UNIVERSITY OF TORONTO
Faculty of Arts & Science

CSC236H1 Y Midterm

Date: June 28th, 2023

Instructor: Harry Sha

Duration: 3hrs

Aids Allowed: None

*Do **not** turn this page until you have received the signal to start.*

In the meantime, please write your full name and student number below—**please do this right now!**—and *carefully* read *all* the information on the rest of this page.

Given Name(s):

Family Name(s):

Student Number (9 or 10 digits):

UTORid (e.g., `shaharry`):

- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. *If it is left in your pocket, it may be an academic offence.*
- When you are done your exam, raise your hand for someone to come and collect your exam. *Do not collect your bag and jacket before your exam is handed in.*
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- This exam consists of 18 pages (including this one), printed on both sides of the paper. There are 36 points, and 2 points for extra credit. *When you receive the signal to start, please make sure that your copy of the examination is complete.*
- Answer each question directly on the examination paper. If you require more space, indicate where you continued your answer. For example, "continued on Extra Space 1", or "continued on page 16"
- Good luck, you got this!

STUDENTS MUST HAND IN ALL EXAMINATION MATERIALS AT THE END

1 Bernoulli's Inequality

Question 1a (4 points). Prove by induction on n that $\forall m \in \mathbb{N}, n \in \mathbb{N}(1+m)^n \geq 1 + mn$.

2 Pairs

Question 2a (4 points). Let f_1, f_2 be defined as follows

- $f_1((a, b)) = (b + 1, a + 1)$
- $f_2((a, b)) = (2b, a + 1)$

Let X be the set generated from $\{(0, 1)\}$, by $\{f_1, f_2\}$. Show

$$\forall (a, b) \in X.(a \cdot b \text{ is even, and } a + b \text{ is odd.})$$

3 Friends and Strangers +

In this problem, you may use any results from HW1.

In HW1, you also showed that if there are at least 17 people, and every pair of people are either friends, strangers, or enemies, then there is always a group of 3 mutual friends, 3 mutual strangers, or 3 mutual enemies.

Now suppose that every pair of people are either friends, strangers, enemies **or soulmates**.

Question 3a (4 points). Find k such that in any group of k people where every pair of people are either friends, strangers, enemies, or soulmates, there are always three people who are mutual friends, mutual strangers, mutual enemies, or mutual soulmates. Prove your choice in k works.

Find the smallest k possible using the same proof technique from hw1 for full credit. You do **not** need to prove that this is, in fact, the smallest possible.

4 Transitive Round Robin

In this problem, you may use any of the results we proved in HW2.

Recall from HW2 that a round-robin tournament is a tournament where every player plays a single game with every other player. There are no draws, so every game ends in a win or a loss.

You can represent a round-robin tournament on n players as a directed graph $G = (V, E)$ where V is the set of players and $(a, b) \in E$ iff a beats b. Then, since every player plays against every other player, for every two distinct players a and b, either $(a, b) \in E$ or $(b, a) \in E$.

Recall that a player p is called a **winner** if for all other players q, either

- p beats q, OR
- p beats someone who beats q.

Question 4a (1 points). Draw a round-robin tournament graph with 4 people where there are multiple winners.

Call a round-robin tournament **transitive** if it satisfies the following property. For any 3 distinct players a, b, c, if a beats b, and b beats c, then a beats c. Equivalently, if $G = (V, E)$ is the graph representing the tournament, then for all distinct players $a, b, c \in V$, $(a, b) \in E \wedge (b, c) \in E \implies (a, c) \in E$.

Question 4b (4 points). Let $G = (V, E)$ be a graph representing a transitive round-robin tournament. Show that $\forall n \geq 3$, if $a_1, ..., a_n$ is a sequence of players in V such that $\forall i \in \{1, 2, ..., n-1\}.(a_i \text{ beats } a_{i+1})$, then a_1 beats a_n.

Question 4c (2 points). Suppose G is a graph representing a transitive round-robin tournament. Show that G is acyclic.

Question 4d (5 points). Let $G = (V, E)$ be a transitive tournament. Let $f : V \to \{0, ..., |V| - 1\}$ be a function mapping each player to the number of games they win. Show that f is bijective.

Question 4e (2 points). Show that every transitive round-robin tournament has a **unique** winner.

5 N-Knights

In chess, a knight moves in an L-shape. I.e., it can move either

- horizontally two squares and vertically one square, OR
- vertically two squares and horizontally one square

In the problem, we're interested in placing knights on the chessboard such that no two knights attack each other (piece a attacks piece b if piece a can move to the square occupied by piece b.)

Question 5a (4 points). Model the problem of placing k knights on a $n \times n$ chess board where no two knights attack each other as one of the graph problems we studied (shortest path, traveling salesman, minimum spanning tree, matching, independent set). For full credit, give a full definition of the graph's vertices and edges and explain why finding a placement of k knights is equivalent to solving the chosen graph problem on the graph.

6 Master Method Recurrences

Solve the following recurrences using The Master Theorem. Here is the Master Theorem.

Theorem (The Master Theorem)

Let $T(n) = aT(n/b) + f(n)$. Define the following cases based on how the root work compares with the leaf work.

1. *Leaf heavy.* $f(n) = O(n^{\log_b(a)-\epsilon})$ *for some constant* $\epsilon > 0$.
2. *Balanced.* $f(n) = \Theta(n^{\log_b(a)})$
3. *Root heavy.* $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *for some constant* $\epsilon > 0$, *and* $af(n/b) \leq cf(n)$ *for some constant* $c < 1$ *for all sufficiently large* n.

Then,

$$T(n) = \begin{cases} \Theta(n^{\log_b(a)}) & \textit{Leaf heavy case} \\ \Theta(f(n)\log(n)) & \textit{Balanced case} \\ \Theta(f(n)) & \textit{Root heavy case} \end{cases}$$

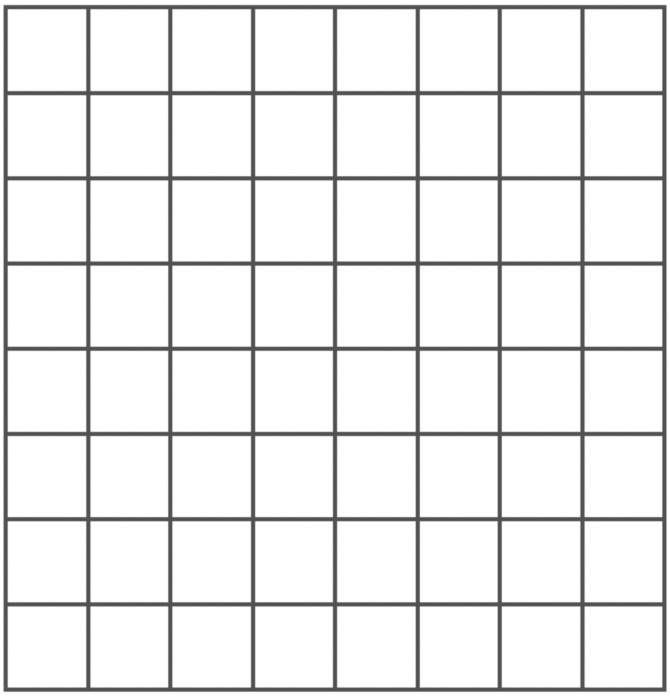
Question 6a (2 points). $T(n) = 8T(n/4) + n^2 \log_2(n)$

Question 6b (2 points). $T(n) = 3T(n/9) + \sqrt{n}$

Question 6c (2 points). $T(n) = 36T(n/6) + n^{1.5}$

7 Extra Credit

Question 7a (1 points). Prove that you can place 32 knights on an 8x8 chess board such that no two knights attack each other. You can draw your solution on the grid below.



Question 7b (1 points). Prove that you cannot place 33 or more knights on an 8x8 chess board such that no two knights attack each other.

Extra Space 1.

Extra Space 2.

Extra Space 3.

Extra Space 4.

Extra Space 5.