Due February 3, 2023 before 1pm

Note: solutions may be incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. [**6 marks**] **Propositional formulas.** For each of the following propositional formulas, find the following two items:

   (i) The truth table for the formula. (You don't need to show your work for calculating the rows of the table.)

   (ii) A logically equivalent formula that only uses the $\neg$, $\land$, and $\lor$ operators; *no* $\Rightarrow$ or $\Leftrightarrow$. You do not need to simplify your formula. (You *should* show your work in arriving at your final result. Make sure you're reviewed the "Additional instructions" for this problem set carefully.)

   (a) [**3 marks**] $(\neg p \Leftrightarrow q) \Rightarrow q$.

Solution

Truth table: (okay to not include intermediate value columns)

p	q	$\neg p$	$(\neg p \Leftrightarrow q)$	$(\neg p \Leftrightarrow q) \Rightarrow q$
False	False	True	False	True
False	True	True	True	True
True	False	False	True	False
True	True	False	False	True

Equivalent formula:

$$
\begin{aligned}
&(\neg p \Leftrightarrow q) \Rightarrow q \\
&\neg(\neg p \Leftrightarrow q) \lor q \qquad \text{(implication rule)} \\
&\neg((\neg p \Rightarrow q) \land (q \Rightarrow \neg p)) \lor q \qquad \text{(biconditional rule)} \\
&(\neg(\neg p \Rightarrow q) \lor \neg(q \Rightarrow \neg p)) \lor q \qquad \text{(DeMorgan rule)} \\
&((\neg p \land \neg q) \lor (q \land \neg\neg p)) \lor q \qquad \text{(implication negation rule)} \\
&((\neg p \land \neg q) \lor (q \land p)) \lor q \qquad \text{(double negation rule)}
\end{aligned}
$$

(okay to stop here and not simplify further)

   (b) [**3 marks**] $\big(p \Rightarrow (q \Rightarrow r)\big) \Rightarrow \big((p \Rightarrow q) \Rightarrow r\big)$.

Solution

Truth table: (okay to not include intermediate value columns)

p	q	r	$q \Rightarrow r$	$p \Rightarrow q$	$p \Rightarrow (q \Rightarrow r)$	$(p \Rightarrow q) \Rightarrow r$	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$
False	False	False	True	True	True	False	False
False	False	True	True	True	True	True	True
False	True	False	False	True	True	False	False
False	True	True	True	True	True	True	True
True	False	False	True	False	True	True	True
True	False	True	True	False	True	True	True
True	True	False	False	True	False	False	True
True	True	True	True	True	True	True	True

Equivalent formula:

$$
\begin{aligned}
&\big(p \Rightarrow (q \Rightarrow r)\big) \Rightarrow \big((p \Rightarrow q) \Rightarrow r\big) \\
&\neg\big(p \Rightarrow (q \Rightarrow r)\big) \lor \big((p \Rightarrow q) \Rightarrow r\big) && \text{(implication rule)} \\
&\big(p \land \neg(q \Rightarrow r)\big) \lor \big((p \Rightarrow q) \Rightarrow r\big) && \text{(implication negation rule)} \\
&\big(p \land (q \land \neg r)\big) \lor \big((p \Rightarrow q) \Rightarrow r\big) && \text{(implication negation rule)} \\
&\big(p \land (q \land \neg r)\big) \lor \big(\neg(p \Rightarrow q) \lor r\big) && \text{(implication rule)} \\
&\big(p \land (q \land \neg r)\big) \lor \big((p \land \neg q) \lor r\big) && \text{(implication negation rule)}
\end{aligned}
$$

(okay to stop here and not simplify further)

2. **[12 marks] Translating statements.**

Consider the following sets and predicates:

Symbol	**Definition**
S	the set of all students
C	the set of all courses
$Study(s, c)$	"student s is taking course c this term," where $s \in S$ and $c \in C$
$Fail(s, c)$	"student s has failed course c," where $s \in S$ and $c \in C$
$CS(s)$	"student s is a Computer Science student" where $s \in S$

Using **only** these sets and predicates, the symbols $=$ and $\neq$, and the standard propositional operators and quantifiers from lecture, translate each of the following English statements into predicate logic.

(a) **[2 marks]** No student in Computer Science has failed any course.

Solution

$\forall s \in S,\ \forall c \in C,\ CS(s) \Rightarrow \neg Fail(s, c)$

Or the contrapositive form, $\forall s \in S,\ \forall c \in C,\ Fail(s, c) \Rightarrow \neg CS(s)$.

(b) **[2 marks]** There is a course that all Computer Science students are taking this term.

Solution

$\exists c \in C,\ \forall s \in S,\ CS(s) \Rightarrow Study(s, c)$

(c) **[2 marks]** Not all students in Computer Science are taking courses this term.

Solution

$\exists s \in S,\ \forall c \in C,\ CS(s) \wedge \neg Study(s, c)$

(d) **[3 marks]** Every course has exactly 1 student. (Hint: use $=$ and/or $\neq$.)

Solution

$\forall c \in C,\ \exists s_1 \in S,\ Study(s_1, c) \wedge \Big(\forall s_2 \in S, Study(s_2, c) \Rightarrow s_1 = s_2\Big)$

(e) **[3 marks]** Every student is taking exactly two different courses this term. (Hint: use $=$ and/or $\neq$.)

Solution

$\forall s \in S,\ \exists c_1, c_2 \in C,\ c_1 \neq c_2 \wedge Study(s, c_1) \wedge Study(s, c_2) \wedge \Big(\forall c \in C,\ Study(s, c) \Rightarrow c = c_1 \vee c = c_2\Big)$

Hint: Be sure to read the "Additional instructions" given on pages 1 and 2.

3. **[9 marks] Choosing a universe and predicates.** This question gets you to investigate some of the subtleties of variable scope and precedence rules that are discussed in pp. 31–33 in the Course Notes.

(a) **[3 marks]** Consider the following two statements:

$$\forall x \in \mathbb{N},\ P(x,165) \lor P(x,0) \qquad \text{(Statement 1)}$$

$$\big(\forall x \in \mathbb{N},\ P(x,165)\big) \Rightarrow \big(\exists x \in \mathbb{N},\ P(x,0)\big) \qquad \text{(Statement 2)}$$

Provide a definition of a binary predicate P, where each parameter has domain $\mathbb{N}$, that makes one of the above statements True and the other statement False. Your predicate may *not* be a constant function (i.e., always True or always False).

Briefly justify your response, but no formal proofs are necessary.

Solution

(There are many possible solutions. Here is one.)

We define the predicate $P(x,y)$: "$x = y$", where $x, y \in \mathbb{N}$. Then we can analyze the truth/falsehood of each statement.

Statement 1 is False.

Statement 1 is False because not every number x is equal to 165 or 0.

Statement 2 is True.

Statement 2 is vacuously True. The hypothesis of the implication, $\forall x \in \mathbb{N},\ P(x,165)$, is False because not every number is equal to 165.

(b) **[2 marks]** Consider once again Statements 1 and 2 from Part (a).

Provide a definition of a binary predicate P, where each parameter has domain $\mathbb{N}$, that makes **both** of the statements True. Your predicate may *not* be a constant function (i.e., always True or always False).

Briefly justify your response, but no formal proofs are necessary.

Solution

(There are many possible solutions. Here is one.)

We define the predicate $P(x,y)$: "$x \geq y$", where $x, y \in \mathbb{N}$. Then we can analyze the truth/falsehood of each statement.

Statement 1 is True.

Statement 1 is True because all natural numbers are greater or equal to 0.

Statement 2 is True.

Statement 2 is vacuously True. The hypothesis of the implication, $\forall x \in \mathbb{N},\ P(x,165)$, is False because not every number is greater or equal to 165.

(c) **[4 marks]** Consider the following two statements:

$$\forall x \in S,\ \Big(\exists y \in T,\ P(x,y)\Big) \Rightarrow Q(x) \qquad \text{(Statement 3)}$$

$$\forall x \in S,\ Q(x) \Rightarrow \Big(\forall y \in T,\ P(x,y)\Big) \qquad \text{(Statement 4)}$$

Provide a definition of *non-empty* sets S and T, a binary predicate P and a unary predicate Q, that makes one of the above statements True and the other statement False. The first parameter of P has domain S and the second parameter of P has domain T. The parameter of Q has domain S. Your sets must be non-empty, and your predicates may *not* be constant functions (i.e., always True or always False).

Briefly justify your response, but no formal proofs are necessary.

Solution

(There are many possible solutions. Here is one.)

We define $S = \mathbb{N}$ and $T = \mathbb{N}$, and the following predicates:
$P(x, y)$: "$x \geq y$", and $Q(x)$: "$x = 0$"

Statement 1 is False.

For any arbitrary $x \in \mathbb{N}$, the expression $\exists y \in \mathbb{N},\ P(x, y)$ is *always* True: just pick $y = 0$! That means that the hypothesis of the implication in Statement 1 is always True, but the conclusion - $Q(x)$ - , is not always True. For example, $x = 0$ is False when $x \neq 0$.

Statement 2 is True. Consider any $x \in \mathbb{N}$. When $x \neq 0$, the hypothesis of Statement 2 is False, which makes the implication vacuously True. When $x = 0$, the hypothesis of Statement 2 is True, but so is the conclusion, and so the implication is True. We can conclude that Statement 2 is True.

4. **[8 marks] Pierre Numbers** A natural number n is said to be a "Pierre number" when it can be expressed as $2^{2^k}+1$ for some integer k.

(a) **[1 mark]** Write a predicate $PierreNumber(n)$ that is True if and only if n is a Pierre number.

> **Solution**
>
> The intended solution is:
> $PierreNumber(n)$: "$\exists k \in \mathbb{Z}, n = 2^{2^k} + 1$", where $n \in \mathbb{N}$
> Given the wording of the question, "$PierreNumber(n) \Leftrightarrow \exists k \in \mathbb{Z}, n = 2^{2^k} + 1$, where $n \in \mathbb{N}$" is fine too.

(b) **[2 marks]** Express the statement "All Pierre numbers are odd." using predicate logic. You may use the predicate $PierreNumber$, the predicate $=$ and logical operators, but may not use other predicates (like Odd or $Even$).

> **Solution**
>
> $\forall n \in \mathbb{N}, PierreNumber(n) \Rightarrow (\exists k \in \mathbb{Z}, n = 2k - 1)$
> Alternate: $\forall n \in \mathbb{N}, \exists k \in \mathbb{Z}, PierreNumber(n) \Rightarrow n = 2k - 1$
> And, of course, $2k + 1$ may be used in place of $2k - 1$.

(c) **[2 marks]** Express the statement "There is not a largest Pierre number." using predicate logic. You may use the predicate $PierreNumber$ and the standard inequality predicates, but may not use other predicates.

> **Solution**
>
> $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, PierreNumber(m) \land m > n$
> (Every number has a Pierre number that is larger than it.)
>
> An alternate solution can be arrived at by considering the negation of "There is a largest Pierre number." That is,
> $\neg\big(\exists n \in \mathbb{N}, PierreNumber(n) \land \big(\forall m \in \mathbb{N}, PierreNumber(m) \Rightarrow m \leq n\big)\big)$. Pushing the negation in gives:
> $\forall n \in \mathbb{N}, \neg PierreNumber(n) \lor \big(\exists m \in \mathbb{N}, PierreNumber(m) \land m > n\big)$.
> Or: $\forall n \in \mathbb{N}, PierreNumber(n) \Rightarrow \big(\exists m \in \mathbb{N}, PierreNumber(m) \land m > n\big)$.
> (Every Pierre number has a Pierre number that is larger than it.)
> We can draw the same conclusion ("There is no largest Pierre number") from both statements, though the first form also includes the requirement that there is a Pierre number.

(d) **[3 marks]** Express the statement "Only the five smallest Pierre numbers are prime." using predicate logic. You may use the predicate $Prime(n)$. The smallest Pierre number is 3. (Hint: you may find this question easier if you don't use your predicate $PierreNumber(n)$. Use the Pierre number definition instead.)

> **Solution**
>
> $\forall k \in \mathbb{N}, Prime(2^{2^k} + 1) \Leftrightarrow k < 5$
> Or we can split the bi-conditional into two parts, and then use the contrapositive to express conclusions of whether or not the Pierre number being discussed is $Prime$.
> $\forall k \in \mathbb{N}, \big((k < 5) \Rightarrow Prime(2^{2^k} + 1)\big) \land \big((k > 4) \Rightarrow \neg Prime(2^{2^k} + 1)\big)$

There are also alternate correct expressions that express the same thing.