## Last time...

Iterative Correctness

- Loop Invariants
- Initialization, Maintenance, Termination

# CSC 236 Lecture 9: Formal Language Theory 1

Harry Sha

July 19, 2023

## Today

Midterm Debrief

Formal languages

Deterministic Finite Automata

Designing DFAs

Non-Deterministic Finite Automata (NFAs)

Regular Languages

## Quick Thoughts

In general...

- Strengths: Proofs by induction were very good. The Master Theorem was applied well (except for the root heavy case)
- Areas for Improvement: Injective/Surjective/Bijective, root heavy case in the Master Theorem.

## Reference

This lecture loosely follows chapter 1 of *Introduction to the Theory of Computation* by Michael Sipser.

The presentation there is a little more formal, but we'll use the same notation so it might be useful to check that out for additional examples.

## Problems

In lecture 1 we used Cantor's Theorem to show that there are problems that can't be solved. In that lecture, we defined a problem for every set.

If $A$ is a set of strings, there is the problem problem of deciding whether a given input is in $A$ or not.

Examples:

$$A = \{w \in \mathrm{Strings} : w \text{ is a palindrome}\},$$

$$A = \{w : w \text{ is a C program with no syntax errors}\}.$$

## Formal Languages

The problems that we consider are still going to be of this type!

This time, we'll make things a little more formal...

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.
- A string $w$ over an alphabet $\Sigma$ is a finite (0 or more) sequence of symbols from $\Sigma$.

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.
- A string $w$ over an alphabet $\Sigma$ is a finite (0 or more) sequence of symbols from $\Sigma$.
- The set of all strings is denoted $\Sigma^*$. (Before now, we've been calling it $\mathrm{Strings}$).

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.
- A string $w$ over an alphabet $\Sigma$ is a finite (0 or more) sequence of symbols from $\Sigma$.
- The set of all strings is denoted $\Sigma^*$. (Before now, we've been calling it $\mathrm{Strings}$).
- A language is any subset of $\Sigma^*$.

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.
- A string $w$ over an alphabet $\Sigma$ is a finite (0 or more) sequence of symbols from $\Sigma$.
- The set of all strings is denoted $\Sigma^*$. (Before now, we've been calling it $\mathrm{Strings}$).
- A language is any subset of $\Sigma^*$.
- Denote the empty string by $\epsilon$ and note that it is the unique string with length 0.

## Definitions

- An alphabet, $\Sigma$ is a non-empty, finite, set of symbols.
- A string $w$ over an alphabet $\Sigma$ is a finite (0 or more) sequence of symbols from $\Sigma$.
- The set of all strings is denoted $\Sigma^*$. (Before now, we've been calling it $\mathrm{Strings}$).
- A language is any subset of $\Sigma^*$.
- Denote the empty string by $\epsilon$ and note that it is the unique string with length 0.

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx\ldots x}_{n \text{ times}}$.

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx \ldots x}_{n \text{ times}}$.
- What is $x^0$?

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx\ldots x}_{n \text{ times}}$.
- What is $x^0$?
- Let $AB = \{ab : a \in A, b \in B\}$.

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx...x}_{n \text{ times}}$.
- What is $x^0$?
- Let $AB = \{ab : a \in A, b \in B\}$.
- Let $A^n = \underbrace{AA...A}_{n \text{ times}}$.

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx \ldots x}_{n \text{ times}}$.
- What is $x^0$?
- Let $AB = \{ab : a \in A, b \in B\}$.
- Let $A^n = \underbrace{AA \ldots A}_{n \text{ times}}$.
- What is $A^0$?

## More definitions

Let $x, y \in \Sigma^*$ be strings, $A, B \subseteq \Sigma^*$ be languages, and $n \in \mathbb{N}$ be a natural number

- Write $xy$ to mean the concatenation of $x$ and $y$.
- Write $x^n$ to mean $\underbrace{xx...x}_{n \text{ times}}$.
- What is $x^0$?
- Let $AB = \{ab : a \in A, b \in B\}$.
- Let $A^n = \underbrace{AA...A}_{n \text{ times}}$.
- What is $A^0$?
- Let $A^* = A^0 \cup A^1 \cup A^2 \cup A^3 \cup ... = \bigcup_{i \in \mathbb{N}} A^i$

## Generated by...

A while ago we studied what it meant for a set to be generated from $B$ by the functions in $F$.

**Question.** How would you generate $\Sigma^*$?

## Generated by...

A while ago we studied what it meant for a set to be generated from $B$ by the functions in $F$.

**Question.** How would you generate $\Sigma^*$?

## Example - English

- $\Sigma = \{a, b, c, ..., z\}$
- $\Sigma^* = \{\epsilon, a, aa, ab, ac, ..., ba, ..., aaa, ...\}$
- $\text{English} \subseteq \Sigma^*$

## Example - Even

- $\Sigma = \{0, 1\}$
- $\Sigma^* = \{\epsilon, 0, 1, 00, 01, ...\}$
- $\mathrm{Even} = \{w \in \Sigma^* : w \text{ has an even number of 1s}\}$

## Alphabet

Any finite set works for the alphabet! However, to make things simple, we'll usually take the alphabet to be $\Sigma = \{0, 1\}$, or $\{a, b\}$.

## The Problem (again)

Given a language $A$ over an alphabet $\Sigma$, come up with a program that decides whether a given string $x \in \Sigma^*$ is in $A$ or not.

## The Problem (again)

Given a language $A$ over an alphabet $\Sigma$, come up with a program that decides whether a given string $x \in \Sigma^*$ is in $A$ or not.

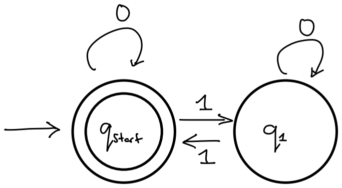
## What is a program?

To talk formally about computation it's important to specify a model of computation.

- What does a program look like?
- What can the program do?

## DFAs

DFAs are one model of computation. They look like this.



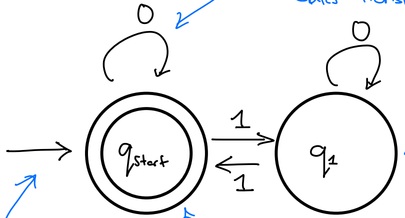
## DFAs

A single DFA corresponds to what we think of as a program.

## Computation of a DFA

Input: a string $w \in \Sigma^*$.
Output: accept/reject.

- The DFA starts at a predefined start state.
- The DFA reads in the input string one character at a time. Depending on the character read and the current state, the DFA deterministically moves to a new state.
- When it has read the entire string, the DFA will be in some state. If that state is one of the accept states, the DFA accepts. Otherwise, the DFA rejects.



arrow are labelled w/ characters $\in \Sigma$ and define how states transition based on the character read.

single circle means the state is rejecting.

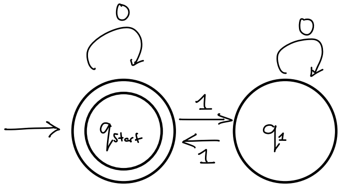
signifies the starting state

double circles mean this is an accepting state. There may be more than 1!

## Language of a DFA

Let $M$ be a DFA, the language of a DFA, denoted $L(M)$, is the set of strings $w \in \Sigma^*$ such that $M$ accepts $w$.

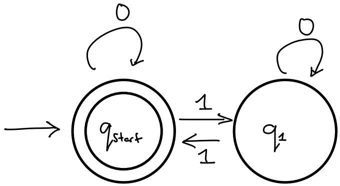
# Example



Does this DFA accept

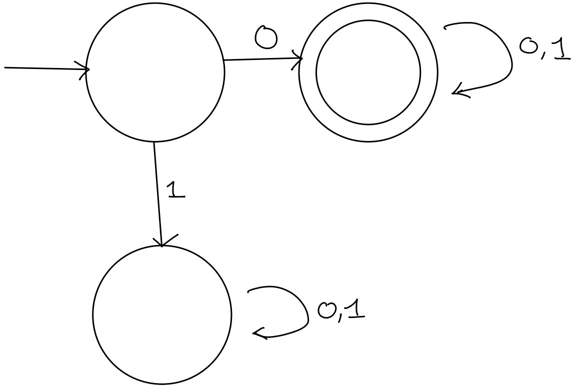
- 001101?
- 100110?
- 111100?
- 011000?
- $\epsilon$?

## Example



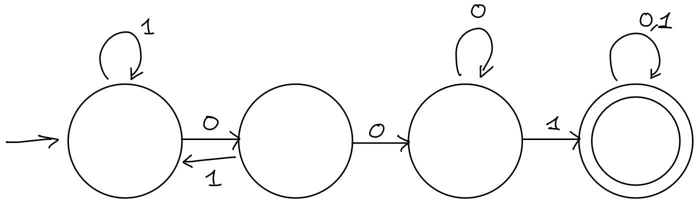
What is the language of the DFA?

## Example



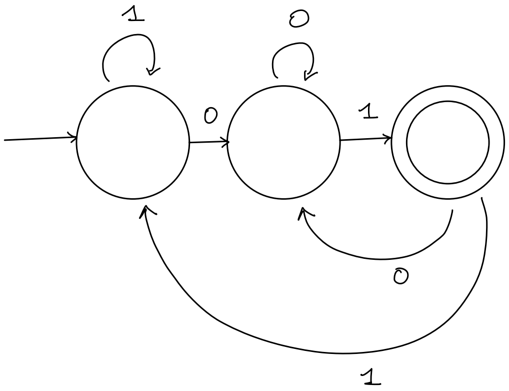
What is the language of the DFA?

## Example



How about this one?

## Example



How about this one?

## DFAs

- Deterministic. Given the current state and character read, the next state is always the same.
- Finite. There are a finite number of states.

States are analogous to “memory”, since we can store information about the input by transitioning to different states.

## Defining a DFA

When defining a DFA, you need to do the following

- Tell me the alphabet, $\Sigma$ of the DFA.
- Define a **finite** set of states, $Q$.
- Tell me a start state $q_{\text{start}} \in Q$.
- Tell me which states are accepting. Formally, find a subset $F \subseteq Q$ of accept states.
- For each state $q$ and each character $x \in \Sigma$, tell me which state to go to next if I read $x$ from state $q$.

In this class, **you may capture all of this information in a drawing.**

## Tips

- States $\iff$ Memory. Use states to capture information about the input read so far! What do you need to remember about the input in order to answer the question?
- It takes some time getting used to the fact that we read the input left to right, and only get to read each character once! Keep this in mind when designing DFAs!
- Common pattern: The garbage state.

## Example

Design a DFA $M$ such that

$$L(M) = \{w \in \{0,1\}^* : w \text{ has at least two 1s}\}$$

## Example

Design a DFA $M$ such that

$$L(M) = \{w \in \{0,1\}^* : w \text{ does not contain the substring } 11\}$$

## Example

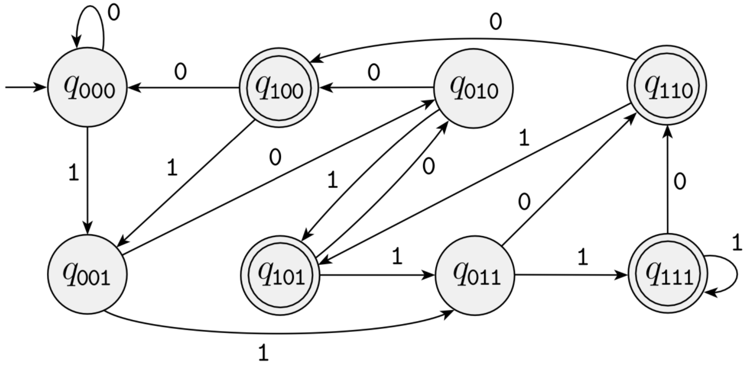
Design a DFA $M$ such that

$$L(M) = \{w \in \{0,1\}^* : \text{ The third last character of } w \text{ is } 1\}$$

## Example

Design a DFA $M$ such that

$$L(M) = \{w \in \{0,1\}^* : \text{ The third last character of } w \text{ is } 1\}$$

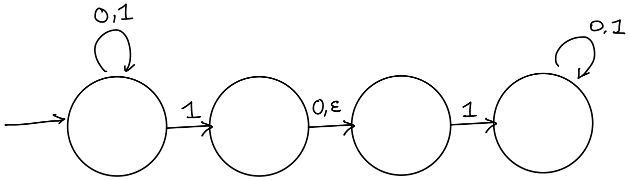


Reference: Sipser

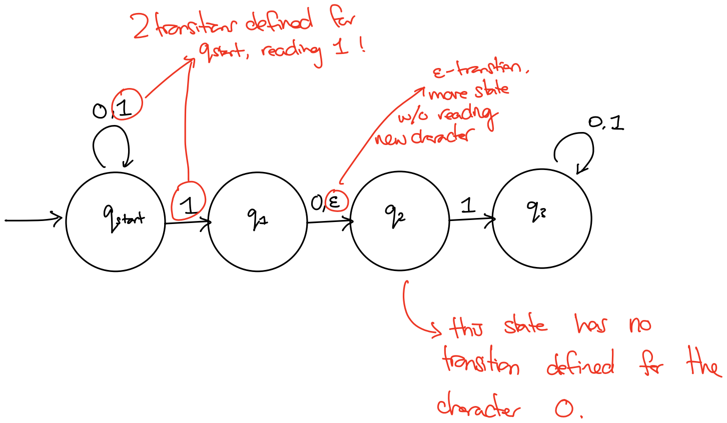
## Another model of computation - Nondeterministic Finite Automata (NFAs)

Here's an example of an NFA

$\Sigma : \{0,1\}$



# Key differences



## Key differences

- For each state, there can be multiple arrows labelled with the SAME character!
- States do not need to have one arrow for each character in $\Sigma$.
- Arrows can be labelled with $\epsilon$.

## Computation of a NFA

Input: a string $w \in \Sigma^*$.

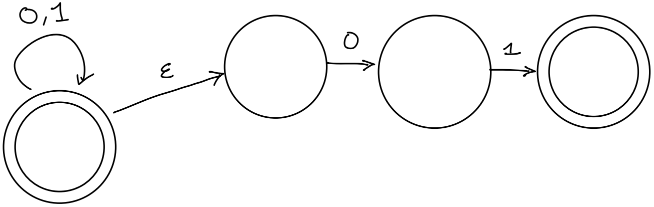
Output: accept/reject.

- The NFA starts at a predefined start state.
- The NFA reads in the input string one character at a time. Let $c$ be the character that was read. There are several cases depending on the current state.
  1. If the state has no arrows coming out of it labelled with $c$, halt execution and immediately reject.
  2. Otherwise, choose one of the arrows labelled with $c$ and follow it and read the next character.
  3. If there is an arrow labelled with $\epsilon$, you may choose to follow it. In this case, you don't read the input character (i.e. the next character to be read is still the same)
- When it has read the entire string, the NFA will be in some state. Depending on the choices made, the final state will either be accepting or rejecting. **The NFA accepts the string if ANY sequence of choices leads to an accept state.**

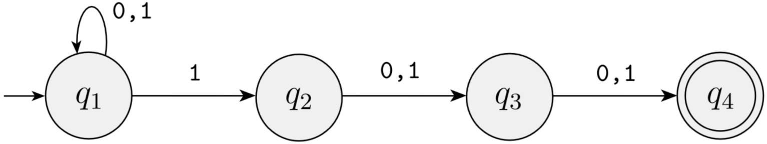
## Language of a NFA

Let $N$ be a NFA, the language of a NFA, denoted $L(N)$, is the set of strings $w \in \Sigma^*$ such that $N$ accepts $w$.

## What is the language of...



## NFA for third last character is a 1


$$\begin{tikzcd} \arrow[r] & q_1 \arrow[loop above, "0,1"] \arrow[r, "1"] & q_2 \arrow[r, "0,1"] & q_3 \arrow[r, "0,1"] & q_4 \end{tikzcd}$$

## Regular Languages

A language $A$ is regular if and only if there is a DFA $M$ such that $L(M) = A$.

## Closure

Suppose $A$ and $B$ are regular languages, then

- $\overline{A}$,
- $AB$,
- $A \cup B$,
- $A \cap B$,
- $A^n$, and
- $A^*$

are also regular.

## Let’s prove this!

## $\overline{A}$

Suppose $A$ is regular. Let $M$ be a DFA for $A$. Let $M'$ be the DFA with all the states of $M$ flipped. Then $L(M') = \overline{A}$.

## $A \cup B$

Let $M$ and $N$ be DFAs for $A$ and $B$ respectively. We need to find a DFA $D$ for $A \cup B$.

## $A \cup B$

Let $M$ and $N$ be DFAs for $A$ and $B$ respectively. We need to find a DFA $D$ for $A \cup B$.

Idea 1: Here's what we'd like to do. Let $w$ be the input. $D$ runs $M$ on $w$ to check if $w \in A$ and then $D$ runs $N$ on $w$ to check if $w \in B$. Accept if either was accept and reject if both were rejected.

## $A \cup B$

Let $M$ and $N$ be DFAs for $A$ and $B$ respectively. We need to find a DFA $D$ for $A \cup B$.

Idea 1: Here's what we'd like to do. Let $w$ be the input. $D$ runs $M$ on $w$ to check if $w \in A$ and then $D$ runs $N$ on $w$ to check if $w \in B$. Accept if either was accept and reject if both were rejected. What's the problem with this approach?

## $A \cup B$

Idea 2: Instead, we need to run the two machines in **parallel**. Thus, each state in $D$ corresponds to a 2-tuple where the first element is the ‘M’ state and the second element is the ‘N’ state.

## $A \cup B$

Idea 2: Instead, we need to run the two machines in **parallel**. Thus, each state in $D$ corresponds to a 2-tuple where the first element is the 'M' state and the second element is the 'N' state.

If we're at state $(x, y)$ and we read a character $\sigma$. Then we transition to $(x', y')$ where $x'$ is the state that $x$ goes to (in $M$) when reading $\sigma$ and $y'$ is the state that $y$ goes to (in $N$) when reading $\sigma$.

## $A \cup B$

Idea 2: Instead, we need to run the two machines in **parallel**. Thus, each state in $D$ corresponds to a 2-tuple where the first element is the 'M' state and the second element is the 'N' state.

If we're at state $(x, y)$ and we read a character $\sigma$. Then we transition to $(x', y')$ where $x'$ is the state that $x$ goes to (in $M$) when reading $\sigma$ and $y'$ is the state that $y$ goes to (in $N$) when reading $\sigma$.

What should the accepting states be?

## $A \cup B$

Idea 2: Instead, we need to run the two machines in **parallel**. Thus, each state in $D$ corresponds to a 2-tuple where the first element is the 'M' state and the second element is the 'N' state.

If we're at state $(x, y)$ and we read a character $\sigma$. Then we transition to $(x', y')$ where $x'$ is the state that $x$ goes to (in $M$) when reading $\sigma$ and $y'$ is the state that $y$ goes to (in $N$) when reading $\sigma$.

What should the accepting states be? $(x, y)$ should be accepting if either $x$ was an accept state in $M$ or $y$ was an accept state in $N$ (or both).

## Example: Starts with 1 or has an even number of 1s

## $A \cap B$

Homework.

## $AB$

The same trick of running the DFAs in parallel doesn't work.

## $AB$

The same trick of running the DFAs in parallel doesn't work.
How would you write code to solve this?