Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the technique of simple induction.
- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

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1. **Induction**. Consider the following statement:

$$\forall n \in \mathbb{N}, n \leq 2^n$$

(a) Suppose we want to prove this statement using induction. Write down the full statement we'll prove (it should be an **AND** of the base case and induction step). Consult your notes if you aren't sure about this!

Solution

$0 \leq 2^0 \land \left(\forall k \in \mathbb{N}, k \leq 2^k \Rightarrow k + 1 \leq 2^{k+1}\right)$

(b) Prove the statement using induction. We strongly recommend reviewing the induction proof template from lecture before working on a proof here.
Hint: $2^{k+1} = 2^k + 2^k$.

Solution

Proof. We will prove this statement using induction on n.

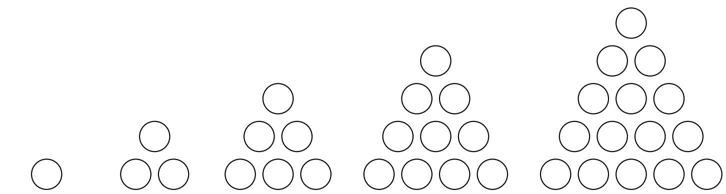
Base case: let $n = 0$.
Then $2^n = 1$, and $n = 0$, so $n \leq 2^n$.

Induction step: let $k \in \mathbb{N}$, and assume that $k \leq 2^k$. We want to prove that $k + 1 \leq 2^{k+1}$.
Since $0 \leq k$, we know that $1 \leq 2^k$ (raising 2 to the power of either side). Then we can add this inequality to our assumption $k \leq 2^k$ to get:

$$k + 1 \leq 2^k + 2^k$$
$$k + 1 \leq 2^{k+1}$$

□

2. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with n marbles on each side, a total of $\sum_{i=1}^{n} i$ marbles will be required. (For convenience, we also define $T_0 = 0$.)



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$ $T_5 = 15$

In the course notes, we prove that $\sum_{i=1}^{n} i = n(n+1)/2$. For each $n \in \mathbb{N}$, let $T_n = n(n+1)/2$; these numbers are usually called the *triangular numbers*. Use induction to prove that

$$\forall n \in \mathbb{N},\ \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$$

Solution

Let us start by defining the predicate

$$P(n) : \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$$

We need to prove that $\forall n \in \mathbb{N}, P(n)$.

Proof. **Base case**: let $n = 0$. We want to prove $P(0)$. Then we can calculate:

$$\begin{aligned} \sum_{j=0}^{n} T_j &= \sum_{j=0}^{0} T_j \\ &= T_0 \\ &= \frac{0(0+1)}{2} \\ &= 0 \end{aligned}$$

And also $\frac{0(0+1)(0+2)}{6} = 0$.

Induction step: Let $k \in \mathbb{N}$ and assume $P(k)$, i.e., that $\sum_{j=0}^{k} T_j = k(k+1)(k+2)/6$. We want to prove $P(k+1)$, i.e., that $\sum_{j=0}^{k+1} T_j = (k+1)(k+2)(k+3)/6$.

We'll calculate starting from the left side and show that it equals the right side.

$$\begin{aligned}
\sum_{j=0}^{k+1} T_j &= \left(\sum_{j=0}^{k} T_j\right) + T_{k+1} && \text{(pulling out the last term)} \\
&= \frac{k(k+1)(k+2)}{6} + T_{k+1} && \text{(by the I.H.)} \\
&= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} && \text{(by the definition of } T_{k+1}\text{)} \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6} \\
&= \frac{(k+1)(k+2)(k+3)}{6}
\end{aligned}$$

□

3. **Induction (inequalities).** Consider the statement:

For every positive real number x and every natural number n, $(1+x)^n \geq (1+nx)$.

We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ (1+x)^n \geq 1+nx$$

Notice that in this statement, there are two universally-quantified variables: n and x.[1] Prove this statement is True using the following approach:

(a) Use the standard proof structure to introduce x.

(b) When proving the $\big(\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx\big)$, do induction on n.[2]

Solution

Proof. Let $x \in \mathbb{R}^+$. We'll prove that for all $n \in \mathbb{N}, (1+x)^n \geq 1+nx$ by induction.

Base case: Let $n = 0$.
Then $(1+x)^n = 1$ and $1+nx = 1$. So then $(1+x)^n \geq 1+nx$.

Induction step: Let $k \in \mathbb{N}$, and assume that $(1+x)^k \geq 1+kx$. We want to prove that $(1+x)^{k+1} \geq 1+(k+1)x$.

We'll start with the quantity on the left, and show that it's $\geq$ the quantity on the right.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) && \text{(by the I.H.)} \\
&= 1+kx+x+kx^2 \\
&\geq 1+kx+x && \text{(since } kx^2 \geq 0\text{)} \\
&= 1+(k+1)x
\end{aligned}$$

□

[1] For extra practice, think about the following questions. First, would the statement still be True with the order of the quantifiers reversed: $\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^+,\ (1+x)^n \geq 1+nx$? Second, if this variation is correct, how would this change the proof?

[2] Your predicate $P(n)$ that you want to prove will also contain the variable x—that's okay, since when we do the induction proof, x has already been defined.

4. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N},\ n \leq 2^n$ using induction.

   (a) First, use trial and error to fill in the blank to make the following statement true—try finding the *smallest natural number* that works!

   $$\forall n \in \mathbb{N}, n \geq \underline{\hspace{3cm}} \Rightarrow 30n \leq 2^n$$

   Solution

   $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n$.

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!

   Solution

   Proof. **Base case**: Let $n = 8$.
   Then $30n = 240$, and $2^n = 256$. So $30n \leq 2^n$.
   Induction step: Let $k \in \mathbb{N}$. Assume that $k \geq 8$, and that $30k \leq 2^k$. We want to prove that $30(k+1) \leq 2^{k+1}$.
   Since $8 \leq k$, we know that $256 \leq 2^k$ (raising 2 to the power of either side). The induction hypothesis tells us that $30k \leq 2^k$. Adding these two inequalities yields:

   $$\begin{aligned} 30k + 256 &\leq 2^k + 2^k \\ 30k + 256 &\leq 2^{k+1} \\ 30k + 30 &\leq 2^{k+1} && \text{(since } 30 \leq 256\text{)} \\ 30(k+1) &\leq 2^{k+1} \end{aligned}$$

   $\square$