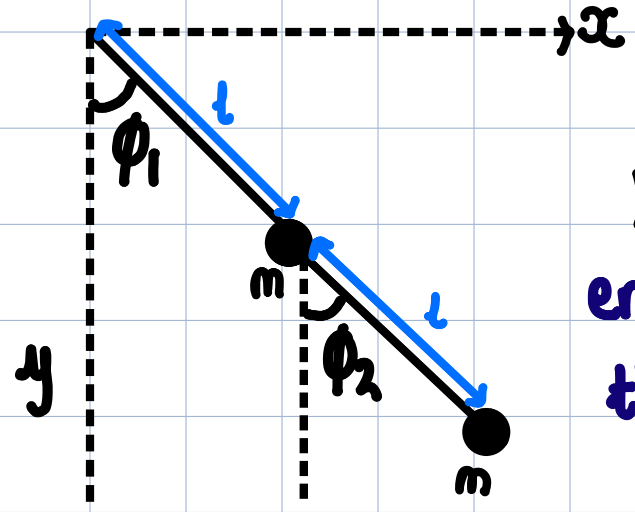
Double Pendulum Supplementary Material - Derivations

Verifying the Lagrangian of the system:



$L = T - V$ where T is the kinetic energy and V is the potential energy of the system.

$T = \frac{1}{2}mv^2 \quad V = -mgy \quad y = y_1 + y_2$

Coordinates of pendulum 1:

$x_1 = l\sin\phi_1 \qquad y_1 = l\cos\phi_1$

$\dot{x}_1 = l\dot{\phi}_1\cos\phi_1 \qquad \dot{y}_1 = -l\dot{\phi}_1\sin\phi_1$

$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2$

$v_1^2 = l^2\dot{\phi}_1^2\cos^2\phi_1 + l^2\dot{\phi}_1^2\sin^2\phi_1$

$v_1^2 = l^2\dot{\phi}_1^2(\cos^2\phi_1 + \sin^2\phi_1)$

$v_1^2 = l^2\dot{\phi}_1^2$

Coordinates of pendulum 2:

$x_2 = l\sin\phi_1 + l\sin\phi_2 \qquad y_2 = l\cos\phi_1 + l\cos\phi_2$

$\dot{x}_2 = l\dot{\phi}_1\cos\phi_1 + l\dot{\phi}_2\cos\phi_2 \qquad \dot{y}_2 = -l\dot{\phi}_1\sin\phi_1 - l\dot{\phi}_2\sin\phi_2$

$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$

$v_2^2 = \underline{l^2\dot{\phi}_1^2\cos^2\phi_1} + \underline{2l^2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2} + \underline{l^2\dot{\phi}_2^2\cos^2\phi_2} + \underline{l^2\dot{\phi}_1^2\sin^2\phi_1}$

$\underline{-2l^2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\,\phi_2} + \underline{l^2\dot{\phi}_2^2\sin^2\phi_2}$

$$v_2^2 = l^2\dot{\phi}_1^2(\cos^2\phi_1 + \sin^2\phi_1) + 2l^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) + l^2\dot{\phi}_2^2(\cos^2\phi_2 + \sin^2\phi_2)$$

$$v_2^2 = l^2\dot{\phi}_1^2 + 2l^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + l^2\dot{\phi}_2^2$$

$$v^2 = v_1^2 + v_2^2 = 2l^2\dot{\phi}_1^2 + 2l^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + l^2\dot{\phi}_2^2$$

$$T = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left[2l^2\dot{\phi}_1^2 + l^2\dot{\phi}_2^2 + 2l^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right]$$

$$= \frac{1}{2}ml^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right]$$

$$V = -mg(l\cos\phi_1 + l\cos\phi_1 + l\cos\phi_2)$$

$$V = -mg(2l\cos\phi_1 + l\cos\phi_2)$$

$$V = -mgl(2\cos\phi_1 + \cos\phi_2)$$

$$L = \frac{1}{2}ml^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right] + mgl(2\cos\phi_1 + \cos\phi_2)$$

Q.E.D.

Deriving Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) - \frac{\partial L}{\partial \phi_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) - \frac{\partial L}{\partial \phi_1} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}_1} = \frac{1}{2}ml^2\left[4\dot{\phi}_1 + 2\dot{\phi}_2\cos(\phi_1-\phi_2)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) = \frac{1}{2}ml^2\left[4\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) - 2\dot{\phi}_2(\dot{\phi}_1-\dot{\phi}_2)\sin(\phi_1-\phi_2)\right]$$

$$= \frac{1}{2}ml^2\left[4\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) - 2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) + 2\dot{\phi}_2^2\sin(\phi_1-\phi_2)\right]$$

$$= 2ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) - ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)$$
$$+ ml^2\dot{\phi}_2^2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_1} = \frac{1}{2}ml^2\left[-2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)\right] - 2mgl\sin\phi_1$$

$$= -ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - 2mgl\sin\phi_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) - \frac{\partial L}{\partial \phi_1} = 0$$

$$\Rightarrow 2ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) - \cancel{ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)}$$
$$+ ml^2\dot{\phi}_2^2\sin(\phi_1-\phi_2) + \cancel{ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)} + 2mgl\sin\phi_1 = 0$$

$$\Rightarrow 2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_2^2\sin(\phi_1-\phi_2) + \frac{2g}{l}\sin\phi_1 = 0 \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) - \frac{\partial L}{\partial \phi_2} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}_2} = \frac{1}{2}ml^2\left[2\dot{\phi}_2 + 2\dot{\phi}_1\cos(\phi_1-\phi_2)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) = \frac{1}{2}ml^2\left[2\ddot{\phi}_2 + 2\ddot{\phi}_1\cos(\phi_1-\phi_2) - 2\dot{\phi}_1\sin(\phi_1-\phi_2)\{\dot{\phi}_1-\dot{\phi}_2\}\right]$$

$$= ml^2\ddot{\phi}_2 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2) - ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_2)$$
$$+ ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)$$

$$\frac{\partial L}{\partial \phi_2} = \frac{1}{2}ml^2\left[-2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)(-1)\right] - mgl\sin\phi_2$$

$$= ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - mgl\sin\phi_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) - \frac{\partial L}{\partial \phi_2} = 0$$

$$\Rightarrow ml^2\ddot{\phi}_2 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2) - ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_2)$$
$$+ \cancel{ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)} - \cancel{ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2)} + mgl\sin\phi_2 = 0$$

$$\Rightarrow \ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1-\phi_2) - \dot{\phi}_1^2\sin(\phi_1-\phi_2) + \frac{g}{l}\sin\phi_2 = 0 \quad (2)$$

From (2): $\ddot{\phi}_2 = -\ddot{\phi}_1\cos\lambda + \dot{\phi}_1^2\sin\lambda - \omega^2\sin\phi_2$ (*)
where we have used $\lambda = \phi_1 - \phi_2$ and $\omega^2 = \frac{g}{l}$ for brevity.

Substituting this into (1):

$$2\ddot{\phi}_1 + \cos\lambda\left[-\ddot{\phi}_1\cos\lambda + \dot{\phi}_1^2\sin\lambda - \omega^2\sin\phi_2\right] + \dot{\phi}_2^2\sin\lambda + 2\omega^2\sin\phi_1 = 0$$

$$2\ddot{\phi}_1 - \ddot{\phi}_1\cos^2\lambda + \dot{\phi}_1^2\cos\lambda\sin\lambda - \omega^2\sin\phi_2\cos\lambda + \dot{\phi}_2^2\sin\lambda + 2\omega^2\sin\phi_1 = 0$$

$$2\ddot{\phi}_1 - \ddot{\phi}_1\cos^2\lambda = \omega^2\sin\phi_2\cos\lambda - \dot{\phi}_1^2\cos\lambda\sin\lambda - \dot{\phi}_2^2\sin\lambda - 2\omega^2\sin\phi_1$$

$$\ddot{\phi}_1 = \frac{\omega^2\sin\phi_2\cos\lambda - \dot{\phi}_1^2\cos\lambda\sin\lambda - \dot{\phi}_2^2\sin\lambda - 2\omega^2\sin\phi_1}{2-\cos^2\lambda}$$

This can be substituted into (*) to find equation for $\ddot{\phi}_2$.

$$\ddot{\phi}_2 = -\cos\lambda\left[\frac{\omega^2\sin\phi_2\cos\lambda - \dot{\phi}_1^2\cos\lambda\sin\lambda - \dot{\phi}_2^2\sin\lambda - 2\omega^2\sin\phi_1}{2-\cos^2\lambda}\right]$$
$$+ \dot{\phi}_1^2\sin\lambda - \omega^2\sin\phi_2$$

$$\ddot{\phi}_2 = \frac{-\omega^2\sin\phi_2\cos^2\lambda + \dot{\phi}_1^2\cos^2\lambda\sin\lambda + \dot{\phi}_2^2\sin\lambda\cos\lambda + 2\omega^2\sin\phi_1\cos\lambda}{2-\cos^2\lambda}$$

$$+ \frac{\dot{\phi}_1^2\sin\lambda(2-\cos^2\lambda) - \omega^2\sin\phi_2(2-\cos^2\lambda)}{2-\cos^2\lambda}$$

$$\ddot{\phi}_2 = \frac{\cancel{-\omega^2\sin\phi_2\cos^2\lambda} \cancel{+ \dot{\phi}_1^2\cos^2\lambda\sin\lambda} + \dot{\phi}_2^2\sin\lambda\cos\lambda + 2\omega^2\sin\phi_1\cos\lambda}{2-\cos^2\lambda}$$

$$+ \frac{2\dot{\phi}_1^2\sin\lambda - \cancel{\dot{\phi}_1^2\sin\lambda\cos^2\lambda} - 2\omega^2\sin\phi_2 + \cancel{\omega^2\sin\phi_2\cos^2\lambda}}{2-\cos^2\lambda}$$

$$\ddot{\phi}_2 = \frac{\dot{\phi}_2^2\sin\lambda\cos\lambda + 2\omega^2\sin\phi_1\cos\lambda + 2\dot{\phi}_1^2\sin\lambda - 2\omega^2\sin\phi_2}{2-\cos^2\lambda}$$

These equations can be solved numerically in Python using the Runge-Kutta method.

Motion for small ϕ_i:

From above we have:

$$2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_2^2\sin(\phi_1-\phi_2) + \frac{2g}{l}\sin\phi_1 = 0 \quad (1)$$

$$\ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1-\phi_2) - \dot{\phi}_1^2\sin(\phi_1-\phi_2) + \frac{g}{l}\sin\phi_2 = 0 \quad (2)$$

For small ϕ, $\sin\phi \simeq \phi$ and $\cos\phi \simeq 1 - \frac{\phi^2}{2}$

If ϕ_1 and ϕ_2 are both small, then $\phi_1 - \phi_2$ is also small

For small ϕ_i:

(1): $2\ddot{\phi}_1 + \ddot{\phi}_2\left[1 - \frac{(\phi_1-\phi_2)^2}{2}\right] + \dot{\phi}_2^2(\phi_1-\phi_2) + \frac{2g}{l}\phi_1 = 0$

$$2\ddot{\phi}_1 + \ddot{\phi}_2\left[1 - \frac{\phi_1^2 - 2\phi_1\phi_2 + \phi_2^2}{2}\right] + \dot{\phi}_2^2(\phi_1-\phi_2) + \frac{2g}{l}\phi_1 = 0$$

neglecting any terms of order greater than 2:

$$2\ddot{\phi}_1 + \ddot{\phi}_2 + \frac{2g}{l}\phi_1 = 0$$

$$\Rightarrow \ddot{\phi}_2 = -2\ddot{\phi}_1 - \frac{2g}{l}\phi_1 \quad (**)$$

(2): $\ddot{\phi}_2 + \ddot{\phi}_1\left[1 - \frac{(\phi_1-\phi_2)^2}{2}\right] - \dot{\phi}_1^2(\phi_1-\phi_2) + \frac{g}{l}\phi_2 = 0$

neglecting terms of order greater than 2:

$$\ddot{\phi}_2 + \ddot{\phi}_1 + \frac{g}{l}\phi_2 = 0$$

Subbing (**) into this:

$$-2\ddot{\phi}_1 - \frac{2g}{l}\phi_1 + \ddot{\phi}_1 + \frac{g}{l}\phi_2 = 0$$

$$\ddot{\phi}_1 = \frac{g}{l}(\phi_2 - 2\phi_1)$$

Subbing this into (**):

$$\ddot{\phi}_2 = -\frac{2g}{l}(\phi_2 - 2\phi_1) - \frac{2g}{l}\phi_1$$

$$\ddot{\phi}_2 = -\frac{2g}{l}\phi_2 + \frac{4g}{l}\phi_1 - \frac{2g}{l}\phi_1$$

$$\ddot{\phi}_2 = \frac{2g}{l}(\phi_1 - \phi_2)$$

motion in the presence of air resistance.

credit to reference [4] in the report.

damping Rayleigh potential: $\mathcal{F} = \frac{1}{2} b v^2$

$$\mathcal{F} = \frac{1}{2} b (\dot{v}_1^2 + \dot{v}_2)$$

$$= \frac{1}{2} b (2l^2 \dot{\phi}_1^2 + l^2 \dot{\phi}_2^2 + 2l^2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2))$$

$$= bl^2 \left[\dot{\phi}_1^2 + \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} \dot{\phi}_2^2 \right]$$

$$= \frac{b}{m} \left[\dot{\phi}_1^2 + \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} \dot{\phi}_2^2 \right]$$

Damping angular frequency is $\frac{b}{m}$

Lagrange's equations of motion become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) - \frac{\partial L}{\partial \phi_i} + \frac{\partial \mathcal{F}}{\partial \dot{\phi}_i}$$

$$\frac{\partial \mathcal{F}}{\partial \dot{\phi}_1} = \frac{b}{m} \left[2\dot{\phi}_1 + \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right]$$

$$\frac{\partial \mathcal{F}}{\partial \dot{\phi}_2} = \frac{b}{m} \left[\dot{\phi}_1 \cos(\phi_1 - \phi_2) + \dot{\phi}_2 \right]$$

So we have:

$$2\ddot{\phi}_1 + \ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \dot{\phi}_2^2 \sin(\phi_1 - \phi_2) + \frac{2g}{l} \sin\phi_1$$
$$+ \frac{b}{m} \left[2\dot{\phi}_1 + \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right] = 0, \quad (3)$$

and

$$\ddot{\phi}_2 + \ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) + \frac{g}{l}\sin\phi_2 + \frac{b}{m}\left[\dot{\phi}_1 \cos(\phi_1 - \phi_2) + \dot{\phi}_2\right] = 0, \quad (4)$$

So

$$\ddot{\phi}_2 = -\ddot{\phi}_1 \cos(\phi_1 - \phi_2) + \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) - \frac{g}{l}\sin\phi_2 - \frac{b}{m}\left[\dot{\phi}_1 \cos(\phi_1 - \phi_2) + \dot{\phi}_2\right] \quad (***)$$

Substituting (***) into (3) using $\lambda = \phi_1 - \phi_2$ and $\omega^2 = \frac{g}{l}$ for brevity.

$$2\ddot{\phi}_1 + \cos\lambda\left[-\ddot{\phi}_1 \cos\lambda + \dot{\phi}_1^2 \sin\lambda - \omega^2 \sin\phi_2 - \frac{b}{m}\left[\dot{\phi}_1 \cos\lambda + \dot{\phi}_2\right]\right] + \dot{\phi}_2^2 \sin\lambda + \omega^2 \sin\phi_1 + \frac{b}{m}\left[2\dot{\phi}_1 + \dot{\phi}_2 \cos\lambda\right] = 0$$

$$2\ddot{\phi}_1 - \ddot{\phi}_1 \cos^2\lambda + \cos\lambda\left[\dot{\phi}_1^2 \sin\lambda - \omega^2 \sin\phi_2 - \frac{b}{m}\left[\dot{\phi}_1 \cos\lambda + \dot{\phi}_2\right]\right] + \dot{\phi}_2^2 \sin\lambda + \omega^2 \sin\phi_1 + \frac{b}{m}\left[2\dot{\phi}_1 + \dot{\phi}_2 \cos\lambda\right] = 0$$

$$\ddot{\phi}_1 = \left\{-\cos\lambda\left[\dot{\phi}_1^2 \sin\lambda - \omega^2 \sin\phi_2 - \frac{b}{m}\left[\dot{\phi}_1 \cos\lambda + \dot{\phi}_2\right]\right] - \dot{\phi}_2^2 \sin\lambda - \omega^2 \sin\phi_1 - \frac{b}{m}\left[2\dot{\phi}_1 + \dot{\phi}_2 \cos\lambda\right]\right\} / (2 - \cos^2\lambda)$$

This can be written first in our computer code, so we can write (***) for $\ddot{\phi}_2$.