CSC 236 Tutorial 5

June 7, 2023

Today

Prime Factorization

For any number $n \in \mathbb{N}$ with $n \geq 2$. n can be written as the product of one or more prime numbers.

- Prove it using (complete induction).
- Prove it using the Well Ordering Principle.

By complete induction

Base case. 2 is prime so the base case holds.

Inductive step. Let $k \in \mathbb{N}$ be any number with $k \geq 2$, and assume that any number between 2 and k can be written as the product of prime numbers. Consider $k+1$. If $k+1$ is prime, then we are done. Otherwise, $k+1 = a \cdot b$ for some numbers a, b with $2 \leq a, b < k+1$. Thus, the inductive hypothesis applies to a and b, so $a = a_1 a_2 ... a_i$ and $b = b_1 b_2 ... b_j$, where each of the a_is and b_is are prime. Therefore $k+1 = a_1 a_1 ... a_i b_1 b_2 ... b_j$ and is also the product of primes. This completes the induction.

By the WOP

By contradiction, assume the claim is false, let
$S = \{n \in \mathbb{N} : n \geq 2, n \text{ is not the product of primes}\}$.

Since S is non-empty, there is some minimal element m.

Since m can not be written as the product of primes, it must not be prime itself. Thus $m = ab$ for some $2 \leq a, b < m$. Since m can not be written as the product of primes, at least one of a or b can not be written as the product of primes. But this contradicts the minimality of m in S!

Generate the following sets

- $\mathrm{Palindrome} = \{w : w \text{ is a palindrome }\}$. Note a palindrome is any string that is the same when read backwards. The empty string, which we'll denote as ϵ, counts as a palindrome. Write a construction sequence for 'tacocat'
- $\mathrm{Even} = \{w : w \text{ is a bitstring with an even number of 1s }\}$. A bit string is a sequence of 0s and 1s. Does $B = \{\epsilon\}$, $F = \{x \mapsto x0, x \mapsto x11\}$ work? If not, explain why and come up with a fix.

Palindrome

$B = \{\epsilon, a, b, ..., z\}$. For each character α, define f_α to be the function that maps $x \mapsto \alpha x \alpha$. Set $F = \{f_\alpha : \alpha \in \{a, ..., z\}\}$

We have

$$\text{tacocat} = f_t(f_a(f_c(o))).$$

So a valid construction sequence for 'tacocat' is [o,coc,acoca,tacocat]. Note that each element of the sequence is either in B or $f_\alpha(x)$ where x is some earlier element in the sequence.

Even

The proposal doesn't work b/c can't generate 100001 for example - 1s are always next to each other! Here is one fix.

$B = \{\epsilon\}$. $F = \{x \mapsto x0, x \mapsto 1x1, x \mapsto 0x\}$

Pairs

Let $B = \{(0, 0)\}$, and f_1, f_2, f_3 be the functions defined as follows

- $f_1(a, b) = (a, b + 1)$
- $f_2(a, b) = (a + 1, b + 1)$
- $f_3(a, b) = (a + 2, b + 1)$

Let X be the set generated from B by $\{f_1, f_2, f_3\}$. Show that for all $(a, b) \in X$, $a \leq 2b$.

Pairs

By structural induction. Let $P(a, b)$ be the predicate that $a \leq 2b$.
Base case. $0 \leq 2 \cdot 0$, so the base case holds.

Inductive Step.

- f_1. Suppose $a \leq 2b$, then $a \leq 2b + 2 = 2(b + 1)$, so $P(f_1(a, b)) = P(a, b + 1)$ holds.
- f_2. Suppose $a \leq 2b$. then $a + 1 \leq 2b + 2$, so $P(f_2(a, b))$ holds.
- f_3. Suppose $a \leq 2b$. then $a + 2 \leq 2b + 2$, so $P(f_3(a, b))$ holds.

This completes the induction.

Nim

Nim is a two player game that works as follows.

- There are two piles of stones. Each pile has the n stones for some $n \in \mathbb{N}$.
- The players alternate taking some (non-zero) number of stones from one pile.
- If at the start of a player's turn, all piles are empty, then that player loses.

Play this game with someone with $n = 15$!

Prove that for all $n \in \mathbb{N}$, there is a winning strategy for the second player.

Proof

The winning strategy for player 2 do whatever player 1 did in the previous turn but to the other pile! We'll show this strategy works for all n by induction.

Base case. For $n = 0$, it is player 1's turn and there are no more stones left in either pile, thus player 1 loses.

Inductive step. Let $k \in \mathbb{N}$ be some natural number. Assume the strategy works for every $i \in \mathbb{N}$ with $i \leq k$. Consider the game with $k + 1$ stones in each pile. Player 1 takes x stones from one pile for some $1 \leq x \leq k + 1$. Applying the strategy, player 2 takes x stones from the other pile. Thus, both piles have $k + 1 - x$ stones. Since $x \geq 1$, $k + 1 - x \leq k$, thus, the inductive hypothesis applies and player 2 wins using this strategy.