CSC263H
Data Structures and Analysis

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Winter 2024 – Week 5

Dictionary ADT:

- **Objects**: A collection of **key-value pairs** (keys are *unique*).
- **Operations**:
  - *Search*(D,k): return x in D s.t. $x.key = k$, or NIL if no such x is in D.
  - *Insert*(D,x): insert x in D; if some y in D has $y.key$ equal to $x.key$, replace y by x.
  - *Delete*(D,x): remove x from D.

Data Structures for Dictionaries: Hash Tables

Problem 1: Read a grade file, where grades are integers between 0 to 99. Keep track of number of occurrences of each grade.
Fastest Solution: Create an array T of size 100. $T[i]$ stores the number of occurrences of grade i.

Problem 2: Read a data file, keep track of number of occurrences of each integer value (from 0 to $2^{32} - 1$).
Fastest Solution: Create an array of size 2^{32}, as above.
Wasteful use of memory, especially when data are files relatively small.

Problem 3: Read a text file, keep track of number of occurrences of each word.
Cannot use keys as indices anymore!

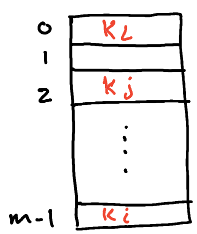
1. We need to be able to convert any type of key to an integer.
2. We need to map the universe of keys into a small number of slots.

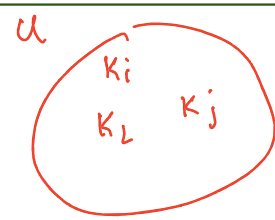
A **hash function** does both!

- **Universe** $\mathcal{U}$: The set of all possible keys.
- **Hash Function**: A function from the set of all possible keys to integers between 0 and $m-1$:
  $h : \mathcal{U} \to \{0, 1, ..., m-1\}$.

Hash Table: A data structure containing an array of length m and a hash function $h : \mathcal{U} \to \{0, 1, ..., m-1\}$.

- $h(k)$ maps a key k to one of the m positions in hash table T.
  That is, $h(k)$ is the the **index** at which the key k is stored.
- Each array location called a *slot* or a *bucket*.





$h(k_i) = m-1$

$h(k_l) = 0$

$h(k_j) = 2$

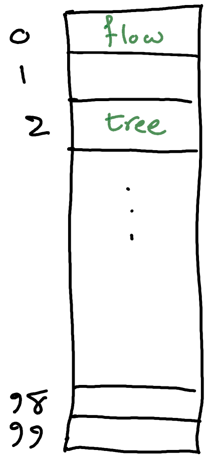
Hash Table: Collisions

- If $m \geq |\mathcal{U}|$, then there exists a hash function h which maps each key to a unique slot (no collisions).
  Such a function is called a **perfect hash function**.
  Typically the number of possible keys is much bigger than the number of array slots.

- If $m < |\mathcal{U}|$, then at least one **collision** occurs.

Example:

Suppose $\mathcal{U}$ is the set of all English words and $m = 100$



$h(\text{tree}) = 2$

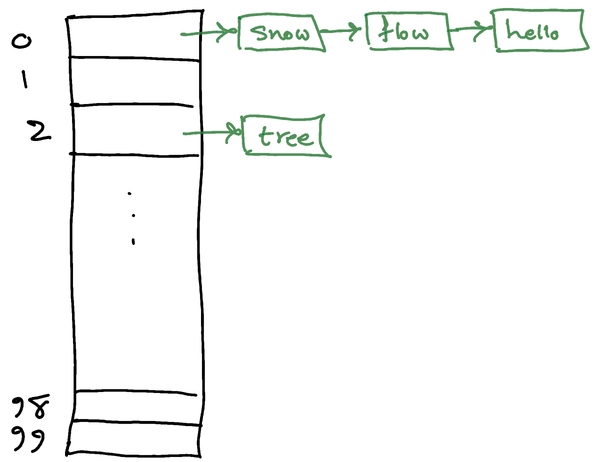
$h(\text{flow}) = 0$

$h(\text{ten}) = 2$

Handling Collisions:

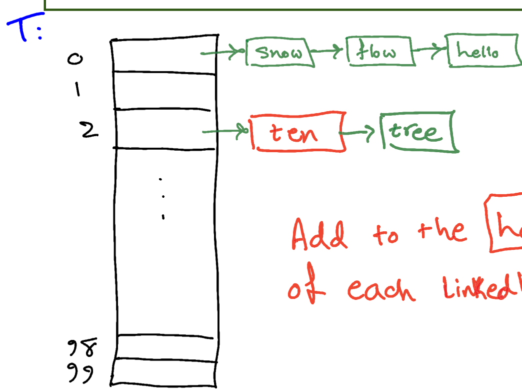
- Chaining (Closed Addressing).
- Open Addressing.

Chaining: Each bucket in the array points to a linked list of key-value pairs.



HashInsert(k,v):

1. Compute $h(k)$. Set $i = h(k)$.
2. Search the linked list stored at $T[i]$ to check whether an element with key k already exists.
3. If so, replace the existing value with v.
   If not, insert a new node to the head of the list.



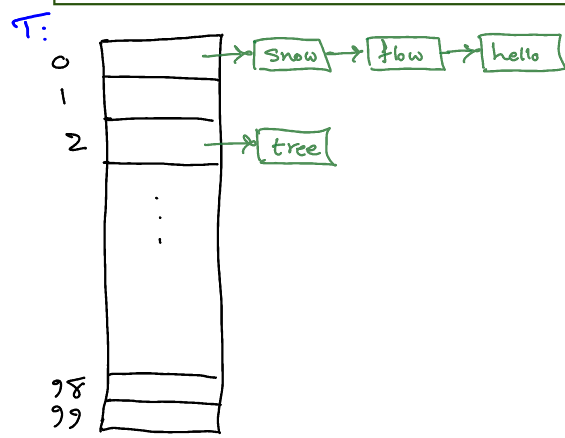
Add to the head of each linkedlist

HashInsert(ten, NIL)

h(ten) = 2

HashSearch(k):

1. Compute $h(k)$. Set $i = h(k)$.
2. Access index i in the table.
3. Search the linked list stored at $T[i]$.



Hashsearch(flow):

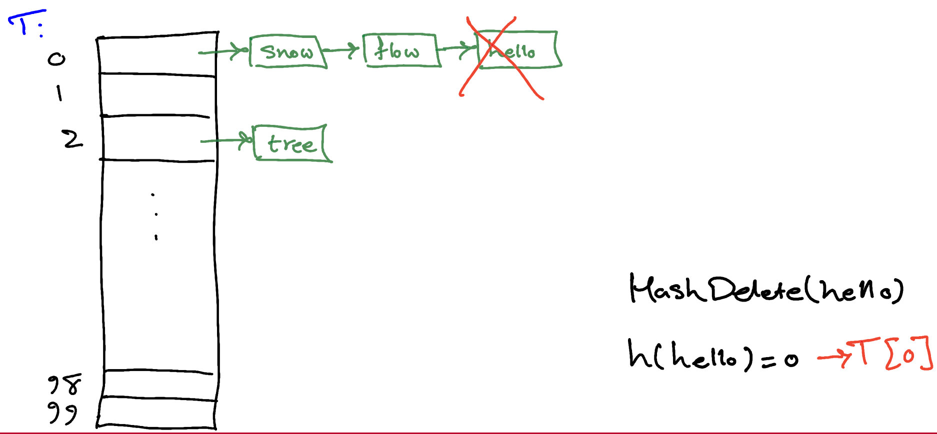
h(flow) = 0 → T[0] ✓

HashSearch(hey)

h(hey) = 2 → T[2] ✗

HashDelete(k):

1. Compute $h(k)$. Set $i = h(k)$.
2. Search the linked list stored at $T[i]$.
3. If an element with key k is found, delete it from the list.



Chaining: Worst-Case Running Time

Worst Case for Search: All elements in the table are hashed into the same bucket

n is the total number of elements stored in the hashtable

Worst-case Running Time:

- *HashSearch*: $\Theta(n)$
- *HashInsert*: $\Theta(n)$
- *HashDelete*: $\Theta(n)$

We assume that the hash value $h(k)$ is computed in constant time.
That is, the run time for computing $h(k)$ is in $\Theta(1)$.

Practical consideration: Hash tables work well in real-world applications. That is because:

- The worst case almost never happens.
- Average case performance is really efficient.

Analyzing Average-Case Running Time – Review

1. Define a random variable $t_n(x)$ denoting the number of steps executed by the algorithm on an input x with size n.

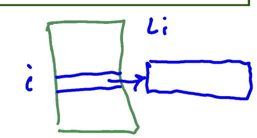
2. Compute $\mathbb{E}[t_n]$.

- Let $t_{m,n}(k)$ denote the number of steps executed by *HashSearch* to find k in a hash table containing m buckets and storing n elements.

- **Simple Uniform Hashing Assumption (SUHA):** Any key equally likely to hash to any bucket.

$i = \underbrace{h(k)}_{\Theta(1)}$

$\mathbb{E}[t_{m,n}(k)] =$ 1 + expected running time of searching for k in L_i



Chaining: Search Average-Case Running Time

Expected Run Time in an Unsuccessful Search (under SUHA):

$\mathbb{E}[t_{m,n}(k)] =$ 1 + expected length of L_i $= 1 + \frac{n}{m}$

- Probability that key k is hashed to bucket i (i.e., the probability of $h(k) = i$):
  There are m candidate position for k, k is equally likely to hash to any of them:

$$Pr[h(k) = i] = \frac{1}{m}$$

- The expected length of the linked list stored at a bucket i:

$$\mathbb{E}[len_i] = \frac{n}{m}$$

where len_i denotes length of the linked list stored in bucket i.

Load Factor of a hash table T: The ratio of the number of keys, denoted by n, stored in T to the number of buckets of T, denoted by m.

$$\alpha = \frac{n}{m}$$

$$E[t_{m,n}(k)] = 1 + \alpha$$
$$\in \Theta(1+\alpha)$$

If $\frac{n}{m} \in \Theta(1)$, then the average-case run time is also in $\Theta(1)$

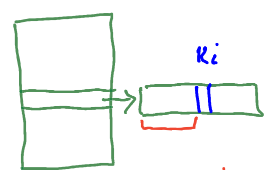
Chaining: Search Average-Case Running Time

Expected Run Time in a Successful Search (under SUHA):
That is k is a key that exists in the hash table.

Let $k_1, k_2, k_3, ..., k_n$ be the *order of insertion* into the hash table.
k could be k_1, or k_2, or k_3, or ...., or k_n.
The probability that k is k_i $(1 \leq i \leq n)$ is: $\frac{1}{n}$

So the expected number of steps to find k is the sum over:
the probability that k is k_i, times the number of steps required to find k_i

$$\mathbb{E}[t_{m,n}(k)] = \frac{1}{n} \times S_1 + \frac{1}{n} \times S_2 + \frac{1}{n} \times S_3 + ... + \frac{1}{n} \times S_n$$
$$= \frac{1}{n} \sum_{i=1}^{n} S_i$$



S_i denotes the <u>*expected number of steps*</u> to find k_i.

S_i = expected number of steps to find k_i
= number of elements examined during search for k_i
$= 1+$ number of elements **before** k_i in the linked list stored at $h(k_i)$
$= 1+$ number of keys that hash **same as** k_i and are inserted **after** k_i.

b.c. we insert at the front

Let X_i be the expected number of keys that hash samely as k_i.
Define **indicator random variables** for X_i:

$$X_{i,j} = \begin{cases} 1 & \text{if } h(k_i) = h(k_j) \\ 0 & \text{otherwise.} \end{cases}$$

Indicator Random Variables – Reminder

Recall that indicator random variables are $X_1, X_2, ..., X_m$ s.t.:

- $X = X_1 + X_2 + ... + X_m$;
- Each X_i has only two possible values: 0 or 1.

Then $\mathbb{E}[X]$ is computed as follows:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_1 + ... + X_m] \\ &= \mathbb{E}[X_1] + ... + \mathbb{E}[X_m] && \text{(by linearity of expectation)} \\ &= Pr[X_1 = 1] + ... + Pr[X_m = 1] \end{aligned}$$

where the last equality holds because for each X_i:
$\mathbb{E}[X_i] = 0 \times Pr[X_i = 0] + 1 \times Pr[X_i = 1] = Pr[X_i = 1]$.

Chaining: Search Average-Case Running Time

Let X_i be the expected number of keys that hash samely as k_i.
Define **indicator random variables** for X_i:

$$X_{i,j} = \begin{cases} 1 & \text{if } h(k_i) = h(k_j) \\ 0 & \text{otherwise.} \end{cases}$$

Example:

4 keys are hashed same as k_1 $\rightarrow$ $X_1 = 4$

k_2 k_{10} k_{15} k_{20}

$X_{1,2} = 1$

$X_{1,10} = 1$

$X_{1,15} = 1$

$X_{1,20} = 1$

$$X_{i,j} = \begin{cases} 1 & \text{if } \boxed{h(k_i) = h(k_j)} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[X_{i,j}] = Pr[X_{i,j}] = \frac{1}{m}$$

$S_i =$ 1 + number of key that are hashed sames as k_i and are inserted after k_i

$$= 1 + \sum_{j=i+1}^{n} E[X_{i,j}]$$

$$= 1 + \sum_{j=i+1}^{n} \frac{1}{m}$$

$$\mathbb{E}[t_{m,n}(k)] = \frac{1}{n}\sum_{i=1}^{n} S_i = \frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n} \frac{1}{m}\right)$$

$$= 1 + \frac{n}{2m} - \frac{1}{m} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2n}$$

$$\in \Theta(1+\alpha)$$

Average-case Running Time:

- *HashSearch*: $\Theta(1+\alpha)$
- *HashInsert*: $\Theta(1+\alpha)$
- *HashDelete*: $\Theta(1+\alpha)$

If the number of hash-table slots is **at least proportional** to the number of elements in the table, then $\alpha \in \Theta(1)$.
That is, all dictionary operations can be implemented with **constant** average run time.

Hash Table: Open Addressing

Open Addressing: Store all items directly in T (no chaining).
If a collision occurred, look for another free spot in some **systematic manner** called **probing**.

Implication: The number of keys stored in the hash table cannot exceed the length of the array.

Insert "tree", where
h(tree) = 2

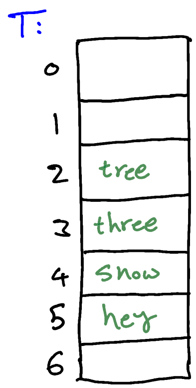
Insert "hey", where
h(hey) = 3

Insert "hello", where
h(hello) = 2 → try T[2] ×
" T[3] ×
" T[4] ✓

Probe Sequence: Sequence of buckets to try.

Search: Follow the same probing approach used for insertion.
Search returns *None* when encounters the first bucket that stores *None*.

Implication: Searching for an item requires examining more than just one spot.



Search for "hey", where

h(hey) = 3 → Try T[3] X

→ Try T[4] X

→ Try T[5] ✓

Search for "hello", where

h(hello) = 4 → Try T[4] X

→ Try T[5] X

→ Try T[6] → empty so return Fail

Open Addressing: Linear probing

Linear Probing: Examine a linear sequence of slots.
Example probe sequence: $h(k)$, $h(k)+1$, $h(k)+2$, $h(k)+3$, ...

Probe sequence: $(h(k)+i) \mod m$, for an integer $i \geq 0$.

Problem: Contiguous blocks of occupied locations (clusters) are created, causing further insertions of keys into any of these locations to take a long time.

Quadratic Probing: Examine a non-linear sequence of slots.
Example probe sequence: $h(k)$, $h(k) + 2$, $h(k) + 6$, $h(k) + 12$, ...

Probe sequence: $(h(k) + c_1 \times i + c_2 \times i^2) \mod m$, for an integer $i \geq 0$.

$c_1, c_2 = 1$

T:

0	
1	hello
2	tree
3	
4	snow
5	
6	

Insert "hello", where $i=0$

$h(hello) = 2 \rightarrow T[2]$ ×

$i=1$

$h(hello) + \underbrace{1}_{c_1} \times \underbrace{1}_{i} + \underbrace{1}_{c_2} \times \underbrace{1}_{i^2} = 4 \rightarrow T[4]$ ×

$(2 + 1 \times 2 + 1 \times 4) \mod 7 = 1$

Problems:

- Collisions still cause a milder form of clustering.
- Need to be careful with the values of c_1 and c_2, as it could jump in such a way that some of the buckets are never reachable.

Double Hashing: Use a second hash function to generate step values that dependents on the key.
Example probe sequence: $h_1(k),\ h_1(k) + h_2(k),\ h_1(k) + 2h_2(k),\ h_1(k) + 3h_2(k)$, ...

Probe sequence: $(h_1(k) + i \times h_2(k)) \mod m$, for an integer $i \geq 0$.

Insert "hello", where $h_1(\text{hello}) = 2$ and $h_2(\text{hello}) = 3$

Try T[2] ×

$h_1(\text{hello}) + 1 \times h_2(\text{hello}) = 2 + 1 \times 3 = 5$

Try T[5] ×

$(h_1(\text{hello}) + 2 \times h_2(\text{hello})) \bmod 7 =$

$(2 + 2 \times 3) \bmod 7 = 1 \rightarrow$ Try T[1] ✓

Open Addressing: Running Time

Under **Simple Uniform Hashing Assumption** (SUHA):

- Average-case number of probes in an **unsuccessful** search: $\frac{1}{1-\alpha}$.

- Average-case number of probes in an **successful** search: $\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$.

(Proof in Section 11.4 of CLRS (optional)).

In **practice** open addressing works best when $\alpha < 0.5$.

Hash Functions

A **good** hash function h should:

- ensure simple uniform hashing.
- h(x) should depends on every part/bit of x (even for complex objects).
- should spreads out values.
- should be possible to be computed in constant time (i.e., in $\Theta(1)$).

In practice, it is difficult to get all of these properties, but there are some **heuristics** that work well.

Important Note: When answering questions in assignments and tests of this course, you can assume that a good hash function exists, unless the question explicitly says otherwise.

Ideas for Implementing Hash Functions

Intuition: Interpreting keys as integers

- Every object stored in a computer can be represented by a bit-string (string of 0's and '1s).
- The bit-string can be considered as base 2 representation of a non-negative integer.
- That is, any type of key can be converted to an integer.
- The integer value, however, is usually very larger than the number of buckets.

So a hash function needs to map larger integers to a small set of integers $\{0, 1, ..., m-1\}$ (0,1,...,$m-1$ represent indexes of the buckets).

Implementing Hash Functions: Division Method

Division method: $h(k) = k \mod m$

Pitfall: Sensitive to the value of m.
Example 1: $k \mod 10$ depends only on last decimal digit of k.

Example 2: $k \mod 8$ depends only on last 3 bits of k.

Example 3: $k \mod 2^p$ depends only on last p bits of k.

Implication: Keys are not spread out.

Good choice for m: A prime not too close to an exact power of 2.
That is, the size of the hash table should be a prime.
Pitfall: Constrains the table size.

Implementing Hash Functions: Multiplication Method

Multiplication Method:

1. Multiply k by a real constant $0 < A < 1$. $\longrightarrow$ mess-up k by multiplying A
2. Let x be the fractional part of $k \times A$ (note that $0 < x < 1$). $\longrightarrow$ take the fractional part of the mess
3. $h(k) = \lfloor m \times x \rfloor$. $\longrightarrow$ multiply m to make sure the result is between 0 and $m - 1$.

- The optimal choice for A depends on the characteristics of the data being hashed.
- Tends to evenly distribute the hash values, because of the mess-up.
- Not sensitive to the value of m (unlike division method).

After Lecture

- Exercises in Chapter 4 of the course notes.
- Problems 11.1-1, 11.2-1, 11.2-2 in CLRS.
- Optional Readings: CLRS Sections 11.1, 11.2, 11.3 (except 11.3.3)