Learning Objectives

By the end of this worksheet, you will:

- Know the definition of bipartite graphs.

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1. **Bipartite graphs.** Let $G = (V, E)$ be a graph. We say that G is **bipartite** when it satisfies the following properties:

   - There exist subsets $V_1, V_2 \subset V$ such that $V_1 \neq \varnothing$, $V_2 \neq \varnothing$, and V_1 and V_2 form a *partition* of V.[1]
   - Every edge in E has exactly one endpoint in V_1 and one in V_2. (Equivalently, no two vertices in V_1 are adjacent, and no two vertices in V_2 are adjacent.)

   When G is bipartite, we call the partitions V_1 and V_2 a **bipartition of G**. TIP: bipartite graphs are typically drawn such that V_1 and V_2 are clearly separated (e.g., with all the vertices of V_1 on the left, and all the vertices of V_2 on the right).

   (a) Prove that the following graph $G = (V, E)$ is bipartite.

   $$V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

   <u>Solution</u>

   Let $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$. Then V_1 and V_2 together provide a partition of V, as $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ and neither V_1 nor V_2 is empty.

   Note that all of the vertex labels in V_1 are odd numbers and all of the vertex labels in V_2 are even numbers.

   Each of the edges $(1, 2)$, $(1, 6)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, and $(5, 6)$, has one endpoint that with a vertex label that is an odd number and one that is an even number.

   (b) Let m and n be positive integers. A **complete bipartite graph on (m, n) vertices** is a graph $G = (V, E)$ that satisfies the following properties:

   i. G is bipartite, with bipartition V_1, V_2 (as defined above).
   ii. (new) $|V_1| = m$ and $|V_2| = n$.
   iii. (new) For all vertices $u \in V_1$ and $w \in V_2$, u and w are adjacent.

   How many edges are in a complete bipartite graph on (m, n) vertices? Your answer will depend on m and n. Explain your answer.

   <u>Solution</u>

   Let $G = (V, E)$ be a complete bipartite graph on (m, n) vertices, with bipartition V_1, V_2, and $|V_1| = m$ and $|V_2| = n$.

   Then each vertex $u \in V_1$ appears as an endpoint in n edges in E, since it has an edge to each of the n vertices in V_2. As there are m vertices in V_1 and the previous statement is true for each of them, we know that there are at least mn edges in E.

   But, since there are no edges between vertices in V_1 and no edges between vertices in V_2, there are no other edges to count.

   And so we can conclude that the number of edges in a complete bipartite graph on (m, n) vertices is mn.

[1] That is, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \varnothing$.

(c) Recall that a *cycle* in a graph $G = (V, E)$ is a sequence of vertices $v_0, v_1, \dots, v_k$ such that $k \geq 3$, $v_k = v_0$, and G contains every edge between consecutive vertices: $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$.

In this question, we will be concerned with the *parity* of the lengths of cycles in bipartite graphs—the parity of an integer is either 0 (when the number is even) or 1 (when the number is odd).

Explore. Draw a few different bipartite graphs and make sure they contain some cycles. What do you notice about the parity of the lengths of these cycles (are they even or odd)? Can you draw a bipartite graph with cycles whose lengths have either parity?

Prove. Make a conjecture about the parity of every cycle length in a bipartite graph, and prove it.

Solution

Conjecture: the length of every cycle in a bipartite graph is even.

Proof. Let $G = (V, E)$ and assume G is bipartite, with bipartition V_1, V_2. Let $C = v_0, \dots, v_k$ be a cycle in G. Without loss of generality, assume $v_0 \in V_1$.* We'll prove that k is even.

Intuition: Because every edge in G is between opposite sides of the bipartition, and $v_0 \in V_1$, the cycle must alternate between V_1 and V_2, where vertex v_i is in V_1 if i is even, and V_2 if i is odd. But then, $v_k = v_0$ is only possible if k is even.

Exercise. Formalize this intuition: write a proof by induction on $k \geq 3$ that for all *paths* $v_0, \dots, v_k$ in a bipartite graph where $v_0 \in V_1$, v_k is in V_1 if k is even, and V_2 if k is odd. Why can we not directly prove this for cycles?

*The phrase "without loss of generality" means that our proof could easily be modified to work if the opposite were true; in this case, simply by switching the roles of V_1 and V_2 in the proof. Rather than repeat the argument twice (when $v_0 \in V_1$ and when $v_0 \in V_2$), with almost identical arguments each time, we write only the first version of the argument.