University of Toronto
Faculty of Arts and Science

CSC165H1S Term Test 1, Version 1

Date: February 13, 2023 **Duration**: 75 minutes **Instructor(s)**: G. Baumgartner, T. Fairgrieve

No Aids Allowed

Given Name(s):

Family Name(s):

Student Number (9 or 10 digits):

UTORid (e.g., `pitfra12`):

- This test has **4** questions. There are a total of **10 pages, DOUBLE-SIDED**.
- Final expressions in predicate logic must have negation symbols ($\neg$) applied **only** to predicates or propositional variables, e.g., $\neg Prime(x)$ or $\neg p$.
- You may not define your own propositional operators, predicates, or sets unless asked for in the question.
- In your proofs, you may always use definitions we have covered in this course. However, you may **not** use any external facts about these definitions unless they are given in the question.
- You may **not** use proofs by induction on this test.
- A list of standard equivalences, predicate definitions and sets is given on Page 2.

Question	Q1	Q2	Q3	Q4	**Total**
Mark					
Out of	10	8	5	11	**34**

On this test you may use the following standard equivalences, predicate definitions and sets. Read $\iff$ as 'is equivalent to'.

Standard Equivalences (p, q, r, $P(x)$, $Q(x)$, *etc.* are arbitrary sentences. D is an arbitrary set)

- *Commutativity*
  $p \land q \iff q \land p$
  $p \lor q \iff q \lor p$
  $p \Leftrightarrow q \iff q \Leftrightarrow p$
- *Associativity*
  $p \land (q \land r) \iff (p \land q) \land r$
  $p \lor (q \lor r) \iff (p \lor q) \lor r$
- *Identity*
  $p \land (q \lor \neg q) \iff p$
  $p \lor (q \land \neg q) \iff p$
- *Absorption*
  $p \land (q \land \neg q) \iff q \land \neg q$
  $p \lor (q \lor \neg q) \iff q \lor \neg q$
- *Idempotency*
  $p \land p \iff p$
  $p \lor p \iff p$
- *Double Negation*
  $\neg\neg p \iff p$
- *Negation rules*
  $\neg(p \land q) \iff \neg p \lor \neg q$
  $\neg(p \lor q) \iff \neg p \land \neg q$
  $\neg(p \Rightarrow q) \iff p \land \neg q$
  $\neg(p \Leftrightarrow q) \iff ((p \land \neg q) \lor (\neg p \land q))$
- *Distributivity*
  $p \land (q \lor r) \iff (p \land q) \lor (p \land r)$
  $p \lor (q \land r) \iff (p \lor q) \land (p \lor r)$
- *Contrapositive*
  $p \Rightarrow q \iff \neg q \Rightarrow \neg p$
- *Implication*
  $p \Rightarrow q \iff \neg p \lor q$
- *Biconditional*
  $p \Leftrightarrow q \iff (p \Rightarrow q) \land (q \Rightarrow p)$
- *Renaming*
  (where $P(x)$ does not contain variable y)
  $\forall x \in D, P(x) \iff \forall y \in D, P(y)$
  $\exists x \in D, P(x) \iff \exists y \in D, P(y)$
- *Quantifier Negation*
  $\neg\forall x \in D, P(x) \iff \exists x \in D, \neg P(x)$
  $\neg\exists x \in D, P(x) \iff \forall x \in D, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x \in D, \forall y \in D, S(x,y) \iff \forall y \in D, \forall x \in D, S(x,y)$
  $\exists x \in D, \exists y \in D, S(x,y) \iff \exists y \in D, \exists x \in D, S(x,y)$
- *Quantifier Distributivity*
  (where S does not contain variable x)
  $S \land \forall x \in D, Q(x) \iff \forall x \in D, S \land Q(x)$
  $S \lor \forall x \in D, Q(x) \iff \forall x \in D, S \lor Q(x)$
  $S \land \exists x \in D, Q(x) \iff \exists x \in D, S \land Q(x)$
  $S \lor \exists x \in D, Q(x) \iff \exists x \in D, S \lor Q(x)$
  $(\forall x \in D, P(x)) \land (\forall x \in D, Q(x)) \iff \forall x \in D, P(x) \land Q(x)$
  $(\exists x \in D, P(x)) \lor (\exists x \in D, Q(x)) \iff \exists x \in D, P(x) \lor Q(x)$

Standard Predicate Definitions

- $Even(n)$: "$\exists k \in \mathbb{Z},\ n = k \cdot 2$", where $n \in \mathbb{Z}$
- $Odd(n)$: "$\exists k \in \mathbb{Z},\ n = k \cdot 2 + 1$", where $n \in \mathbb{Z}$
- $d \mid n$: "$\exists k \in \mathbb{Z},\ n = k \cdot d$", where $d, n \in \mathbb{Z}$
- $Prime(n)$: "$n > 1 \land \big(\forall d \in \mathbb{N},\ d \mid n \Rightarrow (d = 1 \lor d = n)\big)$", where $n \in \mathbb{N}$
- $Atomic(n)$: "$\forall a, b \in \mathbb{N},\ (n \nmid a \land n \nmid b) \Rightarrow n \nmid ab$", where $n \in \mathbb{N}$

Standard Sets

- The set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- The set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The set of positive integers, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.
- The set of real numbers, $\mathbb{R}$. The set of positive real numbers, $\mathbb{R}^+$.

1. **[10 marks] Short answer questions.**

(a) **[2 marks]** Let A, B and C be different **non-empty** sets and consider the (incorrect) claim:

$$(A \cup B) \cap C = A \cup (B \cap C)$$

Show that this claim is not correct. Use suitably constructed sets A, B and C.

Solution

This is one of many possible solutions.
Let $A = \{1\}$, $B = \{2\}$ and $C = \{3\}$.
Then $A \cup B = \{1, 2\}$ and $(A \cup B) \cap C = \varnothing$.
But $B \cap C = \varnothing$ and $A \cup (B \cap C) = \{1\}$, which is differen from $\varnothing$.
And so, $(A \cup B) \cap C \neq A \cup (B \cap C)$.

(b) **[3 marks]** Using a truth table, prove the equivalence rule:

$$\neg(p \vee q) \text{ is equivalent to } (\neg p \wedge \neg q)$$

You must include columns with intermediate results to demonstrate how you determined the rows of the table.

Solution

p	q	$(p \vee q)$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q)$ is equivalent to $\neg(p \vee q)$
True	True	True	False	False	False	False	True
True	False	True	False	False	True	False	True
False	True	True	False	True	False	False	True
False	False	False	True	True	True	True	True

Note that since the fourth and seventh columns are the same, it follows that $\neg(p \vee q)$ is equivalent to $(\neg p \wedge \neg q)$.

(c) **[1 mark]** Rewrite the summation $\sum_{i=5}^{15} (2i+1)$ as an equivalent summation where the index of the summation has lower bound 0. Do **not** determine the numerical value of the resulting sum.

Solution

$\sum_{j=0}^{10} (2j+11)$.

(d) **[2 marks]** Is the statement below True or False? Justify your response.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ xy = 1$$

Solution

This statement is False. Consider $x = 0$. Since $\forall y \in \mathbb{R},\ 0 \cdot y = 0$, there is no real number such that $xy = 1$.

Alternatively, we can consider the negation of the statement, $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy \neq 1$. This can be proven by taking $x = 0$ and noting that $\forall y \in \mathbb{R},\ 0 \cdot y = 0$.

(e) **[2 marks]** In the following, $f : \mathbb{R} \to \mathbb{R}$ is a function, $p \in \mathbb{R}$ and $L \in \mathbb{R}$. The set $\mathbb{R}^+$ is the set of positive real numbers. The term $|x - p|$ means the absolute value of $x - p$ and $|f(x) - L|$ means the absolute value of $f(x) - L$.

Write the negation of the following statement:

$$\forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R},\ (0 < |x-p| \wedge |x-p| < \delta) \Rightarrow |f(x) - L| < \epsilon$$

Solution

$$\exists \epsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R},\ (0 < |x-p| \wedge |x-p| < \delta) \wedge |f(x) - L| \geq \epsilon$$

2. [**8 marks**] **Translations.** Let S be the set of all natural numbers, and recall the following terms:

- A natural number is said to be even when it is divisible by 2.
- A natural number is said to be odd when it is not divisible by 2.
- A natural number greater than 1 is said to be a prime number when its only divisors are 1 and itself.

Suppose we define the following predicates:

- $Even(x)$: "x is even", where $x \in S$.
- $Odd(x)$: "x is odd", where $x \in S$.
- $Prime(x)$: "x is a prime number", where $x \in S$.

In Parts (a) – (e), translate each of the following statements from English into symbolic predicate logic. No explanation is necessary. Do not define any of your own predicates or sets. You may use the predicates $<, \leq, =, >, \geq$ and $\neq$ to compare two numbers.

(a) [**1 mark**] The natural number 13 is an odd prime number.

Solution

$Prime(13) \land Odd(13)$

It is okay to write $13 \in S \land Prime(13) \land Odd(13)$ but it can be taken as a given that 13 is a natural number.

The statement $\forall x \in S, x = 13 \Rightarrow (Prime(x) \land Odd(x))$ is also okay but not concise.

(b) [**1 mark**] Natural numbers are either even or odd.

Solution

$\forall x \in S, Even(x) \lor Odd(x)$

It is okay to interpret as an exclusive-or, that is, $\forall x \in S, (Even(x) \land \neg Odd(x)) \lor (\neg Even(x) \land Odd(x))$.

(c) [**2 marks**] Some odd natural number is not prime.

Solution

$\exists x \in S, Odd(x) \land \neg Prime(x)$

(d) [**2 marks**] Prime numbers greater than 2 are odd.

Solution

$\forall x \in S, (Prime(x) \land x > 2) \implies Odd(x)$

(e) [**2 marks**] The number 2 is the only even prime number.

Solution

$(Even(2) \land Prime(2)) \land (\forall x \in S, Even(x) \land Prime(x) \implies x = 2)$

Alternate solution: $\forall x \in S, (Even(x) \land Prime(x)) \Leftrightarrow x = 2$

3. **[5 marks] A proof about numbers.**

(a) **[1 mark]** Translate the following statement into predicate logic:

"There is a positive integer c such that for every positive integer x, $100x^2 + x + 165 \leq c \cdot (x^5 + 1)$."

Use $\mathbb{Z}^+$ to denote the set of positive integers.

Solution

$\exists c \in \mathbb{Z}^+, \forall x \in \mathbb{Z}^+,\ 100x^2 + x + 165 \leq c \cdot (x^5 + 1)$

(b) **[4 marks]** Prove the statement from Part (a). We have left you space for rough work here but write your formal proof in the box below.

Solution

Proof. Let $c = 1000$ (or some other number greater than or equal to both 100 and 165). Let $x \in \mathbb{Z}^+$.

Then

$$\begin{aligned}
c \cdot (x^5 + 1) &= 1000(x^5 + 1) \\
&= 1000x^5 + 1000 \\
&= 999x^5 + x^5 + 1000 \\
&\geq 999x^5 + x^5 + 165 && \text{(since } 1000 \geq 165\text{)} \\
&\geq 999x^5 + x + 165 && \text{(since } x \geq 1, x^5 \geq x\text{)} \\
&\geq 100x^5 + x + 165 && \text{(since } 999 \geq 100\text{),}
\end{aligned}$$

as required.

(There are many other valid sequences of steps.) □

4. **[11 marks] A proof about rational numbers.**

(a) **[1 mark]** Define a unary predicate *Rational* with domain $\mathbb{R}$ so that $Rational(x)$ means:

x is a rational number.

(That is, x is equal to a quotient of two integers. Do not use the set $\mathbb{Q}$.)

Solution

$Rational(x)$: "$\exists a, b \in \mathbb{Z},\ x = \dfrac{a}{b}$", where $x \in \mathbb{R}$

(b) **[1 mark]** Translate the following statement into predicate logic:

"If a real number x is **not** rational then $x + 1$ is **not** rational."

Do not use the predicate *Rational* in your final statement; use its definition instead.

Solution

$$\forall x \in \mathbb{R},\ \neg\big(\exists a, b \in \mathbb{Z},\ x = \frac{a}{b}\big) \Rightarrow \neg\big(\exists c, d \in \mathbb{Z},\ x + 1 = \frac{c}{d}\big)$$

which becomes

$$\forall x \in \mathbb{R},\ \big(\forall a, b \in \mathbb{Z},\ x \neq \frac{a}{b}\big) \Rightarrow \big(\forall c, d \in \mathbb{Z},\ x + 1 \neq \frac{c}{d}\big)$$

(c) **[1 mark]** Write the contrapositive of the statement from Part (b):

Solution

$$\forall x \in \mathbb{R},\ \big(\exists c, d \in \mathbb{Z},\ x + 1 = \frac{c}{d}\big) \Rightarrow \big(\exists a, b \in \mathbb{Z},\ x = \frac{a}{b}\big)$$

(d) [**4 marks**] Write a direct proof of the statement from Part (**c**). (That is, write a direct proof of the contrapositive of the original statement.) We have left you space for rough work here, but write your formal proof in the box below.

Solution

We want to prove the statement

$$\forall x \in \mathbb{R},\ \left(\exists c, d \in \mathbb{Z},\ x+1 = \frac{c}{d}\right) \Rightarrow \left(\exists a, b \in \mathbb{Z},\ x = \frac{a}{b}\right)$$

Proof. Let $x \in \mathbb{R}$ and assume $\left(\exists c, d \in \mathbb{Z},\ x+1 = \frac{c}{d}\right)$.

We want to prove $\left(\exists a, b \in \mathbb{Z},\ x = \frac{a}{b}\right)$. Then

$$\begin{aligned} x &= x + 1 - 1 \\ &= \frac{c}{d} - 1 \\ &= \frac{c-d}{d} \\ &= \frac{a}{b}, \end{aligned}$$

where we let $a = c - d$ and $b = d$. And so we can conclude $x = \frac{a}{b}$, as required.

(There may well be many other proofs.) □

(e) [**4 marks**] Write a direct proof of the statement from Part (**b**).
This proof can be tricky, so be sure to write out the direct proof outline and any comments (e.g. what you want to show in a part of it, what you may use in a part of it) for partial credit.

We have left you space for rough work here and on the last page, but write your formal proof in the box below.

Solution

We want to prove the statement

$$\forall x \in \mathbb{R},\ \left(\forall a, b \in \mathbb{Z},\ x \neq \frac{a}{b}\right) \Rightarrow \left(\forall c, d \in \mathbb{Z},\ x + 1 \neq \frac{c}{d}\right)$$

Proof. Let $x \in \mathbb{R}$ and assume $\left(\forall a, b \in \mathbb{Z},\ x \neq \frac{a}{b}\right)$.

We want to prove $\forall c, d \in \mathbb{Z},\ x + 1 \neq \frac{c}{d}$.

Let $c, d \in \mathbb{Z}$, with $d \neq 0$. Then $\frac{c}{d} - 1 = \frac{c-d}{d}$. This is the ratio of two integers, and so we know by our assumption that it is not equal to x. And so, $\frac{c}{d} - 1 \neq x$ and $\frac{c}{d} \neq x + 1$, as required.

(There may well be many other proofs.) □