# CSC263H1

## PROBABILITY REVIEW – Sample Solutions

Winter 2024

Work on this problem set right after the first lecture in CSC263. It is based on your Statistics prerequisite material, and you will need this knowledge in CSC263. If you do not feel comfortable with these questions, we *strongly* encourage you to go through this chapter and exercises in their entirety: `http://jeffe.cs.illinois.edu/teaching/algorithms/notes/01-random.pdf`.

1. Two cards are picked uniformly randomly from a deck.

   (a) What is the probability that the first card is a Queen of either Spades(♠) or Clubs(♣)?
   **Solution:** 2/52 because there are 2 accepted cards out of 52 possible cards.

   (b) What is the probability that the first card is either a Queen, or its suit is Spades?
   **Solution:** 16/52. There are 4 Queens and 13 Spades. Since Queen and Spades overlap on Queen of Spade, and the number of accepted cards are 13 + 4 - 1 = 16.

   (c) What is the probability that the first card is Spades?
   **Solution:** 13/52 because there are 13 Spades out of 52 possible cards.

2. **Solution:** 1/6. Let $s_9$ be the sum of the first 9 dice and $d_{10}$ be the result from rolling the last dice. The conditional probability of the event "$(s_9 + d_{10})$ **is divisible by** 6 given a fixed value of $s_9$" is $\frac{1}{6}$ for every possible value of $s_9$. For example, if the first 9 rolls are all 3s, then $s_9 = 3 * 9 = 27$. $s_9 + d_{10}$ is divisible by 6 only if the last roll ($d_{10}$) is 3.

   The probability that $s_9 + d_{10}$ **is divisible by** 6 is:

   $$\sum_{i=9}^{54} P(s_9 + d_{10} \textbf{ is divisible by 6} \mid s_9 = i) P(s_9 = i)$$

   Since $P(s_9 + d_{10} \textbf{ is divisible by 6} \mid s_9 = i) = 1/6$ for every $i$, we conclude:

   $$\sum_{i=9}^{54} P(s_9 + d_{10} \textbf{ is divisible by 6} \mid s_9 = i) P(s_9 = i)$$

   $$= 1/6 * \sum_{i=9}^{54} P(s_9 = i) = 1/6 * 1 = 1/6$$

3. You are given a fair coin.

   (a) **Solution:** We can calculate the expected value directly using:

   $$E[X] = \sum_{i=1}^{\infty} P(X = i)X = \sum_{i=1}^{\infty} P(T)^{i-1} P(H) i = \sum_{i=1}^{\infty} 0.5^i i = 2$$

   Note that $P(X = i)$ is the probability that the $i$th toss is a head, and the 1st - ith tosses are tails. Therefore, $P(X = i) = P(T)^{i-1}P(H)$. Since the coin is fair, $P(T) = P(H) = 0.5$. Therefore, $P(T)^{i-1}P(H) = 0.5^i$.

(b) **Solution:** There are three cases:

i. if the first toss is tails with probability $\frac{1}{2}$, then the new expectation is $1 + E[X]$ since we face the original problem again after the first toss.

ii. if the first toss is heads and the second toss is tails with probability $\frac{1}{4}$, then the new expectation is $2 + E[X]$ since we face the original problem again after the first two tosses.

iii. if the first toss is heads and the second toss is heads with probability $\frac{1}{4}$, then the new expectation is 2 since we have achieved the goal with 2 tosses.

Combining all three cases, we can derive the following equation for the expected value: $E[X] = \frac{1}{2}(E[X] + 1) + \frac{1}{4}(E[X] + 2) + \frac{1}{4} \times 2$. Therefore, $E[X] = 6$.

4. Consider the following probability distribution for chosen fruits during grocery shopping.

(a) **Solution:** $25578 \times 0.22 + 500 \times 0.38 + 125 \times 0.1 + 654 \times 0.14 + 123 \times 0.16 = 5940.9$

(b) **Solution:** Let $S_i$ be an indicator random variable such that

$$S_i = \begin{cases} 1 & \text{if a re-stack is required after adding the } i\text{th fruit} \\ 0 & \text{otherwise} \end{cases}$$

The expected number of re-stacks can be expressed as $E[S] = \sum_{i=1}^{n} E[S_i]$. Since $S_i$ is an indicator random variable, $E[S_i] = P(S_i)$. The probability of $S_i$ is given by the following equation:

$$P(S_i) = \begin{cases} P(watermelon)(1 - P(watermelon)^{i-1}) & \text{if } i \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $E[X] = \sum_{i=2}^{n} 0.22 * (1 - 0.22^{i-1}) = 0.22(n-1) - \sum_{i=2}^{n} 0.22^i$

| Fruit | Watermelon | Grape | Apple | Strawberry | Blueberry |
|---|---|---|---|---|---|
| Weight (g) | 25578 | 500 | 125 | 654 | 123 |
| Probability | 0.22 | 0.38 | 0.1 | 0.14 | 0.16 |