CSC 236 Tutorial 3

May 24, 2023

$\forall n \in \mathbb{N}, n \geq 1.(12^n - 1)$ is a multiple of 11

By induction.

Base case. For the base case, observe that $12^1 - 1 = 11$ which is divisible by 11.
Inductive step. Let $k \in \mathbb{N}$, be any natural number with $k \geq 1$, and assume $12^k - 1 = 11a$ for some $a \in \mathbb{N}$. Then, we have

$$\begin{aligned} 12^{k+1} - 1 &= 12^k \cdot 12 - 1 \\ &= (11a + 1) \cdot 12 - 1 \\ &= 12 \cdot 11a + 11 \\ &= 11 \cdot (12a + 1). \end{aligned}$$

Since a is a natural number so is $12a + 1$, thus 12^{k+1} is divisible by 11, completing the induction.

$\forall n \geq 4. n^4 \leq 4^n$

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By induction.

Base case. Both the RHS and the LHS are 4^4, so the base case holds.
Inductive step. Let $k \in \mathbb{N}$ with $k \geq 4$, and assume $k^4 \leq 4^k$. We then have

$$\begin{aligned}
(k+1)^4 &= k^4 + 4k + 6k^2 + 4k^3 + 1 \\
&\leq k^4 + k^2 + 6k^2 + k^4 + k^4 && (k \geq 4, k^4 \geq 1) \\
&\leq k^4 + k^4 + k^4 + k^4 && (k^2 \geq 7, k^4 \geq k) \\
&= 4k^4 \\
&\leq 44^k && (IH) \\
&= 4^{k+1},
\end{aligned}$$

as required.

$\forall n \in \mathbb{N} \sum_{i=1}^{2^n} \frac{1}{i} \geq 1 + \frac{n}{2}$

Base case. For the base case, the LHS and the RHS are both equal to 1.

Inductive step. Let $k \in \mathbb{N}$ be any natural number and suppose $\sum_{i=1}^{2^k} \frac{1}{i} \geq 1 + \frac{k}{2}$. Then

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq 1 + \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq 1 + \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} \\
&= 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} \\
&= 1 + \frac{k+1}{2},
\end{aligned}$$

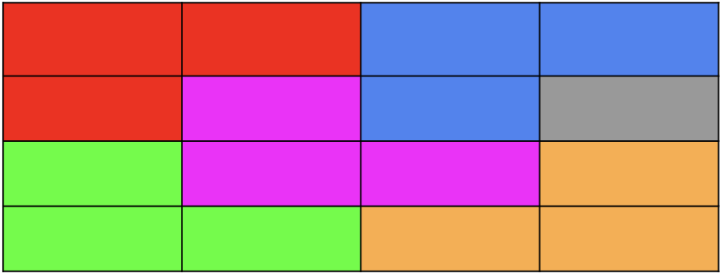
Triominoes

A triomino is a domino that looks like an L.

Consider a $2^n \times 2^n$ grid with one square removed.

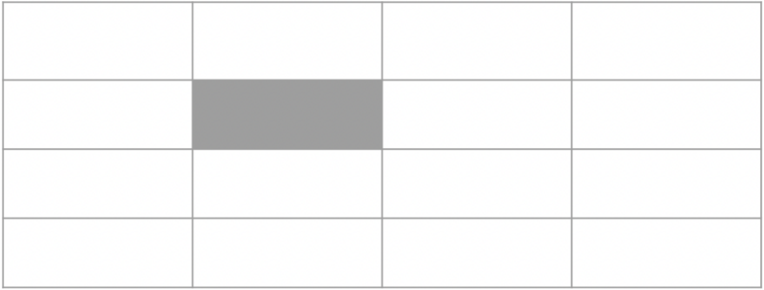
We want to tile the grid using triominoes.

Example



Note the gray square is the one that has been removed.

Try it yourself.



Triominoes

Prove for every $n \in \mathbb{N}$, with $n \geq 1$, you can tile any $2^n \times 2^n$ grid with one square missing using only triominoes (you can rotate them however you wish).

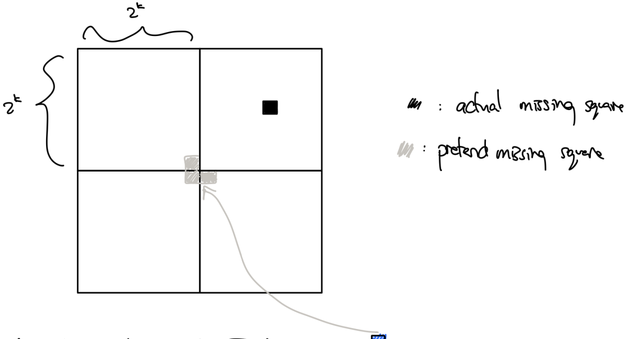
Triominoes

By induction.

Base case. for the $2^1 \times 2^1$ grid, no matter which square you remove, you get exactly a triomino remaining, so the base case holds.

Inductive step. Let $k \in \mathbb{N}, k \geq 1$ be any natural number at least 1, and assume the claim is true for $2^k \times 2^k$ grids with one square removed. We now consider the $2^{k+1} \times 2^{k+1}$ grid with one square removed. Note that this is four $2^k \times 2^k$ grids arranged in a square. If the removed square occurs in the subgrid, we can completely tile the that subgrid by the induction hypothesis. For the remaining 3 subgrids, remove the square in the inner corner of the grid. By the induction hypothesis, these three subgrids can also be tiled using triominoes. Finally, the three squares we removed from the inner corners of the subgrids can be tiled using a single triomino.

Picture



By IH, all subgrids tile-able! Finally put  back.