1 Bernoulli's Inequality

Question 1a.

Let m be an arbitrary natural number, and let $P(n)$ be the predicate $(1+m)^n \geq 1 + mn$. We'll show $\forall n.P(n)$.

By induction on n.

Base case. For $n = 0$, $(1 + m)^0 = 1$, and $1 + m \cdot 0 = 1$.

Inductive step Let $k \in \mathbb{N}$ be an arbitrary natural number, and suppose $P(k)$, we'll show $P(k + 1)$.

$$\begin{aligned}
(1+m)^{k+1} &= (1+m)^k(1+m) \\
&\geq (1+mk)(1+m) && \text{(IH)} \\
&= 1 + mk + m + m^2k \\
&\geq 1 + mk + m && (m^2k \geq 0) \\
&= 1 + m(k+1)
\end{aligned}$$

2 Pairs

Question 2a.

We'll show the following claim. $\forall(a, b) \in X$, a is even and b is odd.

Note that this suffices since an even number times an odd number equals an even number, and an even number plus an odd number is odd.

By structural induction.

Base case. The base case holds because 0 is even and 1 is odd.

Inductive step. Let $(a, b) \in X$ and suppose a is even and b is odd. In particular, suppose $a = 2p$ and $b = 2q + 1$ for some $p, q \in \mathbb{N}$. We will show the claim for $f_1((a, b))$, and $f_2((a, b))$

- $f_1(a, b) = (b + 1, a + 1)$. We have $b + 1 = 2q + 1 + 1 = 2(q + 1)$, so $b + 1$ is even. Also, $a + 1 = 2p + 1$, so a is odd.
- $f_1(a, b) = (2b, a + 1)$. $2b$ is even. Also, $a + 1 = 2p + 1$, so a is odd.

This completes the induction.

3 Friends and Strangers +

Question 3a.

$k = 66$ works. Let p be any person. By the generalized pigeonhole principle, p is either strangers, friends, enemies, or soulmates with at least $\lceil 65/4 \rceil = 17$ other people.

WLOG, p is soulmates with at least 17 other people. Let S be the set of p's soulmates. If there exists $a, b \in S$ where a, b are also soulmates, then there are 3 mutual soulmates, a, b, p. Otherwise, no two people in S are soulmates with each other. Then, S is a set of 17 people who are either friends, strangers, or enemies with one another. By HW 1, we are guaranteed that there exists either 3 mutual friends, 3 mutual strangers, or 3 mutual enemies in S.

4 Transitive Round Robin

Question 4a.

Any graph where there are multiple players with maximal wins.

Question 4b.

Let $P(n)$ be the predicate that is true iff $\forall a_1, ..., a_n \in V$, such that $\forall i \in \{1, 2, ..., n-1\}.(a_i$ beats $a_{i+1})$, then a_1 beats a_n.

By induction on n.

Base case. The base case ($n = 3$) is exactly the transitive property.

Inductive step. Let $k \in \mathbb{N}$ with $k \geq 3$, and assume $P(k)$. We'll show $P(k+1)$. Suppose $a_1, ..., a_{k+1} \in V$ such that a_i beats a_{i+1} for all $1 \leq i \leq k$.

$a_1, ..., a_k$ is a sequence of k elements for which a_i beats a_{i+1} for all $1 \leq i \leq k-1$. Thus, by the inductive hypothesis, a_1 beats a_k. Since a_1 beats a_k, and a_k beats a_{k+1}, by the transitive property, a_1 beats a_{k+1}. This completes the induction.

Question 4c.

By contradiction, suppose there exists a cycle $(p = a_1, ..., a_{k-1}, a_k = p)$. Note that by the definition of the graph,

$a_1, a_2, ..., a_{k-1}$ is a sequence where a_i beats a_{i+1} for all $1 \leq i \leq k-2$. By the previous problem, this implies that $a_1 = p$ beats a_{k-1}. But a_{k-1} beats $a_k = p$. This is a contradiction since only one of the two can be true!

Question 4d.

Injective. Suppose p and q are two players with $p \neq q$. WLOG suppose p beats q. By transitivity, p beats everyone that q beats as well, therefore $f(p) \geq f(q) + 1$, in particular, $f(p) \neq f(q)$, so f is injective.

Surjective. By contradiction, suppose f is not surjective. Then there is some value $k \in \{0, ..., |V|-1\}$ such that $k \neq f(p)$ for any $p \in V$. Then, the codomain of f is $\{0, .., |V|-1\} \setminus \{k\}$ which has size $|V|-1$. Then since f has a domain of size $|V|$, by the pigeonhole principle, f is not injective, which is a contradiction.

Since f is both injective and surjective, f is bijective.

Question 4e.

Suppose $G = (V, E)$ is a transitive tournament.

If p beats someone who beats q, then p beats q by the transitive property. Thus, a winner is simply a person who wins all of their games.

Since the function mapping players to the number of wins is bijective, we know that there is precisely one player with $n-1$ wins (winning all of their games), and this player is the unique winner.

5 N-Knights

Question 5a.

Let $G_n = (V_n, E_n)$, where

- Let $V_n = [n] \times [n]$ be the $n \times n$ grid (identify these with the squares on the chessboard).
- $E_n = \{\{(i,j),(k,l)\} :$ a knight on (i,j) can attack a knight on $(k,l)\}$

Then the problem of placing k knights on an $n \times n$ chess board where no two knights attack each other is equivalent to finding an independent set of size k in G_n.

- If I is an independent set of size k in G_n, then every pair of vertices (i,j), and (a,b) in I are not adjacent, and hence knights placed on these squares do not attack each other.
- If you can place k knights on an $n \times n$ chess board, then vertices corresponding to the squares where the knights are placed form an independent set of size n in G_n since none of the knights can attack any of the other knights.

6 Master Method Recurrences

Question 6a.

$\log_4(8) < 2$, Thus, $f(n) = \Omega(n^{\log_4(8)+\epsilon})$ for some $\epsilon > 0$. So we might be in the root-heavy case; we need to check the regularity condition. $af(n/b) = 8 \cdot (n/4)^2 \log_2(n/4) = n^2/2 \log(n/4) \leq n^2 \log(n)/2 = 0.5f(n)$. Thus, we are in the root heavy case, and hence $T(n) = \Theta(n^2 \log(n))$

Question 6b.

$\log_9(3) = 1/2$, since $n^{1/2} = \sqrt{n}$, we are in the balanced case and $T(n) = \Theta(\sqrt{n} \log(n))$

Question 6c.

$\log_6(32) = 2$, $n^{1.5} = O(n^{2-\epsilon})$, (e.g. for $\epsilon = 0.1$), so we are in the leaf heavy case. Thus $T(n) = \Theta(n^2)$.

7 Extra Credit

Question 7a.

https://en.wikibooks.org/wiki/Chess/Puzzles/Placement/32_Knights/Solution