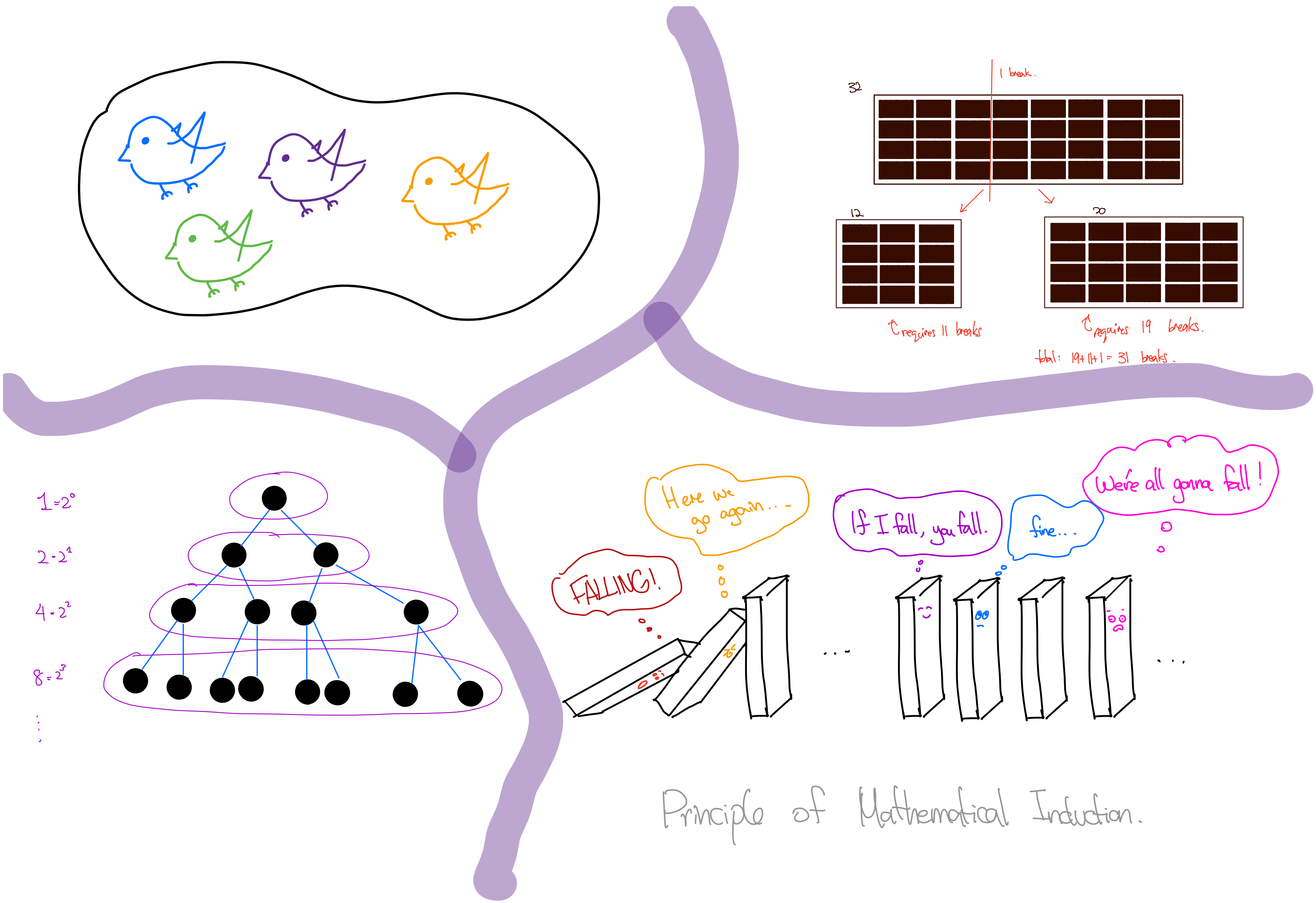
# Last time...



32
1 break.
12
20
requires 11 breaks
requires 19 breaks.
total: 19+11+1 = 31 breaks.
1=2^0
2=2^1
4=2^2
8=2^3
FALLING!
Here we go again...
If I fall, you fall.
fine...
We're all gonna fall!
Principle of Mathematical Induction.

# CSC 236 Lecture 4: Induction 2

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May 31, 2023

# Today

Induction

Structural Induction

Well Ordering Principle and Proof by Infinite Descent

Induction

Structural Induction

Well Ordering Principle and Proof by Infinite Descent

# Induction and Complete Induction

To prove

Base

$$\forall n \in \mathbb{N}.(P(n))$$

it is enough to prove $P(0)$ and one of the following

- $\forall k \in \mathbb{N}.[P(k) \implies P(k+1)]$ regular induction
- $\forall k \in \mathbb{N}.[(P(0) \wedge P(1) \wedge ... \wedge P(k)) \implies P(k+1)]$

complete induction.

# Intuition: why does (regular) induction work again?

Say I managed to show $P(0)$, and $\forall k \in \mathbb{N}.(P(k) \implies P(k+1))$.
Then let $n \in \mathbb{N}$ be any number, here's why $P(n)$ is true:

- $\boxed{P(0) \implies P(1)}$, and $P(0)$, so $P(1)$
- $P(1) \implies P(2)$, and $P(1)$, so $P(2)$
- ...
- $P(n-1) \implies P(n)$, and $P(n)$, so $P(n)$.

# Intuition: why does (complete) induction work again?

Say I managed to show $P(0)$, and $\forall k \in \mathbb{N}.((P(0) \wedge P(1) \wedge \dots P(k)) \implies P(k+1))$. Then let $n \in \mathbb{N}$ be any number, here's why $P(n)$ is true:

- $P(0) \implies P(1)$, and $P(0)$, so $P(1)$
- $P(0) \wedge P(1) \implies P(2)$, and $P(0) \wedge P(1)$, so $P(2)$
- ...
- $(P(0) \wedge \dots. \wedge P(n-1)) \implies P(n)$, and $P(0) \wedge \dots \wedge P(n-1)$, so $P(n)$.

# Postage Stamps

Say that you have an unlimited number of 3 cent and 5 cent postage stamps. Can you make any postage exactly?

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Say that you have an unlimited number of 3 cent and 5 cent postage stamps. Can you make any postage exactly?

No, i.e. 1, 2, 4, 7 can't be made.

Can you make any postage $\geq 8$ cents exactly?

# Rephrasing the problem mathematically

**Claim:** For any $n \geq 8$, there exists $a, b \in \mathbb{N}$ such that $n = 3a + 5b$

# Proof, attempt 1 (wrong!)

**Claim:** For any $n \geq 8$, there exists $a, b \in \mathbb{N}$ such that $n = 3a + 5b$

By complete induction.

**Base case.** We can make an 8 cent postage using one 3 cent stamp and one 5 cent stamp.

**Inductive step.** Let $k \geq 8$ and assume for any $8 \leq i \leq k$, we can make a postage of $i$ cents using only 3 and 5 cent stamps. We'll show that you can also make a $k+1$ postage. Use one 3-cent stamp. We now need to make a $k-2$ postage. By the induction hypothesis, we can make $k-2$ using only 3 cent and 5 cent stamps, so together, we have made a $k+1$ postage.

What's the problem here?

# Proof, attempt 1 (wrong!)

**Claim:** For any $n \geq 8$, there exists $a, b \in \mathbb{N}$ such that $n = 3a + 5b$

By complete induction.

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$k = 8$

$k = 9$

$k = 10$

---

[1] $\boxed{k-2}$ might be 6 which is not covered by the induction hypothesis

# Problem

Out induction hypothesis was $P(8), P(9), ..., P(k)$, and we wanted to show $P(k+1)$. However, when $k = 8$ or 9, $k - 2$ is 6 or 7 which is not covered by the induction hypothesis! So our argument in the inductive step doesn't work for $k = 8$ or $k = 9$.

To fix this, we can just prove $P(k+1)$ directly for these cases for $k = 8$ and $k = 9$.

# Proof 1. multiple base cases

**Claim:** For any $n \geq 8$, there exists $a, b \in \mathbb{N}$ such that $n = 3a + 5b$

Base case: $8 = 3+5$, $9 = 3 \cdot 3$, $10 = 5 \cdot 2$.

Inductive step: suppose $k \in \mathbb{N}$, $\boxed{k \geq 10}$ and assume $\boxed{\forall i,\ 8 \leq i \leq k,\ i = 3a + 5b}$.

then since $k \geq 10$, $k-2 \geq 8$.

furthermore $k-2 \leq k$. Thus we can apply the IH to $k-2$. i.e. $k-2 = 3a + 5b$ for some, $a, b \in \mathbb{N}$. then $k+1 = 3(a+1) + 5b$.

Takeaway: Add more cases in your base case if you find that the assumption that $k$ is large is useful.

# Proof 2. Regular induction

**Claim:** For any $n \geq 8$, there exists $a, b \in \mathbb{N}$ such that $n = 3a + 5b$

Base case: $8 = 3 + 5$.

Inductive step. Let $k \in \mathbb{N}$, where $k \geq 8$. suppose $k = 3a + 5b$. There are 2 cases.

case i: $b \geq 1$. then $k+1 = 3(a+2) + 5(b-1)$.

$b = 0$ then $a \geq 3$. since $k \geq 8$ so.

$k+1 = 3(a-3) + 5(b+2)$.

$\forall n \in \mathbb{N}.(2n = 0)$

complete

By induction.

**Base case.** $2 \cdot 0 = 0$ so the base case holds.

**Inductive step.** Let $k \in \mathbb{N}$ be an arbitrary natural number and assume $2 \cdot k = 0$, we'll show $2 \cdot (k+1) = 0$. Write $k + 1 = i + j$ for some smaller natural numbers $i, j$. Then we have

$$2(k+1) = 2(i+j) = 2i + 2j = 0 + 0,$$

where we used the inductive hypothesis on $i$ and $j$ in the last equality.

# $\forall n \in \mathbb{N}.(2n = 0)$

By induction.

**Base case.** $2 \cdot 0 = 0$ so the base case holds.

**Inductive step.** Let $k \in \mathbb{N}$ be an arbitrary natural number and assume $2 \cdot k = 0$, we'll show $2 \cdot (k+1) = 0$. Write $k + 1 = i + j$ for some smaller natural numbers $i, j$[2]. Then we have

$$2(k+1) = 2(i+j) = 2i + 2j = 0 + 0,$$

where we used the inductive hypothesis on $i$ and $j$ in the last equality.

[2] you can't do this for $k = 0$

Induction

Structural Induction

Well Ordering Principle and Proof by Infinite Descent

# Induction

So far, we've been able to use the powerful tools of induction and complete induction to prove statements of the form.

$$\forall n \in \mathbb{N}.(P(n)).$$

However, in life, we are also interested in objects other than the natural numbers. For example, lists, trees, and logical formulas. I.e., we may want to prove statements like

$$\forall \mathrm{Trees}\, T.(P(T)),$$

and

$$\forall \mathrm{Formulas} f.(P(f)).$$

We "need"[3] a more general tool.

[3] the quotes here will be explained later

# Another view of $\mathbb{N}$

Here's another one way to define $\mathbb{N} = \{0, 1, 2, ...\}$.
Let $\mathrm{AddOne}$ be the function that maps $x \to x + 1$.

**Then, $\mathbb{N}$ is the set of objects can be reached by applying $\mathrm{AddOne}$ to $\{0\}$ a finite number of times.**

# Defining Sets Inductively

- Let $B \subseteq U$ (think $B$ for **b**ase cases)
- Let $F$ be a set of **f**unctions, where each function $f \in F$ has domain $U^m$ and codomain $U$. I.e. $f$ maps a tuple of elements of $U$ to a single element of $U$ (think of $F$ as a set of construction operations)

The set **generated** from $B$ by the functions in $F$ is the set of elements that can be obtained by applying functions in $F$ to elements of $B$ a finite number of times.

# Alternatively

An equivalent way to express

"$A$ is the set of elements that can be obtained by applying functions in $F$ to elements of $B$ a finite number of times."

is to define

1

2.

$A$ is the smallest set satisfying the following conditions.

- $B \subseteq A$
- $\forall a \in A, f \in F, f(a) \in A$.

# Example: $\mathbb{N}$

- $B = \{0\}$
- $F = \{\mathrm{AddOne}\}$

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2. Alternatively, $\mathbb{N}$ is the smallest set that contains 0, and for each $n \in \mathbb{N}$, $\mathbb{N}$ also contains $\mathrm{AddOne}(n)$.

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1. $\mathbb{Z}$ is the set generated from $\{0\}$ by $\{\mathrm{AddOne}, \mathrm{MinusOne}\}$
2. Alternatively, $\mathbb{Z}$ is the smallest set that contains 0, and for each $z \in \mathbb{Z}$, $\mathbb{Z}$ also contains $\mathrm{AddOne}(z)$, and $\mathrm{MinusOne}(z)$.

# Example: $\mathrm{List}[X]$

Let $X$ be some set, and let $\boxed{\mathrm{List}[X]}$ be the set of lists of elements in $X$.

$$B = \{[x] : x \in X\} \cup \{[\,]\}.$$

$$F = \{\text{concat}\}.$$

# Example: $\mathrm{List}[X]$

Let $X$ be some set, and let $\mathrm{List}[X]$ be the set of lists of elements in $X$.

For each $x \in X$ define the function $\mathrm{Append}_x$ be the function that takes in a $l$ and appends $x$ to $l$.

- $B = \{[]\}$
- $F = \{\mathrm{Append}_x : x \in X\}$

$\mathrm{List}[X]$ is the set generated from $B$ by functions in $F$.

# Propositional logic

no $P(x)$,

$\forall x$ . $\exists x$ .

Propositional logic is logic without predicates or quantifiers. For example $((A \wedge B) \vee (\neg C))$ is a propositional formula. Let $\mathrm{Prop}$ be the set of propositional formulas. Define $\mathrm{Prop}$ inductively.

Intuition for the base elements: they should be the simplest possible thing

$$\mathcal{B} = \{A, B, C \ldots\}$$

$$F = \{\mathcal{E}_\neg, \mathcal{E}_\wedge, \mathcal{E}_\vee\}.$$

$$\mathcal{E}_\neg(f) = (\neg f) \qquad \mathcal{E}_\wedge(f, g) = (f \wedge g)$$

# Propositional logic

Propositional logic is logic without predicates or quantifiers. For example $((A \wedge B) \vee (\neg C))$ is a propositional formula. Let $\mathrm{Prop}$ be the set of propositional formulas. Define $\mathrm{Prop}$ inductively.

- $B = \{A, B, C, ...\}$ be a set of variables
- $F = \{\mathbf{E}_\neg, \mathbf{E}_\wedge, \mathbf{E}_\vee\}$

Where $\mathbf{E}_\neg(A) = (\neg A)$, $\mathbf{E}_\wedge(A, B) = (A \wedge B)$, and $\mathbf{E}_\vee(A, B) = (A \vee B)$.

# Structural Induction

Let $C$ be a set generated from $B$ by the functions in $F$.

If

- for every $b \in B$, $P(b)$, Base case.
- and for every $f \in F$ on $m$ inputs, for every $a_1, \ldots, a_m \in C$,
  $(P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_m)) \implies P(f(a_1, \ldots, a_m))$

inductive step.

Then $\forall x \in C.(P(x))$

# Structural Induction in English

Let $P$ be any predicate.

- If I can show $P$ is true of all the base cases,
- and I can show that for every construction function, if $P$ holds for the the inputs to the construction function then $P$ must hold for the output of the construction function,

Then $P$ holds for every element constructed from the bases cases and the construction functions.

# Recovering regular induction

$\mathbb{N}$ is generated by $\{0\}$ and $\mathrm{AddOne}$. So substituting $\mathbb{N}$ for $C$, $\{0\}$ for $B$ and $\{\mathrm{AddOne}\}$ for $F$ in structural induction, we get

- for every $b \in \{0\}$, $P(b)$, this is just $P(0)$
- and for every $f \in \{\mathrm{AddOne}\}$ on $m$ inputs, for every $a_1, \ldots, a_m \in \mathbb{N}$,

$$(P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_m)) \implies P(f(a_1, \ldots, a_m))$$

this is just $\forall k \in \mathbb{N}.(P(k) \implies P(k+1))$

Then $\forall x \in \mathbb{N}.(P(x))$.

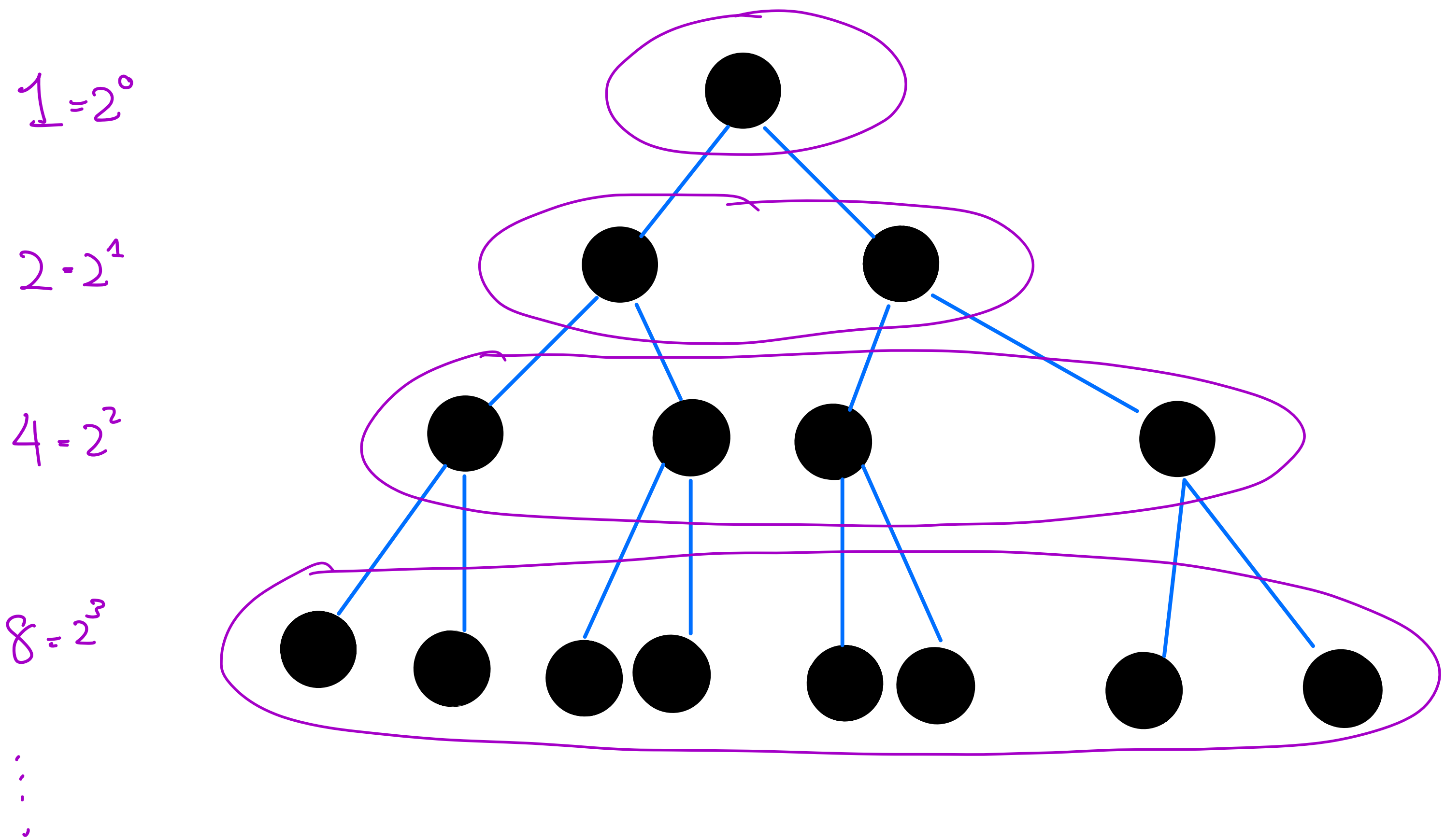
# Flexibility

This more general version formalizes the intuition for why were were able to change the base cases when trying to prove, for example, $\forall n \in \mathbb{N}, n \geq 4.(P(n))$.

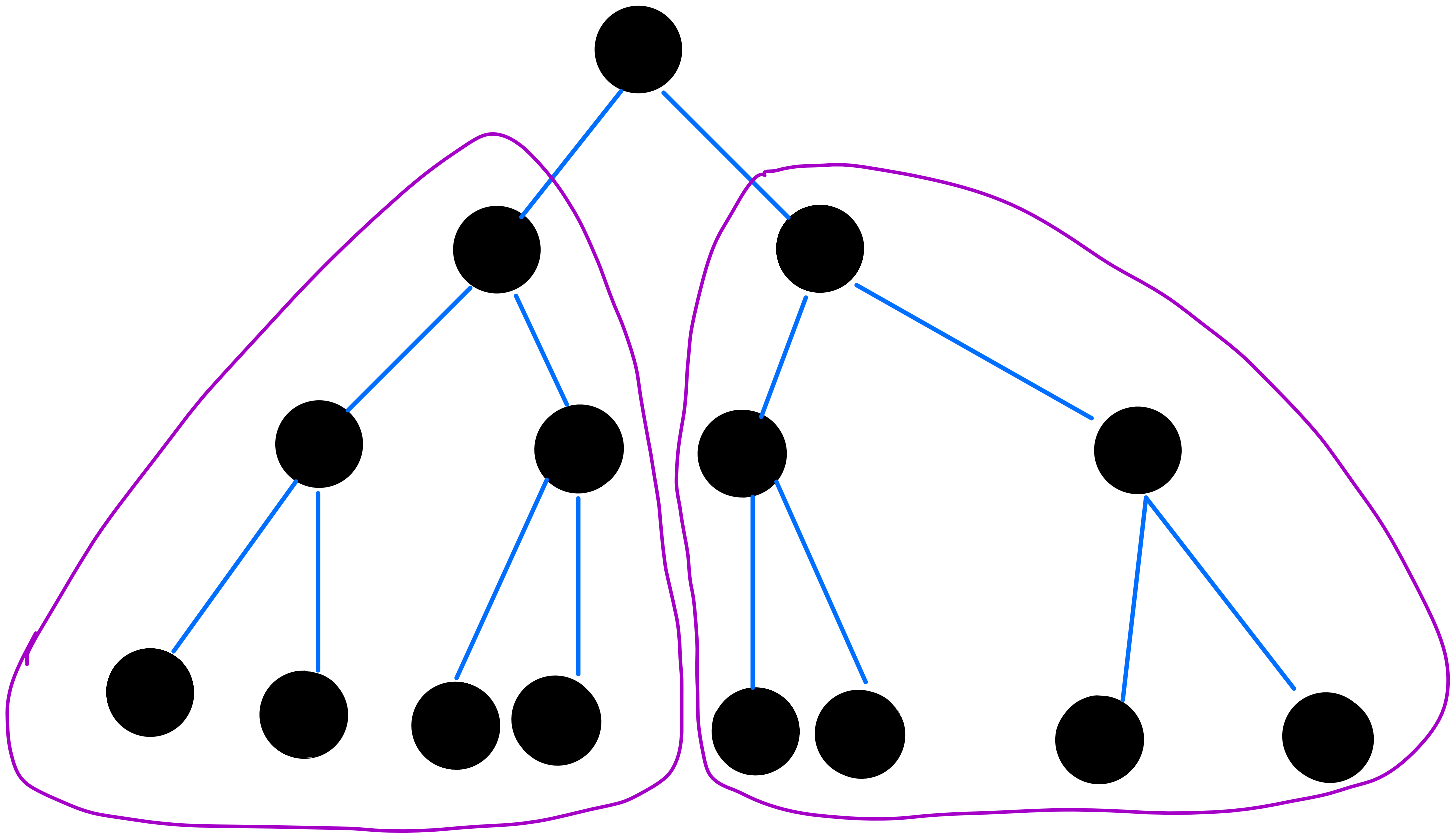
We really just showed $P$ holds for every number in the set $\mathbb{N}_{\geq 4} = \{4, 5, 6, ...\}$ which is generated from the singleton set $\{4\}$, and the function $\mathrm{AddOne}$.

# Perfect Binary Trees (Again)

Last time we showed a perfect binary tree of height $h$ has $2^{h+1} - 1$ vertices. By showing $2^0 + 2^1 + \ldots + 2^h = 2^{h+1} - 1$ for all $h \in \mathbb{N}$.



# An alternate way of looking at things



# Perfect Binary Trees

Let $\mathrm{PerfectBinaryTrees}$ be the set of perfect binary trees, and let's write it as being generated from a set by some function.

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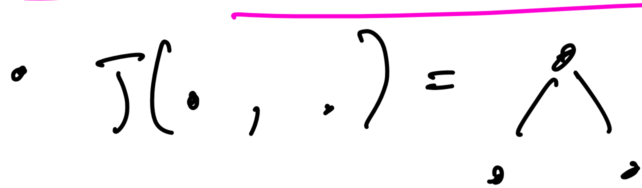
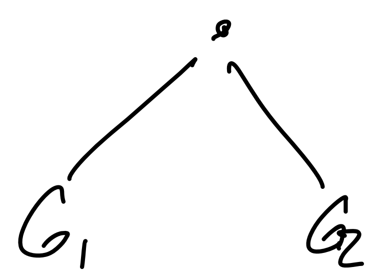
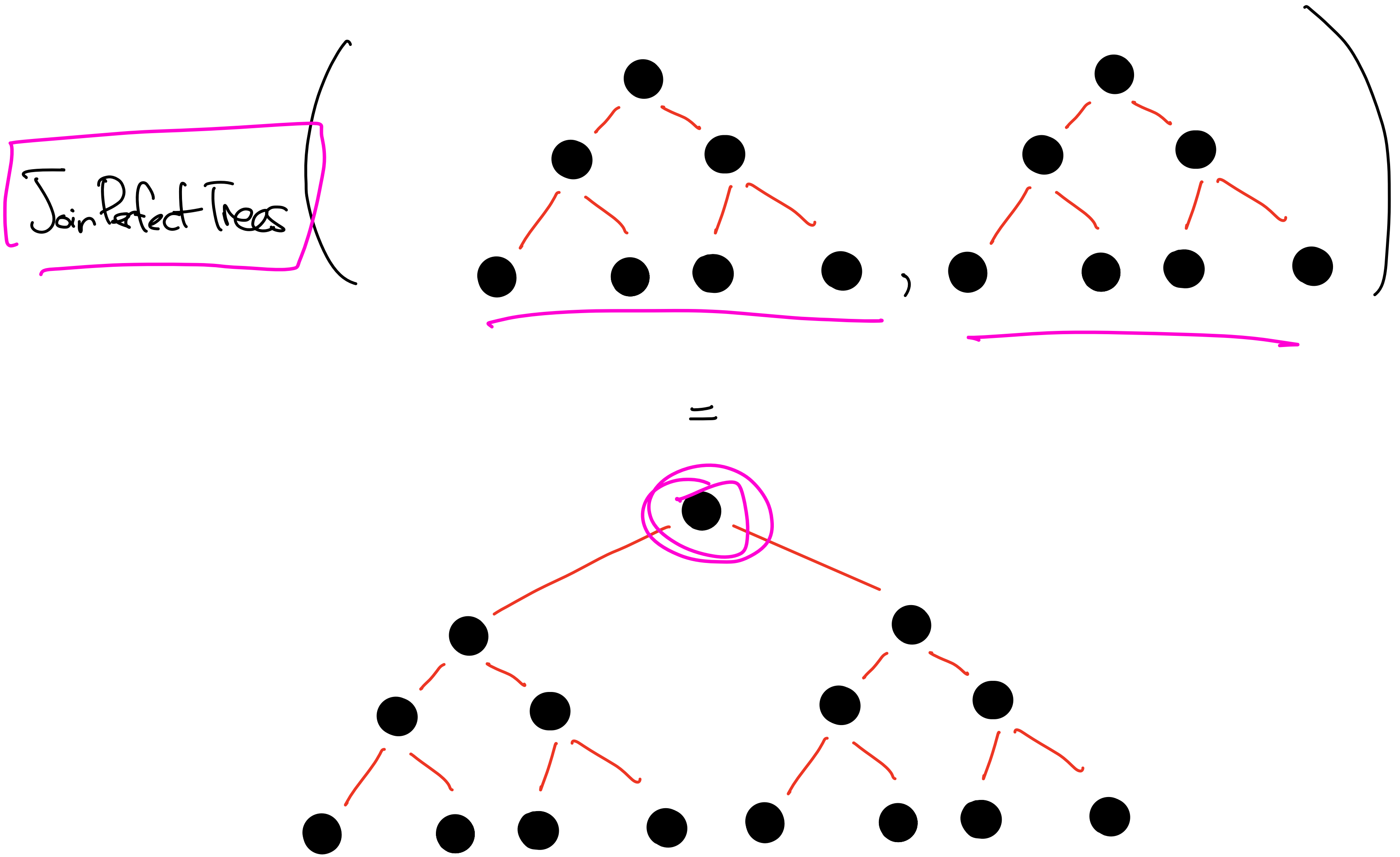
Let $\mathrm{PerfectBinaryTrees}$ be the set of perfect binary trees, and let's write it as being generated from a set by some function.

- $U$ (for example might be the set of all graphs).
- $B = \{\text{single node}\}$
- $\mathrm{JoinPerfectTrees} : U \times U \to U$ maps $\boxed{(G_1, G_2)}$ to the tree with $G_1$ as left subtree and $G_2$ as right subtree if and only if $G_1$ and $G_2$ are perfect binary trees of the same height. Otherwise, map to the graph with a single node.



# JoinPerfectTrees



# A perfect binary tree of height $h$ has $2^{h+1} - 1$ vertices

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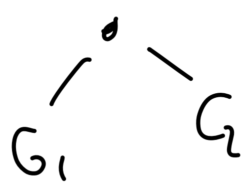
By structural ind.

Base case: $\bullet$ has $2^{0+1} - 1$ vertices ✓

Inductive step: WTS $\forall G_1, G_2$, if $\boxed{P(G_1), P(G_2)}$, then $P(J(G_1, G_2))$. If $G_1$ and $G_2$ have different heights, then $J(G_1, G_2) = \bullet$ and is covered by the base case. Else $G_1, G_2$ have the <u>same</u> height $k$. and $J(G_1, G_2)$ has height $k+1$




By IH: $G_1$ has $2^{k+1} - 1$ vertices, and so does $G_2$.

$\Rightarrow J(G_1, G_2)$ has $\underline{1 + 2(2^{k+1} - 1)} = 2^{k+2} - 1$ ∎

## Postage Stamps (Again)

$\mathbb{N}_{\geq 8}$ is generated from $\{8, 9, 10\}$ by $\{Add3\}$.

$\forall n \in \mathbb{N}_{\geq 8}$ $\quad n = 3a + 5b$. By structural Induction.

Base cases: $8 = 5 + 3$, $\quad 9 = 3 \cdot 3$, $\quad 10 = 5 \cdot 2$.

Inductive step: Let $n \in \mathbb{N}_{\geq 8}$. then WTS.

$$P(n) \Rightarrow P(Add3(n)). \quad P(n) \Rightarrow P(n+3).$$

$$P(n) \Rightarrow n = 3a + 5b.$$

$$n + 3 = 3(a+1) + 5b$$

# Structural vs. Complete Induction

If you prefer complete induction to structural induction, you can always opt to use complete induction instead. The following slides will detail why.

# Construction Sequences

Let $C$ be the set generated from $B$ by the functions in $F$. Define a construction sequence of length $n$, to be a sequence of elements $(x_0, .., x_n)$ where for each $x_i$ in the sequence, either

- $\boxed{x_i \in B,}$
- or $x_i = \boxed{f(x_{j_1}, ..., x_{j_m})}$ for some $f \in F$, and $j_1, ..., j_m < i$.

I.e., **every element in the sequence is either in the base set $B$ or is constructed by applying a construction function to earlier elements in the sequence.**

# Example - construction sequence

Construction Fns : $\mathcal{E}_{\neg}$ , $\mathcal{E}_{\wedge}$ , $\mathcal{E}_{\vee}$

$\mathcal{E}_{\neg}(x) = (\neg x)$ $\quad \mathcal{E}_{\wedge}(x,y) = (x \wedge y)$ $\quad \mathcal{E}_{\vee}(x,y) = (x \vee y)$

Construct: $\boxed{((A \vee B) \wedge (\neg (A \wedge B)))}$

$$\boxed{A,\ B,\ (A \vee B),\ (A \wedge B),\ (\neg (A \wedge B)),\ ((A \vee B) \wedge (\neg (A \wedge B))),}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & (A \vee B) \arrow[rr] & & ((A \vee B) \wedge (\neg (A \wedge B))) \\
B \arrow[ur] \arrow[r] & (A \wedge B) \arrow[r] & (\neg (A \wedge B)) \arrow[ur] &
\end{tikzcd}$$

# Structural vs. Complete Induction

Define $C_i$ be the set where $x \in C_i$ if there exists some construction sequence of length at most $i$ ending in $x$. Then $C = C_0 \cup C_1 \cup \ldots$.

Instead of doing structural induction, we can do induction on the length of the construction sequence. I.e., show that if $P$ holds for every element with construction sequences of at most $k$, then $P$ also holds for elements with construction sequences of length at most $k + 1$.

Usually, *length of construction sequence* is represented by some measure of complexity of the object, for example, 'height of a tree' or 'number of parenthesis,' or 'length of the list.'

# Perfect Binary Trees (again again)

$\forall h \in \mathbb{N}$, if $G$ has a PBT of height $h$, then, $G$ has $2^{h+1}-1$ vertices.

Base case: $\bullet$ has $2^{0+1}-1 = 1$ vertex.

Inductive step: Let $k \in \mathbb{N}$, suppose all PBT of height $i$ ~~have~~ where $0 \leq i \leq k$ have $2^{i+1}-1$ vertices. Consider a PBT of height $k+1$. This is just 2 PBTs of height $k$, joined by a single vertex. Thus, in total, it has

$$2(2^{k+1}-1)+1 = 2^{k+2}-1 \text{ vertices}.$$

# Level of Formality

So far, we have seen many examples of proof by induction. You can use any approach you wish.

You don't need to talk about construction sequences in your proofs and can instead say, for example, 'by induction on the height of the tree.'

Structural induction is usually trickier to get right, so I'd recommend sticking to complete/regular induction whenever possible. We present it here since

1. It allows us to introduce iterative/recursive definitions.
2. Its generality allows us to explain some variants of regular induction (e.g., why we can start at $n = 4$ if we want to.)

Induction

Structural Induction

Well Ordering Principle and Proof by Infinite Descent

# The Well Ordering Principle

Let $S \subseteq \mathbb{N}$ be a non-empty subset, $a$ is a minimal element of $S$ if $\forall b \in S.(a \leq b)$

The Well Ordering Principle states that for any non-empty subset $S \subseteq \mathbb{N}$, $S$ has a minimal element.

In particular, this is true even for infinite subsets.

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In particular, this is true even for infinite subsets.

Thoughts? Is this obvious?

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What if we replace $\mathbb{N}$ with $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$? Is it still true?

# Well Ordering Principle

Well Ordering Principle: For any non-empty subset $S \subseteq \mathbb{N}$, $S$ has a minimal element.

What if we replace $\mathbb{N}$ with $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$? Is it still true?
No, for example $\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ has no minimal element!

# Proofs using the Well Ordering Principle

The Well Ordering Principle also lets us prove statements of the form $\boxed{\forall n \in \mathbb{N}.(P(n))}$. Here's how:

- Check $P(0)$ is true.
- By contradiction, assume $\exists n \in \mathbb{N}.(\neg P(n))$. So the set $S = \{n \in \mathbb{N} : \neg P(n)\}$ is non-empty.
- By the Well Ordering Principle, $S$ has a minimal element, $m$ (i.e. $m$ is the smallest natural number for which $P$ doesn't hold.) Since we know $P(0)$, $m \geq 1$.
- Derive a contradiction by showing $P(m)$, or by finding a $m' < m$, for which $\neg P(m')$.

## For all $n \in \mathbb{N}$, $n \geq 2$. $n$ has a prime divisor

$\hookrightarrow \exists p$, prime. s.t $n = p \cdot k$
$k \in \mathbb{N}$

$P(2)$: 2 has a prime divisor, 2.

By contr. assume $\exists\, n \geq 2$, s.t. $n$ does not have a prime divisor. $\Rightarrow S = \{n : \neg P(n)\}$, is non-empty. By WOP. $S$ has a minimal element $m$.

$m$ can't be prime since o/w it is its own prime divisor.

$\Rightarrow m = ab$ for some $a, b < m$. Since $m$ is minimal, $a, b \notin S$,
$\Rightarrow$ $a$ and $b$ have prime divisors. but this implies $m$ has prime divisors. $\Rightarrow\Leftarrow$.

# $\sqrt{2}$ is irrational (a classic)

$\forall n \in \mathbb{N}, \neg \exists m \in \mathbb{N}$ s.t. $n/m = \sqrt{2}$. $P(0)$, b/c $0/\text{anything} = 0 \neq \sqrt{2}$.

$P(n): \neg \exists m \in \mathbb{N}$ s.t. $n/m = \sqrt{2}$. WTS $\forall n\, P(n)$.

By contr. suppose $\exists n\ \neg P(n)$. $\Rightarrow S = \{n : \neg P(n)\}$ is non-empty. $\xRightarrow{\text{WOP}}$ $\exists$ a minimal element $x$ s.t. $\neg P(x)$. $\Rightarrow \exists y$ s.t. $\boxed{x/y = \sqrt{2}}$. $\Rightarrow$

$x^2 = 2y^2 \Rightarrow x$ is even $\Rightarrow x = 2z$. $\Rightarrow (2z)^2 = 2y^2$

$\Rightarrow 4z^2 = 2y^2 \Rightarrow 2z^2 = y^2 \Rightarrow y$ is even $\Rightarrow y = 2w$.

$\sqrt{2} = x/y = \frac{2z}{2w} = \underline{\frac{z}{w}}$ but $z = x/2$, $\boxed{\text{so } z < x}$ true since $x > 0$

$\Rightarrow \Leftarrow$ since this means $\neg P(z)$, so $z \in S$ but $z < x$.

# Induction in disguise

Let's take another look at the complete induction. We want to show that $P(0)$ and

$$\boxed{(\forall k \in \mathbb{N}.(P(0), ..., P(k))) \implies P(k+1)}$$

Usually, we prove the inductive step directly by picking an arbitrary $k \in \mathbb{N}$ and assuming $P(0) \wedge ... \wedge P(k)$, and then showing $P(k+1)$.

If we instead chose to do it by contradiction, it might look like this. Let $k \in \mathbb{N}$ be any natural number, and assume $P(0) \wedge ... \wedge P(k)$, by contradiction, assume $\neg P(k+1)$. At this point, our assumptions are

$$P(0) \wedge ... \wedge P(k) \wedge \neg P(k+1).$$

# Induction in disguise

But

$$P(0) \wedge ... \wedge P(k) \wedge \neg P(k+1)$$

is exactly what it means for $k+1$ to be the minimal element of the set $S = \{n \in \mathbb{N} : \neg P(n)\}$.

Thus, proving the inductive step for complete induction by contradiction amounts to finding a contradiction by assuming there was a minimal element of the set $S = \{n \in \mathbb{N} : \neg P(n)\}$, which is exactly the same as what we'd do in a proof using the WOP.

# Additional notes

- This presentation of structural induction loosely follows the one in *A Mathematical Introduction to Logic* by Herbert Enderton. So check that out as supplementary reading.
- Our approach for proving a mathematical statement using the Well Ordering Principle is sometimes called ‘proof by infinite descent’. Read all about that here.

# A trusty toolkit