034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
A.Y. 2023-2024
Part X: Design of Feedback Control Systems

Gianfranco Fenu, Thomas Parisini

Department of Engineering and Architecture

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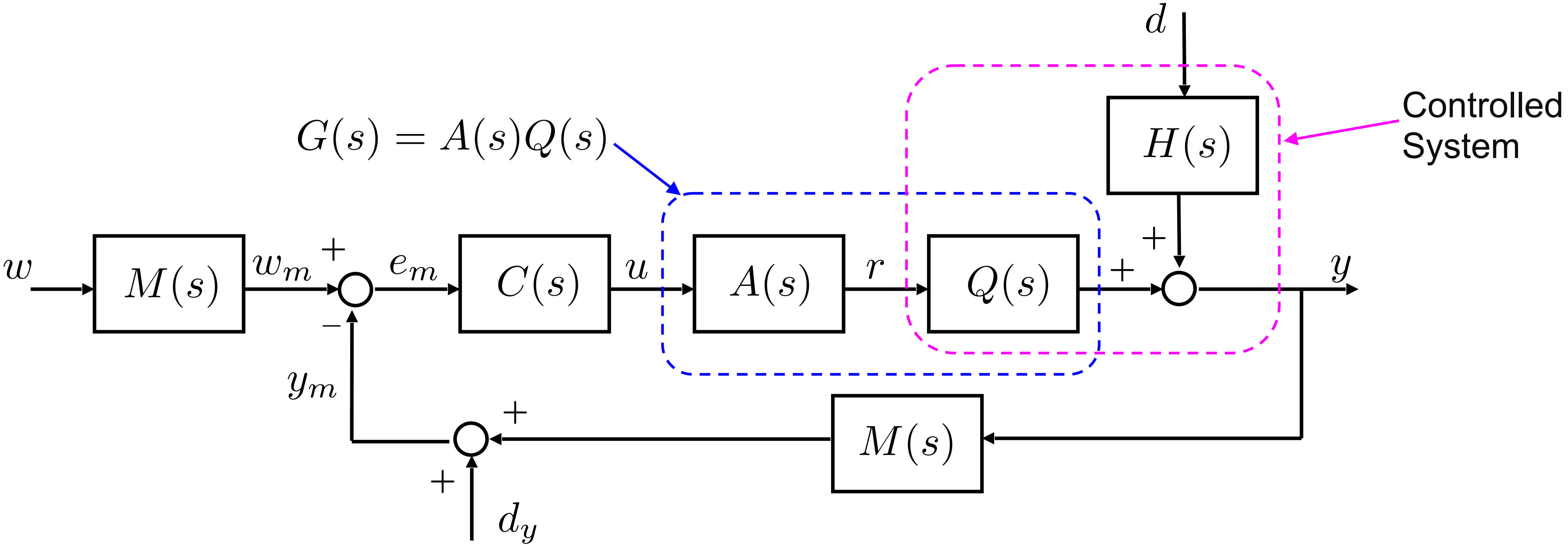
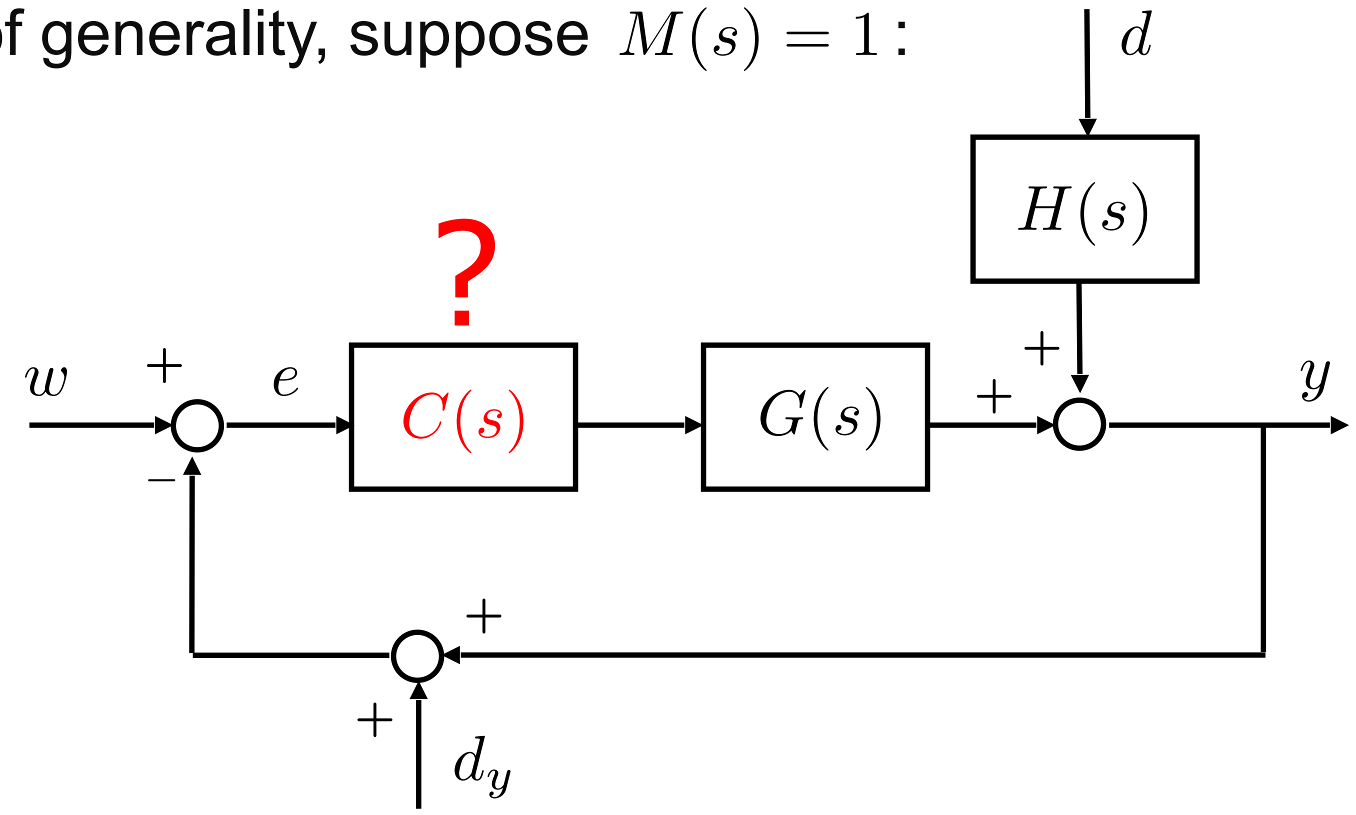
Recall from Part 9, slide 10 (adding the output disturbance d_y):



Design of the Controller

Without loss of generality, suppose $M(s) = 1$:



- **Controller Design**: Determine the transfer function $C(s)$ of the controller such that the **closed-loop requirements are satisfied**
- **Very different** from the analysis dealt with in Part 9: now the open-loop transfer function $L(s) = C(s)G(s)$ is **unknown** because $C(s)$ is **unknown**!

Closed-Loop Qualitative Requirements on Bode Diagrams

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- Closed-loop asymptotic stability $\Rightarrow$ $\begin{cases} \mu > 0 \\ \varphi_m > 0 \end{cases}$

- Static precision $\Rightarrow$ $\begin{cases} g > 0 \text{ (pole(s) in 0)} \\ \text{and/or} \\ \mu \text{ "large enough"} \end{cases}$

- Dynamic precision
  - speed of the closed-loop response $\Rightarrow$ ω_c "large enough"
  - closed-loop damping ratio $\Rightarrow$ φ_m "large enough"

Closed-Loop Qualitative Requirements on Bode Diagrams (contd.)

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- Disturbance rejection on direct path $\Longrightarrow$ $\begin{cases} \omega_c \text{ “large enough”} \\ |L(j\omega)|_{\mathrm{dB}}\,,\ \omega < \omega_c \text{ “large enough”} \end{cases}$

- Disturbance rejection on feedback path $\Longrightarrow$ $\begin{cases} \omega_c \text{ “not too large”} \\ |L(j\omega)|_{\mathrm{dB}}\,,\ \omega < \omega_c \text{ “small”} \end{cases}$

- Robust stability $\Longrightarrow$ $\left.\begin{matrix} \varphi_m \\ K_m \end{matrix}\right\}$ “large enough”

Design Specifications on Bode Diagrams

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Starting from closed-loop requirements, precise specifications on Bode diagrams are devised

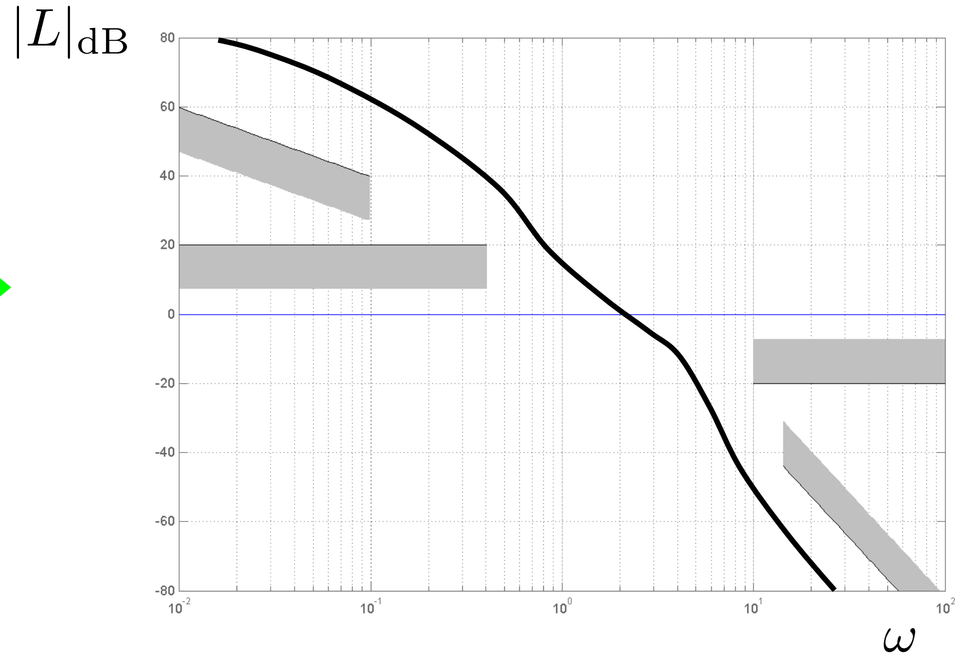
Example

- Static Specifications

$$|e(\infty)| \leq \bar{e}\,, \;\; \text{with } w, d \text{ specified}$$

- Dynamic Specifications
  - $\omega_{\min} \leq \omega_c \leq \omega_{\max}$
  - $\varphi_m \geq \bar{\varphi}_m$
  - $K_m \geq \bar{K}_m$

Constraints on **open-loop** Bode diagrams of $|L(j\omega)|_{\mathrm{dB}}$



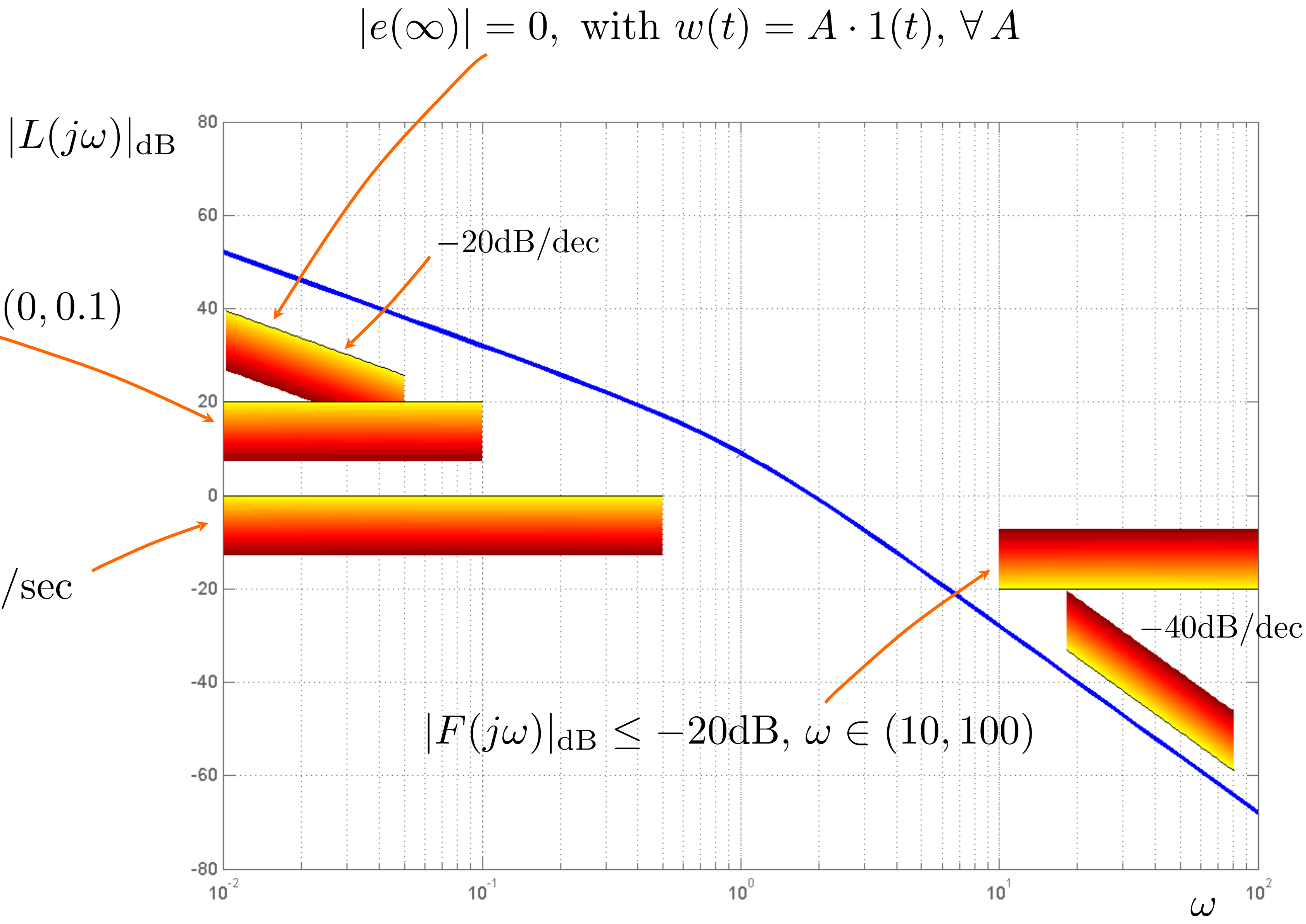
Design Specifications on Bode Diagrams: Example

$$L(s) = \frac{4}{s(s+1)}$$

$|S(j\omega)|_{\text{dB}} \leq -20\text{dB},\ \omega \in (0, 0.1)$

$\Delta\% \leq 3 \rightarrow \xi \geq 0.75$

$t_s \leq 10\,\text{sec} \rightarrow \omega_c \geq 0.5\text{rad/sec}$



Design Specifications on Root Locus

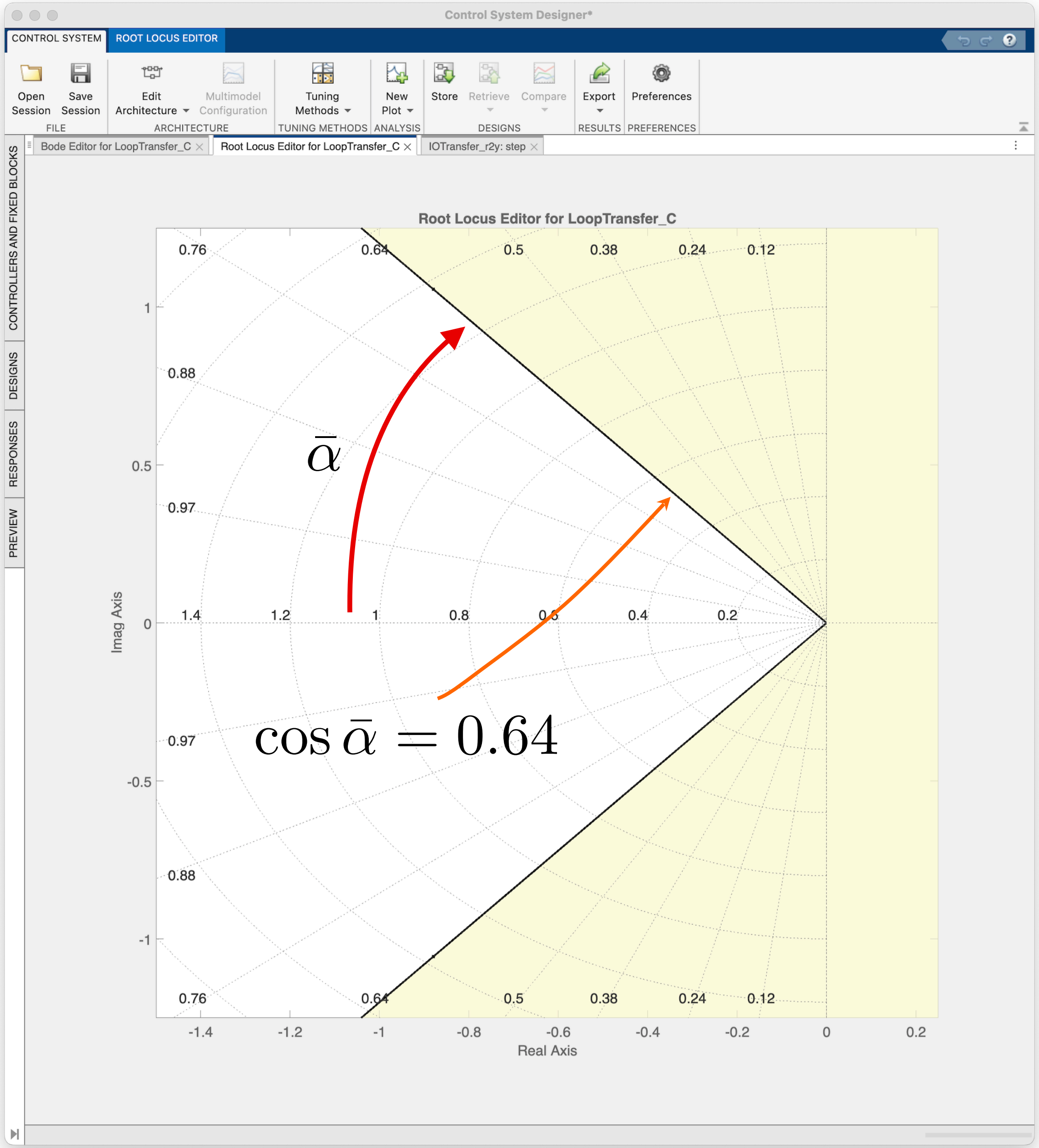
Closed-loop requirement:

- the **damping ratio** ξ has to satisfy the constraint

$$\xi \geq \bar{\xi}$$

- The region of the complex plane where the constraint $\xi \geq \bar{\xi}$ is satisfied can be drawn as a **graphical constraint** on the RL:

$$s \in \mathbb{C} : \ \xi = \cos\alpha \geq \bar{\xi} = \cos\bar{\alpha}$$

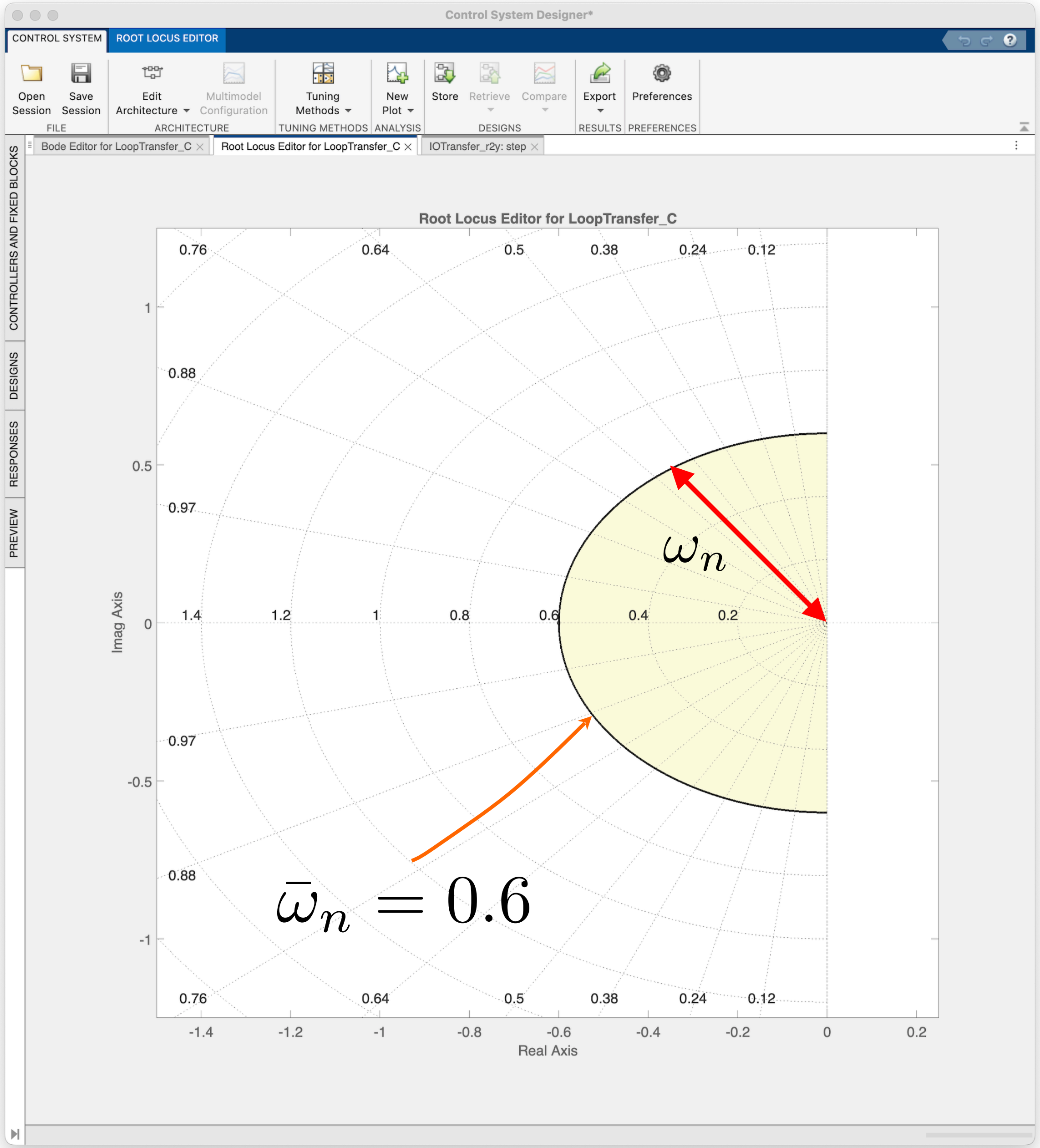


Design Specifications on Root Locus (contd.)

Closed-loop requirement:

- the **natural angular frequency** ω_n has to satisfy the constraint
$$\omega_n \geq \bar{\omega}_n$$
- The region of the complex plane where the constraint $\omega_n \geq \bar{\omega}_n$ is satisfied can be drawn as a **graphical constraint** on the RL:

$$s \in \mathbb{C} : \ \omega_n \geq \bar{\omega}_n$$



Design Specifications on Root Locus (contd.)

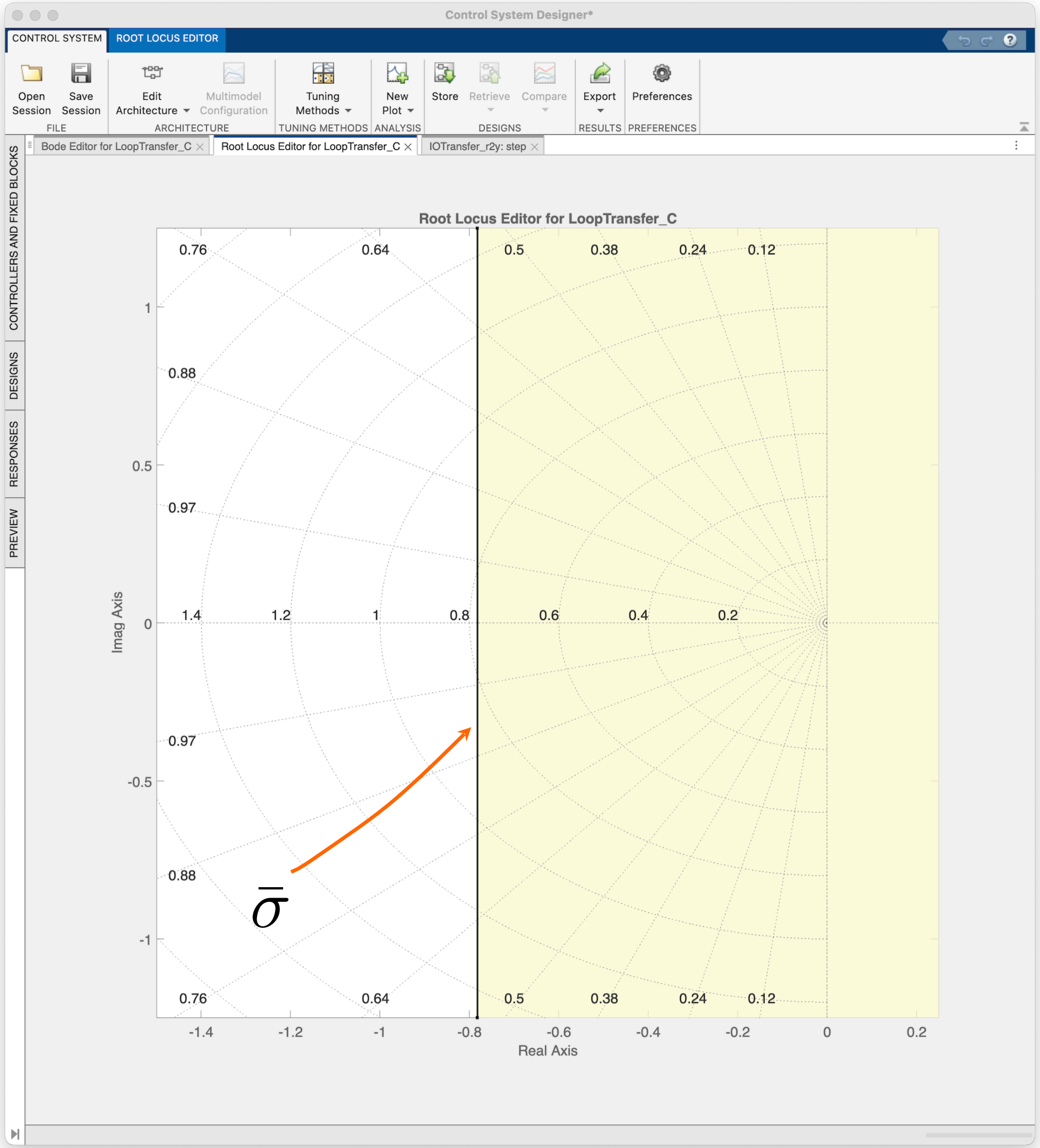
Closed-loop requirement:

- the (negative) **real part of the poles** σ has to satisfy the constraint

$$\sigma \leq \bar{\sigma}$$

- The region of the complex plane where the constraint $\sigma \leq \bar{\sigma}$ is satisfied can be drawn as a **graphical constraint** on the RL:

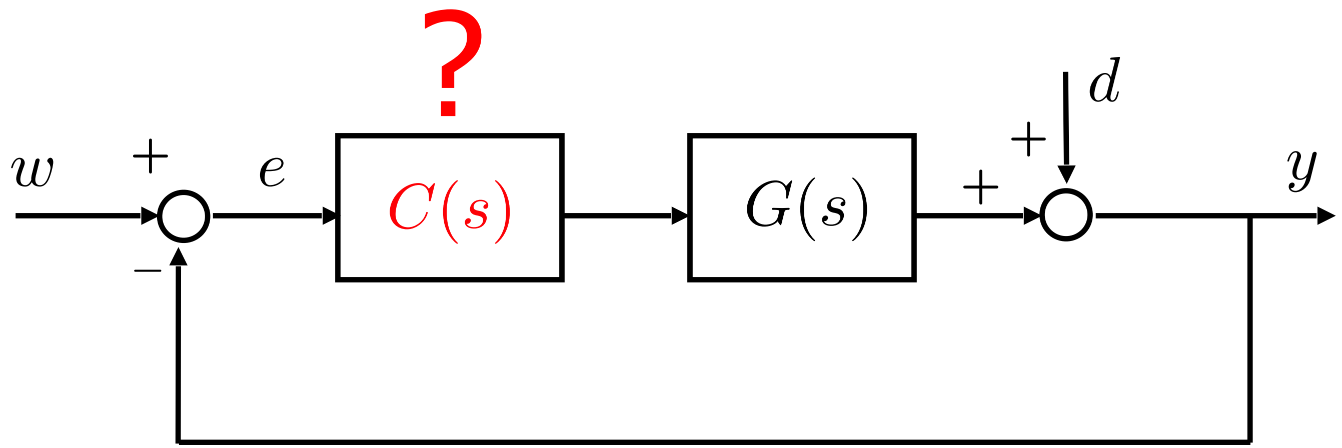
$$s \in \mathbb{C} : \ \sigma = \mathrm{Re}(s) \leq \bar{\sigma}$$



Loop-Shaping Iterative Controller Design

- Analyse the **closed-loop requirements** and translate them to **design specifications on the Bode diagrams**
- Select an **initial attempt** for the controller $C(s)$ using a very simple structure. For example, a typical initial choice is an algebraic controller $C_0(s) = \mu_C$
- Check whether the chosen controller satisfies **all specifications**. If not, consider another attempt using a more complex structure of the controller and proceed with the design trying to meet all specifications
- **Iterate** the procedure till when all specifications are met

Loop-Shaping Iterative Controller Design: Example 1



$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

- **Design specifications**:
  - $|e(\infty)| \leq 0.1 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), & |A| \leq 1 \\ d(t) = B \cdot 1(t), & |B| \leq 5 \end{cases}$
  - $\omega_c \geq 0.2$
  - $\varphi_m \geq 60°$

used to meet **dynamic** specifications

used to meet **static** specifications

- **Controller Structure**:

$$C(s) = C_1(s) \cdot C_2(s) \quad \text{where} \;\; C_1(s) = \frac{\mu_C}{s^g}\,; \qquad C_2(s) = \frac{\prod (1+sT_i)}{\prod (1+s\tau_i)}$$

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

$$L(s) = C(s)G(s) = C_1(s)C_2(s)\frac{10}{(1+10s)(1+5s)(1+s)}$$

Static Design:

Since $C_2(0) = 1$ the **static design is not influenced by**

gain: $10\mu_C > 0$

type: g

Due to the **linearity** of the closed-loop system:

$$e(\infty) = e_w(\infty) + e_d(\infty)$$

where:

$$e_w(t) = e(t)\,|_{d(t)=0} \; ; \quad e_d(t) = e(t)\,|_{w(t)=0}$$

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Hence:

$$e(\infty) = e_w(\infty) + e_d(\infty)$$

$$\Rightarrow \quad |e(\infty)| \leq |e_w(\infty)| + |e_d(\infty)|$$

and:

$$|e_w(\infty)| = \begin{cases} \dfrac{|A|}{1+10\mu_C} \leq \dfrac{1}{1+10\mu_C}\,, & \text{if } g = 0 \\ 0\,, \ \text{if } g > 0 \end{cases}$$

$$|e_d(\infty)| = \begin{cases} \dfrac{|B|}{1+10\mu_C} \leq \dfrac{5}{1+10\mu_C}\,, & \text{if } g = 0 \\ 0\,, \ \text{if } g > 0 \end{cases}$$

- **Option 1**: static design with $g = 0$ (no poles in the origin introduced by $C_1(s)$)

$$|e(\infty)| \leq \frac{1}{1+10\mu_C} + \frac{5}{1+10\mu_C} = \frac{6}{1+10\mu_C}$$

Imposing $|e(\infty)| \leq 0.1$:

$$\frac{6}{1+10\mu_C} \leq 0.1 \quad \Rightarrow \quad \mu_C \geq 5.9$$

Possible choice: $\boxed{C_1(s) = 8}$

- **Option 2**: static design with $g = 1$ (one pole in the origin introduced by $C_1(s)$)

$$e(\infty) = 0\,, \ \ \forall\, \mu_C \quad \Rightarrow \quad \boxed{C_1(s) = \frac{\mu_C}{s}}$$

Dynamic Design, Option (A):

Pick the static controller: $C_1(s) = 8$

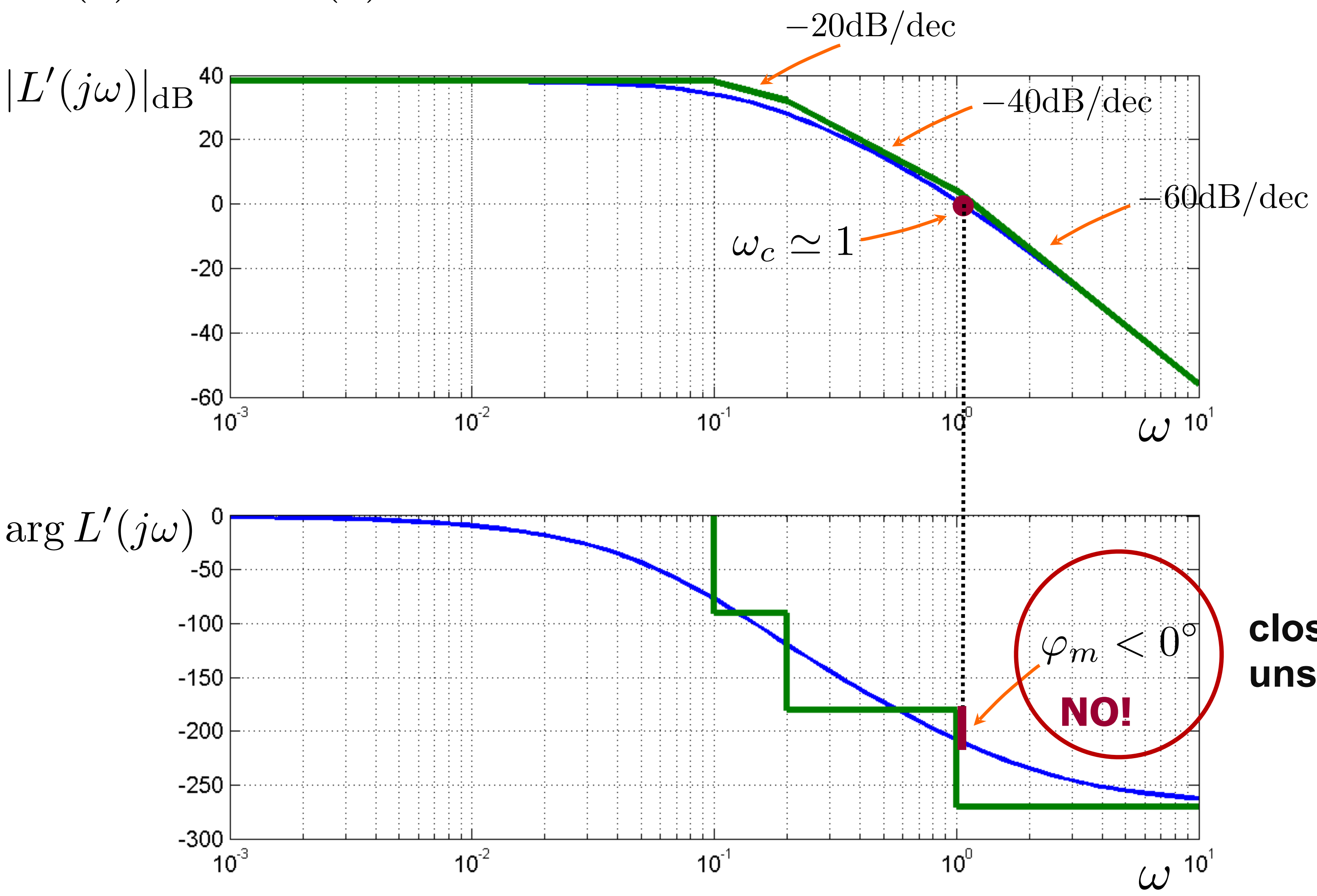
$$L(s) = 8\, G(s)\, C_2(s) = L'(s)\, C_2(s)$$

$$\text{with} \quad L'(s) = \frac{80}{(1+10s)(1+5s)(1+s)}$$

We proceed by a **sequence of attempts** with increasing complexity of the structure of controller $C_2(s)$ by following the **iterative procedure** outlined in Slide 8:

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Attempt 1: $C_1(s) = 8,\ C_2(s) = 1$



closed-loop unstable

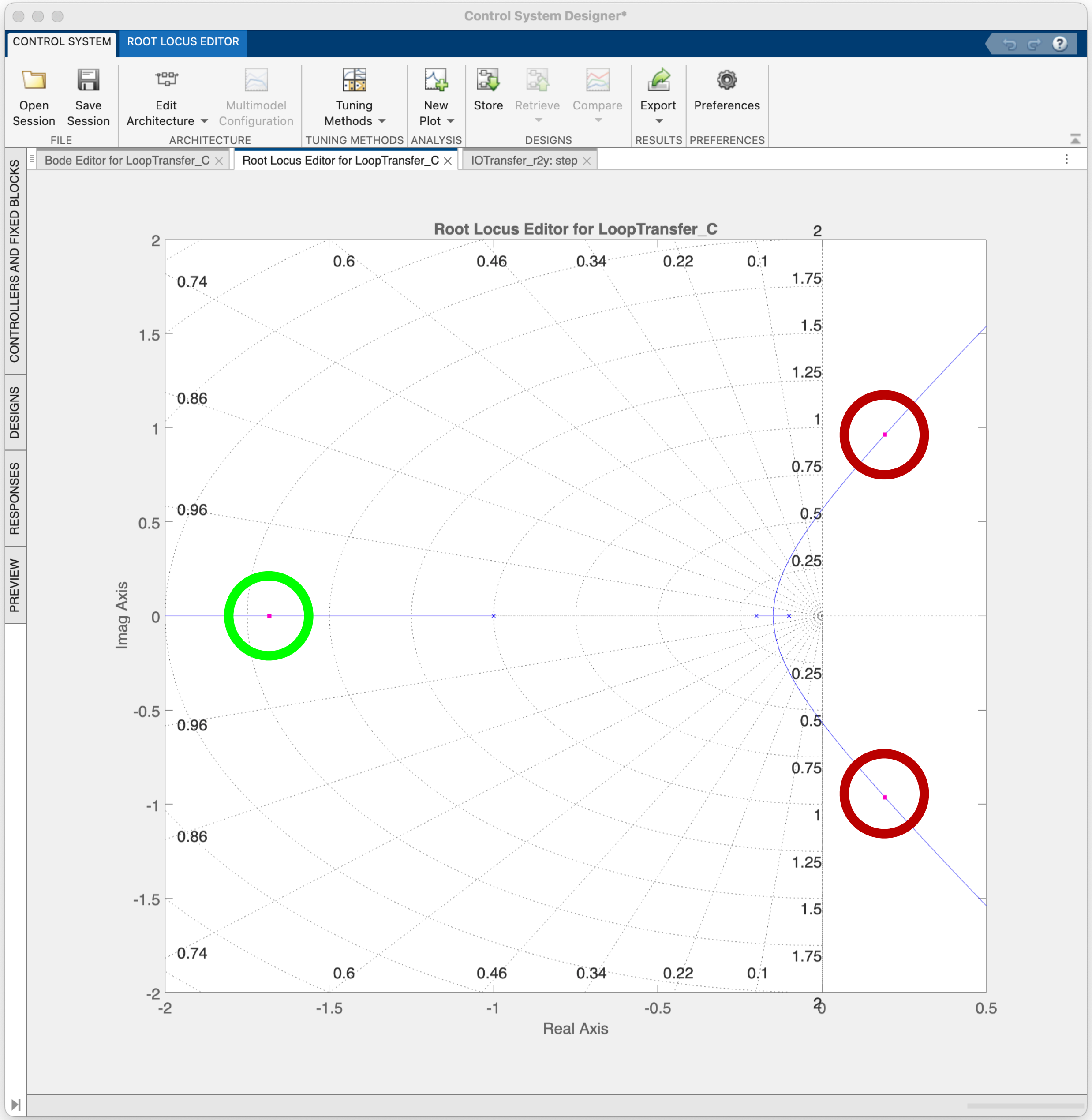
Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Attempt 1: $C_1(s) = 8,\ C_2(s) = 1$

unstable **closed-loop** pole

asymptotically stable **closed-loop** pole

- The choice of controller $C_1(s) = 8$ to satisfy the static requirement is **not feasible** because the closed-loop system would be **unstable**
- Decreasing the gain in $C_1(s)$ would "*move*" the close-loop poles into the open left-plane but this way the static requirement would **not** be satisfied



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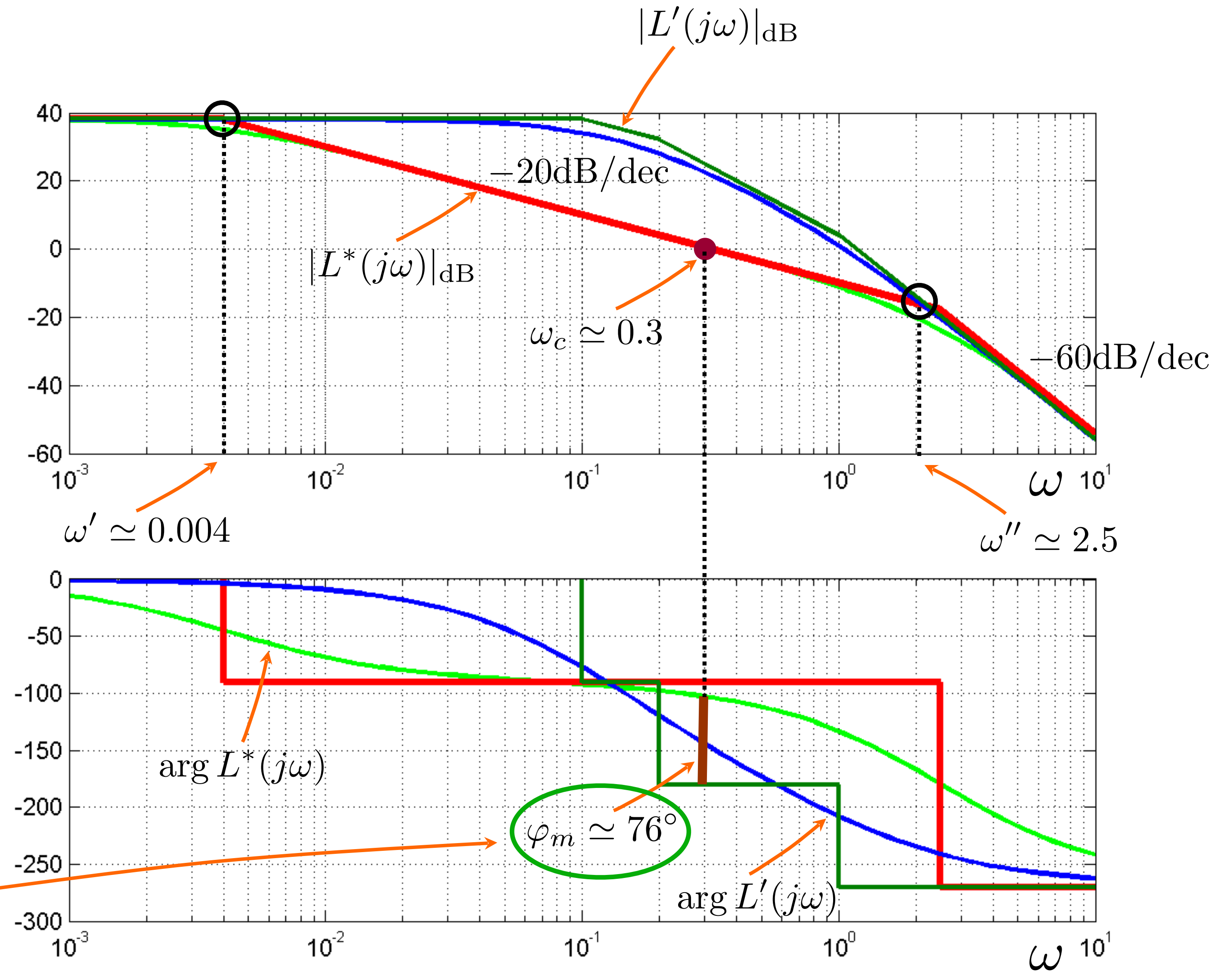
Attempt 2:



closed-loop asympt. stable

Procedure followed in Attempt 2 of Dynamic Design Option (A) :

- Set a satisfactory critical angular frequency: $\omega_c = 0.3 \geq 0.2$
- Draw a straight line on the magnitude Bode plot with slope $-20\mathrm{dB/dec}$ and passing through the point $(\omega_c, 0\mathrm{dB})$; find the frequencies ω', ω'' where the straight line intersects the diagram of $|L'(j\omega)|_{\mathrm{dB}}$
- Construct the asymptotic Bode diagram of $|L^*(j\omega)|_{\mathrm{dB}}$ which coincides with the drawn straight line for $\omega \in [\omega', \omega'']$ and with the diagram of $|L'(j\omega)|_{\mathrm{dB}}$ for $\omega < \omega'$ and $\omega > \omega''$
- The resulting asymptotic diagram of $|L^*(j\omega)|_{\mathrm{dB}}$ gives:

$$L^*(s) = \frac{80}{(1+s/0.004)(1+s/2.5)^2} \quad \Longrightarrow \quad C_2(s) = \frac{L^*(s)}{L'(s)} = \frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$

Analysis of the controller obtained with Design Option (A):

guarantees the **required static precision**

zeros introduced to "cancel" the poles -0.1, -0.2, -1 thus **increasing** $\arg L(j\omega)$

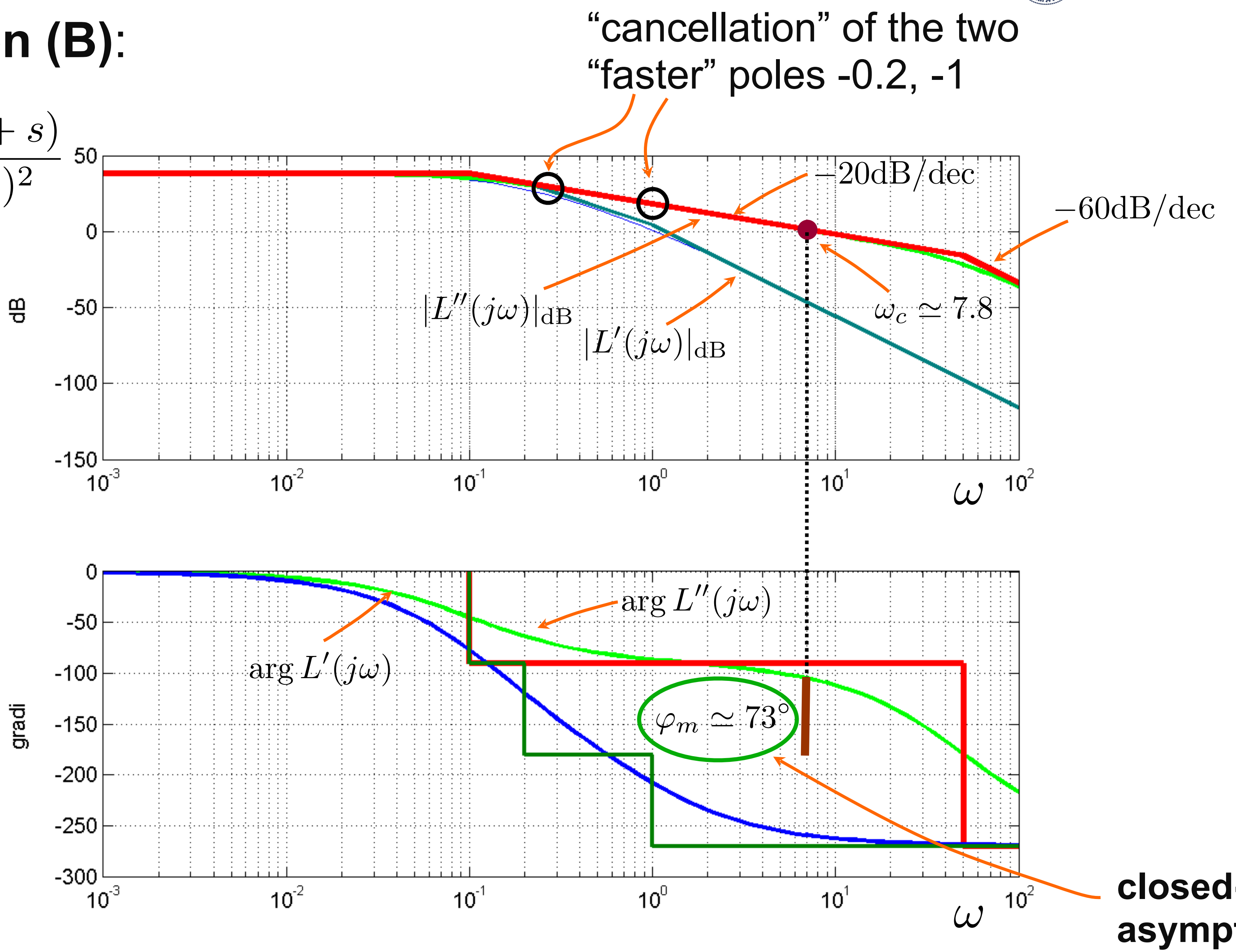
$$C(s) = C_1(s)C_2(s) = 8\,\frac{\boxed{(1+10s)(1+5s)(1+s)}}{\boxed{(1+250s)}\boxed{(1+0.4s)^2}}$$

"moves" to low frequency the slope of $-20\mathrm{dB/dec}$ in order to cross the $0\mathrm{dB}$ axis at the desired critical frequency with a slope $-20\mathrm{dB/dec}$ to guarantee the desired phase margin (the system is minimum phase and hence **the Bode criterion can be used**)

high-frequency poles:
- physical realisability
- attenuation of high-frequency output disturbances

Dynamic Design, Option (B):

$$C(s) = 8\,\frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$



closed-loop asympt. stable

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Dynamic Design, Option (B):

$$C(s) = 8\,\frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$

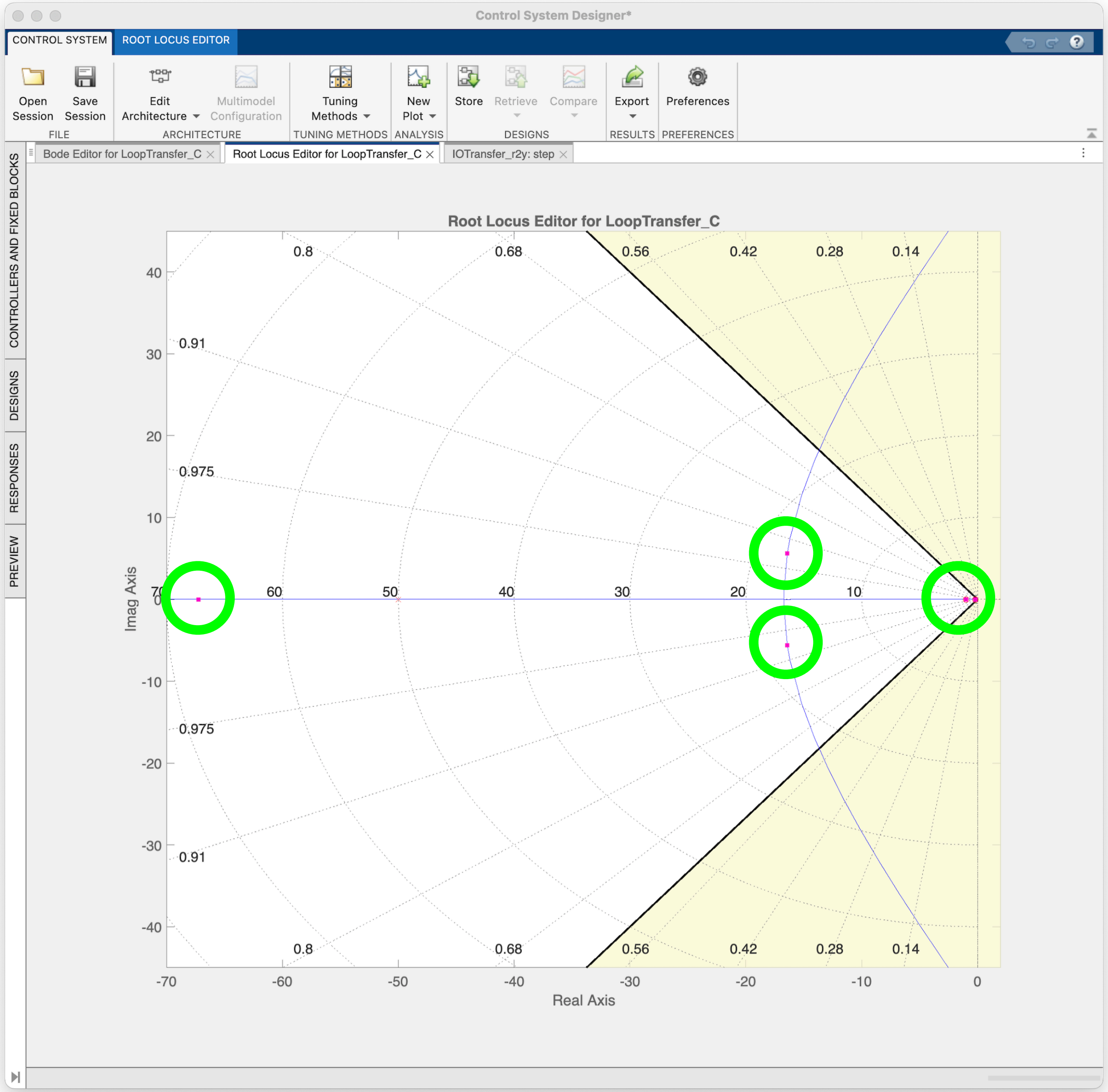
closed-loop pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\omega_c \geq 0.2$ we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.2$$

- From the Requirement $\varphi_m \geq 60^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{60^\circ}{100} = 0.6$$



Procedure followed in Option (B) :

- The static design is kept unchanged with $C_1(s) = 8$
- The "slower" pole in -0.1 is kept whereas the "faster" poles in -0.2, -1 are cancelled by respective zeros introduced by the controller to ensure a slope $-20\mathrm{dB/dec}$ when crossing the point $(\omega_c, 0\mathrm{dB})$
- Two sufficiently fast (at frequency sufficiently larger than ω_c) poles are introduced by the controller
- The resulting asymptotic diagram of $|L''(j\omega)|_{\mathrm{dB}}$ gives:

$$L''(s) = \frac{80}{(1+10s)(1+0.02s)^2} \quad \Longrightarrow \quad C_2(s) = \frac{L''(s)}{L'(s)} = \frac{(1+5s)(1+s)}{(1+0.02s)^2}$$

Static and Dynamic Design, Option (C):

Pick the static controller (thus we also modify the static design): $C_1(s) = \dfrac{\mu_C}{s}$

$$L(s) = \frac{G(s)}{s}\,\mu_C\,C_2(s) = L_1(s)\,\mu_C\,C_2(s)$$

$$\text{with} \quad L_1(s) = \frac{10}{s(1+10s)(1+5s)(1+s)}$$

Again, we proceed by a **sequence of attempts** with increasing complexity of the structure of controller $C_2(s)$ by following the **iterative procedure** outlined in Slide 8:

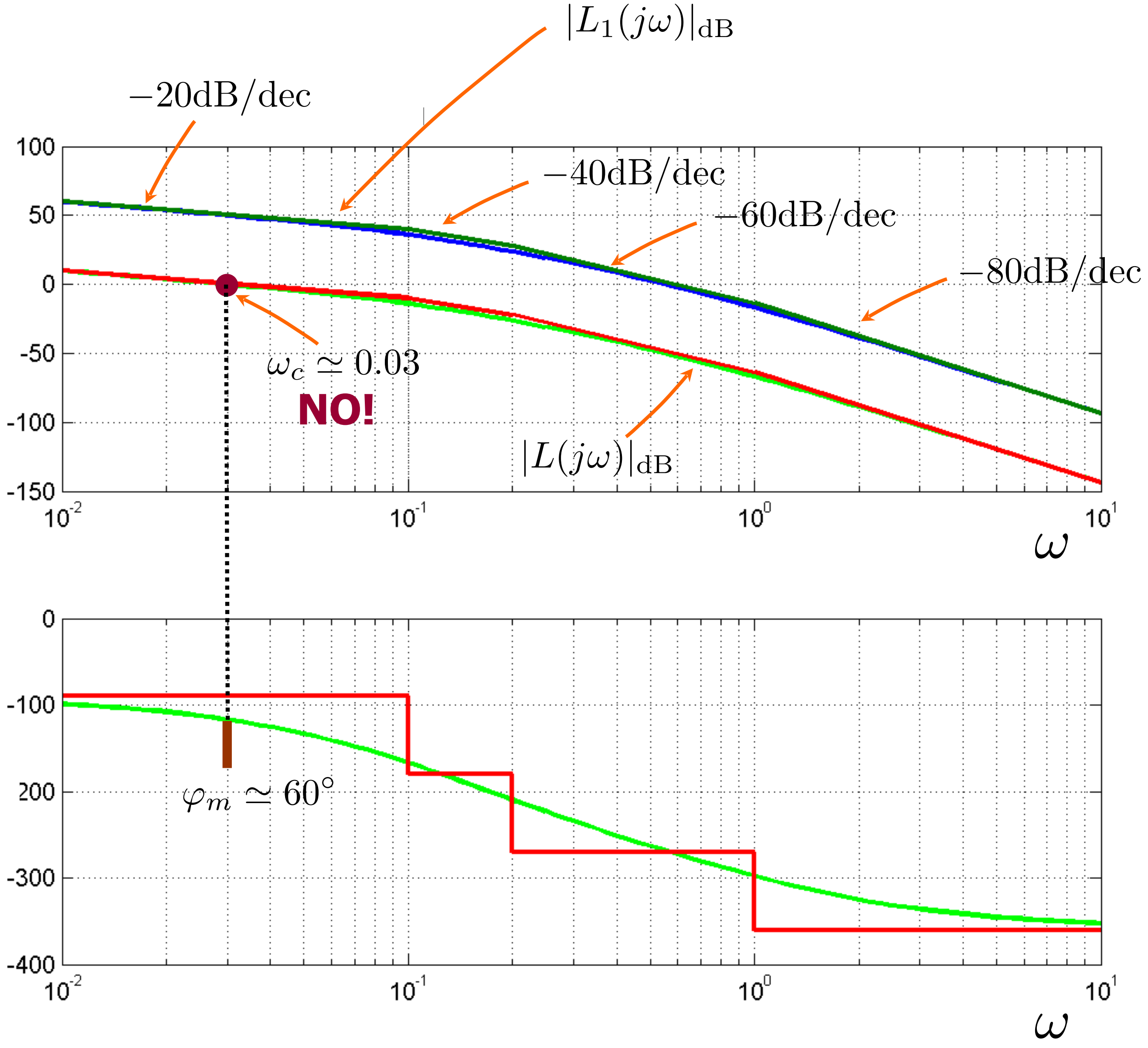
Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Attempt 1:

$$C_2(s) = 1$$

$$L(s) = \mu_C L_1(s)$$

$$\mu_C = 10^{-5/2}$$



Loop-Shaping Iterative Controller Design: Example 1 (contd.)

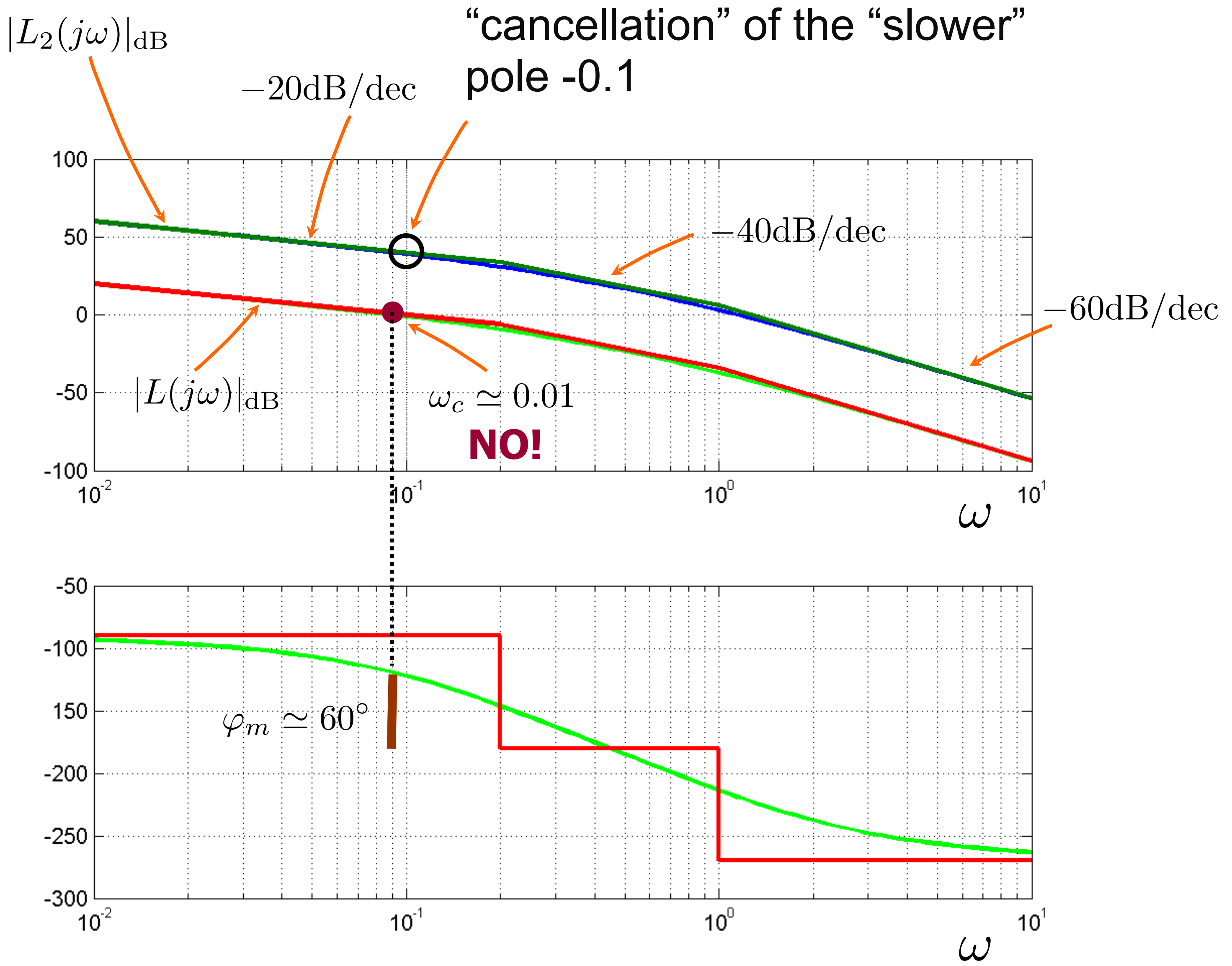
Attempt 2:

$$C_2(s) = (1 + 10s)$$

$$\Rightarrow L_2(s) = \frac{10}{s(1+5s)(1+s)}$$

$$\Rightarrow L(s) = \mu_C L_2(s)$$

$$\Rightarrow \mu_C \simeq 0.01$$



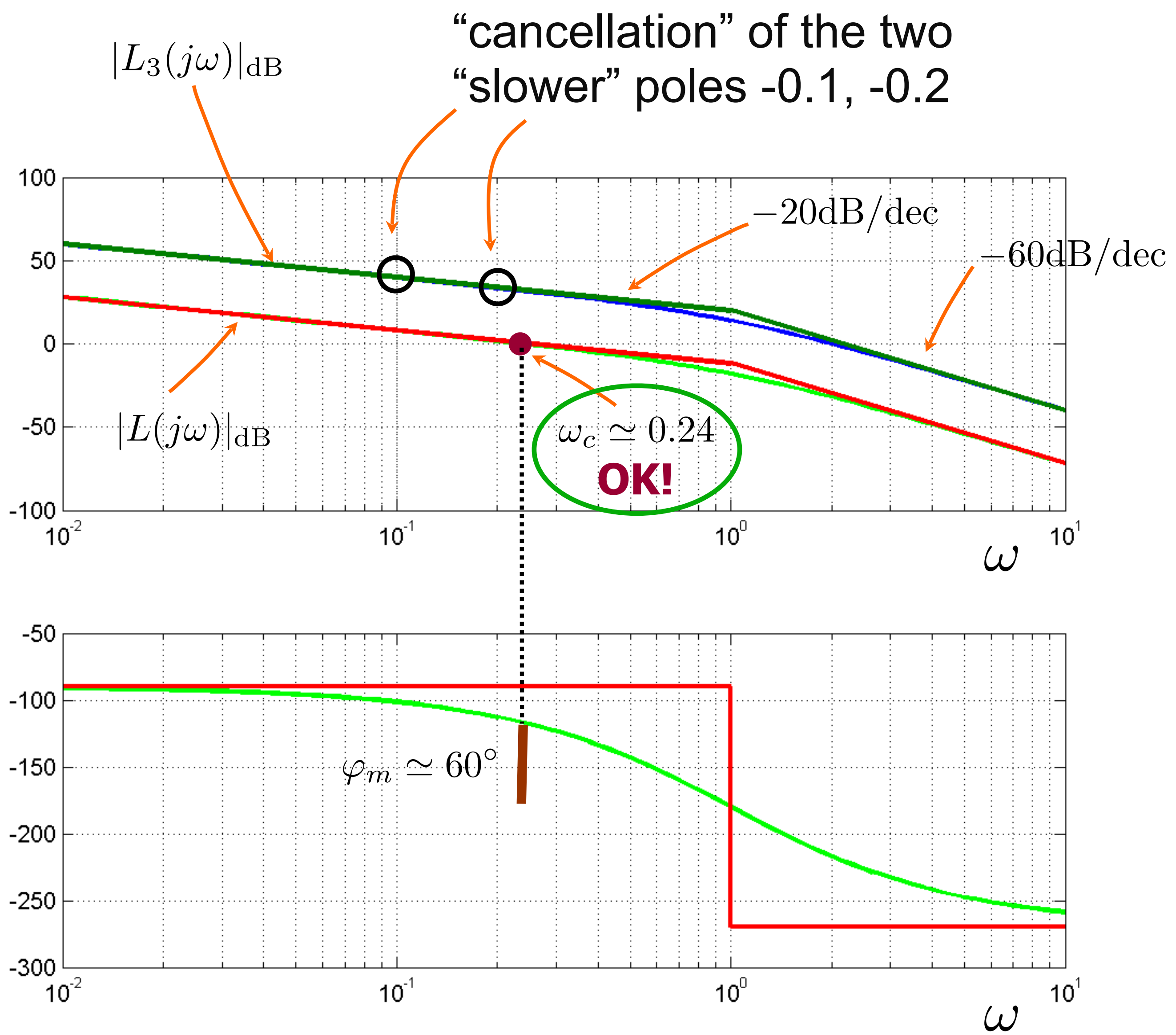
Attempt 3:

$$C_2(s) = \frac{(1+10s)(1+5s)}{1+s}$$

$$L_3(s) = \frac{10}{s(1+s)^2}$$

$$L(s) = \mu_C L_3(s)$$

$$\mu_C \simeq 0.0254$$



Dynamic Design, Option (C):

$$C(s) = 0.0254\frac{(1+10s)(1+5s)}{s(1+s)}$$

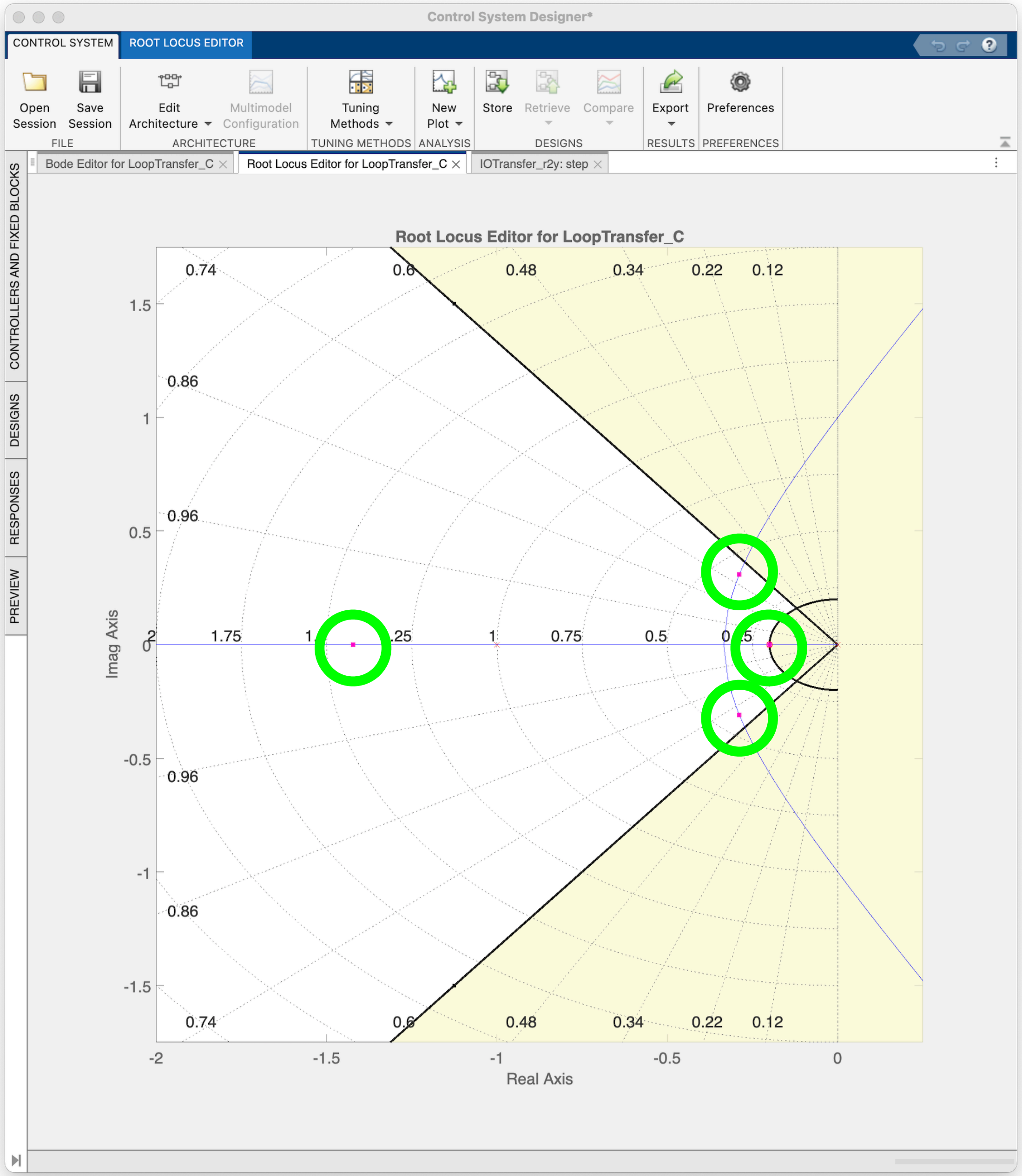
closed-loop pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\omega_c \geq 0.2$ we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.2$$

- From the Requirement $\varphi_m \geq 60^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{60^\circ}{100} = 0.6$$



Example 1: Comparison Between the Designed Controllers

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$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

- **Design specifications**:
  - $|e(\infty)| \leq 0.1 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), \ |A| \leq 1 \\ d(t) = B \cdot 1(t), \ |B| \leq 5 \end{cases}$
  - $\omega_c \geq 0.2$
  - $\varphi_m \geq 60°$

(A): $C(s) = 8\frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$ $\quad \begin{cases} \omega_c \simeq 0.3 \\ \varphi_m \simeq 76° \end{cases}$

(B): $C(s) = 8\frac{(1+5s)(1+s)}{(1+0.02s)^2}$ $\quad \begin{cases} \omega_c \simeq 7.8 \\ \varphi_m \simeq 73° \end{cases}$

(C): $C(s) = 0.0254\frac{(1+10s)(1+5s)}{s(1+s)}$ $\quad \begin{cases} \omega_c \simeq 0.24 \\ \varphi_m \simeq 63° \end{cases}$ and $\quad e(\infty) = 0$

Example 1: Comparison Between the Designed Controllers (contd.)

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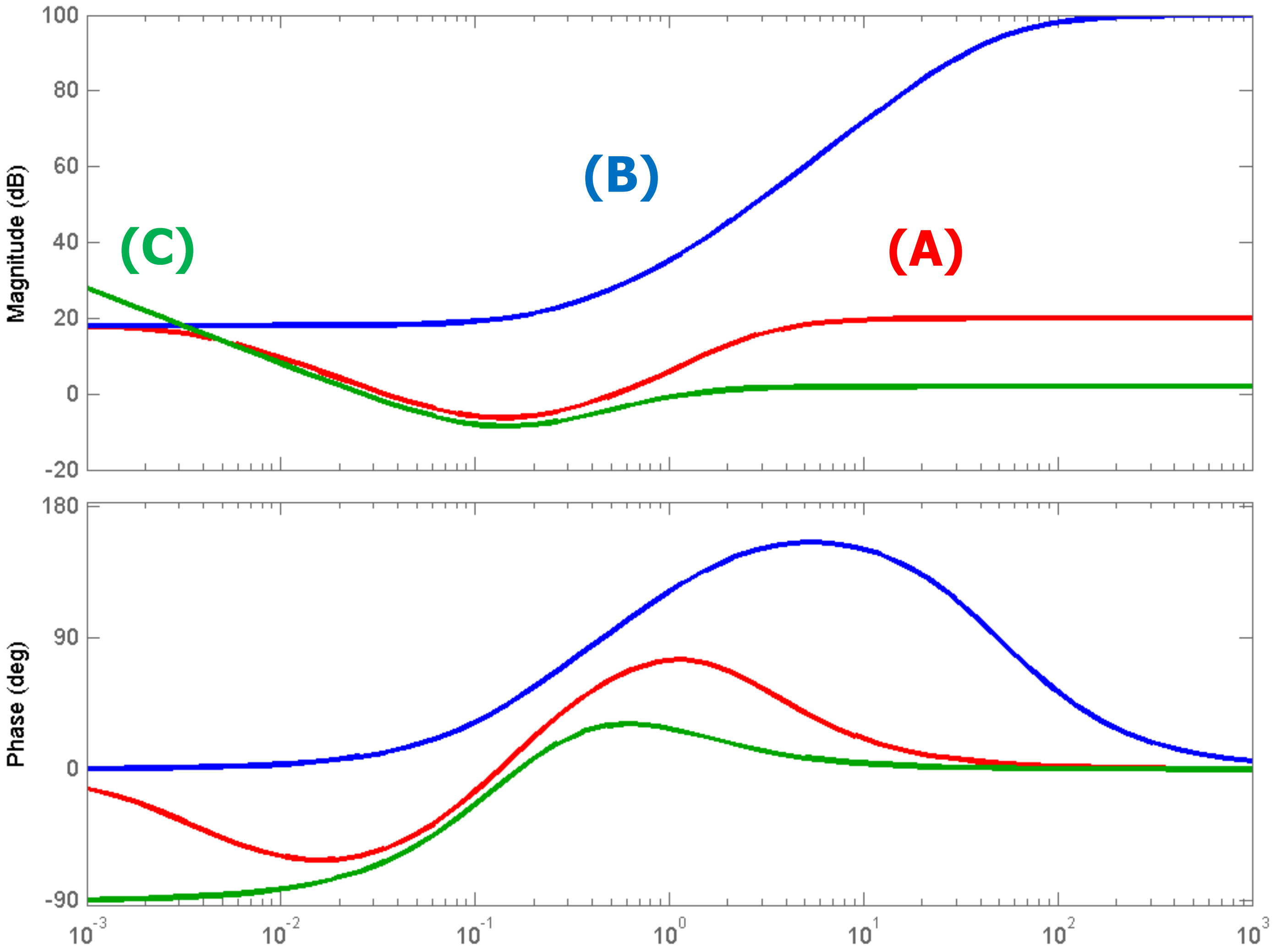
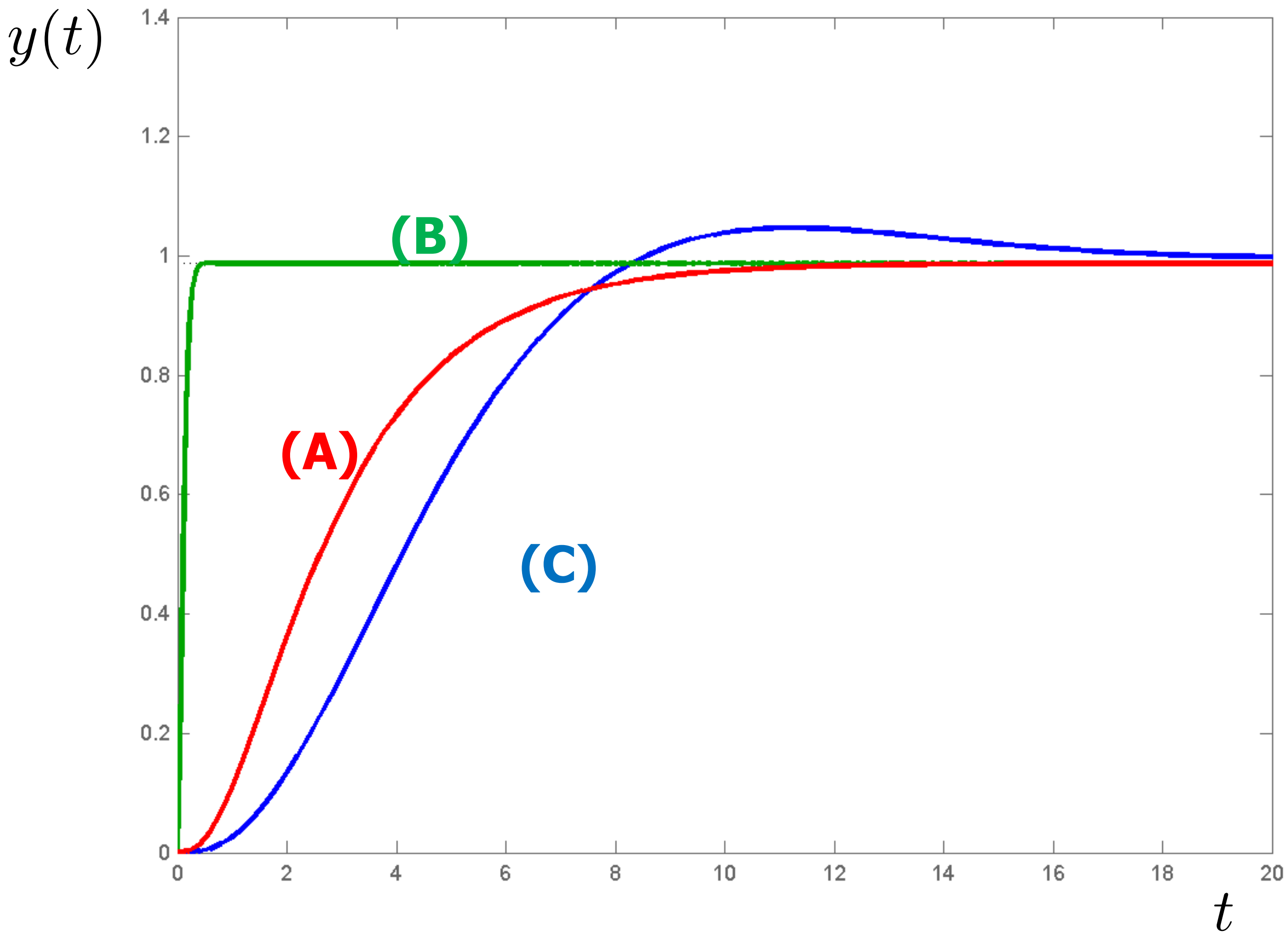
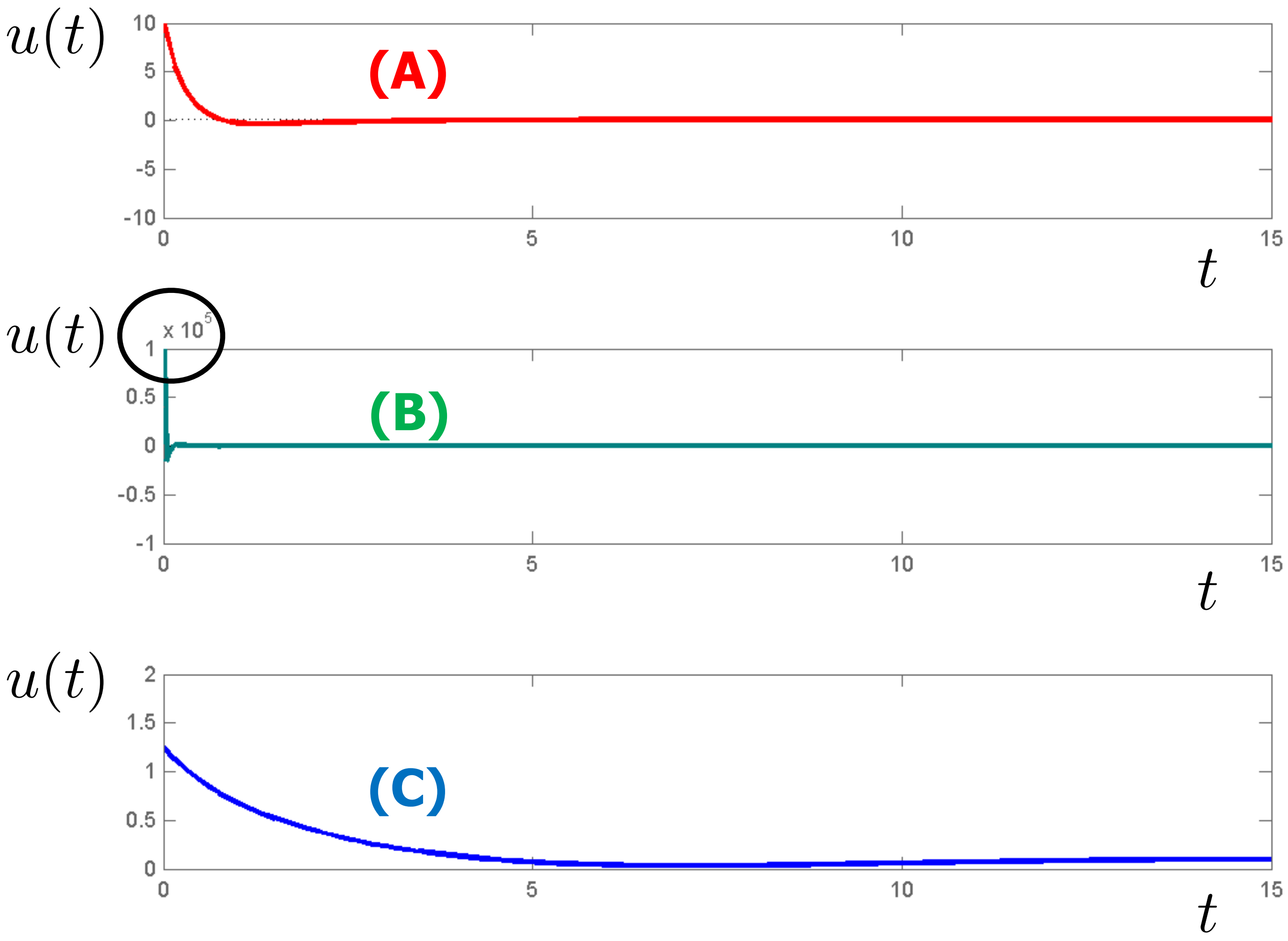
Bode diagrams of the controllers



Closed-loop step response



Closed-loop step response



Control Design for Non-Minimum Phase Zeros/Unstable Systems

In scenarios where the system to be controlled has **non-minimum phase zeros** or is **unstable**:

– the **Bode criterion cannot be used**

– the **"cancellation"** of zeros and poles with non-negative real part is **not allowed** not to generate unstable hidden dynamics

– the presence of zeros and/or poles with non-negative real part introduces **inherent limitations on the achievable performance**

Example 2 - System with a Non-Minimum Phase Zero

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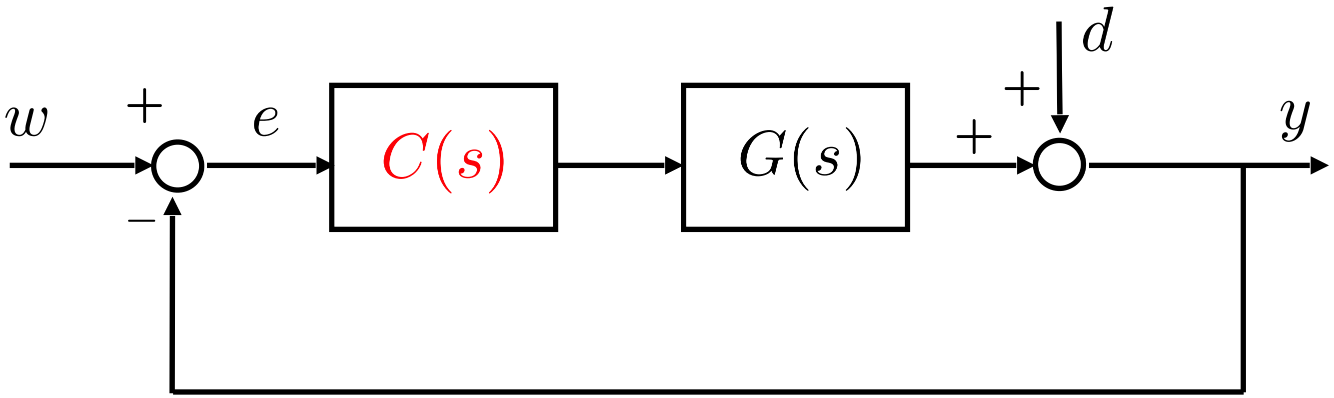


$$G(s) = \frac{10(1-2s)}{s(1+10s)(1+0.1s)}$$

Design specifications: $\varphi_m \geq 40^\circ$

The term $\dfrac{(1-2s)}{s}$:

- cannot be “canceled”
- it gives a **negative contribution** to the phase: $-90^\circ - \mathrm{arcgtg}\,(2\omega)$

↳ for $\omega = 0.5$ the negative contribution to the phase is -135°

↳ the critical frequency ω_c cannot exceed "too much" $\omega_c = 0.5$

↳ strong limitation to the achievable speed of response

Example 2 - System with a Non-Minimum Phase Zero (contd.)

Note that:

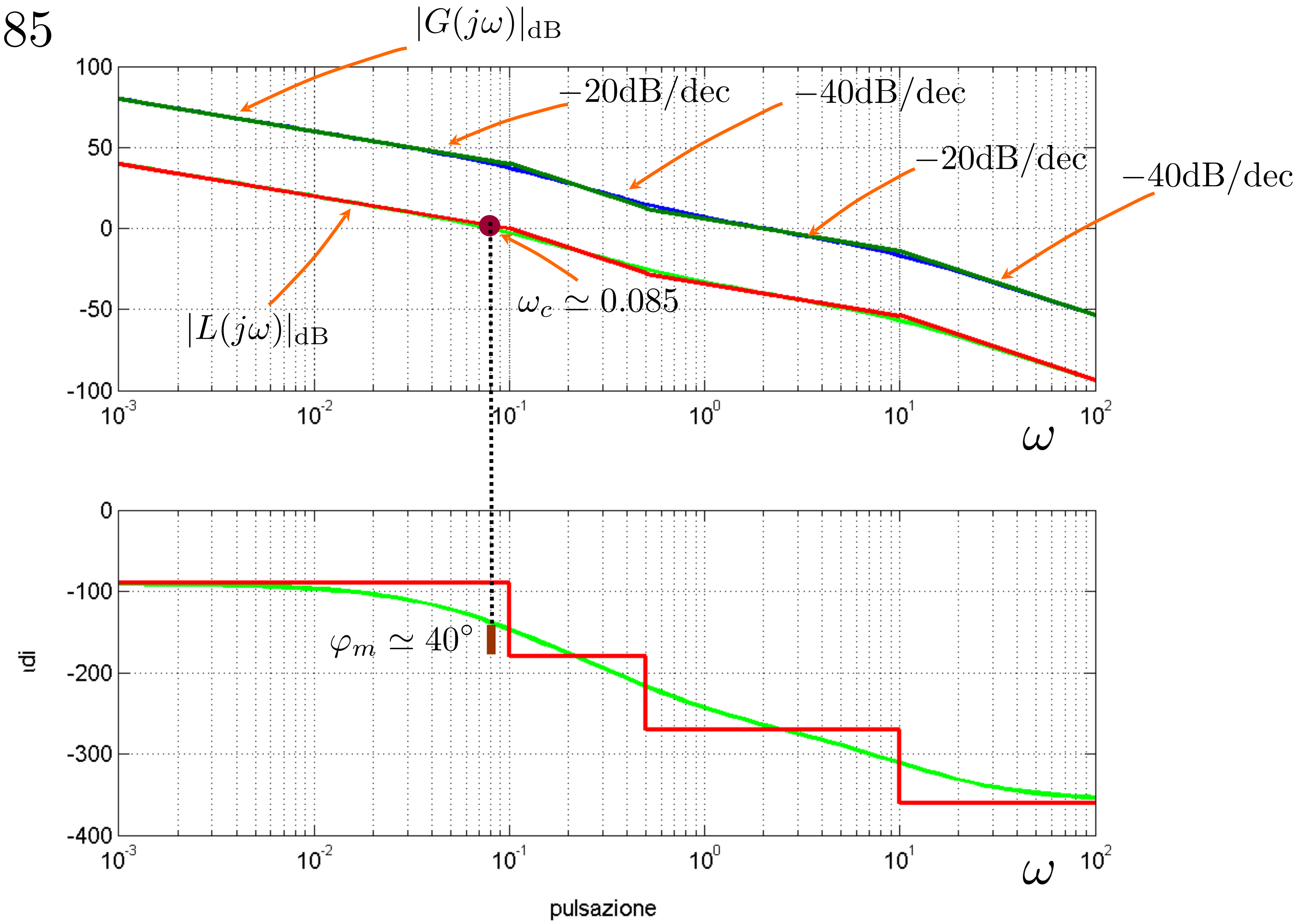
$$\arg G(j\omega) = -140^\circ \text{ for } \omega \simeq 0.085$$

Attempt 1:

$$C(s) = \mu_C\,;\ L(s) = \mu_C G(s)$$

$$\omega_{c\max} = 0.085$$

$$\mu_C \simeq 0.01$$



Attempt 2:

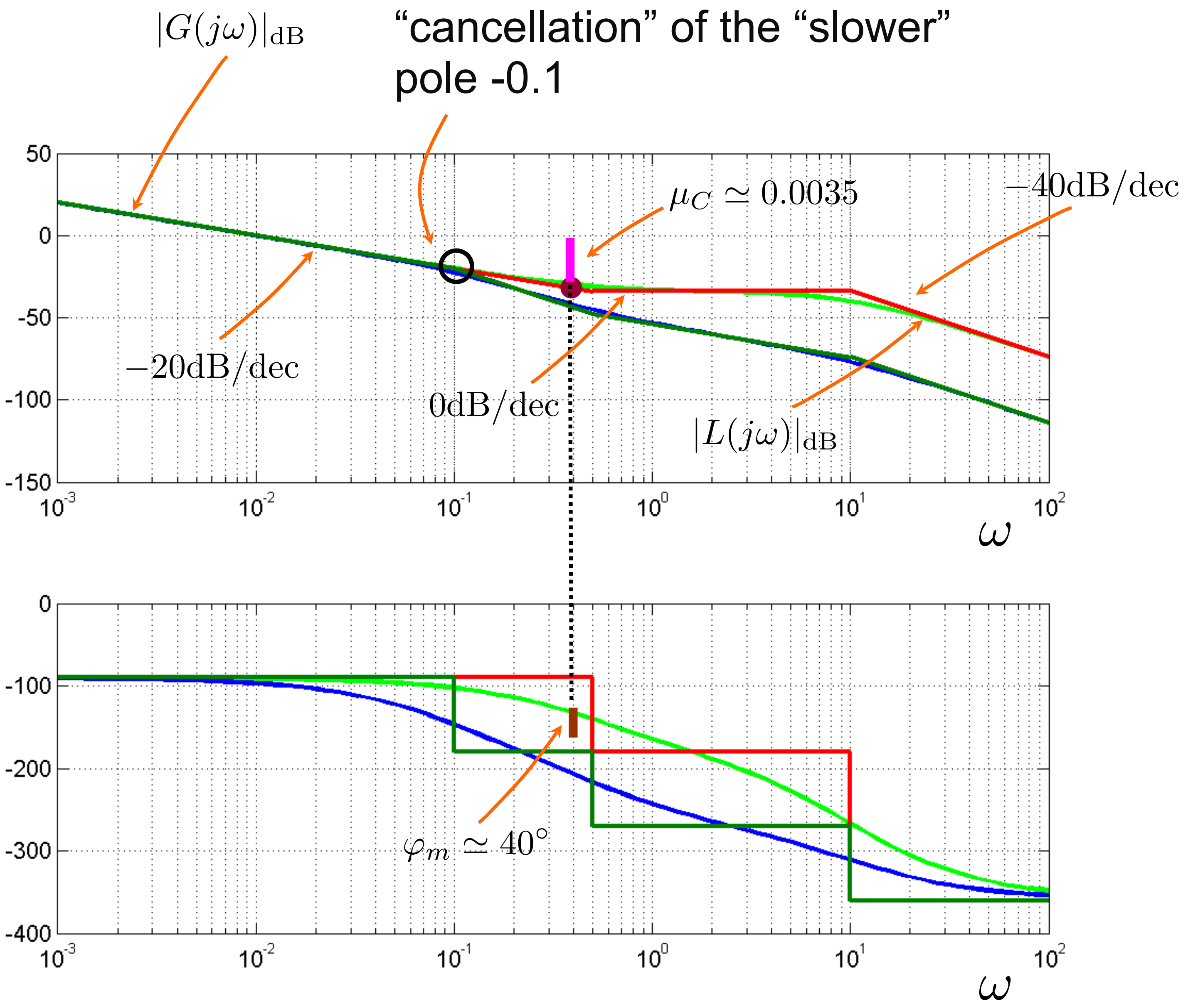
$$C(s) = \mu_C \frac{1+10s}{1+0.1s}$$

$$\Rightarrow \quad L(s) = \mu_C \frac{10(1-2s)}{s(1+0.1s)^2}$$

Note that:

$$\arg L(j\omega) = -140^\circ \text{ for } \omega \simeq 0.5$$

$$\Rightarrow \quad \mu_C \simeq 0.0035$$



Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Dynamic Design, Attempt 2:

$$C(s) = 0.0035\,\frac{1+10s}{1+0.1s}$$

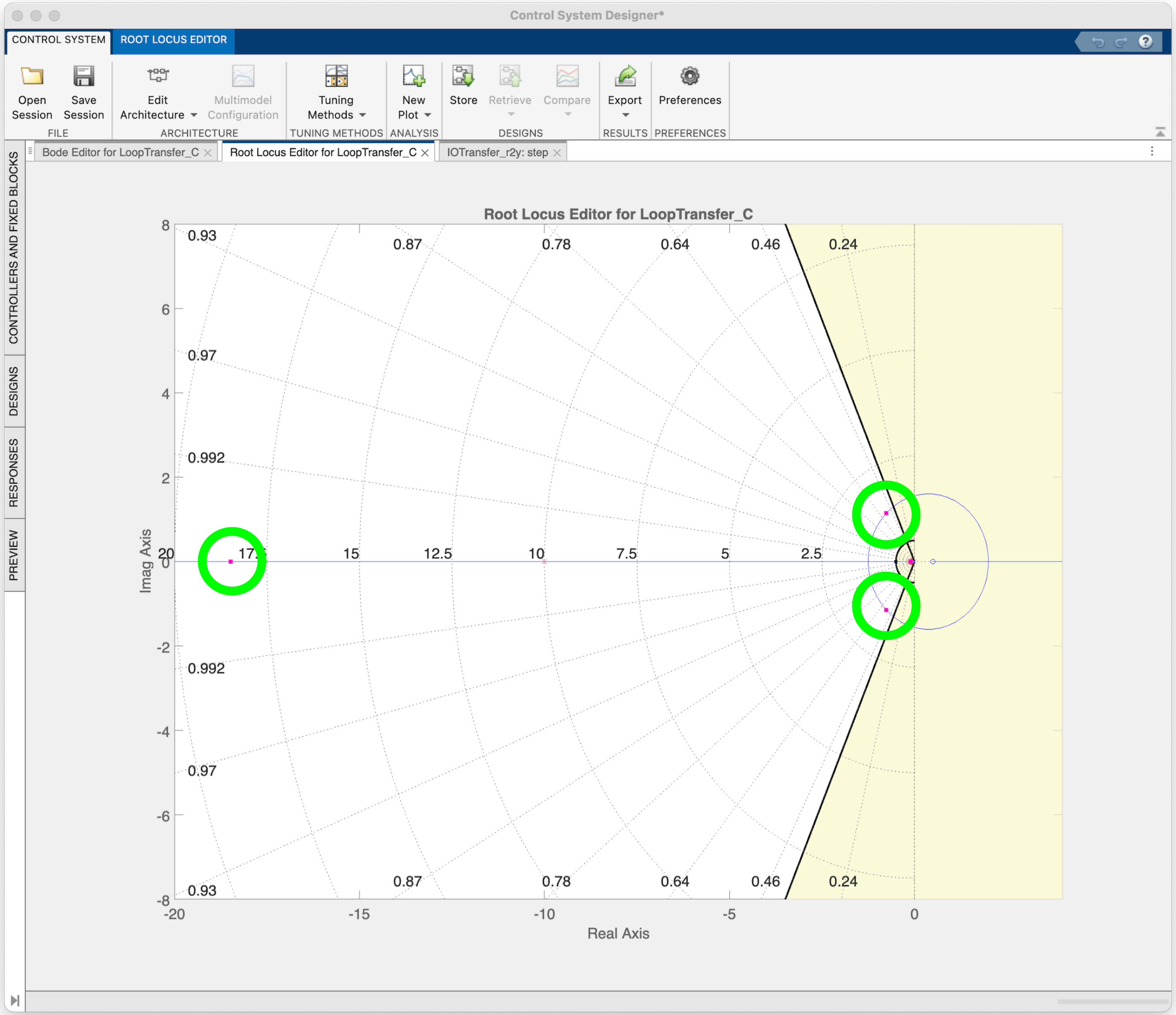
closed-loop pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\varphi_m \geq 40^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{40^\circ}{100} = 0.4$$

- Even if it is not a requirement, to get a controller that "pushes" the performance to the limits, we impose $\omega_c \geq 0.5$, thus we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.5$$



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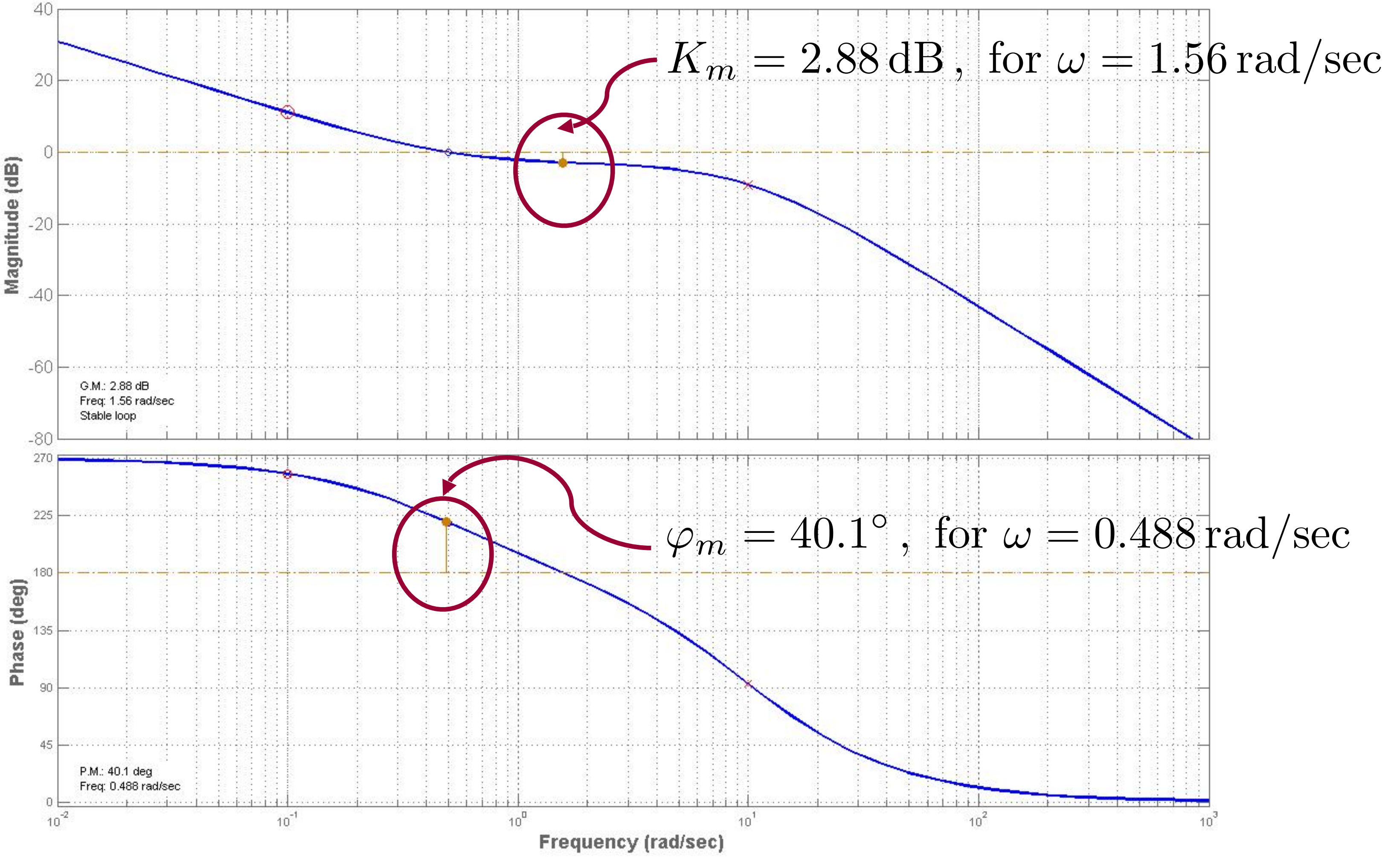
Actual performance with controller obtained by Attempt 2:

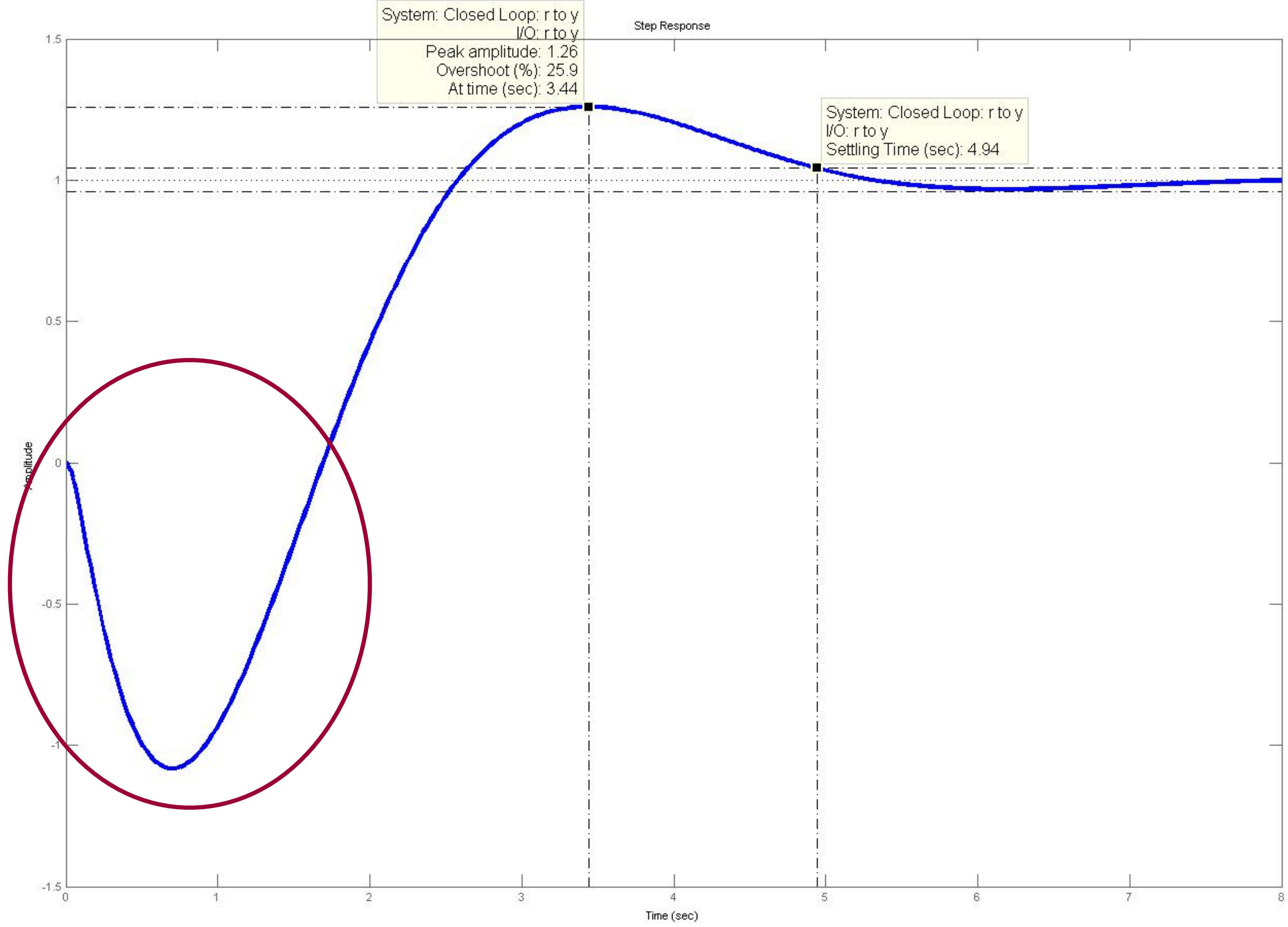
Note that the **gain margin** K_m is rather small



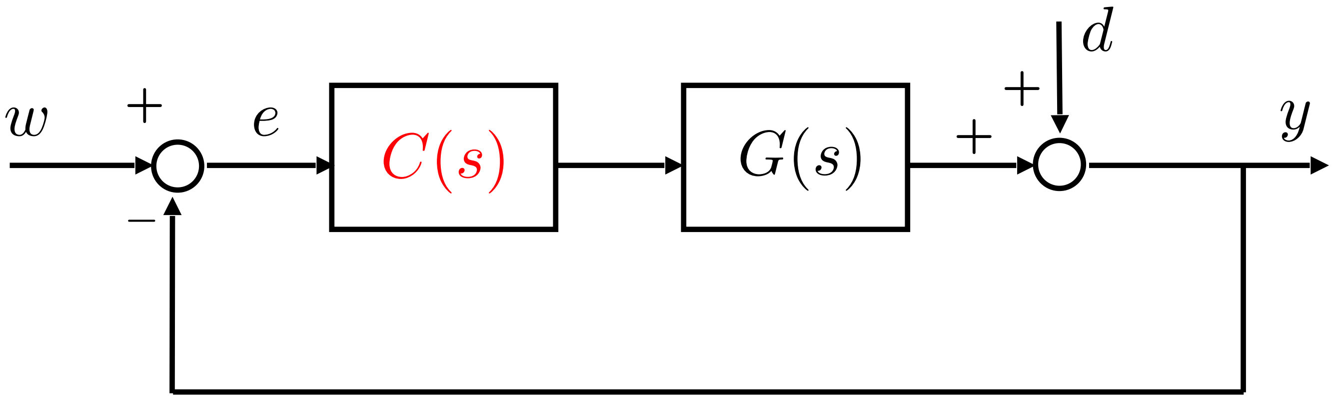
Example 2 - System with a Non-Minimum Phase Zero (contd.)

Actual performance with controller obtained by Attempt 2:

Also, note the rather significant **under-shoot** (typical for non-minimum phase systems)



Example 3 - Unstable System



$$G(s) = \frac{1}{s-1}$$

Design specifications:

- $|e(\infty)| = 0 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), \ \forall A \\ d(t) = 0\,, \ \forall t \end{cases}$
- $\omega_c \geq 0.5$
- $\varphi_m \geq 45°$

Example 3 - Unstable System (contd.)

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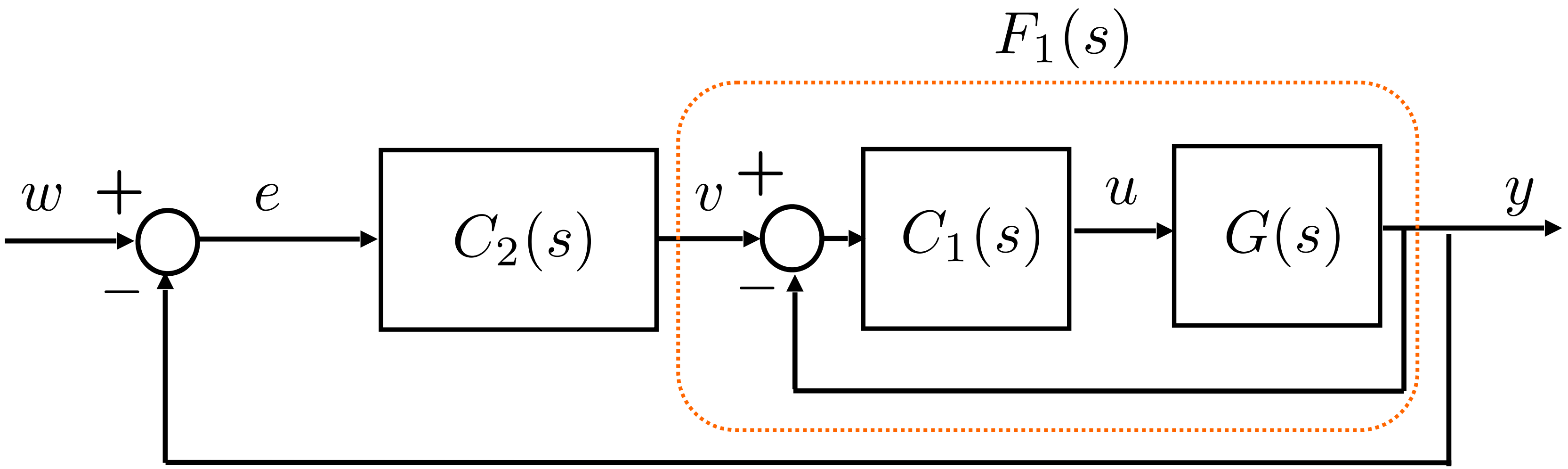
The simplest and frequently used approach is to consider a **dual-loop controller architecture**



- Controller $C_1(s)$ is designed to **stabilise** the system described by the inner closed-loop transfer function $F_1(s)$
- Controller $C_2(s)$ is designed to **meet the specifications** for the whole feedback control system

Example 3 - Unstable System (contd.)

The design of the inner controller $C_1(s)$ can be carried out as:

$$C_1(s) = \mu_1 \Rightarrow L_1(s) = C_1(s) \cdot G(s) = \frac{\mu_1}{s-1}$$

$$\Rightarrow F_1(s) = \frac{L_1(s)}{1 + L_1(s)} = \frac{\mu_1}{s - 1 + \mu_1}$$

By choosing (for example): $\mu_1 = 11$

$$\Rightarrow F_1(s) = \frac{1.1}{1 + 0.1s}$$ **asymptotically stable**

Example 3 - Unstable System (contd.)

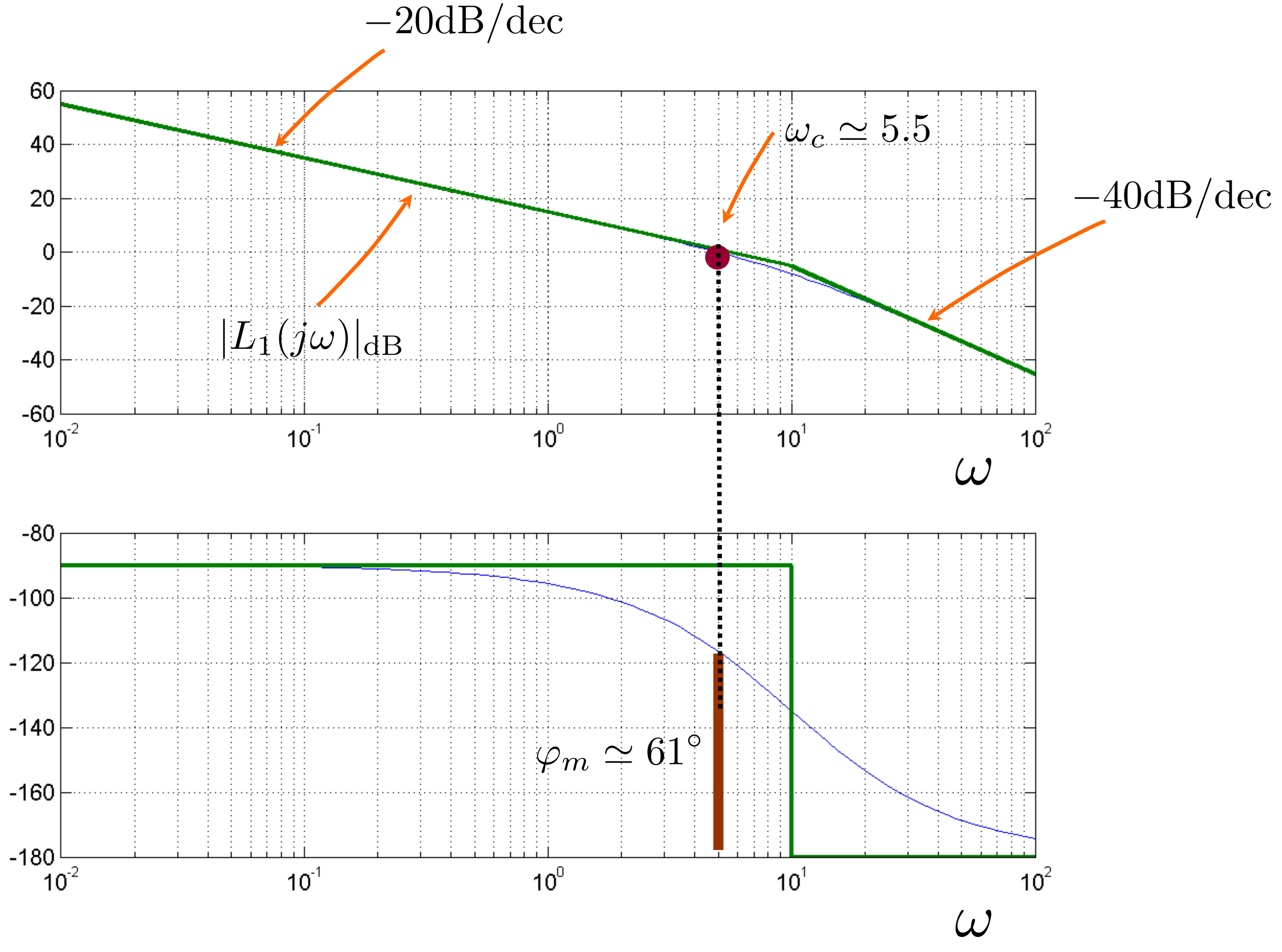
- Now, the design of the outer controller $C_2(s)$ is very simple:

- To meet the specification on the static error, an **integrator** on the direct path has to be introduced.
- Hence, by choosing:

$$C_2(s) = \frac{5}{s}$$

$$L_1(s) = C_2(s) \cdot F_1(s) = \frac{5.5}{s(1+0.1s)}$$

All specifications are met



Fundamentals of Automatic Control

... The End ...