Due Wednesday 22 March, *before* 10:00am

**Note**: solutions may be incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

**1. [8 marks] Number representation.**

For each $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, define $C(n,k)$ to be:

$$\exists a_1, \ldots, a_k \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq k \Rightarrow a_i \leq i) \wedge (n = \sum_{i=1}^{k} a_i \cdot i!)$$

Prove, using Induction, that: $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}^+, n < (k+1)! \Rightarrow C(n,k)$.

Here we use the factorial function, where for each $m \in \mathbb{N}$, $m! = \prod_{j=1}^{m} j$.

**Solution**

The statement is equivalent to: $\forall k \in \mathbb{Z}^+, \forall n \in \mathbb{N}, n < (k+1)! \Rightarrow C(n,k)$.

We prove it by Induction on $k$.

**Base Case $k = 1$:** Let $n \in \mathbb{N}$ and assume $n < (k+1)! = (1+1)! = 2$, then $n = 0$ or $n = 1$.

Let $a_1 = n$. Then $a_1 \in \mathbb{N}$, $a_1 \leq 1$, and $n = a_1 = a_1 \cdot 1!$.

**Inductive Step:** Let $k \in \mathbb{Z}^+$ and assume $\forall n_0 \in \mathbb{N}, n_0 < (k+1)! \Rightarrow C(n_0, k)$.

Let $n \in \mathbb{N}$ and assume $n < (k+1+1)! = (k+2)!$.

Since $(k+1)! \in \mathbb{Z}^+$ we can divide $n$ by $(k+1)!$ to get a quotient and remainder $q, r \in \mathbb{N}$.

The quotient and remainder have the properties: $n = q(k+1)! + r$ and $r < (k+1)!$.

(Exercise: define these explicitly as $q = \lfloor n/(k+1)! \rfloor$ and $r = n - q(k+1)!$ and show those properties.)

So from the Inductive Hypothesis for $n_0 = r < (k+1)!$ there are $a_1, ..., a_k \in \mathbb{N}$ with each $a_i \leq i$ and $r = a_1 \cdot 1! + \cdots + a_k \cdot k!$.

Now, $q = (n-r)/(k+1)! \leq n/(k+1)! < (k+2)!/(k+1)! = k+2$, so $q \leq k+1$ since $q$ is an integer.

So let $a_{k+1} = q < k+1$, then $n = r + q(k+1)! = a_1 \cdot 1! + \cdots + a_k \cdot k! + a_{k+1} \cdot (k+1)!$.

**2. [12 marks] Induction.**

For each $m, n \in \mathbb{N}$, let $A_m = \{a \mid a \in \mathbb{N} \wedge a \leq m\}$ and $B_n = \{b \mid b \in \mathbb{N} \wedge b \leq n\}$, and define $F_{m,n}$ to be:

$$\{f : A_m \to B_n \mid [\ \forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)\ ] \wedge f(m) = n\}$$

For each $m, n \in \mathbb{N}$, define $P(m,n)$ to be:

$$|F_{m,n}| = \frac{(m+n)!}{m! \cdot n!}$$

**(a) [6 marks]**

Prove each of the following statements:

i. $\forall m \in \mathbb{N}, P(m, 0)$.

**Solution**

Let $m \in \mathbb{N}$. $F_{m,0} = \{f : A_m \to \{0\} \mid [\ \forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)\ ] \wedge f(m) = 0\}$.
The co-domain contains only one point, so the only function from $A_m$ to $\{0\}$ is defined by $f(a) = 0$ for each $a \in A_m$, which satisfies the first condition since each $f(k) = f(l)$ and clearly satisfies the second condition.
So $|F_{m,0}| = 1 = \dfrac{m!}{m!} = \dfrac{(m+0)!}{m! \cdot 0!}$.

ii. $\forall n \in \mathbb{N}, P(0, n)$.

**Solution**

Let $n \in \mathbb{N}$. $F_{0,n} = \{f : \{0\} \to B_n \mid [\ \forall k, l \in \{0\}, k \leq l \Rightarrow f(k) \leq f(l)\ ] \wedge f(0) = n\}$.
The domain contains only one point, so the only function satisfying the second condition is defined by $f(0) = n \in B_n$, and the first condition is just $f(0) \leq f(0)$ which is clearly true.
So $|F_{0,n}| = 1 = \dfrac{n!}{n!} = \dfrac{(0+n)!}{0! \cdot n!}$.

iii. $\forall m, n \in \mathbb{N}, P(m, n+1) \wedge P(m+1, n) \Rightarrow P(m+1, n+1)$.

**Solution**

Let $m, n \in \mathbb{N}$.
First, let's show that $|F_{m,n+1}| + |F_{m+1,n}| = |F_{m+1,n+1}|$.
The conditions on $f : \{0, ..., m, m+1\} \to \{0, ..., n, n+1\}$ to be in $F_{m+1,n+1}$ are:

$$f(0) \leq \cdots \leq f(m) \leq f(m+1) = n+1$$

By the co-domain, each $f(0), ..., f(m+1) \leq n+1$, so in particular we can simplify to:

$$f(0) \leq \cdots \leq f(m) \wedge f(m+1) = n+1$$

Since $f(m) \leq n+1$, $F_{m+1,n+1}$ can be partitioned into the subset (call it $S$) of elements with $f(m) = n+1$ and the subset (call it $T$) of elements with $f(m) \leq n$.
In particular, $|F_{m+1,n+1} = |S| + |T|$.

The conditions for $f \in S$ are:

$$f(0) \leq \cdots \leq f(m-1) \leq f(m) = n+1 \wedge f(m+1) = n+1$$

Since the condition $f(m+1) = n+1$ is fixed, there is one $f \in S$ for each way to satisfy:

$$f(0) \leq \cdots \leq f(m-1) \leq f(m) = n+1$$

Since $f \in S$ has the same co-domain as functions in $F_{m,n+1}$, those are also the conditions for a function $f$ from just $A_m$ to $B_{n+1}$ to be in $F_{m,n+1}$. So $|S| = |F_{m,n+1}|$.

The conditions for $f \in T$ are:

$$f(0) \leq \cdots \leq f(m-1) \leq f(m) \leq n \wedge f(m+1) = n+1$$

Since the condition $f(m+1) = n+1$ is fixed, there is one $f \in T$ for each way to satisfy instead:

$$f(0) \leq \cdots \leq f(m-1) \leq f(m) \leq n = f(m+1)$$

Since those conditions restrict the range to $B_n$, those are also the conditions for a function $f$ from $A_{m+1}$ to just $B_n$ to be in $F_{m+1,n}$. So $|T| = |F_{m+1,n}|$.

Thus $|F_{m+1,n+1}| = |S| + |T| = |F_{m,n+1}| + |F_{m+1,n}|$.

Now assuming $P(m, n+1) \wedge P(m+1, n)$:

$$\begin{aligned}
|F_{m+1,n+1}| &= \frac{(m+n+1)!}{m! \cdot (n+1)!} + \frac{(m+1+n)!}{(m+1)! \cdot n!} \\
&= \frac{(m+1)(m+n+1)!}{(m+1)! \cdot (n+1)!} + \frac{(n+1)(m+n+1)!}{(m+1)! \cdot (n+1)!} \\
&= \frac{(m+1+n+1)(m+n+1)!}{(m+1)! \cdot (n+1)!} \\
&= \frac{(m+1+n+1)!}{(m+1)! \cdot (n+1)!}
\end{aligned}$$

Thus $P(m+1, n+1)$ is true.

**(b)** [**2 marks**]

Prove, using the results from part (a), that: $P(1,1) \wedge P(2,2)$.

**Solution**

From the first two sub-parts of (a): $P(0,1)$, $P(1,0)$, $P(0,2)$, and $P(2,0)$ are true.
From the third part of (a), $P(0,1) \wedge P(1,0) \Rightarrow P(1,1)$ is true, so now $P(1,1)$ is also true.
Again from the third part of (a), $P(0,2) \wedge P(1,1) \Rightarrow P(1,2)$ is true, so now $P(1,2)$ is also true.
Again from the third part of (a), $P(1,1) \wedge P(2,0) \Rightarrow P(2,1)$ is true, so now $P(2,1)$ is also true.
And one more time: $P(1,2) \wedge P(2,1) \Rightarrow P(2,2)$ is true, so now $P(2,2)$ is also true.

**(c)** [**3 marks**]

For each $t \in \mathbb{N}$, define $Q(t)$ to be: $\forall m, n \in \mathbb{N}, m + n = t \Rightarrow P(m, n)$.
Prove, using Induction and the results from part (a), that: $\forall t \in \mathbb{N}, Q(t)$.

**Solution**

**Base Case $Q(0)$:** Let $m, n \in \mathbb{N}$ and assume $m + n = 0$.
Then $m = n = 0$, so $P(m, n)$ is true by the first (or second) sub-part of part (a).
**Inductive Step:** Let $t \in \mathbb{N}$ and assume $Q(t)$: $\forall m_0, n_0 \in \mathbb{N}, m_0 + n_0 = t \Rightarrow P(m_0, n_0)$.
Let $m, n \in \mathbb{N}$ and assume $m + n = t + 1$.
If $m = 0$ or $n = 0$, then $P(m, n)$ is true by the second or first sub-part of part (a), respectively.
Otherwise $m \geq 1$ and $n \geq 1$. Then $m-1, n-1 \in \mathbb{N}$, $(m-1) + n = t$, and $m + (n-1) = t$.
So using the Inductive Hypothesis twice, with $m_0 = m-1, n_0 = n$ and with $m_0 = m, n_0 = n-1$, we get $P(m-1, n)$ and $P(m, n-1)$.
Then by the third sub-part of part (a), with $m-1$ and $n-1$: $P((m-1)+1, (n-1)+1)$ is true, i.e. $P(m, n)$, is true.

**(d)** [**1 mark**]

Prove, using the result from part (c), that: $\forall m, n \in \mathbb{N}, P(m, n)$.

**Solution**

Let $m, n \in \mathbb{N}$. Then $m + n \in \mathbb{N}$.
Part (c) says $P(m_0, n_0)$ is true for every $m_0, n_0 \in \mathbb{N}$ with sum $m + n$, in particular for $m$ and $n$.

Let's see that formally, for comparison.
Since $m + n \in \mathbb{N}$, $Q(m + n)$ is true: $\forall m_0, n_0 \in \mathbb{N}, m_0 + n_0 = m + n \Rightarrow P(m_0, n_0)$.
Letting $m_0 = m$ and $n_0 = n$ satisfies the hypothesis of the conditional, so $P(m, n)$ is true.

**3.** [**8 marks**] **Asymptotic notation.**

For the following questions use the definitions of $\mathcal{O}$, $\Omega$, and $\Theta$, *not* our various results about them.

**(a)** [**3 marks**]

Prove or disprove that $n^n \in \mathcal{O}(n!)$.

**Solution**

We provide a disproof of this statement. That is, we prove $n^n \notin \mathcal{O}(n!)$.
Unpacking the definition of $\mathcal{O}$, we have to prove

$$\neg\Big(\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^n \leq c \cdot n!\Big)$$

or

$$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^n > c \cdot n!$$

*Proof.* Let $c$ and $n_0$ be arbitrary positive real numbers, and then let $n = \max(\lceil c \rceil, \lceil n_0 \rceil)$.
Then we have that $n \geq n_0$.
Unpacking both $n^n$ and $c \cdot n!$, we have:

$$\begin{aligned} n^n &= n \cdot n \cdot n \cdot \dots \cdot n \cdot n \\ c \cdot n! &= c \cdot n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 \\ &= n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot c \end{aligned}$$

There $n$ terms in the expression for $n^n$, and each of the terms is equal to $n$.
There $n$ terms in the expression for $c \cdot n!$. The first term is equal to $n$. The next $n - 2$ term are $< n$. And, since $n \geq c$, the last term is $\leq n$. Hence the product of these $n$ terms is $< n^n$, and we can conclude that $n^n > c \cdot n!$.
We have shown $n \geq n_0 \wedge n^n > c \cdot n!$, as required.

$\square$

**(b)** [**5 marks**]

Prove that if $a, b \in \mathbb{R}$ and $b > 0$, then $(n + a)^b \in \Theta(n^b)$.

**Solution**

We want to prove the statement

$$\forall a, b \in \mathbb{R}, b > 0 \Rightarrow (n+a)^b \in \Theta(n^b)$$

or, after unpacking the $\Theta$ expression,

$$\forall a, b \in \mathbb{R}, b > 0 \Rightarrow \Big(\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \cdot n^b \leq (n+a)^b \leq c_2 \cdot n^b\Big).$$

*Proof.* (One could consider the separate cases of $a \leq 0$ and $a > 0$, but instead we deal with the $a$ term by working with $|a|$.)
Let $c_1 = \left(\frac{1}{2}\right)^b$, $c_2 = 2^b$ and $n_0 = 2|a|$.
Let $n$ be an arbitrary integer that is $\geq n_0$.
Since $n \geq n_0$ and $n_0 = 2|a|$, we can conclude that $n - 2|a| \geq 0$.
Then, we have

$$\begin{aligned}
(n+a) &\geq (n - |a|) \\
&= \left(\frac{1}{2}n + \frac{1}{2}n - |a|\right) \\
&= \left(\frac{1}{2}n + \frac{1}{2}(n - 2|a|)\right) \\
&\geq \left(\frac{1}{2}n\right)
\end{aligned}$$

and so, since for a fixed $b > 0$ and $m \in \mathbb{N}$, $m^b$ in a non-decreasing function of $m$,

$$\begin{aligned}
(n+a)^b &\geq \left(\frac{1}{2}n\right)^b \\
&= \left(\frac{1}{2}\right)^b \cdot n^b \\
&= c_1 \cdot n^b
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
(n+a)^b &\leq (n + |a|)^b \\
&\leq (2n)^b \qquad \text{(since } |a| \leq n\text{)} \\
&= 2^b \cdot n^b \\
&= c_2 \cdot n^b
\end{aligned}$$

Hence, $c_1 \cdot n^b \leq (n+a)^b \leq c_2 \cdot n^b$, as required. □

**4. [7 marks] More asymptotic notation.**

For the following questions use the definitions of $\mathcal{O}$, $\Omega$, and $\Theta$, *not* our various results about them.

**(a) [3 marks]**

Prove or disprove that: if $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $k \in \mathbb{R}^+$, and $f(n) \in \mathcal{O}(n^k)$, then $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$.

<u>**Solution**</u>

We will provide a proof of this statement.

*Proof.* Let $f$ be an arbitrary function from $\mathbb{N}$ to $\mathbb{R}^{\geq 0}$, and let $k$ be an arbitrary positive real number.
Assume further that $f(n) \in \mathcal{O}(n^k)$. That is, assume

$$\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq c_0 \cdot n^k.$$

We wish to show that $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$.
Let $c_1 = |\log_2(c_0)| + k$ and $n_1 = \max(2, \lceil n_0 \rceil)$.
Assume that $n$ is an arbitrary natural number and that $n \geq n_1$.
Since $n \geq n_0$, $f(n) \leq c_0 \cdot n^k$. Also, since for $x \in \mathbb{R}^+$, $\log_2(x)$ in an increasing function of $x$,

$$\begin{aligned}
\log_2(f(n))) &\leq \log_2(c_0 \cdot n^k) \\
&= \log_2(c_0) + \log_2(n^k) \\
&= \log_2(c_0) + k \cdot \log_2(n) \\
&\leq |\log_2(c_0)| + k \cdot \log_2(n).
\end{aligned}$$

Since $\log_2(2) = 1$ and $n \geq 2$, we know that $\log_2(n) \geq 1$.
Hence

$$\begin{aligned}
\log_2(f(n))) &\leq |\log_2(c_0)| + k \cdot \log_2(n) \\
&\leq |\log_2(c_0)| \cdot \log_2(n) + k \cdot \log_2(n) \\
&= (|\log_2(c_0)| + k) \cdot \log_2(n) \\
&= c_1 \cdot \log_2(n).
\end{aligned}$$

We have shown

$$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow \log_2(f(n)) \leq c_1 \cdot \log_2(n),$$

and so $\log_2(f(n)) \in \mathcal{O}(\log_2(n))$, as required.

□

**(b) [4 marks]**
Prove that: if $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_1 \in \mathcal{O}(g_1)$, and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$.
Here, $(f_1 + f_2)(n) = f_1(n) + f_2(n)$ and $\max(g_1, g_2)(n) = \max(g_1(n), g_2(n))$.

**<u>Solution</u>**

*Proof.* Let $f_1, f_2, g_1, g_2$ be arbitrary functions from $\mathbb{N}$ to $\mathbb{R}^{\geq 0}$ and assume $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$.
We wish to show that $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$.
Since $f_1 \in \mathcal{O}(g_1)$, we know

$$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f_1(n) \leq c_1 \cdot g_1(n)$$

and since $f_2 \in \mathcal{O}(g_2)$, we know

$$\exists c_2, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow f_2(n) \leq c_2 \cdot g_2(n).$$

Let $n_3 = \max(n_1, n_2)$ and $c_3 = 2 \cdot \max(c_1, c_2)$.
Let $n$ be an arbitrary natural number and assume that $n \geq n_3$.
Then

$$\begin{aligned}
(f_1 + f_2)(n) &= f_1(n) + f_2(n) \\
&\leq c_1 \cdot g_1(n) + c_2 \cdot g_2(n) \\
&\leq \max(c_1, c_2) \cdot g_1(n) + \max(c_1, c_2) \cdot g_2(n) \\
&= \max(c_1, c_2) \cdot \big(g_1(n) + g_2(n)\big) \\
&\leq \max(c_1, c_2) \cdot \big(\max(g_1(n), g_2(n)) + \max(g_1(n), g_2(n))\big) \\
&= \max(c_1, c_2) \cdot 2 \cdot \max(g_1(n), g_2(n)) \\
&= c_3 \cdot \max(g_1(n), g_2(n)) \\
&= c_3 \cdot \max(g_1, g_2)(n)
\end{aligned}$$

Hence, $(f_1 + f_2)(n) \leq c_3 \cdot \max(g_1, g_2)(n)$ and $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$, as required. $\square$