of FV):



Now, we need to find $FV = x$. Note that we also have $VP = x$. By definition of parabola, $AQ = AF = 20$. Now, this is just a geometry problem. We notice that $FX = VY = y$. Now, by Pythagorean Theorem, we have the equations:

$$(20-x)^2 + y^2 = 21^2$$
$$(20-2x)^2 + y^2 = 20^2$$

Now, we can solve for x by putting both equations in terms of y^2 and substituting. This gives $3x^2 - 40x + 41 = 0$, so by Vieta's the answer is $\boxed{\frac{40}{3}}$. □

Remark 6.2. An alternative way to solve this is to consider the equation of the parabola. You can see more solutions here.

Example 6.3 (Summer MAT 2021/8)
An ellipse with a focus at $(6, 2)$ is tangent to the positive x axis at $(a, 0)$. There is a maximum value of b such that for all $0 < a < b$, it is possible for this ellipse to be tangent to the y axis. If this maximal value of b can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

This is a good example of synthetic geometry in ellipses.

Solution. Let $F_1 = (6, 2)$ and let the focus of the other ellipse be F_2. Finally, let A be the point $(a, 0)$. We are motivated to relate A with F_2 and F_1. A good trick that is useful due to the reflective property of ellipses (Theorem 4.4) is to reflect F_1 over the x-axis and y-axis to points P and Q respectively. Now, because $F_1X + F_2X$ is constant for any point X on the ellipse, and due to Theorem 4.4, we can deduce that $F_2P = F_2Q$. This also means that F_2 lies on the perpendicular bisector of PQ! This is useful because we have related the other focus to A. In particular, F_2 is the intersection of the perpendicular bisector of PQ and AP. This intersection is in the first quadrant since we want the ellipse to be tangent to the y-axis. This tells us that the slope of AP is less than the slope of the perpendicular bisector of PQ. So, let us consider the extreme point $(b, 0)$. It can be found as the line through P parallel to the perpendicular bisector of PQ, which has a slope 3 so the answer is $\boxed{\frac{20}{3}}$. □

Example 6.4 (1985 AIME/11)
An ellipse has foci at $(9, 20)$ and $(49, 55)$ in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

Solution. Let $F_1 = (9, 20)$ and $F_2 = (49, 55)$. Let T be the tangency point on the x-axis. The T is the unique point on the x-axis such that the distance $F_1T + F_2T$ is minimal. To see this, note that for any other point, the ellipse will cross the x-axis.

Consider the constant sum of the ellipse for any point P, call it d The standard way to find t would be to let $T = (t, 0)$ and use the distance formula with both foci. However, we will present a more interesting solution. Reflect F_2 over the x-axis to F_2'. Note that $F_2T = F_2'T$. Then

$$d = F_1F_2' = \sqrt{(49-9)^2 + (20-(-55))^2} = 85.$$

Clearly, the major axis is $\boxed{085}$. □

Example 6.5 (PuMAC 2012 Geometry A4/B6)
A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form $a + b\sqrt{c}$ where a, b, and c are integers, and c is not divisible by the square of any prime. Find the sum $a + b + c$.

Solution. As usual, let one the foci be F_1 and F_2, and let the square be $ABCD$ such that F_1 is on AD and F_2 is on BC. Let The major axis have length M, and let the minor axis have length m. Finally, let $x = 2$ be the distance from the foci to the center of the ellipse.

Using our known properties, we know that F_1 and F_2 are midpoints of the square's sides, and we know that $F_1A + F_2A = 2M$. By Pythagorean Theorem, we have

$$2M = F_1A + F_2A = F_1A + \sqrt{F_1A^2 + 4^2} = 2 + \sqrt{20} = 2 + 2\sqrt{5}.$$

Recall the relation

$$M^2 = x^2 + m^2.$$

mentioned earlier. Since $x = 2$, we can solve for b:

$$\begin{aligned} M^2 &= x^2 + m^2 \\ (1+\sqrt{5})^2 &= 4 + m^2 \\ \sqrt{2+\sqrt{5}} &= m. \end{aligned}$$

Since we are looking for $(2m)^2 = \sqrt{8 + 8\sqrt{5}}$, the answer is $\boxed{21}$. □

Example 6.6 (2006 AMC 12A/21)
Let

$$S_1 = \{(x,y) | \log_{10}(1 + x^2 + y^2) \le 1 + \log_{10}(x+y)\}$$

and

$$S_2 = \{(x,y) | \log_{10}(2 + x^2 + y^2) \le 2 + \log_{10}(x+y)\}.$$

What is the ratio of the area of S_2 to the area of S_1?

Solution. We need to manipulate this into some graph we know how to find the area of. The two graphs we have are

$$\begin{aligned} \log_{10}(1 + x^2 + y^2) &\le 1 + \log_{10}(x+y) \\ \log_{10}(2 + x^2 + y^2) &\le 2 +_{10} (x+y), \end{aligned}$$

and we want to simplify these inequalities. Consider the first one. The pesky 1 term is making it hard to progress here. But notice that $\log_{10} 10 = 1$. Also, because $\log_{10} z$ is strictly increasing as z increases, we have that if $\log_{10} p = \log_{10} q$, then $p = q$. So, we have

$$\begin{aligned}
\log_{10}(1 + x^2 + y^2) &\leq 1 + \log_{10}(x + y) \\
\log_{10}(1 + x^2 + y^2) &\leq \log_{10}(10x + 10y) \\
1 + x^2 + y^2 &\leq 10x + 10y \\
(x^2 - 10) + (y^2 - 10) &\leq -1 \\
(x - 5)^2 + (y - 5)^2 &\leq 49,
\end{aligned}$$

where we add $2 \cdot 25$ in the last step to complete the square. Similarly, the other inequalities is

$$(x - 50)^2 + (y - 50)^2 \leq 4998.$$

Thus, the ratio of the areas if $\frac{4998\pi}{49\pi} = \boxed{102}$. □

7 Problems

Problem 1 (Intermediate Algebra). Hyperbola $\mathcal{H}$ is the graph of the equation

$$\frac{(x-3)^2}{a^2} = \frac{y^2}{b^2} = 1.$$

One of the points of intersection of $\mathcal{H}$ and the line $y = ax$ has x-coordinate 0. Find the x-coordinates of the points of intersection of the line $y = bx$ and $\mathcal{H}$.

Problem 2 (2021 CMIMC Algebra & Number Theory). Let $f(x) = \frac{x^2}{8}$. Starting at point $(7,3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?

Problem 3 (2017 AMC 10B). The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?

Problem 4 (2021 AMC 12A). The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$

may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$ where m and n are relatively prime positive integers. What is $m+n$? (Recall that the eccentricity of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the is the distance between its two foci.)

Problem 5 (2012 MPFG). Say that a complex number z is three-presentable if there is a complex number w of absolute value 3 such that $z = w - \frac{1}{w}$. Let T be the set of all three-presentable complex numbers. The set T forms a closed curve in the complex plane. What is the area inside T?

Problem 6 (2017 CMIMC Algebra Tiebreaker). The parabola $\mathcal{P}$ given by equation $y = x^2$ is rotated some acute angle θ clockwise about the origin such that it hits both the x and y axes at two distinct points. Suppose the length of the segment $\mathcal{P}$ cuts the x-axis is 1. What is the length of the segment $\mathcal{P}$ cuts the y-axis?

Problem 7 (2020 CMIMC Geometry). Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Parabola $\mathcal{P}$, having vertex F_1 and focus F_2, intersects $\mathcal{E}$ at two points X and Y. Suppose the tangents to $\mathcal{E}$ at X and Y intersect on the directrix of $\mathcal{P}$. Compute the eccentricity of $\mathcal{E}$.

(A parabola $\mathcal{P}$ is the set of points which are equidistant from a point, called the focus of $\mathcal{P}$, and a line, called the directrix of $\mathcal{P}$. An ellipse $\mathcal{E}$ is the set of points P such that the sum $PF_1 + PF_2$ is some constant d, where F_1 and F_2 are the foci of $\mathcal{E}$. The eccentricity of $\mathcal{E}$ is defined to be the ratio F_1F_2/d.)

Problem 8 (2016 HMMT Guts). Find the smallest possible area of an ellipse passing through $(2,0)$, $(0,3)$, $(0,7)$, and $(6,0)$.

Problem 9 (2020 Fall OMO). Among all ellipses with center at the origin, exactly one such ellipse is tangent to the graph of the curve $x^3 - 6x^2y + 3xy^2 + y^3 + 9x^2 - 9xy + 9y^2 = 0$ at three distinct points. The area of this ellipse is $\frac{k\pi\sqrt{m}}{n}$, where k, m, and n are positive integers such that $\gcd(k,n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

Problem 10 (2018 Fall OMO). In triangle ABC, $AB = 13, BC = 14, CA = 15$. Let Ω and ω be the circumcircle and incircle of ABC respectively. Among all circles that are tangent to both Ω and ω, call those that contain ω inclusive and those that do not contain ω exclusive. Let $\mathcal{I}$ and $\mathcal{E}$ denote the set of centers of inclusive circles and exclusive circles respectively, and let I and E be the area of the regions enclosed by $\mathcal{I}$ and $\mathcal{E}$ respectively. The ratio $\frac{I}{E}$ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Compute $100m + n$.