UNIVERSITY OF TORONTO
Faculty of Arts & Science

CSC236H1 Y Midterm

Date: July 5th, 2023

Instructor: Harry Sha

Duration: 3hrs

Aids Allowed: None

*Do **not** turn this page until you have received the signal to start.*

In the meantime, please write your full name and student number below—**please do this right now!**—and *carefully* read *all* the information on the rest of this page.

Given Name(s):

Family Name(s):

Student Number (9 or 10 digits):

UTORid (e.g., `shaharry`):

- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. *If it is left in your pocket, it may be an academic offence.*
- When you are done your exam, raise your hand for someone to come and collect your exam. *Do not collect your bag and jacket before your exam is handed in.*
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- This exam consists of 16 pages (including this one), printed on both sides of the paper. There are 36 points, and 2 points for extra credit. *When you receive the signal to start, please make sure that your copy of the examination is complete.*
- Answer each question directly on the examination paper. If you require more space, indicate where you continued your answer. For example, "continued on Extra Space 1", or "continued on page 16"
- Good luck, you got this!

STUDENTS MUST HAND IN ALL EXAMINATION MATERIALS AT THE END

1 Propositional Formulas

Recall that a **propositional formula** is built up of **propositional variables** (letters that represent arbitrary **propositions**, statements that are either True or False) using **propositional connectives** like negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\implies$), etc. Also recall that two propositional formulae ϕ_1, ϕ_2 are said to be **logically equivalent** (denoted $\phi_1 \equiv \phi_2$) when they have the same value for every assignment of True/False to their variables.

Let B be the set of propositional variables. Define the following functions $\mathbf{E}_\neg, \mathbf{E}_\vee, \mathbf{E}_\implies$ as follows,

- $\mathbf{E}_\neg(f) = \neg f$
- $\mathbf{E}_\vee(f, g) = f \vee g$
- $\mathbf{E}_\implies(f, g) = f \implies g$

Let G be the set generated from B by the functions $\{\mathbf{E}_\neg, \mathbf{E}_\vee, \mathbf{E}_\implies\}$.

As a reminder, $f \implies g$ False when f is True and g is False, and True otherwise.

Question 1a (4 points). Prove that for every $f \in G$, there exists $f' \in G$ such that f and f' are logically equivalent, and f' does not contain the $\vee$ symbol. Note that this is very similar to a problem you had in HW3.

2 Roommate Matching

Suppose you're in charge of assigning roommate pairs for the incoming class of students in a dorm room. The students have filled out a survey with the following three questions

1. When are you typically asleep: ____-____.
2. Rate your cleanliness out of 5: ____/5.
3. Circle your interests from the following list
   - Sports
   - Music
   - ...

You would like to assign roommate pairs such that everyone has a roommate, and pairs have similar living preferences, and a high overlap in hobbies.

Question 2a (4 points). Model the problem of assigning roommate pairs as one of the graph problems that we studied (shortest path, traveling salesman, minimum spanning tree, matching, independent set). For full credit, give a full definition of the vertices and edges, and weights (if any) of the graph, and an explanation of why finding good roommate pairing is equivalent to solving the graph problem.

Note there might be some flexibility in your model, we'll give credit to anything that makes sense.

Question 2b (2 points). Suppose instead of selecting roommate pairs with a high overlap in hobbies, you want to select roommate pairs with a large diversity of hobbies (but still with compatible living preferences), how do you change your model?

3 Recurrences

Use any method to solve the following recurrences. By 'solve the recurrence', I mean find f such that $T(n) = \Theta(f)$.

Question 3a (4 points).

$$T(n) = 2T(n/5) + T(2n/5) + n$$

4 Induction

Define the function $f : \mathbb{N}_{\geq 1} \to \mathbb{N}$ recursively as follows

$$f(n) = \begin{cases} 0 & n = 1 \\ 3 & n = 2 \\ 2f(n-1) - f(n-2) + 2 & n > 2 \end{cases}$$

Your friend is tasked with showing the following: $\forall n \in \mathbb{N}, n \geq 1.(f(n) \leq n^2)$.

They proposes the following (incorrect) proof.

Let $P(n)$ be the predicate that is true when $f(n) \leq n^2$. By complete induction.

Base case. We'll show $P(1)$, $P(2)$. For $n = 1$, $1^2 = 1$, and $f(1) = 0$, and $0 \leq 1$. For $n = 2$, $2^2 = 4$, and $f(2) = 3$, and $3 \leq 4$. Thus, the base cases hold.

Inductive step. Let $k \in \mathbb{N}$ with $k \geq 2$, and suppose for the inductive hypothesis that for all $i \in \mathbb{N}$ with $1 \leq i \leq k$, $P(i)$. We'll show $P(k+1)$.

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) + 2 \\ &\leq 2k^2 - (k-1)^2 + 2 && \text{(IH)} \\ &= 2k^2 - (k^2 - 2k + 1) + 2 \\ &= k^2 + 2k - 1 + 2 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

This completes the induction

Question 4a (2 points). Point out the invalid step in the proof and explain why it is invalid.

It turns out that although the above proof is invalid - the result is still true. I.e. it is true that $\forall n \in \mathbb{N}, n \geq 1. f(n) \leq n^2$.

Question 4b (4 points). Prove $\forall n \in \mathbb{N}, n \geq 1. f(n) \leq n^2$.

5 Sandwich

- Let $\mathrm{Chars} = \{a, b, c, ..., z\}$ and use ϵ to denote the empty string (which is "" in Python).
- Let $\mathrm{Strings} = \{\epsilon, a, b, ..., aa, ab, ...\}$ be the set of all strings over the alphabet Chars. I.e. it is the set of sequences of 0 or more characters in Chars.
- Let $f : \mathrm{Strings} \times \mathrm{Strings} \rightarrow \mathrm{Strings}$ be the function that maps $(x, y) \rightarrow xyx$. For example,

$$f(\text{'bread'}, \text{'peanutbutter'}) = \text{'breadpeanutbutterbread'}.$$

Question 5a (2 points). Is f injective? Prove your answer

Question 5b (1 points). Is f surjective? Prove your answer

Let $B = \mathrm{Chars} \cup \{\epsilon\}$, and let X be the set generated from B by $\{f\}$.

Question 5c (4 points). Prove that $\forall w \in X$, at most 1 character in Chars appears an odd number of times.

- Let $\text{Reverse} : \text{Strings} \to \text{Strings}$ be the function that maps a string to its reverse. For example,

$$\text{Reverse}(\text{'hello'}) = \text{'olleh'}.$$

- Let $P = \{w \in \text{Strings} : w = \text{Reverse}(w)\}$. Note that this is the set of Palindromes.

Question 5d (5 points). Show that $X = P$.

Question 5e (4 points). Suppose we changed the domain and codomain of f. Each row of the table below represents one possiblility. For a given row, write a ✓ in the injective column if the function is injective on that domain/codomain, and a ✗ otherwise. No justification is required.

Domain	**Codomain**	**Injective?**	**Surjective?**
$P \times P$	P		
$\text{Chars} \times \text{Strings}$	Strings		
$\text{Chars} \cup \{\epsilon\} \times \text{Strings}$	Strings		
$P \times \text{Chars} \cup \{\epsilon\}$	P		

6 Extra Credit: NOR

Let $\mathbf{E}_\neg, \mathbf{E}_\vee, \mathbf{E}_{\implies}$ be as defined in the Problem 1. Define the NOR connective such that $f \mathsf{~NOR~} g \iff \neg(f \vee g)$. That is $f \mathsf{~NOR~} G$ is True precisely when both f and g are False. Similarly, define $\mathbf{E}_{\mathsf{NOR}}$ to be the function that maps two propositional formulas f and g to $f \mathsf{~NOR~} g$.

Let B be the set of propositional variables.

Let G be the set of generated from B by $\{\mathbf{E}_\neg, \mathbf{E}_\vee, \mathbf{E}_{\implies}, \mathbf{E}_{\mathsf{NOR}}\}$

Question 6a (2 points). Prove that for every $f \in G$, there exists $f' \in G$ such that f and f' are logically equivalent, and f' contains only NOR, i.e. f' does not contain any of $\vee$, $\implies$, $\neg$.

Extra Space 1.

Extra Space 2.

Extra Space 3.

Extra Space 4.