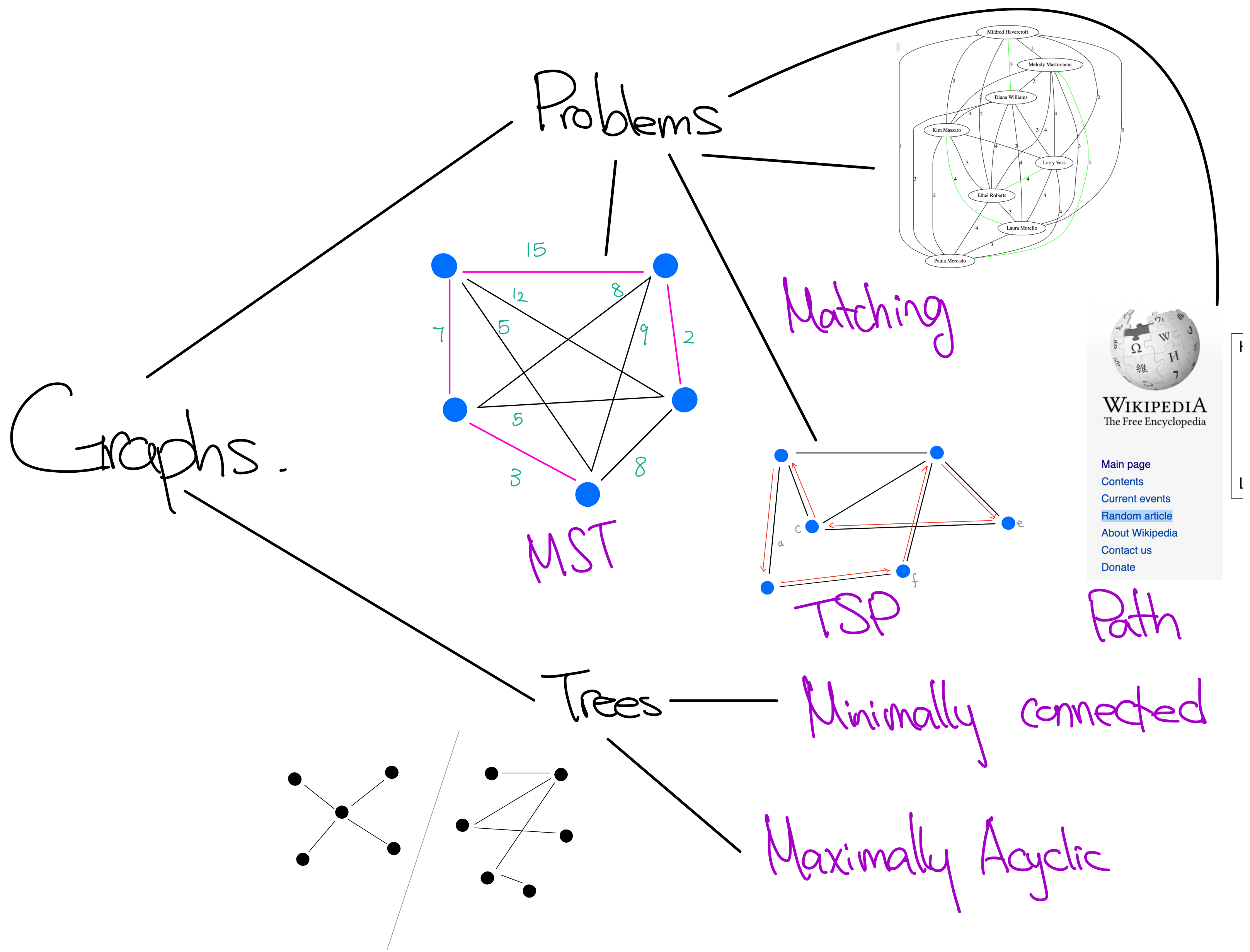
Announcements

- Sign up for check in 1 by midnight!
- HW2 is out.

Last time...



Problems
Graphs.
Mildred Havercroft
Melody Mastroianni
Diana Williams
Kim Massaro
Larry Vass
Ethel Roberts
Laura Morello
Paula Mercado
Matching
15
12
8
7
5
9
2
5
3
8
MST
a
c
e
f
TSP
WIKIPEDIA
The Free Encyclopedia
Main page
Contents
Current events
Random article
About Wikipedia
Contact us
Donate
Path
Trees
Minimally connected
Maximally Acyclic

CSC 236 Lecture 3: Induction

Harry Sha

May 24, 2023

Today

Induction

Examples of Proofs by Induction

Complete Induction

Induction

Examples of Proofs by Induction

Complete Induction

What is induction used for

Induction is used to prove statements of the following form:

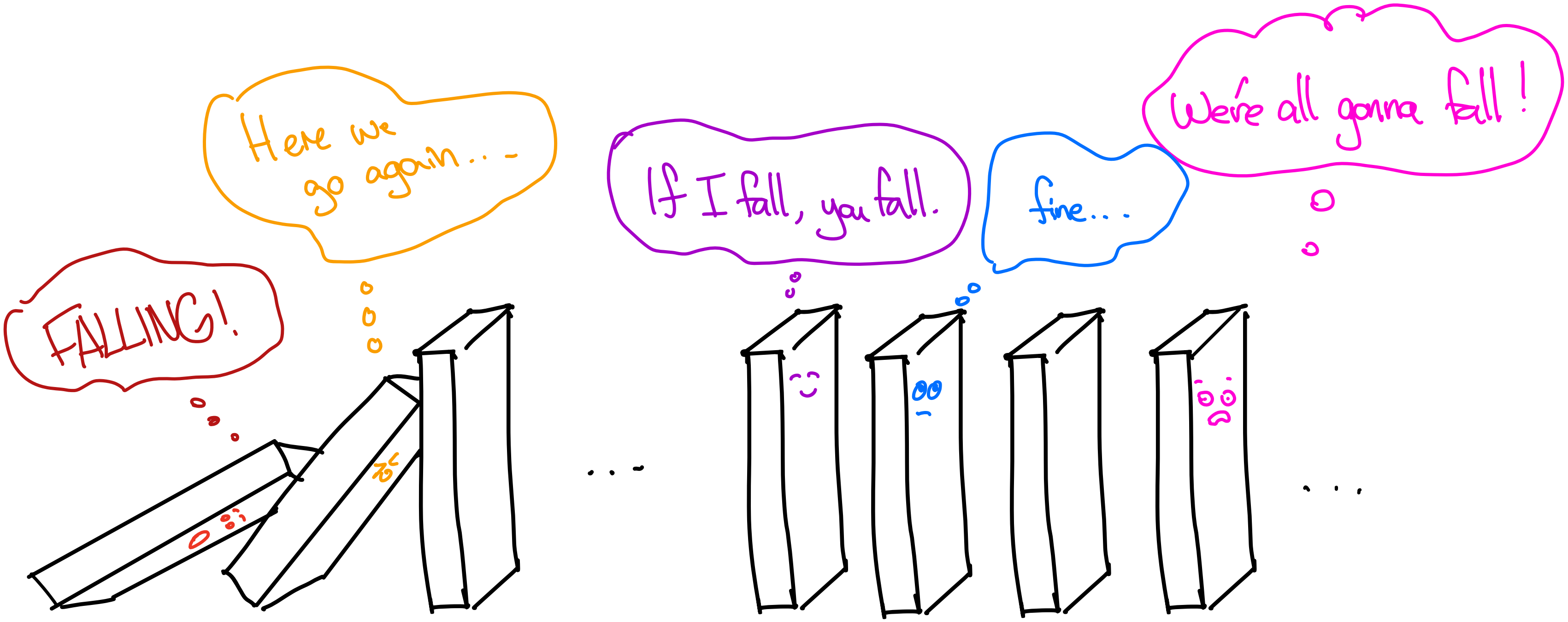
$$\forall n \in \mathbb{N}.(P(n)).$$

Note that P is a predicate on the natural numbers. I.e. for any natural number n, $P(n)$ is either true or false. For example, $P(n)$ might be

- The sum of the first n odd numbers is n^2.
- $(12^n - 1)$ is divisible by 11.
- Trees with n vertices have $n - 1$ edges.

Induction is super useful when analyzing the correctness and runtime of algorithms.

Induction



Principle of Mathematical Induction.

Induction

$$(P(0) \land \forall k \in \mathbb{N}.(P(k) \implies P(k+1))) \implies \forall n \in \mathbb{N}.(P(n))$$

Induction

$$(P(0) \wedge \forall k \in \mathbb{N}.(P(k) \implies P(k+1))) \implies \forall n \in \mathbb{N}.(P(n))$$

"If I can show that the first domino falls, and I can show that for any domino, if that domino falls, the next one falls, every domino falls".

Here, the kth domino falling is analogous to $P(k)$ being true.

Proofs by induction

$$\overbrace{(P(0) \wedge \forall k \in \mathbb{N}.(P(k) \implies P(k+1)))} \implies \underline{\forall n \in \mathbb{N}.(P(n))}$$

If we want to prove a statement of the form $\forall n \in \mathbb{N}.(P(n))$, it suffices to prove

1. $P(0)$ (base case)
2. $\forall k \in \mathbb{N}.(P(k) \implies P(k+1))$ (inductive step)

Induction template

Say we wanted to prove $\forall n \in \mathbb{N}.(P(n))$. Here is the template:

By induction.

Base case. [Prove $P(0)$ is true]

Inductive step. Let $k \in \mathbb{N}$ be an arbitrary natural number, and assume $P(k)$. We'll show $P(k+1)$. [Prove $P(k+1)$ assuming $P(k)$]. This completes the induction.

The assumption, $P(k)$, in the inductive step is called the inductive hypothesis (IH).

Flexibility

There is some flexibility in the proof by induction template. For example, sometimes, we want to prove a statement is true for all $n \geq 1$, so the base case starts at 1 instead of 0. Sometimes, for the inductive step, we need to show that if the previous **two** dominoes fall, then the next one falls. In this case, we would need to prove two base cases. These will come up in the examples, and you will see why we can do this next week.

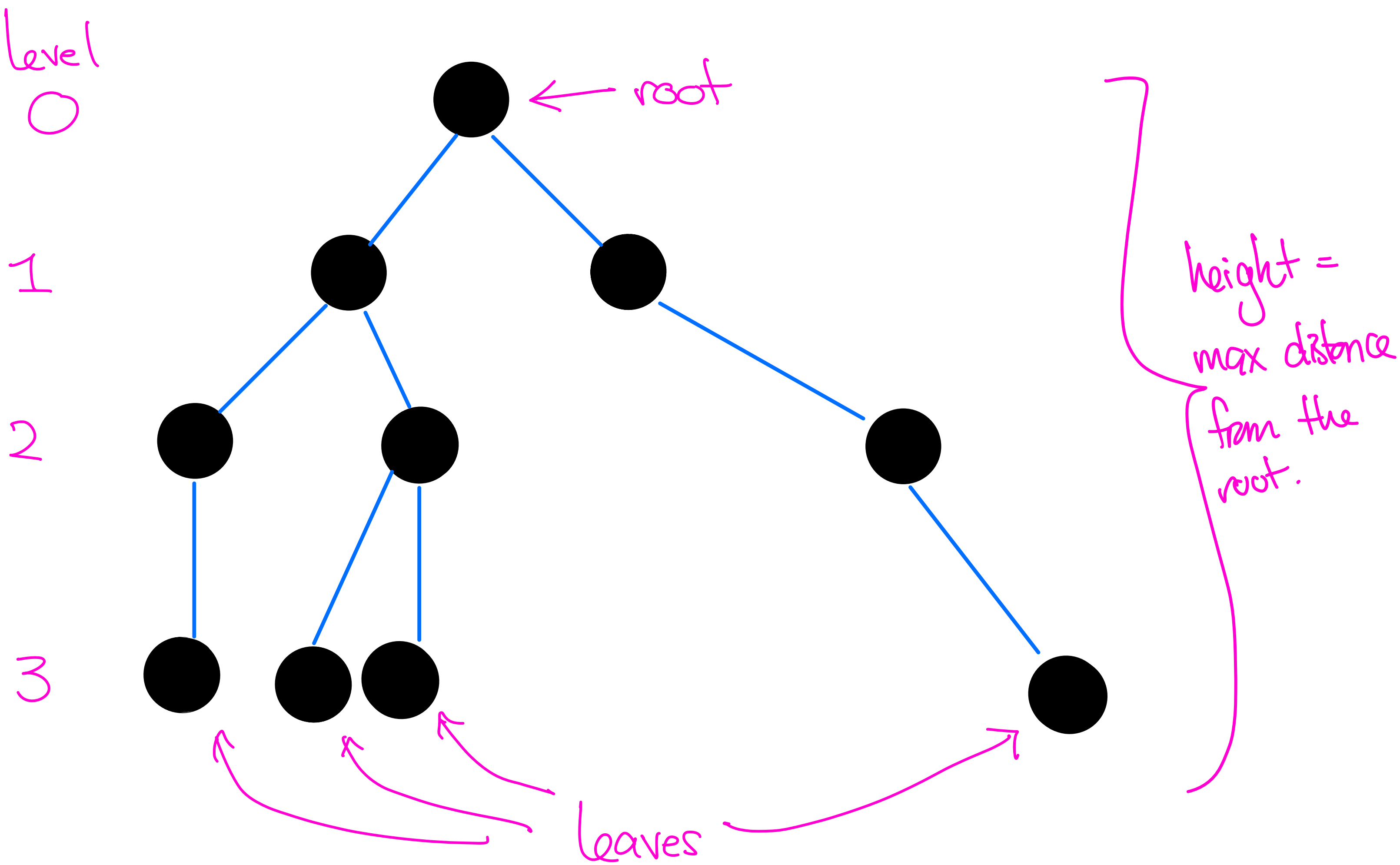
You should be okay as long as you showed enough so that "all the dominoes fall".

Induction

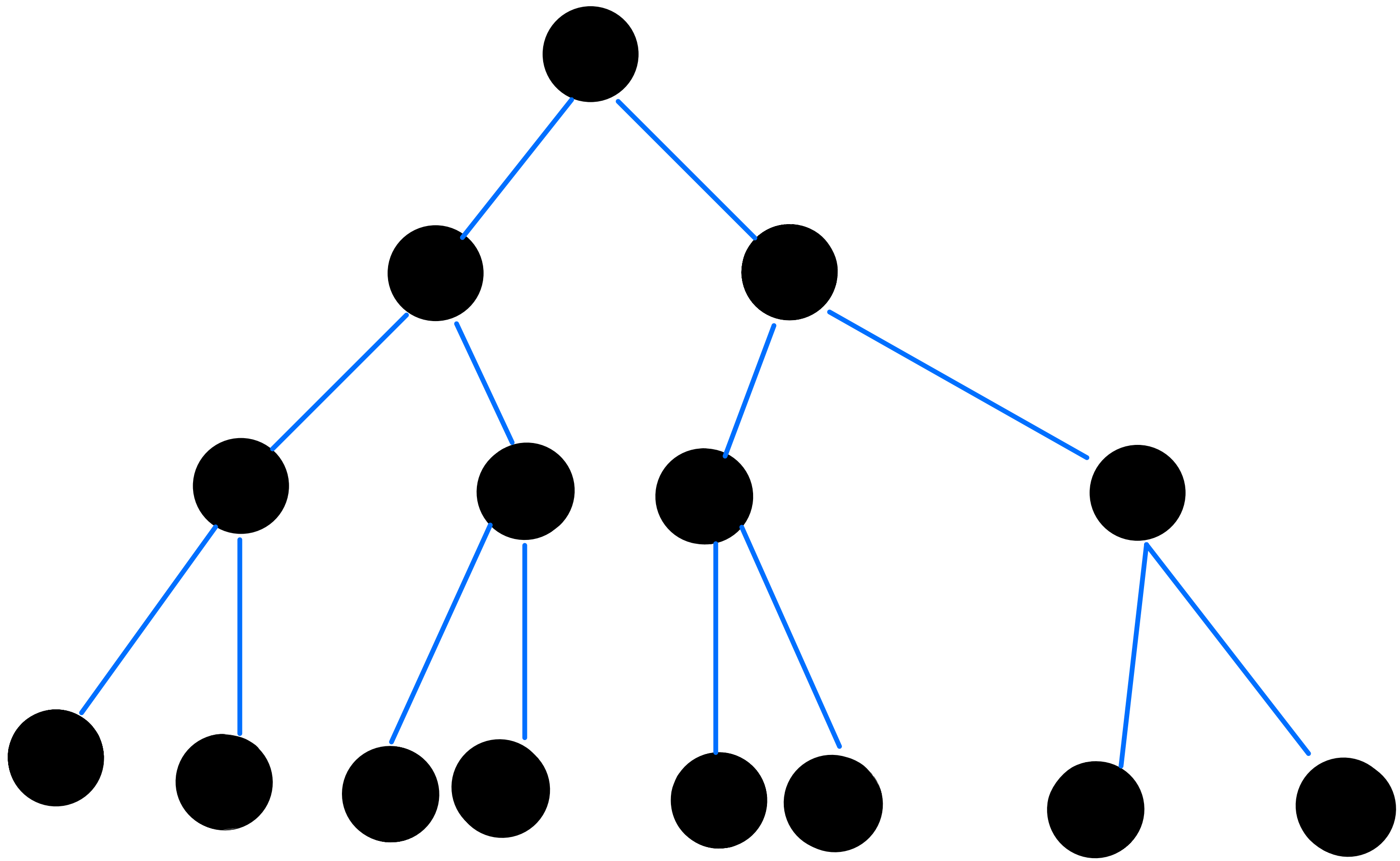
Examples of Proofs by Induction

Complete Induction

Binary Trees

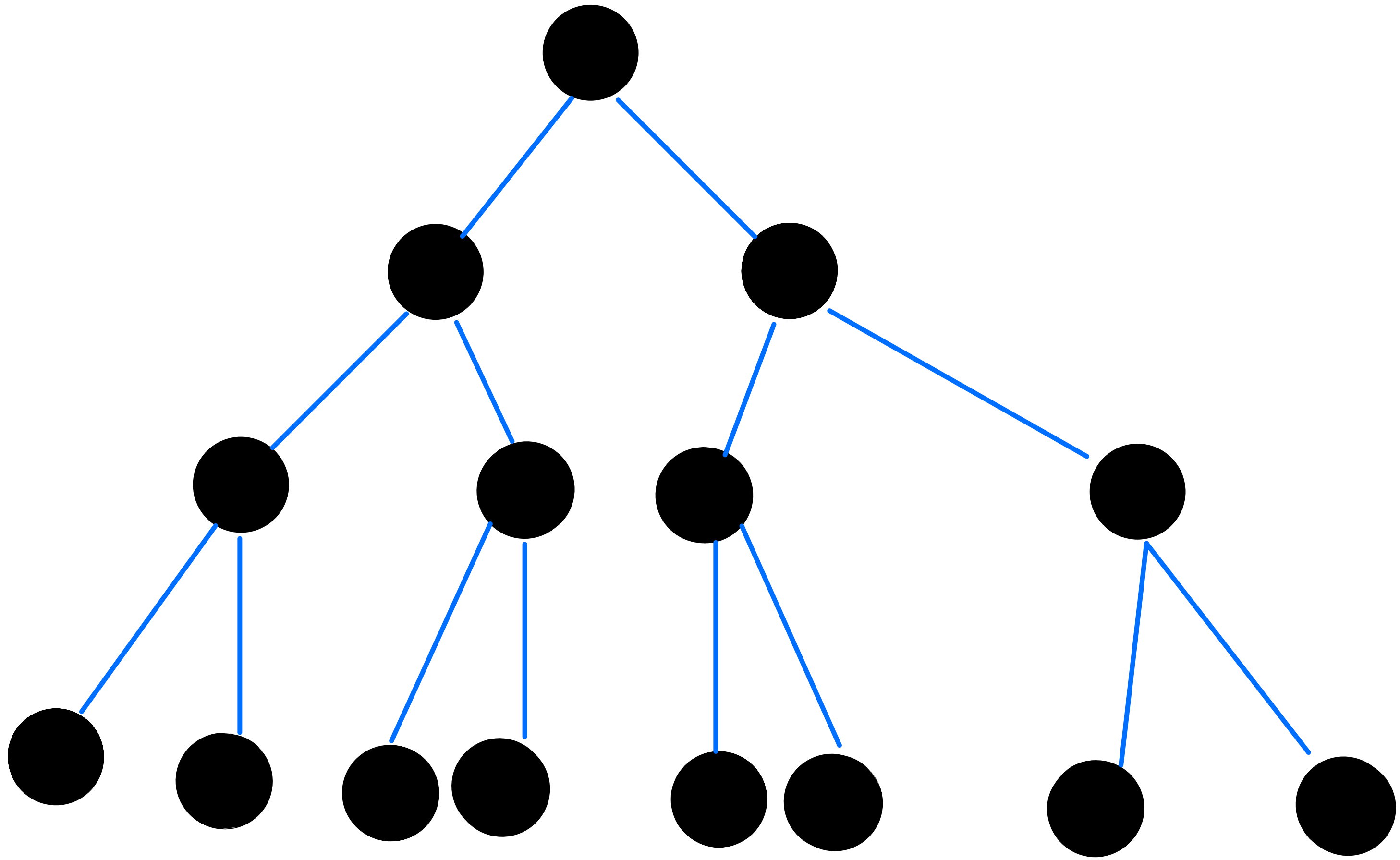


Perfect binary trees



In a perfect binary tree, all leaves are at the same level, and every other vertex has two children and one parent (except for the root, which does not have a parent).

Number of vertices of perfect binary trees



How many vertices does perfect binary tree of height n have?

Number of vertices of perfect binary trees

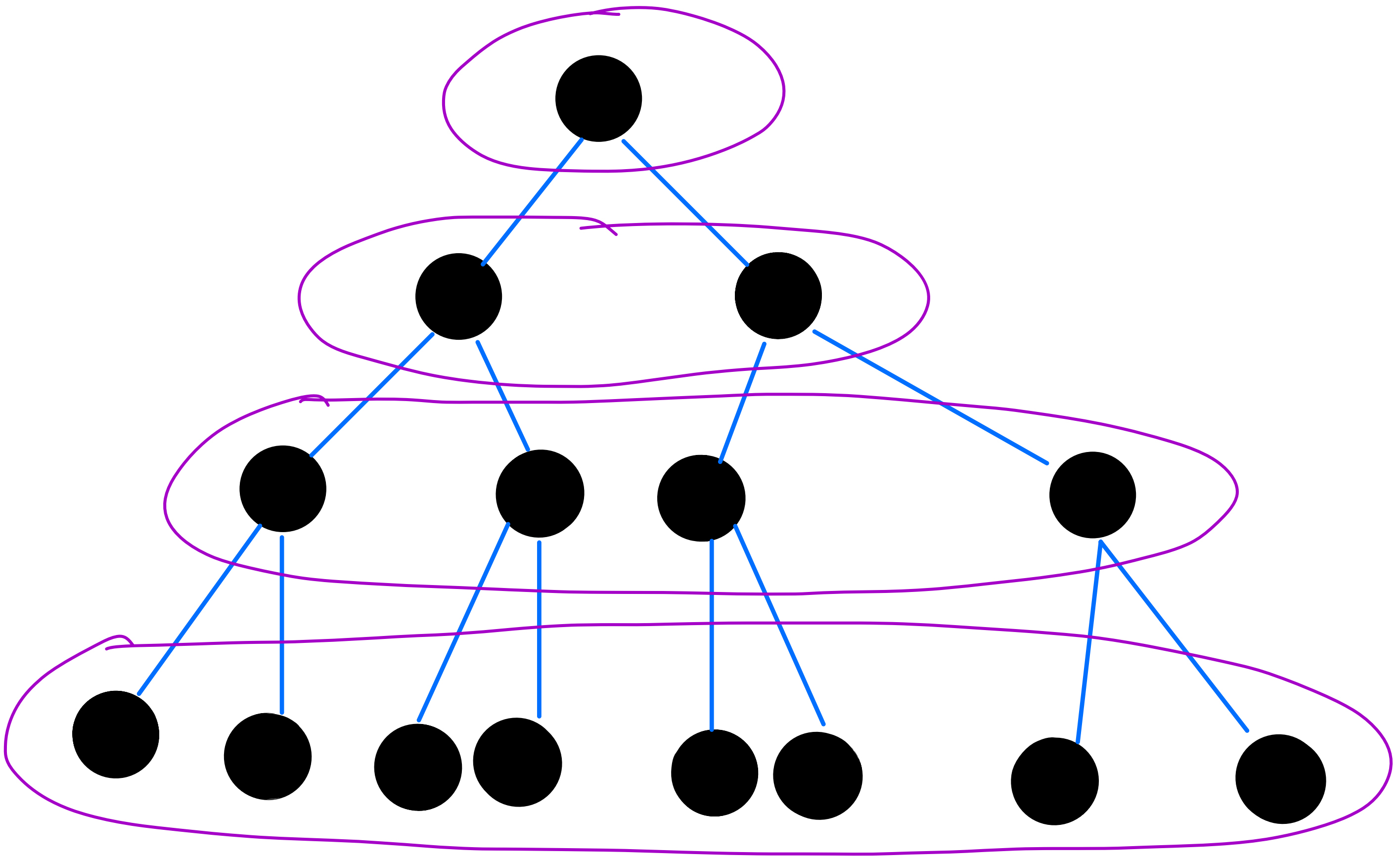
$1 = 2^0$

$2 = 2^1$

$4 = 2^2$

$8 = 2^3$

$\vdots$



$\forall n \in \mathbb{N}.(2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1)$

Base case: $2^0 = \underbrace{2^{0+1} - 1}_{\text{both just } 1 \ldots}$

Ind step. Let $k \in \mathbb{N}$, assume $\underbrace{\sum_{i=0}^{k} 2^i = 2^{k+1} - 1}_{\text{IH.}}$

Consider $\sum_{i=0}^{k+1} 2^i = \underbrace{\left(\sum_{i=0}^{k} 2^i\right)} + 2^{k+1}$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$= 2^{k+2} - 1$, this completes the induction

Sum of first n odd numbers is the nth square

Define $P(n)$ to be the predicate that is true iff. $\sum_{i=1}^{n} 2i-1 = n^2$

Base: $P(1)$: $1 = 1^2$ ✓.

Inductive step. Let $k \in \mathbb{N}$, assume $P(k)$.

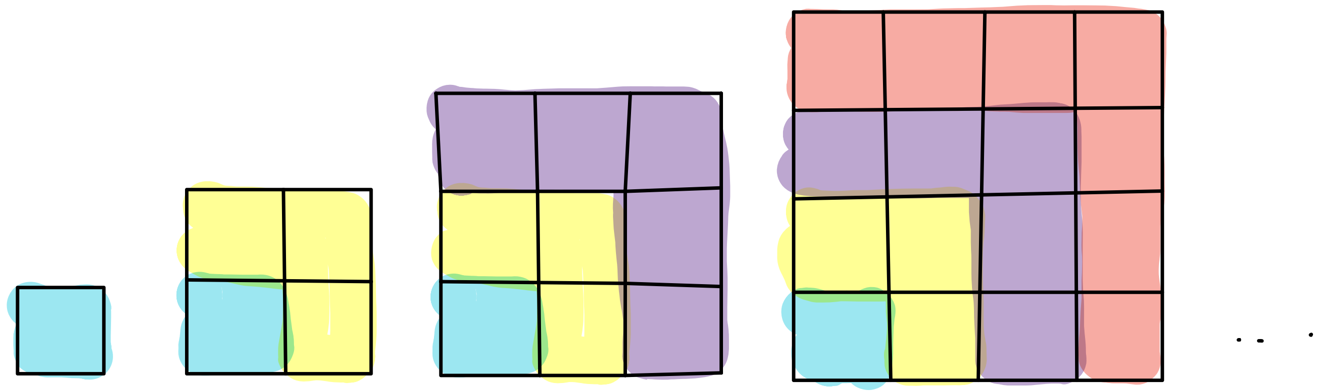
consider $P(k+1)$.

$$\sum_{i=1}^{k+1} 2i-1 = \underbrace{\sum_{i=1}^{k} (2i-1)}_{} + 2(k+1)-1$$

$$\overset{(IH)}{=} k^2 + 2k + 2 - 1 = (k+1)^2$$

This ...

Sum of first n odd numbers is the nth square



Sum of first n odd numbers is the nth square

We want to prove $\forall n \in \mathbb{N}, n \geq 1.(\sum_{i=1}^{n} 2i - 1 = n^2)$

$\forall n \in \mathbb{N}.(n^3 - n + 3)$ is divisible by 3

$0 \in \mathbb{N}$!!

Base case: $0^3 - 0 + 3 = 3$. ✓

Inductive step: let $k \in \mathbb{N}$. assume.

$$k^3 - k + 3 = 3p \quad \text{for some } p \in \mathbb{N}.$$

now consider

$$(k+1)^3 - (k+1) + 3$$

$$= \underline{k^3} + 3k^2 + 3k + \cancel{1} \underline{- k} - \cancel{1} \underline{+ 3}$$

$$= 3p + 3k^2 + 3k.$$

$$= 3(p + k^2 + k). \quad \blacksquare$$

$\forall n \in \mathbb{N}$. the units digit of 7^n is 1, 3, 7, or 9, or 8.

Base: $7^0 = 1 \in \{1, 3, 7, 9\}$ ✓.

Inductive: Assume 7^k has unit dig. 1, 3, 7, 9, or 8.

consider 7^{k+1}.

$7^k \% 10 = 1$, 7^{k+1} has unit digit $= 7$

3, .. $= 1$

7, .. $= 9$

9, .. $= 3$

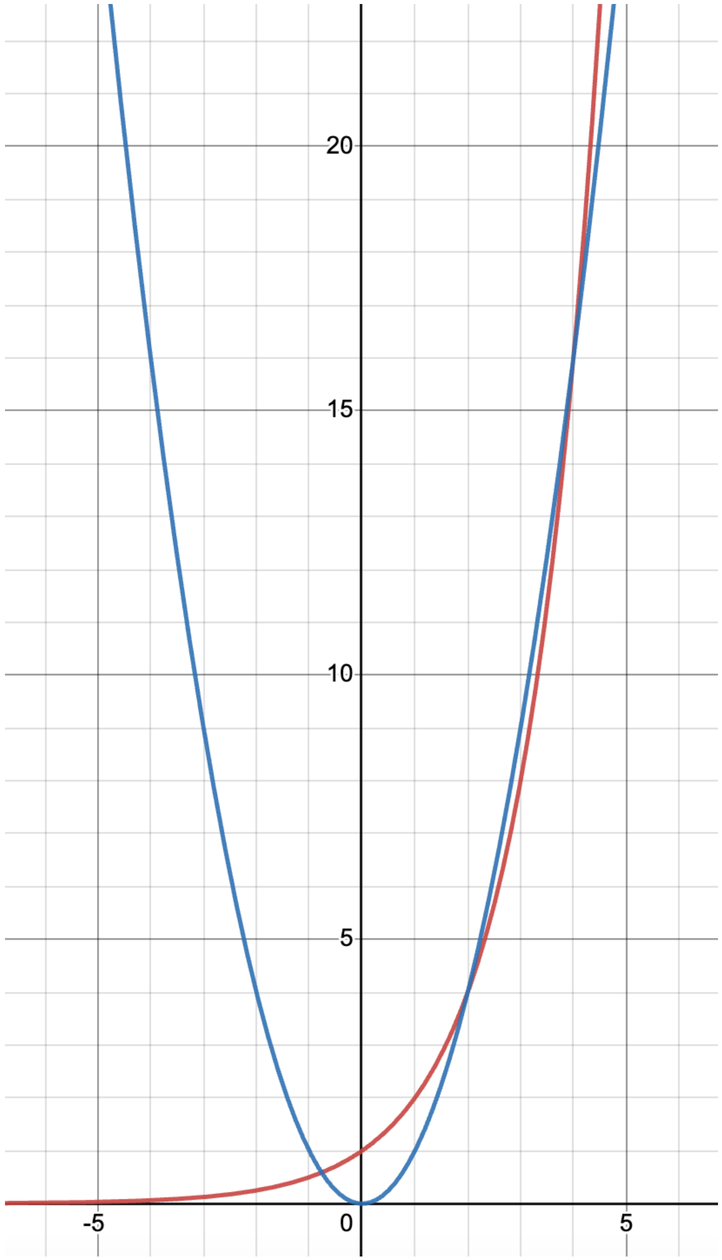
8 $= 6 \notin \{1, 3, 7, 9, 8\}$

$$\forall n \in \mathbb{N}.(n^2 \leq 2^n)$$

$\forall n \in \mathbb{N}.(n^2 \leq 2^n)$

False! E.g. for $n = 3$, the LHS is 9 and the RHS is 8.

$$\forall n \in \mathbb{N}, n \geq 4.(n^2 \leq 2^n)$$



$\forall n \in \mathbb{N}, n \geq 4.(n^2 \leq 2^n)$

Notice we start @ 4!

Base case: $4^2 = 16$, $2^4 = 16$ ✓.

Inductive step: Let $k \in \mathbb{N}$, $k \geq 4$, and assume

$k^2 \leq 2^k$. (IH), consider $k+1$...

$$(k+1)^2 = k^2 + 2k + 1 \overset{IH}{\leq} 2^k + 2k + 1.$$

$$\leq 2^k + (k-2)\cdot k + 1$$

$$= 2^k + \underbrace{\boxed{k^2 - 2k + 1}}_{=(k-1)^2}$$

$$\leq 2^k + 2^k \underbrace{- 2k + 1}$$

$$\leq 2^k + 2^k = 2^{k+1}$$

$k \geq 4 \Rightarrow k - 2 \geq 2$

$k \geq 4 \Rightarrow -2k + 1 \leq 0$

All birds have the same color

Claim. $\forall n \in \mathbb{N}$, a set of n birds will all have the same color.

Is this claim true?

[1] This example is usually "all **horses** have the same color," but I do not know how to draw horses - hence "all birds have the same color"

All birds have the same color

Claim. $\forall n \in \mathbb{N}$, a set of n birds will all have the same color.

Is this claim true?

No, of course not![1]

[1] This example is usually "all **horses** have the same color," but I do not know how to draw horses - hence "all birds have the same color"

All birds have the same color - "proof"

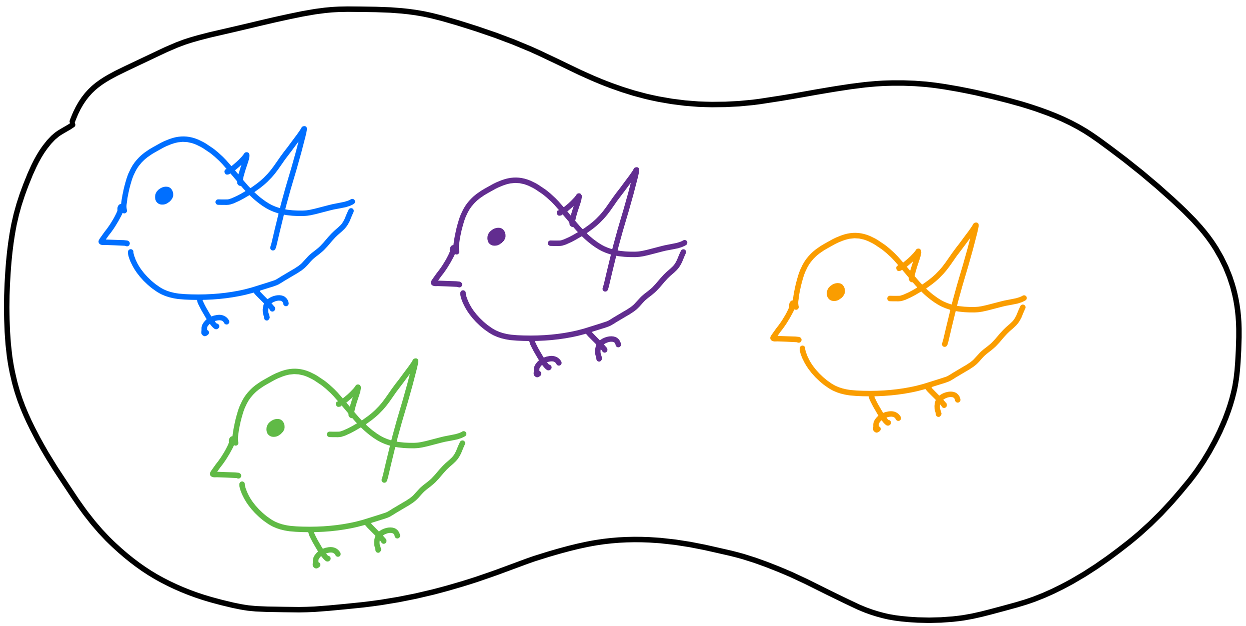
Claim. $\forall n \in \mathbb{N}$, a set of n birds will all have the same color.

Base Case. For $n = 0$, the claim is vacuously true.

All birds have the same color - “proof”

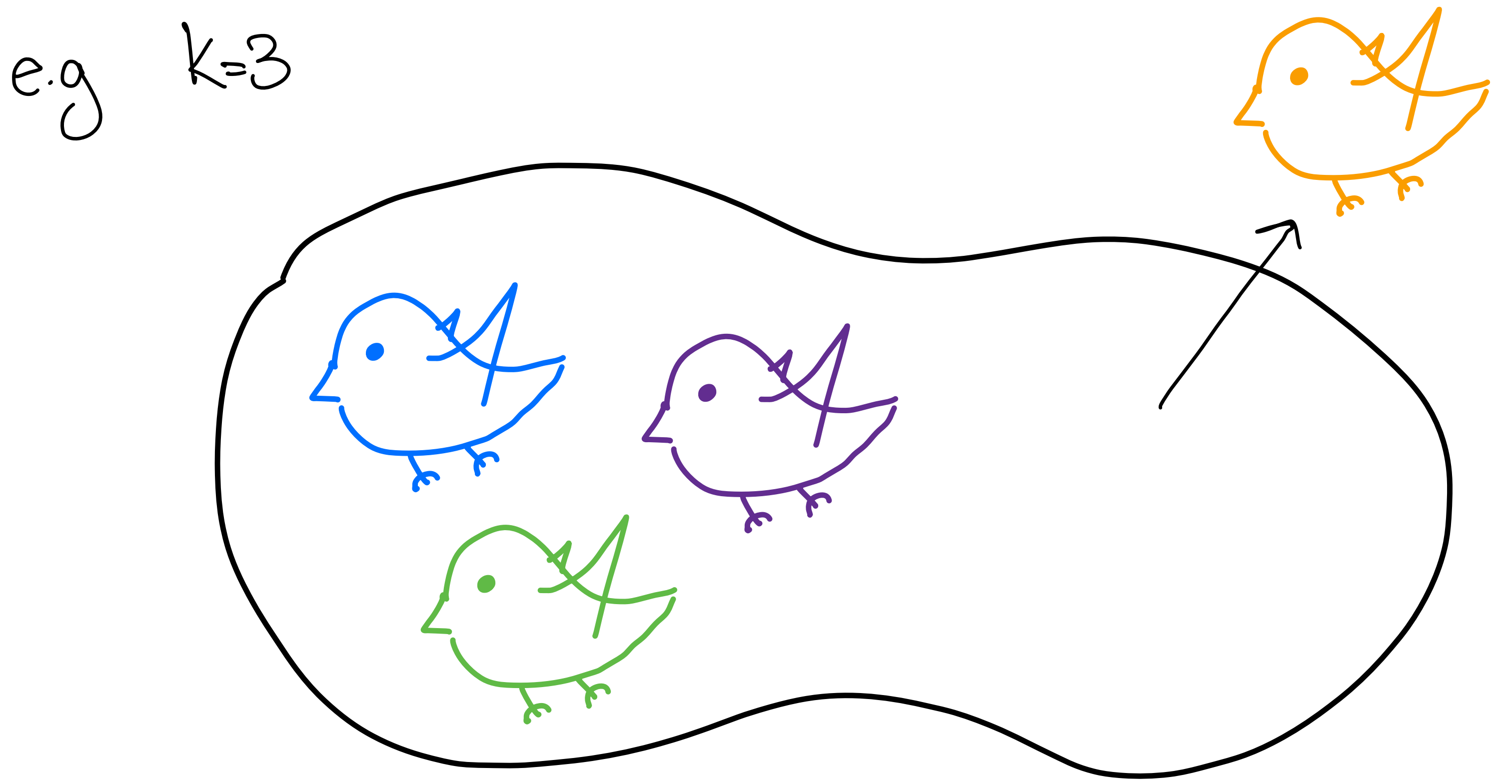
e.g $k=3$

S :



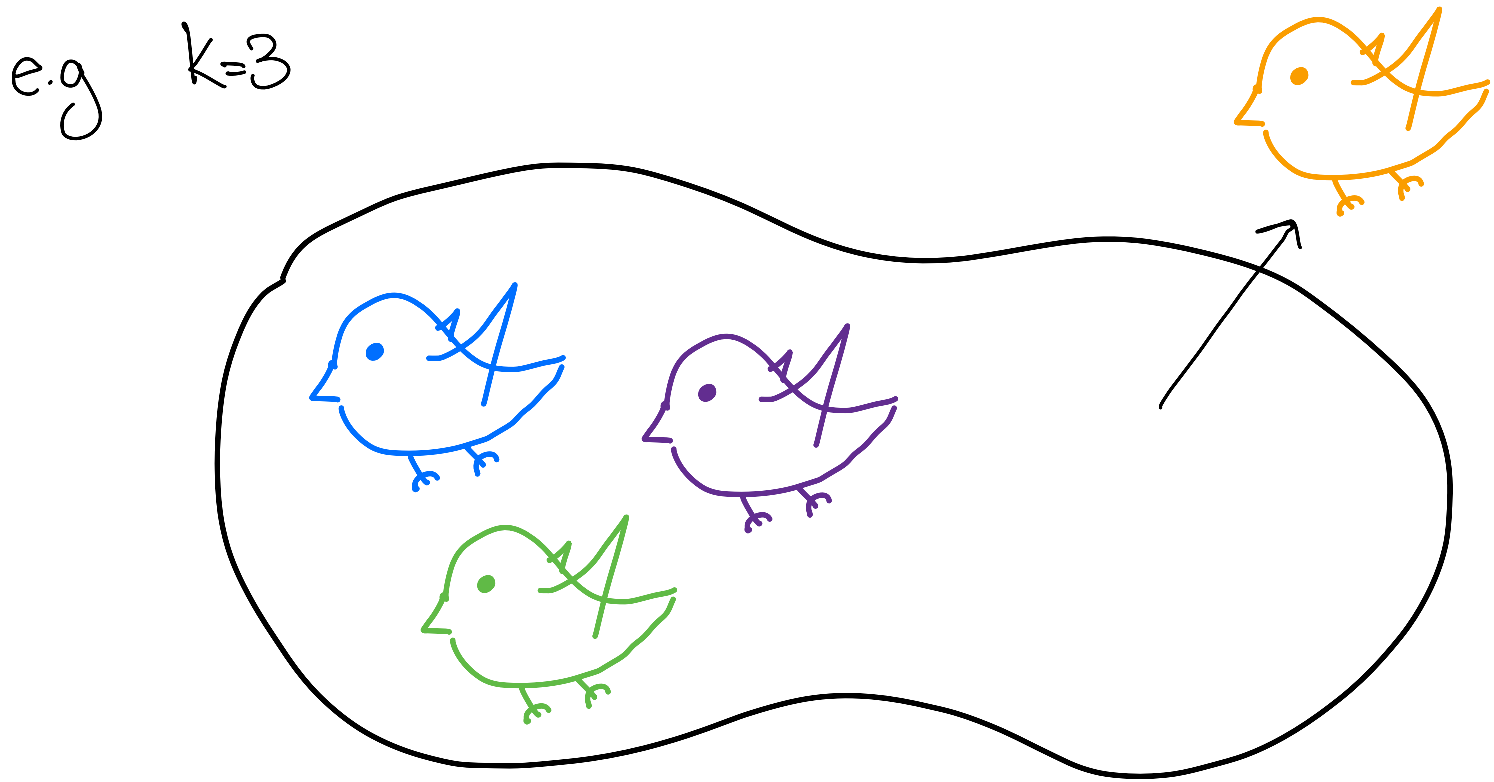
Inductive step. Let $k \in \mathbb{N}$ be any number and assume a set of k birds will all have the same color. Let S be a set of $k+1$ birds, we’ll show that all the birds in S have the same color.

All birds have the same color - “proof”



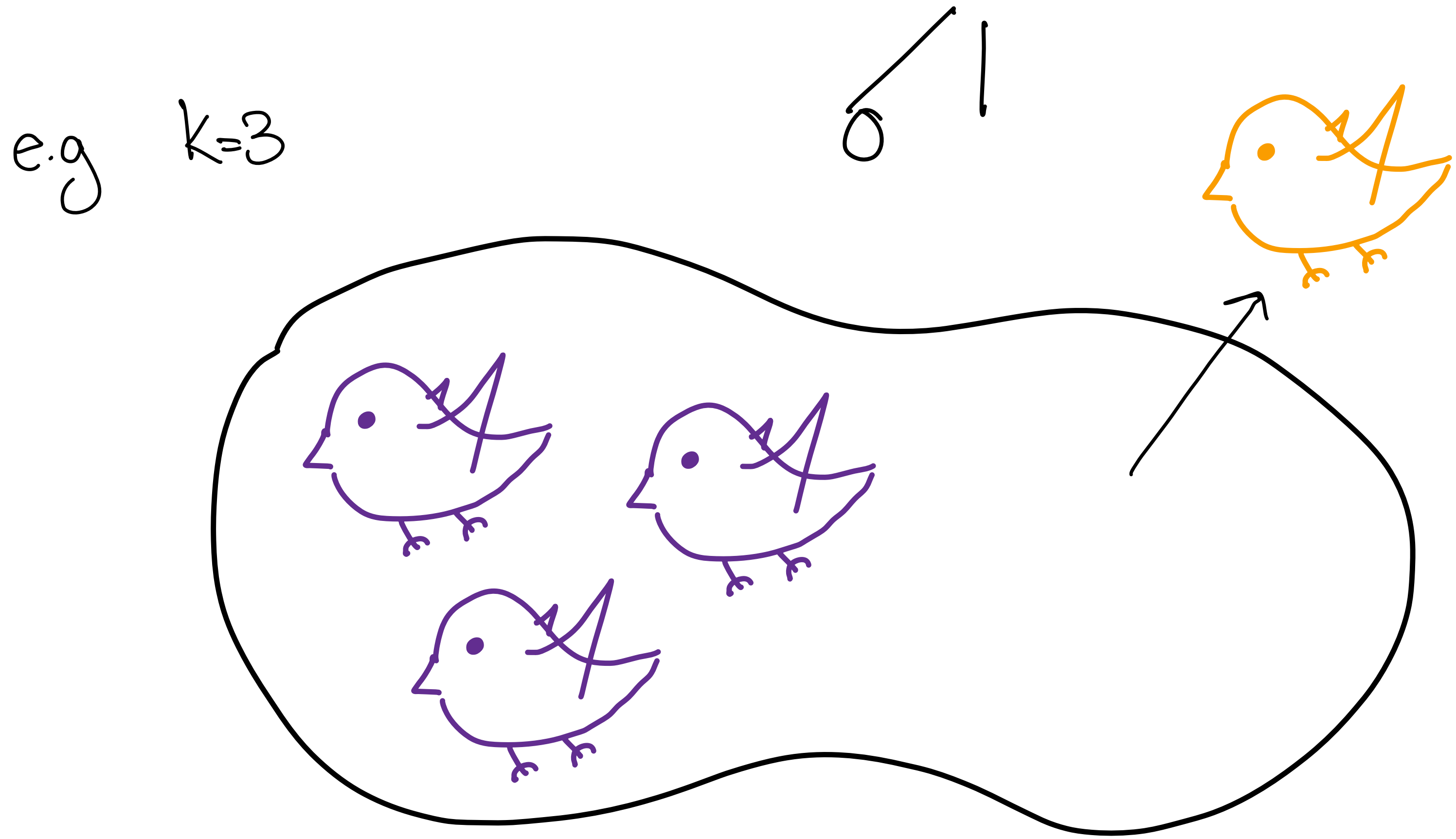
Remove an arbitrary bird b_1.

All birds have the same color - “proof”



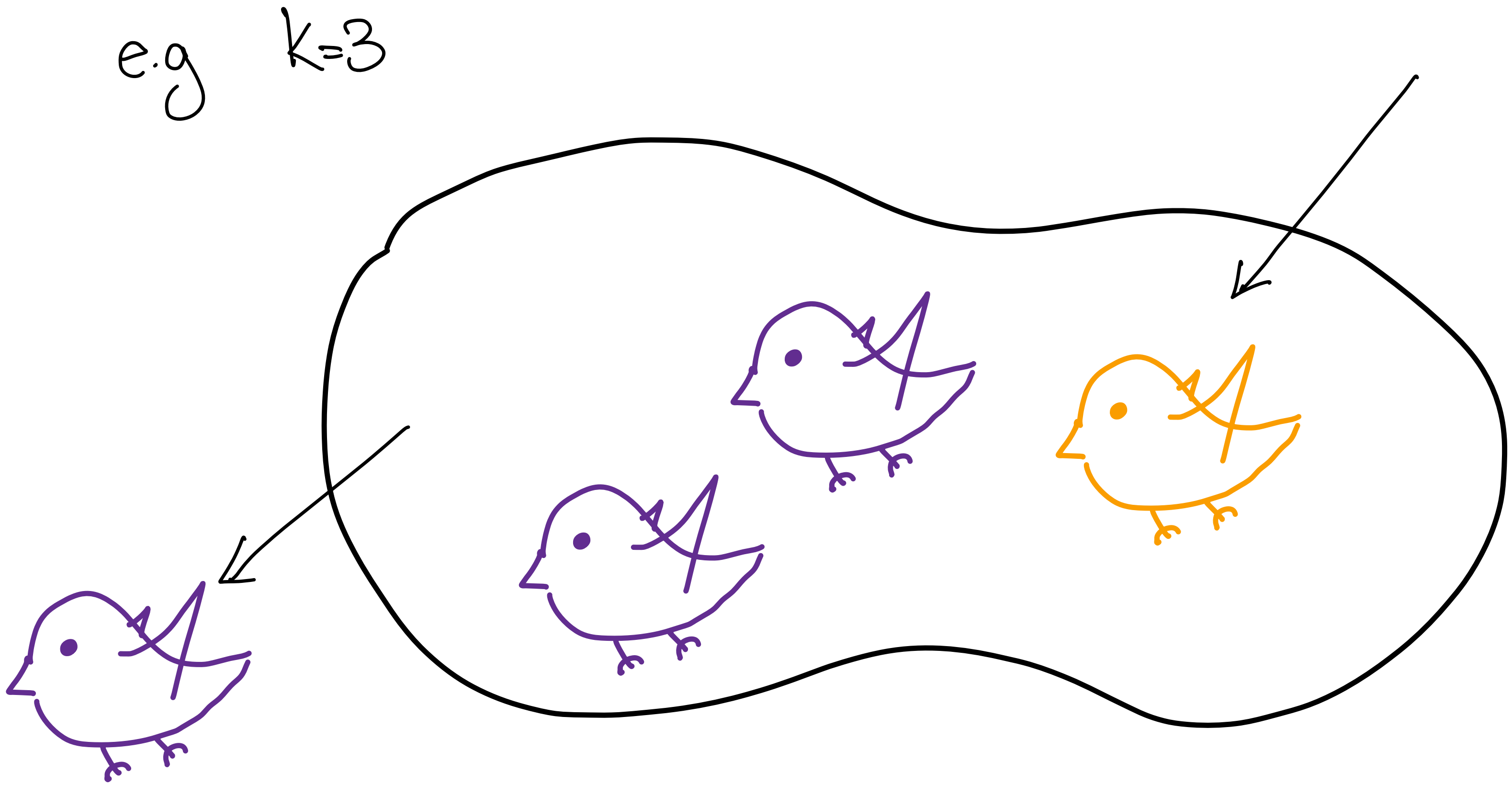
The remaining k birds must have the same color by the inductive hypothesis.

All birds have the same color - “proof”



The remaining k birds must have the same color by the inductive hypothesis.

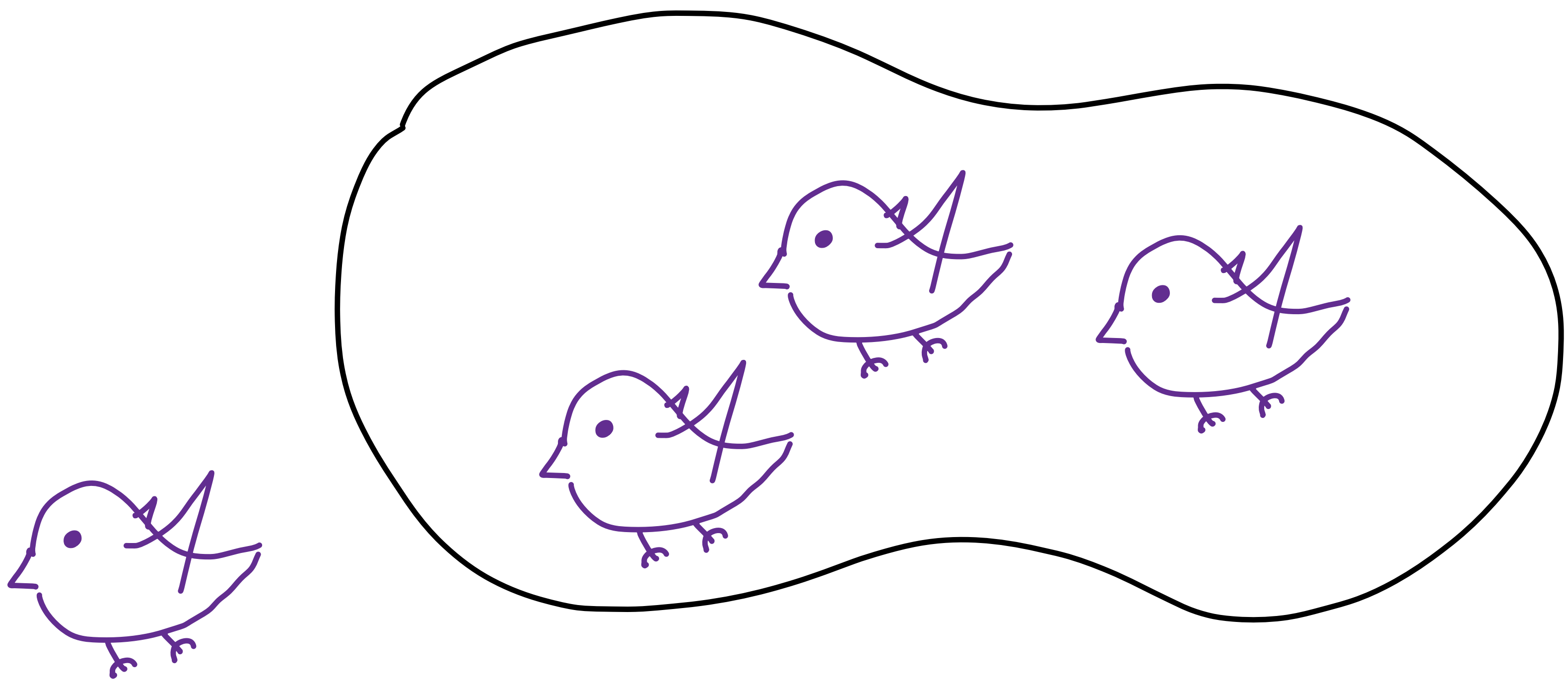
All birds have the same color - “proof”



Add back the removed bird and now remove a different bird, b_2.

All birds have the same color - “proof”

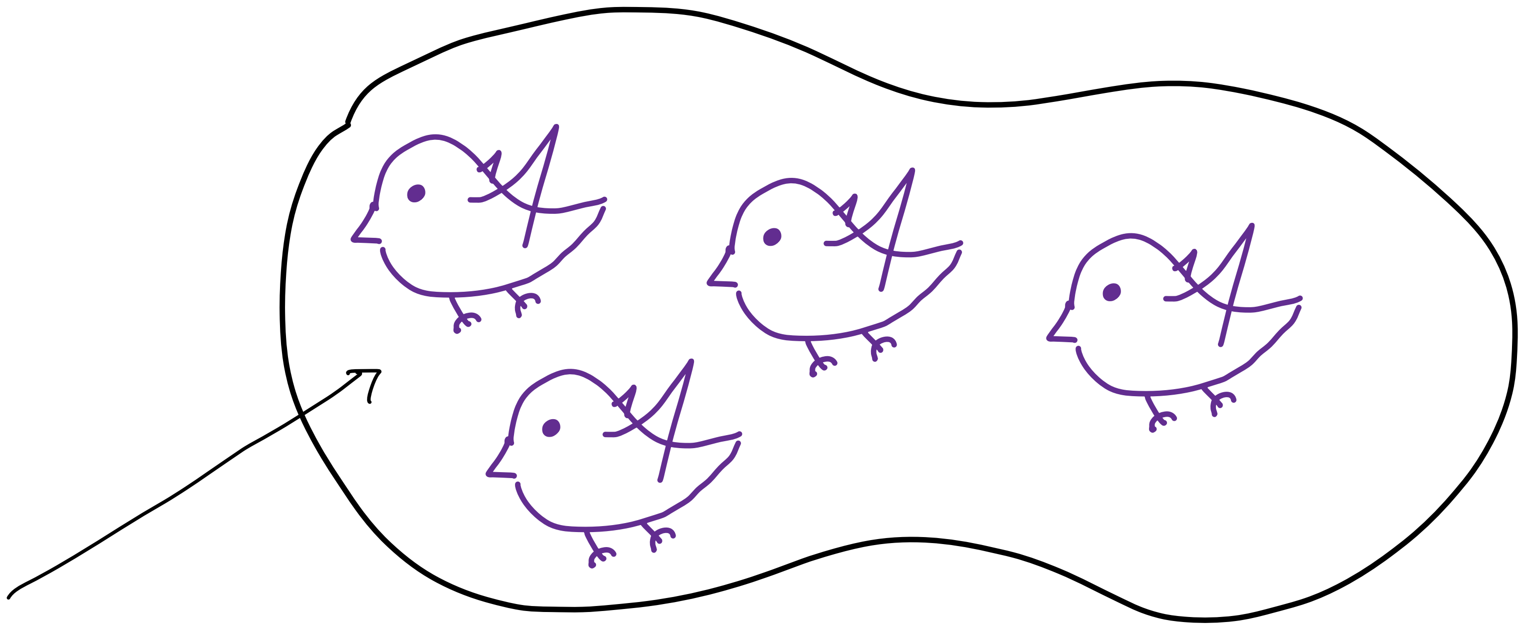
e.g $k=3$



By the same reasoning, the remaining k birds must have the same color. Therefore b_1 has the same color as birds which have not been removed which in turn have the same color as b_2.

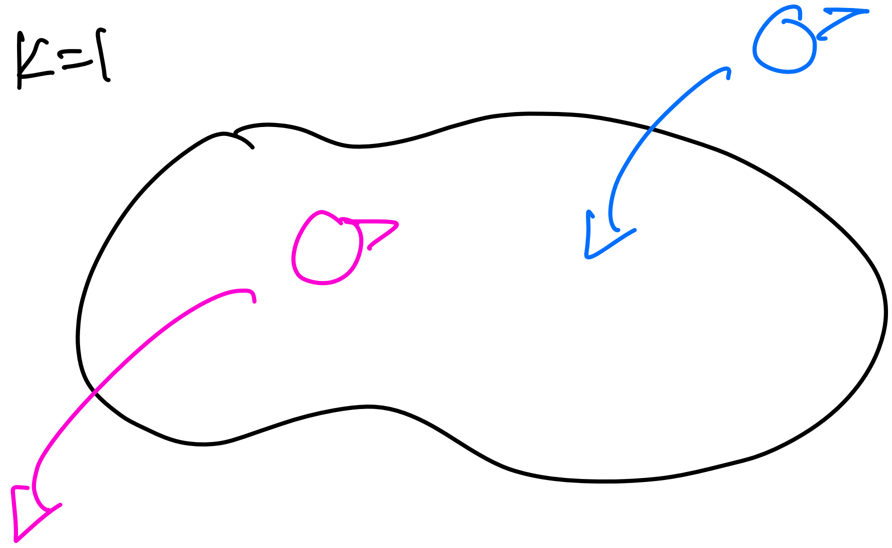
All birds have the same color - “proof”

e.g $k=3$



All birds have the same color. $\square$

What went wrong?

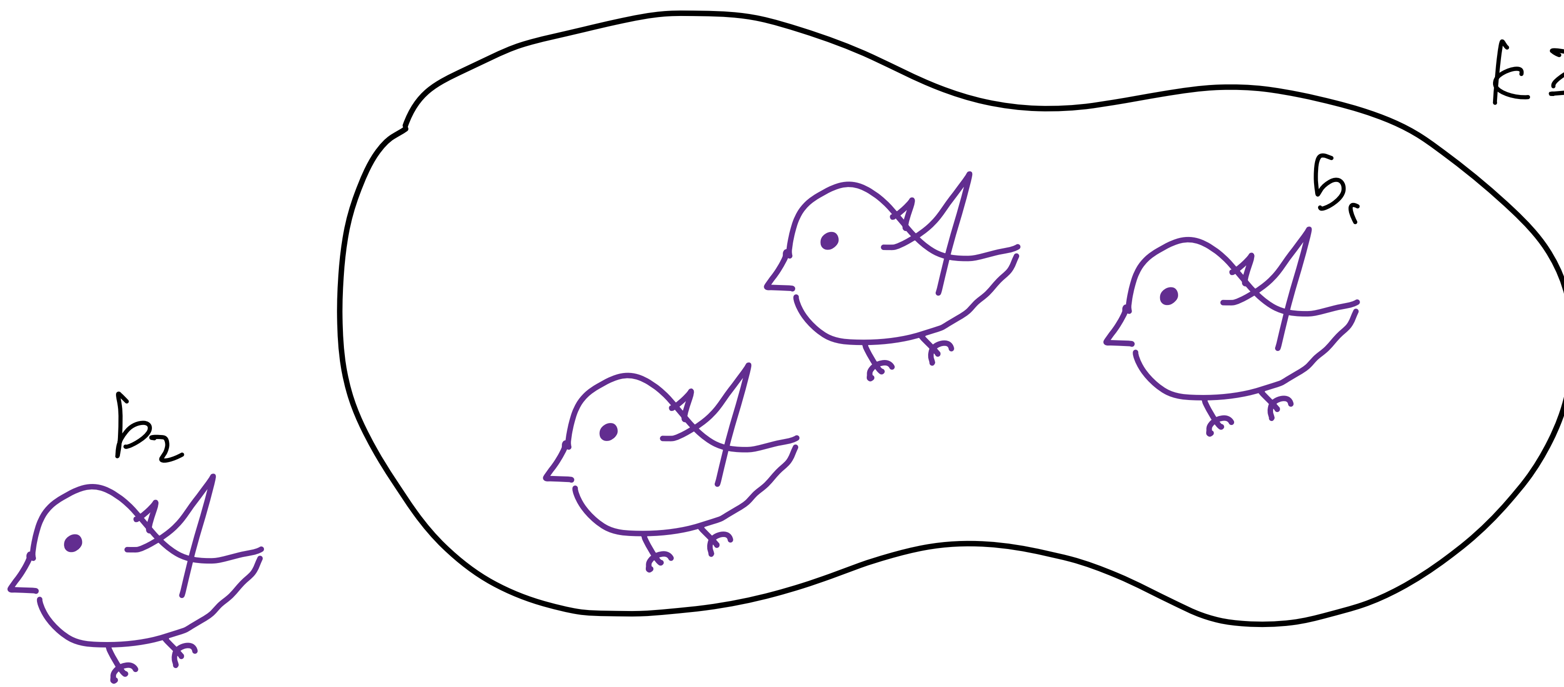


What went wrong?

$\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1).$

e.g $k=3$

$k \geq 2.$



By the same reasoning, the remaining k birds must have the same color. Therefore b_1 has the same color as the birds which have not been removed which in turn have the same color as b_2.[a]

---

[a] There is an implicit assumption here that there is another bird other than b_1 and b_2! I.e. for this to hold, we need $|S| \geq 3$! In particular, it does not hold for $k = 1$, $(|S| = 2)$.

Takeaway

Induction can be tricky! Make sure your inductive step does not assume anything more than what you claim! For example, in the $n^2 \leq 2^n$ example, we could assume $k \geq 4$ since we were restricting to the case where $k \geq 4$, but we couldn't do the same for the "all birds have the same color" example.

Induction

Examples of Proofs by Induction

Complete Induction

Complete Induction

Complete induction is another way to prove statements of the form $\forall n \in \mathbb{N}.(P(n))$.

Another way to get all the dominoes to fall

If I want to show $\forall n \in \mathbb{N}.(P(n))$, it suffices to prove

- $P(0)$
- $\cancel{\forall k \in \mathbb{N}.(P(k) \implies P(k+1))}$
- $\forall k \in \mathbb{N}.((P(0) \wedge P(1) \wedge ... \wedge P(k)) \implies P(k+1))$

"If I can show that the first domino falls, and I can show that for any domino, if that domino falls **and all previous dominoes fall**, that the next one also falls, every domino falls".

Complete Induction template

Say we wanted to prove $\forall n \in \mathbb{N}.(P(n))$. Here is the template:

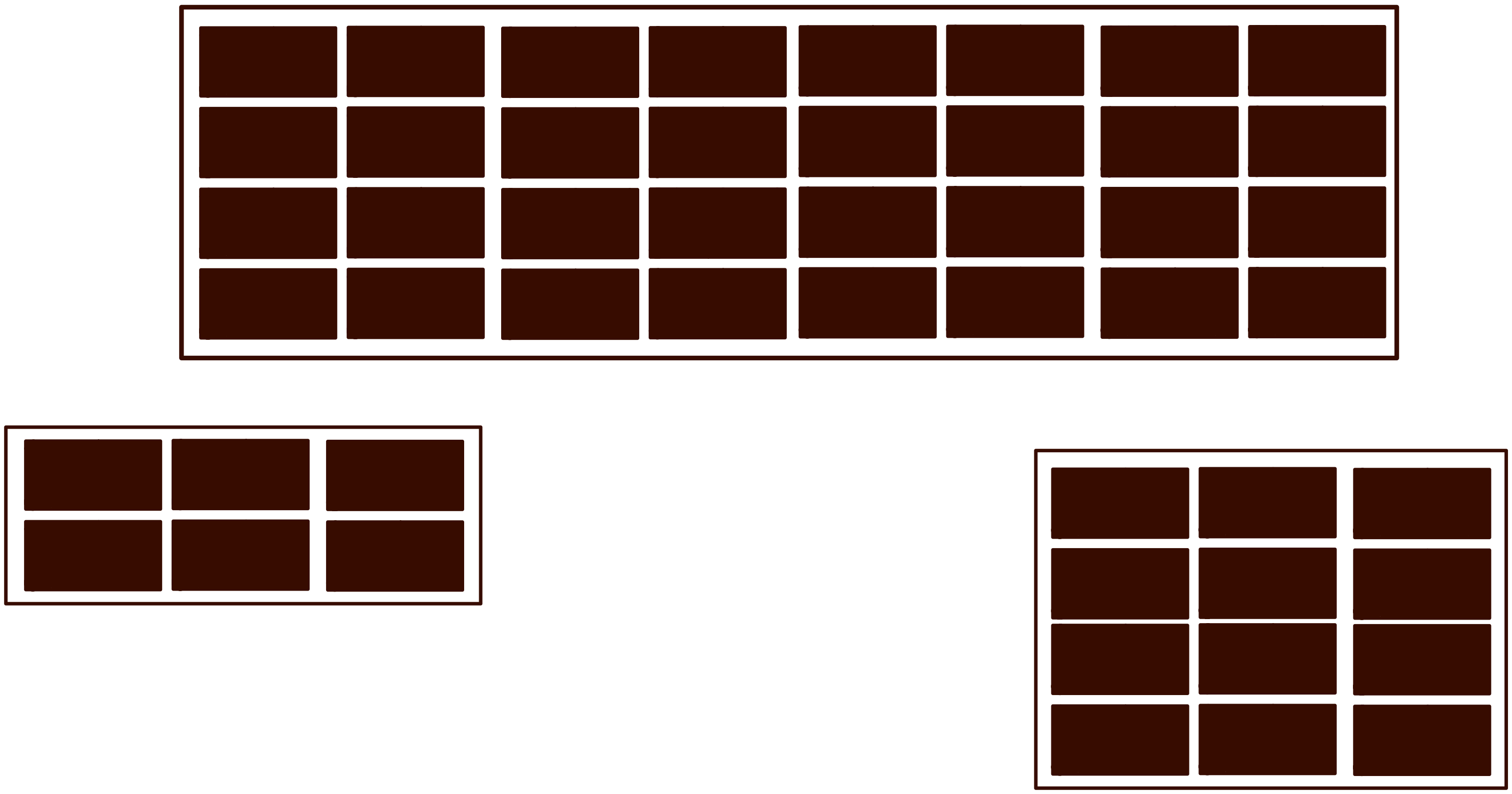
By complete induction.

Base case. [Prove $P(0)$ is true]

Inductive step. Let $k \in \mathbb{N}$ be an arbitrary natural number, and assume for every $i \in \mathbb{N}$ with $i \leq k$, $P(i)$ holds. We'll show $P(k+1)$. [Prove $P(k+1)$ using this assumption]

Note. $\forall i \in \mathbb{N}, i \leq k.(P(k))$ is just another way of writing $P(0) \wedge P(1) \wedge ... \wedge P(k)$. This is again called the inductive hypothesis.

Chocolate



How many breaks do you need to split the chocolate bar into individual pieces?

Chocolate - Attempted proof by regular induction

Claim: Let $n \in \mathbb{N}, n \geq 1$ be any natural number. A chocolate bar with n individual pieces requires $n-1$ breaks to split the bar into individual pieces.

Let $P(n)$ be the predicate: A bar of chocolate composed of n individual pieces requires $n-1$ breaks.

Base case. For $n=1$, the chocolate bar is already a single piece of chocolate and so requires $1-1=0$ breaks.

Inductive step. Let $k \in \mathbb{N}, k \geq 1$ be an arbitrary natural number at least 1, and assume $P(k)$ is true. We need to show $P(k+1)$ is true. Let B be a bar of chocolate with $k+1$ pieces and pick a way break a chocolate. We are left with two blocks of chocolate of size a and b respectively where $a+b=k+1$.

Chocolate - Attempted proof by regular induction

Claim: Let $n \in \mathbb{N}, n \geq 1$ be any natural number. A chocolate bar with n individual pieces requires $n-1$ breaks to split the bar into individual pieces.

Let $P(n)$ be the predicate: A bar of chocolate composed of n individual pieces requires $n-1$ breaks.

Base case. For $n = 1$, the chocolate bar is already a single piece of chocolate and so requires $1-1 = 0$ breaks.

Inductive step. Let $k \in \mathbb{N}, k \geq 1$ be an arbitrary natural number at least 1, and assume $P(k)$ is true. We need to show $P(k+1)$ is true. Let B be a bar of chocolate with $k+1$ pieces and pick a way break a chocolate. We are left with two blocks of chocolate of size a and b respectively where $a + b = k+1$. What now? We can't apply the IH since a, b might not be k!

$a-1 + b-1 + 1 = a+b-1$
$= k+1-1$

Chocolate - Proof by Complete Induction

Base case: $n=1$. already individual pieces so requires $1-1=0$ breaks.

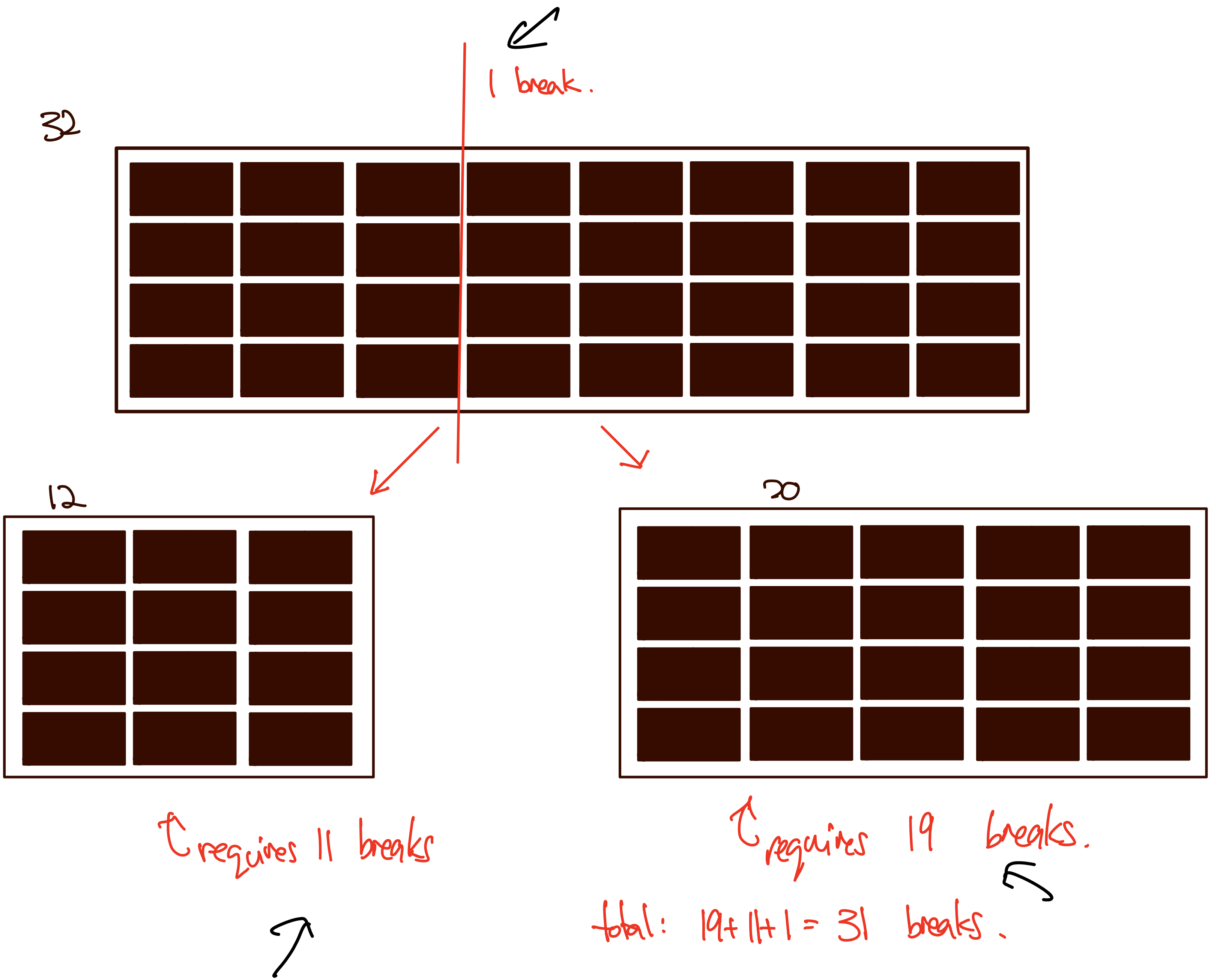
Inductive step: Let $k \in \mathbb{N}, k \geq 1$. assume $\forall i \leq k$, a chocolate bar w/ i pieces requires $i-1$ breaks. IH

consider a chocolate bar w/ $k+1$ pieces.

choose any way to break the chocolate bar into two pieces. The two pieces have size, a, b, where $1 \leq a, b \leq k$, $a+b=k+1$ since $1 \leq a, b \leq k$, we can apply the IH, so the two pieces require $(a-1)$, and $(b-1)$ breaks respectively.

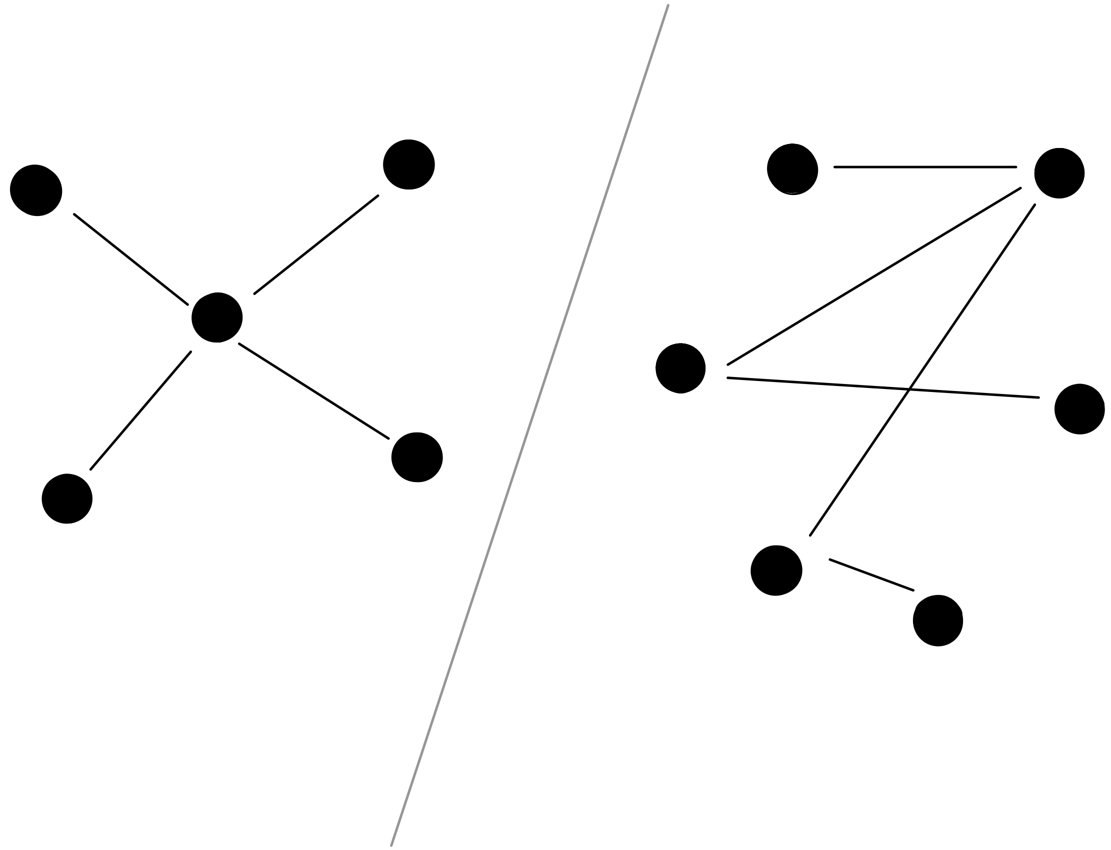
In total, we needed $a-1+b-1+1 = a+b-1$ breaks.

Chocolate - Proof by Complete Induction



Number of Edges in a Tree

As a reminder, a tree is a graph $G = (V, E)$ that is both acyclic (has no cycles) and connected (every pair of vertices is connected by some path). What was our conjecture from last time?



A tree has $|V|$ - 1 edges

By Complete induction.

Base case: ✓

Inductive step. IH Let $k \in \mathbb{N}, k \geq 1$, assume $\forall i \leq k$, $P(i)$.

a

Consider a tree on $k+1$ vertices. Take any edge $\{u, v\}$, consider what happens if you remove the edge $\{u, v\}$.

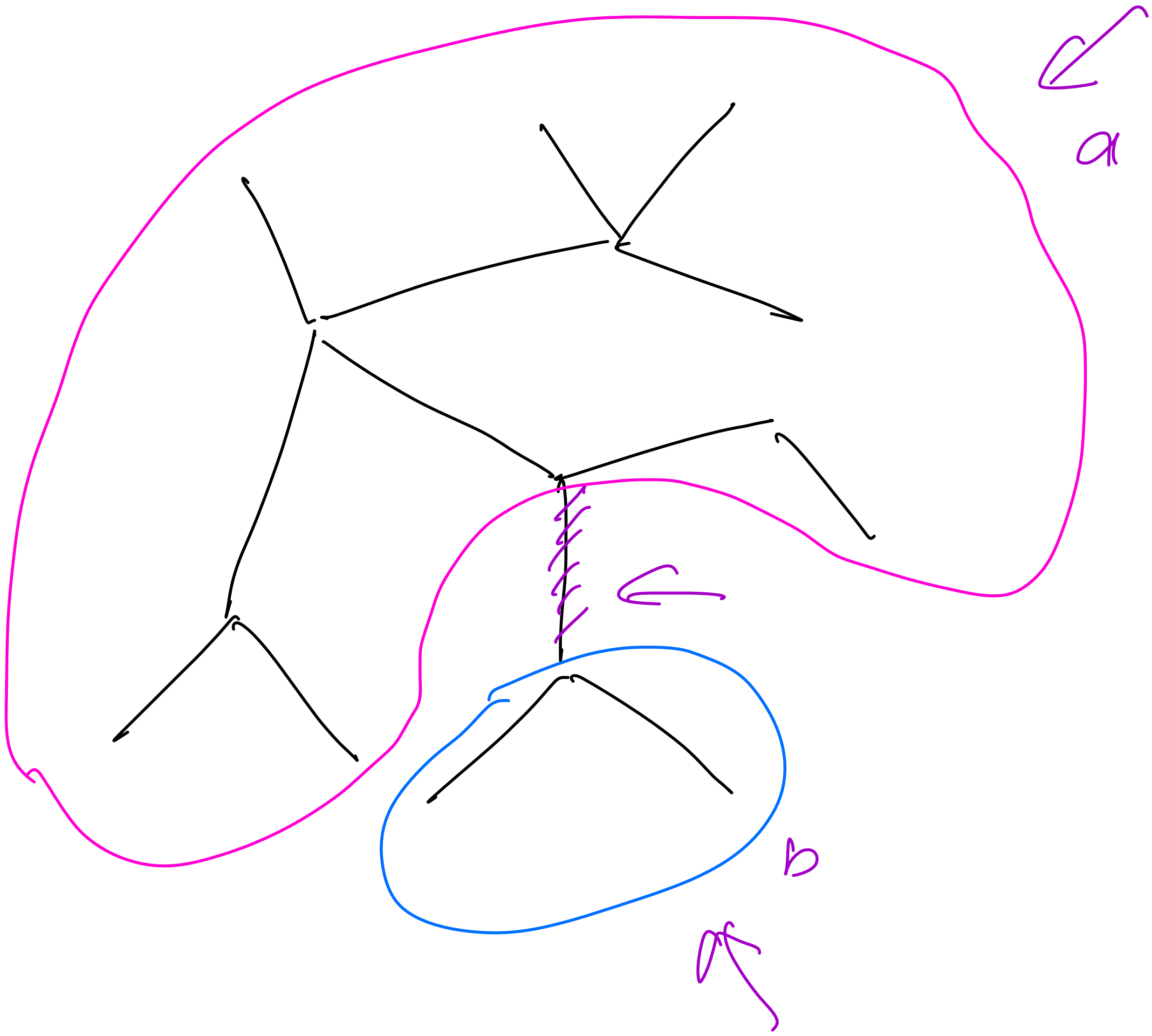
V_1 = all vertices that have a path to u,

V_2 = all vertices that have a path to v

argue V_1, V_2 are trees

$a = |V_1|$, $b = |V_2|$. $a + b = k+1$ $\boxed{1 \leq a, b \leq k}$

so total # of edges is $(a-1) + (b-1) + 1$



C
a
C
b
A

G is a Tree, I removed $\{u,v\}$.

$V_1 = \{x : \exists$ a path from x to u in $G - \{u,v\}\}$.

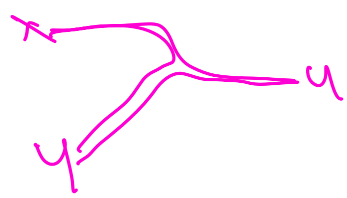
$V_2 = \{x : \exists$ a path from x to v in $G - \{u,v\}\}$.

1.) argue that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$.

2.) show that V_1, V_2 are trees.

↳ acyclic: didn't add any edges ✓.

↳ connected: $\forall x, y \in V_1$,



Why would I ever use regular induction if I can just use complete induction?

Why would I ever use regular induction if I can just use complete induction?

You can always use complete induction if you wish! The inductive hypothesis is stronger (I.e., you get to assume $P(0) \wedge \ldots \wedge P(k)$ instead of just $P(k)$), but still lets you prove the same statement: $\forall n \in \mathbb{N}.(P(n))$.

That being said, some mistakes are easier to make when using complete induction, and sometimes regular induction is easier to work with.

Induction and Algorithms

Proof by induction is an incredibly powerful technique that will allow us to prove strong guarantees about the runtime and correctness of algorithms.

I.e. Induction let's us prove $\forall n \in \mathbb{N}.(P(n))$ consider what this means when

- $P(n)$ is: Algorithm X is correct on inputs of size n.
- $P(n)$ is: Algorithm X is correct if the `for` loop runs for at most n iterations.
- $P(n)$ is: The runtime of Algorithm X is $\Theta(n^2)$.
- ..etc.

A note about style

Sometimes you might find it easier to define a predicate in the following way: "Let $P(n)$ be the predicate..." for example, in the chocolate example. This approach allows you to refer to the predicate easily. For example, defining P allows you to say "Assume $P(k)$ is true..."

Other times, you might find it easier to directly work with the predicate without giving it a name, for example, in the divisibility example. This approach makes stating the inductive hypothesis a little more troublesome but reminds the reader of exactly what you're trying to prove.

Both are fine!

Additional Notes

- Induction and recursive algorithms are closely linked. Think of how! We will explore this in future classes.
- Induction is a hard concept to grasp. In particular, it takes a little bit of faith to believe that simply showing a base case and an inductive step allows us to prove a statement is true for all natural numbers. It's good to keep the domino analogy in mind, i.e., when writing your proofs, ask - *'did I show all the dominoes fall?'*
- Although intuition is important, at the end of the day, remember that the base case and inductive step are both mathematical statements that you need to prove. I.e., you should approach proving $\forall k \in \mathbb{N}.(P(k) \implies P(k+1))$ like you would approach proving any other FOL statement.