Learning Objectives

By the end of this worksheet, you will:

- Prove statements about sets using induction.

So far, we have only used induction to prove statements about numbers. In CSC236, you'll see many more applications of the principle of induction beyond just numbers, but in this problem session you'll get a taste of what's to come by learning how to use induction on another familiar entity: sets.

1. **A first example.** Consider the following statement: "Every finite set S has exactly $\dfrac{|S|(|S|-1)}{2}$ subsets of size 2."[1]

   This is a universally-quantified statement, but given an arbitrary set S, it isn't obvious how to make an argument for its number of subsets. It isn't obvious how to use induction either! After all, we only know (right now) how to apply induction to prove statements about the natural numbers.

   Here's the big insight into how to apply induction to (finite) sets: every set has a simple corresponding natural number, its *size*. We can rewrite the initial statement to emphasize the size of the sets, as

   $$\text{For every } n \in \mathbb{N}, \text{ every set of size } n \text{ has } \frac{n(n-1)}{2} \text{ subsets of size 2.}$$

   If we define the predicate $P(n)$: "every set S of size n has $\dfrac{n(n-1)}{2}$ subsets of size 2," then this statement is exactly the kind of statement we can prove using induction!

   (a) Check your understanding: write down, in English, what $P(0)$ means. (This is the base case of the induction proof.)

   > **Solution**
   >
   > Every set S of size 0 has 0 subsets of size 2.

   (b) Prove $P(0)$. We have started the proof for you.

   Proof. **Base case**: Let $n = 0$. Let S be an arbitrary set, and assume S has size 0.

   > **Solution**
   >
   > We want to prove that S has 0 subsets of size 2.
   >
   > Since S has no elements, it doesn't have any subsets of size 2 (in fact, its only subset is itself, which has size 0).

   $\square$

   (c) Now we'll prove the induction step, in a series of parts. Please read through the following and complete the proof.

   Proof. **Induction step**. Let $k \in \mathbb{N}$, and assume $P(k)$. That is, assume that:
   [*Write down, in English, the induction hypothesis (what $P(k)$ means).*]

   > **Solution**
   >
   > Every set S of size k has exactly $\dfrac{k(k-1)}{2}$ subsets of size 2.

   We want to prove $P(k+1)$, that is,
   [*Write down, in English, what you want to prove (what $P(k+1)$ means).*]

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[1]For example, a set containing 4 elements has $\dfrac{4 \cdot 3}{2} = 6$ subsets of size 2.

> **Solution**
>
> Every set S of size $k + 1$ has exactly $\dfrac{(k+1)k}{2}$ subsets of size 2.

Let S be an arbitrary set, assume S has size $k + 1$, and let the elements in the set be $\{s_1, s_2, \ldots, s_k, s_{k+1}\}$. The **key idea** of most proofs by induction on sets is that you can take a set of size $k + 1$, and split it up into a single element and another set of size k.[2] Let $S' = \{s_1, s_2, \ldots, s_k\}$, so that $S = S' \cup \{s_{k+1}\}$. Note that S' has size k. Now, we'll treat s_{k+1} as "special," and use it to divide up the possible subsets of S in a really nice way!

Part 1: counting subsets of S of size 2 that contain s_{k+1}.

> **Solution**
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> We'll prove that the number of subsets of S of size 2 that contain s_{k+1} is exactly k.
>
> Every subset of S of size 2 that contains s_{k+1} must contain exactly one element from S'; there are k choices of elements from S' (since $|S'| = k$), and so k subsets of S of size 2 that contain s_{k+1}.

Part 2: counting subsets of S of size 2 that don't contain s_{k+1}.

[*Use the induction hypothesis to determine the number of subsets of S of size 2 that don't contain s_{k+1}.*][3]

> **Solution**
>
> Every subset of S of size 2 that doesn't contain s_{k+1} must contain 2 elements from $\{s_1, \ldots, s_k\}$. That is, these subsets are exactly the subsets of S' of size 2. Since S' has size k, the induction hypothesis tells us that S' has exactly $\dfrac{k(k-1)}{2}$ subsets of size 2.

Part 3: putting the counts together.
[*Finish off this proof by adding up your results from Parts 1 and 2.*]

> **Solution**
>
> By combining the two counts from Parts 1 and 2, the total number of subsets of size 2 of S is
>
> $$k + \frac{k(k-1)}{2} = \frac{2k + k(k-1)}{2} = \frac{k(k+1)}{2}$$

$\square$

2. **Extending your work.** Use the technique presented in the previous question to prove the following statement: "Every finite set S has exactly $\dfrac{|S|(|S|-1)(|S|-2)}{6}$ subsets of size 3." You can (and should) use the statement you used in the previous question as an external fact in your proof.

> **Solution**
>
> The only difference between this proof and the previous one is that here, the subsets of size 3 that contain the "last" element s_{k+1} correspond to the subsets of **size 2** from a set of size k. That's why you need to use what you proved in the previous question to count the number of subsets of size 3 in this part.

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[2]This is the exact same idea as taking a summation with range 1 to $(k+1)$, and splitting it up into a summation with range 1 to k, plus the $(k+1)$-st term.

[3]Note that the induction hypothesis only applies to sets of size k; do you have a set of size k in this proof to work with?

3. **Counting all subsets.** We'll wrap up this worksheet by counting one more quantity: *all* the subsets of a given set. Recall that for a set S, the *power set of S*, denoted $\mathcal{P}(S)$, is the set of all subsets of S.[4] For example,

$$\mathcal{P}(\{1,2\}) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$$

Your goal for this question is to prove the following statement: "For every natural number n, every set S of size n satisfies $|\mathcal{P}(S)| = 2^n$."

(a) (Warm-up) Before trying to prove this directly, let's try out a small example to gain some intuition about why this statement is true. Consider a set $S = \{1,2,3\}$, with size 3. The above statement predicts that S will have **eight** subsets.

In the space below, write down the subsets of S in two groups:

- The subsets of S that contain the element 3.
- The subsets of S that do not contain the element 3.

(Can you find a relationship between the two groups?)

> **Solution**
>
> Contain 3: $\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.
> Do not contain 3: $\{\}, \{1\}, \{2\}, \{1,2\}$.

(b) Prove the above statement using induction. Clearly define the predicate $P(n)$ that is relevant to this statement (this is to help you make sure you understand exactly what it is you're proving).

> **Solution**
>
> The proof is actually quite similar to the previous two proofs, with the major difference being that the number of subsets that contain the last element s_{k+1} is 2^k, and the number of subsets that do not contain the last element is also 2^k (and $2^k + 2^k = 2^{k+1}$).
>
> Please consult the Course Notes, page 76-77, for a complete proof.

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[4]Remember that the empty set is a subset of every set, and every set is also a subset of itself.