UNIVERSITÀ DEGLI STUDI DI TRIESTE

034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL A.Y. 2023-2024

Part IX: Analysis of Feedback Control Systems

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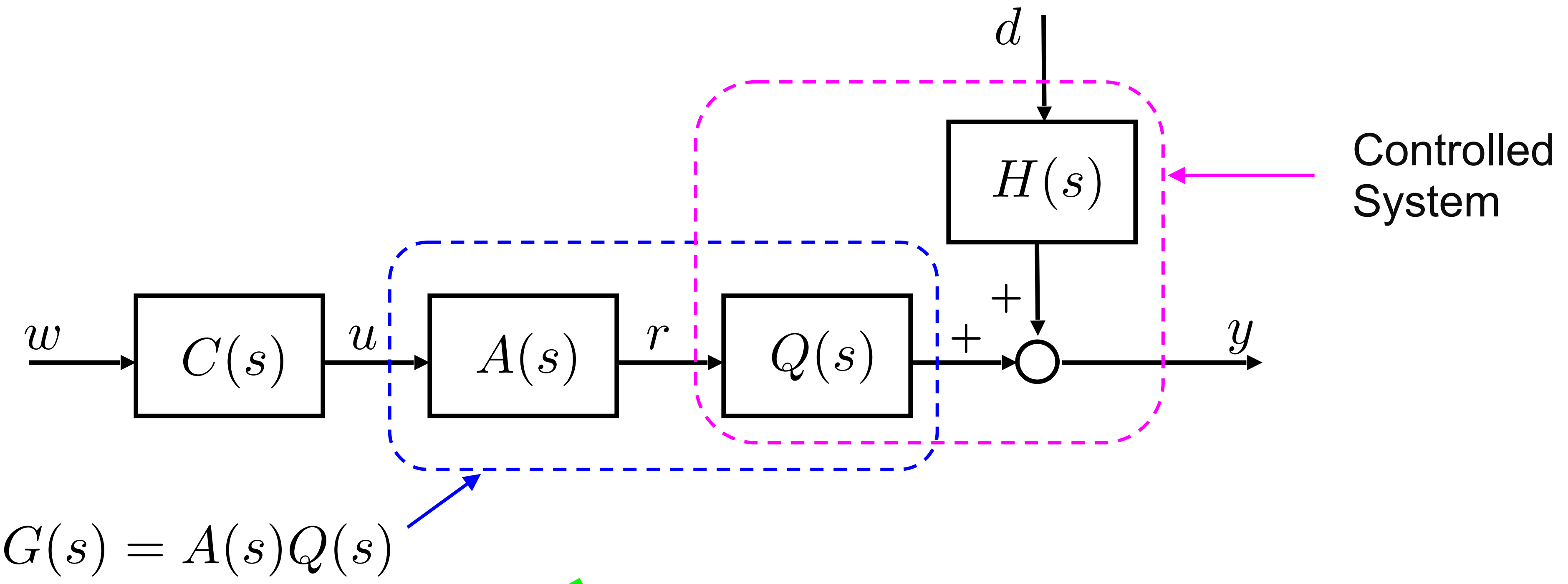
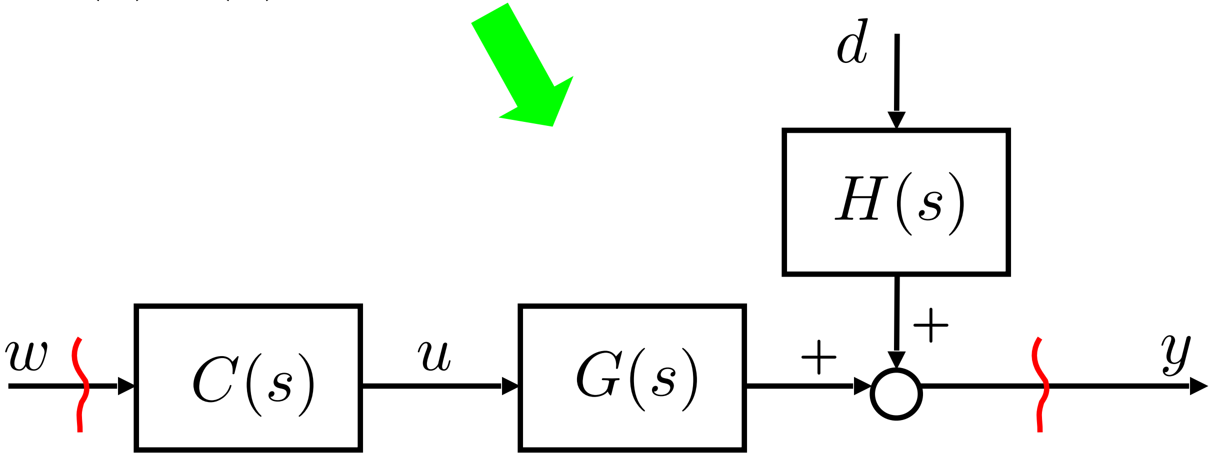
Open-Loop Control Systems





Open-Loop Control Systems – “Ideal” Performance

$$y(t) = w(t),\, t \geq 0 \quad \Rightarrow \quad \frac{Y(s)}{W(s)} = 1 \quad \Rightarrow \quad C(s)G(s) = 1$$

$$\boxed{C(s) = G(s)^{-1}}$$

The “ideal” open-loop controller “inverts” the system’s dynamics

Hence:

- pole-zero ”cancellations” in the right half-plane may occur causing **unstable hidden dynamics**
- open-loop stabilisation of unstable systems is not possible
- the controller $C(s)$ may result in having **more zeros than poles**, thus not being physically implementable
- **uncertainty** in $G(s)$ makes the ideal performance anyway not achievable

Examples

Consider the controlled system $G(s) = \dfrac{10(1+s)}{(1+2s)(1+0.1s)}$

$$C_0(s) = \frac{0.1(1+2s)(1+0.1s)}{1+s}$$

The "ideal" open-loop controller shows two zeros and one pole and hence it is not physically implementable

- Consider a first candidate of physically implementable open-loop controller:

$$C_1(s) = \frac{0.1(1+2s)(1+0.1s)}{(1+s)(1+0.01s)}$$

$$F_1(s) = \frac{Y(s)}{W(s)} = C_1(s)G(s) = \frac{1}{1+0.01s}$$

Low-pass filter with bandwidth $B_1 \simeq [0,\ 100]$

Examples (contd.)

- Consider a second **simpler** candidate of physically implementable open-loop controller:

$$C_2(s) = \frac{0.1(1+2s)}{1+s}$$

$$\Rightarrow F_2(s) = \frac{Y(s)}{W(s)} = C_2(s)G(s) = \frac{1}{1+0.1s}$$

Low-pass filter with bandwidth $B_2 \simeq [0,\ 10]$

- Consider a third **much simpler (not even dynamic)** candidate of physically implementable open-loop controller:

$$C_3(s) = 0.1$$

$$\Rightarrow F_3(s) = \frac{Y(s)}{W(s)} = C_3(s)G(s) = \frac{1+s}{(1+2s)(1+0.1s)}$$

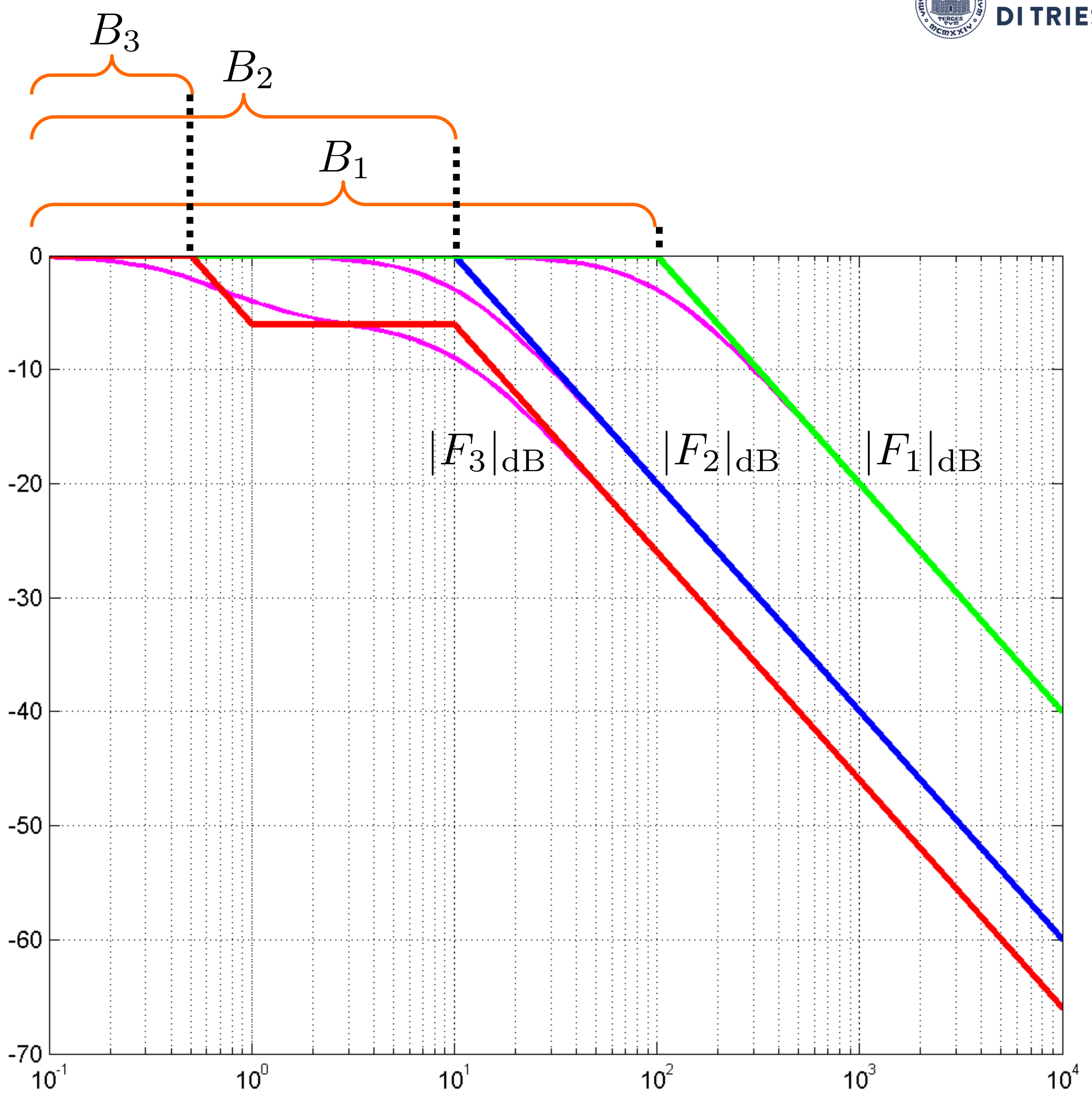
Low-pass filter with bandwidth $B_3 \simeq [0,\ 0.5]$

Examples - Comparison

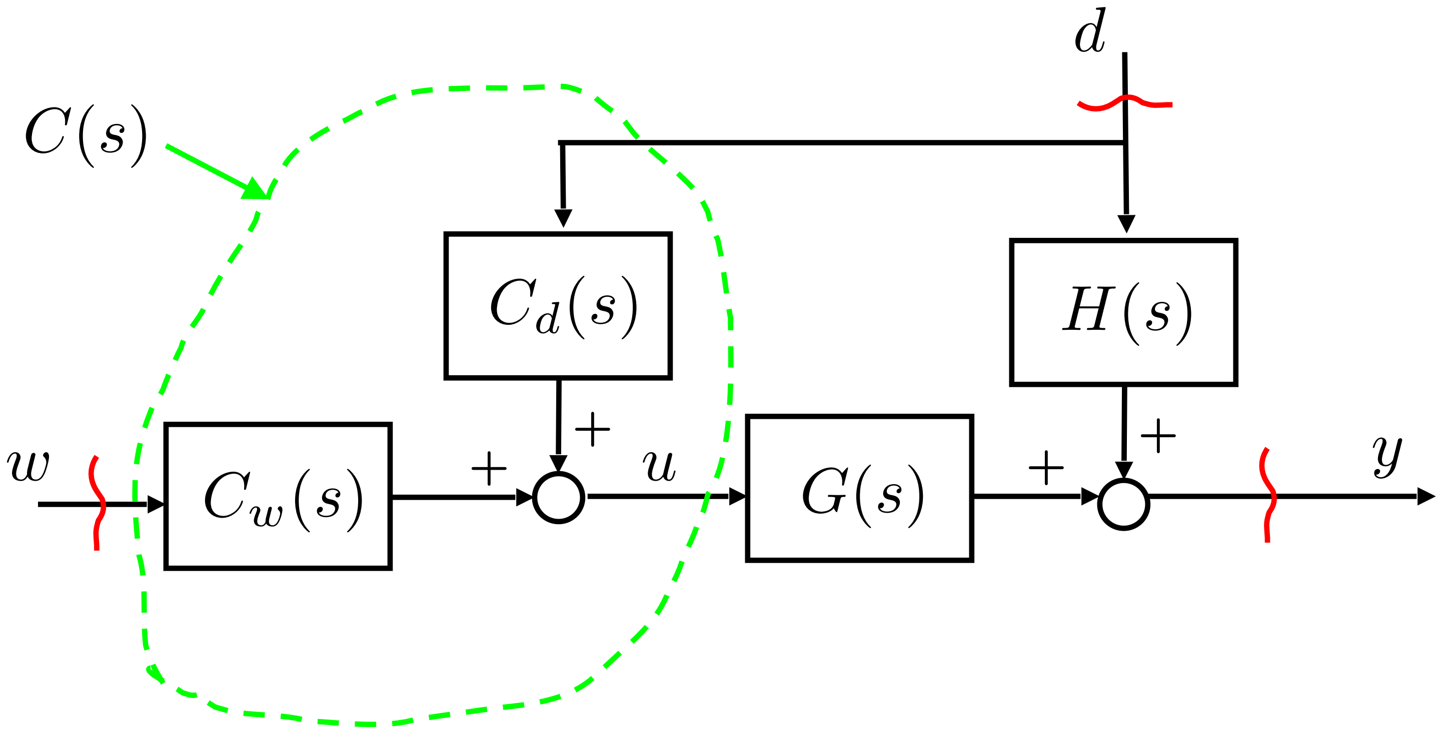
$$F_1(s) = \frac{1}{1+0.01s}$$

$$F_2(s) = \frac{1}{1+0.1s}$$

$$F_3(s) = \frac{1+s}{(1+2s)(1+0.1s)}$$



Open-Loop Control Systems with Disturbance Rejection



Assumption: the disturbance $d(t)$ is **accessible** for measurement

Open-Loop Control Systems – "Ideal" Performance

$$y(t) = w(t),\ t \geq 0 \quad \Rightarrow \quad \frac{Y(s)}{W(s)} = 1 \quad \Rightarrow \quad C_w(s)G(s) = 1$$

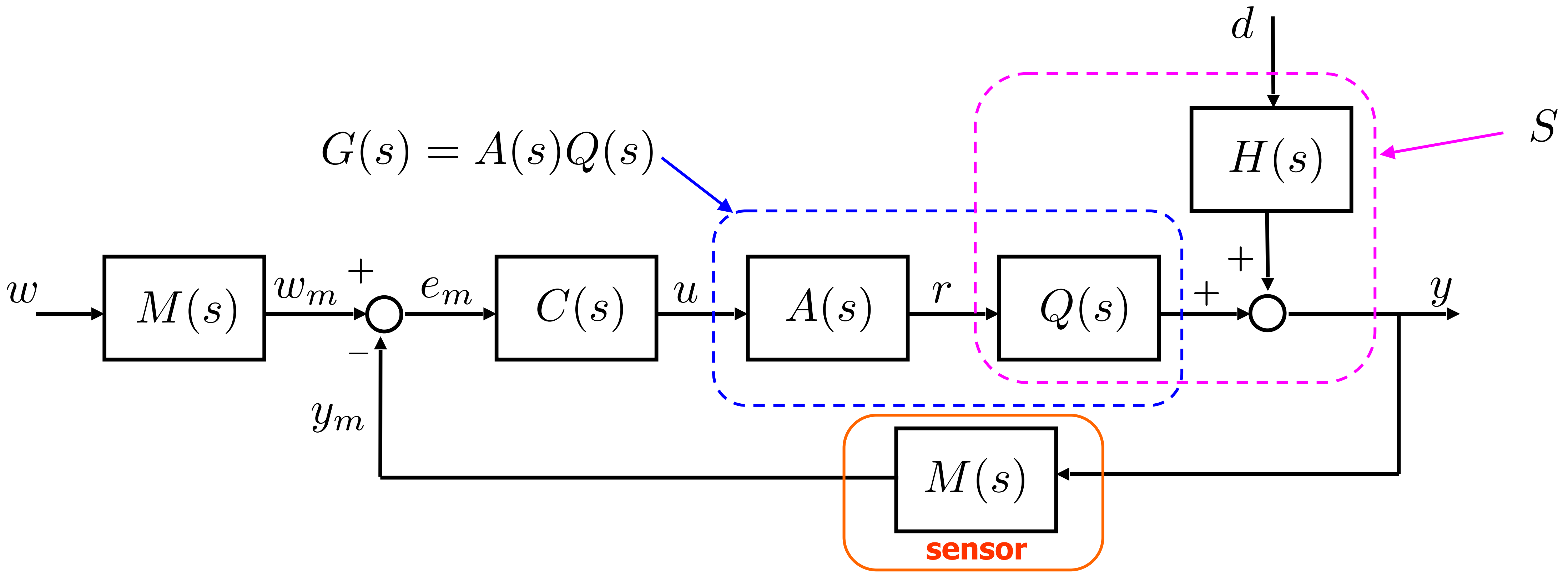
$$\boxed{C_w(s) = G(s)^{-1}}$$

$$y(t) = 0,\ t \geq 0 \text{ if } w(t) = 0,\ t \geq 0 \quad \Rightarrow \quad \frac{Y(s)}{D(s)} = 0 = H(s) + C_d(s) \cdot G(s)$$

$$\boxed{C_d(s) = -G(s)^{-1} H(s)}$$

Hence, the **same limitations** we have seen in the no-disturbance case apply here as well because of the need of "inverting" the system's dynamics

Feedback (Closed-Loop) Control Systems



$$
\begin{aligned}
E_m(s) &= W_m(s) - Y_m(s) \\
&= M(s)[W(s) - Y(s)] = M(s) \cdot E(s)
\end{aligned}
$$

Feedback (Closed-Loop) Control Systems (contd.)

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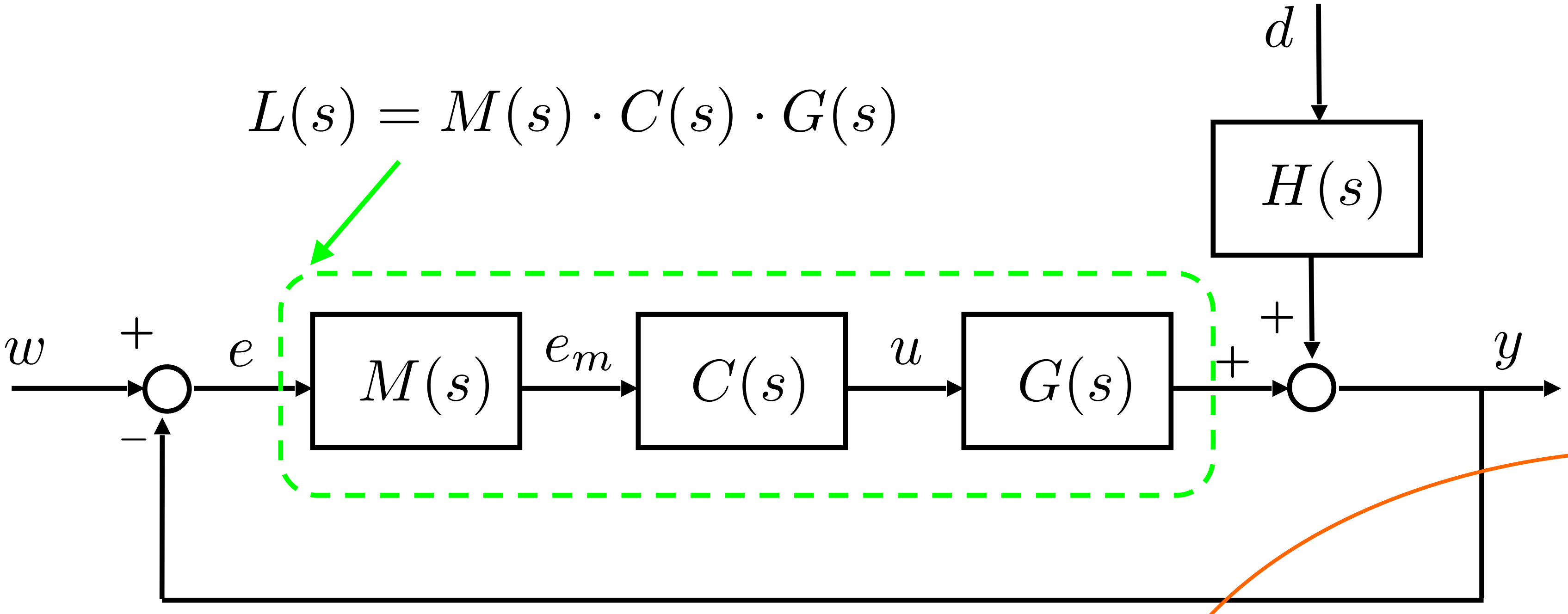
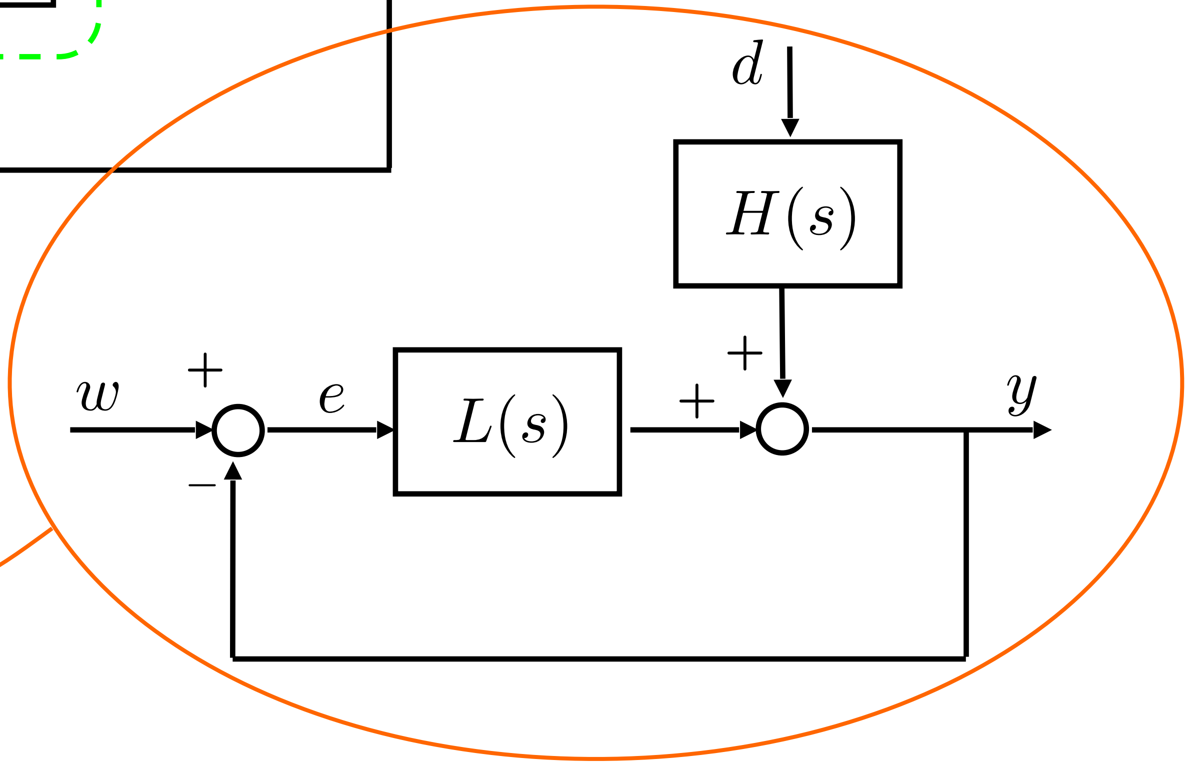
$$L(s) = M(s) \cdot C(s) \cdot G(s)$$





Block scheme used for **analysis** of the feedback control system without loss of generality

Analysis of Feedback Control Systems
Methods and Tools in the Frequency Domain

Feedback Control Systems – "Ideal" Performance

$$y(t) = w(t),\ t \geq 0 \quad \Rightarrow \quad F(s) = \frac{Y(s)}{W(s)} = 1$$

$$y(t) = 0,\ t \geq 0 \text{ if } w(t) = 0,\ t \geq 0 \quad \Rightarrow \quad R(s) = \frac{Y(s)}{D(s)} = 0$$

However, in general:

$$F(s) = \frac{L(s)}{1 + L(s)} \neq 1$$

$$R(s) = \frac{H(s)}{1 + L(s)} \neq 0$$

The **realistic scenario to be achieved** is:

- $F(s)$ low-pass filter with a sufficiently large bandwidth and gain $\mu_F = 1$
- $|R(j\omega)| \simeq 0$ in the frequency range where the spectrum of $d(t)$ is significant

Nyquist Diagram

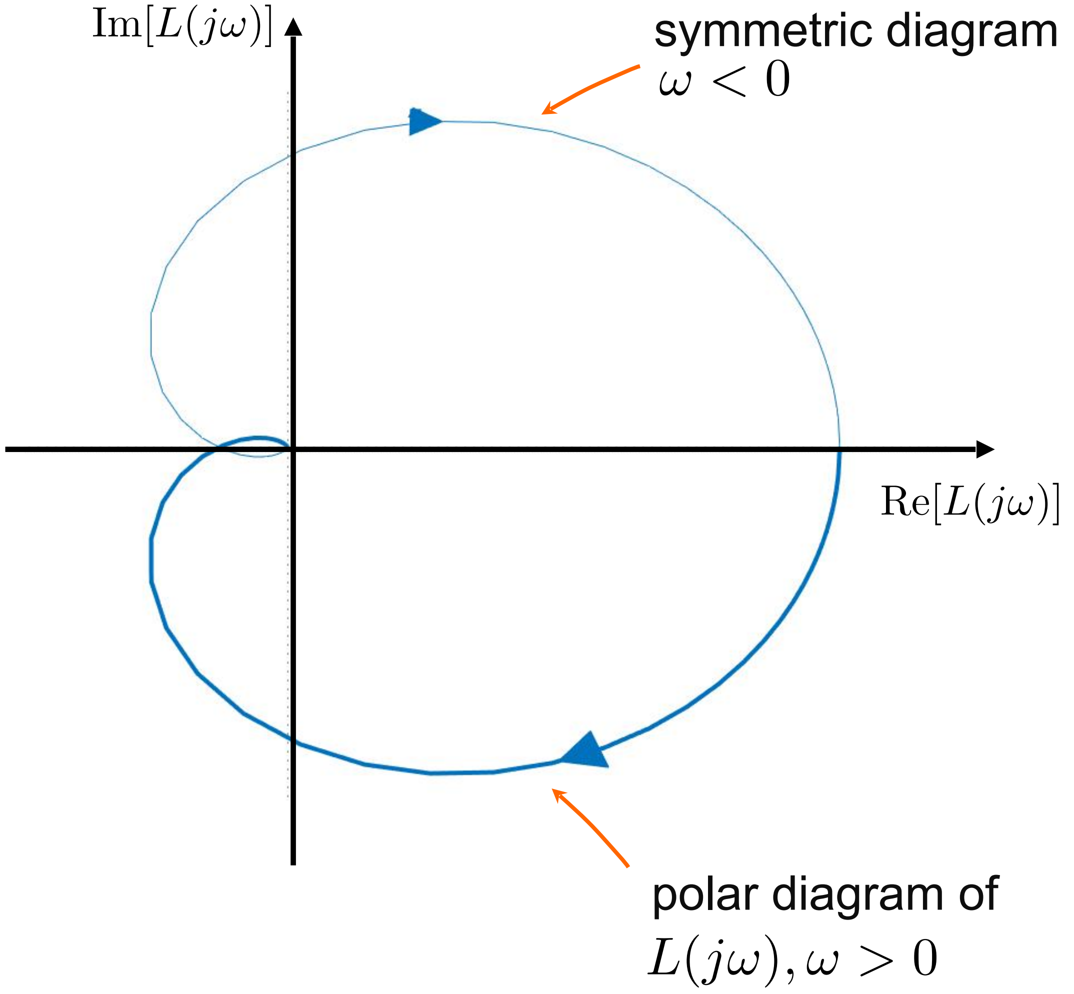
- The **Nyquist diagram** Γ is an extension of the conventional polar diagram to the angular frequencies range $-\infty < \omega < +\infty$.
- The following property holds:

$$L(-j\omega) = L^*(j\omega)$$

- The Nyquist diagram Γ is:
  - polar diagram of $L(j\omega),\ \omega \geq 0$

$$+$$

  - the diagram symmetric to the polar diagram with respect to the real axis



Nyquist Diagram (contd.)

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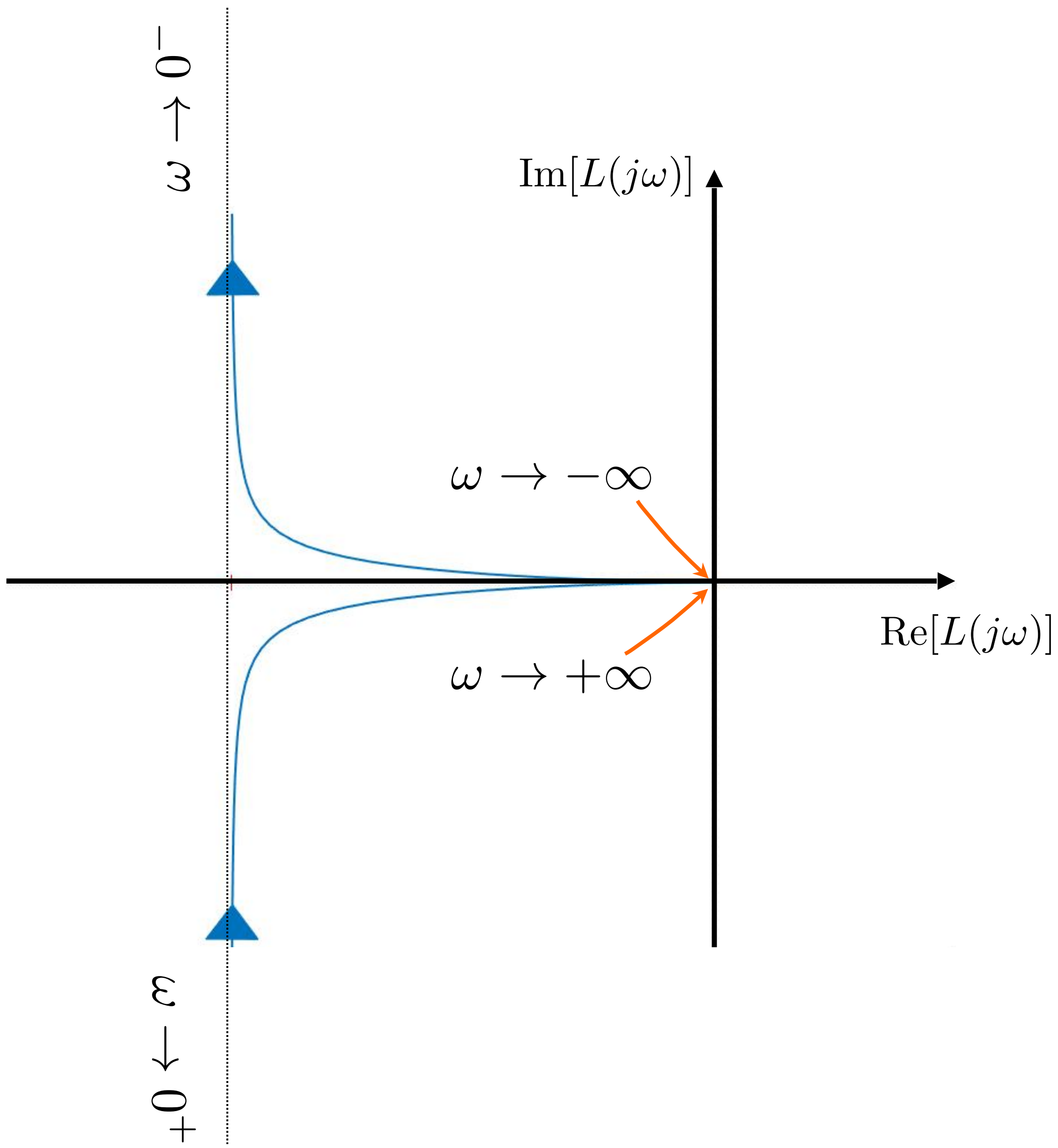
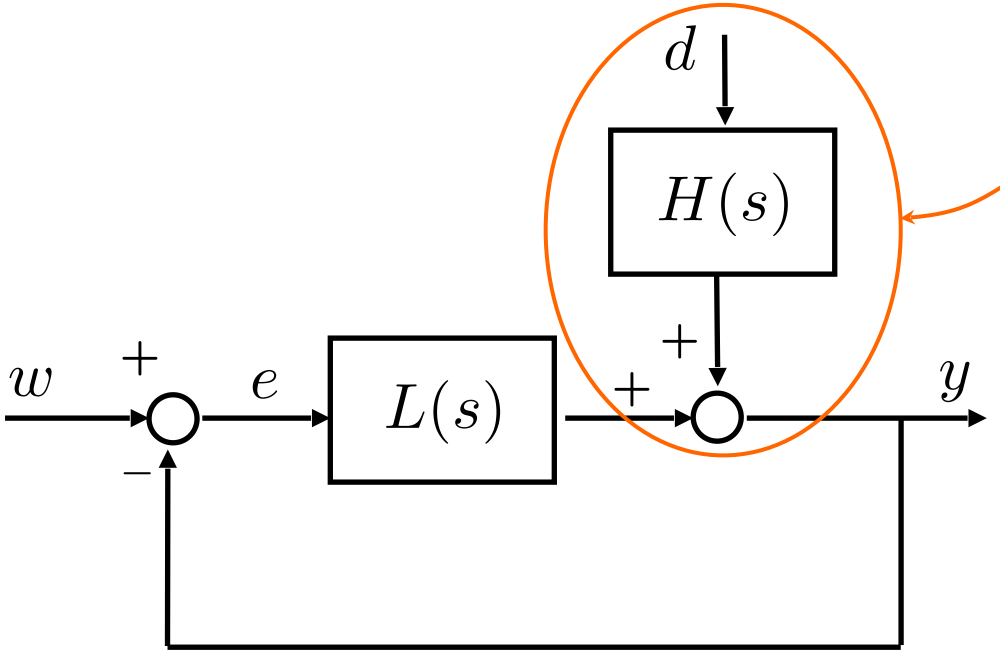
If the polar diagram of $L(j\omega),\ \omega \geq 0$ is not closed and bounded a **clock-wise "infinite-width" closure diagram** must be added to obtain the Nyquist diagram Γ



Nyquist Closed-Loop Stability Criterion



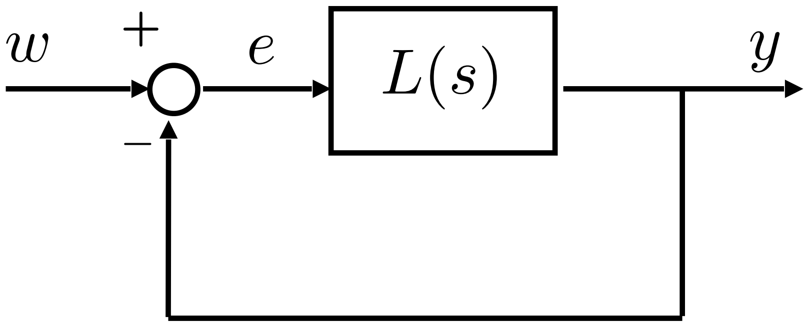
Open-loop block not affecting closed-loop stability analysis

$$F(s) = \frac{L(s)}{1+L(s)} \; ; \; R(s) = \frac{H(s)}{1+L(s)}$$

Closed-loop stability analysis boils down to establishing the location of the **zeros** of $1+L(s)$ in the complex plane, that is, analysing the solutions of

$$\boxed{1 + L(s) = 0}$$

Nyquist Closed-Loop Stability Criterion (contd.)



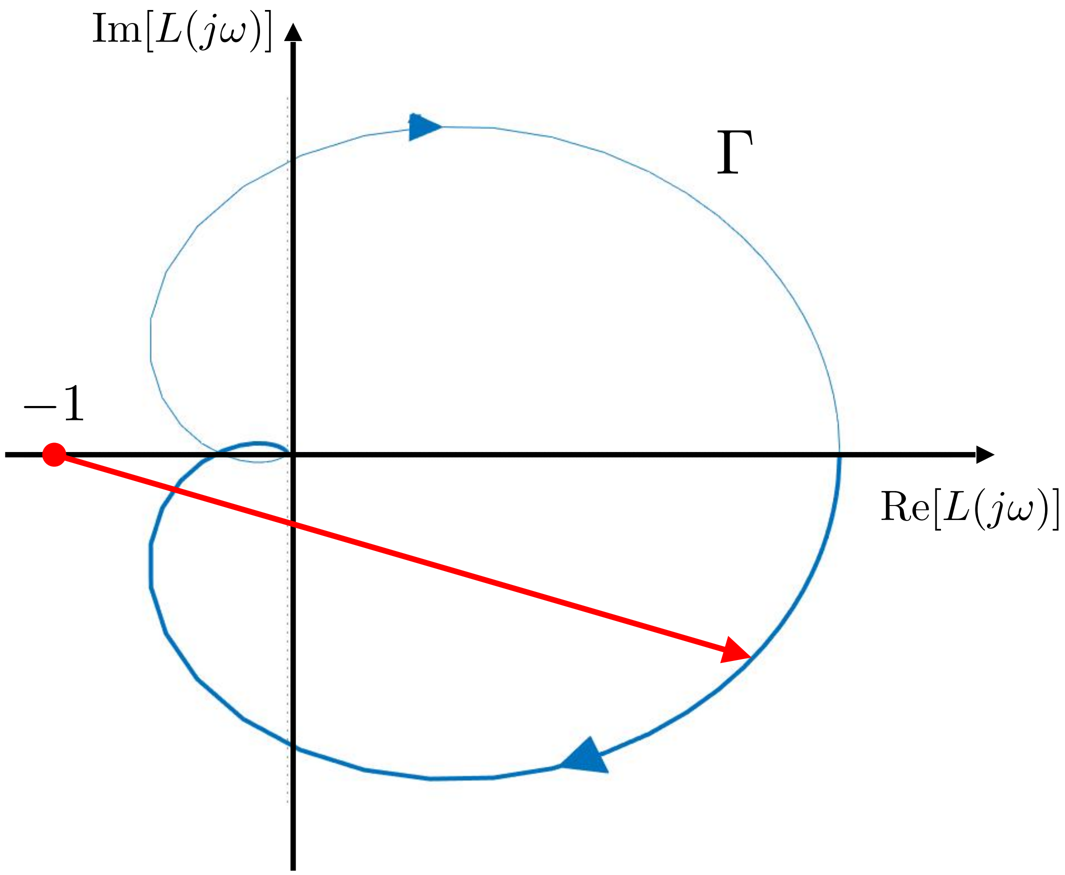
$$1 + L(s) = 0$$

- Nyquist diagram Γ of $L(s)$
- Number N of **counterclockwise** rotations of Γ around point $(-1, 0) \in \mathbb{C}$
- Number $n_{p>0}$ of poles of $L(s)$ in the right half-plane

Closed-loop Asymptotic Stability $\Longleftrightarrow$

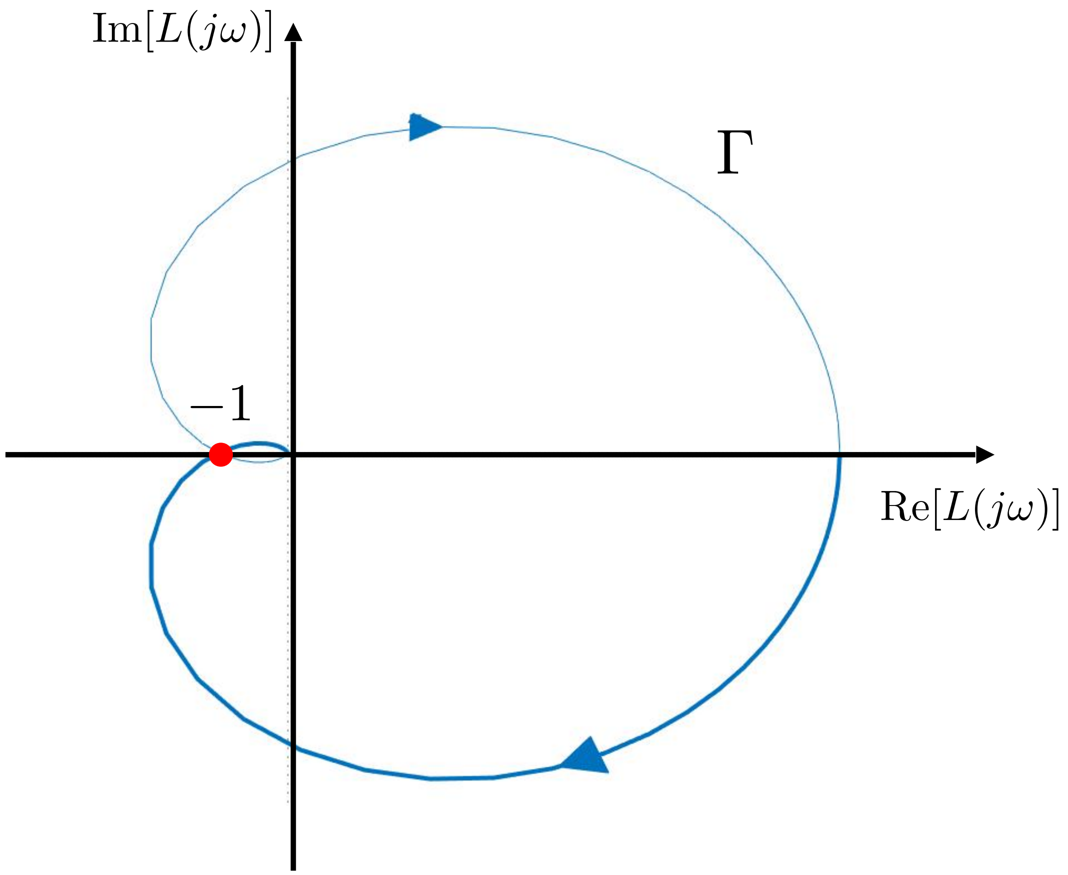
- N "well-defined"
- $N = n_{p>0}$

Remarks



The point $(-1,0)$ lies outside of the Nyquist diagram Γ

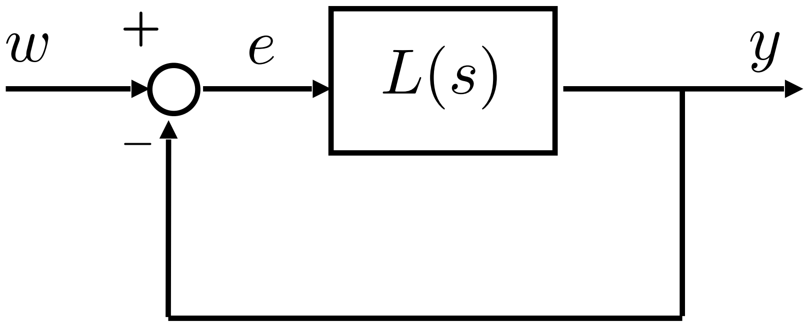
$N = 0$

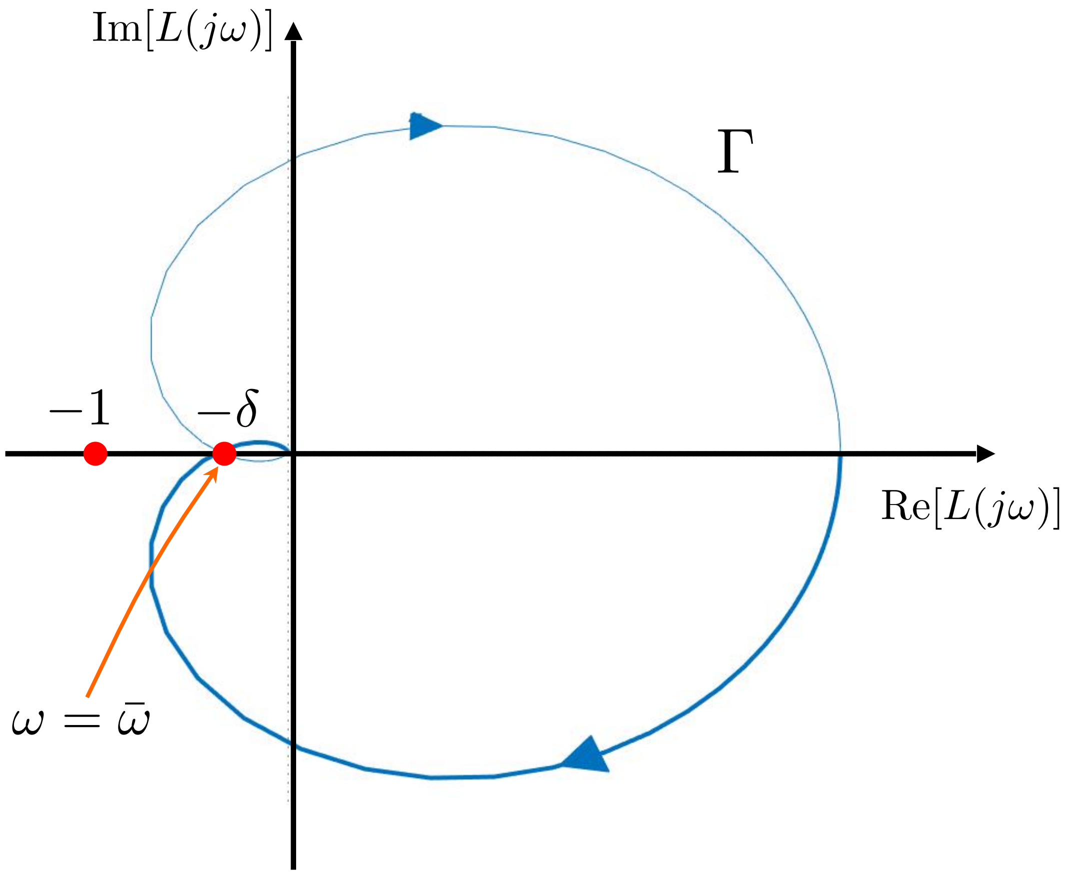


The point $(-1,0)$ lies on the Nyquist diagram Γ

N undefined

Nyquist Closed-Loop Stability Criterion - Intuitive Justification

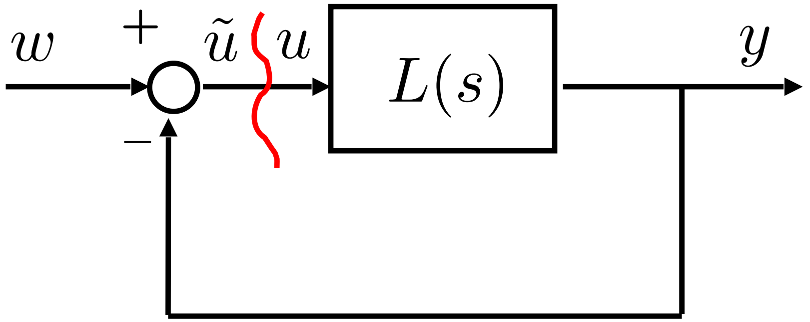




Suppose for simplicity that the open-loop system $L(s)$ is asymptotically stable, hence $n_{p>0} = 0$

$$L(j\bar{\omega}) = -\delta \quad \Longleftrightarrow \quad \begin{array}{l} |L(j\bar{\omega})| = \delta \\ \arg L(j\bar{\omega}) = -180^\circ \end{array}$$

Hypothetically, suppose to “break the loop” in an arbitrary point:



$$u(t) = \sin(\bar{\omega} t)$$

$$\Rightarrow \quad y(t) \simeq |L(j\bar{\omega})| \sin[\bar{\omega} t + \arg(L(j\bar{\omega}))]$$

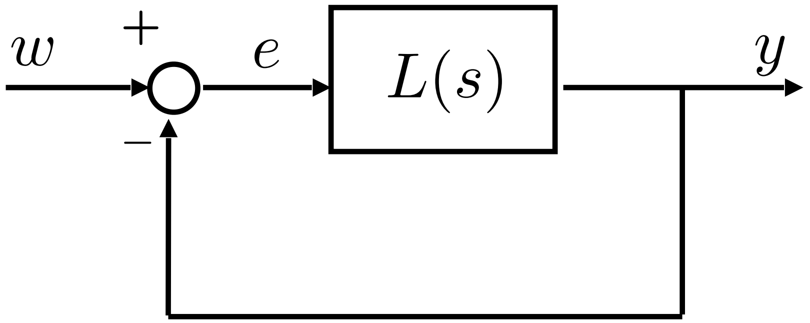
$$= \delta \sin(\bar{\omega} t - \pi) = -\delta \sin(\bar{\omega} t)$$

$$\Rightarrow \quad \tilde{u}(t) = \delta \sin(\bar{\omega} t)$$

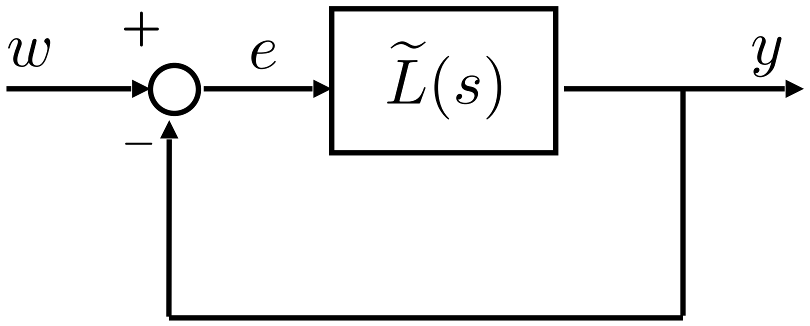
Hence:

- $\delta > 1$ ⇒ unstable [$N \neq 0\,(= n_{p>0})$]
- $\delta < 1$ ⇒ asymptotically stable [$N = 0\,(= n_{p>0})$]
- $\delta = 1$ ⇒ not asymptotically stable [N undefined]

Closed-Loop Robust Stability



nominal model



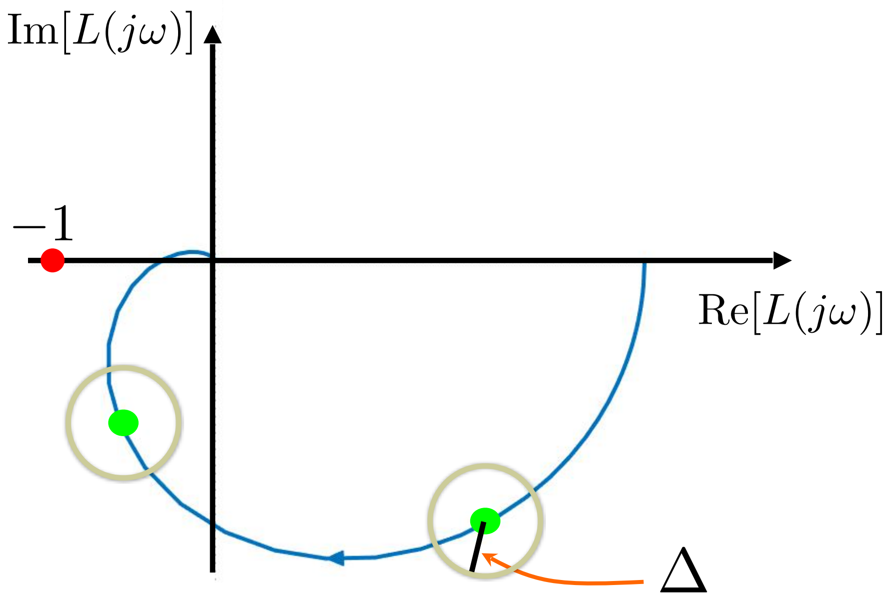
true model

- In practical engineering contexts $\widetilde{L}(s) \neq L(s)$
- The aim is guaranteeing the **closed-loop robust stability**, that is, the **stability in the presence of uncertainty on the open-loop nominal model** $L(s)$
- A **mathematical characterisation of the uncertainty** is needed

Typical Uncertainty Models

- **Unstructured Uncertainty**:

$$\widetilde{L}(s) = L(s) + \delta L(s)\,;\ \ |\delta L(j\omega)| \leq \Delta$$



- **Open-Loop Gain Uncertainty**

$$\widetilde{L}(s) = K \cdot L(s)\,;\ \ 0 < K < \bar{K}$$

Robust Stability Indicators

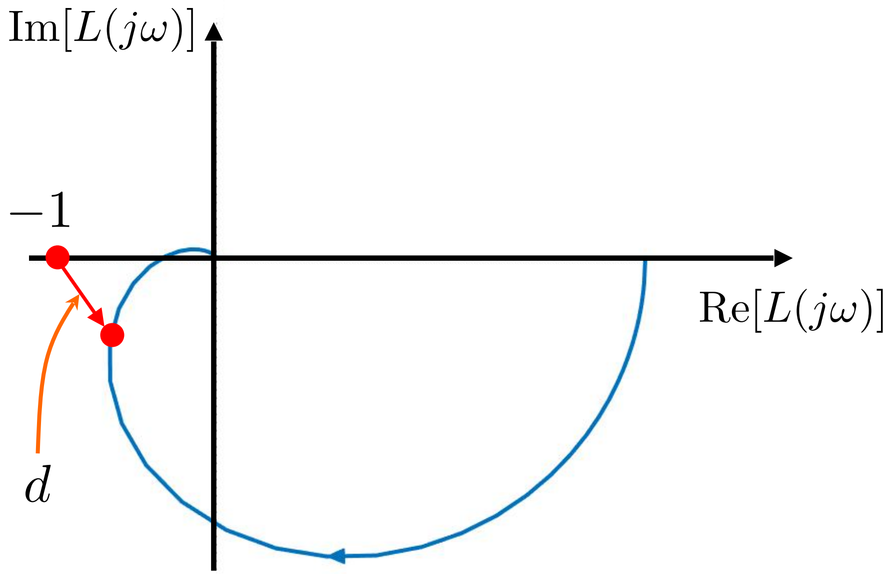
- We suppose that the **nominal** open-loop model $L(s)$ is asymptotically stable
- Hence, to guarantee closed-loop asymptotic stability for the nominal model, the nominal Nyquist diagram must **not encircle the point** $(-1,0)$)
- **Robust stability indicators** quantify the magnitude of the uncertainty for which closed-asymptotic stability is preserved when the nominal model is **replaced with the true model** $\widetilde{L}(s)$ in the closed-loop scheme
- Robust stability indicators also quantify the **“distance”** of the nominal closed-loop system **from the "instability scenario"**

A Candidate Robust Stability Indicator

The more natural choice as robust stability indicator is the Euclidean distance of the polar diagram $L(j\omega)$ from the point $(-1, 0)$

vector stability margin:

$$d = \min_{\omega} |1 + L(j\omega)|$$



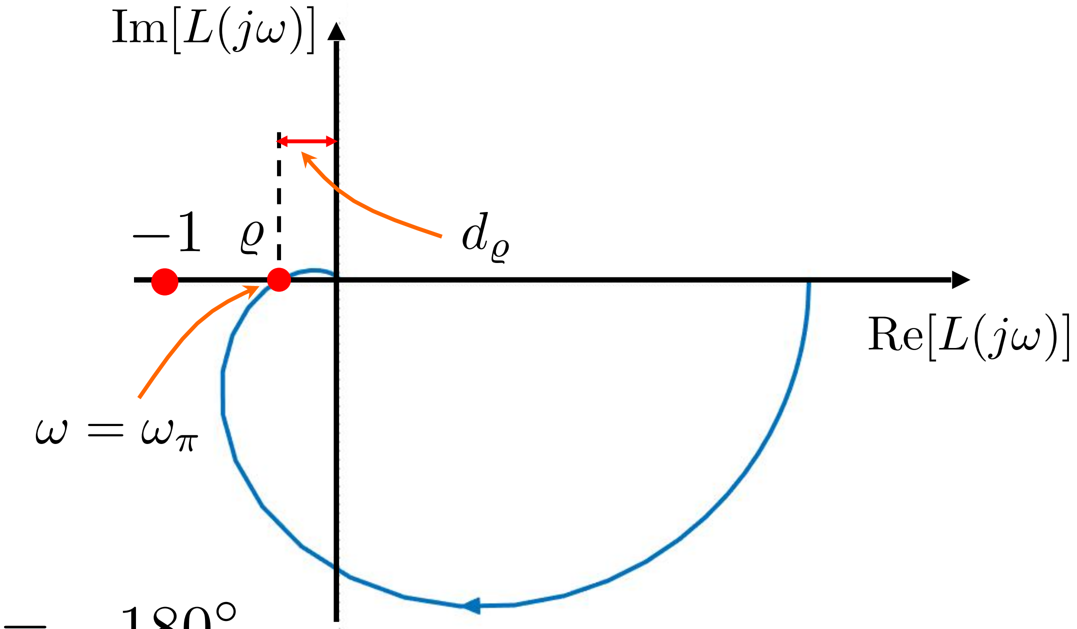
The vector stability margin is well-defined mathematically, but it is **not very useful in practice** since it cannot be evaluated using the Bode diagrams of $L(j\omega)$

Gain Margin

The **gain margin** is a robust stability indicator defined as:

gain margin:

$$K_m = \frac{1}{d_\varrho}$$



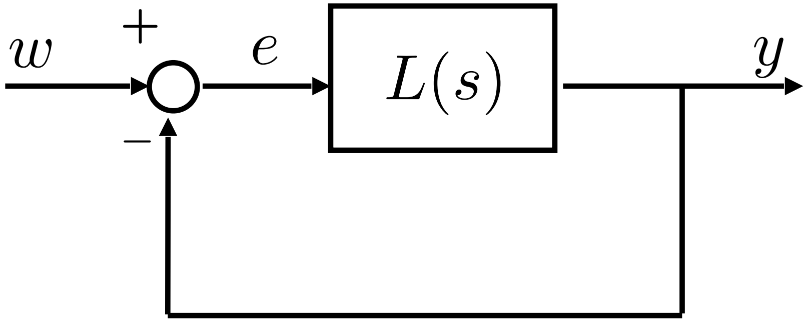
Hence:

$$\omega_\pi : \ \omega \ \text{ such that } \ \arg L(j\omega) = -180^\circ$$

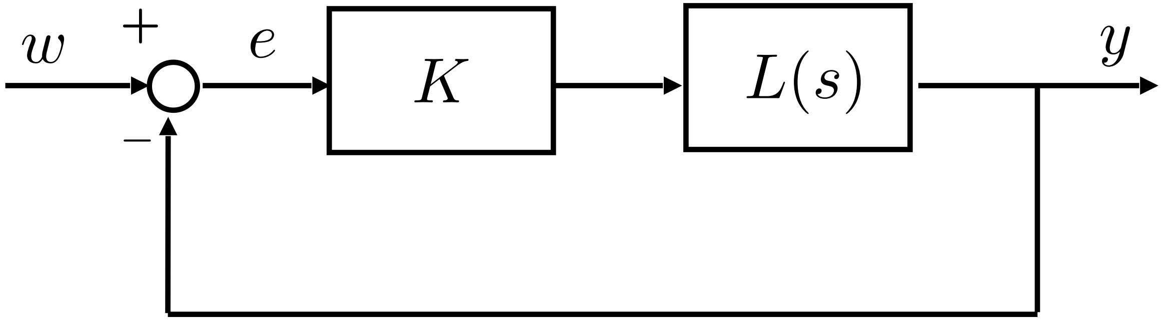
$$K_m = \frac{1}{|L(j\omega_\pi)|} = -|L(j\omega_\pi)|_{\mathrm{dB}}$$

K_m can be evaluated from the Bode diagrams

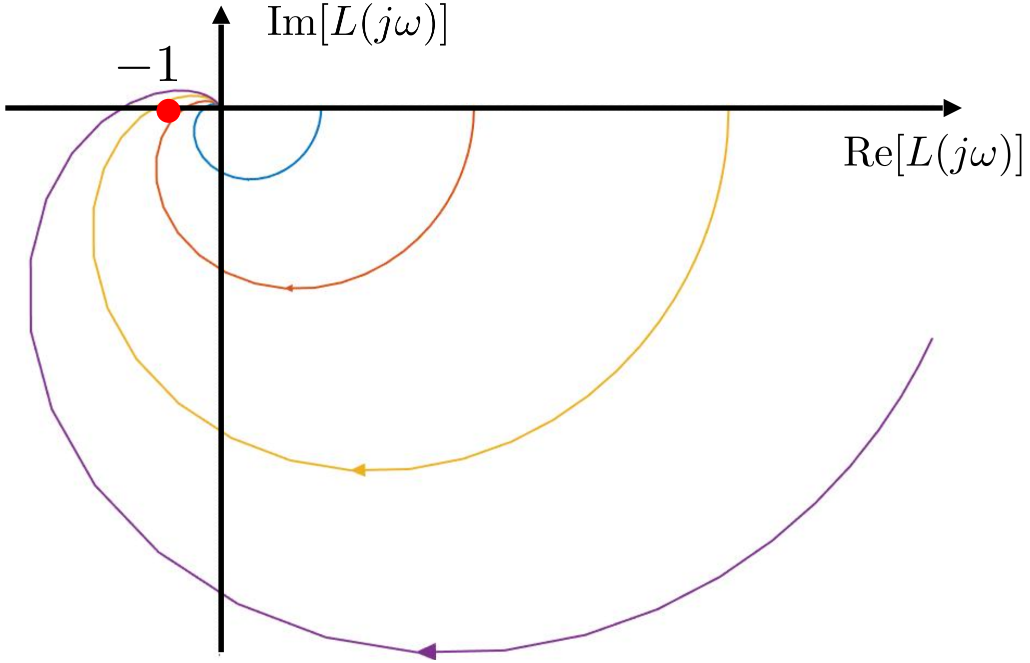
Gain Margin - Interpretation



nominal model



$0 < K < K_m$ → asymptotically stable



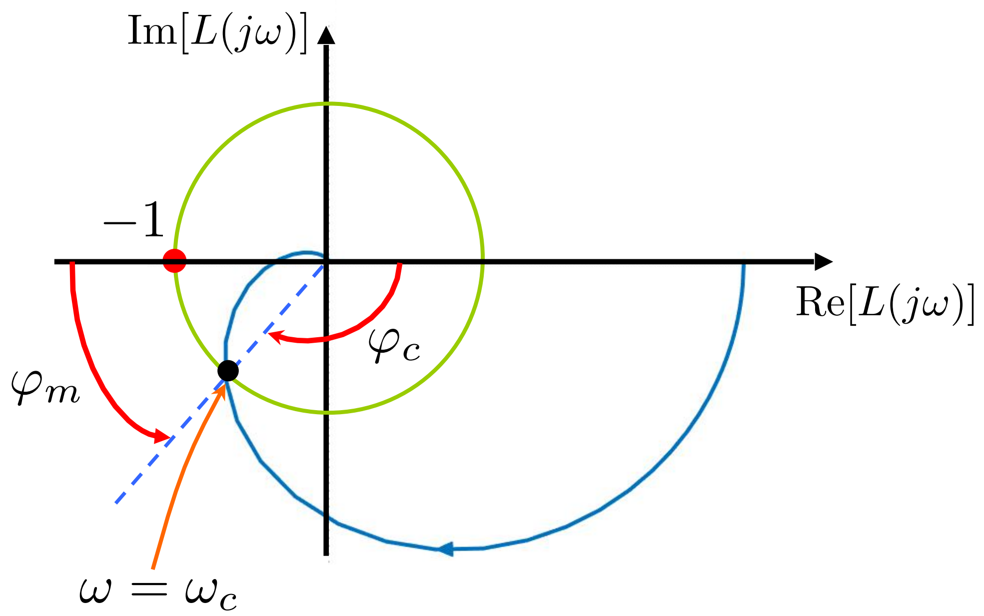
The gain margin K_m is a robustness indicator referring to **uncertainty on the loop gain**

The **phase margin** is a robust stability indicator defined as:

phase margin:

$$\varphi_m = \varphi_c - (-180°)$$

Remark: recall that $\varphi_c < 0$ in the figure on the left consistently with the conventions we have used in the Nyquist and Bode diagrams (see Part 8)



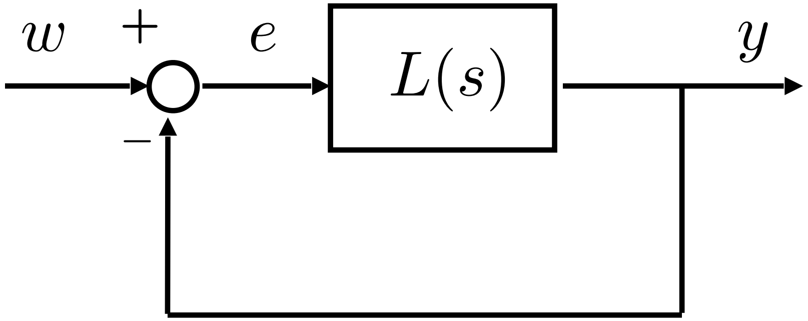
Hence:

$$\omega_c : \omega \text{ such that } |L(j\omega_c)| = 1 = 0\text{dB}$$

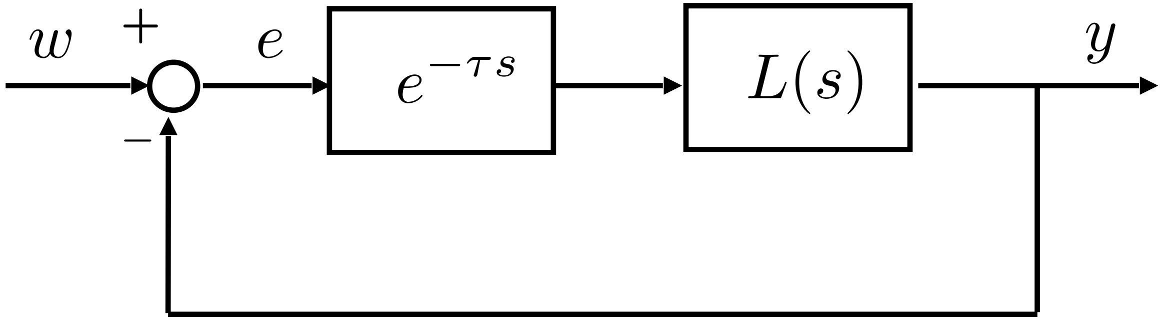
$$\varphi_c = \arg L(j\omega_c)$$

$\varphi_m = \varphi_c - (-180°)$ can be evaluated from the Bode diagrams

Phase Margin - Interpretation



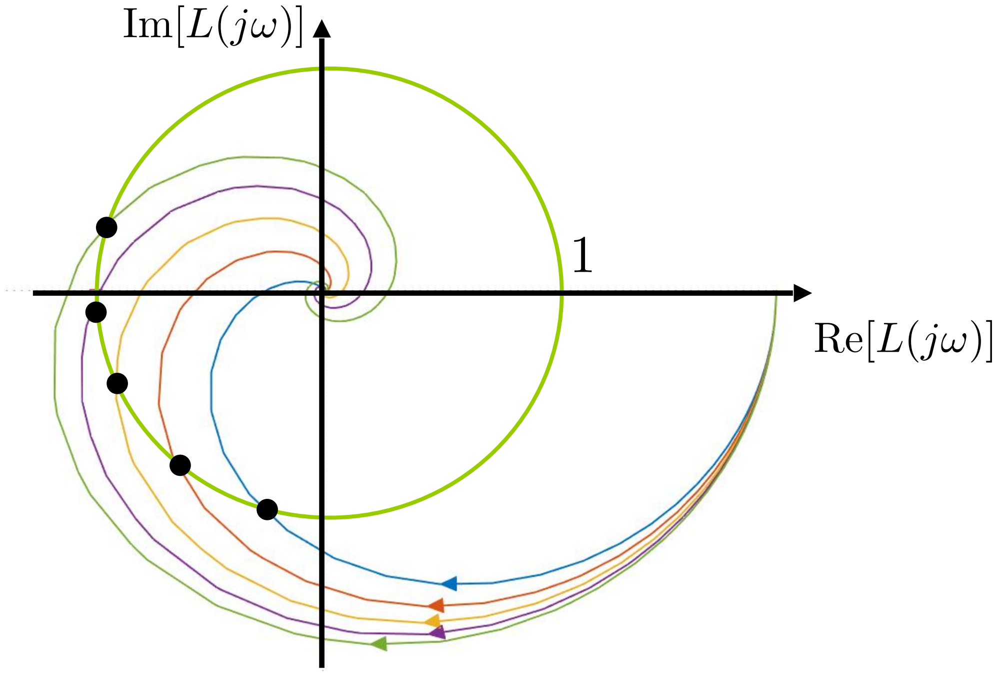
nominal model



Since $\omega_c \tau = \varphi_m \cdot \dfrac{\pi}{180}$

$$0 < \tau < \frac{\varphi_m}{\omega_c} \cdot \frac{\pi}{180}$$

asymptotically stable



The phase margin φ_m is a robustness indicator referring to **uncertainty on the loop "delay"**

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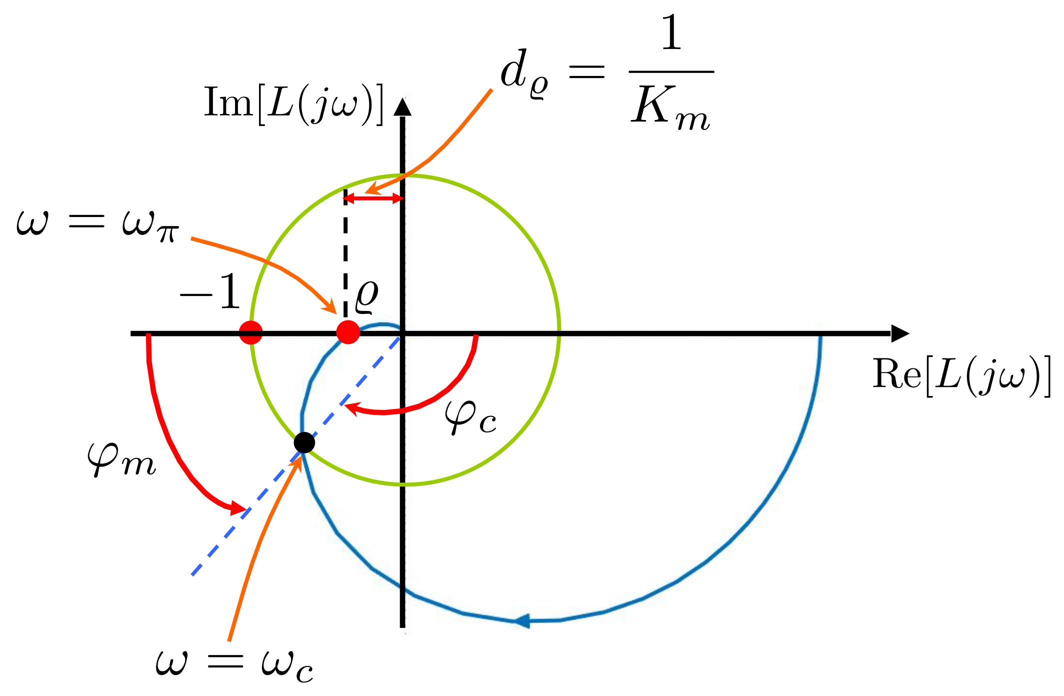
- **gain margin**:

$$K_m = \frac{1}{d_\varrho}$$

$$K_m = \frac{1}{|L(j\omega_\pi)|} = -|L(j\omega_\pi)|_{\mathrm{dB}}$$

$$\arg L(j\omega_\pi) = -180^\circ$$



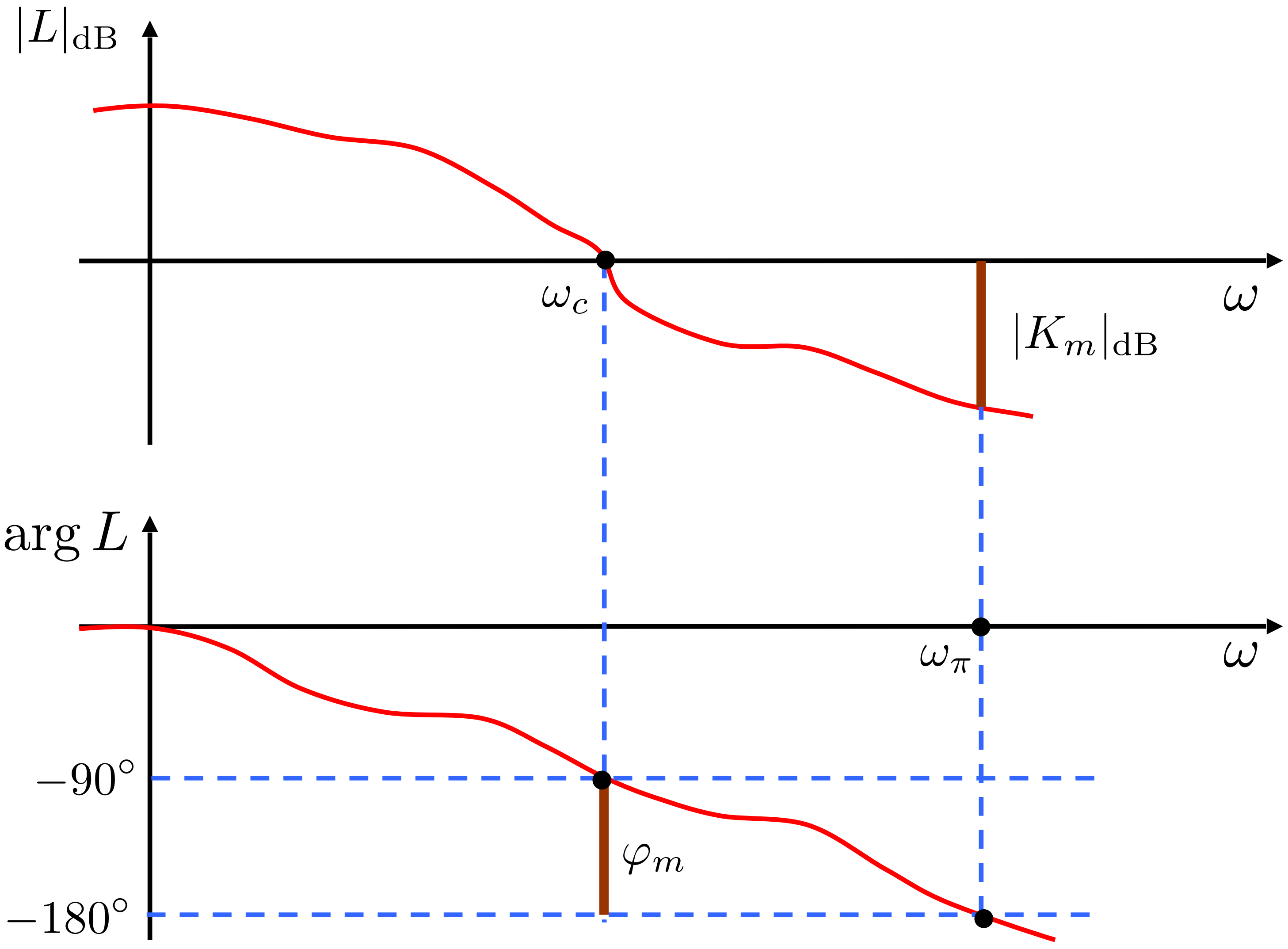
- **phase margin**:

$$\varphi_m = \varphi_c - (-180^\circ)$$

$$\omega_c : \ \omega \ \text{ such that } \ |L(j\omega_c)| = 1 = 0\mathrm{dB}$$

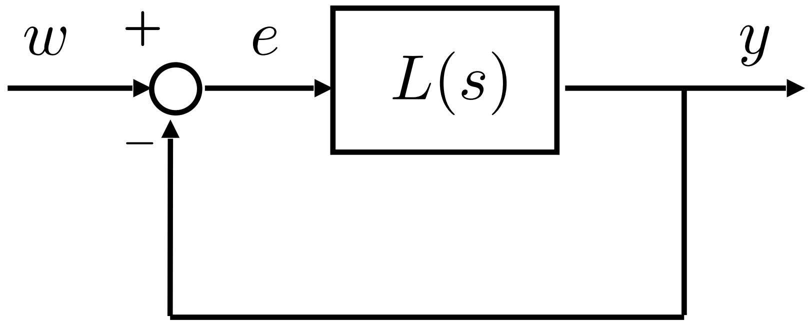
$$\varphi_c = \arg L(j\omega_c)$$

Gain and Phase Margin – Evaluation from Bode Diagrams



Bode Closed-Loop Stability Criterion

The Bode stability criterion is very useful for controller design even if it is applicable in a more restricted scenario than the Nyquist stability criterion:



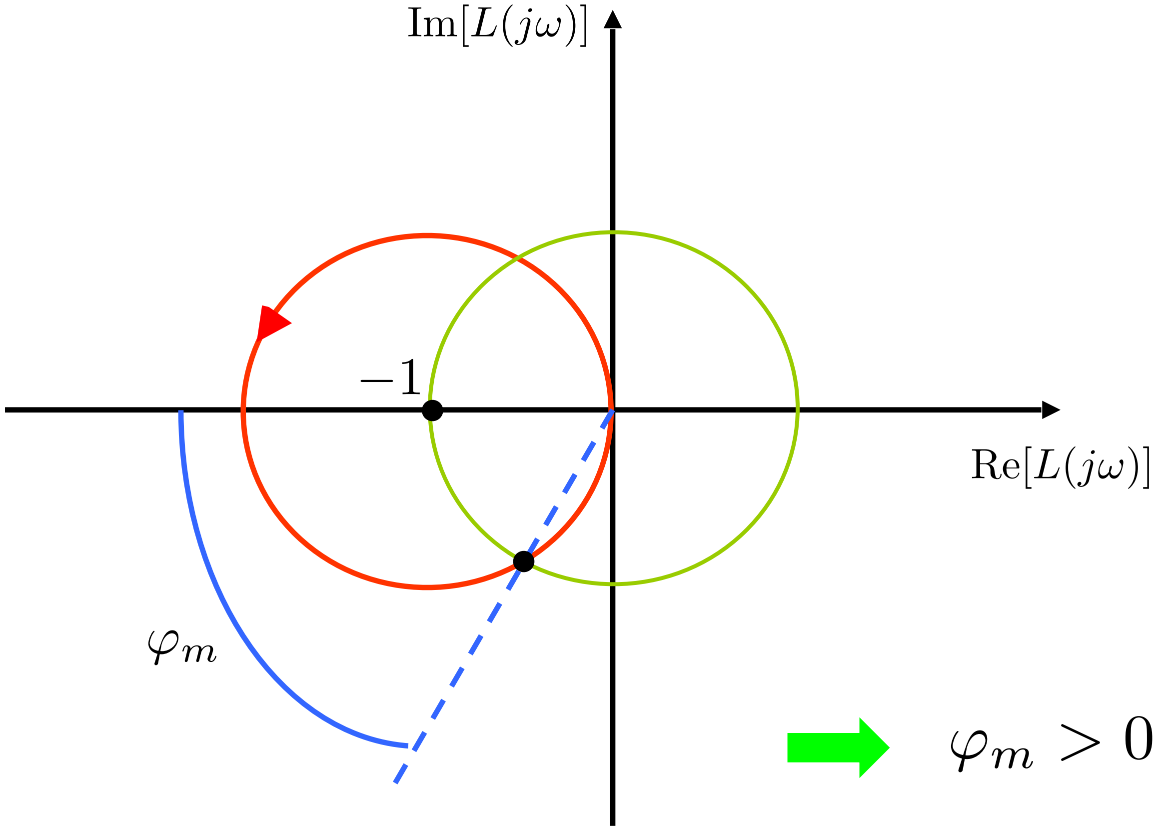
- Loop gain: μ
- Phase Margin: φ_m

Assumptions:

- Open-loop asymptotic stability: $n_{p>0} = 0$
- The magnitude Bode diagram $|L(j\omega)|_{\mathrm{dB}}$ crosses the 0dB axis "only one time from top to bottom"

Closed-loop Asymptotic Stability $\Longleftrightarrow$
- $\mu > 0$
- $\varphi_m > 0$

The assumption on **positivity of the loop-gain** $\mu > 0$ is important to avoid cases like:



$\varphi_m > 0$ **BUT** closed-loop unstable

A Useful Empirical Criterion for Minimum-Phase Systems

Recall from Part 8, slide 19:

- **Minimum Phase Systems** are characterized by:
  - positive gain: $\mu > 0$
  - all poles and zeros located in the left half-plane
- For minimum phase systems there is a **direct relation among the approximate Bode diagrams of magnitude and phase**:

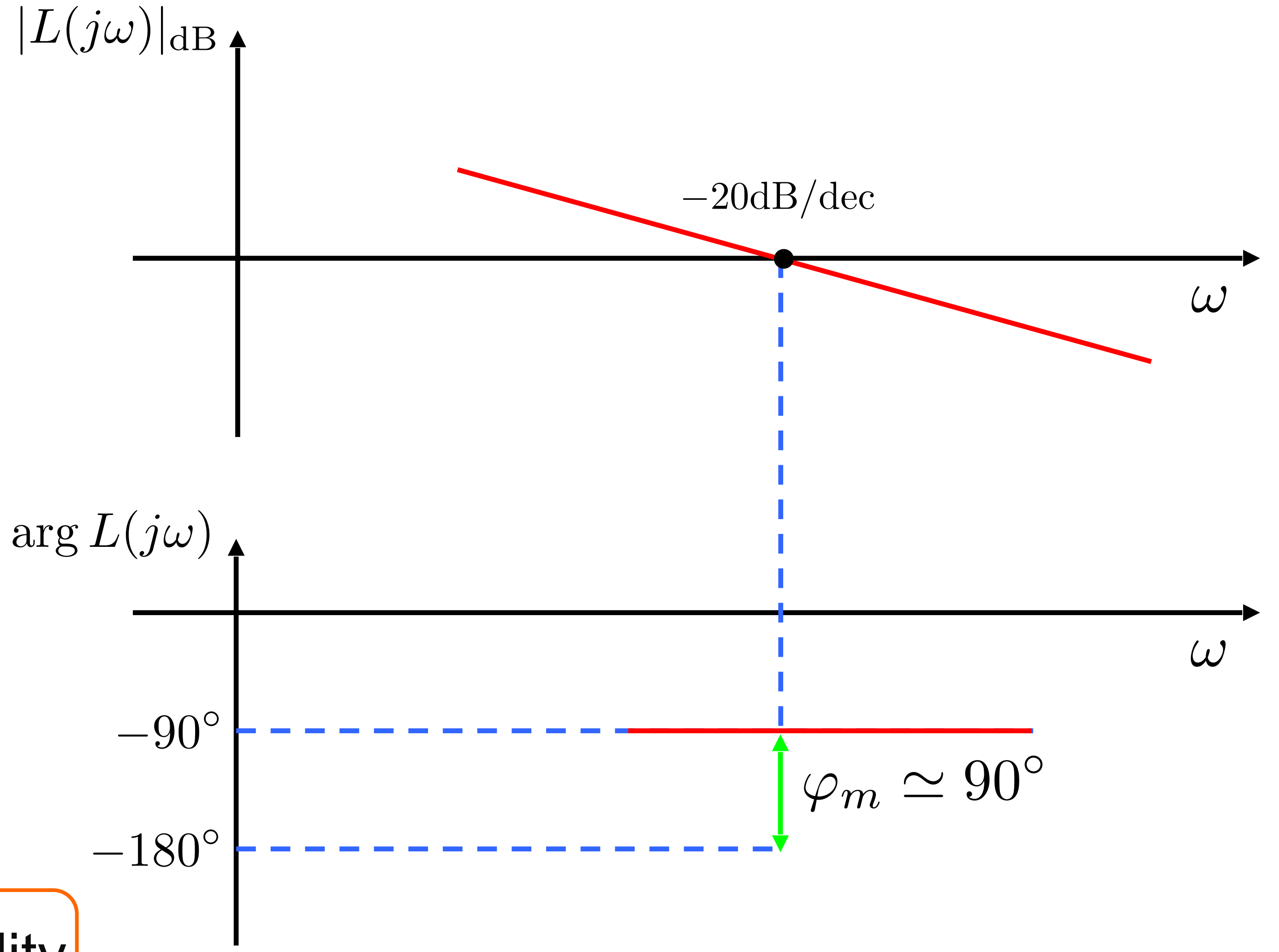
	Slope of $\lvert L(j\omega)\rvert_{\mathrm{dB}}$	Value of $\arg L(j\omega)$
pole	$-20\mathrm{dB/dec}$	-90°
zero	$20\mathrm{dB/dec}$	90°

A Useful Empirical Criterion for Minimum-Phase Systems (contd.)

If:

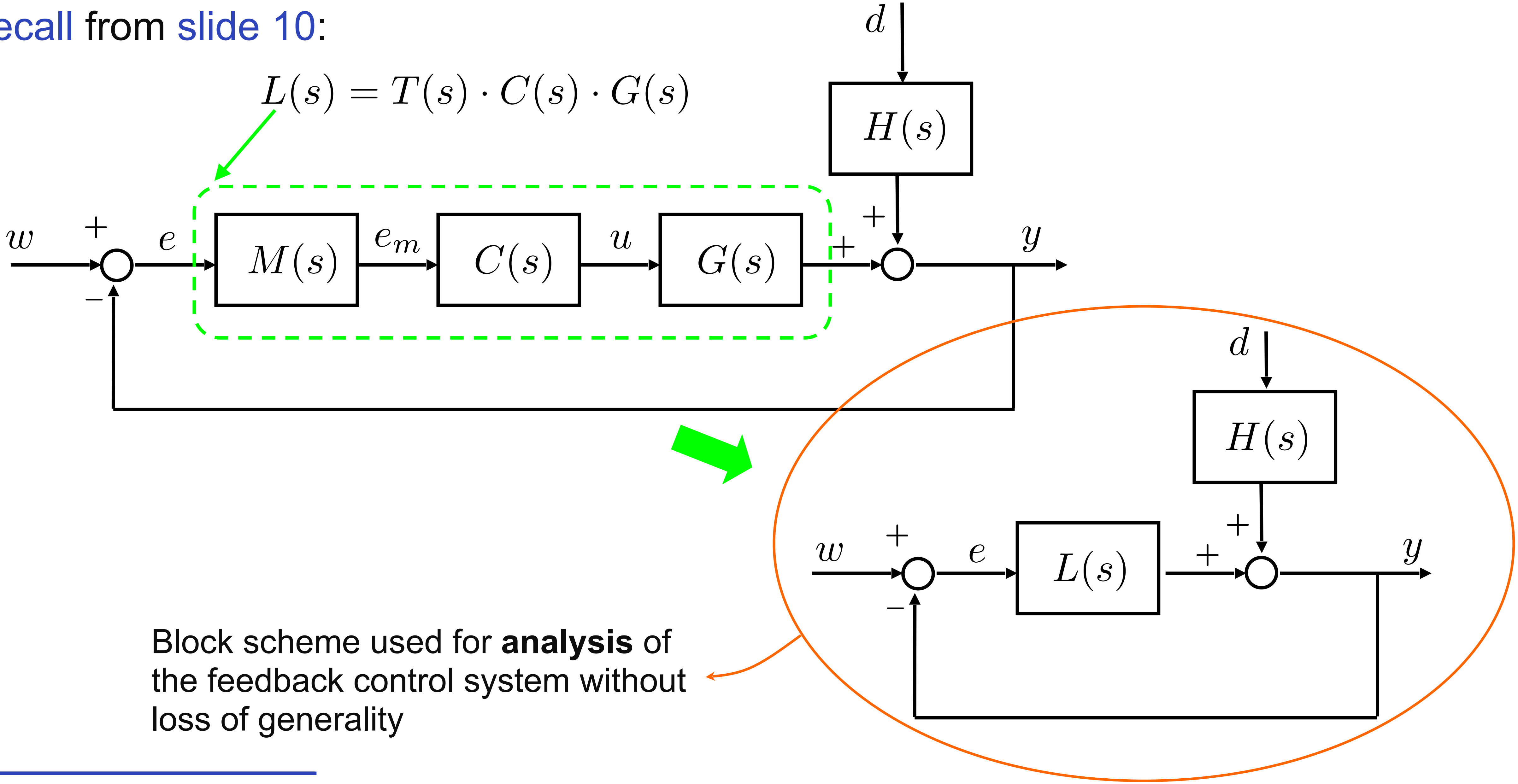
- The open-loop system is **minimum phase**
- The magnitude diagram $|L(j\omega)|_{\mathrm{dB}}$ **crosses the 0dB axis only one time from top to bottom**
- The **slope** of the asymptotic magnitude Bode diagram is **-20 dB/dec** in a **sufficiently large** frequency range around the 0dB crossing

↳ Closed-loop Asymptotic Stability



Analysis of Feedback Control Systems

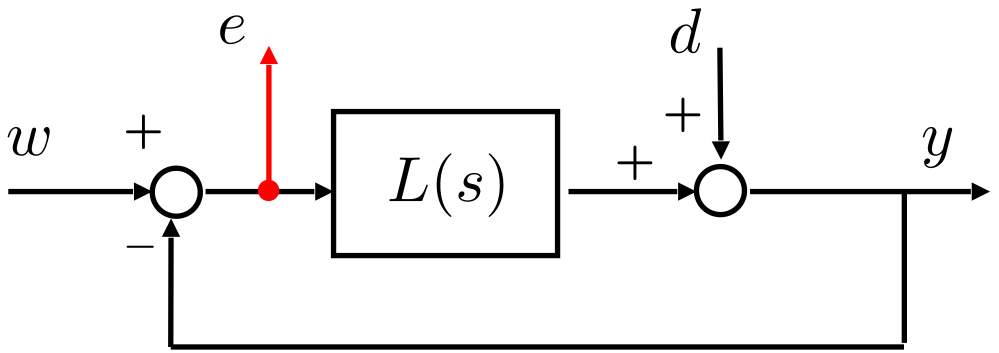
Recall from slide 10:



Block scheme used for **analysis** of the feedback control system without loss of generality

Analysis of Feedback Control Systems (contd.)

For the time being suppose that $H(s) = 1$:



Hence:

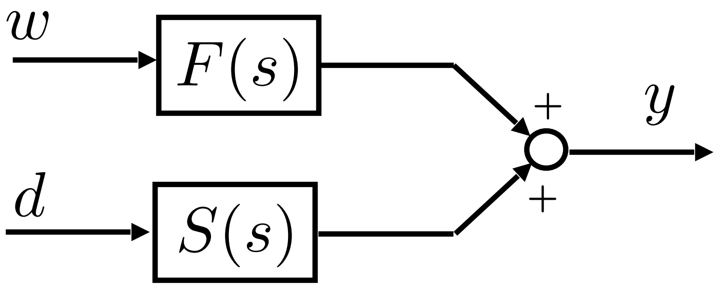
$$\frac{Y(s)}{W(s)} = \frac{L(s)}{1+L(s)} = F(s)\,; \qquad \frac{Y(s)}{D(s)} = \frac{1}{1+L(s)} = S(s)$$

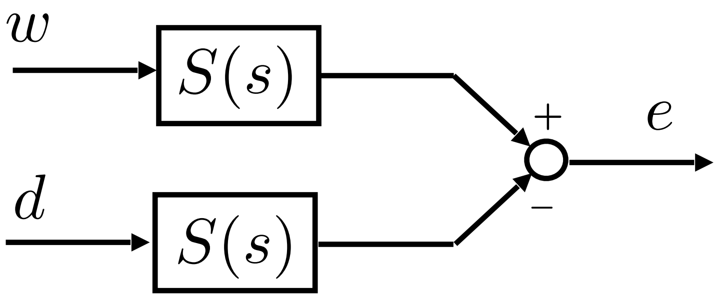
$$\frac{E(s)}{W(s)} = \frac{1}{1+L(s)} = S(s)\,; \qquad \frac{E(s)}{D(s)} = \frac{-1}{1+L(s)} = -S(s)$$

Define:

- **Complementary Sensitivity Function**: $F(s)$
- **Sensitivity Function**: $S(s)$

Hence:





Ideal Performance

$$F(s) = \frac{L(s)}{1+L(s)} \simeq 1$$

$$S(s) = \frac{1}{1+L(s)} \simeq 0$$

Moreover:

$$\boxed{F(s) + S(s) = 1}$$

Static Analysis of Complementary Sensitivity Function

Using the parametrisation of the open-loop t.f. $L(s)$ in slides 32-33 Part IV:

$$L(s) = \mu \frac{1}{s^g} \boxed{\frac{\prod_l \left(1+\frac{s}{z_l}\right)}{\prod_i \left(1+\frac{s}{p_i}\right)} \frac{\prod_h \left(1+\frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_k \left(1+\frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}} \xrightarrow[s \to 0]{} 1$$

Asymptotic value of the **closed-loop step-response**:

$$w(t) = A \cdot 1(t) \quad \Rightarrow \quad y(\infty) = \lim_{s\to 0} \cancel{s} \cdot F(s) \cdot \frac{A}{\cancel{s}} = A \lim_{s\to 0} F(s)$$

$$= A \lim_{s\to 0} \frac{L(s)}{1+L(s)} = A \lim_{s\to 0} \frac{\frac{\mu}{s^g}}{1+\frac{\mu}{s^g}}$$

$$= A \lim_{s\to 0} \frac{\mu}{s^g+\mu}$$

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Therefore:

$$y(\infty) = \begin{cases} A \cdot \dfrac{\mu}{1+\mu} \ \ (\text{hence } \mu_F \simeq 1 \text{ if } \mu \gg 1) & \text{if } g = 0 \\ A \ \ (\text{hence } \mu_F = 1) & \text{if } g > 0 \\ 0 & \text{if } g < 0 \end{cases}$$

Presence of an integrator (pole = 0) in the direct path of the feedback loop

We have:

$$L(s) = \frac{N(s)}{\varphi(s)} \quad \Longrightarrow \quad F(s) = \frac{L(s)}{1 + L(s)} = \frac{N(s)}{\varphi(s) + N(s)}$$

Therefore:

- the zeros of $F(s)$ coincide with the zeros of $L(s)$
- the poles of $F(s)$ coincide with the roots of $\varphi(s) + N(s)$

Frequency Response Analysis of Complementary Sensitivity Function

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We have:

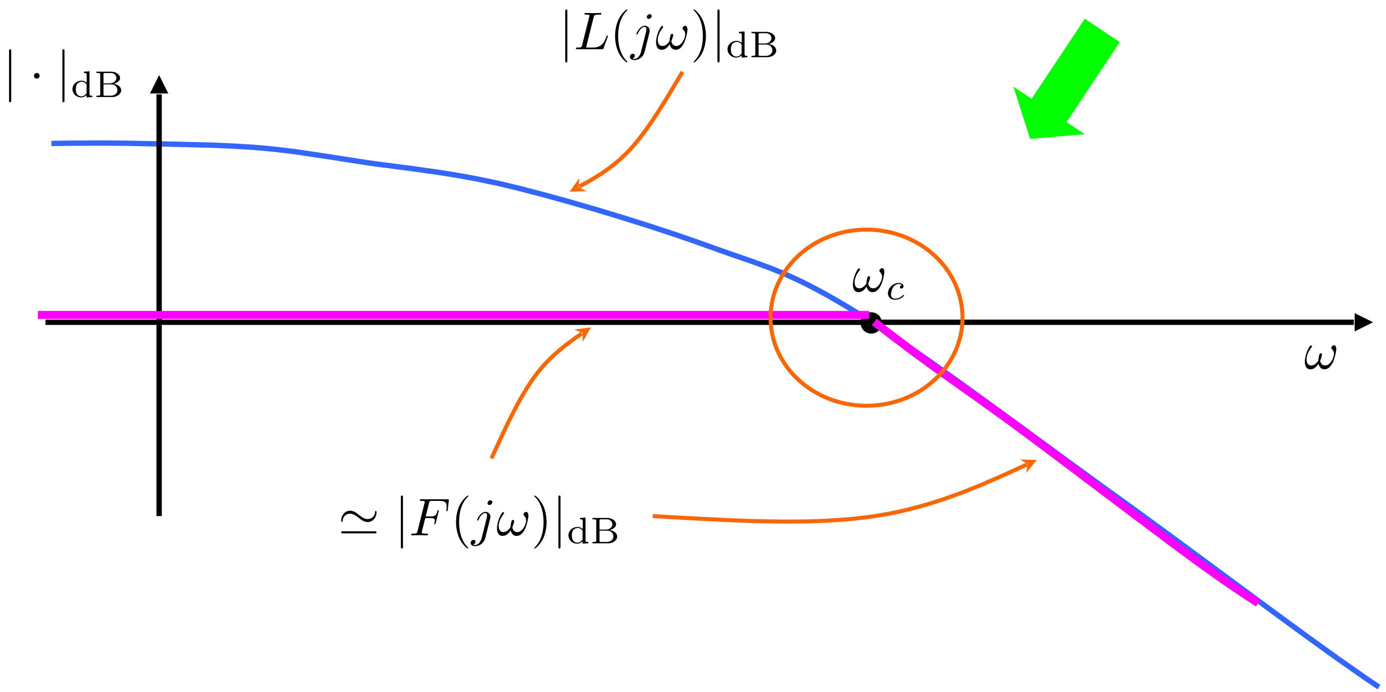
$$|F(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} 1 & \text{if } |L(j\omega)| \gg 1 \\ |L(j\omega)| & \text{if } |L(j\omega)| \ll 1 \end{cases}$$

- $F(s)$ **low-pass** filter with bandwidth $B \simeq [0, \omega_c]$
- Gain:

$$\mu_F = \begin{cases} 1 & \text{if } g > 0 \\ \dfrac{\mu}{1+\mu} & \text{if } g = 0 \end{cases}$$

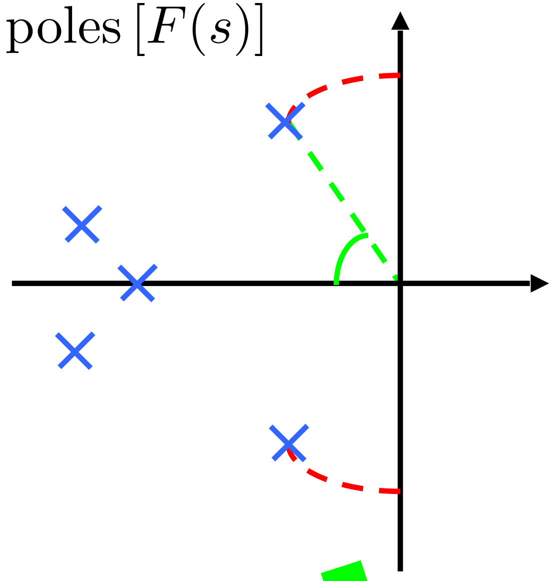
- Dominant poles:
  - if real: ➡ $\tau \simeq \dfrac{1}{\omega_c}$
  - if complex: ➡ $\omega_n \simeq \omega_c$

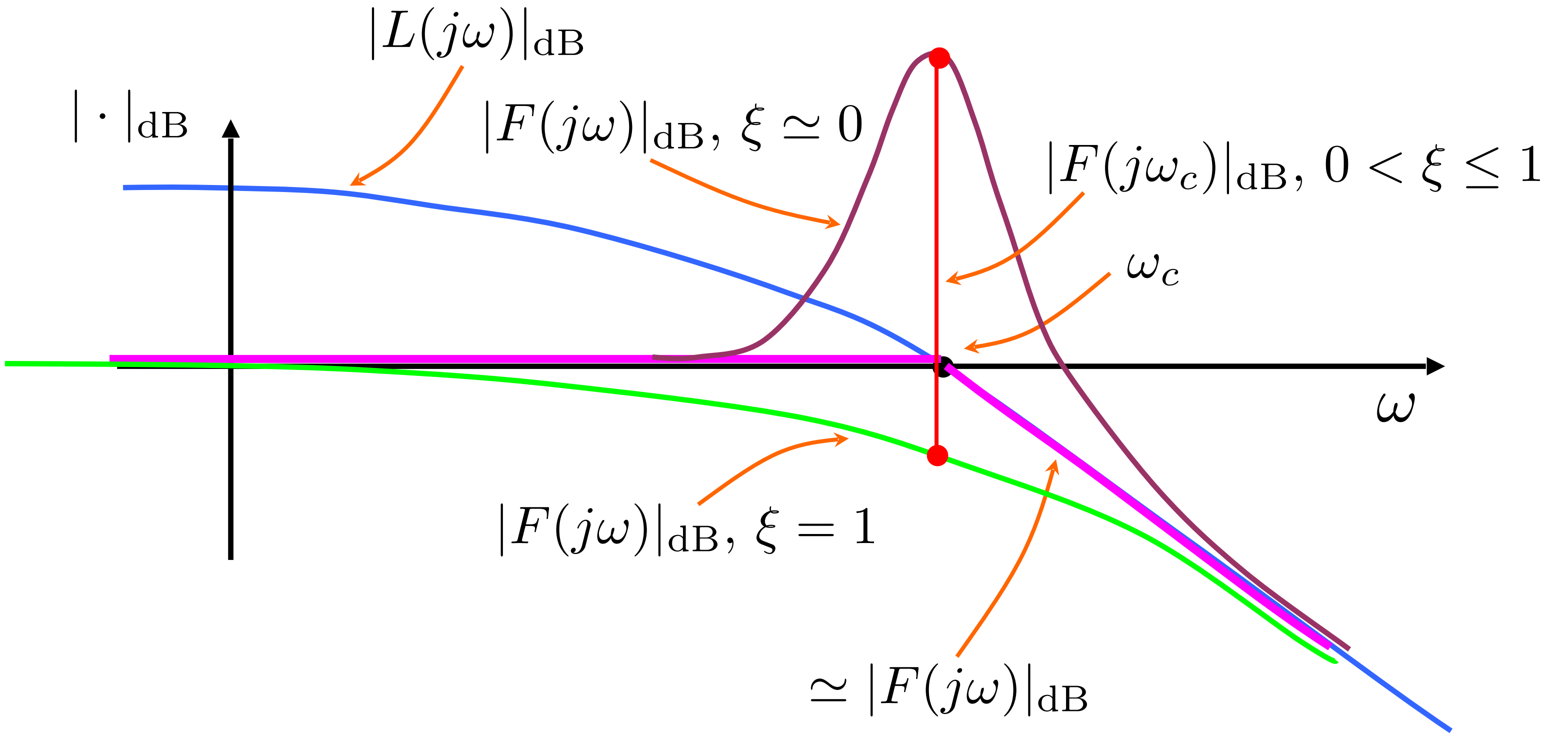
a typical scenario



Closed-Loop Damping Ratio and Phase Margin

But: how about the damping ratio of the dominant poles of $F(s)$?





small damping ratio $\xi \simeq 0$ ➡ "fragile" stability ➡ $\varphi_m \simeq 0$

Question: $\varphi_m \simeq 0 \xrightarrow{?} \xi \simeq 0$

Closed-Loop Damping Ratio and Phase Margin (contd.)

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Let us determine $|F(j\omega_c)|$:

$$|L(j\omega_c)| = 1 \Rightarrow L(j\omega_c) = 1 \cdot e^{j\varphi_c} \quad \text{with} \quad \varphi_c = \arg L(j\omega_c)$$

$$|F(j\omega_c)| = \frac{|L(j\omega_c)|}{|1 + L(j\omega_c)|} = \frac{1}{|1 + e^{j\varphi_c}|} = \frac{1}{|1 + \cos\varphi_c + j\sin\varphi_c|}$$

$$= \frac{1}{\sqrt{(1 + \cos\varphi_c)^2 + \sin^2\varphi_c}} = \frac{1}{\sqrt{1 + \cos^2\varphi_c + 2\cos\varphi_c + \sin^2\varphi_c}}$$

$$= \frac{1}{\sqrt{2(1 + \cos\varphi_c)}} = \frac{1}{\sqrt{2(1 - \cos\varphi_m)}}$$

$$= \frac{1}{2\sin(\frac{\varphi_m}{2})} \quad \text{where} \quad \varphi_m = \varphi_c - (-180^\circ)$$

Closed-Loop Damping Ratio and Phase Margin (contd.)

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Suppose that $F(s)$ takes on the form of a **second-order** system:

$$F(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

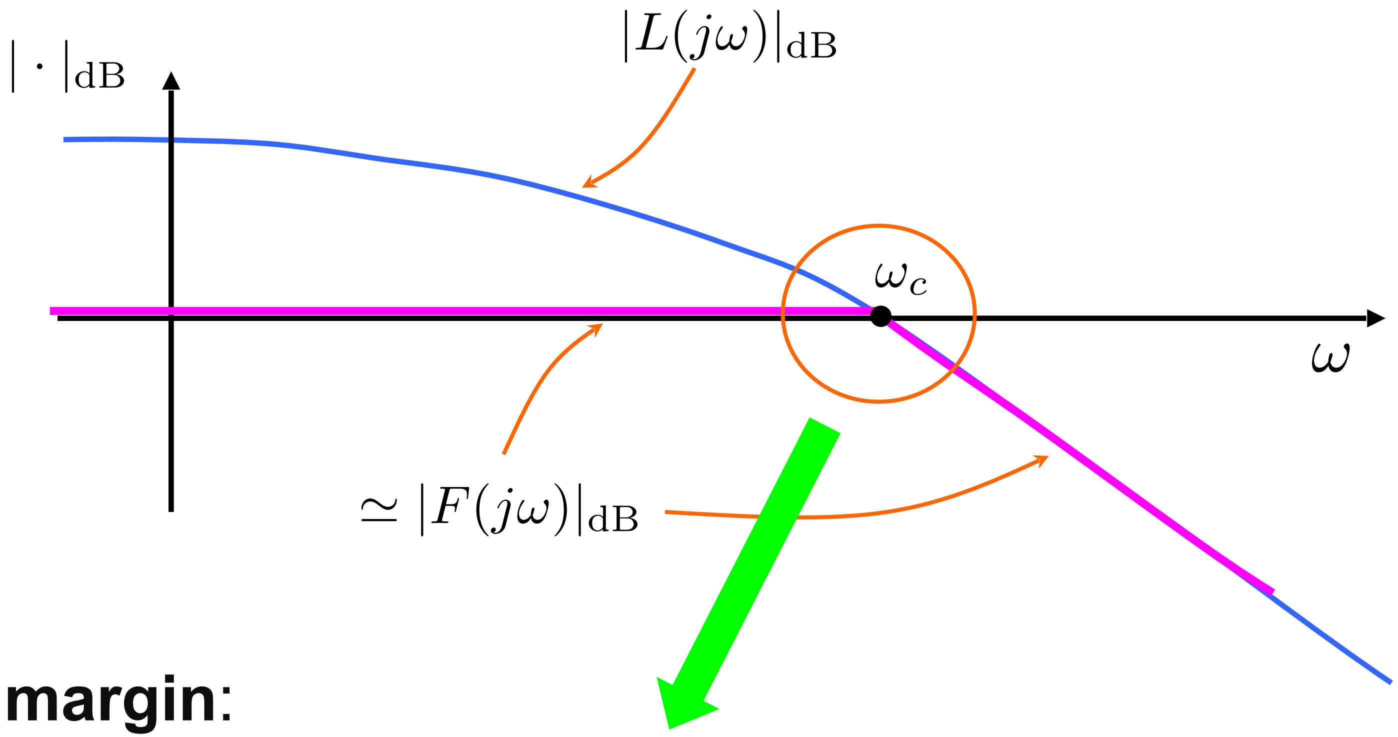
$$|F(j\omega_n)| = \left| \frac{\cancel{\omega_n^2}}{-\cancel{\omega_n^2} + 2j\xi\cancel{\omega_n^2} + \cancel{\omega_n^2}} \right| = \left| \frac{1}{j2\xi} \right| = \frac{1}{2\xi}$$

Consider the **approximation** $\omega_n \simeq \omega_c$

$$|F(j\omega_c)| = \frac{1}{2\sin(\frac{\varphi_m}{2})} \simeq |F(j\omega_n)| = \frac{1}{2\xi}$$

$$\boxed{\xi \simeq \sin\left(\frac{\varphi_m}{2}\right) \simeq \frac{\varphi_m}{2} \cdot \frac{\pi}{180} \simeq \frac{\varphi_m}{100}}$$

Empirical/Practical Rule



Phase margin:

- $\varphi_m > 75^\circ$: one real dominant pole with $\tau \simeq \dfrac{1}{\omega_c}$
- $\varphi_m < 75^\circ$: two complex conjugate poles with $\omega_n \simeq \omega_c\,;\ \xi \simeq \dfrac{\varphi_m}{100}$

Static Analysis of Sensitivity Function

Using the parametrisation of the open-loop t.f. $L(s)$ in slides 32-33 Part IV:

$$L(s) = \mu \, \frac{1}{s^g} \boxed{\frac{\prod_l \left(1 + \frac{s}{z_l}\right)}{\prod_i \left(1 + \frac{s}{p_i}\right)} \, \frac{\prod_h \left(1 + \frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_k \left(1 + \frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}} \xrightarrow[s \to 0]{} 1$$

Asymptotic value of the **closed-loop step-response**:

$$d(t) = A \cdot 1(t) \quad \Rightarrow \quad y(\infty) = \lim_{s \to 0} \cancel{s} \cdot S(s) \cdot \frac{A}{\cancel{s}} = A \lim_{s \to 0} S(s)$$

$$= A \lim_{s \to 0} \frac{1}{1 + L(s)} = A \lim_{s \to 0} \frac{1}{1 + \dfrac{\mu}{s^g}}$$

$$= A \lim_{s \to 0} \frac{s^g}{s^g + \mu}$$

Static Analysis of Sensitivity Function (contd.)

Therefore:

$$y(\infty) = \begin{cases} A \cdot \dfrac{1}{1+\mu} \;\; (\text{hence } \mu_S \simeq 0 \text{ if } \mu >> 1) & \text{if } g = 0 \\ 0 & \text{if } g > 0 \\ A \;\; (\text{hence } \mu_S = 1) & \text{if } g < 0 \end{cases}$$

Presence of an integrator (pole = 0) in the direct path of the feedback loop

Pole-Zero Analysis of Sensitivity Function

We have:

$$L(s) = \frac{N(s)}{\varphi(s)} \quad \Rightarrow \quad S(s) = \frac{1}{1+L(s)} = \frac{\varphi(s)}{\varphi(s)+N(s)}$$

Therefore:

- the zeros of $S(s)$ coincide with the poles of $L(s)$
- the poles of $S(s)$ coincide with the roots of $\varphi(s) + N(s)$

  ↳ same as for $F(s)$

Frequency Response Analysis of Sensitivity Function

We have:

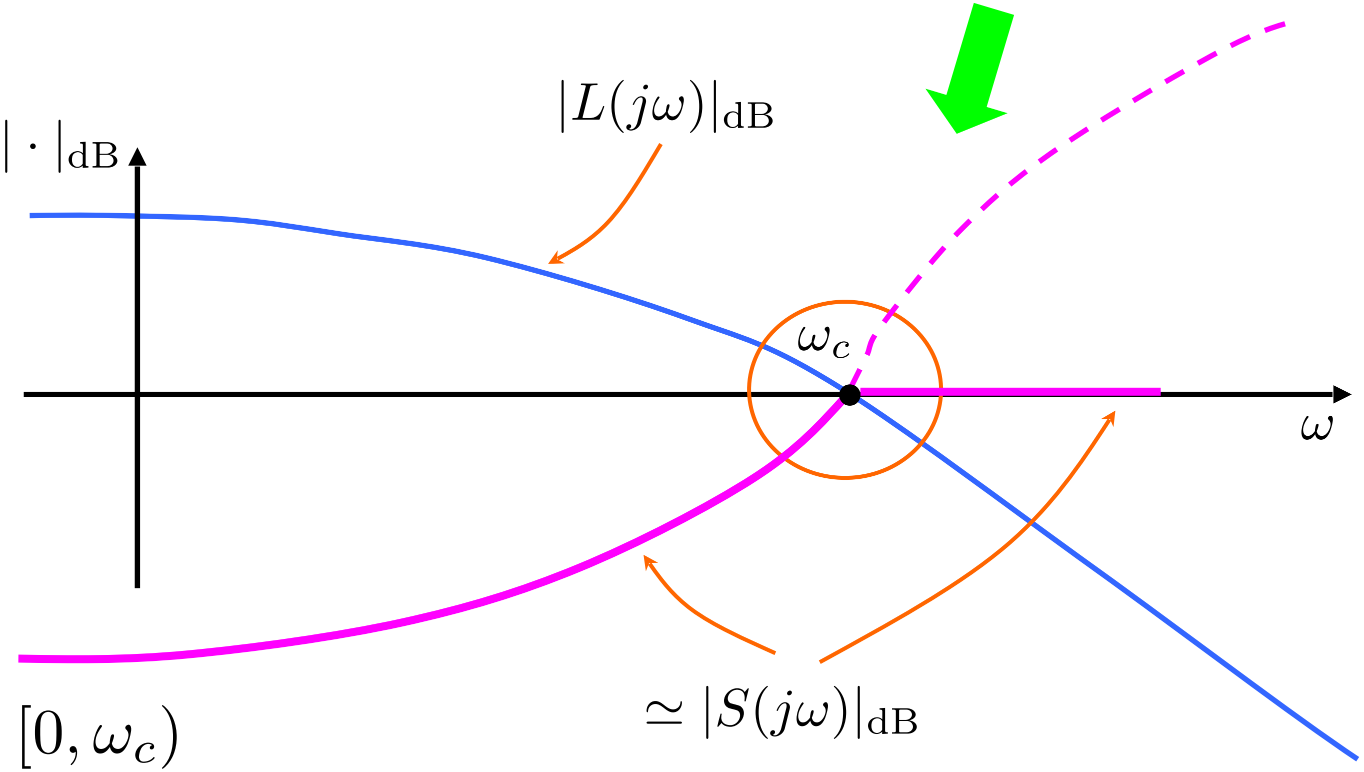
$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \simeq \begin{cases} 1 & \text{if } |L(j\omega)| \ll 1 \\ \dfrac{1}{|L(j\omega)|} = -|L(j\omega)|_{\mathrm{dB}} & \text{if } |L(j\omega)| \gg 1 \end{cases}$$

- $S(s)$ **high-pass** filter with bandwidth $B \simeq [\omega_c, \infty)$
- **Gain**:

$$\mu_S = \begin{cases} 0 & \text{if } g > 0 \\ \dfrac{1}{1+\mu} & \text{if } g = 0 \end{cases}$$

- **Disturbance attenuation**:
  - in $B \simeq [0, \omega_c)$
  - improve attenuation by:
    - increasing ω_c
    - increasing $|L(j\omega)|_{\mathrm{dB}}$, $\omega \in [0, \omega_c)$

a typical scenario



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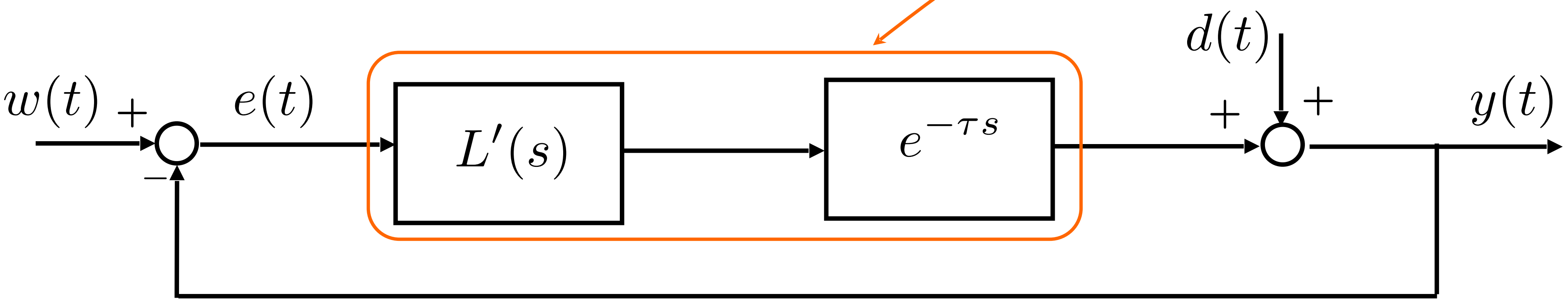
Recalling Part 8, slide 31, we get:

$$L(s) = L'(s) \cdot e^{-\tau s}$$



$$\begin{cases} |L(j\omega)| = |L'(j\omega)| \cdot |e^{j\omega\tau}| = |L'(j\omega)| \\ \arg L(j\omega) = \arg L'(j\omega) - \omega\tau \dfrac{180}{\pi} \end{cases}$$

The presence of a **delay** in a closed-loop system has a significant impact on **stability** and **dynamic performance.**

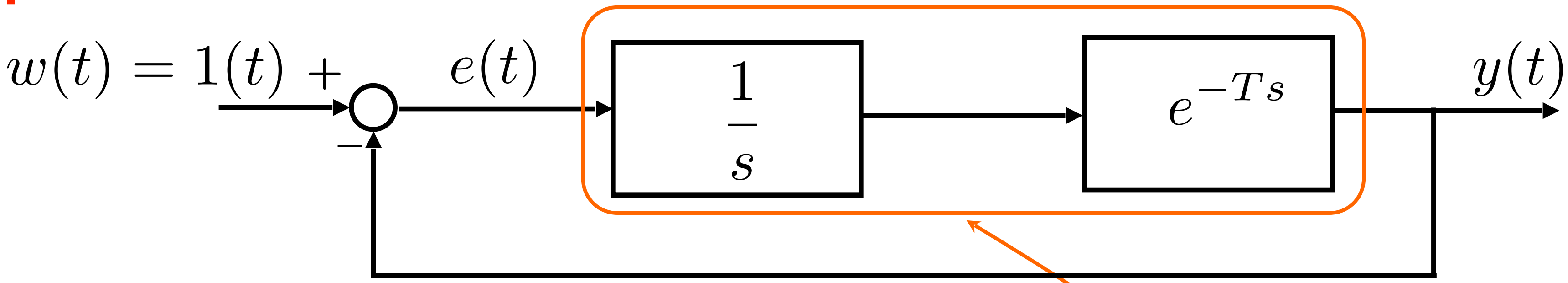
- The **static analysis is unchanged:**

$$\lim_{s \to 0} e^{-\tau s} = 1$$

- The delay **modifies the dynamic analysis**:
  - the critical angular frequency ω_c does not change (the magnitude Bode diagram does not change)
  - the critical phase φ_c decreases
  - the phase margin φ_m decreases $\Longrightarrow$ ξ decreases

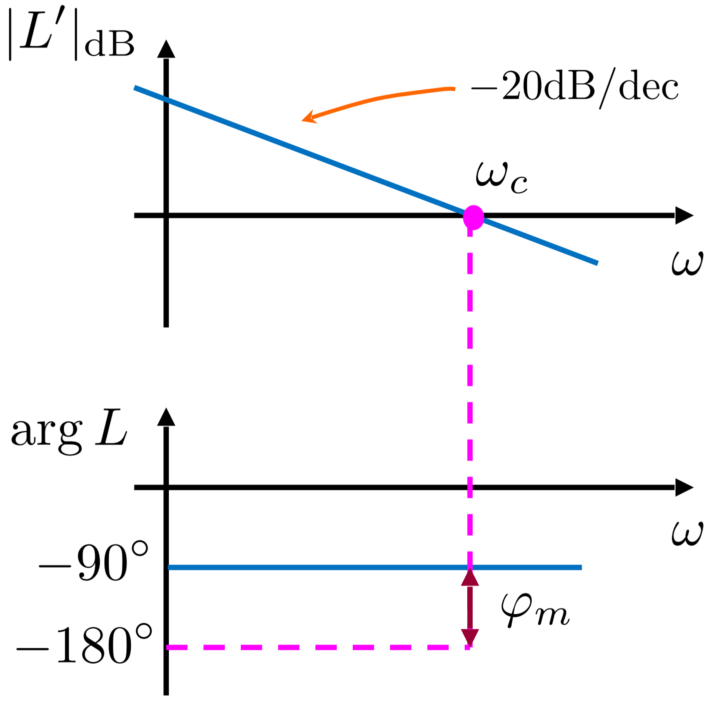
$\Rightarrow$ **oscillations**

Example



$$L(s) = L'(s) \cdot e^{-Ts} = \frac{1}{s} \cdot e^{-Ts}$$

- Case 1: $T = 0\,,\;\; y(\infty) = 1$



$$\begin{cases} \omega_c = 1 \\ \varphi_m = 90° \;\; (\varphi_c = -90°) \end{cases}$$

dominant real pole with $\tau \simeq \dfrac{1}{\omega_c} = 1$

Example (contd.)

- Case 2: $T > 0\,,\;\; y(\infty) = 1$

$$\begin{aligned} \omega_c &= 1 \\ \varphi_c &= -90^\circ - \omega_c T \cdot \frac{180^\circ}{\pi} \\ &= -90^\circ - T \cdot \frac{180^\circ}{\pi} \end{aligned}$$

$$\varphi_m = 90^\circ - T \cdot \frac{180^\circ}{\pi}$$

Hence:

closed-loop asymptotic stability $\Longleftrightarrow$ $T < \dfrac{\pi}{2} \simeq 1.57\,\mathrm{sec}$

Example (contd.)

Consider the specific case: $T = 1$

$$\varphi_m = 90^\circ - T \cdot \frac{180^\circ}{\pi} = 90^\circ - 57^\circ = 33^\circ$$

Hence, recalling slide 43:

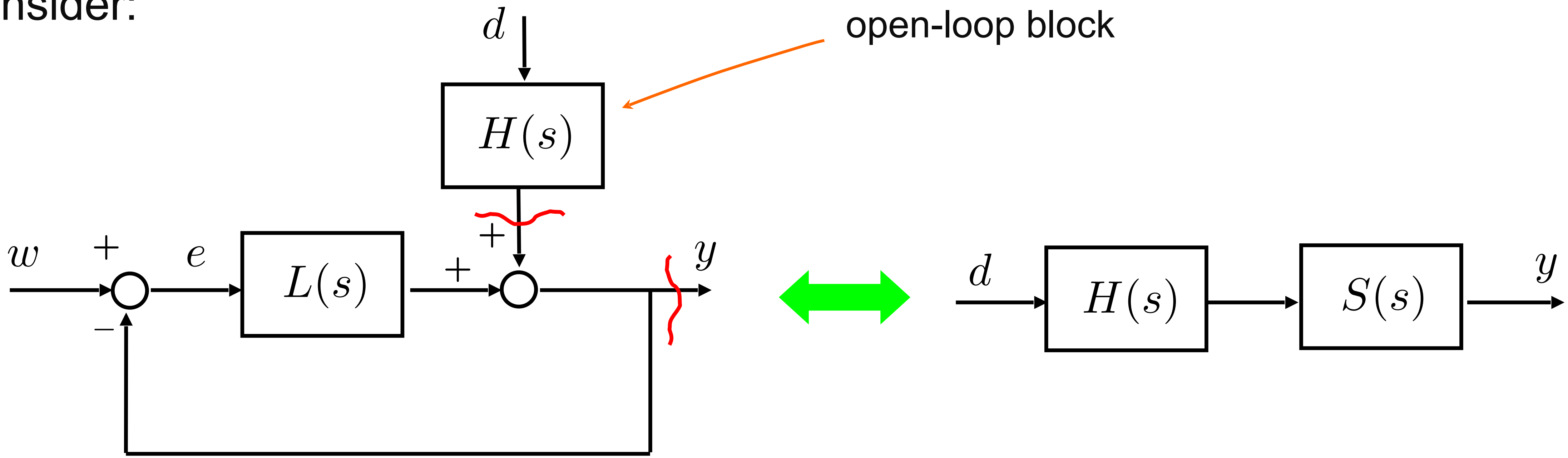
- $\varphi_m > 75^\circ$: one dominant real pole with $\tau \simeq \dfrac{1}{\omega_c}$

- $\varphi_m < 75^\circ$: two dominant complex conjugate poles with $\omega_n \simeq \omega_c\,;\ \xi \simeq \dfrac{\varphi_m}{100}$

$$\begin{cases} \omega_n \simeq \omega_c = 1 \\ \xi \simeq \dfrac{\varphi_m}{100} \simeq 0.33 \end{cases} \longrightarrow \quad t_s \simeq \frac{4.6}{\xi \omega_n} \simeq 14\,\text{sec !!!}$$

Presence of Open-Loop Blocks

Consider:



Clearly:

closed-loop asymptotic stability $\Longleftrightarrow$ $\begin{cases} H(s) \text{ asymptotically stable} \\ \quad \rightarrow \ \mathrm{Re}\ (\text{poles}) < 0 \\ S(s) \text{ asymptotically stable} \end{cases}$

$\rightarrow$ Bode, Nyquist

Analysis of the Effect of the Disturbance on the Output

Clearly: $R(s) = \dfrac{H(s)}{1+L(s)} = H(s) \cdot S(s)$

- **Static** Analysis:

  Without loss of generality, suppose $g_H = 0\,,\ g_L = 0\ :$

$$d(t) = A \cdot 1(t) \implies y(\infty) = \lim_{s\to 0} \cancel{s} \cdot R(s) \cdot \frac{A}{\cancel{s}} = A \lim_{s\to 0} R(s)$$

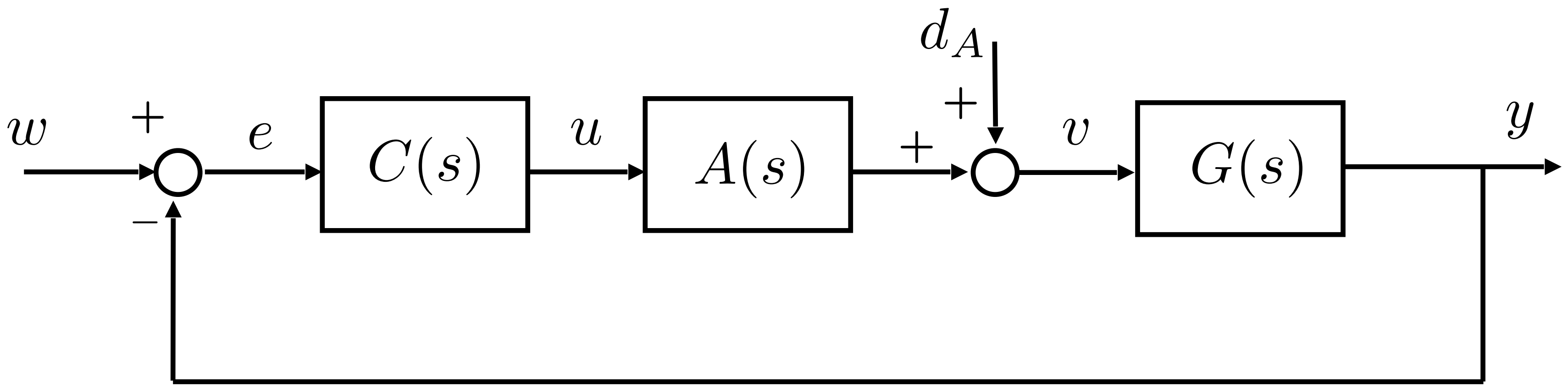
$$= A \cdot \frac{\mu_H}{1+\mu_L} = A \cdot \mu_H \cdot \mu_S$$

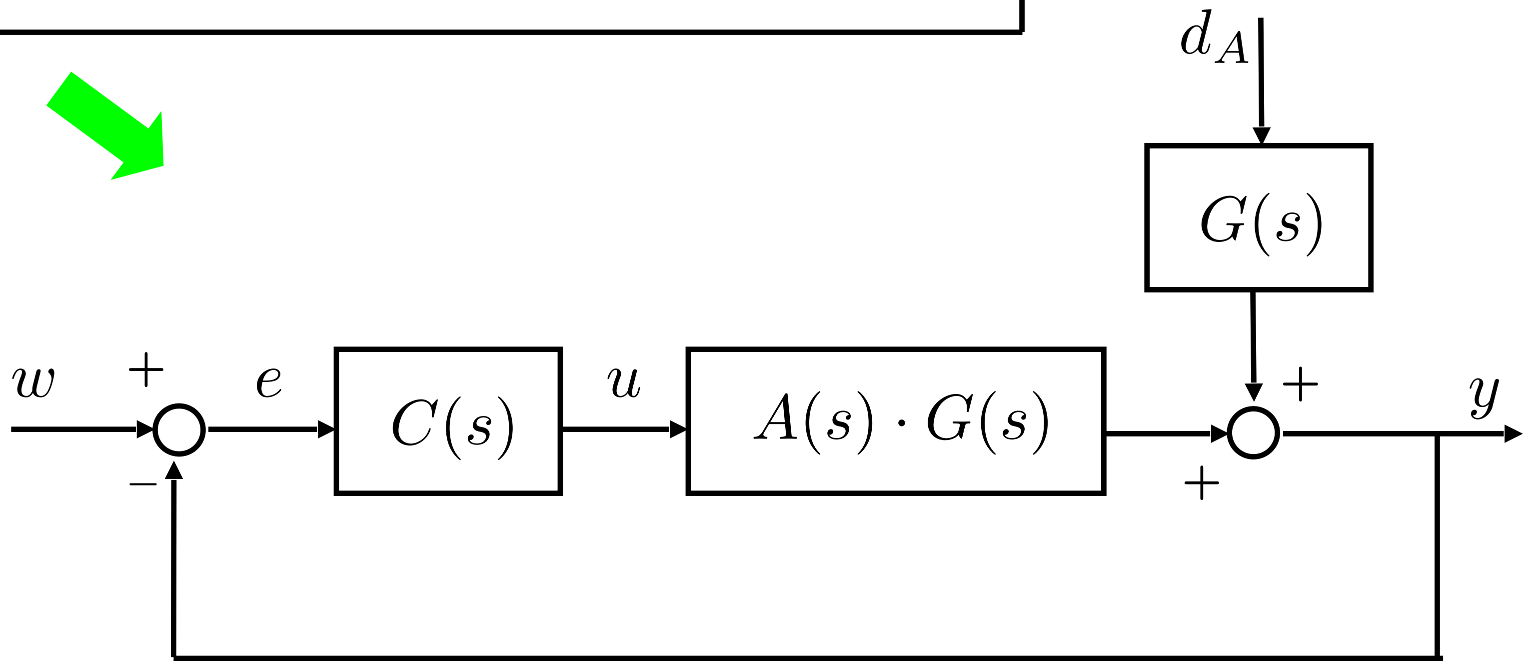
- **Dynamic** Analysis:

$$|R(j\omega)| = |H(j\omega)| \cdot |S(j\omega)| \implies |R(j\omega)|_{\mathrm{dB}} = |H(j\omega)|_{\mathrm{dB}} + |S(j\omega)|_{\mathrm{dB}}$$

Disturbance on the Actuators

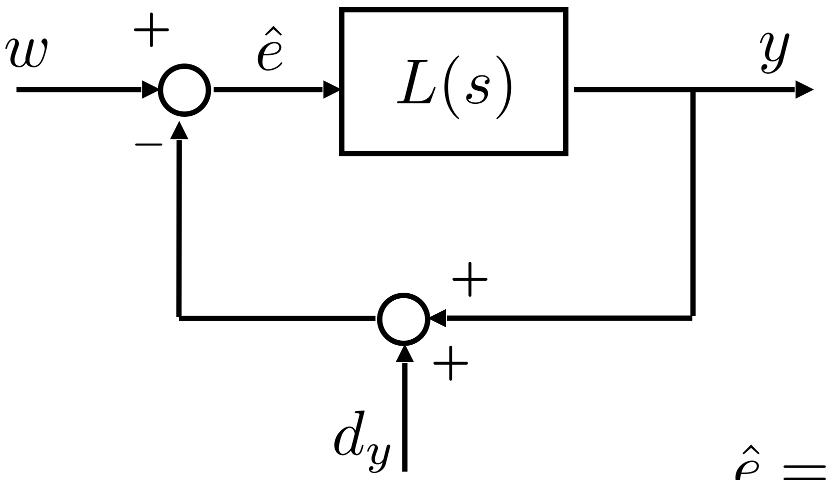
Recalling slide 10 (suppose $M(s) = 1$ for simplicity):

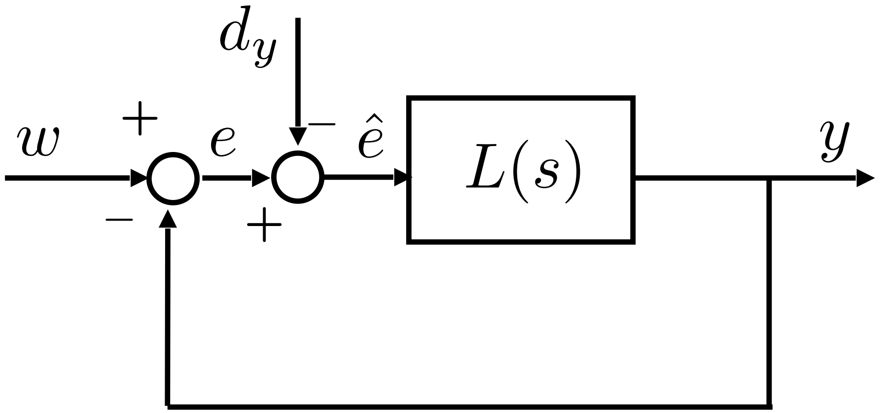




Disturbance on the Output Measurements

This scenario represent the presence of disturbances on sensors, very frequent in applications





$$\hat{e} = w - (y + d_y) = w - y - d_y = e - d_y$$

Hence:

$$\frac{E(s)}{D_y(s)} = \frac{L(s)}{1 + L(s)} = F(s)$$

low-pass filter with bandwidth $B \simeq [0, \omega_c]$

low-frequency disturbances $d_y(t)$ influence **directly** the error $e(t)$

Disturbance on the Output Measurements (contd.)

- **Static** Analysis:

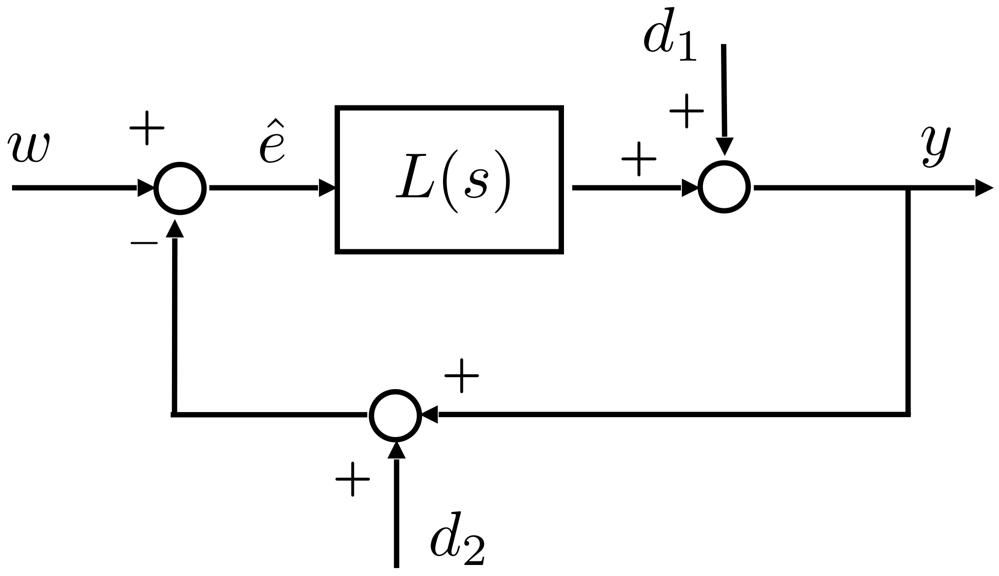
$$d_y(t) = A \cdot 1(t)$$

$$e(\infty) = \lim_{s \to 0} \cancel{s} \cdot F(s) \frac{1}{\cancel{s}} = \begin{cases} A & \text{if } g > 0 \\ A \cdot \dfrac{\mu}{1+\mu} & \text{if } g = 0 \end{cases}$$

Hence, rejection of disturbances on the direct path is in contrast with rejection of disturbances on the feedback path

Disturbances Rejection – Concluding Remarks

Consider:



A standard feedback control system allows to reject simultaneously:

- **low-frequency** disturbances d_1 on the **direct** path with $\omega \ll \omega_c$
- **high-frequency** disturbances d_2 on the **feedback** path with $\omega \gg \omega_c$

Analysis of Feedback Control Systems Methods and Tools in the s-Domain: The Root-Locus Technique

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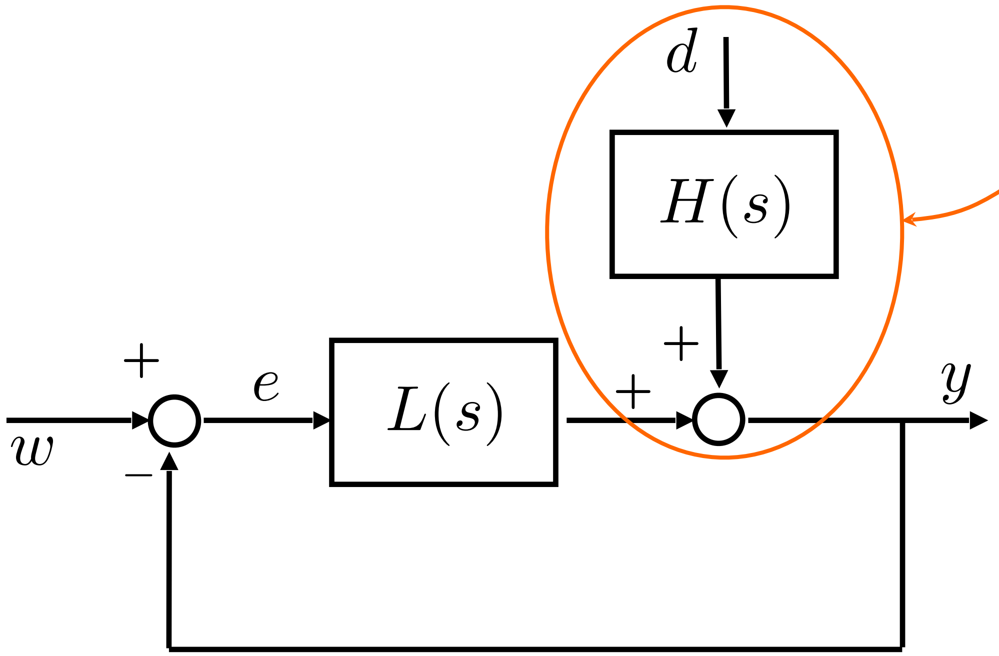
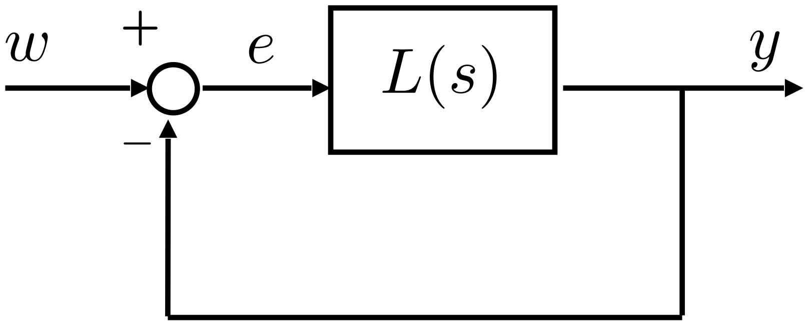
Refer to the usual scheme feedback control scheme:



Open-loop block not affecting closed-loop stability analysis



The root locus technique is a *graphical method* relating the **location in the complex plane of the closed-loop poles,** that is, **the roots of**

$$\boxed{1 + L(s) = 0}$$

with the location in the complex plane of the open-loop zeros and poles and the **open-loop gain**.

The Root Locus Technique - Advantages and Disadvantages

The RL technique has distinct advantages and disadvantages in analysing and designing feedback control systems with respect to frequency domain tools:

- RL enables to obtain a graphical evidence on the location of **all** closed-loop poles and not only the location of the closed-loop **dominant** poles as is for frequency domain techniques (see slides Part 9, 40-44)
- RL technique also can be applied in the presence of **unstable open-loop poles**
- More in general, the RL tool can be applied in many scenarios in which the Bode criterion is not applicable
- The RL cannot be applied when the open-loop transfer function $L(s)$ is not rational (i.e., a fraction of polynomial numerator and denominator). The typical case is when $L(s)$ contains **delay blocks**

We refer to the following parameterisation of $L(s)$ using poles and zeros (see slide Part 4, 28):

$$L(s) = \varrho \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \varrho \frac{\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)}$$

The RL diagram shows in the complex plane for all $\varrho \in \mathbb{R}$, $\varrho \neq 0$ the roots of

$$1 + L(s) = 1 + \varrho \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 1 + \varrho \frac{\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)} = 0$$

- The **direct RL** refers to all values $\varrho > 0$
- The **inverse RL** refers to all values $\varrho < 0$
- For $\varrho = 0$ the feedback loop ``degenerates" and the closed-loop poles coincide with the open-loop ones

Illustrative Example:

Consider the open-loop transfer function $L(s) = \varrho \dfrac{s+2}{s(s+1)}$. We have:

$$1 + L(s) = 1 + \varrho \frac{s+2}{s(s+1)} = \frac{s(s+1) + \varrho\,(s+2)}{s(s+1)} = \frac{s^2 + (1+\varrho)s + 2\varrho}{s(s+1)}$$

Hence, the closed-loop poles coincide with the roots of the equation

$$s^2 + (1+\varrho)s + 2\varrho = 0$$

and the location in the complex plane of these roots changes when ϱ changes thus giving rise to **changes** in **closed-loop stability** properties, as well as **changes** in characteristics of the **closed-loop transient behaviours**.

We have:

$$1 + L(s) = 0 \Rightarrow 1 + \varrho \frac{\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)} = 0 \Rightarrow \frac{\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)} = -\frac{1}{\varrho}$$

Thus:

- $$\frac{\prod_{j=1}^{m}|s+z_j|}{\prod_{i=1}^{n}|s+p_i|} = \frac{1}{|\varrho|}$$
- $$\arg\prod_{j=1}^{m}(s+z_j) - \arg\prod_{i=1}^{n}(s+p_i) = \begin{cases} (2k+1)\cdot 180^\circ,\ k\in\mathbb{Z}, & \text{if } \varrho > 0 \ \textbf{(Direct RL)} \\ 2k\cdot 180^\circ,\ k\in\mathbb{Z}, & \text{if } \varrho < 0 \ \textbf{(Inverse RL)} \end{cases}$$

- Relationship $\dfrac{\prod_{j=1}^{m} |s+z_j|}{\prod_{i=1}^{n} |s+p_i|} = \dfrac{1}{|\varrho|}$ associates every RL point with the values of ϱ

- Relationships

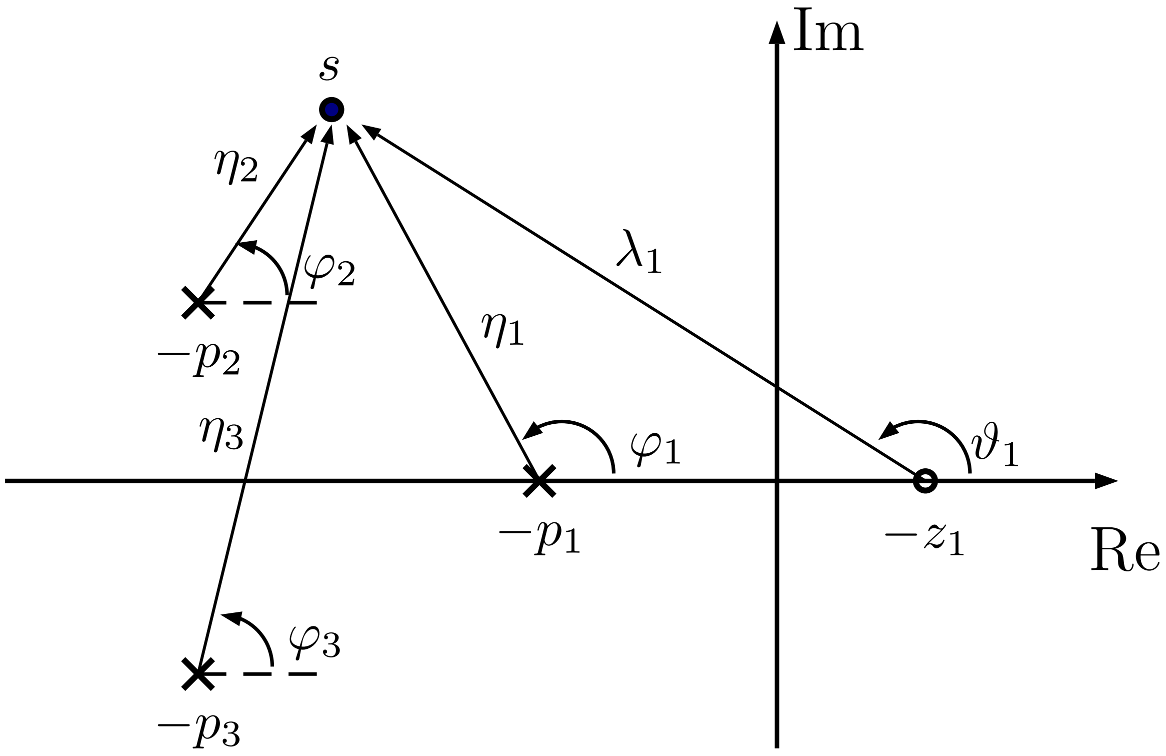
$$\arg \prod_{j=1}^{m} (s+z_j) - \arg \prod_{i=1}^{n} (s+p_i) = \begin{cases} (2k+1)\cdot 180^\circ,\ k \in \mathbb{Z}, & \text{if } \varrho > 0 \ \textbf{(Direct RL)} \\ 2k \cdot 180^\circ,\ k \in \mathbb{Z}, & \text{if } \varrho < 0 \ \textbf{(Inverse RL)} \end{cases}$$

fully characterise the shape of the Direct and Inverse RLs in the complex plane

The Root Locus Technique - Geometric Interpretation

From
$$\frac{\prod_{j=1}^{m} |s+z_j|}{\prod_{i=1}^{n} |s+p_i|} = \frac{1}{|\varrho|}$$

we get
$$\frac{\prod_{j=1}^{m} |s+z_j|}{\prod_{i=1}^{n} |s+p_i|} = \frac{\prod_{j=1}^{m} \lambda_j}{\prod_{i=1}^{n} \eta_i} = \frac{1}{|\varrho|}$$

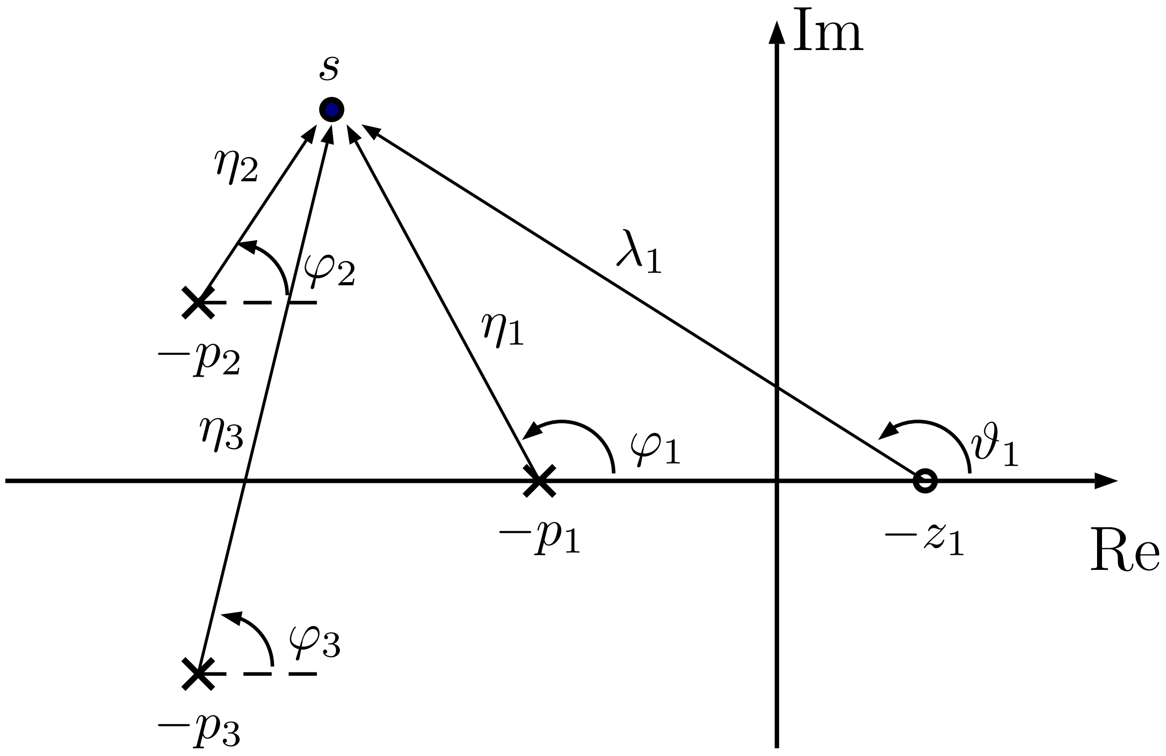


Moreover, from

$$\arg \prod_{j=1}^{m} (s+z_j) = \sum_{j=1}^{m} \arg(s+z_j) = \sum_{j=1}^{m} \theta_j$$

and

$$\arg \prod_{i=1}^{n} (s+p_i) = \sum_{i=1}^{n} \arg(s+p_i) = \sum_{j=1}^{n} \varphi_i$$



we get

$$\sum_{j=1}^{m} \theta_j - \sum_{i=1}^{n} \varphi_i = \begin{cases} (2k+1)\cdot 180^\circ,\ k \in \mathbb{Z}, & \text{if } \varrho > 0 \ \textbf{(Direct RL)} \\ 2k \cdot 180^\circ,\ k \in \mathbb{Z}, & \text{if } \varrho < 0 \ \textbf{(Inverse RL)} \end{cases}$$

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Illustrative Example:

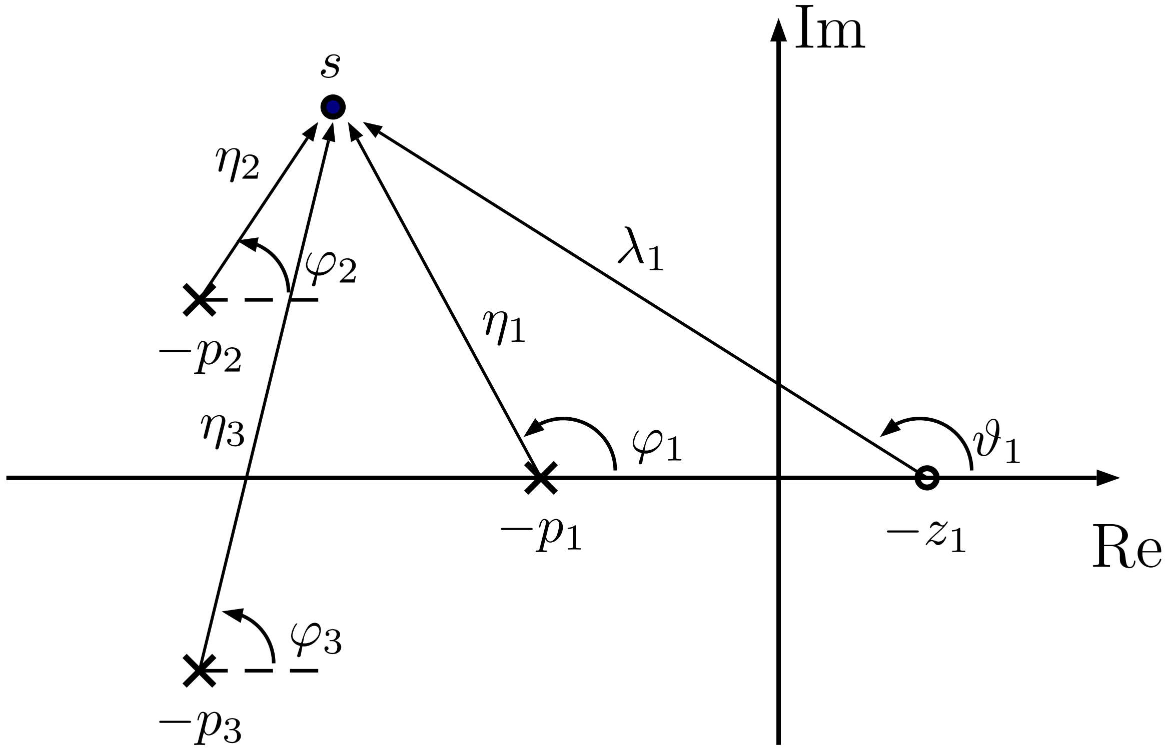
Consider the open-loop transfer function:

$$L(s) = \varrho \frac{s-1}{(s+2)(s^2+6s+13)}$$

The value of ϱ for the specific point s in the complex plane is:

$$|\varrho| = \frac{1}{\lambda_1} \prod_{i=1}^{3} \eta_i$$

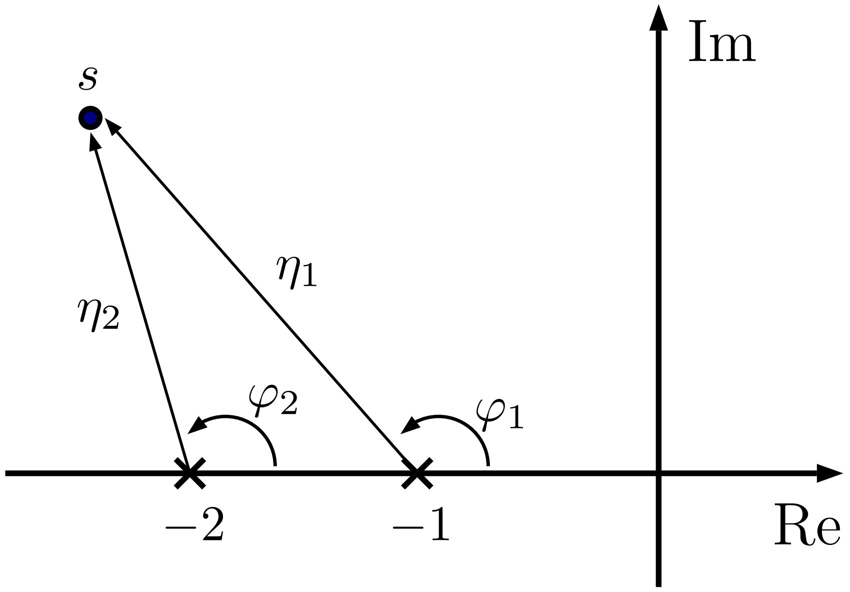
and



- the specific point s in the complex plane belongs to the Direct RL if $\theta_1 - \sum_{i=1}^{3} \varphi_i = (2k+1) \cdot 180^\circ,\ k \in \mathbb{Z}$
- the specific point s in the complex plane belongs to the Inverse RL if $\theta_1 - \sum_{i=1}^{3} \varphi_i = 2k \cdot 180^\circ,\ k \in \mathbb{Z}$

A Simple Example

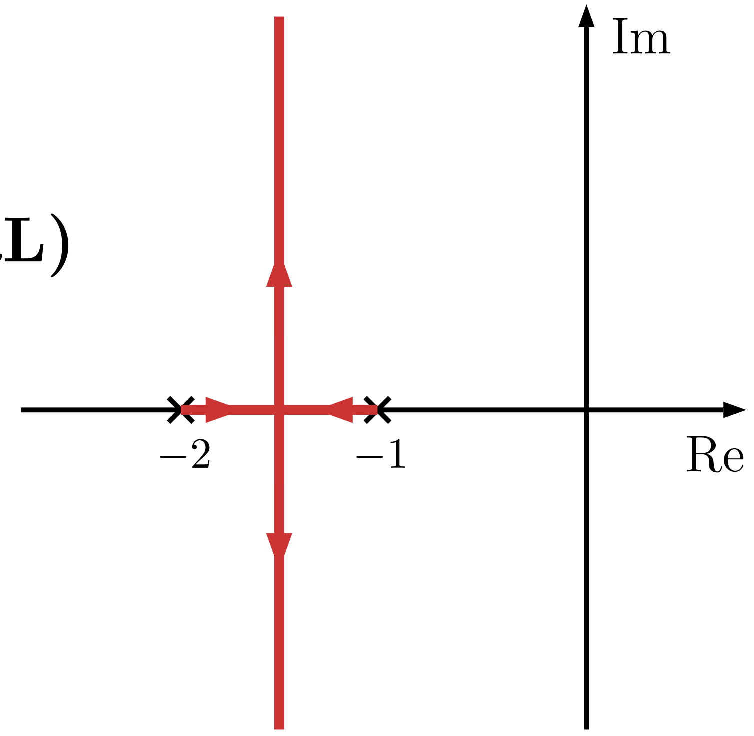
$$L(s) = \varrho \, \frac{1}{(s+1)(s+2)}$$



We have:

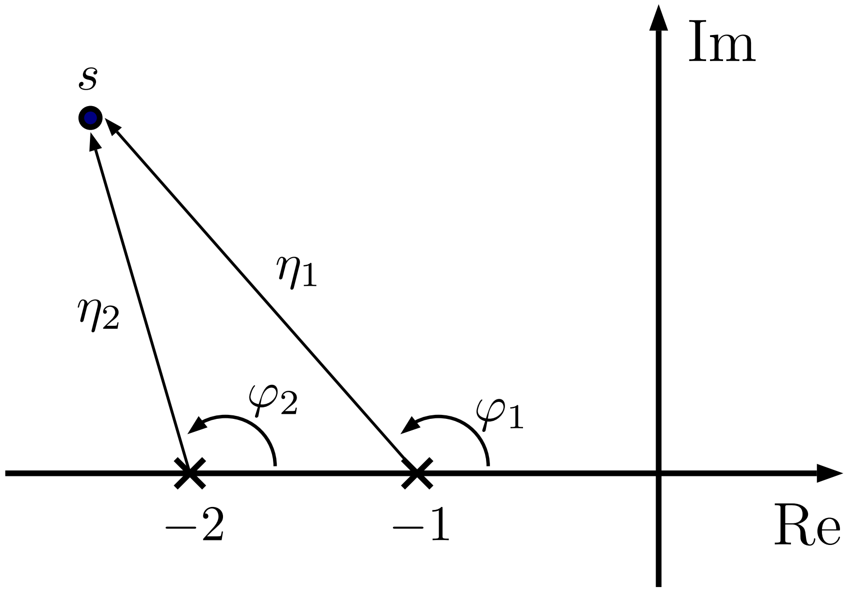
$$-\varphi_1 - \varphi_2 = (2k+1)\cdot 180^\circ,\ k \in \mathbb{Z},\ \text{if } \varrho > 0 \ \textbf{(Direct RL)}$$

The arrows on the Direct RL branches depicted in **red** show the directions of movement of the closed-loop poles for increasing values of $\varrho > 0$



A Simple Example (contd.)

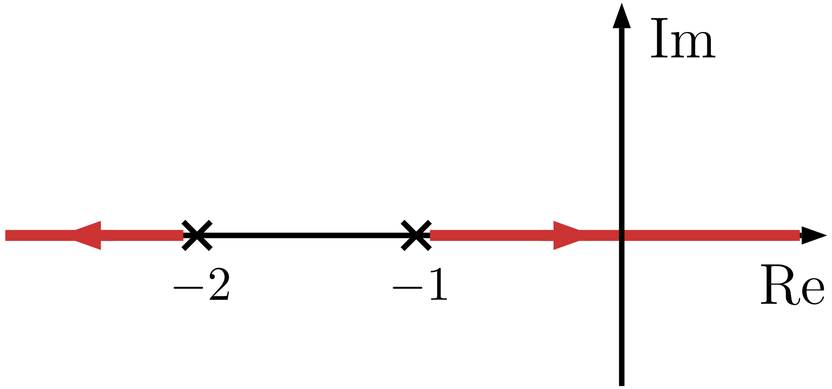
$$L(s) = \varrho \frac{1}{(s+1)(s+2)}$$



We have:

$$-\varphi_1 - \varphi_2 = 2k \cdot 180°,\ k \in \mathbb{Z},\ \text{if } \varrho < 0\ \textbf{(Inverse RL)}$$

The arrows on the Inverse RL branches depicted in **red** show the directions of movement of the closed-loop poles for decreasing values of $\varrho < 0$ (that is, increasing values of $|\varrho|$).



A Simple Example (contd.)

Starting from $L(s) = \varrho \dfrac{1}{(s+1)(s+2)}$ we have

$$1 + L(s) = 1 + \varrho \frac{1}{(s+1)(s+2)} = \frac{(s+1)(s+2)+\varrho}{(s+1)(s+2)} = \frac{s^2+3s+(2+\varrho)}{(s+1)(s+2)}$$

that gives the solutions $s_{1,2} = -\dfrac{3}{2} \pm \sqrt{1-4\varrho}$

- However, this is **<u>not</u>** the procedure that is typically carried out to draw the RL.
- Instead, the construction of the RL typically follows some **basic properties** shown in the subsequent slides.

The Root Locus Technique - Basic Properties

Consider again:

$$1 + L(s) = 0 \quad \Rightarrow \quad 1 + \varrho \frac{\prod_{j=1}^{m}(s+z_j)}{\prod_{i=1}^{n}(s+p_i)} = 0 \quad \Rightarrow \quad \prod_{i=1}^{n}(s+p_i) + \varrho \prod_{j=1}^{m}(s+z_j) = 0$$

- **Property 1:** the RL is made of $2n$ branches: n branches constitute the **Direct RL** and n branches constitute the **Inverse RL**
- **Property 2:** the RL is symmetric with respect to the real axis
- **Property 3:** when $|\varrho| \to 0$ the RL branches "originate" from the open-loop poles, that is the poles of $L(s)$
- **Property 4:** when $|\varrho| \to \infty$, m branches of the RL "terminate" on the open-loop zeros, that is the zeros of $L(s)$ and $n-m$ branches diverge to infinity

- **Property 5:** the $n-m$ branches diverging to infinity asymptotically approach asymptotes having the following characteristics:
  - the $n-m$ asymptotes intersect the real axis on the point x_a (named **centroid**) as follows:

$$x_a = \frac{1}{n-m}\left(\sum_{j=1}^{m} z_j - \sum_{i=1}^{n} p_i\right)$$

  - the $n-m$ asymptotes form the following angles with the real axis:

$$\psi_a = \begin{cases} \dfrac{(2k+1)\cdot 180^\circ}{n-m}, \; k = 0,\ldots,n-m-1, & \text{if } \varrho > 0 \; \textbf{(Direct RL)} \\[2ex] \dfrac{2k\cdot 180^\circ}{n-m}, \; k = 0,\ldots,n-m-1, & \text{if } \varrho < 0 \; \textbf{(Inverse RL)} \end{cases}$$

The Root Locus Technique - Basic Properties (contd.)

- **Property 6:** all points on the real axis belong to the RL (Direct and Inverse), excluding the open-loop zeros and poles of $L(s)$
- **Property 7:** all points on the real axis having on their right-hand-side an **odd** number of open-loop zeros and poles of $L(s)$ belong to the **Direct RL**
- **Property 8:** all points on the real axis having on their right-hand-side an **even** number of open-loop zeros and poles of $L(s)$ belong to the **Inverse RL**
- **Property 9:** possible intersections $\bar{x} \in \mathbb{R}$ of the RL with the real axis can be obtained as follows:

$$\bar{x} \in \mathbb{R} \text{ such that } \left.\frac{\mathrm{d}\gamma(x)}{\mathrm{d}x}\right|_{x=\bar{x}} = 0 \quad \text{where} \quad \gamma(x) = -\frac{\prod_{i=1}^{n}(x+p_i)}{\prod_{j=1}^{m}(x+z_j)}$$

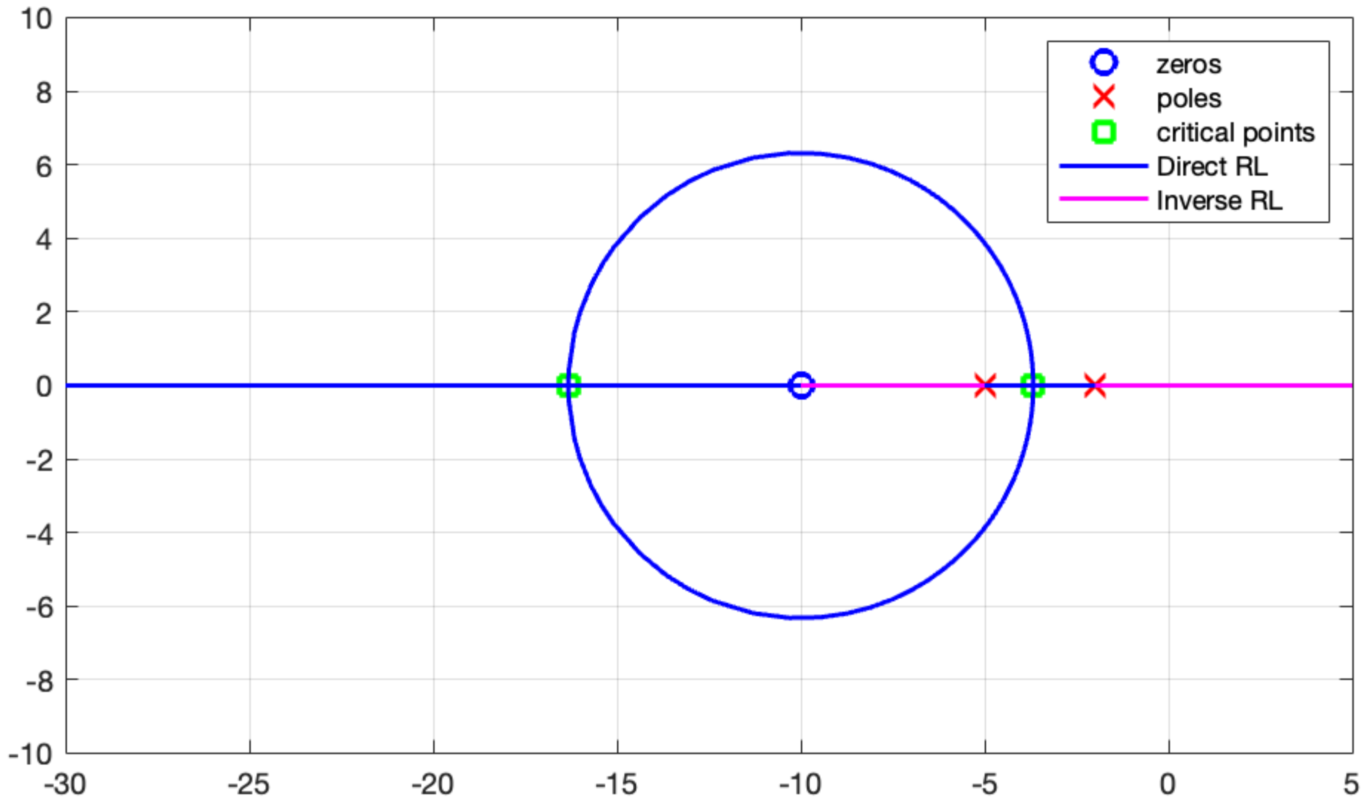
that is, $\bar{x} \in \mathbb{R}$ are **stationary points** (if any) of $\gamma(x)$

Example 1

$$L(s) = \frac{s+10}{(s+5)(s+2)}$$

Remarks:

- The open-loop zero and poles are all located in the left half-plane. Hence, as shown by the **Direct RL,** the closed-loop systems is asymptotically stable for any $\varrho > 0$
- As shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable
- In the **Direct RL,** the zero "attracts" the RL for increasing $\varrho > 0$
- In the **Direct RL,** increasing $\varrho > 0$ causes first a decrease, then an increase of the damping ratio, eventually yielding real closed-loop poles for high values of $\varrho > 0$



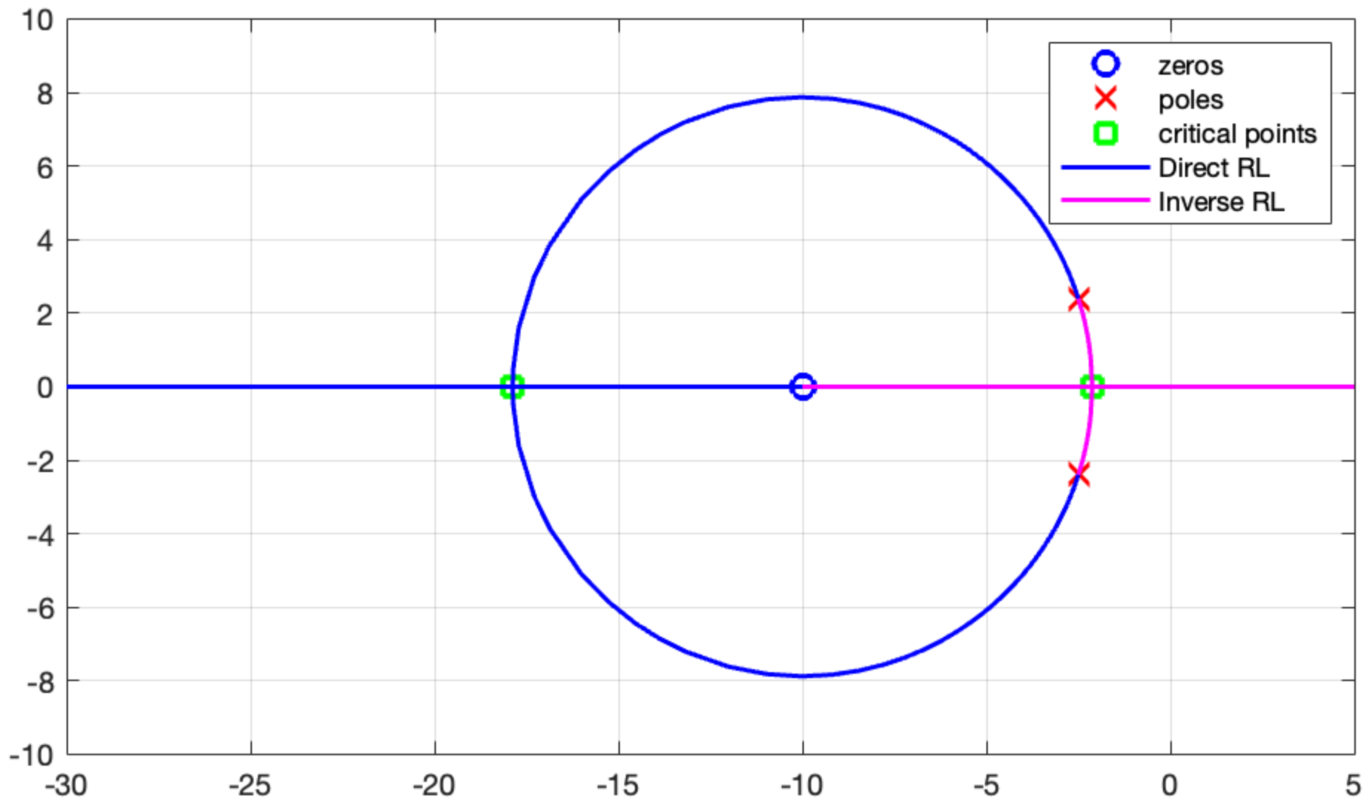
- Both critical points belong to the **Direct RL**

Example 2

$$L(s) = \frac{s+10}{s^2+5s+12}$$

Remarks:

- The open-loop zero and poles are all located in the left half-plane. Hence, as shown by the **Direct RL,** the closed-loop systems is asymptotically stable for any $\varrho > 0$
- As shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable
- In the **Direct RL,** the zero "attracts" the RL for increasing $\varrho > 0$
- In the **Direct RL,** increasing $\varrho > 0$ causes first a decrease, then an increase of the damping ratio, eventually yielding real closed-loop poles for high values of $\varrho > 0$



- One critical point belongs to the **Direct RL** and one to the **Inverse RL**

Example 3

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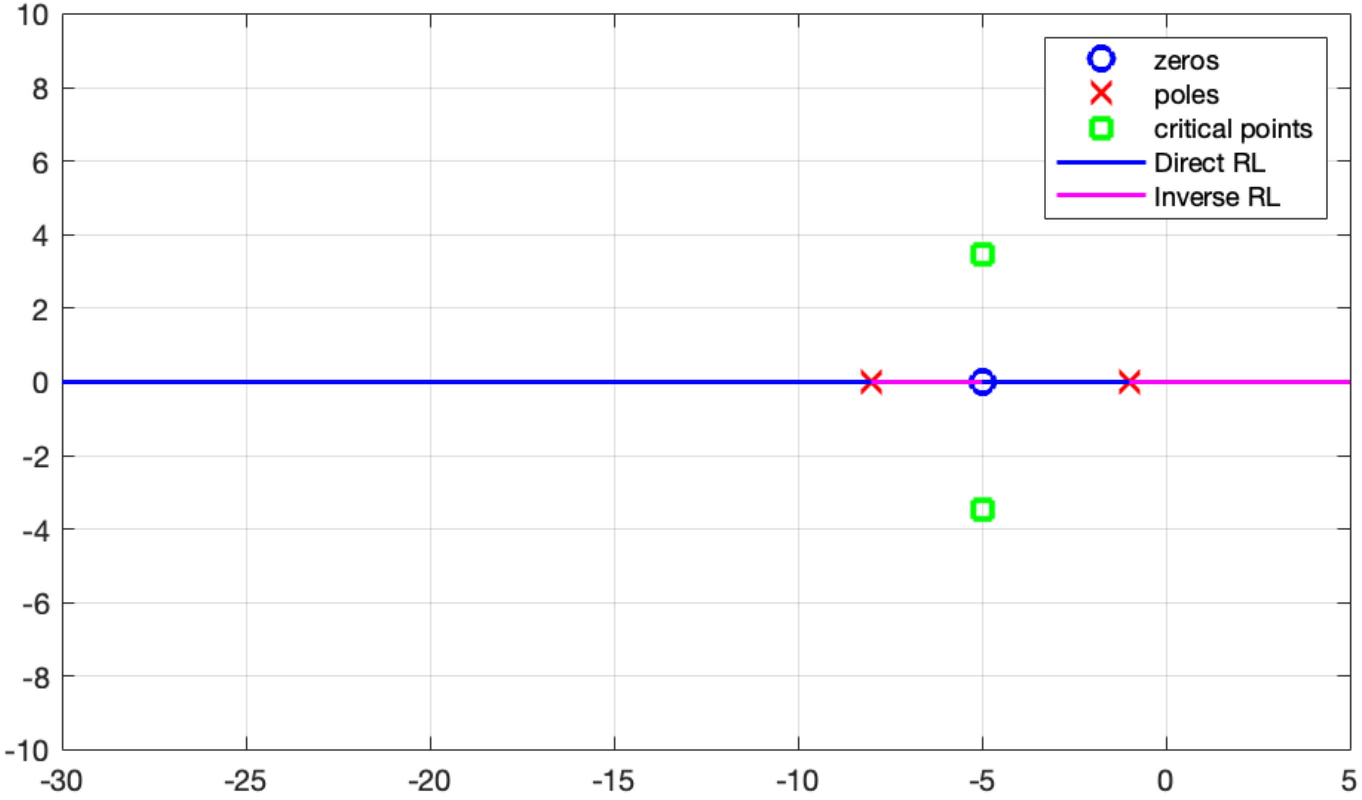
$$L(s) = \frac{s+5}{(s+1)(s+8)}$$

Remarks:

- The open-loop zero and poles are all located in the left half-plane. Hence, as shown by the **Direct RL,** the closed-loop systems is asymptotically stable for any $\varrho > 0$
- As shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable
- In the **Direct RL,** the zero "attracts" the RL for increasing $\varrho > 0$
- **Direct** and **Inverse RLs** develop only on the real axis
- Critical points do **not** belong to the RL (neither **Direct** nor **Inverse**)

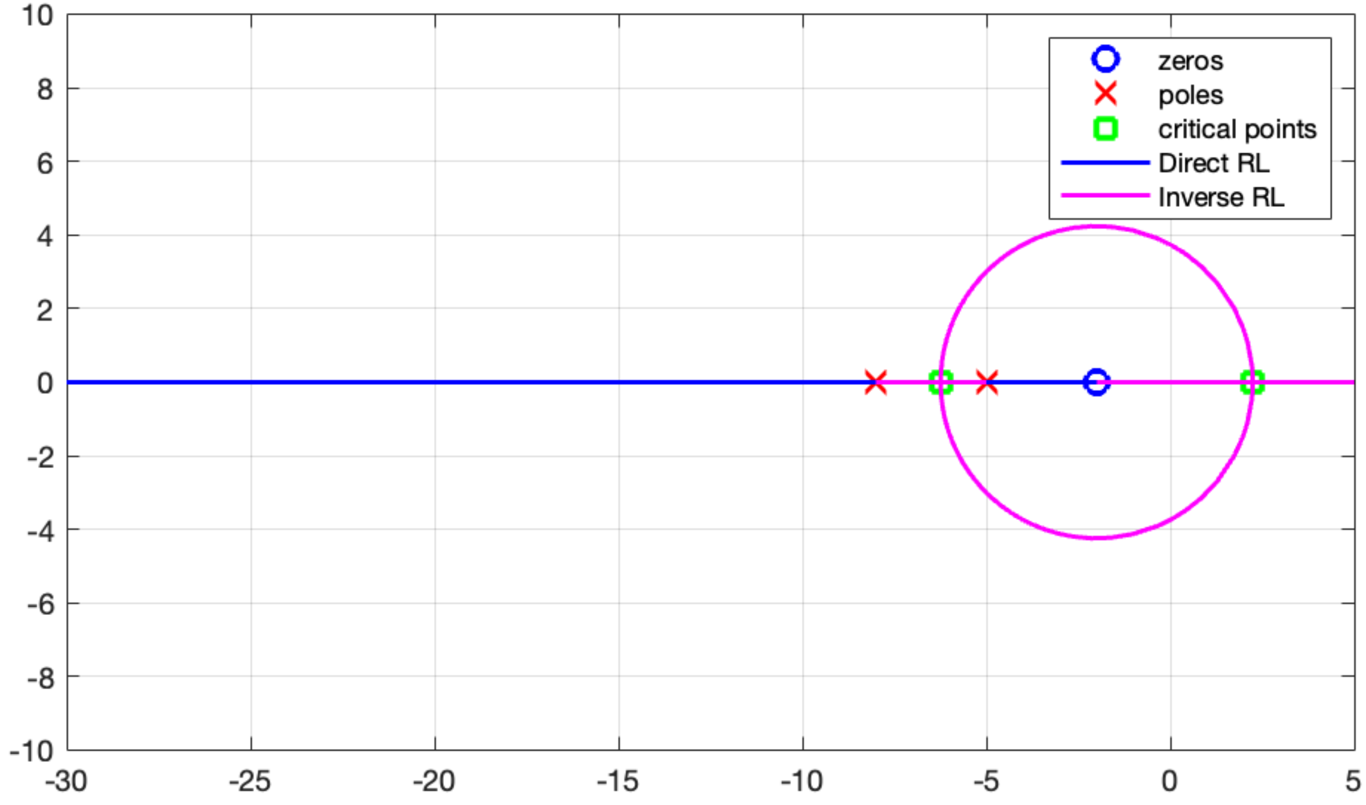


Example 4

$$L(s) = \frac{s+2}{(s+5)(s+8)}$$

Remarks:

- The open-loop zero and poles are all located in the left half-plane. Hence, as shown by the **Direct RL,** the closed-loop systems is asymptotically stable for any $\varrho > 0$
- As shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable
- In the **Direct RL,** the zero "attracts" the RL for increasing $\varrho > 0$
- The **Direct RL,** fully belongs to the real axis, hence closed-loop poles are all real and negative for any value of $\varrho > 0$



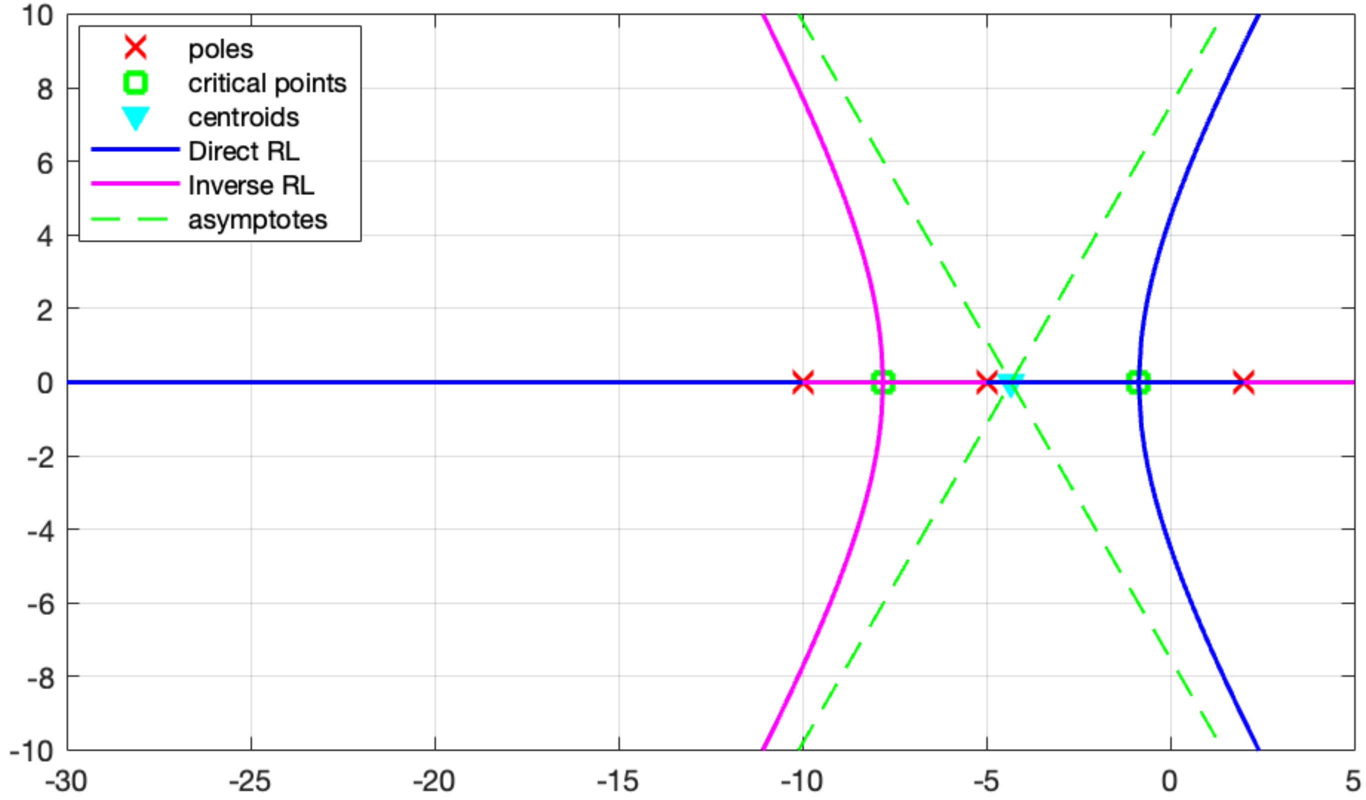
- Unlike Example 1, both critical points now belong to the **Inverse RL**

Example 5

$$L(s) = \frac{1}{(s-2)(s+5)(s+10)}$$

Remarks:

- There are no open-loop zeros and one of the open-loop poles is located in the right half-plane.
- Both the **Direct RL** and the **Inverse RL** have three asymptotes forming angles of 120° between them
- As shown by the **Direct RL**, increasing $\varrho > 0$ makes the closed-loop system unstable
- One branch of the **Inverse RL,** fully belongs to the positive real axis, hence for any $\varrho < 0$, the closed-loop system is unstable



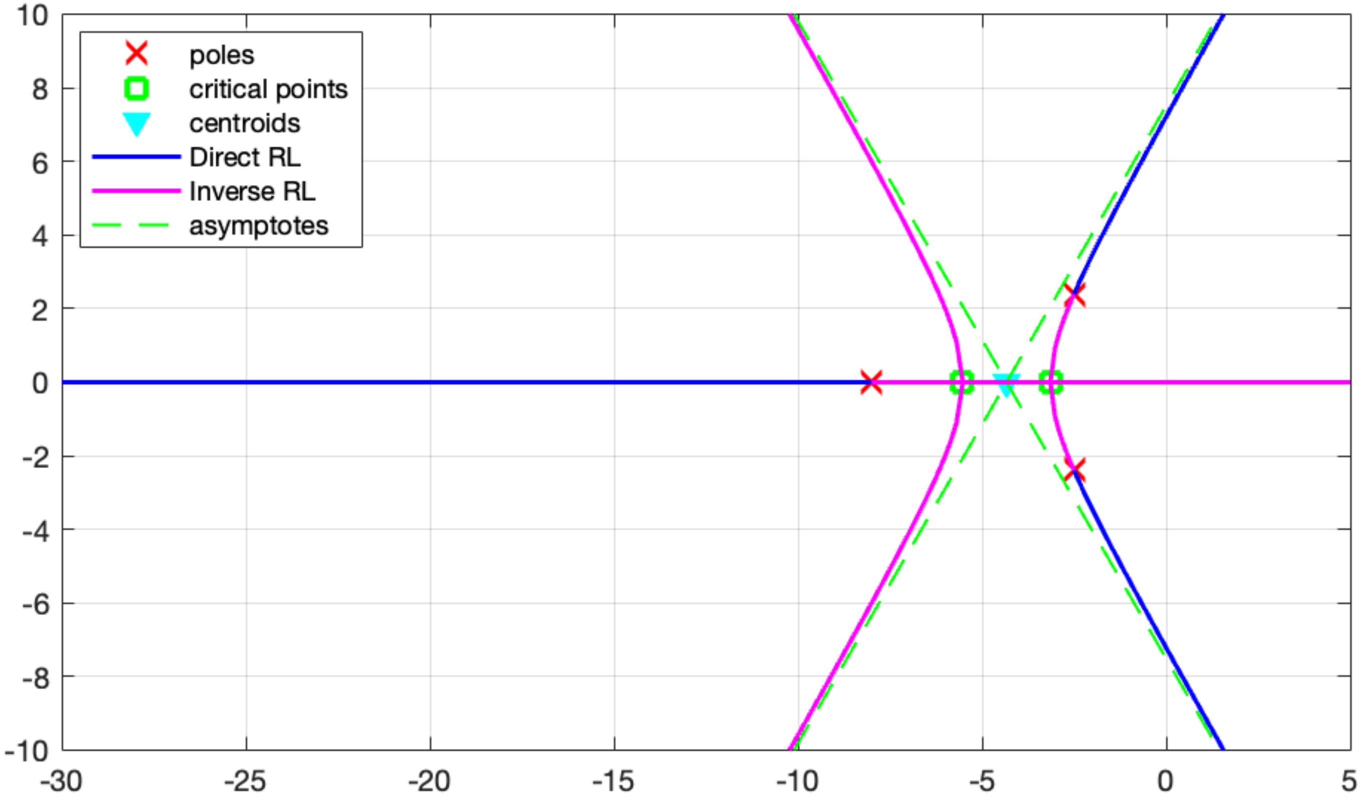
- One critical point belongs to the **Direct RL** and one to the **Inverse RL**

Example 6

$$L(s) = \frac{1}{(s+8)(s^2+5s+12)}$$

Remarks:

- There are no open-loop zeros and all the open-loop poles are located in the left half-plane. The real open-loop pole is rather "distant" from the pair of complex-conjugate open-loop poles.
- Both the **Direct RL** and the **Inverse RL** have three asymptotes forming angles of 120° between them
- As shown by the **Direct RL**, increasing $\varrho > 0$ makes the closed-loop system unstable
- Likewise, as shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable



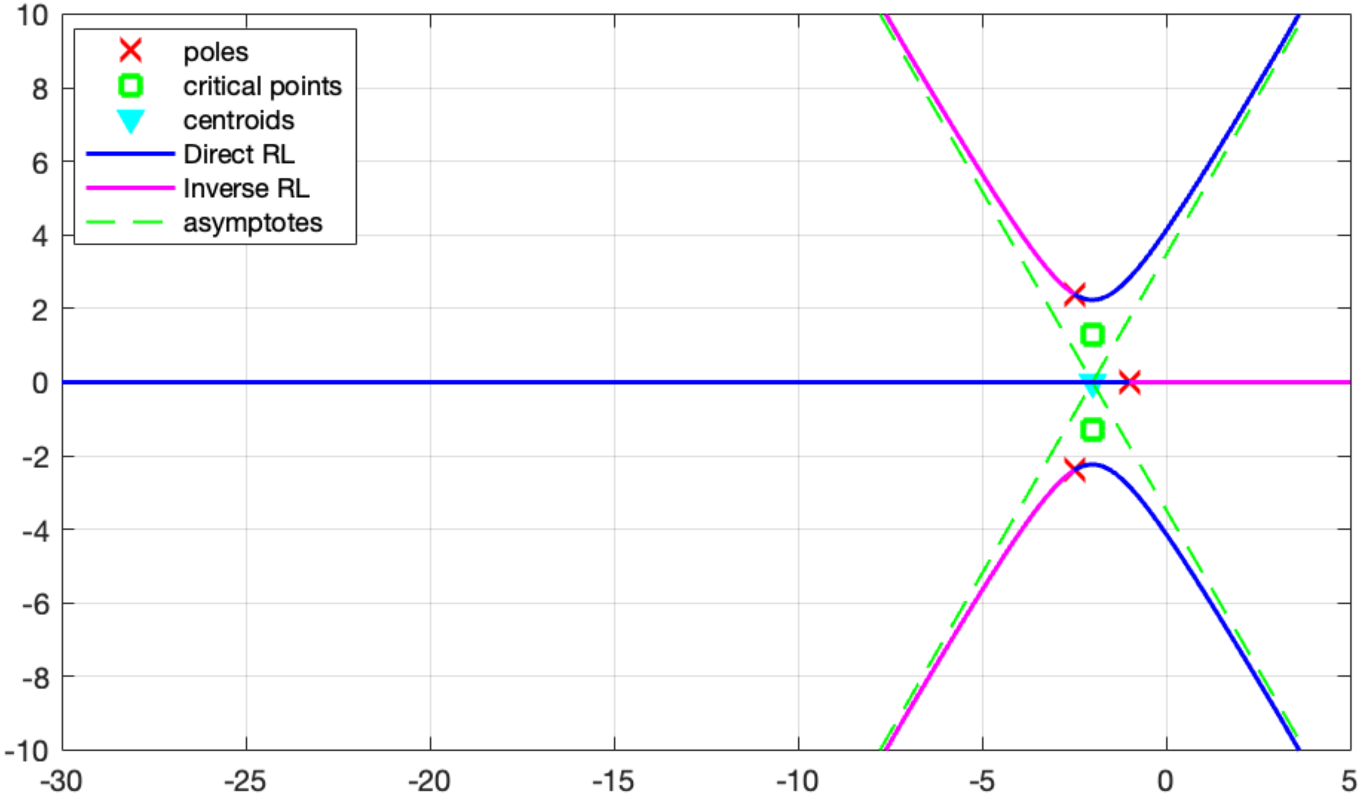
- Both critical points belong to the negative real axis as part of the **Inverse RL**

Example 7

$$L(s) = \frac{1}{(s+1)(s^2+5s+12)}$$

Remarks:

- There are no open-loop zeros and all the open-loop poles are located in the left half-plane. Unlike Example 6, the real open-loop pole is rather "close" to the pair of complex-conjugate open-loop poles.
- Both the **Direct RL** and the **Inverse RL** have three asymptotes forming angles of 120° between them
- As shown by the **Direct RL**, increasing $\varrho > 0$ makes the closed-loop system unstable
- Likewise, as shown by the **Inverse RL**, decreasing $\varrho < 0$ makes the closed-loop system unstable



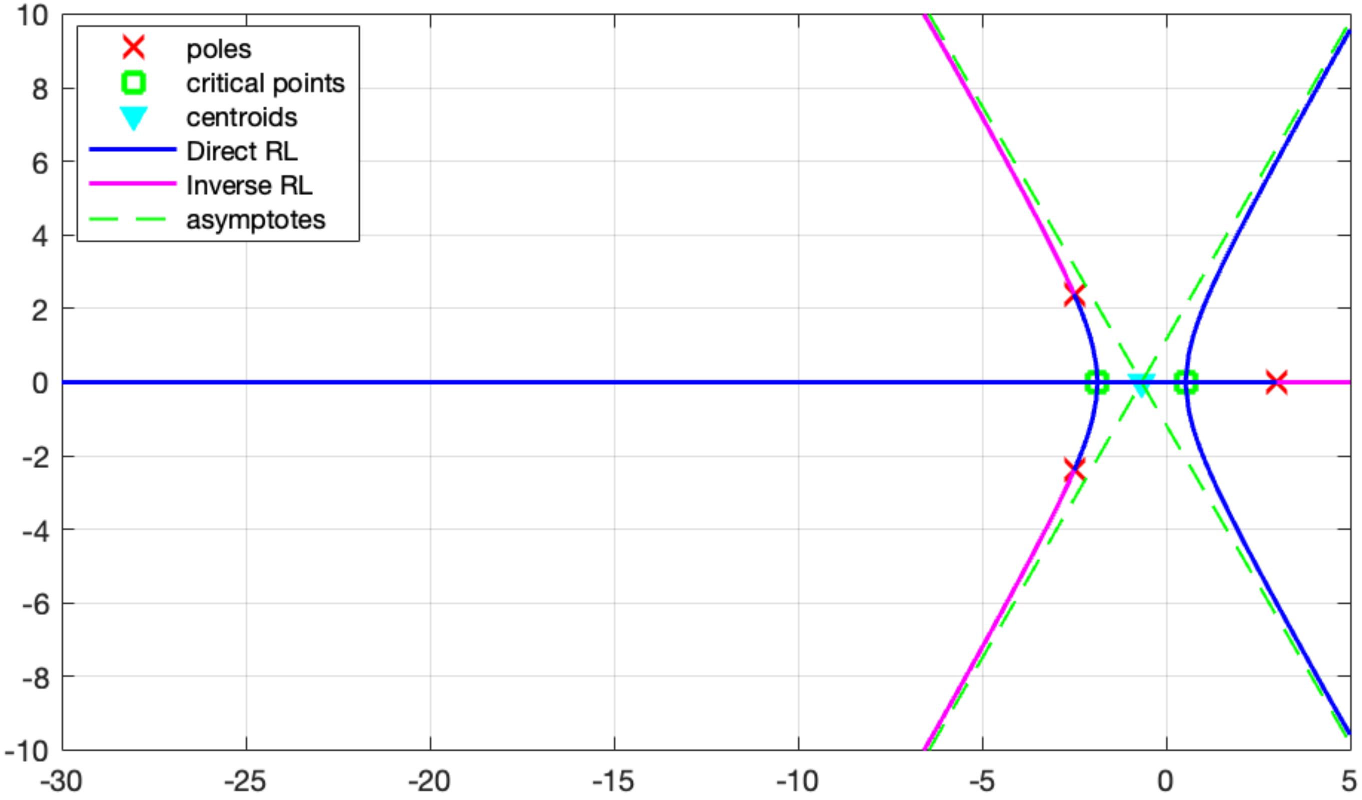
- Both critical points do **not** belong to the **RL** (neither the **Direct** nor the **Inverse**)

Example 8

$$L(s) = \frac{1}{(s-3)(s^2+5s+12)}$$

Remarks:

- There are no open-loop zeros and one of the open-loop poles is located in the right half-plane. The real open-loop pole is rather "distant" from the pair of complex-conjugate open-loop poles.
- Both the **Direct RL** and the **Inverse RL** have three asymptotes forming angles of 120° between them
- For any $\varrho > 0$ the **Direct RL** shows that the closed-loop system is unstable
- Likewise, for any $\varrho < 0$ the **Inverse RL** shows that the closed-loop system is unstable

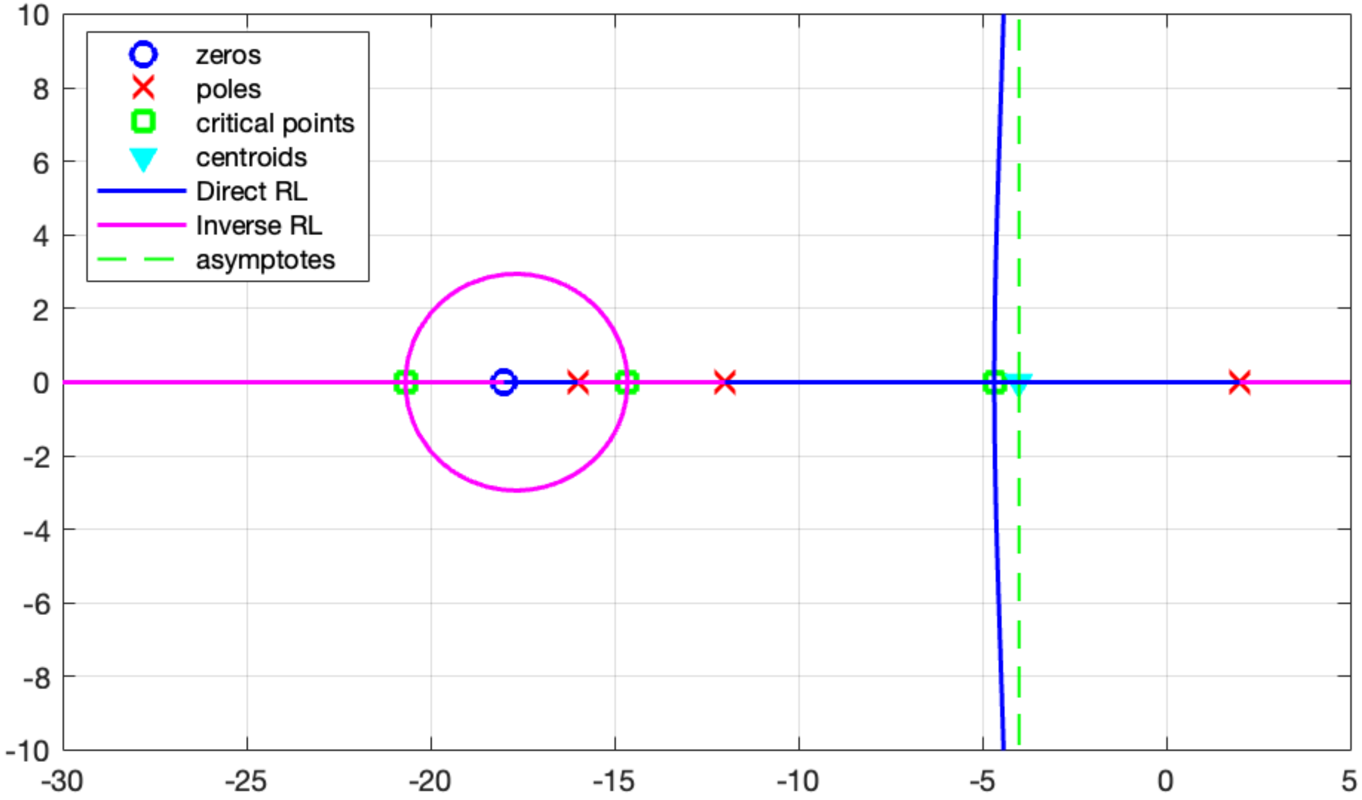


- The "form" of the RL is very similar to the one in Example 6
- But, both critical points belong to the **Direct RL:** one belongs to the negative real axis and the other to the positive real axis

Example 9

$$L(s) = \frac{s+18}{(s-2)(s+12)(s+16)}$$

Remarks:
- There is one open-loop zero located on the left of all open-loop poles, one of which is located in the right half-plane.
- In the region close to the zero and the negative poles, the **Inverse RL** has a form similar to the **Direct RL** in Example 1.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- Increasing $\varrho > 0$ makes the closed-loop system as. stable as shown by the **Direct RL**
- For any $\varrho < 0$ the **Inverse RL** shows that the closed-loop system is unstable



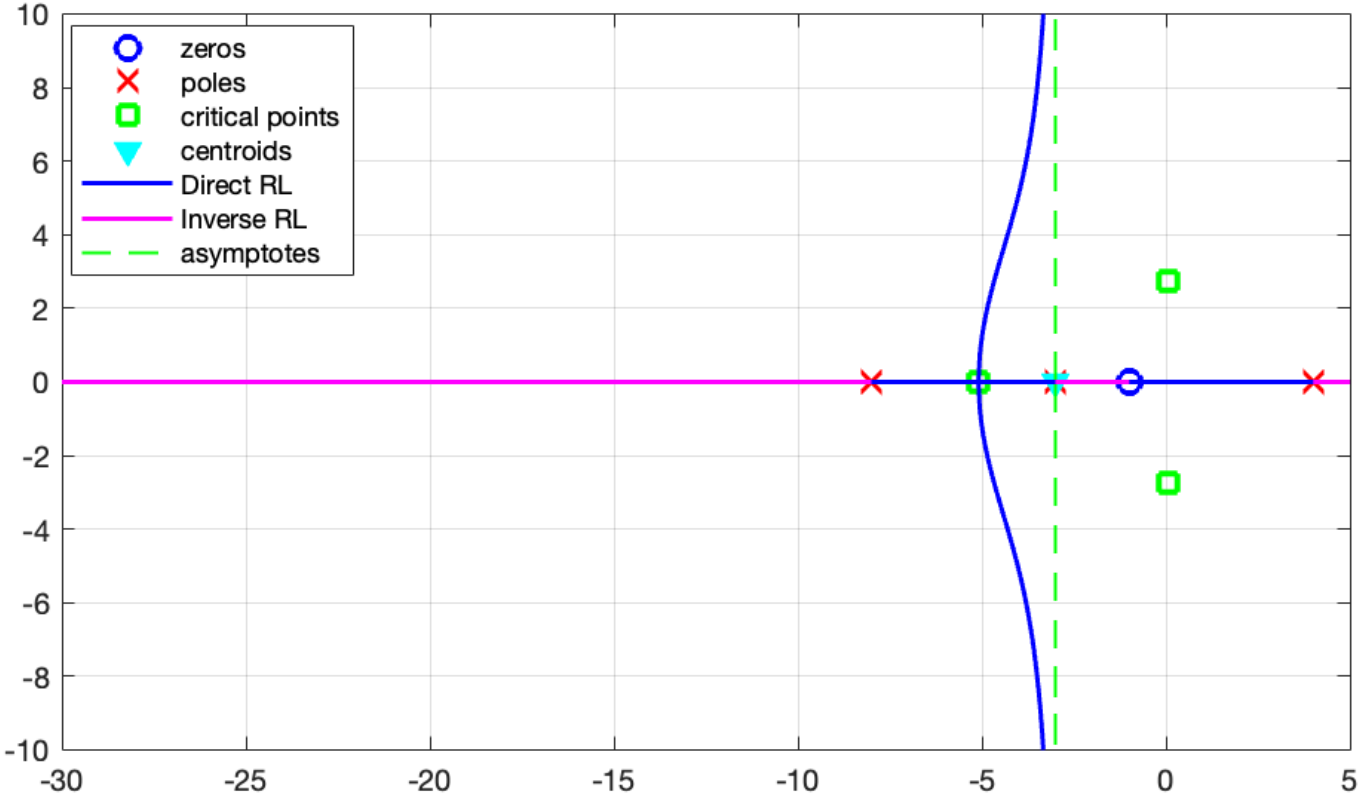
- One critical point belongs to the **Direct RL** and one to the **Inverse RL**
- The open-loop zero has a beneficial effect because its presence means having two asymptotes in the left half-plane

Example 10

$$L(s) = \frac{s+1}{(s-4)(s+3)(s+8)}$$

Remarks:

- There is one open-loop zero located on the right of the negative open-loop poles. There is one positive open-loop pole.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- The zero "attracts" the centroid.
- Increasing $\varrho > 0$ makes the closed-loop system as. stable as shown by the **Direct RL**
- For any $\varrho < 0$ the **Inverse RL** shows that the closed-loop system is unstable



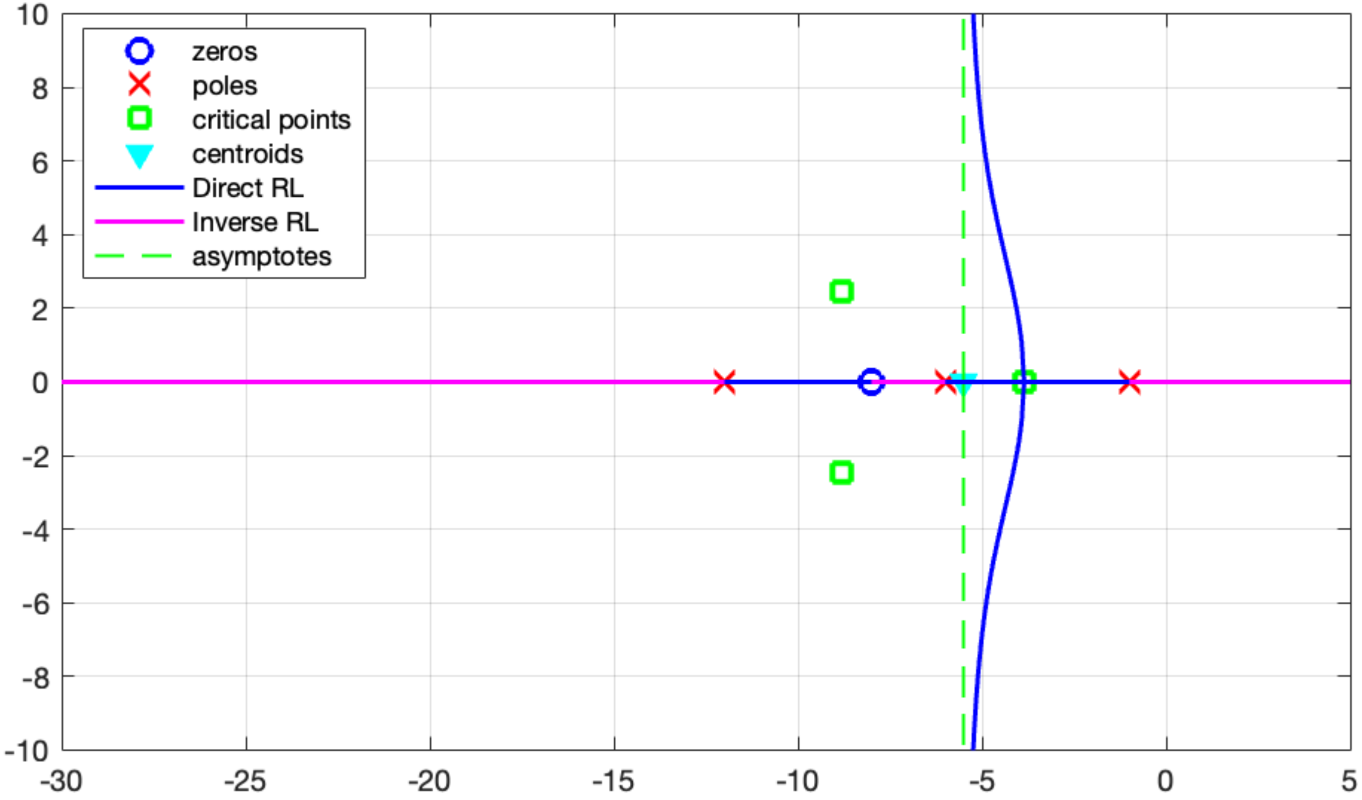
- One critical point belongs to the **Direct RL** but the other two do not belong to the **RL**

Example 11

$$L(s) = \frac{s+8}{(s+1)(s+6)(s+12)}$$

Remarks:

- All the open-loop zeros and poles are located in the left half-plane.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- The zero "attracts" the centroid.
- For any $\varrho > 0$ the **Direct RL** shows that the closed-loop system is as. stable
- Decreasing $\varrho < 0$ makes the closed-loop system unstable as shown by the **Inverse RL**
- One critical point belongs to the **Direct RL** but the other two do not belong to the **RL**

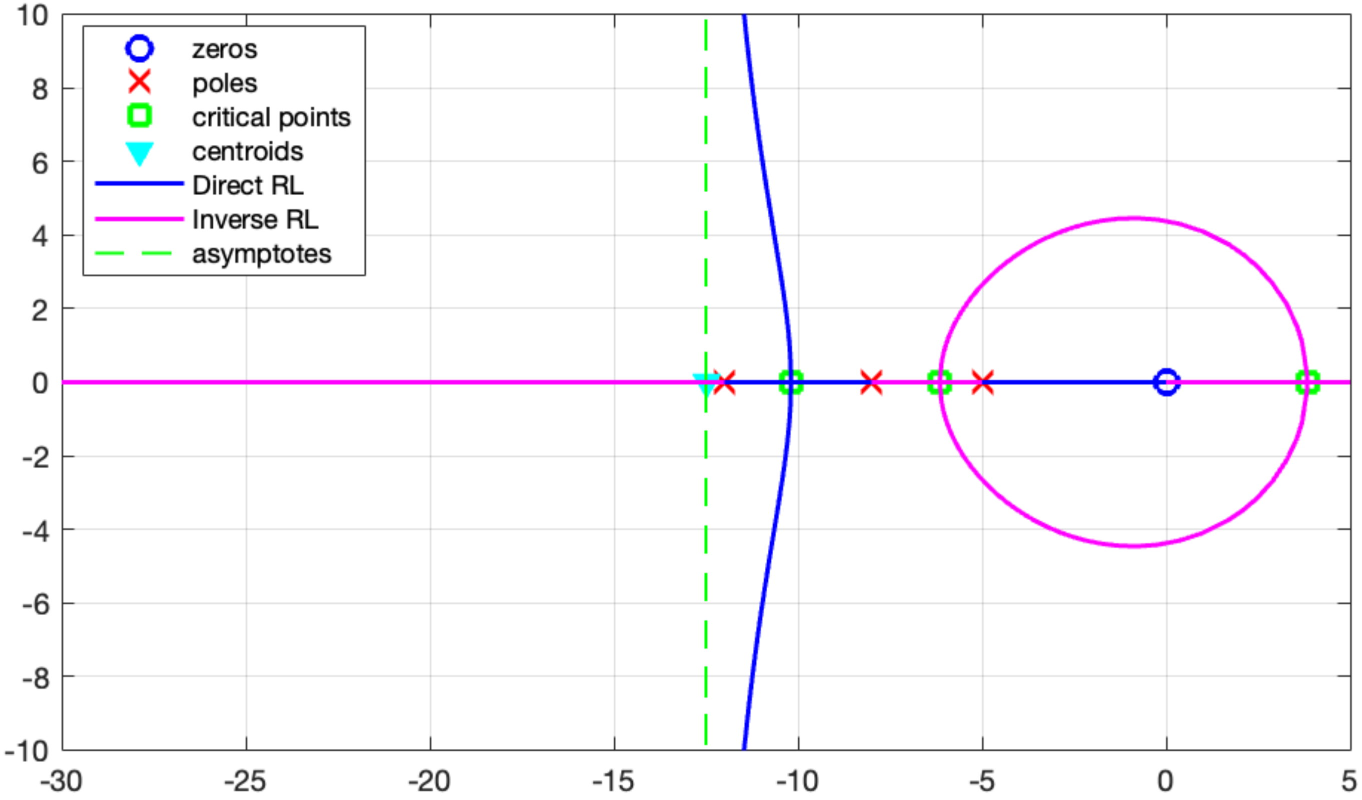


Example 12

$$L(s) = \frac{s}{(s+5)(s+8)(s+12)}$$

Remarks:

- The open-loop zero is located on the imaginary axis and all open-loop poles are located in the left half-plane.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- The zero does not "attract" the centroid in this case because the pole in -5 is too close
- For any $\varrho > 0$ the **Direct RL** shows that the closed-loop system is as. stable
- Decreasing $\varrho < 0$ makes the closed-loop system unstable as shown by the **Inverse RL**



- One critical point belongs to the **Direct RL** and the other two belong to the **Inverse RL**

Example 13

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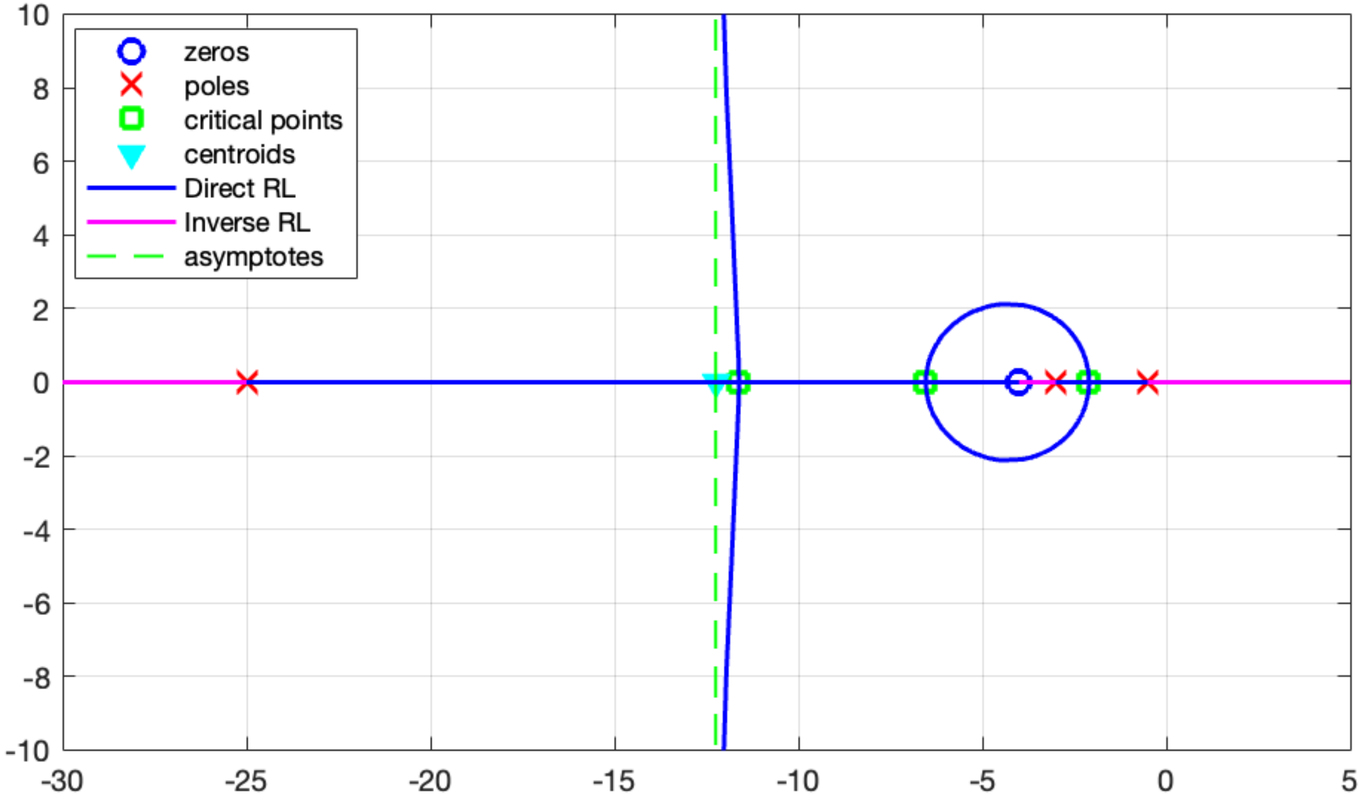
$$L(s) = \frac{s+4}{(s+1/2)(s+3)(s+25)}$$

Remarks:

- All the open-loop zeros and poles are located in the left half-plane.
- The zero is very close with the pole in -3. Hence the zero does not "attract" the centroid.
- In the region of the zero and the two close negative poles, the form of the RL is similar to Example 1.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- For any $\varrho > 0$ the **Direct RL** shows that the closed-loop system is as. stable
- Decreasing $\varrho < 0$ makes the closed-loop system unstable as shown by the **Inverse RL**



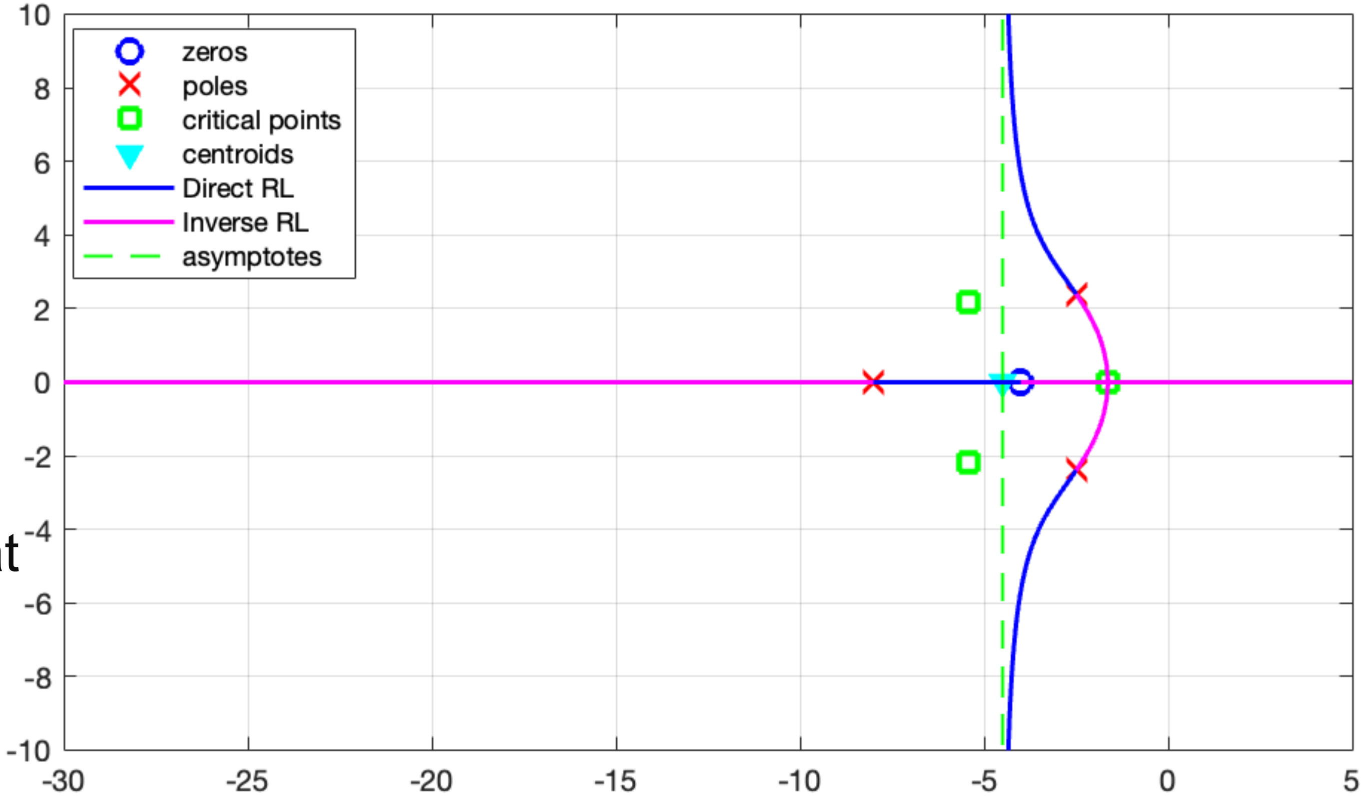
- All three critical points belong to the **Direct RL**

Example 14

$$L(s) = \frac{s+4}{(s+8)(s^2+5s+12)}$$

Remarks:

- All the open-loop zeros and poles are located in the left half-plane.
- Compared to Example 6 (no open-loop zero), the form of the RL is modified by the zero.
- Both the **Direct RL** and the **Inverse RL** have two asymptotes
- For any $\varrho > 0$ the **Direct RL** shows that the closed-loop system is as. stable. Compared with Example 6, the zero has a stabilising effect.
- Decreasing $\varrho < 0$ makes the closed-loop system unstable as shown by the **Inverse RL**



- Only one critical point belongs to the RL