

034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL

A.Y. 2023-2024

Part VIII: Sinusoidal-Response Analysis & Frequency Response

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- Consider an **asymptotically stable LTI** continuous-time that is **fully described** (that is, no common factors) by the transfer function:

$$G(s) = \frac{N(s)}{\varphi(s)}$$

- Impose a **generic input** $u(t)$ that admits a rational Laplace transform. Hence (initial conditions are zero):

$$Y(s) = \frac{N(s)}{\varphi(s)} \cdot U(s)$$

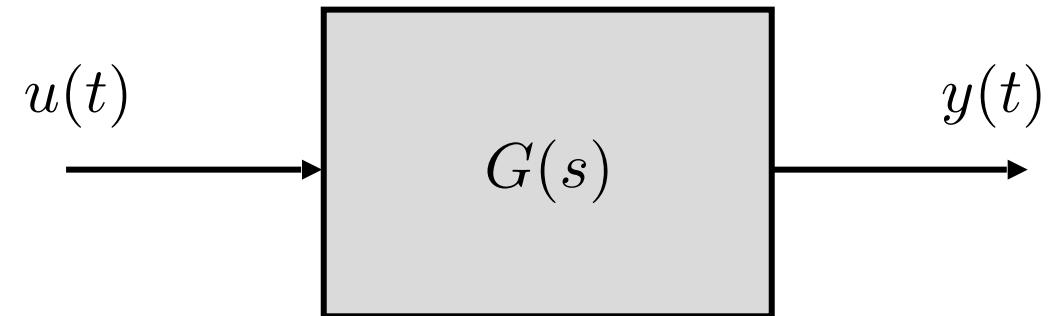
Example



Consider:

$$G(s) = \frac{2(s+1)}{(s+2)(s+10)}$$

$$u(t) = 4t \cdot 1(t)$$



Hence, carrying out the Laplace transform:

→
$$Y(s) = \frac{2(s+1)}{(s+2)(s+10)} \cdot \frac{4}{s^2}$$

Therefore:

- **Transient Response:**

$$Y_{\text{trans.}}(s) = -\frac{1}{4} \cdot \frac{1}{s+2} + \frac{9}{100} \cdot \frac{1}{s+10}$$

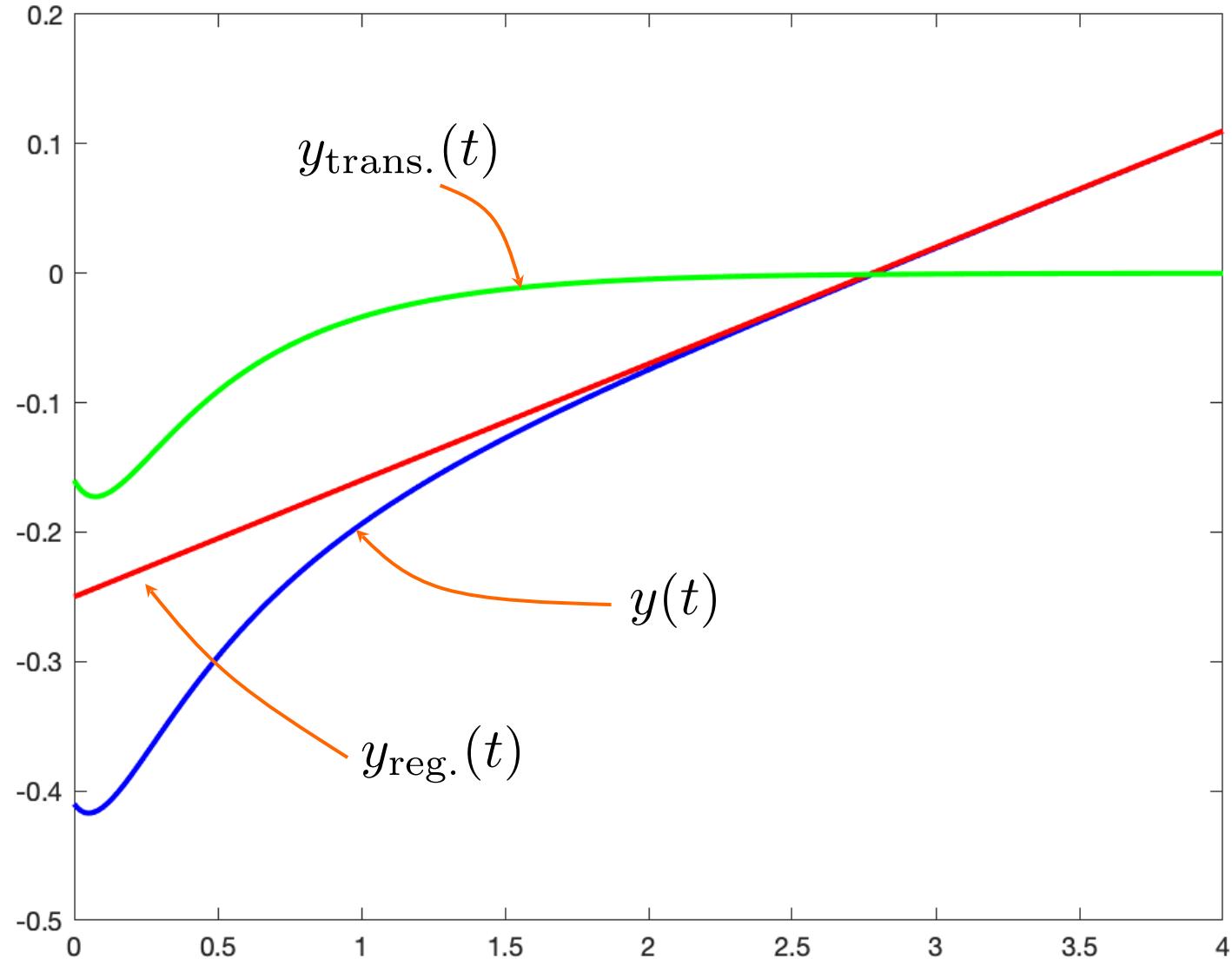
↳ $y_{\text{trans.}}(t) = \left[-\frac{1}{4} e^{-2t} + \frac{9}{100} e^{-10t} \right] \cdot 1(t)$

- **Regime (Steady-State) Response:**

$$Y_{\text{reg.}}(s) = \frac{4}{25} \cdot \frac{1}{s} + \frac{2}{5} \cdot \frac{1}{s^2}$$

↳ $y_{\text{reg.}}(t) = \left[\frac{4}{25} + \frac{2}{5} t \right] \cdot 1(t)$

Example (contd.)



- Carrying out the partial fraction expansion and the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1} \left[\sum_{i=1}^{n_{a.s.}} \frac{P_i}{s - p_i} \right] + \dots$$

Contribution due to asymptotically stable poles of $Y(s)$

$$+ \mathcal{L}^{-1} \left[\sum_{j=1}^{n_{m.s.}} \frac{Q_j}{s - p_j} \right] + \dots$$

Contribution due to marginally stable poles of $Y(s)$

$$+ \mathcal{L}^{-1} \left[\sum_{k=1}^{n_{unst.}} \frac{R_k}{s - p_k} \right]$$

Contribution due to unstable poles of $Y(s)$

Therefore:

- **Transient Response:**

$$y_{\text{trans.}}(t) = \mathcal{L}^{-1} \left[\sum_{i=1}^{n_{a.s.}} \frac{P_i}{s - p_i} \right] \xrightarrow[t \rightarrow \infty]{} 0$$

The transient response contribution **vanishes asymptotically**

- **Regime (Steady-State) Response:**

$$y_{\text{reg.}}(t) = \mathcal{L}^{-1} \left[\sum_{j=1}^{n_{m.s.}} \frac{Q_j}{s - p_j} \right] + \mathcal{L}^{-1} \left[\sum_{k=1}^{n_{unst.}} \frac{R_k}{s - p_k} \right]$$

The regime response contribution does not vanish over time and is **only caused by the non-vanishing input**

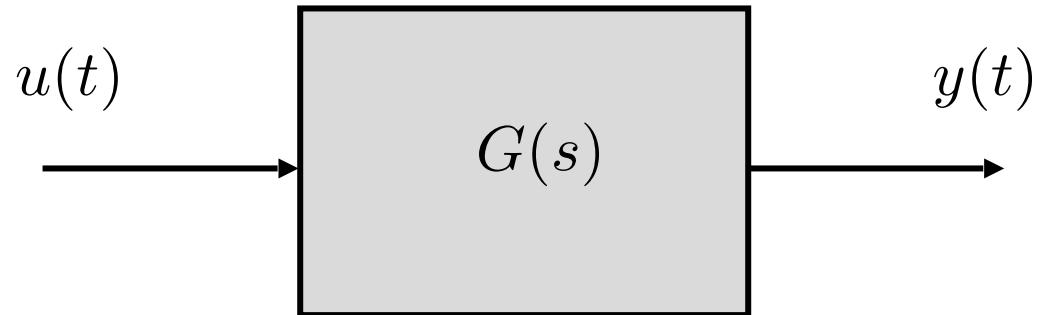
Sinusoidal Response



- Consider an **asymptotically stable** LTI continuous-time that is **fully described** (that is, no common factors) by the transfer function $G(s)$
- Moreover, consider:

$$u(t) = A \sin(\omega t) 1(t)$$

↳ $U(s) = \frac{A\omega}{s^2 + \omega^2}$



- For simplicity, assume that all poles of $G(s)$ are **real** and **distinct**. From the asymptotic stability assumption, it follows that:

$$G(s) = \mu \frac{\prod_{i=1}^m (1 + sT_i)}{\prod_{i=1}^n (1 + s\tau_i)}$$

Sinusoidal Response (contd.)



Hence:

$$Y(s) = G(s) \frac{A\omega}{s^2 + \omega^2} = G(s) \frac{A\omega}{(s - j\omega)(s + j\omega)}$$

↳
$$Y(s) = \underbrace{\frac{\alpha_1}{1 + s\tau_1} + \frac{\alpha_2}{1 + s\tau_2} + \cdots + \frac{\alpha_n}{1 + s\tau_n}}_{Y_1(s)} + \underbrace{\frac{\beta s + \gamma}{s^2 + \omega^2}}_{Y_2(s)}$$

$$\mathcal{L}^{-1} \quad \text{↳ } y(t) = \boxed{y_1(t)} + y_2(t)$$

$$t \rightarrow \infty$$

0

↳ **for $t \rightarrow \infty$ (in practice $t > t_s$)** $\rightarrow y(t) \simeq y_2(t)$

Regime ("steady-state") Sinusoidal Response



$$Y(s) = G(s) \frac{A\omega}{s^2 + \omega^2} = G(s) \frac{A\omega}{(s - j\omega)(s + j\omega)}$$

$$\mathcal{L}^{-1} \rightarrow y(t) \simeq y_2(t) = k_1 e^{j\omega t} + k_2 e^{-j\omega t}, \quad t \geq 0$$

$$k_1 = \left. G(s) \frac{A\omega}{s + j\omega} \right|_{s=j\omega} = \frac{A}{2j} G(j\omega)$$

$$k_2 = \left. G(s) \frac{A\omega}{s - j\omega} \right|_{s=-j\omega} = -\frac{A}{2j} G(-j\omega)$$

Regime ("steady-state") Sinusoidal Response (contd.)



It is easy to show that: $G(s^*) = G^*(s)$ $\rightarrow G(j\omega) = G^*(-j\omega)$

Since $G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}$ with $\varphi(\omega) := \arg G(j\omega)$

$\hookrightarrow G(-j\omega) = G^*(j\omega) = |G(j\omega)|e^{-j\varphi(\omega)}$

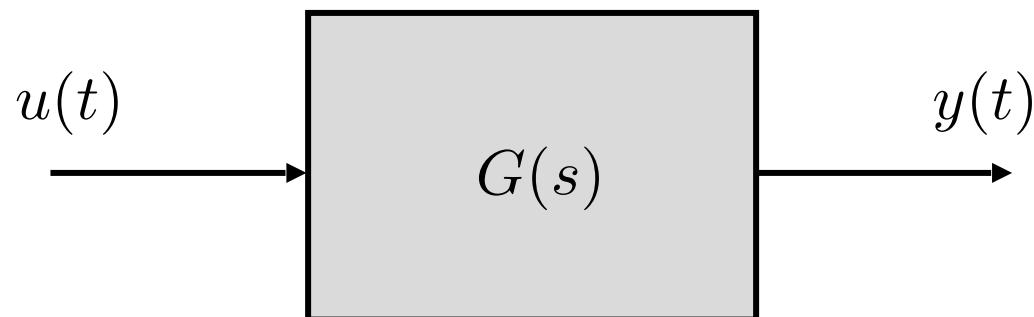
Hence

$$\begin{aligned} y(t) \simeq y_2(t) &= A|G(j\omega)| \frac{e^{j\omega t} \cdot e^{j\varphi(\omega)} - e^{-j\omega t} \cdot e^{-j\varphi(\omega)}}{2j} \\ &= A|G(j\omega)| \frac{e^{j(\omega t + \varphi(\omega))} - e^{-j(\omega t + \varphi(\omega))}}{2j} \\ &= A \cdot |G(j\omega)| \sin[\omega t + \varphi(\omega)], \quad t \geq 0 \end{aligned}$$

- Consider an **asymptotically stable LTI** continuous-time that is **fully described** (that is, no common factors) by the transfer function $G(s)$
- Moreover, consider:

$$u(t) = A \sin(\omega t) \mathbf{1}(t)$$

↳ $U(s) = \frac{A\omega}{s^2 + \omega^2}$



The regime sinusoidal response (approximately equal to $y(t)$ for $t > t_s$) is:

$$y_{\text{reg.}}(t) = B \sin(\omega t + \varphi)$$

where: $B = |G(j\omega)| \cdot A$
 $\varphi = \arg G(j\omega)$

same angular frequency
of the input sinusoid

The **frequency response** is defined as: $G(j\omega)$, $\omega \geq 0$

Example 1



$$G(s) = \frac{1}{1+s}; \quad u(t) = 10 \sin(2t)1(t)$$

↳ $A = 10; \omega = 2\text{rad/s}$

$$G(j2) = \frac{1}{1+2j} = \frac{1-2j}{(1-2j)(1+2j)} = \frac{1-2j}{5} = \frac{1}{5} - j\frac{2}{5}$$

↳ $|G(j2)| = \sqrt{\frac{1}{25} + \frac{4}{25}} = \frac{1}{\sqrt{5}} \simeq 0.447$

$$\arg G(j2) = \arctg(-2) = -63^\circ = -63^\circ \frac{\pi}{180} \simeq -1.1$$

↳ $y(t) \simeq \frac{10}{\sqrt{5}} \sin(2t - 1.1), \quad t > t_s \simeq 4.6\text{sec}$

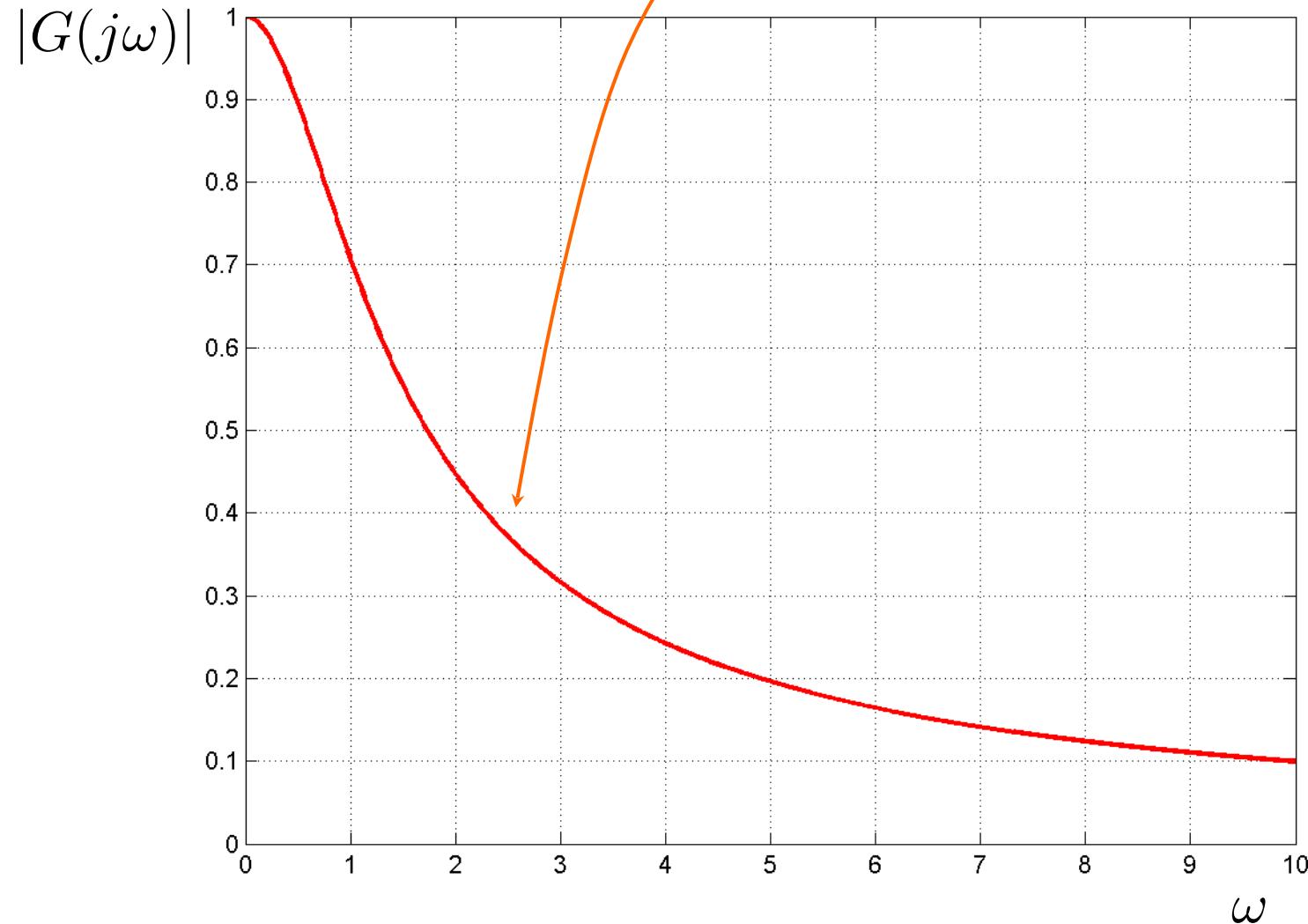
Example 1 (contd.)

$|G(j\omega)|$ decreases when ω increases



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the larger ω the smaller the amplitude of the regime response



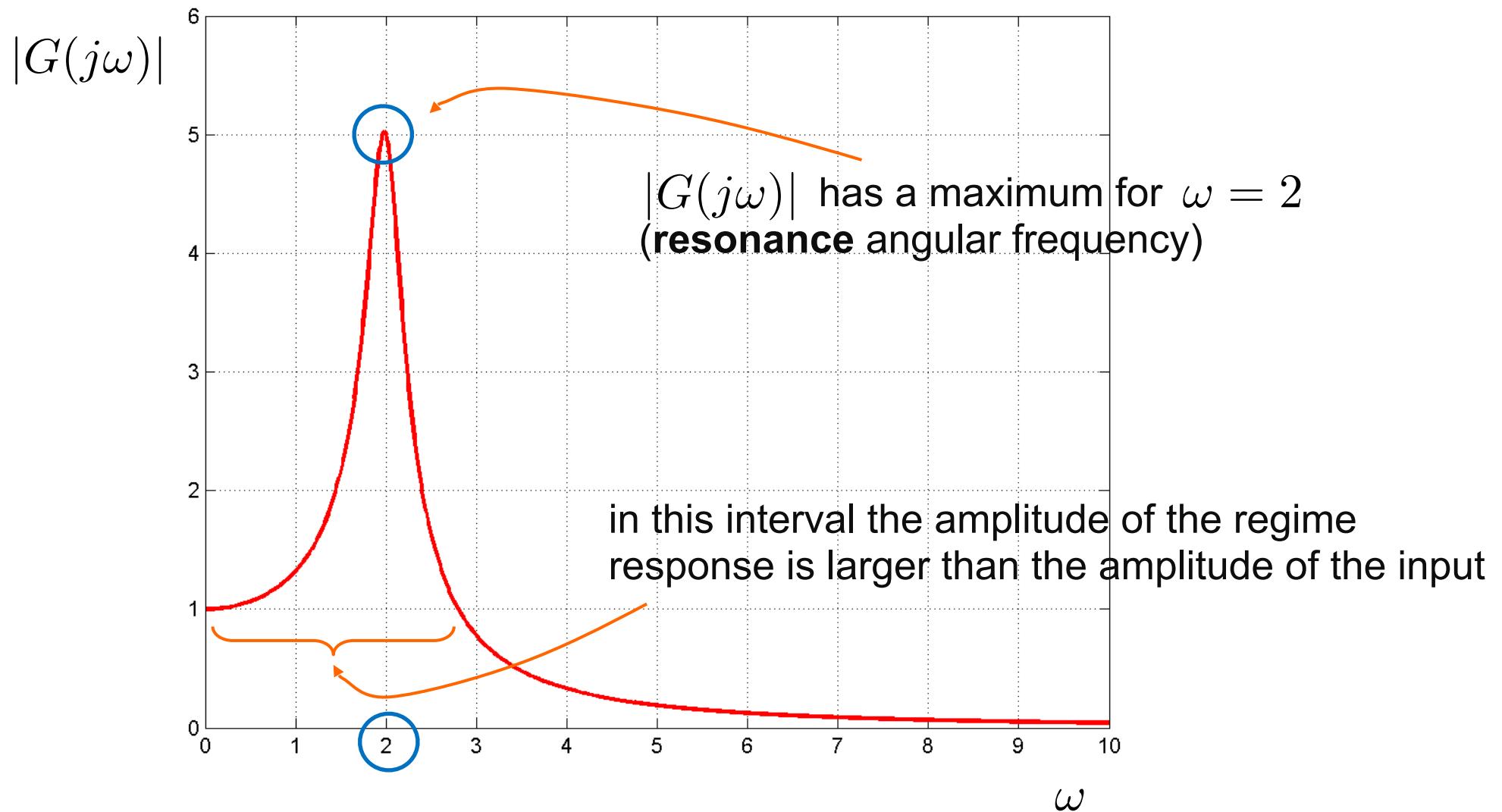
Example 2



$$G(s) = \frac{1}{1 + 0.1s + \frac{s^2}{4}}$$

↳ $|G(j\omega)| = \frac{1}{\left|1 + j0.1\omega + \frac{(j\omega)^2}{4}\right|} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{4}\right)^2 + 0.01\omega^2}}$

Example 2 (contd.)

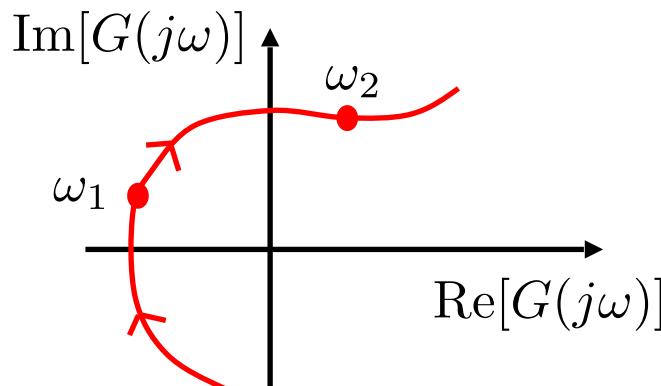


Graphical Representations of the Frequency Response



- **Polar Diagrams:**

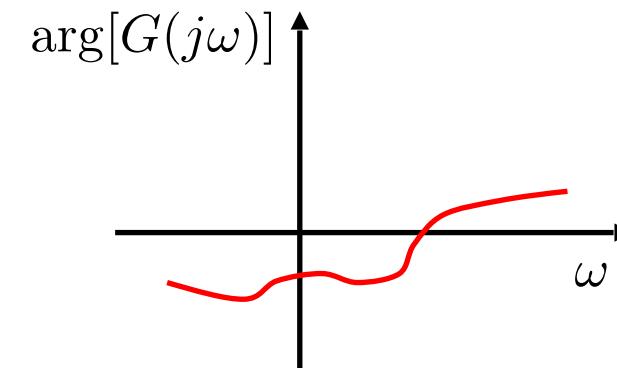
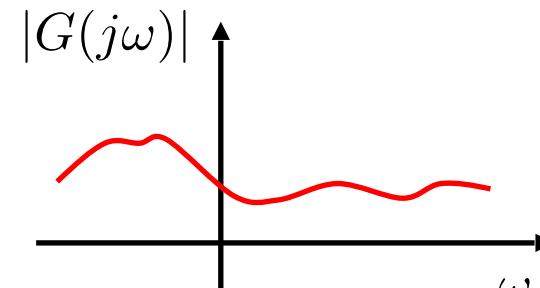
$$G(j\omega), \omega \geq 0$$



- **Bode Diagrams:**

$$|G(j\omega)|, \omega \geq 0$$

$$\varphi(\omega) = \arg G(j\omega), \omega \geq 0$$



Polar and Bode diagrams are related:

$$G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}$$



- **Magnitude:**

- on the “horizontal axis”: $\log \omega$
- on the “vertical axis”: $|G(j\omega)|_{\text{dB}}$ ($x_{\text{dB}} := 20 \log x$)

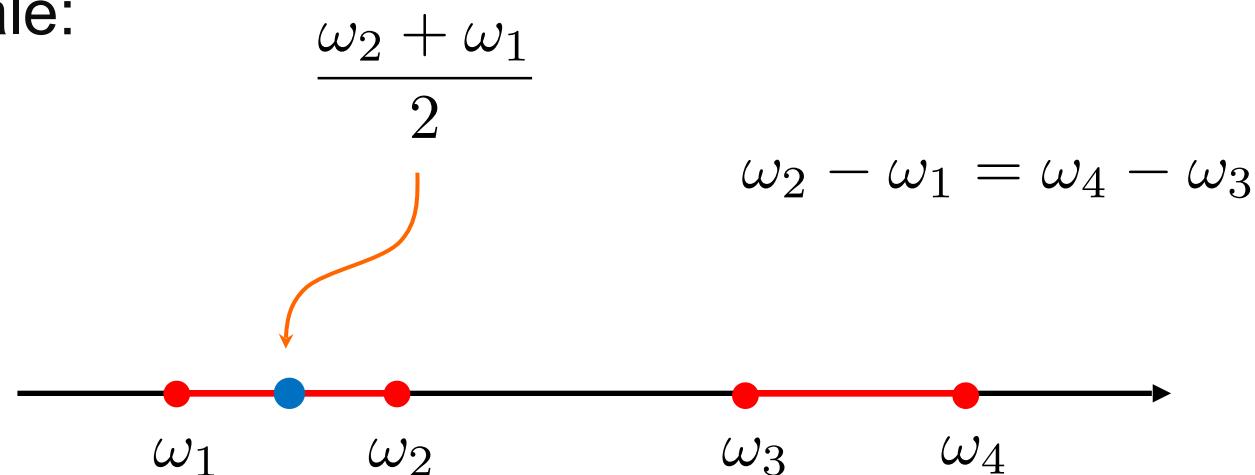
- **Phase:**

- on the “horizontal axis”: $\log \omega$
- on the “vertical axis”: $\arg G(j\omega)$ in degrees

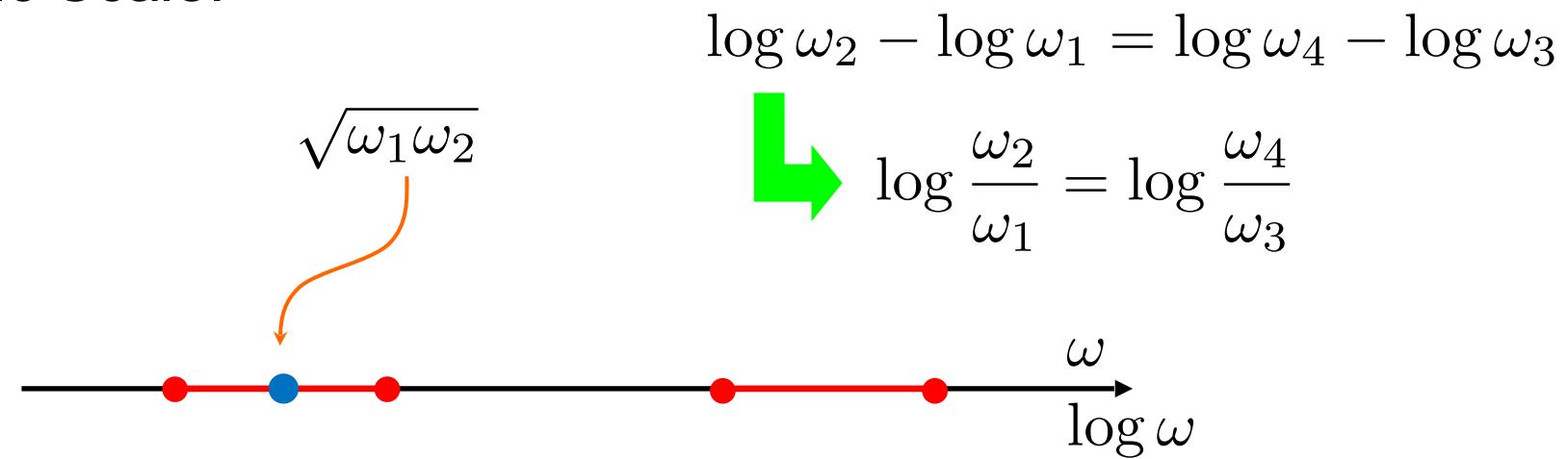
Bode Diagrams: Linear and Logarithmic Scale



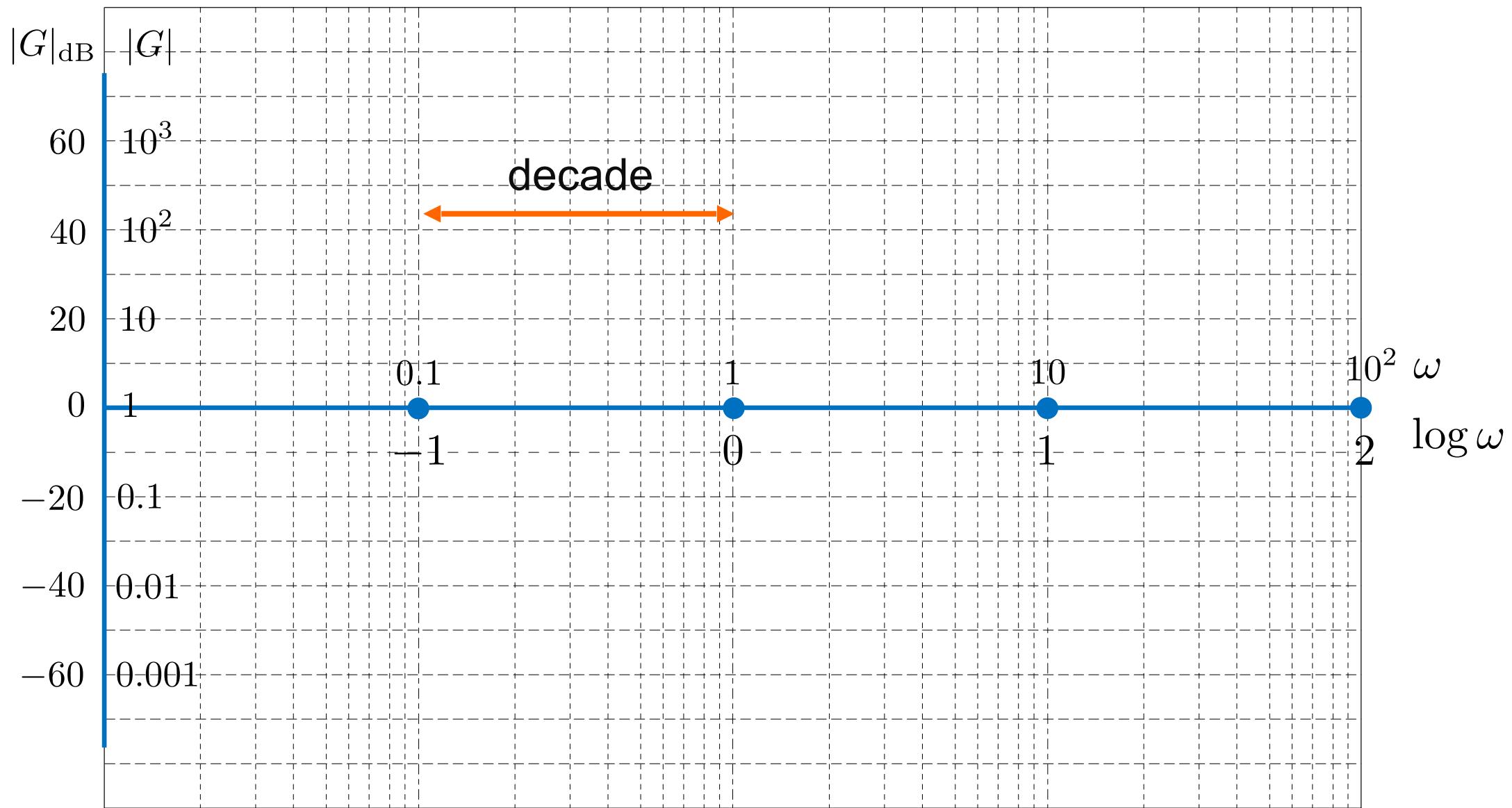
- Linear Scale:



- Logarithmic Scale:



Bode Semi-Logarithmic Chart



Bode Diagrams: Magnitude



$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^m (1 + sT_i)}{\prod_{i=1}^n (1 + s\tau_i)}$$

↳ $|G(j\omega)| = \frac{|\mu|}{|j\omega|^g}$

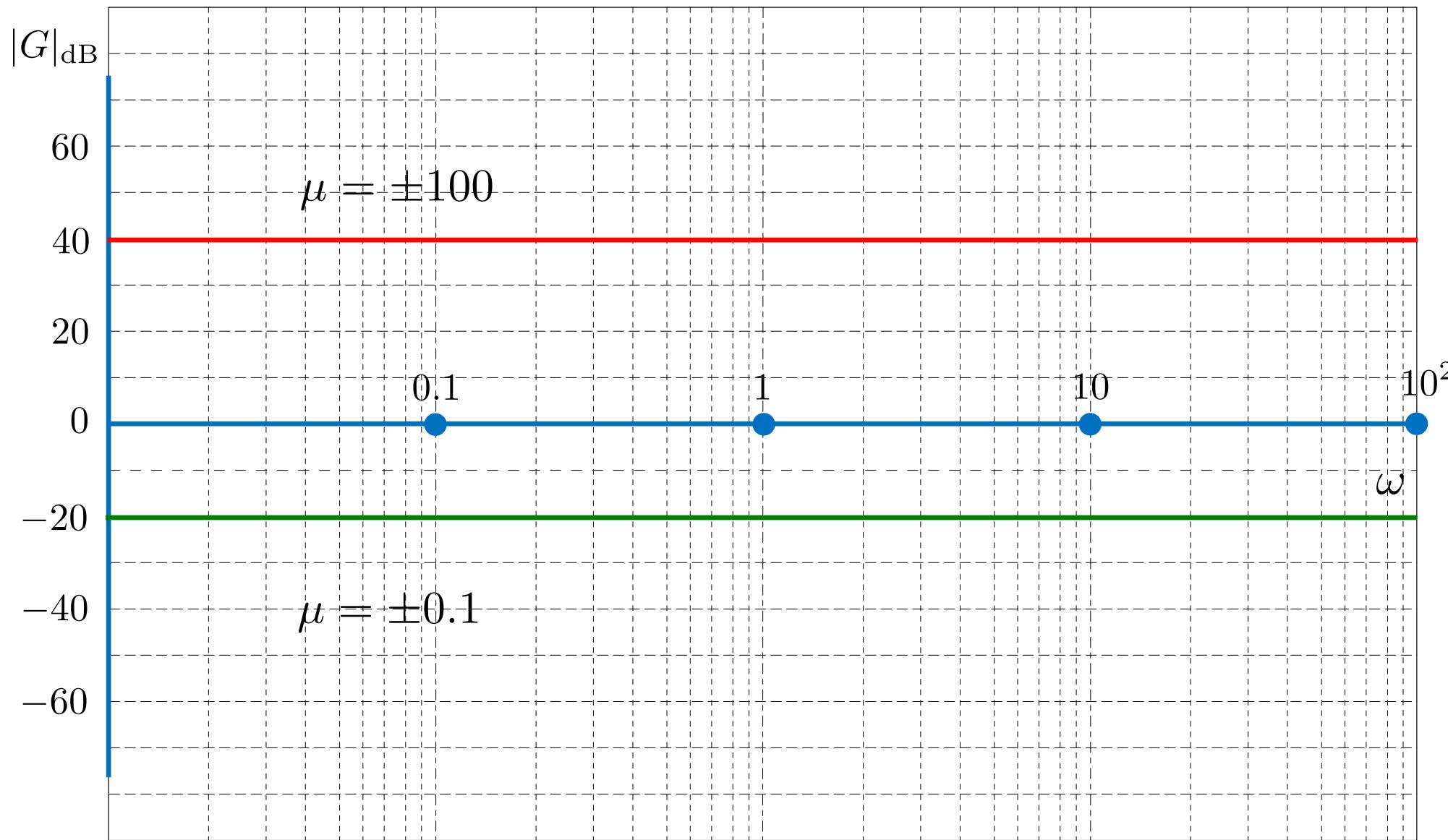
↳ $|G(j\omega)|_{\text{dB}}$

$$\begin{aligned} & \frac{\prod_{i=1}^m |1 + j\omega T_i|}{\prod_{i=1}^n |1 + j\omega \tau_i|} \\ &= \boxed{20 \log |\mu|} \quad (\text{A}) \\ & \quad \boxed{-20 \log |j\omega|^g} \quad (\text{B}) \\ & \quad + \sum_i \boxed{20 \log |1 + j\omega T_i|} \quad (\text{C}), (\text{D}) \\ & \quad - \sum_i \boxed{20 \log |1 + j\omega \tau_i|} \end{aligned}$$

(A)

20 log | μ |

constant horizontal line



(B) $-20 \log |j\omega|^g$

 $-20 \log |j\omega|^g = -20g \log \omega$

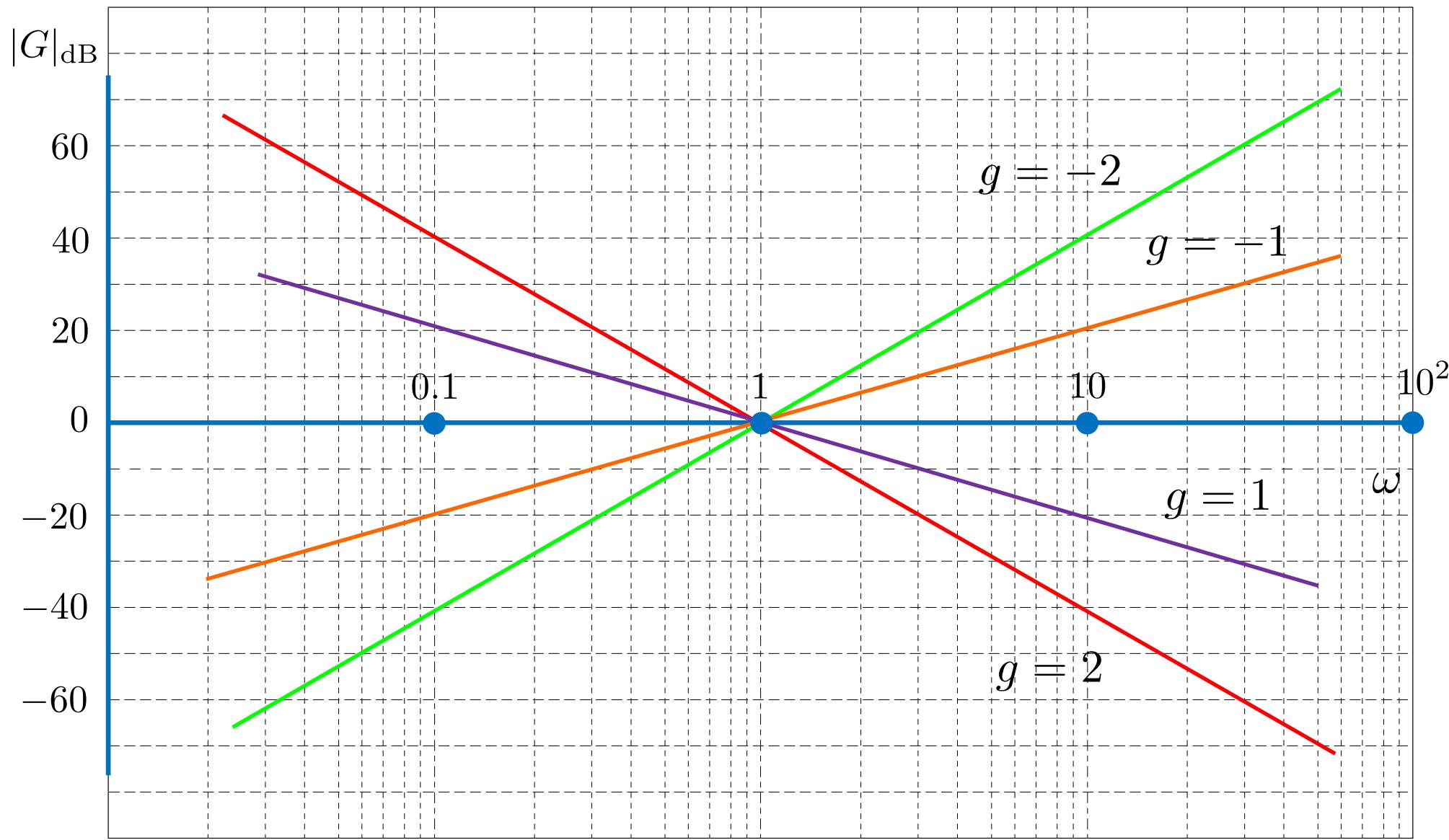


in logarithmic coordinates it is a straight line with slope $-20 \cdot g$ dB/dec

Conventionally, the straight line is set to pass through the point $\omega = 1$ with 0dB gain

(B)

$$-20 \log |j\omega|^g$$



(C), (D) $20 \log |1 + j\omega T|, T \in \mathbb{R}$

↳ $20 \log |1 + j\omega T| = 20 \log \sqrt{1 + \omega^2 T^2}$

- If $\omega^2 T^2 \ll 1$, that is $\omega \ll \frac{1}{|T|}$

↳ $20 \log |1 + j\omega T| \simeq 0$

- If $\omega^2 T^2 \gg 1$, that is $\omega \gg \frac{1}{|T|}$

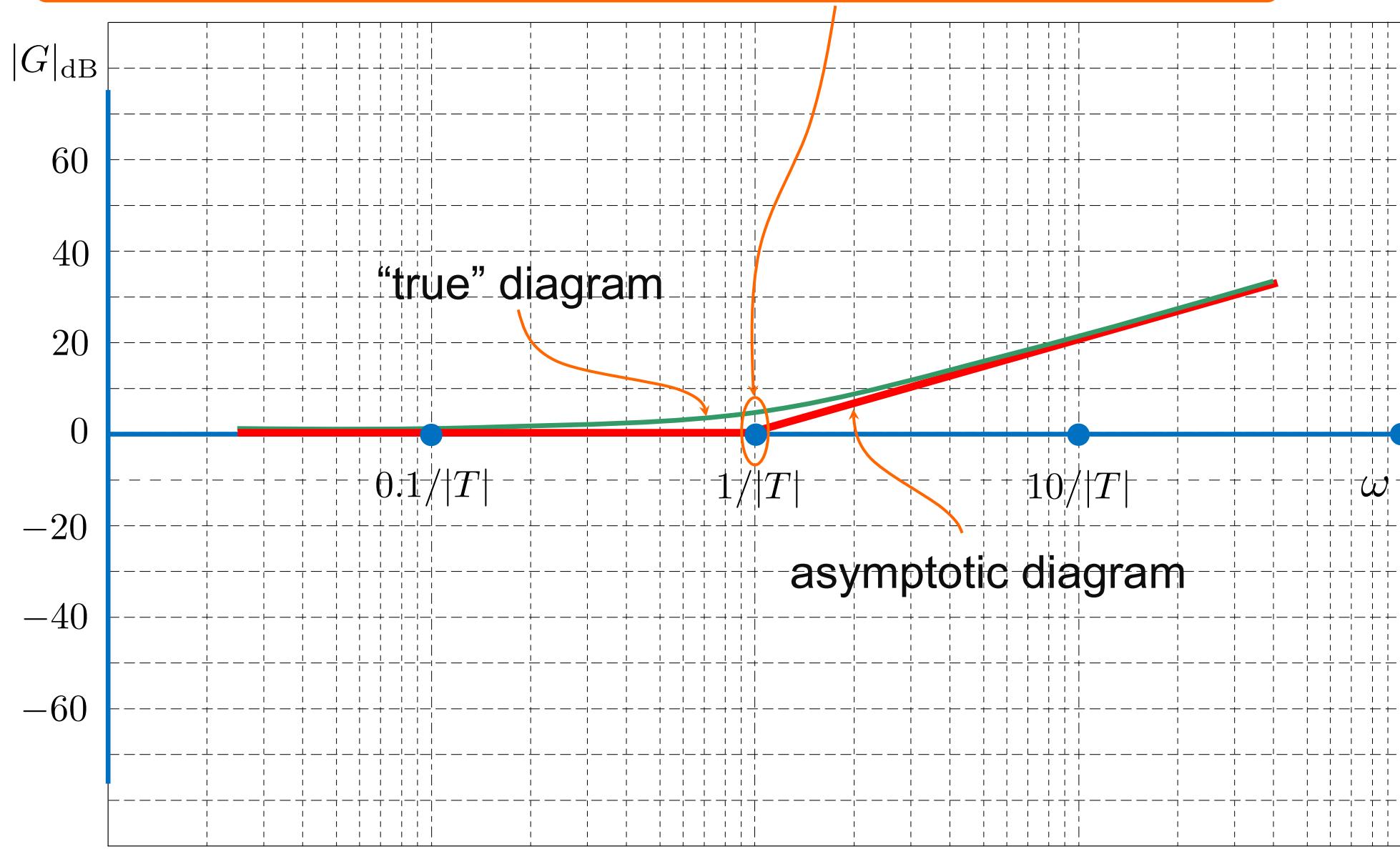
↳ $20 \log |1 + j\omega T| \simeq 20 \log |j\omega T|$

$$= 20 \log \omega |T|$$

$$= 20 \log \omega + 20 \log |T|$$

in logarithmic coordinates
it is a straight line with
slope $20 \cdot \text{dB/dec}$

maximum approximation error for $\omega = 1/|T| : 20 \log \sqrt{2} = 10 \log 2 \simeq 3\text{dB}$



(C),(D) $20 \log |1 + j\omega T| + 20 \log |1 + j\omega T^*|, T, T^* \in \mathbb{C}$

Recall:

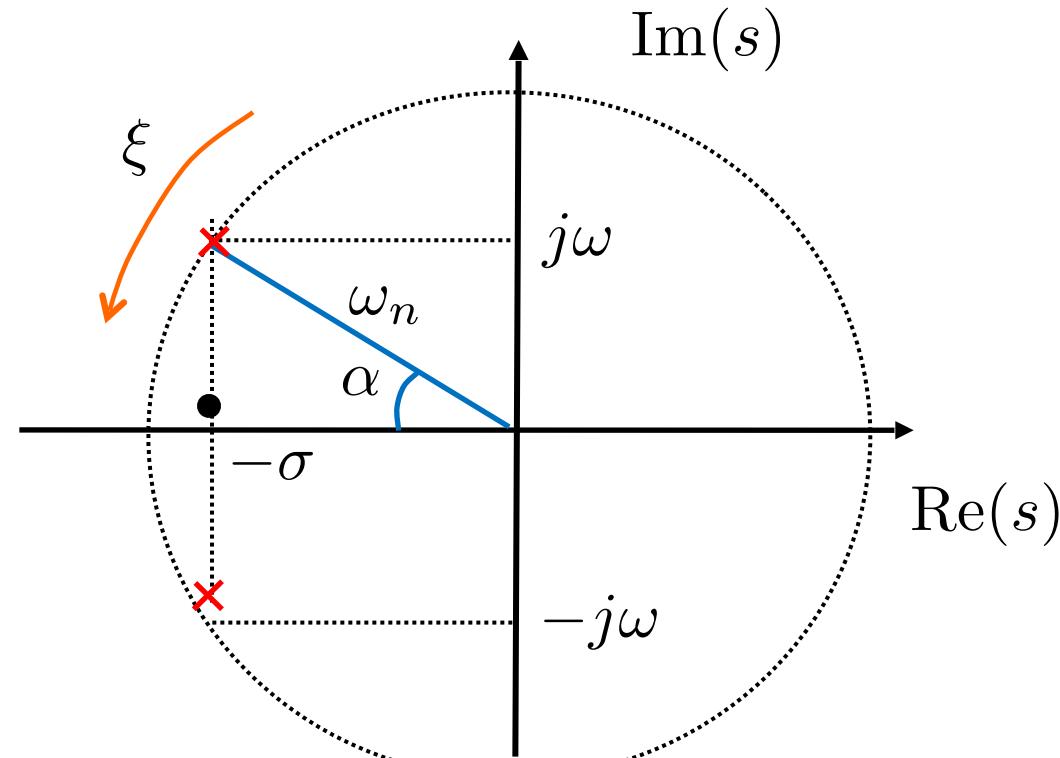
$$(1 + sT)(1 + sT^*)$$

$$= 1 + \frac{2\xi}{\omega_n} s + \frac{1}{\omega_n^2} s^2$$

where:

$0 < \xi < 1$  left half-plane

$-1 < \xi < 0$  right half-plane



$$20 \log |1 + j\omega T| + 20 \log |1 + j\omega T^*|$$

$$= 20 \log \left| 1 + \frac{2\xi}{\omega_n} j\omega - \frac{1}{\omega_n^2} \omega^2 \right| = 20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \frac{4\xi^2 \omega^2}{\omega_n^2}}$$

- If $\omega \rightarrow 0$

↳

$$20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \frac{4\xi^2 \omega^2}{\omega_n^2}} \simeq 0$$

- If $\omega \rightarrow \infty$

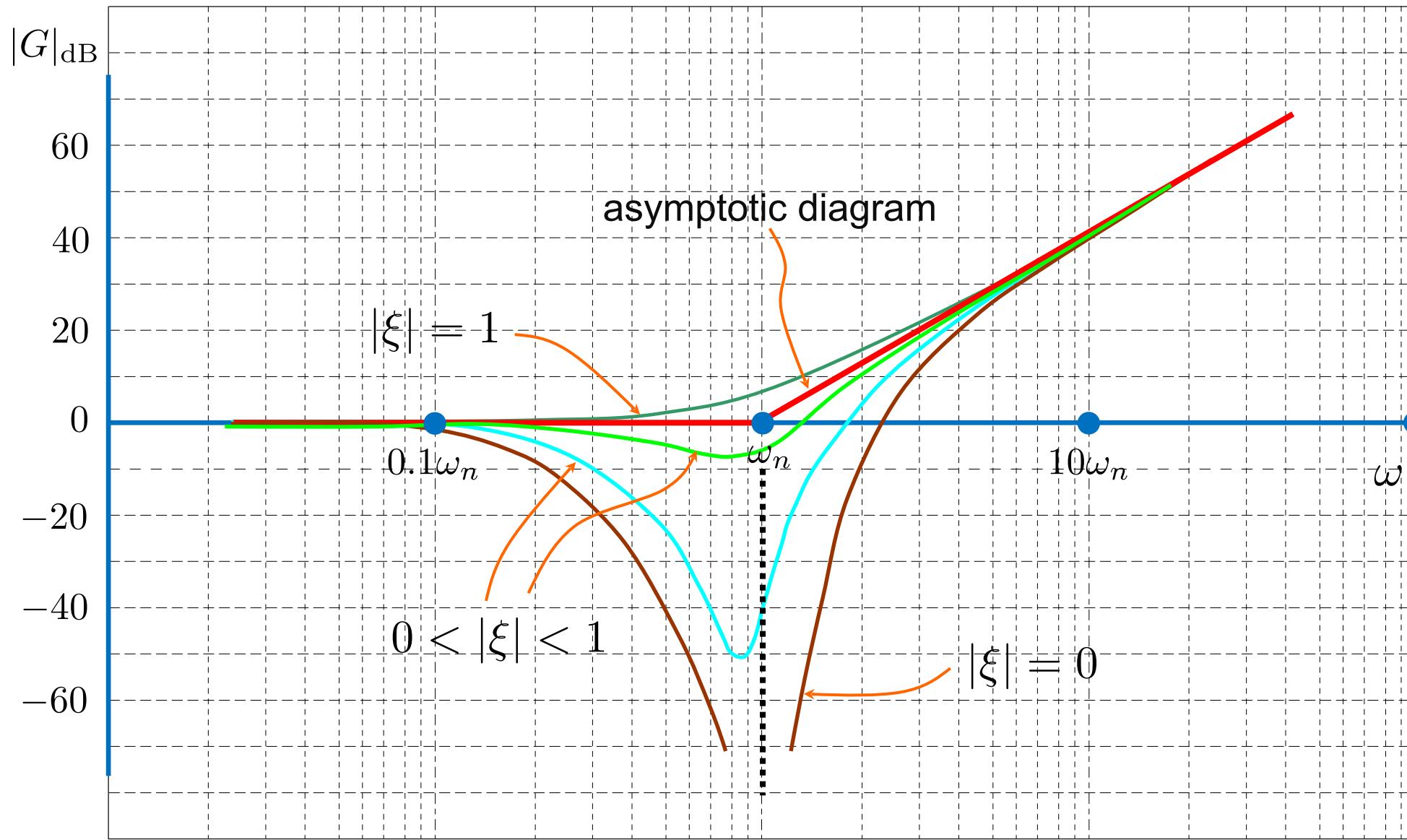
↳

$$20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \frac{4\xi^2 \omega^2}{\omega_n^2}}$$

$$\simeq 20 \log \sqrt{\left(\frac{\omega^2}{\omega_n^2} \right)^2} = 20 \log \frac{\omega^2}{\omega_n^2} = 40 \log \frac{\omega}{\omega_n} = 40 \log \omega - 40 \log \omega_n$$

in logarithmic coordinates
it is a straight line with
slope $40 \cdot \text{dB/dec}$





Maximum approximation error for $\omega = \omega_n$:

↳ $20 \log \sqrt{4\xi^2} = 20 \log(2|\xi|)$

- If $|\xi| = 1$

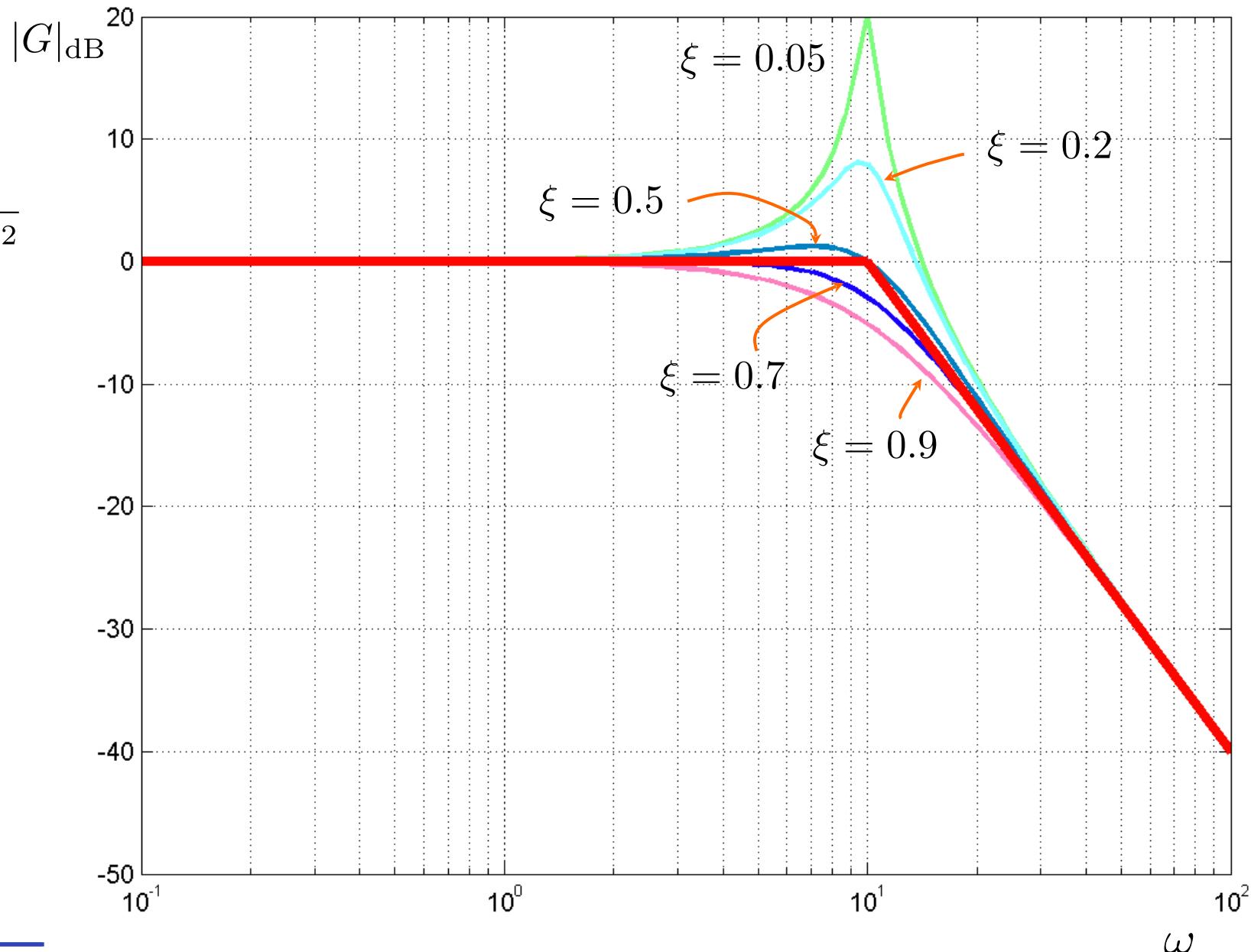
↳ $20 \log(2) \simeq 6\text{dB}$

- If $|\xi| \rightarrow 0$

↳ $\rightarrow -\infty$

Example 1

$$G(s) = \frac{1}{1 + \frac{2\xi}{10}s + \frac{1}{10^2}s^2}$$



Example 2



$$G(s) = \frac{100(1 + 10s)}{s(1 + 2s)(1 + 0.4s + s^2)}$$

$$g = 1$$

$$\mu = 100 \implies \mu_{\text{dB}} = 40 \text{ dB}$$

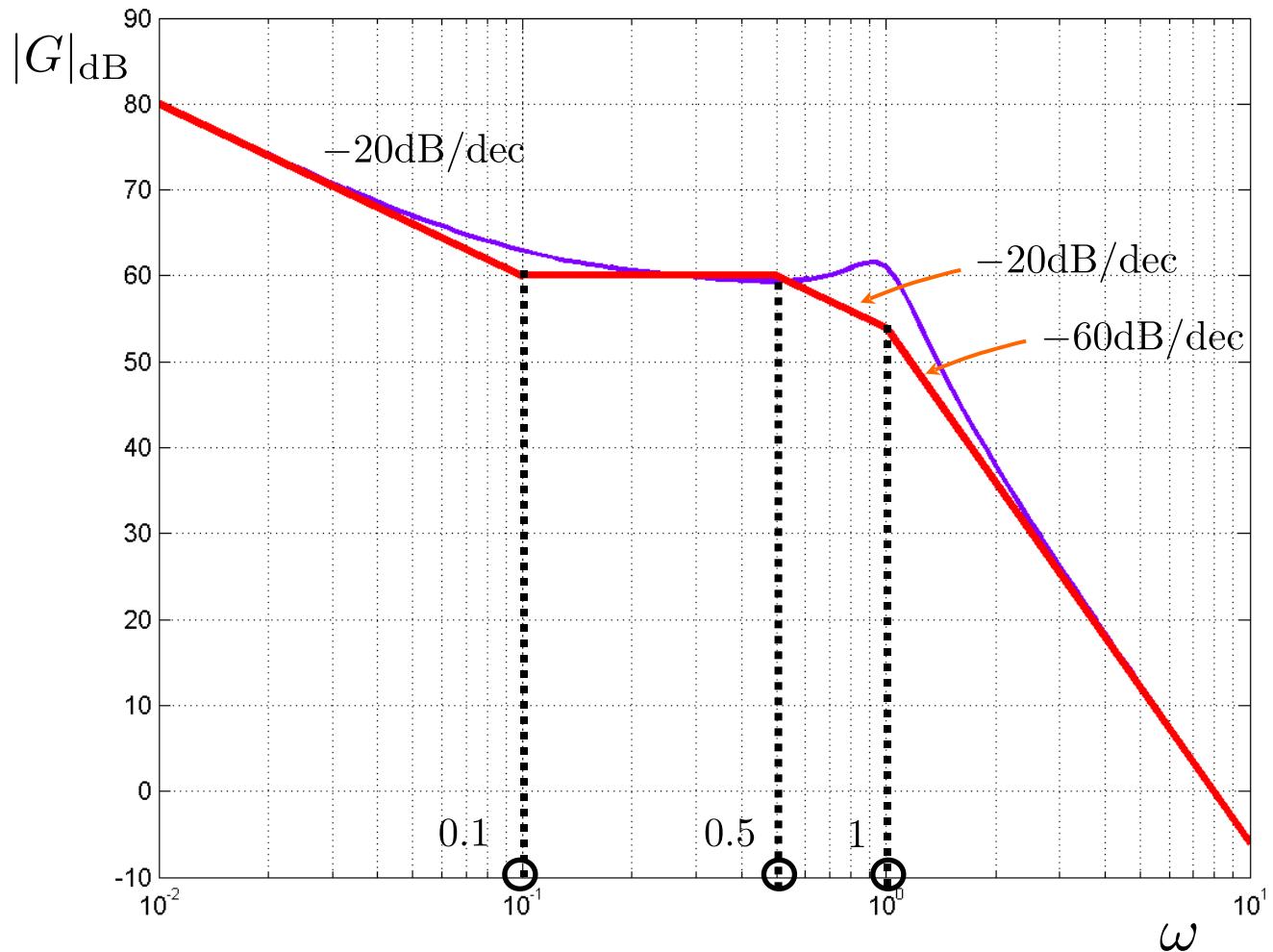
$$z_1 = -0.1$$

$$p_1 = 0$$

$$p_2 = -0.5$$

$$p_{3,4} = -0.2 \pm j\sqrt{0.96}$$

$$\omega_n = 1; \xi = 0.2$$



Example 3

$$G(s) = \frac{100(1 + 10s)}{s(1 + s)^2}$$

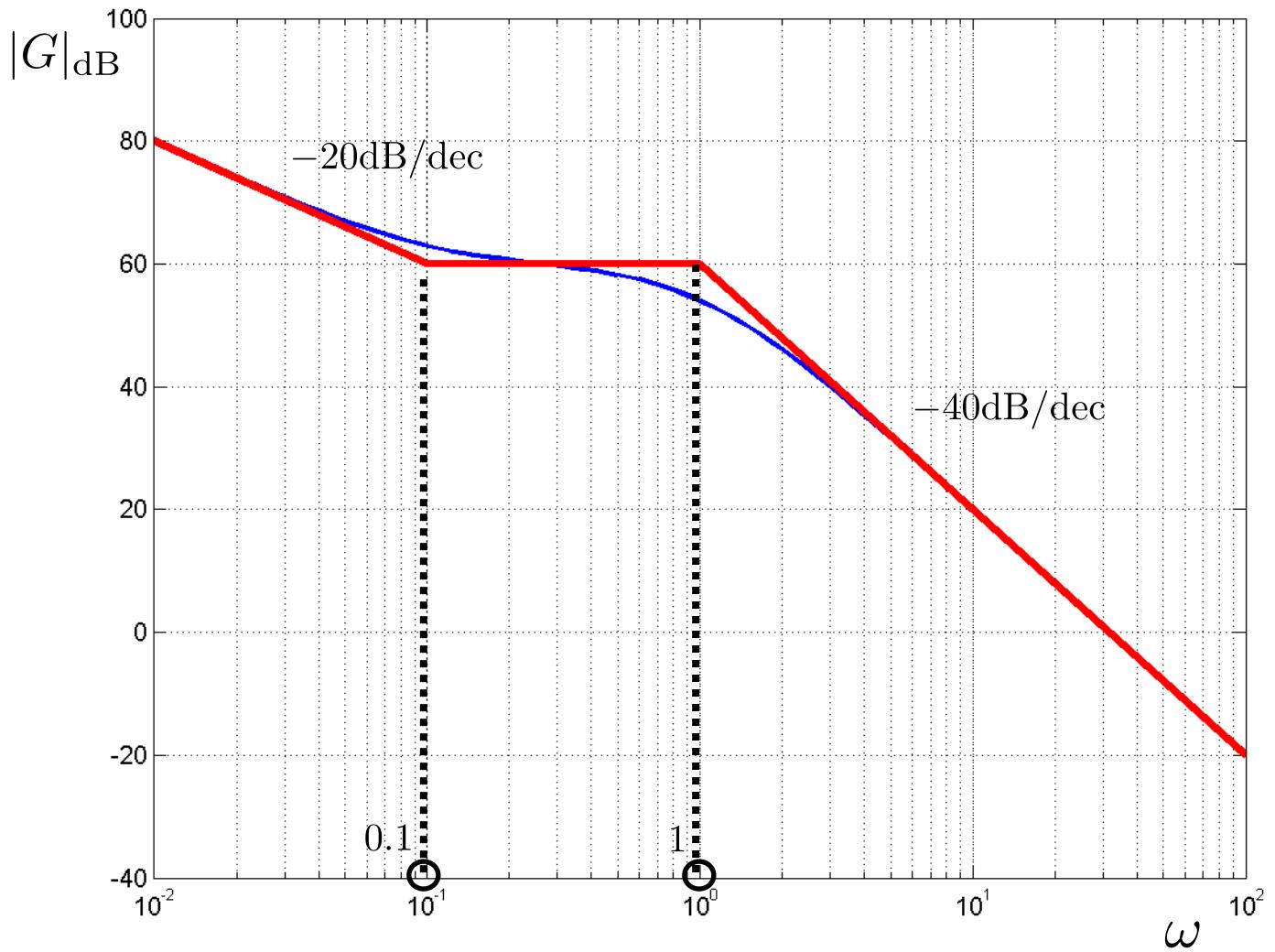
$$g = 1$$

$$\mu = 100 \implies \mu_{\text{dB}} = 40 \text{dB}$$

$$z_1 = -0.1$$

$$p_1 = 0$$

$$p_2 = p_3 = -1$$



Example 4

$$G(s) = \frac{0.1s(1+s)}{(1+5s)^2(1+0.2s)(1-0.1s)}$$

$$g = -1$$

$$\mu = 0.1 \implies \mu_{\text{dB}} = -20 \text{ dB}$$

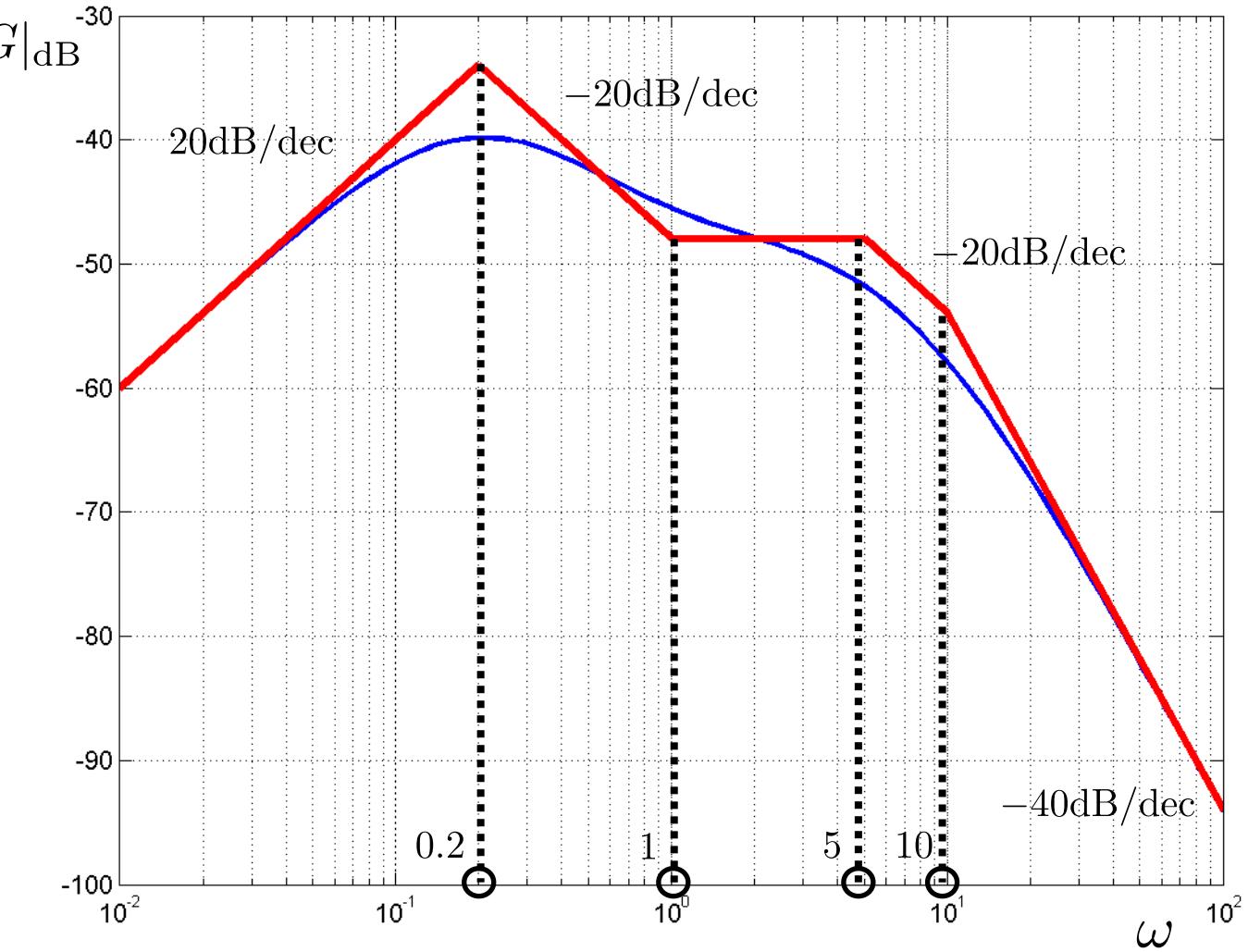
$$z_1 = 0$$

$$z_2 = -1$$

$$p_1 = p_2 = -0.2$$

$$p_3 = -5$$

$$p_4 = 10$$



Example 5

$$G'(s) = \frac{50(1 + 0.4s)}{(1 + 10s)(1 + 0.2s + s^2)}$$

$$G''(s) = \frac{50(1 + 0.4s)}{(1 + 10s)(1 + 1.6s + s^2)}$$

$$g = 0$$

$$\mu = 50 \implies \mu_{\text{dB}} \simeq 34 \text{ dB}$$

$$z_1 = -2.5$$

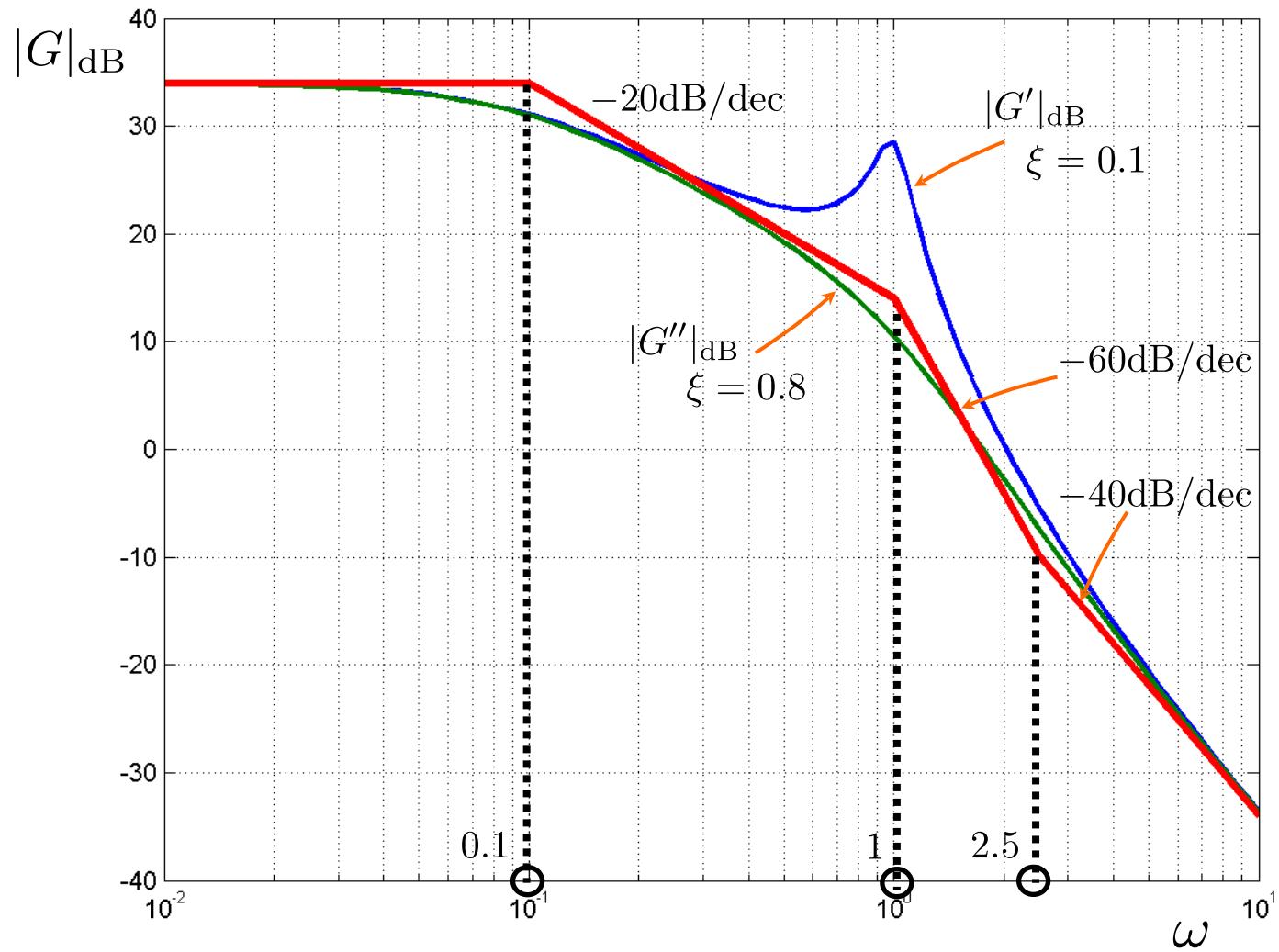
$$p_1 = -0.1$$

$$p'_{2,3} = -0.1 \pm j\sqrt{0.99}$$

$$\omega_n = 1, \xi = 0.1$$

$$p''_{2,3} = -0.8 \pm j\sqrt{0.36}$$

$$\omega_n = 1, \xi = 0.8$$



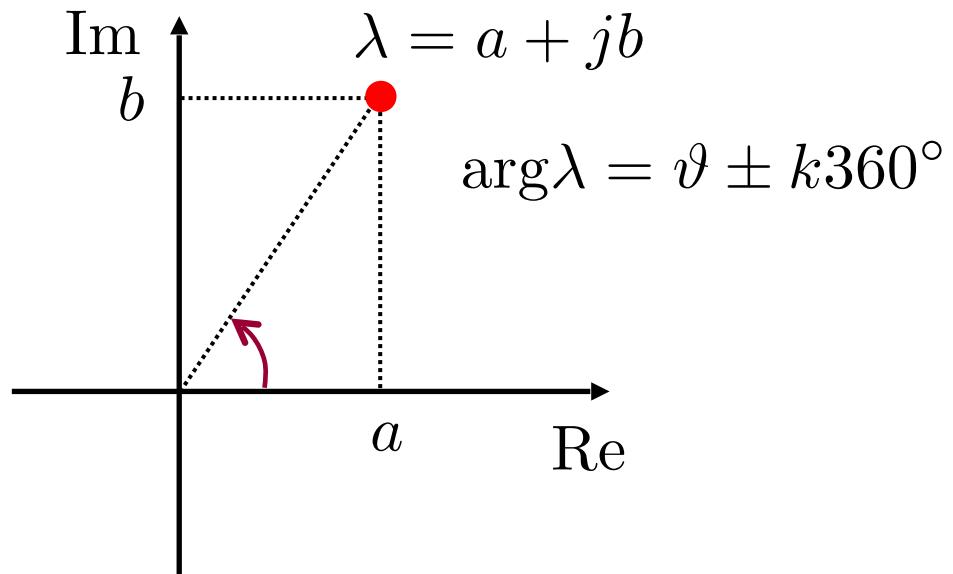


First, recall the **conventions** and **assumptions** for the frequency response diagram of the phase:

- **Phase:**
 - on the “horizontal axis”: $\log \omega$
 - on the “vertical axis”: $\arg G(j\omega)$ in degrees

Care has to be exercised on the **calculation of the phase of a complex number**:

Conventions:



Remark: the Matlab Control Systems Toolbox uses a different convention since it represents the phase in the Bode diagrams in the interval $(-180^\circ, +180^\circ]$. Hence, the Bode diagrams of the phase provided by Matlab in some cases differ from the ones with our convention by a constant angle. This issue will be discussed in Part 10 in a case-by-case way.

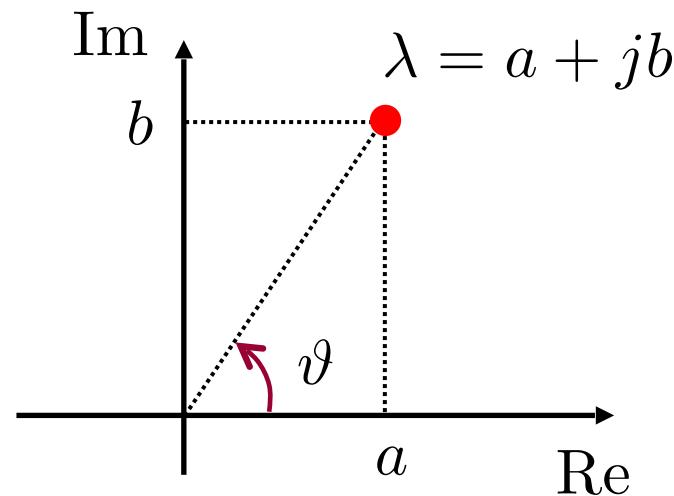
- $\vartheta > 0$: counterclockwise rotations
- $\arg \lambda \in [-180^\circ, 180^\circ)$
- If $a > 0$
 - ↳ $\arg \lambda = \arctg \frac{b}{a}$
- If $a < 0 ; b > 0$
 - ↳ $\arg \lambda = \arctg \frac{b}{a} + 180^\circ$
- If $a < 0 ; b < 0$
 - ↳ $\arg \lambda = \arctg \frac{b}{a} - 180^\circ$

Bode Diagrams: Phase (contd.)

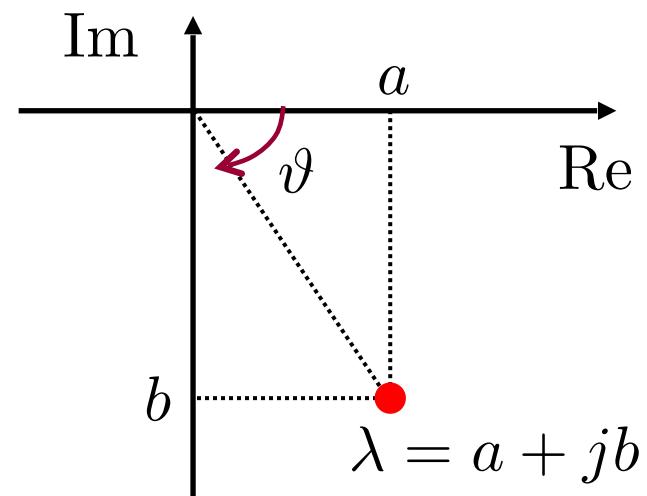


More specifically:

- If $a > 0, b \neq 0$  $\arg \lambda = \arctg \frac{b}{a}$



$$0^\circ < \vartheta < +90^\circ$$



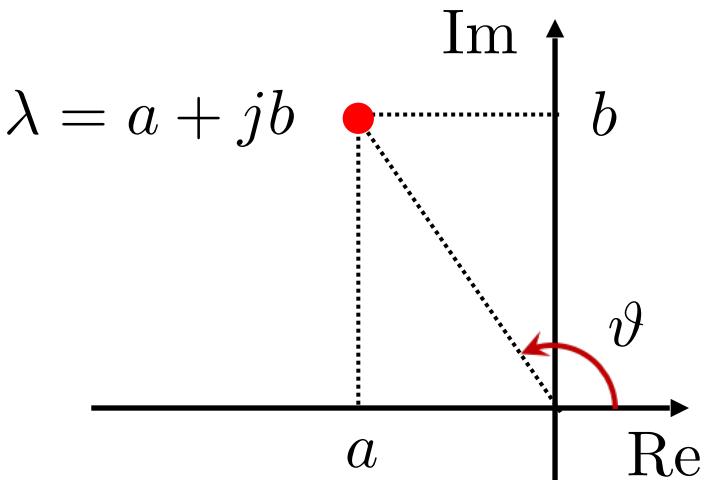
$$-90^\circ < \vartheta < 0^\circ$$

Bode Diagrams: Phase (contd.)



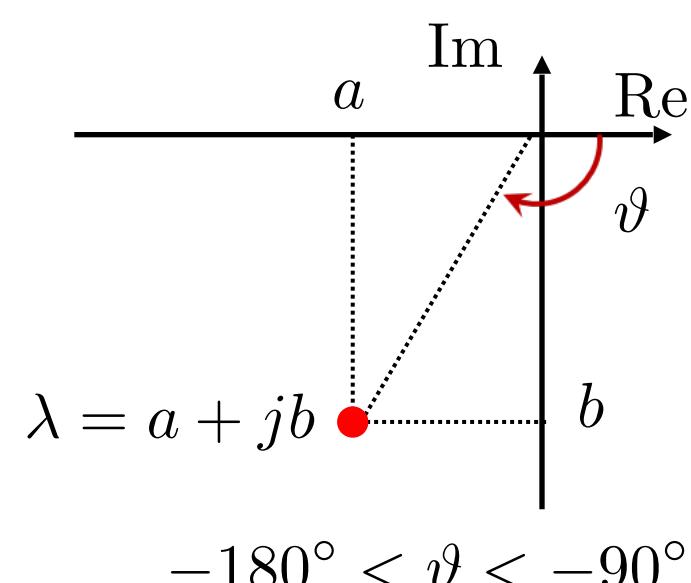
- If $a < 0 ; b > 0$

↳ $\arg \lambda = \operatorname{arctg} \frac{b}{a} + 180^\circ$



- If $a < 0 ; b < 0$

↳ $\arg \lambda = \operatorname{arctg} \frac{b}{a} - 180^\circ$

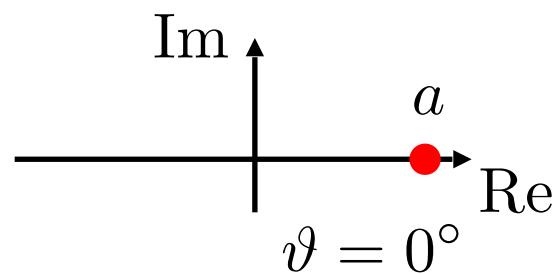


Bode Diagrams: Phase (contd.)



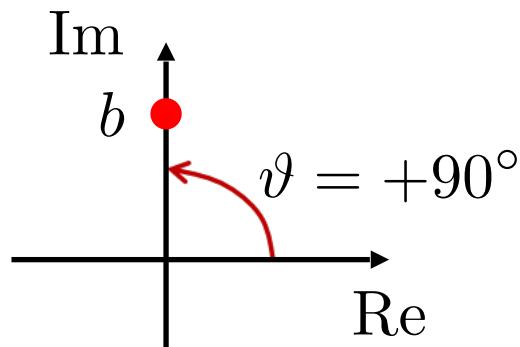
- If $a > 0, b = 0$

↳ $\arg \lambda = 0^\circ$



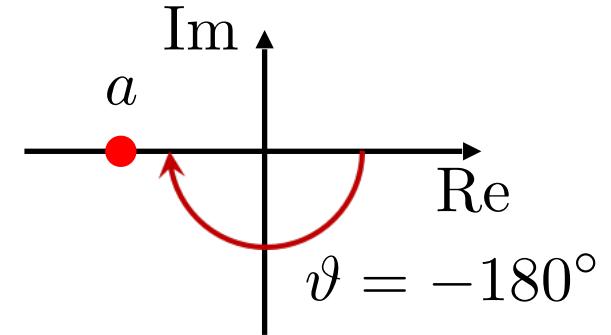
- If $a = 0, b > 0$

↳ $\arg \lambda = +90^\circ$



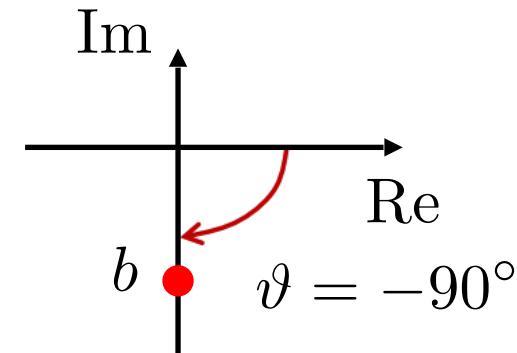
- If $a < 0, b = 0$

↳ $\arg \lambda = -180^\circ$



- If $a = 0, b < 0$

↳ $\arg \lambda = -90^\circ$





The phase of a complex number satisfies some useful properties:

- $\arg(\lambda\eta) = \arg(\lambda) + \arg(\eta)$
- $\arg(\lambda^k) = k \arg(\lambda)$
- $\arg\left(\frac{\lambda}{\eta}\right) = \arg(\lambda) - \arg(\eta)$



The calculation of the phase of $G(j\omega)$ satisfies properties that are analogous to the ones of the logarithm in the case of the calculation of $20 \log |G(j\omega)|$

Bode Diagrams: Phase (contd.)

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^m (1 + sT_i)}{\prod_{i=1}^n (1 + s\tau_i)}$$



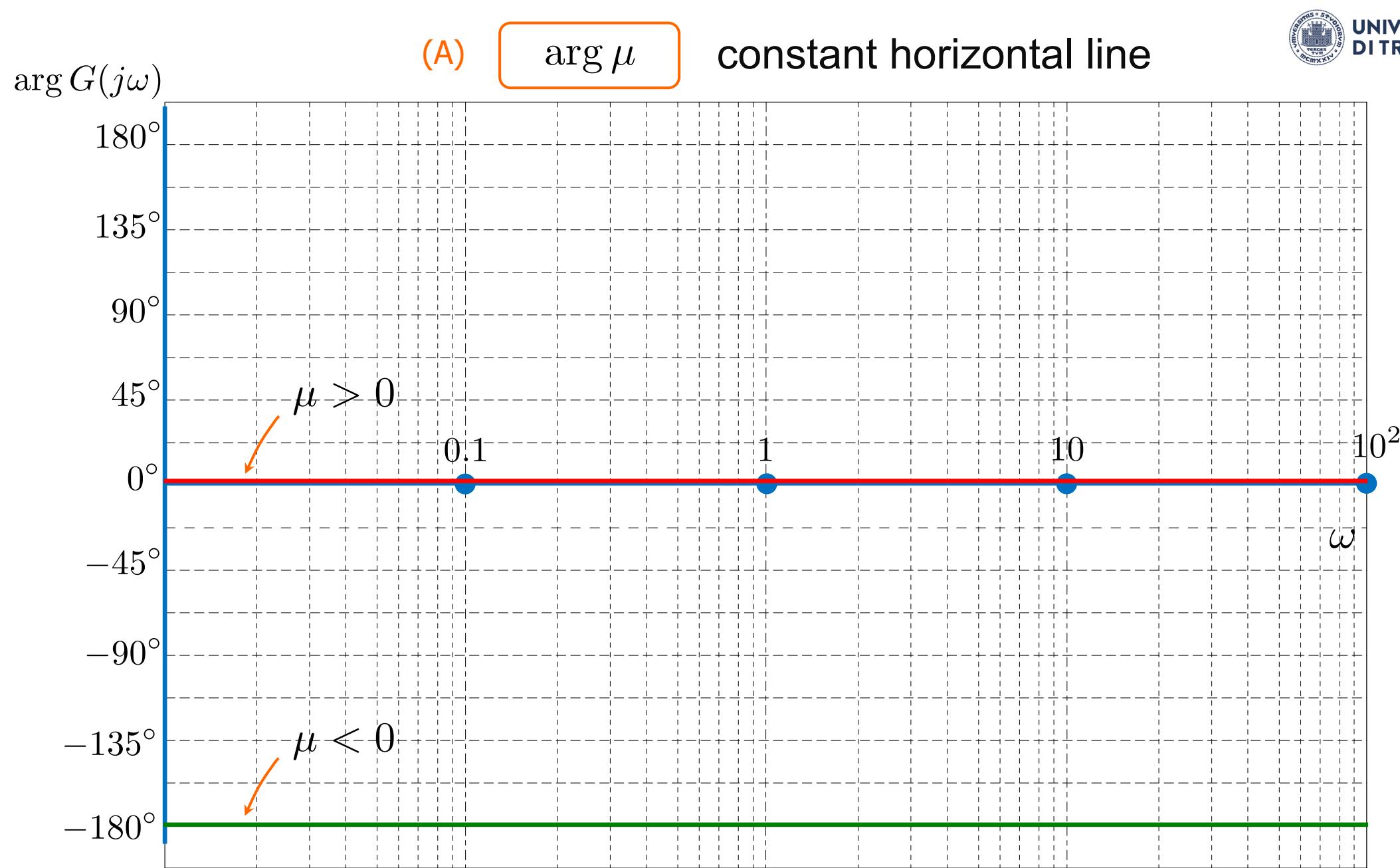
$$\arg G(j\omega) = \boxed{\arg \mu} \quad (\text{A})$$

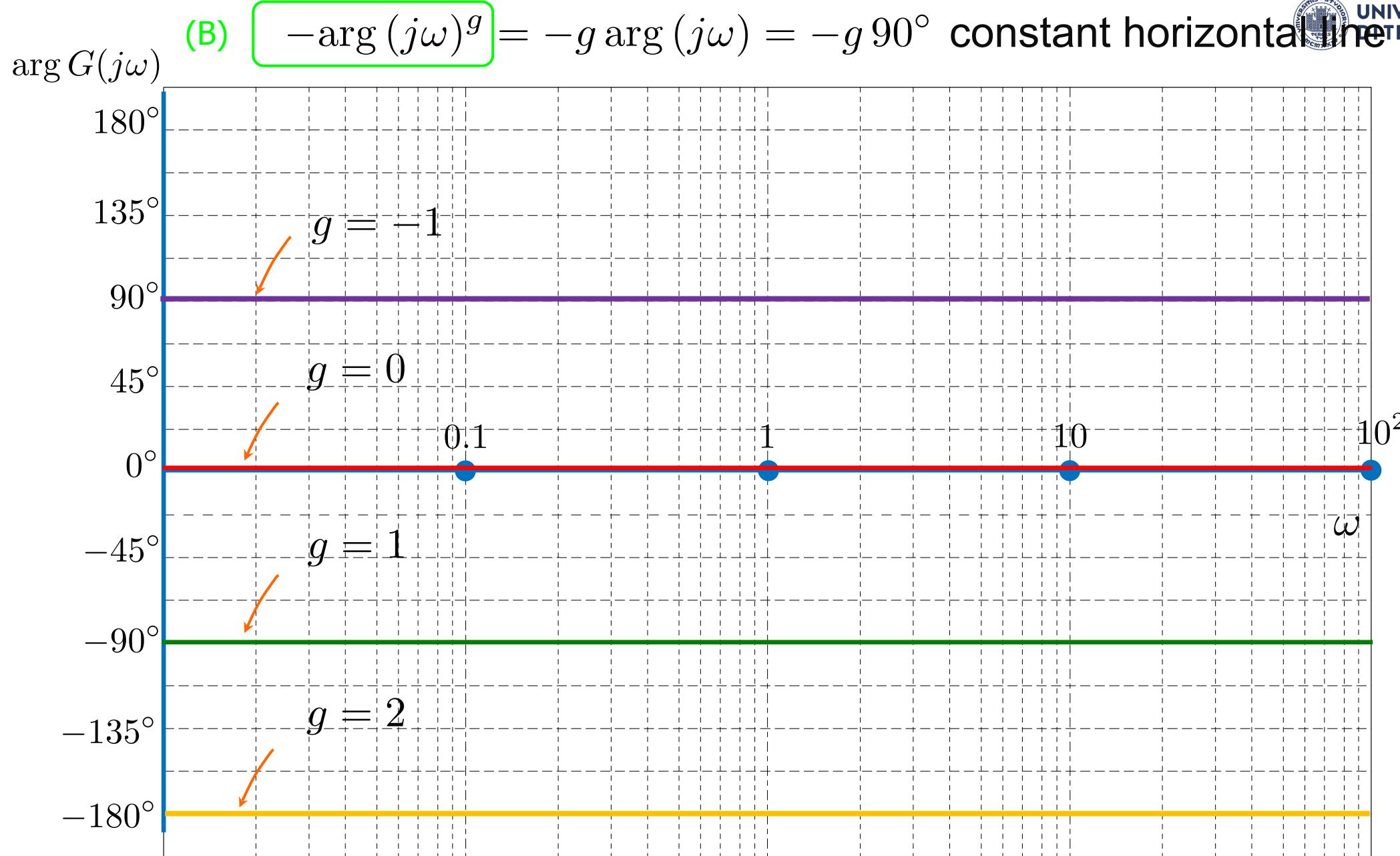
$$- \boxed{\arg (j\omega)^g} \quad (\text{B})$$

$$+ \sum_i \boxed{\arg (1 + j\omega T_i)} \quad (\text{C}), (\text{D})$$

$$- \sum_i \boxed{\arg (1 + j\omega \tau_i)}$$

- The conventions have to be applied separately on each single element (A), (B), (C), and (D)
- All such phase contributions have to be added along with their sign
- The total phase at each value of the angular frequency ω may well take on a value outside $[-180^\circ, 180^\circ]$





(C) $\boxed{\arg(1 + j\omega T), T \in \mathbb{R}}$

↳ $\arg(1 + j\omega T) = \operatorname{arctg} \omega T$

- If $\omega \rightarrow 0$

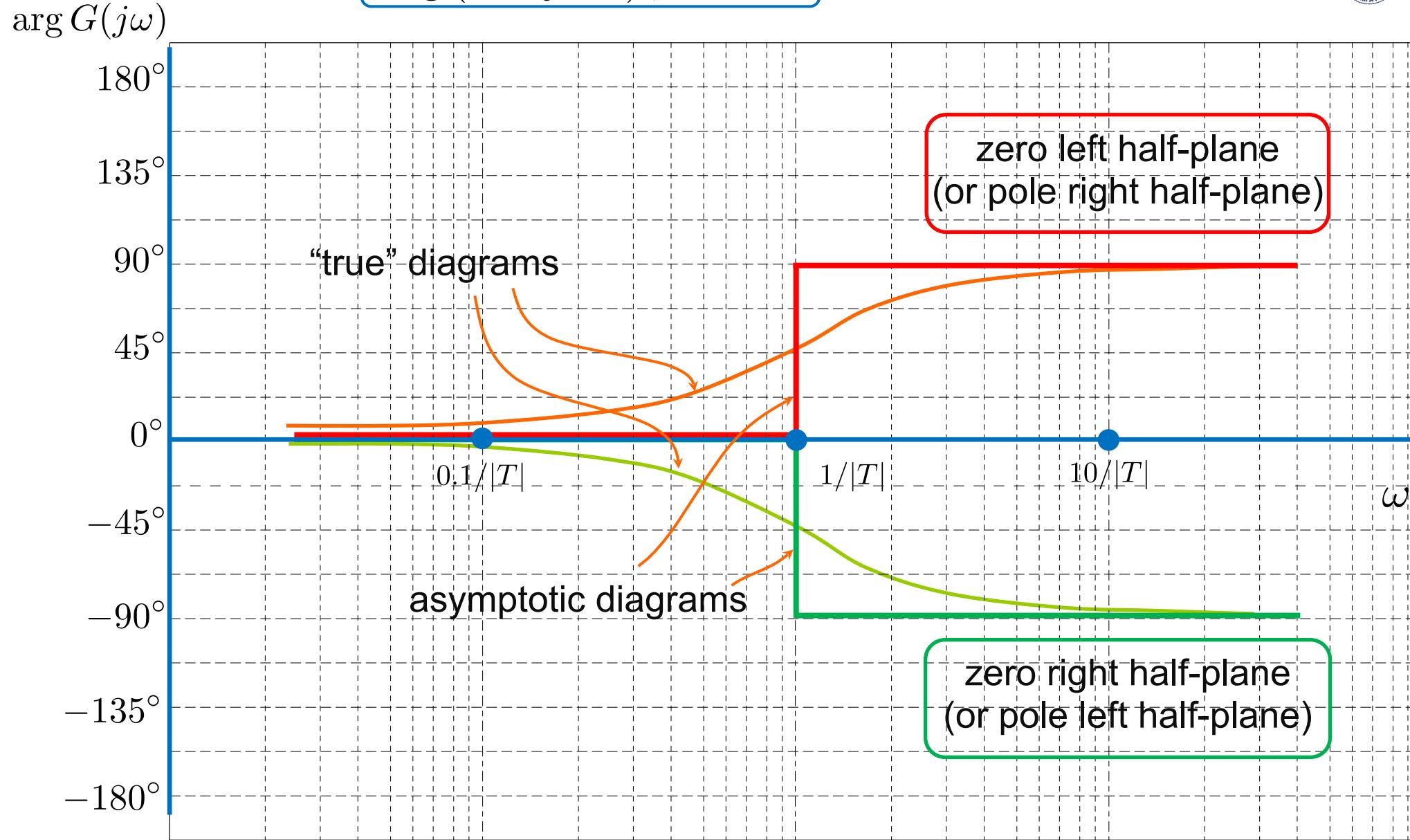
↳ $\arg(1 + j\omega T) \rightarrow 0^\circ$

- If $\omega \rightarrow \infty$

↳ $\arg(1 + j\omega T) \begin{cases} \rightarrow 90^\circ & \text{if } T > 0 \\ \rightarrow -90^\circ & \text{if } T < 0 \end{cases}$

- If $\omega = \frac{1}{|T|}$

↳ $\arg(1 + j\omega T) = \begin{cases} 45^\circ & \text{if } T > 0 \\ -45^\circ & \text{if } T < 0 \end{cases}$

(C) $\arg(1 + j\omega T), T \in \mathbb{R}$


(D) $\arg(1 + j\omega T) + \arg(1 + j\omega T^*)$, $T \in \mathbb{C}$



↳ $\arg(1 + j\omega T) + \arg(1 + j\omega T^*) = \arg\left(1 + \frac{2\xi}{\omega_n}j\omega - \frac{1}{\omega_n^2}\omega^2\right)$

- If $\omega \rightarrow 0$

↳ $\arg\left(1 + \frac{2\xi}{\omega_n}j\omega - \frac{1}{\omega_n^2}\omega^2\right) \simeq \arg(1) = 0^\circ$

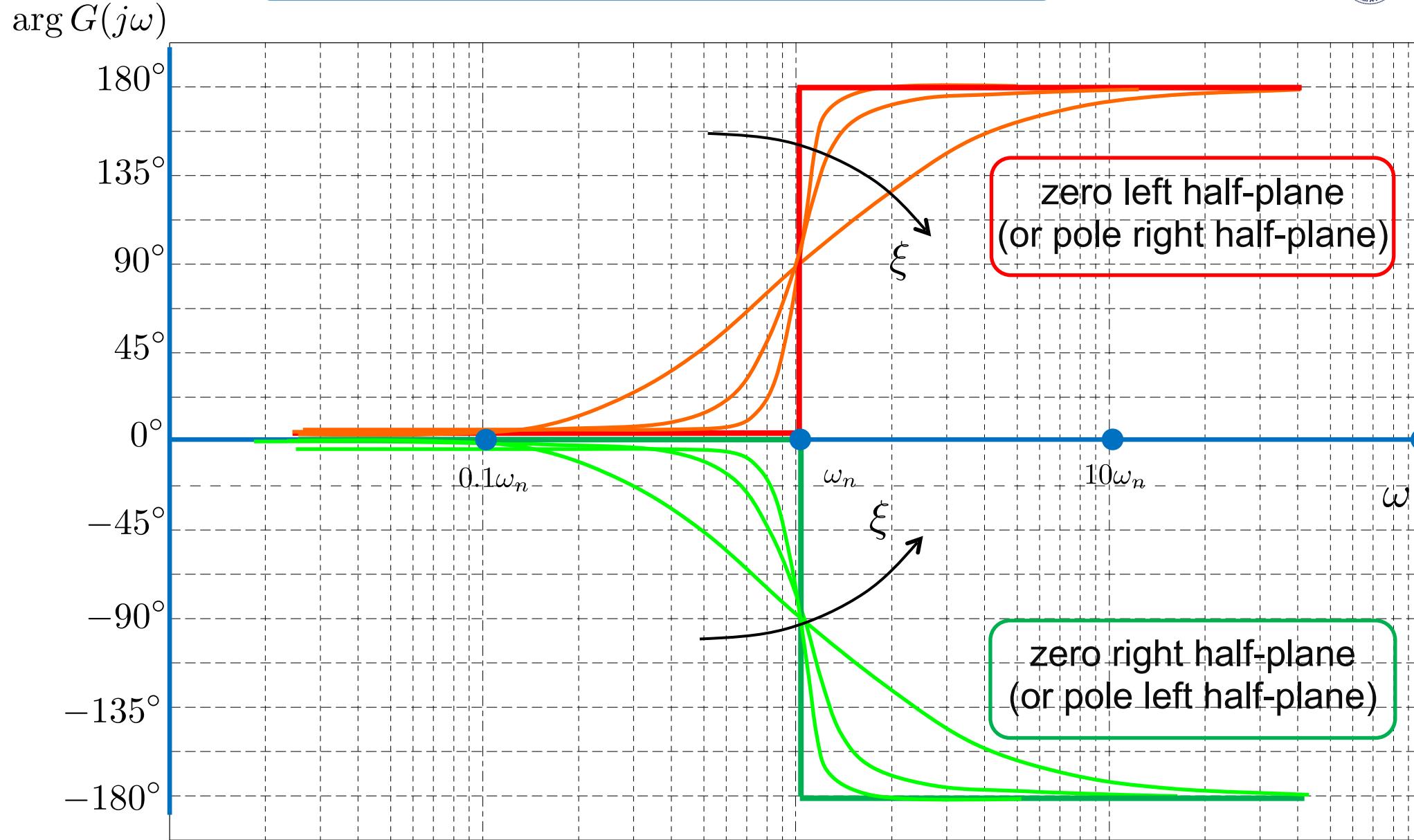
- If $\omega \rightarrow \infty$

↳ $\lim_{\omega \rightarrow \infty} \operatorname{arctg} \frac{2\xi\omega/\omega_n}{1 - \omega^2/\omega_n^2} = 0^\circ \begin{cases} +180^\circ & \text{if } \xi > 0 \\ -180^\circ & \text{if } \xi < 0 \end{cases}$

- If $\omega = \omega_n$

↳ $\arg\left(j \frac{2\xi\omega_n}{\omega_n}\right) = \begin{cases} 90^\circ & \text{if } \xi > 0 \\ -90^\circ & \text{if } \xi < 0 \end{cases}$

(D) $\arg(1 + j\omega T) + \arg(1 + j\omega T^*)$, $T \in \mathbb{C}$



Example 2 (contd.)

$$G(s) = \frac{100(1 + 10s)}{s(1 + 2s)(1 + 0.4s + s^2)}$$

$$g = 1$$

$$\mu = 100 \implies \mu_{\text{dB}} = 40 \text{dB}$$

$$z_1 = -0.1$$

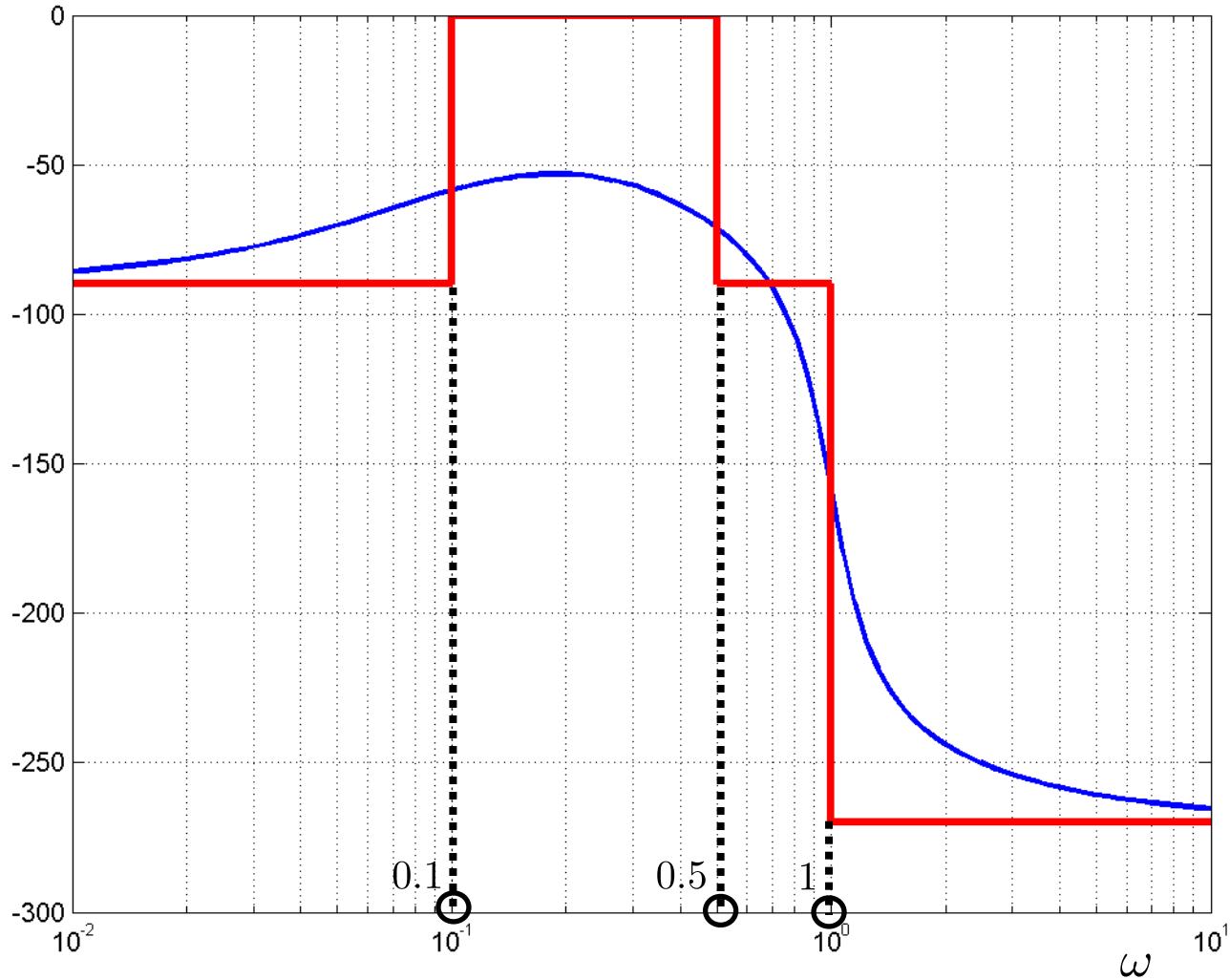
$$p_1 = 0$$

$$p_2 = -0.5$$

$$p_{3,4} = -0.2 \pm j\sqrt{0.96}$$

$$\omega_n = 1; \xi = 0.2$$

$\arg G(j\omega)$



Example 3 (contd.)

$$G(s) = \frac{100(1 + 10s)}{s(1 + s)^2}$$

$$g = 1$$

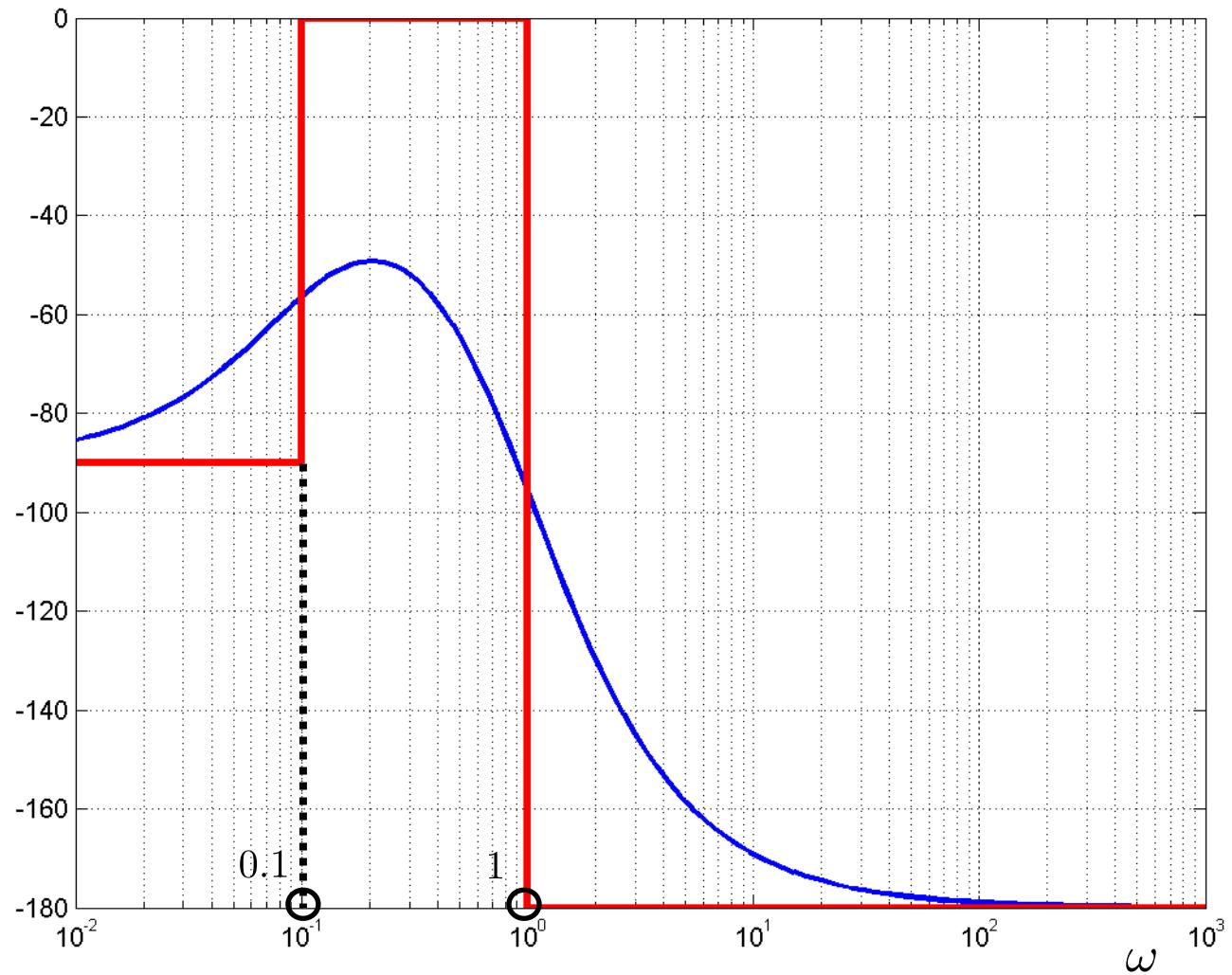
$$\mu = 100 \implies \mu_{\text{dB}} = 40\text{dB}$$

$$z_1 = -0.1$$

$$p_1 = 0$$

$$p_2 = p_3 = -1$$

$\arg G(j\omega)$



Example 4 (contd.)

$$G(s) = \frac{0.1s(1+s)}{(1+5s)^2(1+0.2s)(1-0.1s)}$$

$$g = -1$$

$$\mu = 0.1 \implies \mu_{\text{dB}} = -20 \text{dB}$$

$$z_1 = 0$$

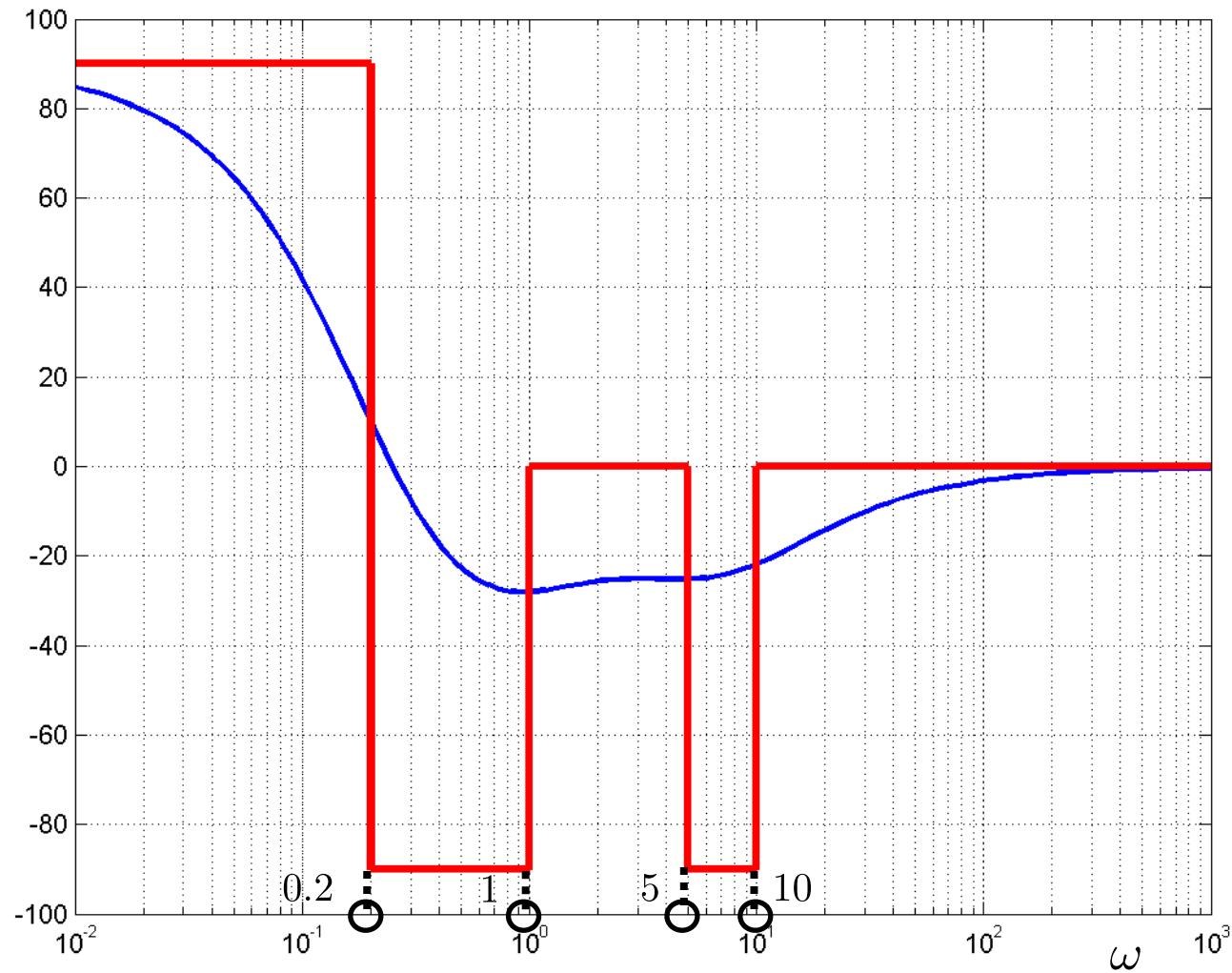
$$z_2 = -1$$

$$p_1 = p_2 = -0.2$$

$$p_3 = -5$$

$$p_4 = 10$$

$\arg G(j\omega)$



Example 5 (contd.)

$$G'(s) = \frac{50(1 + 0.4s)}{(1 + 10s)(1 + 0.2s + s^2)}$$

$$G''(s) = \frac{50(1 + 0.4s)}{(1 + 10s)(1 + 1.6s + s^2)}$$

$$g = 0$$

$$\mu = 50 \implies \mu_{\text{dB}} \simeq 34 \text{dB}$$

$$z_1 = -2.5$$

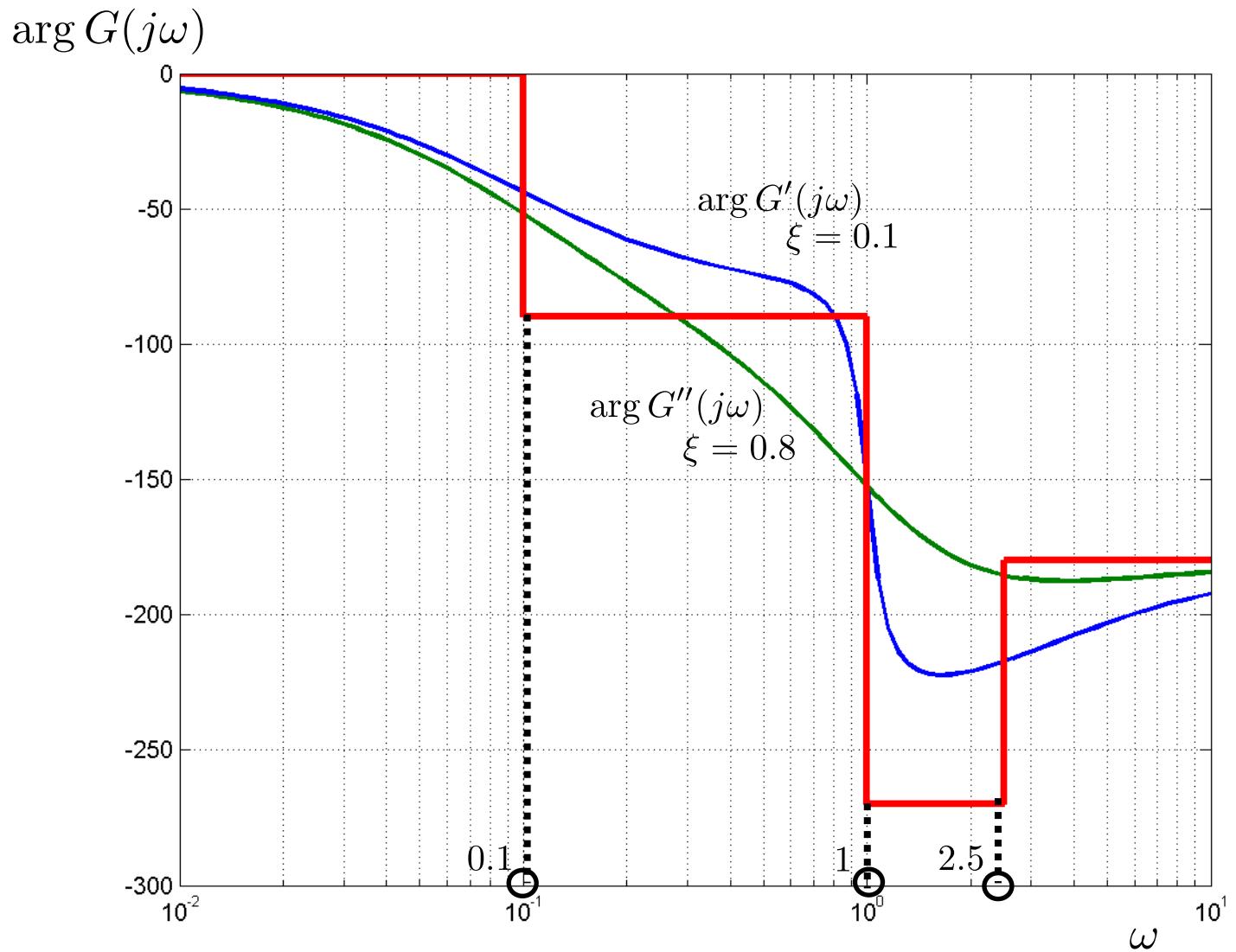
$$p_1 = -0.1$$

$$p'_{2,3} = -0.1 \pm j\sqrt{0.99}$$

$$\omega_n = 1, \xi = 0.1$$

$$p''_{2,3} = -0.8 \pm j\sqrt{0.36}$$

$$\omega_n = 1, \xi = 0.8$$

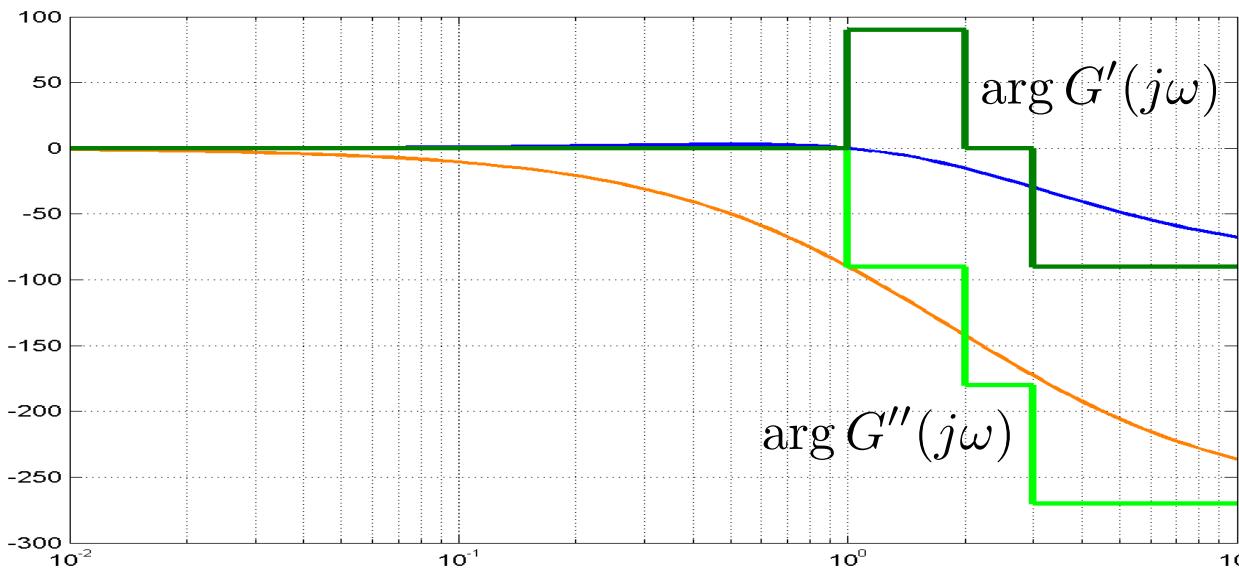
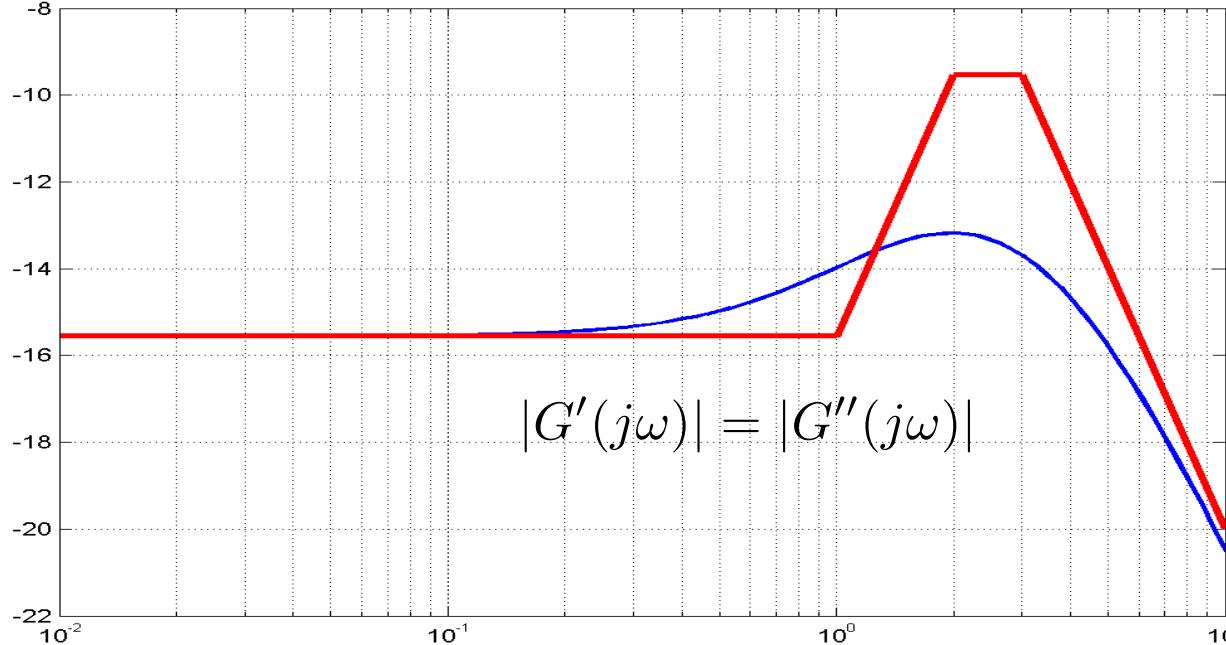


Example - Non Minimum Phase



$$G'(s) = \frac{1+s}{s^2 + 5s + 6}$$

$$G''(s) = \frac{1-s}{s^2 + 5s + 6}$$





- **Magnitude:**
 - initial slope: $-g \cdot 20\text{dB/dec}$
 - initial segment intersects the point $\omega = 1, |\mu|_{\text{dB}}$
 - slope changes are located in correspondence of poles and zeros:
 - zero: $+20\text{dB/dec}$
 - pole: -20dB/dec
 - final slope: $(\text{nr. zeros} - \text{nr. poles}) \cdot 20\text{dB/dec}$
- **Phase:**
 - initial value: $\arg(\mu) - g 90^\circ$
 - value changes are located in correspondence of poles and zeros:
 - zero in the left half-plane: $+90^\circ$
 - zero in the right half-plane: -90°
 - pole in the left half-plane: -90°
 - pole in the right half-plane: $+90^\circ$

Relationship Between Magnitude and Phase



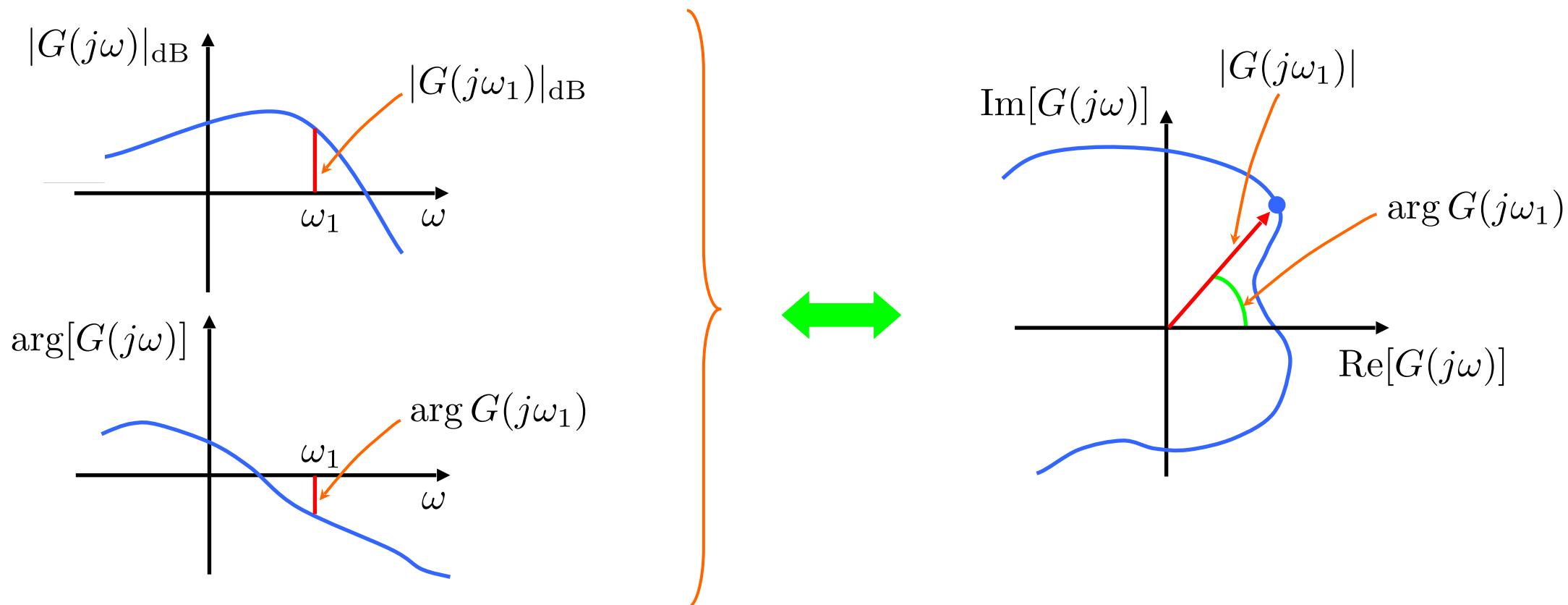
- In general there is no relation between the magnitude $|G(j\omega)|_{\text{dB}}$ and the phase $\arg G(j\omega)$
- An exception is represented by **Minimum Phase Systems**:
 - positive gain: $\mu > 0$
 - all poles and zeros located in the left half-plane
- For minimum phase systems there is a direct relation among the approximate Bode diagrams of magnitude and phase:

	Slope of $ G(j\omega) _{\text{dB}}$	Value of $\arg G(j\omega)$
pole	-20dB/dec	-90°
zero	20dB/dec	90°

Polar Diagram of the Frequency Response



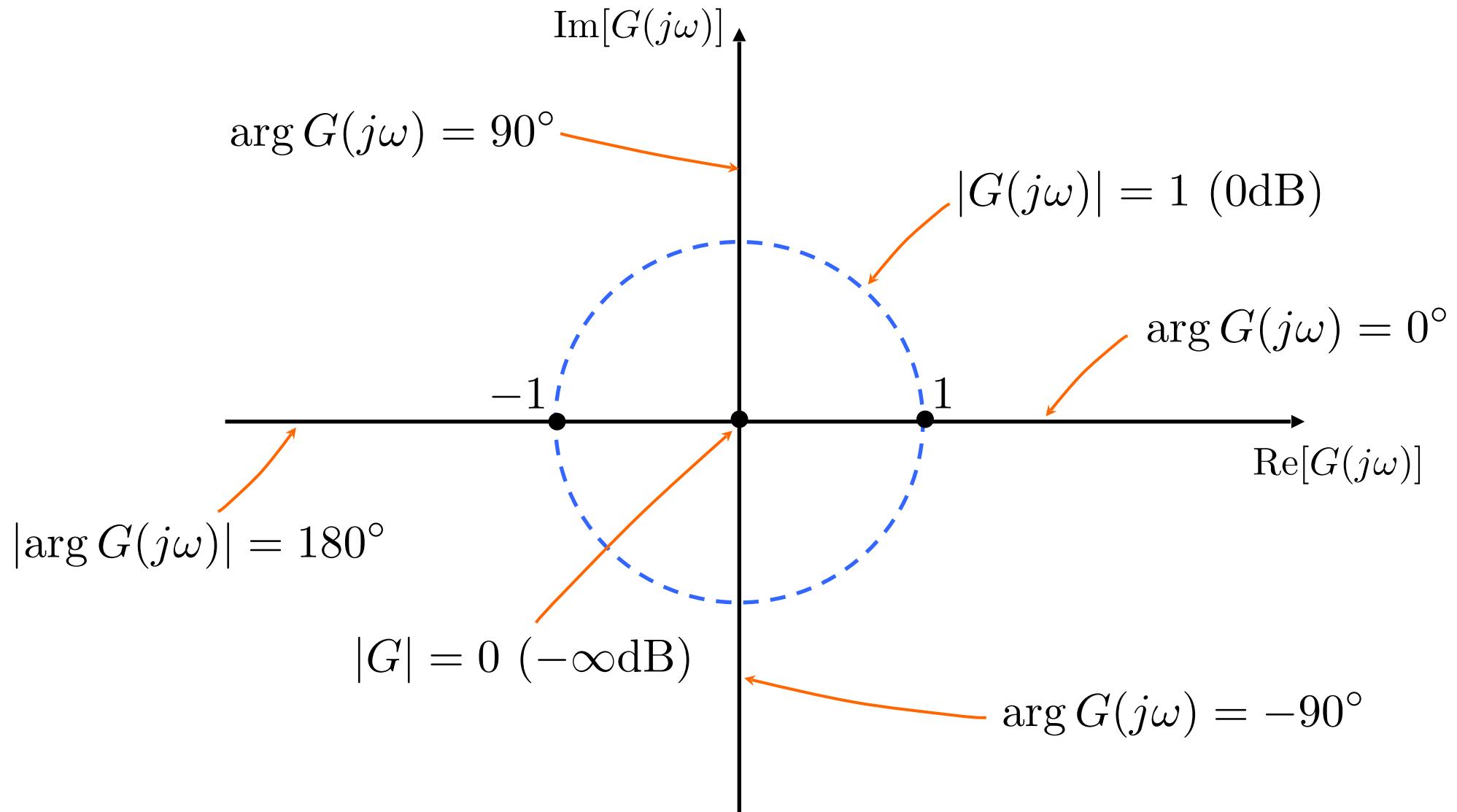
Polar Diagram: $G(j\omega)$, $\omega \geq 0$



Polar and Bode diagrams are related:

$$G(j\omega) = |G(j\omega)| e^{j\arg G(j\omega)}$$

Polar Diagram: Key Elements



Example 1

$$G(s) = \frac{10}{(1 + 10s)(1 + 2s)}$$

$$g = 0$$

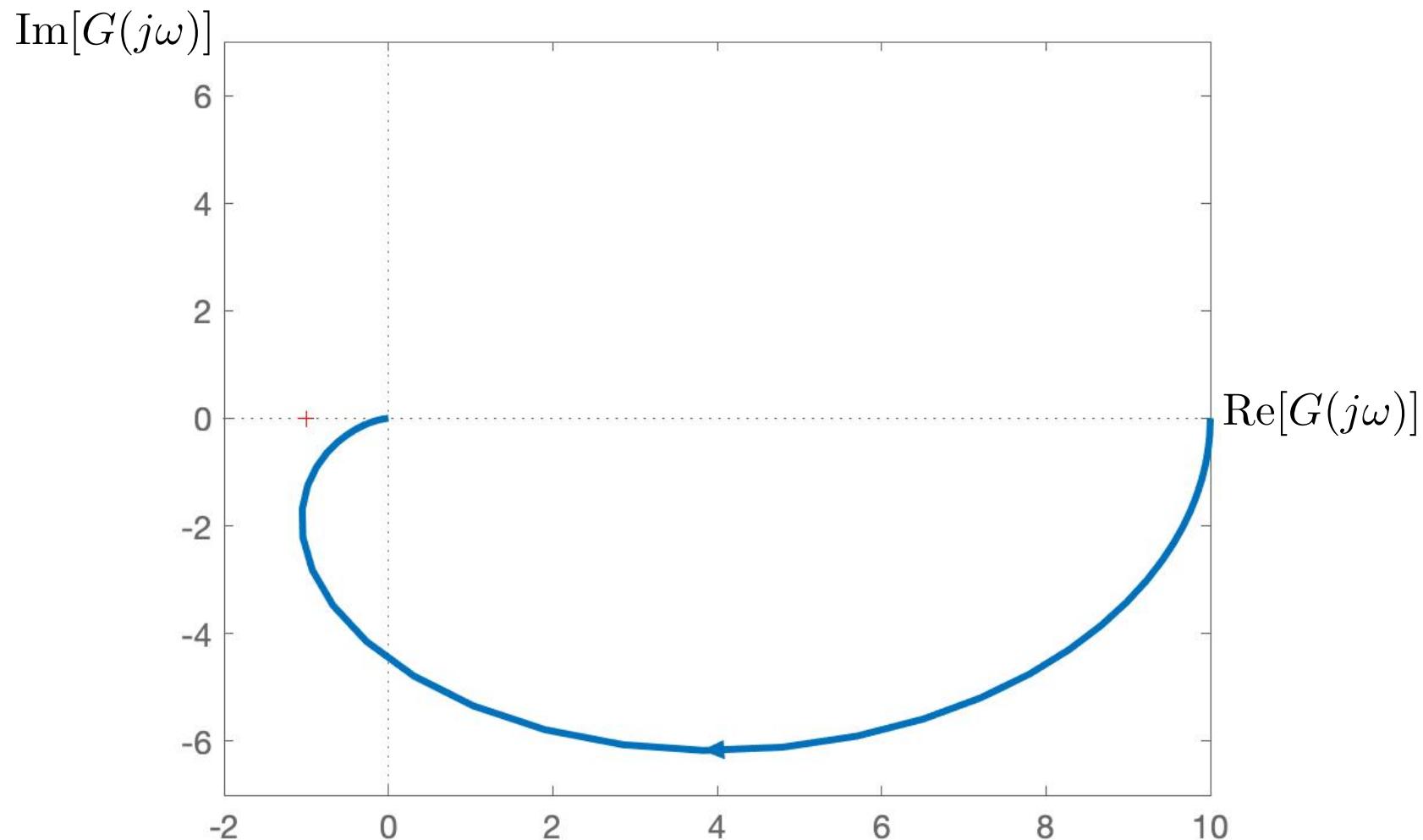
$$\mu = 10 \implies \mu_{\text{dB}} = 20 \text{dB}$$

$$\tau_1 = 10$$

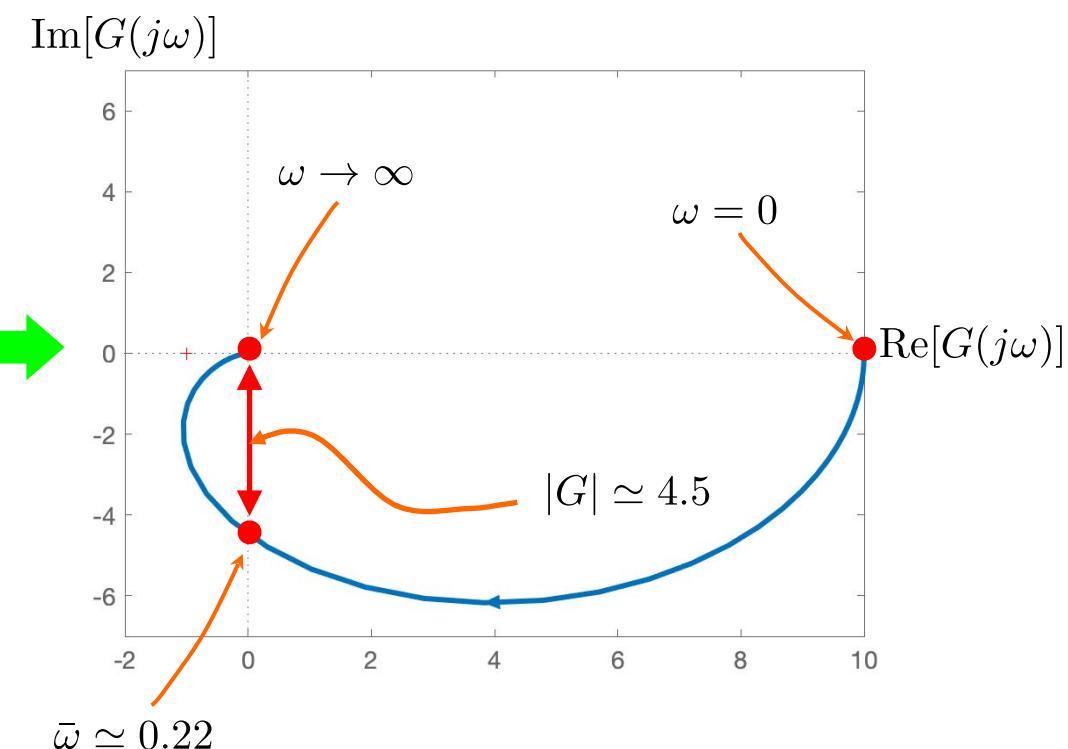
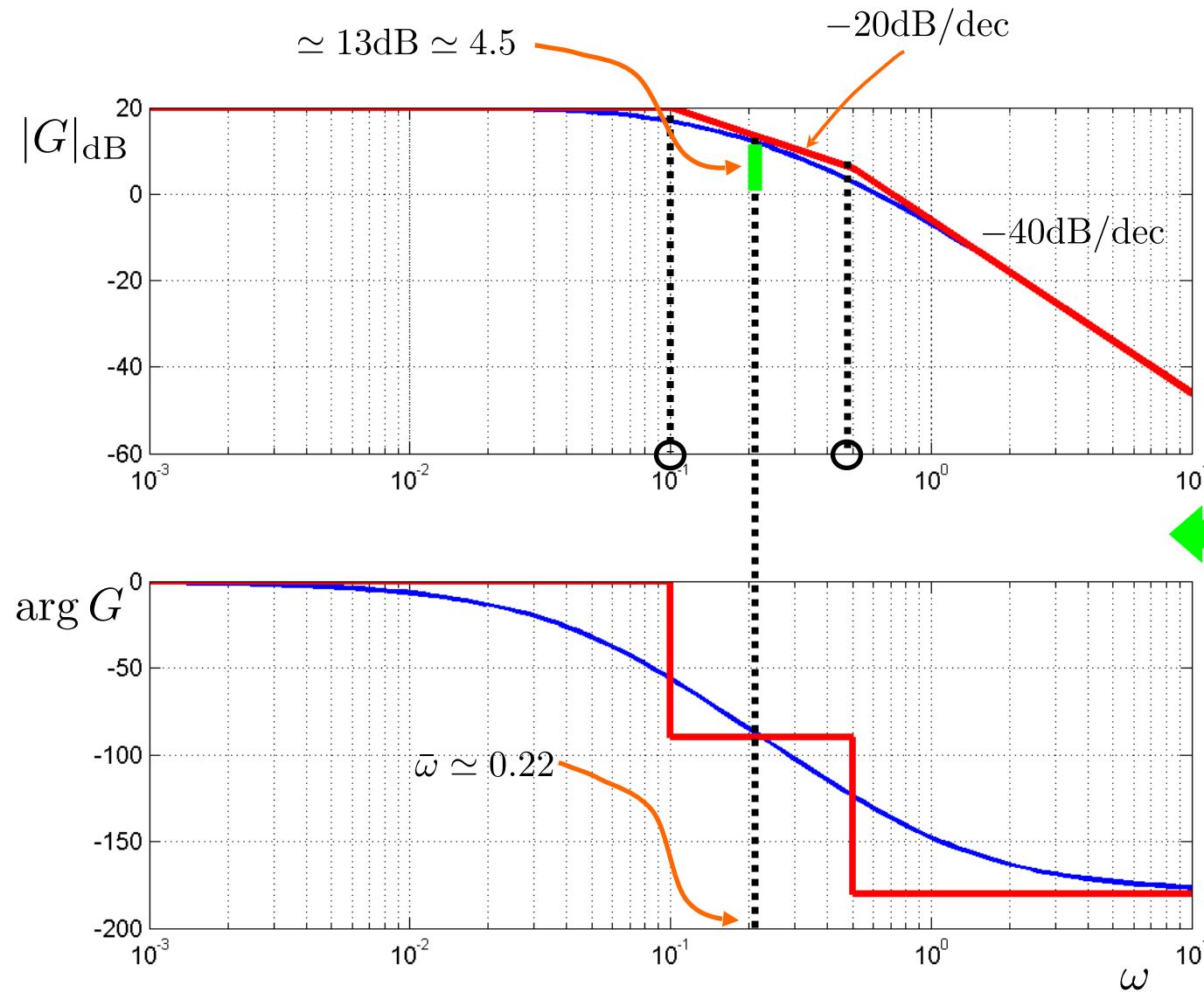
$$\tau_2 = 2$$

$$\omega_1 = 0.1$$

$$\omega_2 = 0.5$$



Example 1 (contd.)



Example 2

$$G(s) = \frac{25}{s^2 + 10\xi s + 25}$$

$$g = 0$$

$$\mu = G(0) = 1 \implies \mu_{\text{dB}} = 0 \text{dB}$$

Consider five possible values of the damping factor:

$$\xi = 0.9$$

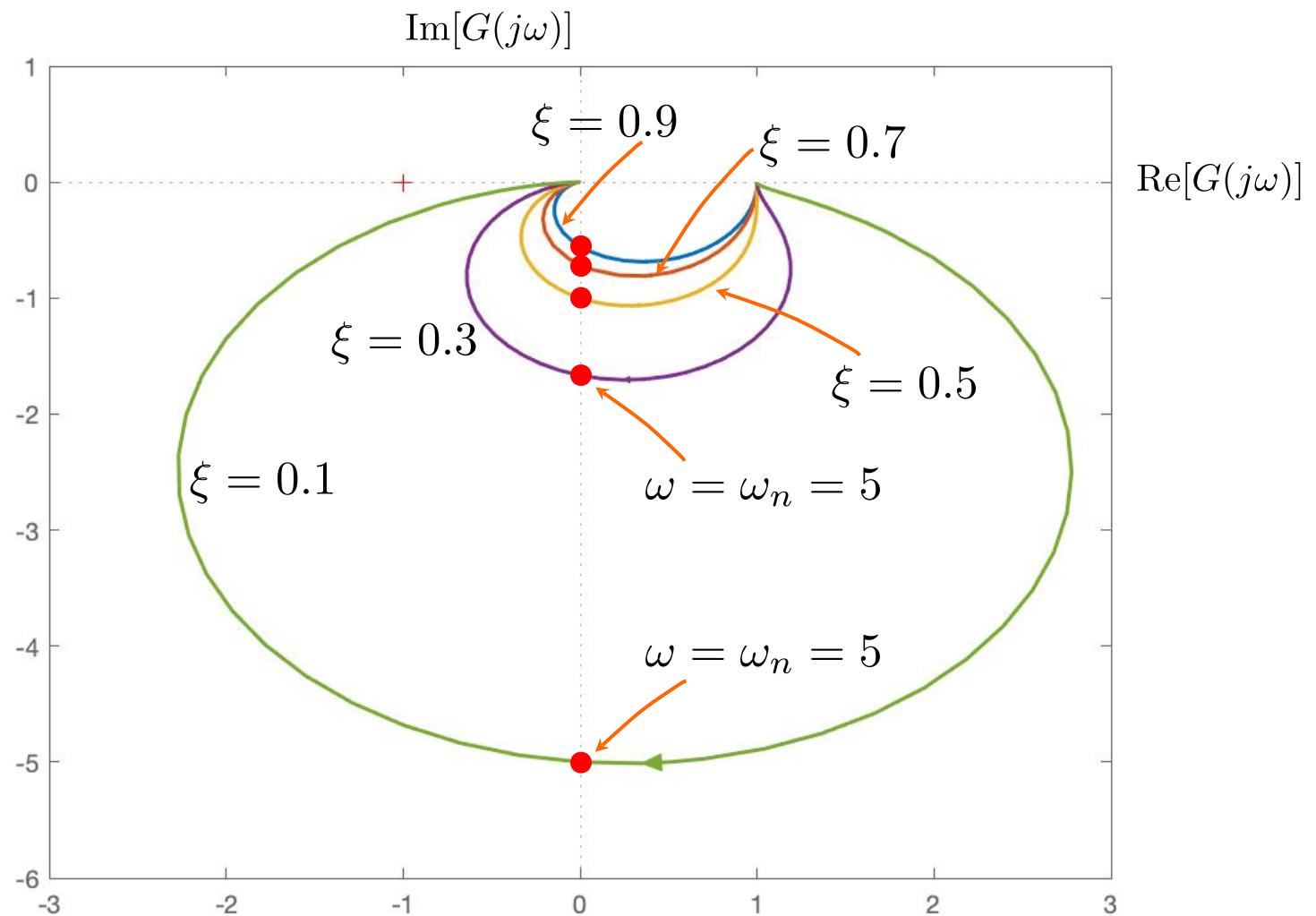
$$\xi = 0.7$$

$$\xi = 0.5$$

$$\xi = 0.3$$

$$\xi = 0.1$$

Example 2 (contd.)



Example 3

$$G(s) = \frac{s + 10}{s(s + 1)} = \frac{10(1 + \frac{s}{10})}{s(1 + s)}$$

$$g = 1$$

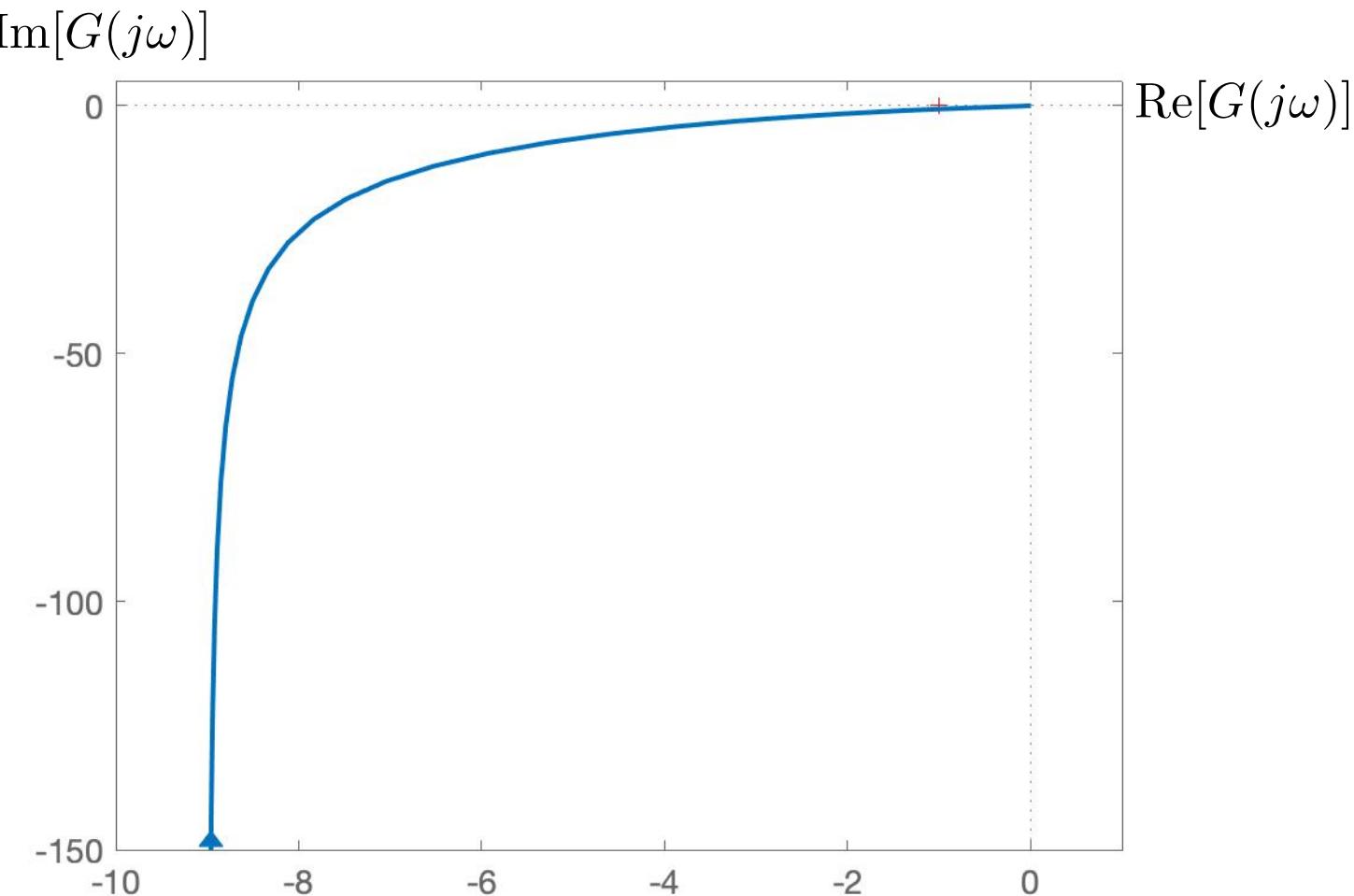
$$\mu = 10 \implies \mu_{\text{dB}} = 20 \text{dB}$$

$$T_1 = 1/10$$

$$\tau_1 = 1$$

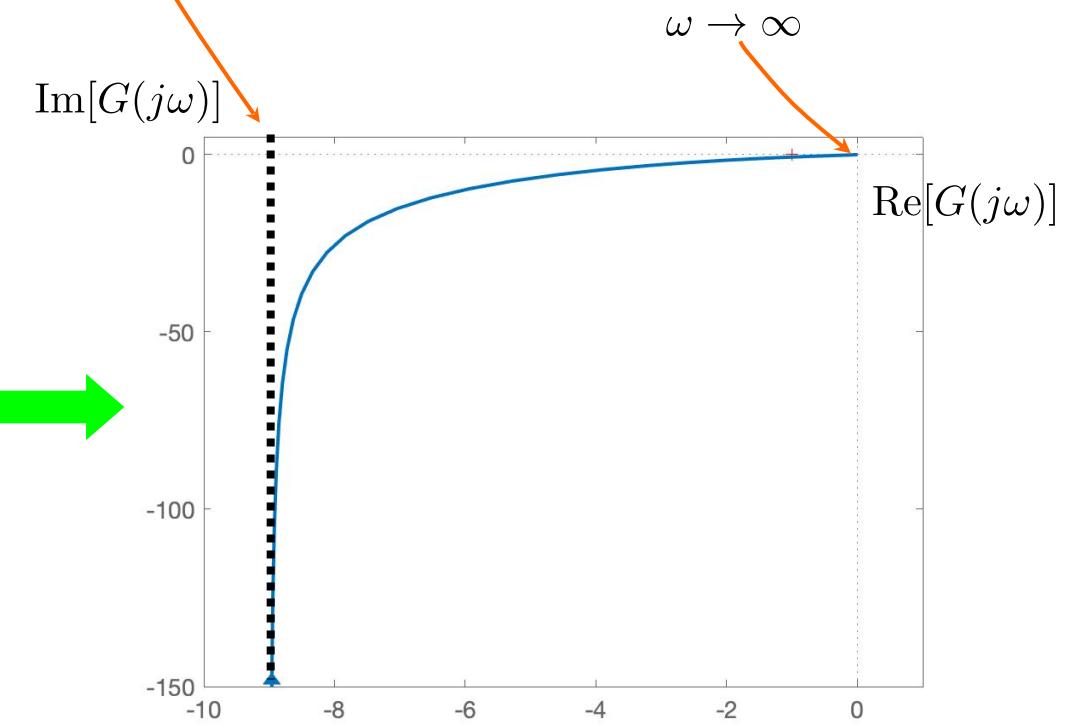
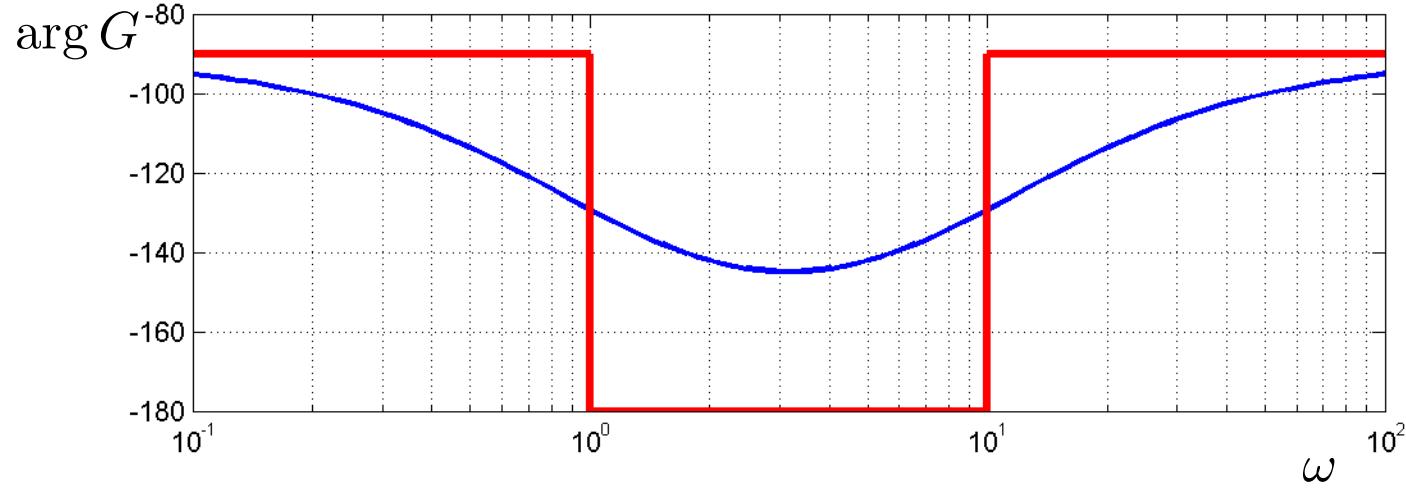
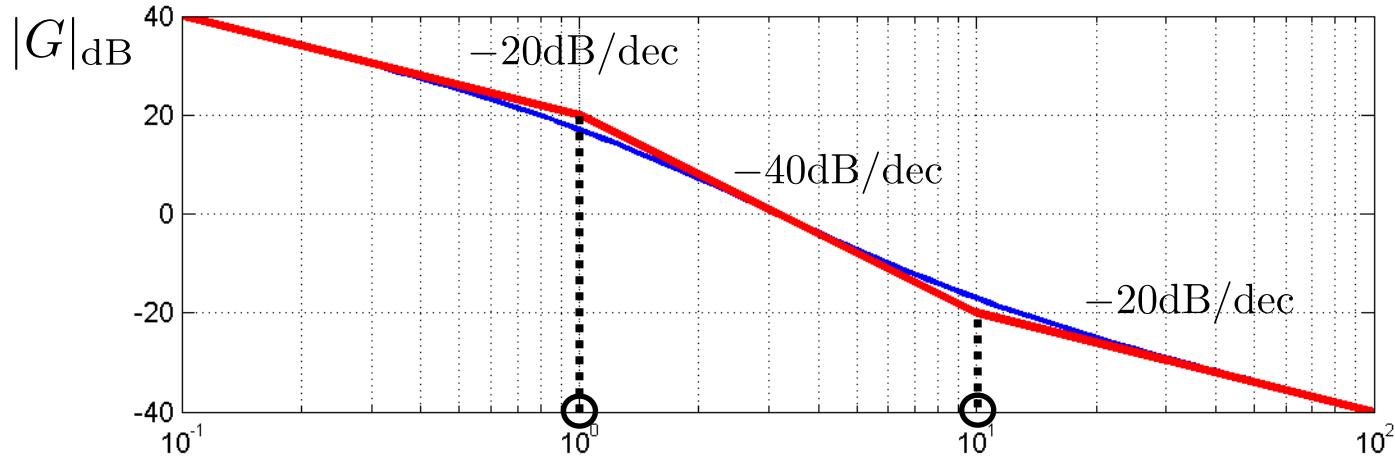
$$\omega_1 = 1$$

$$\omega_2 = 10$$



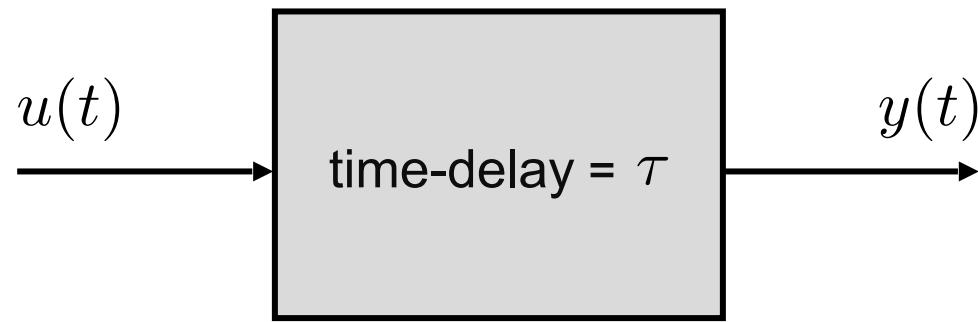
$$G(j\omega) = \frac{j\omega + 10}{j\omega(j\omega + 1)} = \frac{(-\omega + 10j)(1 - j\omega)}{-\omega(1 + \omega^2)} = \frac{9}{-(1 + \omega^2)} + j\frac{10 + \omega^2}{-\omega(1 + \omega^2)}$$

L $\text{Re}[G(j\omega)] = \frac{9}{-(1 + \omega^2)}$ $\omega \rightarrow 0^+$

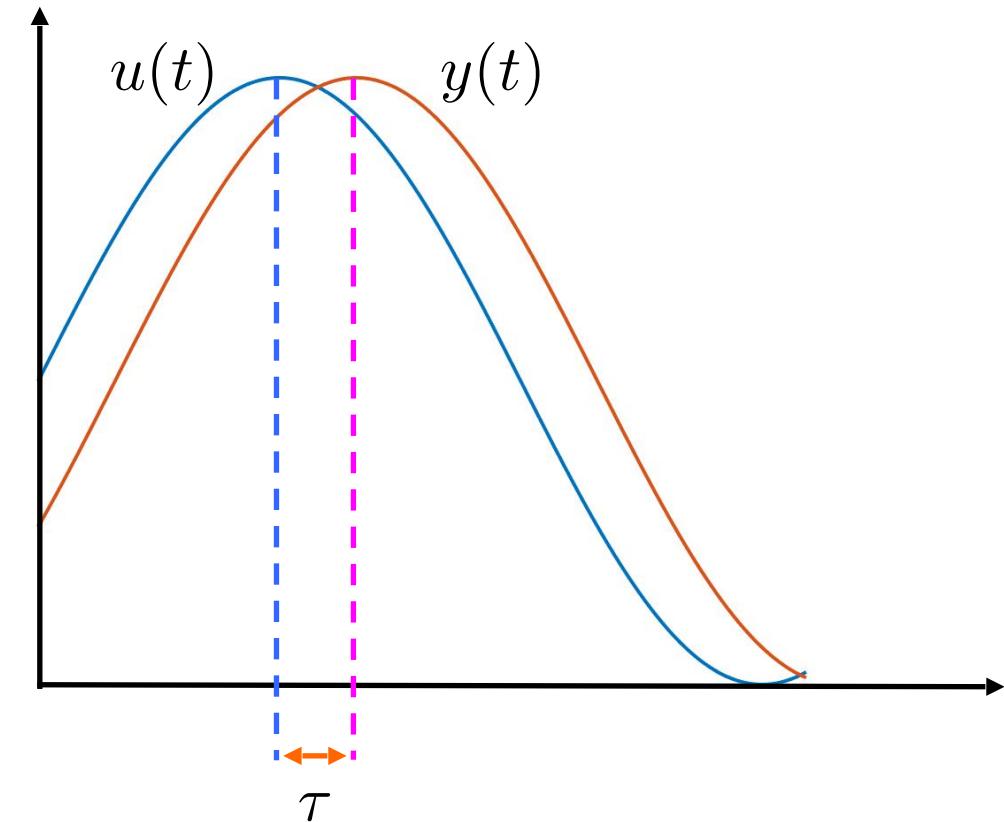


Time-Delay and Frequency Response

Consider:



$$y(t) = u(t - \tau)$$

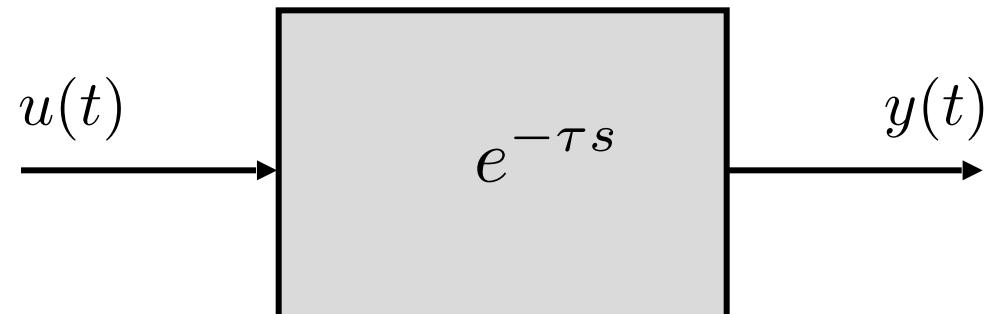


Hence:

$$Y(s) = \mathcal{L}[y(t)] = \mathcal{L}[u(t - \tau)] = e^{-\tau s} \cdot U(s)$$

Letting:

$$G(s) := e^{-\tau s}$$



we are able to treat $G(s)$ as the usual transfer function (not rational though) with $G(0) = 1$

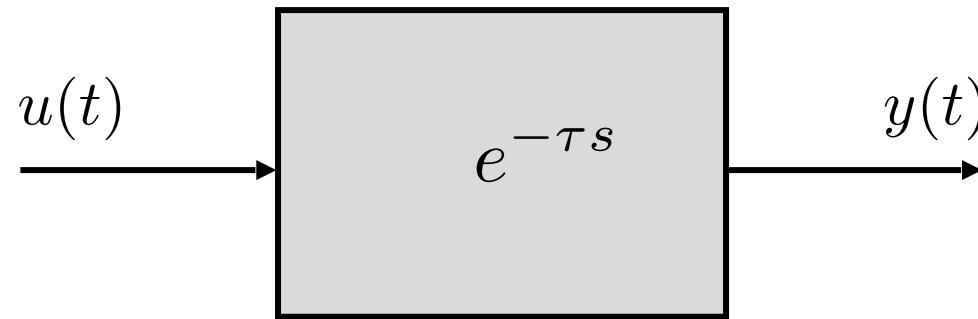
Moreover, if $u(t) = A \sin(\omega t)$

$$\Downarrow y(t) = A \sin[\omega(t - \tau)] = \underbrace{1}_{|G(j\omega)|} \cdot A \sin(\omega t - \omega\tau) \quad \arg G(j\omega)$$

The frequency response result holds!

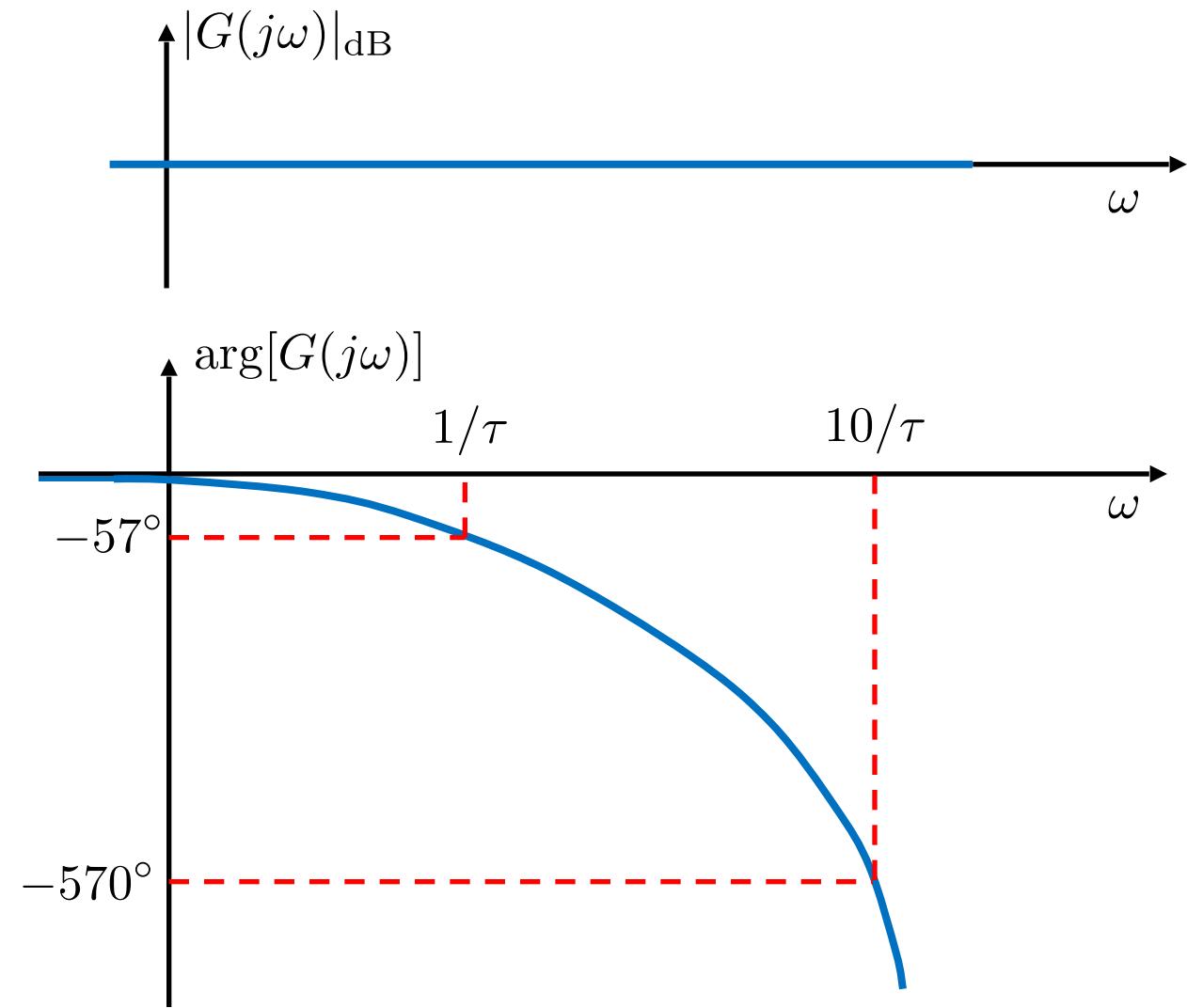
Bode diagram of the time-delay block:

$$G(s) := e^{-\tau s}$$

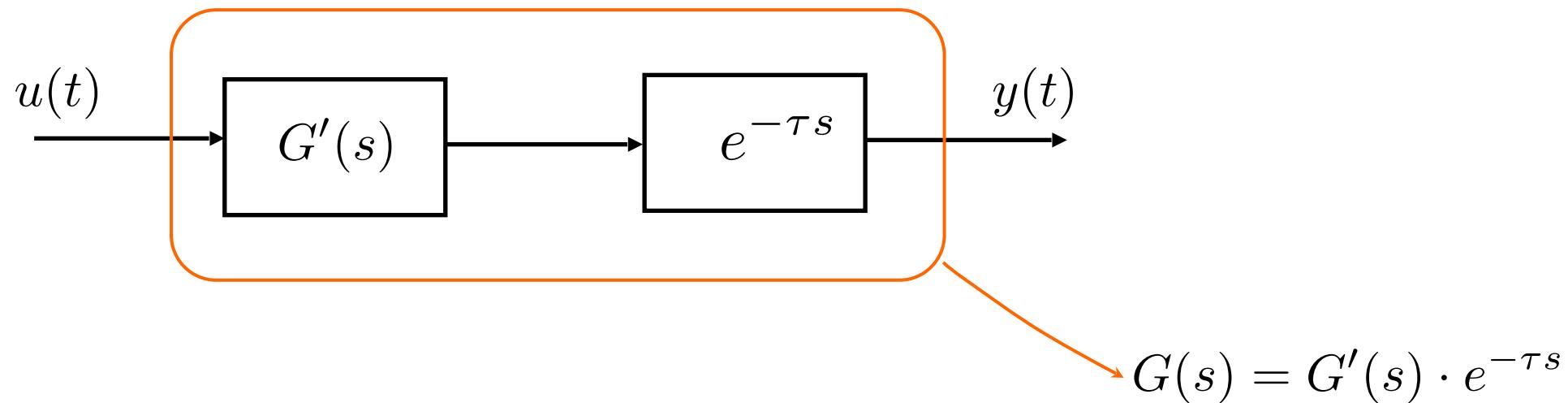


$$|G(j\omega)|_{\text{dB}} = 0$$

$$\arg G(j\omega) = -\omega\tau \frac{180}{\pi}$$



Systems with Time-Delay Blocks



$$|G(j\omega)| = |G'(j\omega)| \cdot \underbrace{|e^{j\omega\tau}|}_{1} = |G'(j\omega)|$$

$$\arg G(j\omega) = \arg G'(j\omega) - \omega\tau \frac{180}{\pi}$$