CSC 236 Tutorial 1

May 10th 2023

Today

Injective, Surjective, Bijective

Applications of the Pigeonhole Principle

Generalized Pigeonhole Principle

Too few pigeons

Tutorial

There are several problems in each tutorial to work on $\approx$ 30mins.

Use the remaining time as Q/A (office hour).

Injective, Surjective, Bijective

Applications of the Pigeonhole Principle

Generalized Pigeonhole Principle

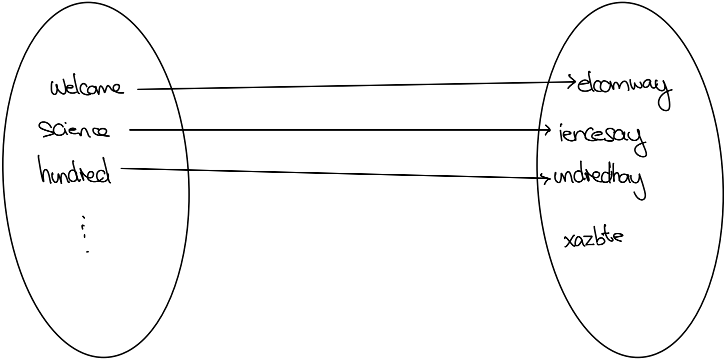
Too few pigeons

Injective, Surjective, Bijective

Here are the example functions from lecture. Are they injective/surjective/bijective?

Is this injective/surjective/bijective?

PigLatin: English $\longrightarrow$ Strings



Is this injective/surjective/bijective?

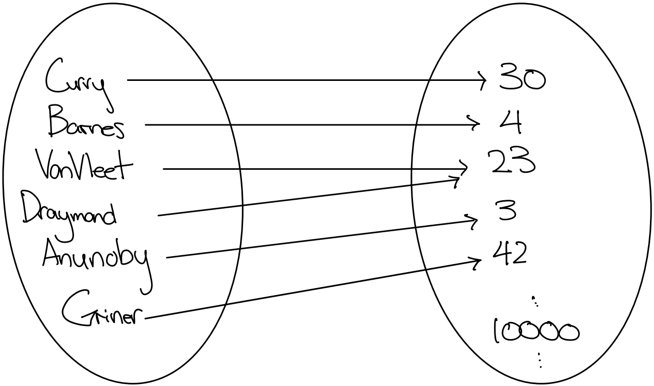
```python
def product(xs: List[int]) -> int:
    accumulator = 1
    for x in xs:
        accumulator *= x
    return accumulator
```

Is this injective/surjective/bijective?

(\mathbb{N})

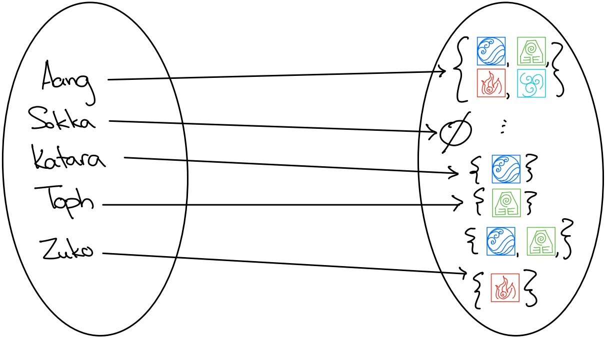
Jersey Number : Player $\longrightarrow \mathbb{N}$ $\rightsquigarrow \{0, 1, 2, \ldots\}$

"natural numbers"



Is this injective/surjective/bijective?

$$\text{Bends} : \text{Gaang} \to \mathcal{P}(\text{Elements})$$



Sock Picking

Imagine you have ten colors of socks in your sock drawer. You pull out one sock from the drawer at a time. What is the minimum number of socks you pull out before you are guaranteed two socks of the same color?

11. Apply the Pigeonhole Principle with socks as pigeons and colors as pigeonholes.

Hand Shaking

Consider a social event with n people. Each person shakes hands a certain number of times throughout the event. Prove that there are always two people that have shaked hands the same number of times.

You can try this out in small groups! You don't actually have to shake hands - just pretend.

Proof

A person can shake hands with between 0 and $n-1$ (inclusive) other people. Think of these numbers as the pigeonholes. Think of the people as pigeons. Note that there are n pigeonholes and n pigeons, so the Pigeonhole Principle doesn't directly apply.

If you shook hands with $n-1$ people, you shook everyone's hand (except for your own). Therefore, no one shakes hands with 0 people because they at least shook your hand! Similarly, if you shook hands with 0 people, then no one shook hands with $n-1$ people because they at least have not shook your hand! Therefore, at least one of the pigeonholes corresponding to 0 and $n-1$ is empty.

Now apply the Pigeonhole Principle with $n-1$ pigeonholes and n pigeons.

Birthdays

There are around 145 in CSC236 (including staff). Show that at least 2 of us are born in the same month.

Apply the Pigeonhole Priciple to 145 pigeons and 12 months.

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Can we do better?

There are around 145 in CSC236 (including staff). Show that at least **13** of us are born in the same month.

Generalized Pigeonhole Principle

Theorem
If there are m pigeons and n pigeonholes, not matter how we assign pigeons to pigeonholes, at least one pigeonhole will have at least $\lceil m/n \rceil$ pigeons

Notation: $\lceil x \rceil$ means x rounded up to the nearest integer.

Proof

Abbreviate $\{1, ..., n\}$ by $[n]$. Assign the m pigeons into the n pigeonholes.

Let for $i \in [n]$ let x_i denote the number of pigeons in the ith pigeonhole. Note that $m = x_1 + x_2 + ... + x_n$.

By contradiction, assume that for every $i \in [n]$, $x_i < \lceil m/n \rceil$. Since each of the x_i are integral (no fractional part), $x_i < \lceil m/n \rceil$ implies $x < m/n$. Then we have

$$\begin{aligned} m &= x_1 + x_2 + ... + x_n \\ &< \underbrace{m/n + m/n + ... + m/n}_{(n \text{ times})} \\ &= m. \end{aligned}$$

$m < m$ is a contradiction, so we are done.

Number of birthdays in the same month

There are around 145 in CSC236 (including staff). Show that at least **13** of us are born in the same month.

Think of the 145 people as pigeons and the 12 months as pigeonholes.

Apply the Generalized Pigeonhole Principle: No matter how we assign people to months, at least one month has at least $\lceil 145/12 \rceil = 13$ people. In particular, this holds true when we assign people to their birth months.

Too few pigeons

The Pigeonhole Principle says that if there are more pigeons than pigeonholes, some pigeonhole will have at least two pigeons.

What if there are fewer pigeons than pigeonholes?

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No matter how you assign pigeons to pigeonholes, some pigeonhole will not have any pigeons!

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Reverse Pigeonhole Principle*

Theorem

Let A, B be finite sets such that $|A| < |B|$. Then there is no surjection from A to B.

* This is not standard terminology

Contrapositives

Let A, B be finite sets

- Pigeonhole Principle: $|A| > |B| \implies \neg\exists$ injection $f : A \to B$.
- Reverse Pigeonhole Principle: $|A| < |B| \implies \neg\exists$ surjection $f : A \to B$.

Question: What are the contrapositives of the Pigeonhole Principle and the Reverse Pigeonhole Principle?

Solution

Pigeonhole Principle: $\exists$ injection $f : A \to B \implies |A| \leq |B|$

Reverse Pigeonhole Principle: $\exists$ surjection
$f : A \to B \implies |A| \geq |B|$

A new way to compare sizes of sets

Given this, If I wanted to show that a set A is smaller than another set B without calculating the size of the sets, I could just find an injective function from A to B.

In fact existing an injective function from A to B is how we define $|A| \leq |B|$ for general (potentially infinite) sets.