Network Analysis and Simulation - AY 2020/2021
Homework 4

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Exercise 1

We want to study the delay in function of the utilization factor of an M/M/1 queue with Poisson arrivals with rate λ and service time of one or two time units with same probability, i.e., $\mathbb{P}[\text{service time 1 time unit}] = \mathbb{P}[\text{service time 2 time units}] = a = 0.5$. Hence, the mean service is $\mu = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$, which means that the utilization factor is $\rho = \lambda/\mu = 3\lambda/2$. Now, if we let vary λ in the interval $[0.1, 1.5]$ we will have ρ varying in the interval $(0, 1]$. In Fig. 1 is depicted the average delay, expressed in minuted, in function of ρ.

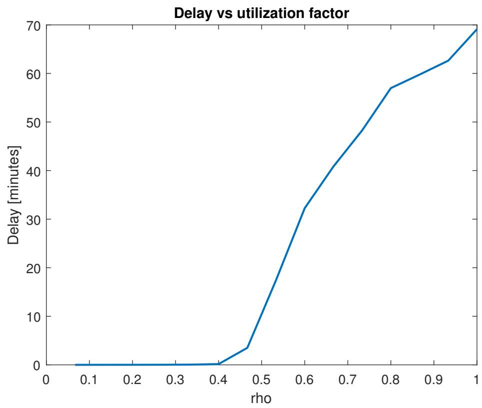

Figure 1: Delay vs ρ

Exercise 2

If we have X_1, X_2 iid random variables, then

$$Var\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}\left[Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)\right] \quad (1)$$

The key idea of the antithetic variables method is that, if we drop the assumption of X_1, X_2 being independent, and instead we try to make them negatively correlated, then

$$Cov(X_1, X_2) \leq 0 \quad (2)$$

and the variance of the estimator $\frac{X_1+X_2}{2}$ is reduced.
Now, suppose that we want to estimate

$$\theta = \mathbb{E}[\mathrm{U}] = \int_0^1 e^x dx, \quad with\ \mathrm{U} \sim \mathcal{U}[0,1] \tag{3}$$

With Monte Carlo simulation, using the classic method, we do

Algorithm 1: Raw estimator

```
i = 0;
n = 10000;
initialize empty array a;
while i ≤ n do
    draw u_1 and u_2 from U[0,1] independently;
    compute (e^{u1} + e^{u2})/2 and append the result to array a;
    i = i + 1;
end
```

Instead, with the antithetic variables method, we do

Algorithm 2: Antithetic variables

```
i = 0;
n = 10000;
initialize empty array b;
while i ≤ n do
    draw u from U[0,1];
    compute (e^u + e^{1-u})/2 and append the result to array b;
    i = i + 1;
end
```

In both cases, we obtained as result $\theta = e - 1$, as expected. But the sample variance of the raw estimator is $V_1 = 0.1212$, while the sample variance of the estimator taking advantage of antithetic variables is $V_2 = 0.0039$. Hence, we experience a variance reduction of

$$\Delta_V = \frac{V_1 - V_2}{V_1} = 0.9674 = 96.74\% \tag{4}$$

This result is in compliance with the theory results in Example 8d in book [1, p. 144].

References

[1] Sheldon M. Ross. *Simulation, Fourth Edition.* Academic Press, Inc., USA, 2006.