

Chapter-13

আংশিক ভগ্নাংশের সাহায্যে ইন্টিগ্রেশন

INTEGRATION BY PARTIAL FRACTION

যদি $f(x)$ এবং $g(x)$ বহুপদী হয় তবে $\frac{f(x)}{g(x)}$ আকারের ফাংশনকে মূলদীয় ফাংশন বলে। এইরূপ ফাংশনের জন্য ফাংশনটিকে আংশিক ভগ্নাংশে পরিণত করিয়া নিতে হইবে।

$f(x)$ যদি $g(x)$ এর সমান বা অধিক মাত্রার হয়, তবে সাধারণ ভাগ প্রক্রিয়ার সাহায্যে $f(x)$ কে $g(x)$ দ্বারা এমন ভাবে ভাগ করিতে হইবে যেন অবশিষ্টের লবের মাত্রা $g(x)$ এর মাত্রা হইতে ছোট হয়।

যখন $f(x)$ এর মাত্রা $g(x)$ এর মাত্রা অপেক্ষা ক্ষুদ্রতর তখন মূলদীয় ফাংশনকে আংশিক ভগ্নাংশে পরিণত করার সাধারণ সমাধানের নিয়ম :

(i) প্রত্যেক $ax + b$ উৎপাদকের জন্য প্রতিষঙ্গী আংশিক ভগ্নাংশ $\frac{A}{ax + b}$.

(ii) প্রত্যেক $(ax + b)^n$ উৎপাদকের জন্য প্রতিষঙ্গী আংশিক ভগ্নাংশ

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}.$$

(iii) প্রত্যেক $ax^2 + bx + c$ দ্বিঘাত উৎপাদকের জন্য প্রতিষঙ্গী আংশিক ভগ্নাংশ

$$\frac{Ax + B}{ax^2 + bx + c}.$$

(iv) প্রত্যেক $(ax^2 + bx + c)^n$ পুনরাবৃত্ত দ্বিঘাত উৎপাদকের জন্য প্রতিষঙ্গী আংশিক ভগ্নাংশ $\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$.

একইভাবে এই নিয়মটি সম্প্রসারণ করা যায়।

13.1. কভার আপ রুল (Cover up rule) :

মূলদীয় ফাংশনকে কভার আপ রুল (Cover-up rule) পদ্ধতির সাহায্যে সহজে আংশিক ভগ্নাংশে পরিণত করা যায়। নিম্নে উদাহরণের মাধ্যমে নিয়মটি আলোচনা করা হইল :

উদাহরণ-1. কভার আপ রুল পদ্ধতির সাহায্যে $\frac{x+4}{(x-2)(x-3)}$ কে আংশিক ভগ্নাংশে পরিণত হয়।

$$\text{সমাধান : } \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad \dots\dots (1)$$

$$\text{এখানে } A = \left[\frac{x+4}{x-3} \right]_{x=2} = \frac{2+4}{2-3} = -6 \text{ এবং } B = \left[\frac{x+4}{x-2} \right]_{x=3} = \frac{3+4}{3-2} = 7$$

$$\text{সুতরাং } (1) \Rightarrow \frac{x+4}{(x-2)(x-3)} = \frac{-6}{x-2} + \frac{7}{x-3}.$$

উদাহরণ-2. $\frac{x^2 + 5x - 7}{(x-1)(x-2)(x+4)}$ কে কভার-আপ রূল পদ্ধতির সাহায্যে
আংশিক ভগ্নাংশে পরিণত কর।

$$\begin{aligned} \text{সমাধান : } & \frac{x^2 + 5x - 7}{(x-1)(x-2)(x+4)} = \frac{1^2 + 5 \cdot 1 - 7}{(x-1)(1-2)(1+4)} \\ & + \frac{2^2 + 5 \cdot 2 - 7}{(2-1)(x-2)(2+4)} + \frac{(-4)^2 + 5(-4) - 7}{(-4-1)(-4-2)(x+4)} \\ & = \frac{-1}{-5(x-1)} + \frac{7}{6(x-2)} + \frac{11}{30(x+4)} = \frac{1}{5} \frac{1}{x-1} + \frac{7}{6} \frac{1}{x-2} + \frac{11}{30} \frac{1}{x+4} \end{aligned}$$

উদাহরণ-3. $\int \frac{dx}{(x-a)(x-b)}$ এর মান নির্ণয় কর। [NUH-94, DUH-62]

$$\begin{aligned} \text{সমাধান : } & \int \frac{dx}{(x-a)(x-b)} = \int \left[\frac{1}{(b-a)(x-b)} + \frac{1}{(x-a)(a-b)} \right] dx \\ & = \frac{1}{b-a} \int \frac{1}{x-b} dx + \frac{1}{a-b} \int \frac{1}{x-a} dx \\ & = \frac{1}{b-a} \ln|x-b| - \frac{1}{b-a} \ln|x-a| + c \\ & = \frac{1}{b-a} [\ln|x-b| - \ln|x-a|] + c = \frac{1}{b-a} \ln \left| \frac{x-b}{x-a} \right| + c. \end{aligned}$$

উদাহরণ-4. $\int \frac{x-1}{(x-2)(x-3)} dx$ এর মান নির্ণয় কর।

[DUH-1986, CUH-1982]

$$\begin{aligned} \text{সমাধান : } & \int \frac{x-1}{(x-2)(x-3)} dx = \int \left[\frac{3-1}{(3-2)(x-3)} + \frac{2-1}{(x-2)(2-3)} \right] dx \\ & = \int \left(\frac{2}{x-3} - \frac{1}{x-2} \right) dx = 2 \ln|x-3| - \ln|x-2| + c. \end{aligned}$$

উদাহরণ-5. $\int \frac{x^2 + 1}{(x-1)(x-2)(x+2)} dx$ এর মান নির্ণয় কর।

$$\begin{aligned} \text{সমাধান : } & \int \frac{x^2 + 1}{(x-1)(x-2)(x+2)} dx \\ & = \int \left[\frac{1^2 + 1}{(x-1)(1-2)(1+2)} + \frac{2^2 + 1}{(2-1)(x-2)(2+2)} \right. \\ & \quad \left. + \frac{(-2)^2 + 1}{(-2-1)(-2-2)(x+2)} \right] dx \\ & = \int \left[\frac{2}{-3(x-1)} + \frac{5}{4(x-2)} + \frac{5}{12(x+2)} \right] dx \end{aligned}$$

$$\begin{aligned}
 &= -\frac{2}{3} \int \frac{1}{x-1} dx + \frac{5}{4} \int \frac{1}{x-2} dx + \frac{5}{12} \int \frac{dx}{x+2} \\
 &= -\frac{2}{3} \ln|x-1| + \frac{5}{4} \ln|x-2| + \frac{5}{12} \ln|x+2| + c.
 \end{aligned}$$

উদাহরণ-6. $\int \frac{7x+4}{x^3-4x} dx$ এর মান নির্ণয় কর। [DUH-1988]

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{7x+4}{x^3-4x} dx = \int \frac{7x+4}{x(x^2-4)} dx = \int \frac{7x+4}{x(x+2)(x-2)} dx \\
 &= \left[\frac{7.0+4}{x(0+2)(0-2)} + \frac{7(-2)+4}{(-2)(x+2)(-2-2)} + \frac{7.2+4}{2.(2+2)(x-2)} \right] dx \\
 &= -\int \frac{1}{x} dx - \frac{5}{4} \int \frac{1}{x+2} dx + \frac{9}{4} \int \frac{1}{x-2} dx \\
 &= -\ln|x| - \frac{5}{4} \ln|x+2| + \frac{9}{4} \ln|x-2| + c.
 \end{aligned}$$

উদাহরণ-7. $\int \frac{4x+3}{x(2x+1)(2x+3)} dx$ এর মান নির্ণয় কর।

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{4x+3}{x(2x+1)(2x+3)} dx \\
 &= \left[\frac{0+3}{x(0+1)(0+3)} + \frac{-2+3}{-\frac{1}{2}(2x+1)(-1+3)} + \frac{-6+3}{-\frac{3}{2}(-3+1)(2x+3)} \right] dx \\
 &= \int \left(\frac{1}{x} - \frac{1}{2x+1} - \frac{1}{2x+3} \right) dx \\
 &= \int \frac{1}{x} dx - \frac{1}{2} \int \frac{2}{2x+1} dx - \frac{1}{2} \int \frac{2}{2x+3} dx \\
 &= \ln|x| - \frac{1}{2} \ln|2x+1| - \frac{1}{2} \ln|2x+3| + c.
 \end{aligned}$$

উদাহরণ-8. $\int \frac{dx}{x^3-x^2-9x+9}$ এর মান নির্ণয় কর। [DUH-1955]

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{dx}{x^3-x^2-9x+9} \\
 &= \int \frac{dx}{x^2(x-1)-9(x-1)} = \int \frac{dx}{(x-1)(x^2-9)} \\
 &= \int \frac{dx}{(x-1)(x+3)(x-3)} \\
 &= \int \left[\frac{1}{(x-1)(-2).4} + \frac{1}{2(x-3).6} + \frac{1}{(-4)(-6)(x+3)} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{8} \int \frac{1}{x-1} dx + \frac{1}{12} \int \frac{1}{x-3} dx + \frac{1}{24} \int \frac{1}{x+3} dx \\
 &= -\frac{1}{8} \ln|x-1| + \frac{1}{12} \ln|x-3| + \frac{1}{24} \ln|x+3| + c.
 \end{aligned}$$

উদাহরণ-9. $\int \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}$ এর মান নির্ণয় কর। [DUH-1962]

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} \\
 &= \int \left[\frac{-a^2}{(x^2 + a^2)(-a^2 + b^2)} + \frac{-b^2}{(-b^2 + a^2)(x^2 + b^2)} \right] dx \\
 &= \frac{a^2}{a^2 - b^2} \int \frac{1}{x^2 + a^2} dx - \frac{b^2}{a^2 - b^2} \int \frac{1}{x^2 + b^2} dx \\
 &= \frac{a^2}{a^2 - b^2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right) + c \\
 &= \frac{a}{a^2 - b^2} \tan^{-1}\left(\frac{x}{a}\right) - \frac{b}{a^2 - b^2} \tan^{-1}\left(\frac{x}{b}\right) + c.
 \end{aligned}$$

উদাহরণ-10. $\int \frac{dx}{(x^2 + 9)(x^2 + 16)}$ এর মান নির্ণয় কর। [RUH-1956]

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{dx}{(x^2 + 9)(x^2 + 16)} \\
 &= \int \left[\frac{1}{(x^2 + 9)(-9 + 16)} + \frac{1}{(-16 + 9)(x^2 + 16)} \right] dx \\
 &= \int \left[\frac{1}{7(x^2 + 9)} - \frac{1}{7(x^2 + 16)} \right] dx = \frac{1}{7} \int \frac{1}{x^2 + 9} dx - \frac{1}{7} \int \frac{1}{x^2 + 16} dx \\
 &= \frac{1}{7} \int \frac{1}{x^2 + 3^2} dx - \frac{1}{7} \int \frac{1}{x^2 + 4^2} dx \\
 &= \frac{1}{7} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) - \frac{1}{7} \cdot \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + c \\
 &= \frac{1}{21} \tan^{-1}\left(\frac{x}{3}\right) - \frac{1}{28} \tan^{-1}\left(\frac{x}{4}\right) + c.
 \end{aligned}$$

উদাহরণ-11. $\int \frac{x^2 dx}{x^4 - x^2 - 12}$ এর মান নির্ণয় কর। [RUH-1990]

$$\begin{aligned}
 \text{সমাধান : } &\int \frac{x^2 dx}{x^4 - x^2 - 12} = \int \frac{x^2 dx}{(x^2 - 4)(x^2 + 3)} \\
 &= \int \left[\frac{4}{(x^2 - 4)(4 + 3)} + \frac{-3}{(-3 - 4)(x^2 + 3)} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{7} \int \frac{1}{x^2 - 4} dx + \frac{3}{7} \int \frac{1}{x^2 + 3} dx = \frac{4}{7} \int \frac{dx}{x^2 - 2^2} + \frac{3}{7} \int \frac{dx}{x^2 + (\sqrt{3})^2} \\
 &= \frac{4}{7} \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{x-2}{x+2} \right| + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left| \frac{x}{\sqrt{3}} \right| + c \\
 &= \frac{1}{7} \ln \left| \frac{x-2}{x+2} \right| + \frac{\sqrt{3}}{7} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + c.
 \end{aligned}$$

উদাহরণ-12. $\int \frac{x \, dx}{(x^2 + a^2)(x^2 + b^2)}$ এর মান নির্ণয় কর।

$$\begin{aligned}
 \text{সমাধান : } & \int \frac{x \, dx}{(x^2 + a^2)(x^2 + b^2)} && \text{ধরি } x^2 = z \\
 &= \frac{1}{2} \int \frac{dz}{(z + a^2)(z + b^2)} && \therefore 2x \, dx = dz \\
 &= \frac{1}{2} \int \left[\frac{1}{(z + a^2)(-a^2 + b^2)} + \frac{1}{(-b^2 + a^2)(z + b^2)} \right] dz \\
 &= \frac{1}{2(b^2 - a^2)} \left[\int \frac{1}{z + a^2} dz - \int \frac{1}{z + b^2} dz \right] \\
 &= \frac{1}{2(b^2 - a^2)} [\ln|z + a^2| - \ln|z + b^2|] + c \\
 &= \frac{1}{2(b^2 - a^2)} \ln \left| \frac{z + a^2}{z + b^2} \right| + c \\
 &= \frac{1}{2(b^2 - a^2)} \ln \left| \frac{x^2 + a^2}{x^2 + b^2} \right| + c.
 \end{aligned}$$

উদাহরণ-13. $\int \frac{x^3 \, dx}{(x^2 + 4)(x^2 + 9)}$ এর মান নির্ণয় কর।

$$\begin{aligned}
 \text{সমাধান : } & \int \frac{x^3 \, dx}{(x^2 + 4)(x^2 + 9)} && \text{ধরি } x^2 = z \\
 &= \frac{1}{2} \int \frac{z \, dz}{(z + 4)(z + 9)} && \therefore 2x \, dx = dz \\
 &= \frac{1}{2} \int \left[\frac{-4}{(z + 4)(-4 + 9)} + \frac{-9}{(-9 + 4)(z + 9)} \right] dz \\
 &= \frac{1}{10} \left[-4 \int \frac{1}{z + 4} dz + 9 \int \frac{1}{z + 9} dz \right] \\
 &= \frac{1}{10} [-4 \ln|z + 4| + 9 \ln|z + 9|] + c \\
 &= -\frac{2}{5} \ln|x^2 + 4| + \frac{9}{10} \ln|x^2 + 9| + c.
 \end{aligned}$$

উদাহরণ-14. $\int \frac{\cos x \, dx}{(a + \sin x)(b - \sin x)}$ এর মান নির্ণয় কর।

[RUH-1981, CUH-1980]

সমাধান : $\int \frac{\cos x \, dx}{(a + \sin x)(b - \sin x)}$ ধরি $\sin x = z$

$$\therefore \cos x \, dx = dz$$

$$= \int \frac{dz}{(a+z)(b-z)} = \int \left[\frac{1}{(a+z)(b+a)} + \frac{1}{(a+b)(b-z)} \right] dz$$

$$= \int \frac{1}{(a+b)(a+z)} dz + \int \frac{1}{(a+b)(b-z)} dz$$

$$= \frac{1}{a+b} [\ln|a+z| - \ln|b-z|] + c = \frac{1}{a+b} \ln \left| \frac{a+\sin x}{b-\sin x} \right| + c.$$

উদাহরণ-15. $\int \frac{dx}{x(2 + \ln x)(3 - \ln x)}$ এর মান নির্ণয় কর।

সমাধান : $\int \frac{dx}{x(2 + \ln x)(3 - \ln x)}$ ধরি $\ln x = z$

$$\therefore \frac{1}{x} dx = dz$$

$$= \int \left[\frac{1}{(2+z)(3+2)} + \frac{1}{(2+3)(3-z)} \right] dz$$

$$= \frac{1}{5} \left[\int \frac{1}{2+z} dz + \int \frac{1}{3-z} dz \right]$$

$$= \frac{1}{5} [\ln|2+z| - \ln|3-z|] + c$$

$$= \frac{1}{5} \ln \left| \frac{2+\ln x}{3-\ln x} \right| + c.$$

উদাহরণ-16. $\int \frac{e^x \, dx}{e^x - 3e^{-x} + 2}$ এর মান নির্ণয় কর।

সমাধান : $\int \frac{e^x \, dx}{e^x - 3e^{-x} + 2}$ ধরি $e^x = z$

$$\therefore e^x \, dx = dz$$

$$= \int \frac{e^x \cdot e^x \, dx}{(e^x)^2 + 2e^x - 3} = \int \frac{z \, dz}{z^2 + 2z - 3} = \int \frac{z \, dz}{(z+3)(z-1)}$$

$$= \int \left[\frac{-3}{(z+3)(-3-1)} + \frac{1}{(1+3)(z-1)} \right] dz$$

$$= \frac{3}{4} \int \frac{1}{z+3} dz + \frac{1}{4} \int \frac{1}{z-1} dz = \frac{3}{4} \ln|e^x+3| + \frac{1}{4} \ln|e^x-1| + c.$$

উদাহরণ-17. $\int \frac{(x-1)(x-2)(x-3)}{(x-5)(x-6)(x-7)} dx$ এর মান নির্ণয় কর।

সমাধান : ধরি $I = \int \frac{(x-1)(x-2)(x-3)}{(x-5)(x-6)(x-7)} dx$

$$= \int \left[1 + \frac{(5-1)(5-2)(5-3)}{(x-5)(5-6)(5-7)} + \frac{(6-1)(6-2)(6-3)}{(6-5)(x-6)(6-7)} \right. \\ \left. + \frac{(7-1)(7-2)(7-3)}{(7-5)(7-6)(x-7)} \right] dx$$

$$= \int \left(1 + \frac{12}{x-5} - \frac{60}{x-6} + \frac{60}{x-7} \right) dx$$

$$= x + 12 \ln|x-5| - 60 \ln|x-6| + 60 \ln|x-7| + c.$$

উদাহরণ-18. $\int \frac{x^4 dx}{(x^2 + a^2)(x^2 + b^2)}$ এর মান নির্ণয় কর। [DUH-1991]

সমাধান : ধরি $I = \int \frac{x^4}{(x^2 + a^2)(x^2 + b^2)}$

$$= \int \left[1 + \frac{(-a^2)^2}{(x^2 + a^2)(-a^2 + b^2)} + \frac{(-b^2)^2}{(-b^2 + a^2)(x^2 + b^2)} \right] dx$$

$$= \int \left[1 - \frac{a^4}{a^2 - b^2} \cdot \frac{1}{x^2 + a^2} + \frac{b^4}{a^2 - b^2} \cdot \frac{1}{x^2 + b^2} \right] dx$$

$$= x - \frac{a^4}{a^2 - b^2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + \frac{b^4}{a^2 - b^2} \cdot \frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right) + c$$

$$= x - \frac{a^3}{a^2 - b^2} \tan^{-1}\left(\frac{x}{a}\right) + \frac{b^3}{a^2 - b^2} \tan^{-1}\left(\frac{x}{b}\right) + c.$$

উদাহরণ-19. $\int \frac{2x^2 - 1}{(x+1)^2(x-2)} dx$ এর মান নির্ণয় কর।

[NUH-1995, RUH-1979, 1983]

সমাধান : ধরি $I = \int \frac{2x^2 - 1}{(x+1)^2(x-2)} dx \dots\dots (1)$

আংশিক ভগ্নাংশের কভার-আপ রুল হইতে পাই

$$\frac{2x^2 - 1}{(x+1)^2(x-2)} = \frac{2 \cdot 2^2 - 1}{(2+1)^2(x-2)} + \frac{2(-1)^2 - 1}{(x+1)^2(-1-2)} + \frac{A}{x+1}$$

$$x = 0 \text{ বসিয়ে পাই } \frac{1}{2} = -\frac{7}{18} - \frac{1}{3} + A \therefore A = \frac{11}{9}$$

$$\begin{aligned}\therefore \frac{2x^2 - 1}{(x+1)^2(x-2)} &= \frac{7/9}{x-2} - \frac{1/3}{(x+1)^2} + \frac{11/9}{x+1} \\ \therefore (1) \Rightarrow I &= \int \left\{ \frac{7/9}{x-2} - \frac{1/3}{(x+1)^2} + \frac{11/9}{x+1} \right\} dx \\ &= \frac{7}{9} \ln|x-2| + \frac{1}{3} \cdot \frac{1}{x+1} + \frac{11}{9} \ln|x+1| + C.\end{aligned}$$

উদাহরণ-20. $\int \frac{2x^2 + x + 17}{(x-1)(x^2 + 2x - 3)} dx$ এর মান নির্ণয় কর। [DUH-87]

$$\text{সমাধান : } \text{ধরি } I = \int \frac{2x^2 + x + 17}{(x-1)(x^2 + 2x - 3)} dx = \int \frac{2x^2 + x + 17}{(x-1)^2(x+3)} dx$$

আংশিক ভগ্নাংশের কভার-আপ রূল অনুযায়ী পাই

$$\begin{aligned}\frac{2x^2 + x + 17}{(x-1)^2(x+3)} &= \frac{2+1+17}{(x-1)^2 \cdot 4} + \frac{A}{x-1} + \frac{18-3+17}{16(x+3)} \\ x=0 \text{ বসিয়ে পাই } \frac{17}{3} &= 5 - A + \frac{2}{3} \quad \therefore A=0\end{aligned}$$

$$\therefore \frac{2x^2 + x + 17}{(x-1)^2(x+3)} = \frac{5}{(x-1)^2} + \frac{2}{x+3}$$

$$\begin{aligned}(1) \Rightarrow I &= \int \frac{2x^2 + x + 17}{(x-1)^2(x+3)} dx = \int \left\{ \frac{5}{(x-1)^2} + \frac{2}{(x+3)} \right\} dx \\ &= -\frac{5}{x-1} + 2 \ln|x+3| + C.\end{aligned}$$

উদাহরণ-21. $\int \frac{x^2 + 2}{(x+2)^3(x-1)} dx$ এর মান নির্ণয় কর। [NUH-2006]

$$\text{সমাধান : } \text{ধরি } I = \int \frac{x^2 + 2}{(x+2)^3(x-1)} dx \dots\dots (1)$$

আংশিক ভগ্নাংশের কভার-আপ রূল অনুযায়ী পাই,

$$\begin{aligned}\frac{x^2 + 2}{(x+2)^3(x-1)} &= \frac{1^2 + 2}{(1+2)^3(x-1)} + \frac{(-2)^2 + 2}{(x+2)^3(-2-1)} + \frac{A}{(x+2)^2} + \frac{B}{x+2} \\ \Rightarrow x^2 + 2 &= \frac{1}{9}(x+2)^3 - 2(x-1) + A(x-1)(x+2) + B(x-1)(x+2)^2\end{aligned}$$

উভয়পক্ষ হইতে x^3 ও x^2 এর সহগ সমীকৃত করিয়া পাই

$$\begin{aligned}0 &= \frac{1}{9} + B \text{ এবং } 1 = \frac{2}{3} + A + 3B \quad \therefore B = -\frac{1}{9} \quad \therefore \frac{1}{3} = A - \frac{1}{3} \Rightarrow A = \frac{2}{3} \\ \therefore \frac{x^2 + 2}{(x+2)^3(x-1)} &= \frac{\frac{1}{9}}{x-1} - \frac{\frac{2}{3}}{(x+2)^3} + \frac{\frac{2}{3}}{(x+2)^2} - \frac{\frac{1}{9}}{x+2}\end{aligned}$$

$$(1) \Rightarrow I = \int \left\{ \frac{1}{9} - \frac{2}{(x+2)^3} + \frac{2}{(x+2)^2} - \frac{1}{9(x+2)} \right\} dx \\ = \frac{1}{9} \ln|x-1| + \frac{1}{(x+2)^2} - \frac{2}{3(x+2)} - \frac{1}{9} \ln|x+2| + c.$$

উদাহরণ-22. $\int \frac{2x^2 - 1}{(x+1)^2(x-2)} dx$ এর মান নির্ণয় কর। [NUH-2005]

সমাধান : ধরি $I = \int \frac{2x^2 - 1}{(x+1)^2(x-2)} dx \dots\dots (1)$

$$\text{এবং } \frac{2x^2 - 1}{(x+1)^2(x-2)} = \frac{-\frac{1}{3}}{(x+1)^2} + \frac{A}{x+1} + \frac{\frac{7}{9}}{x-2} \\ \Rightarrow 2x^2 - 1 = -\frac{1}{3}(x-2) + A(x+1)(x-2) + \frac{7}{9}(x+1)^2$$

x^2 এর সহগ সমীকৃত করিয়া পাই, $2 = A + \frac{7}{9}$

$$\Rightarrow A = \frac{11}{9}$$

$$\therefore \frac{2x^2 - 1}{(x+1)^2(x-2)} = \frac{-\frac{1}{3}}{(x+1)^2} + \frac{\frac{11}{9}}{x+1} + \frac{\frac{7}{9}}{x-2}$$

$$(1) \Rightarrow I = \int \left\{ \frac{-\frac{1}{3}}{(x+1)^2} + \frac{\frac{11}{9}}{x+1} + \frac{\frac{7}{9}}{x-2} \right\} dx \\ = \frac{1}{3(x+1)} + \frac{11}{9} \ln|x+1| + \frac{7}{9} \ln|x-2| + c$$

উদাহরণ-23. $\int \frac{dx}{(x-1)^2(x-2)^3}$ এর মান নির্ণয় কর। [NUH(NM)-2002]

সমাধান : ধরি $I = \int \frac{dx}{(x-1)^2(x-2)^3} \dots\dots (1)$

$$\text{এবং } \frac{1}{(x-1)^2(x-2)^3} = \frac{-1}{(x-1)^2} + \frac{A}{x-1} + \frac{1}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

$$\Rightarrow 1 = -(x-2)^3 + A(x-1)(x-2)^3 + (x-1)^2 + B(x-1)^2(x-2) \\ + C(x-1)^2(x-2)^2 \\ = -(x^3 - 6x^2 + 12x - 8) + A(x-1)(x^3 - 6x^2 + 12x - 8) \\ + (x^2 - 2x + 1) + B(x^2 - 2x + 1)(x-2) \\ + C(x^2 - 2x + 1)(x^2 - 4x + 4)$$

উভয় পক্ষ হতে x^4, x^3, x^2 এর সহগ সমীকৃত করিয়া পাই,

$$0 = A + C \dots\dots (2)$$

$$0 = -1 - 7A + B - 6C \dots\dots (3)$$

$$\text{এবং } 0 = 6 + 18A + 1 - 4B + 5C$$

$$\Rightarrow 0 = 7 + 18A - 4B + 5C \dots\dots (4)$$

$$(4) + (3) \cdot 4 \Rightarrow 0 = 3 - 10A - 19C \dots\dots (5)$$

$$(5) + (2) \cdot 10 \Rightarrow 0 = 3 - 9C$$

$$\Rightarrow C = \frac{1}{3}$$

$$\therefore (2) \Rightarrow A = -\frac{1}{3}$$

$$\therefore (3) \Rightarrow B = 1 - \frac{7}{3} + \frac{6}{3}$$

$$= \frac{2}{3}$$

$$\therefore \frac{1}{(x-1)^2(x-2)^3} = \frac{-1}{(x-1)^2} + \frac{-\frac{1}{3}}{x-1} + \frac{1}{(x-2)^3} + \frac{\frac{2}{3}}{(x-2)^2} + \frac{\frac{1}{3}}{x-2}$$

$$(1) \Rightarrow I = \int \left\{ -\frac{1}{(x-1)^2} - \frac{1/3}{x-1} + \frac{1}{(x-2)^3} + \frac{2/3}{(x-2)^2} + \frac{1/3}{x-2} \right\} dx \\ = \frac{1}{x-1} - \frac{1}{3} \ln|x-1| - \frac{1}{2(x-2)^2} - \frac{2}{3(x-2)} + \frac{1}{3} \ln|x-2| + C$$

উদাহরণ-24. $\int \frac{dx}{x^3+1}$ এর মান নির্ণয় কর।

[NUH-2009]

$$\text{সমাধান : ধরি } I = \int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} \dots\dots (1)$$

$$\text{এবং } \frac{1}{(x+1)(x^2-x+1)} = \frac{1/3}{x+1} + \frac{Ax+B}{x^2-x+1}$$

$$\Rightarrow 1 = \frac{1}{3}(x^2-x+1) + (Ax+B)(x+1)$$

উভয়পক্ষ হইতে x^2 ও x এর সহগ সমীকৃত করিয়া পাই,

$$0 = \frac{1}{3} + A \text{ এবং } 0 = -\frac{1}{3} + A + B$$

$$\therefore A = -\frac{1}{3} \text{ এবং } B = \frac{2}{3}$$

$$\therefore \frac{1}{(x+1)(x^2-x+1)} = \frac{\frac{1}{3}}{x+1} - \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}$$

$$\begin{aligned}
 (1) \Rightarrow I &= \frac{1}{3} \int \frac{1}{1+x} dx - \frac{1}{3} \int \frac{x-2}{x^2-x+1} dx \\
 &= \frac{1}{3} \ln(1+x) - \frac{1}{3} \int \frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1} dx \\
 &= \frac{1}{3} \ln(x+1) - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{dx}{x^2-x+1} \\
 &= \frac{1}{3} \ln(x+1) - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{2} \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
 &= \frac{1}{3} \ln(x+1) - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C \\
 &= \frac{1}{3} \ln(x+1) - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + C.
 \end{aligned}$$

উদাহরণ-25. সমাকলন কর [Integrate] : $\int \frac{x \, dx}{(x-1)(x^2+1)}$.

[NUH(NM)-2012]

সমাধান : ধরি $I = \int \frac{x \, dx}{(x-1)(x^2+1)} \dots\dots (1)$

$$\text{এবং } \frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{Ax+B}{x^2+1}$$

$$\Rightarrow x = \frac{1}{2}(x^2+1) + (Ax+B)(x-1)$$

উভয়পক্ষ হইতে x^2 ও x এর সহগ সমীকৃত করিয়া পাই,

$$0 = \frac{1}{2} + A \quad \text{এবং} \quad 1 = -A + B$$

$$\Rightarrow A = -\frac{1}{2} \quad \text{এবং} \quad B = \frac{1}{2}$$

$$\therefore \frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\begin{aligned}
 (1) \Rightarrow I &= \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{dx}{x^2+1} \\
 &= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x + C
 \end{aligned}$$

উদাহরণ-26. $\int \frac{x^3 + 1}{x(x^2 + 1)^2} dx$ এর মান নির্ণয় কর।

$$\text{সমাধান : } \text{ধরি } I = \int \frac{x^3 + 1}{x(x^2 + 1)^2} dx \dots\dots (1)$$

$$\text{এবং } \frac{x^3 + 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$\Rightarrow x^3 + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$$

উভয় পক্ষ হইতে x^4 , x^3 , x^2 , x এর সহগ এবং ক্রিবপদ তুলনা করিয়া পাই

$$A = 1, A + B = 0 \Rightarrow B = -1$$

$$C = 1, C + E = 0 \Rightarrow E = -1 \text{ এবং } 2A + B + D = 0 \quad \therefore D = -1$$

$$\therefore \frac{x^3 + 1}{x(x^2 + 1)^2} = \frac{1}{x} + \frac{-x + 1}{x^2 + 1} + \frac{-x - 1}{(x^2 + 1)^2}$$

$$(1) \Rightarrow I = \int \frac{1}{x} dx - \int \frac{x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx - \int \frac{x}{(x^2 + 1)^2} dx - \int \frac{1}{(x^2 + 1)^2} dx$$

$$\therefore I = \ln|x| - \frac{1}{2} \ln(x^2 + 1) + \tan^{-1}x - \frac{1}{2} \int \frac{d(x^2 + 1)}{(x^2 + 1)^2} - \int \frac{dx}{(x^2 + 1)^2}$$

$$= \ln|x| - \frac{1}{2} \ln(x^2 + 1) + \tan^{-1}x + \frac{1}{2(x^2 + 1)} - I_1 \dots \quad (2)$$

$$\text{येथाने } I_1 = \int \frac{dx}{(x^2 + 1)^2} \quad \text{धरि } x = \tan \theta \\ dx = \sec^2 \theta d\theta$$

$$\sin x = \tan \theta$$

$$dx = \sec^2\theta \, d\theta$$

$$= \int \frac{\sec^2 \theta \, d\theta}{(1 + \tan^2 \theta)^2}$$

$$= \int \cos^2 \theta \, d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} \theta + \frac{\sin 2\theta}{4} + C$$

$$= \frac{1}{2}\theta + \frac{1}{4} \cdot \frac{2\tan\theta}{1+\tan^2\theta} + C$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \cdot \frac{x}{1+x^2} + C$$

$$(2) \Rightarrow I = \ln x - \frac{1}{2} \ln(x^2 + 1) + \tan^{-1} x + \frac{1}{2(x^2 + 1)} - \frac{1}{2} \tan^{-1} x$$

$$s - \frac{x}{2(x^2 + 1)} + c$$

$$= \ln x - \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \tan^{-1} x + \frac{1 - x}{2(x^2 + 1)} + C.$$

Brief / Quiz Questions with Answers [উত্তরসহ অতি সংক্ষিপ্ত প্রশ্ন]

13.1. যদি $f(x)$ এবং $g(x)$ বহুপদী হয় তবে $\int \frac{f(x)}{g(x)} dx$ এর মান নির্ণয় করার নিয়ম কি? [If $f(x)$ and $g(x)$ are polynomials then what is the rule to find the value of $\int \frac{f(x)}{g(x)} dx?$]

Ans : প্রদত্ত ইনটিগ্র্যালের মান নির্ণয়ের জন্য $\frac{f(x)}{g(x)}$ ফাংশনটিকে আংশিক ভগ্নাংশে প্রকাশ করিতে হয়।

13.2. $\frac{x+4}{(x-2)(x-3)}$ কে আংশিক ভগ্নাংশে প্রকাশ কর। [Express $\frac{x+4}{(x-2)(x-3)}$ as partial fractions.]

$$\begin{aligned}\text{Ans : } \frac{x+4}{(x-2)(x-3)} &= \frac{2+4}{(x-2)(2-3)} + \frac{3+4}{(x-2)(x-3)} \\ &= \frac{-6}{x-2} + \frac{7}{x-3}.\end{aligned}$$

13.3. $\frac{x^2+5x-7}{(x-1)(x-2)(x+4)}$ কে আংশিক ভগ্নাংশে প্রকাশ কর। [Express $\frac{x^2+5x-7}{(x-1)(x-2)(x+4)}$ as partial fractions.]

$$\begin{aligned}\text{Ans : } \frac{x^2+5x-7}{(x-1)(x-2)(x+4)} &= \frac{1+5-7}{(x-1)(1-2)(1+4)} \\ &\quad + \frac{4+10-7}{(x-2)(2-1)(2+4)} + \frac{16-20-7}{(x+4)(-4-1)(-4-2)} \\ &= \frac{1/5}{x-1} + \frac{7/6}{x-2} + \frac{11/30}{x+4}.\end{aligned}$$

13.4. $\int \frac{(x+4) dx}{(x-2)(x-3)} = ?$

$$\begin{aligned}\text{Ans : } \int \frac{x+4}{(x-2)(x-3)} dx &= \int \left(\frac{7}{x-3} - \frac{6}{x-2} \right) dx \\ &= 7 \ln|x-3| - 6 \ln|x-2| + c.\end{aligned}$$

EXERCISE-13

Part-A : Brief Questions [অতি সংক্ষিপ্ত প্রশ্ন]

1. যদি $f(x)$ এবং $g(x)$ বহুপদী হয় তবে $\int \frac{f(x)}{g(x)} dx$ এর মান নির্ণয় করার নিয়ম কি?
 [If $f(x)$ and $g(x)$ are polynomials then what is the rule to find the value of $\int \frac{f(x)}{g(x)} dx?$]

Part-B : Short Questions [সংক্ষিপ্ত প্রশ্ন]

মান নির্ণয় কর।

1. $\int \frac{x \, dx}{(x-a)(x-b)}$
2. $\int \frac{dx}{x^2 + x - 42}$ [RUH-1958]
3. $\int \frac{dx}{x^2 + 7x - 18}$ [RUH-59]
4. $\int \frac{3x \, dx}{x^2 - x - 2}$ [RUH-1990]
5. $\int \frac{x-1}{(x+2)(x-3)} dx$
6. $\int \frac{x \, dx}{x^2 - 12x + 35}$
7. $\int \frac{2x-3}{(x-1)(x-2)} dx$
8. $\int \frac{x \, dx}{(3-x)(3+2x)}$
9. $\int \frac{dx}{(x+1)(x+2)(x+3)}$
10. $\int \frac{x+4}{(x^2-1)(x+2)} dx$ [RUH-86]
11. $\int \frac{x^2 \, dx}{(x-1)(x-2)(x-3)}$
12. $\int \frac{x^2+1}{x(x^2-1)} dx$
13. $\int \frac{x \, dx}{(x-a)(x-b)(x-c)}$
14. $\int \frac{x^2+1}{(x^2-1)(2x+1)} dx$
15. $\int \frac{2x+3}{x^3+x^2-2x} dx$
16. $\int \frac{x^2}{1-x^4} dx$
17. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$
18. $\int \frac{dx}{x^2(x^2+a^2)}$ [DUH-1955]
19. $\int \frac{x^2 \, dx}{x^4-x^2-2}$ [DUH-85]
20. $\int \frac{dx}{x^4-1}$
21. $\int \frac{x \, dx}{x^4-4}$
22. $\int \frac{x \, dx}{x^4-x^2-2}$
23. $\int \frac{x^3 \, dx}{(x^2+a^2)(x^2+b^2)}$
24. $\int \frac{x \, dx}{x^8-1}$ [CUH-1985]

25.
$$\int \frac{e^x dx}{(e^x - 1)(e^x + 1)}$$

26.
$$\int \frac{e^x}{2e^{2x} + 3e^x + 1} dx$$

27.
$$\int \frac{\sec^2 \theta d\theta}{(a + \tan \theta)(b - 2 \tan \theta)}$$

28.
$$\int \frac{dx}{2e^{2x} + 3e^x + 1}$$
 [DUH-53]

29.
$$\int \frac{dx}{(e^x - 1)(e^x + 3)}$$
 [DUH-90]

30.
$$\int \frac{(x - 1)(x - 5)}{(x - 2)(x - 4)} dx$$

31.
$$\int \frac{(x - a)(x - b)}{(x - c)(x - d)} dx$$

32.
$$\int \frac{x^3 dx}{(x - a)(x - b)(x - c)}$$

33.
$$\int \frac{x^3 + 3}{x^3 - 3x} dx$$

34.
$$\int \frac{dx}{\sin x + \sin 2x}$$
 [NUH-02]

35.
$$\int \frac{dx}{\sin x(3 + 2 \cos x)}$$
 [Engg-1993]

36.
$$\int \frac{dx}{x^2(x + 1)^2}$$

37.
$$\int \frac{x dx}{(x + 1)(x + 2)^2}$$

[DUA-1989]

[DUH-1969, 1977]

38.
$$\int \frac{dx}{x^3 - x^2 - x + 1}$$

39.
$$\int \frac{x^2 + 1}{(x - 3)(x - 1)^2} dx$$

40.
$$\int \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx$$

41.
$$\int \frac{x^4}{(x - 1)^2(x + 2)} dx$$

42.
$$\int \frac{x + 1}{(x - 1)^2(x + 2)} dx$$

43.
$$\int \frac{x dx}{(x - a)^2(x - b)}$$

44.
$$\int \frac{x^3 + 2}{(x - 1)(x - 2)^3} dx$$

45.
$$\int \frac{x dx}{(x - 1)^3(x + 1)}$$

46.
$$\int \frac{x dx}{(x - 1)(x^2 + 4)}$$
 [NUH-08]

Part-C : Broad Questions [বড় প্রশ্ন]

1.
$$\int \frac{dx}{(x^2 + 1)(x^2 + x + 1)}$$

2.
$$\int \frac{2x^2 + 3}{(x^2 + 1)^2} dx$$

3.
$$\int \frac{dx}{x^2(x^2 + 1)^2}$$
 [DUH-1963]

4.
$$\int \frac{2x^3 + x + 3}{(x^2 + 1)^2} dx.$$

5.
$$\int \frac{dx}{(e^x - 1)^2}$$
 [NUH-2008]

6.
$$\int \frac{dx}{\sin x + \tan x}$$
 [NUH(NM)-08]

ANSWER-13 : Part-B

1. $\frac{1}{a-b} [a \ln|x-a| - b \ln|x-b|] + c$

2. $\frac{1}{13} \ln \left| \frac{x-6}{x+7} \right| + c$

3. $\frac{1}{11} \ln \left| \frac{x-2}{x+9} \right| + c$

4. $2 \ln|x-2| + \ln|x+1| + c$ 5. $\frac{3}{5} \ln|x+2| + \frac{2}{5} \ln|x-3| + c$

6. $\frac{7}{2} \ln|x-7| - \frac{5}{2} \ln|x-5| + c$ 7. $\ln|x-1| \ln|x-2| + c$

8. $-\frac{1}{3} \ln|3-x| - \frac{1}{6} \ln|3+2x| + c$

9. $\frac{1}{2} \ln|x+1| \ln|x+3| - \ln|x+2| + c$

10. $\frac{5}{6} \ln|x-1| - \frac{3}{2} \ln|x+1| + \frac{2}{3} \ln|x+2| + c$

11. $\frac{1}{2} \ln|x-1| - 4 \ln|x-2| + \frac{9}{2} \ln|x-3| + c$ 12. $\ln \left| \frac{x^2-1}{x} \right| + c$

13.
$$\begin{aligned} & \frac{a}{(a-b)(a-c)} \ln|x-a| + \frac{b}{(b-a)(b-c)} \ln|x-b| \\ & + \frac{c}{(c-a)(c-b)} \ln|x-c| + c_1 \end{aligned}$$

14. $\frac{1}{3} \ln|x-1| + \ln|x+1| - \frac{5}{6} \ln|2x+1| + c$

15. $-\frac{3}{2} \ln|x| + \frac{5}{3} \ln|x-1| - \frac{1}{6} \ln|x+2| + c$

16. $\frac{1}{4} \ln \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + c$

17. $\frac{1}{a^2-b^2} \left[\frac{1}{b} \tan^{-1} \frac{x}{b} - \frac{1}{a} \tan^{-1} \frac{x}{a} \right] + c$

18. $\frac{1}{a^2} \left[-\frac{1}{x} - \frac{1}{a} \tan^{-1} \frac{x}{a} \right] + c$

19. $\frac{1}{3\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| + \frac{1}{3} \tan^{-1} x + c$

20. $\frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + c$

21. $\frac{1}{8} \ln|x^2 - 2| - \frac{1}{8} \ln|x^2 + 2| + c$

22. $\frac{1}{6} [\ln|x^2 - 2| - \ln|x^2 + 1|] + c$

23. $\frac{1}{2(a^2 - b^2)} [a^2 \ln(x^2 + a^2) - b^2 \ln(x^2 + b^2)] + c$

24. $\frac{1}{4} \left[\frac{1}{2} \ln \left| \frac{x^2 - 1}{x^2 + 1} \right| - \tan^{-1} x^2 \right] + c$

25. $\frac{1}{2} \ln \left| \frac{e^x - 1}{e^x + 1} \right| + c$

26. $\ln|2e^x + 1| - \ln|e^x + 1| + c$

27. $\frac{1}{2a+b} [\ln|a + \tan \theta| - \ln|b - 2 \tan \theta|] + c$

28. $x + \ln|1 + e^x| - 2 \ln|1 + 2e^x| + c$

29. $-\frac{1}{3}x + \frac{1}{4} \ln|e^x - 1| + \frac{1}{12} \ln|e^x + 3| + c$

30. $x + \frac{3}{2} [\ln|x-2| - \ln|x-4|] + c$

31. $x + \frac{(c-a)(c-b)}{c-d} \ln|x-c| + \frac{(d-a)(d-b)}{d-c} \ln|x-d| + c_1$

32. $x + \frac{a^3}{(a-b)(a-c)} \ln|x-a| + \frac{b^3}{(b-a)(b-c)} \ln|x-b| + \frac{c^3}{(c-a)(c-b)} \ln|x-c| + c_1$

33. $x - \ln|x| + \frac{1+\sqrt{3}}{2} \ln|x-\sqrt{3}| + \frac{1-\sqrt{3}}{2} \ln|x+\sqrt{3}| + c$

34. $\frac{1}{6} \ln|1 - \cos x| + \frac{1}{2} \ln|1 + \cos x| - \frac{2}{3} \ln|1 + 2 \cos x| + c$

35. $\frac{1}{10} \ln|1 - \cos x| - \frac{1}{2} \ln|1 + \cos x| + \frac{2}{5} \ln|3 + 2 \cos x| + c$

36. $-2 \ln|x| - \frac{1}{x} + 2 \ln|x+1| - \frac{1}{x+1} + c$

37. $-\ln|x+1| + \ln|x+2| - \frac{2}{x+2} + c$

38. $-\frac{1}{2(x-1)} + \frac{1}{4} \ln|x+1| - \frac{1}{4} \ln|x-1| + c$

39. $\frac{5}{2} \ln|x-3| - \frac{3}{2} \ln|x-1| + \frac{1}{x-1} + c$

40. $3 \ln|x+2| - 2 \ln|x+1| - \frac{1}{x+1} + c$

41. $\frac{x^2}{2} + \frac{11}{9} \ln|x-1| + \frac{16}{9} \ln|x+2| - \frac{1}{3(x-1)} + c$

42. $\frac{1}{9} \ln|x-1| - \frac{1}{9} \ln|x+2| - \frac{2}{3|x-1|} + c$

43. $\frac{b}{(a-b)^2} \ln \left| \frac{x-b}{x-a} \right| - \frac{a}{(a-b)(x-a)} + c$

44. $-3 \ln|x-1| + 4 \ln|x-2| - \frac{2}{x-2} - \frac{5}{(x-2)^2} + c$

45. $\frac{1}{8} \ln|x+1| - \frac{1}{8} \ln|x-1| - \frac{1}{4}/(x-1) - \frac{1}{4}/(x-1)^2 + c$

46. $\frac{1}{5} \ln|x-1| - \frac{1}{10} \ln|x^2+4| + \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + c$

Part-C

1. $\frac{1}{2} \ln\left(\frac{x^2+x+1}{x^2+1}\right) + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

2. $\frac{5}{2} \tan^{-1}x + \frac{x}{2(1+x^2)} + c$

3. $-\frac{1}{2} \cdot \frac{x}{1+x^2} - \frac{1}{x} - \frac{3}{2} \tan^{-1}x + c$

4. $\ln(x^2+1) + \frac{1+3x}{2(x^2+1)} + \frac{3}{2} \tan^{-1}x + c.$

5. $\ln|e^x| - \frac{1}{e^x-1} - \ln|e^x-1| + c$

6. $\frac{1}{4} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| - \frac{1}{2(\cos x + 1)} + c$