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Theoretical and experimental analysis of the physics of water rockets

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Abstract

A simple rocket can be made using a plastic bottle filled with a volume of water and pressurized air. When opened, the air pressure pushes the water out of the bottle. This causes an increase in the bottle momentum so that it can be propelled to fairly long distances or heights. Water rockets are widely used as an educational activity, and several mathematical models have been proposed to investigate and predict their physics. However, the real equations that describe the physics of the rockets are so complicated that certain assumptions are usually made to obtain models that are easier to use. These models provide relatively good predictions but fail in describing the complex physics of the flow. This paper presents a detailed theoretical analysis of the physics of water rockets that concludes with the proposal of a physical model. The validity of the model is checked by a series of field tests. The tests showed maximum differences with predictions of about 6%. The proposed model is finally used to investigate the temporal evolution of some significant variables during the propulsion and flight of the rocket. The experience and procedure described in this paper can be proposed to graduate students and also at undergraduate level if certain simplifications are assumed in the general equations.

1. Introduction

Rocket motion has been traditionally used in the classroom as an application of some basic principles of physics to a real world situation [1–3]. The teaching of these principles becomes more illustrative if the students can observe or even conduct experiments supporting theory. In this regard, model rocketry has played a significant role (especially in the USA) as a means to introduce students to basic investigation and to the procedures used in aeronautics for larger rockets [4, 5]. The beautiful film 'October Sky' is a good example of how model rocketry can be used to train future technicians.

Water rockets could be considered as the economic alternative to model rocketry. They can be as simple as a plastic bottle with a cap containing a bicycle valve. A volume of water is poured into the bottle and a mass of pressurized air is subsequently introduced through the valve. The bottle is placed upside down (i.e. with the nozzle at the bottom) so that the air cannot escape until it pushes the entire volume of water out of the bottle. A jet of water is expelled through the nozzle when the bottle is opened due to the effect of air pressure, which causes an increase in bottle momentum as deduced from Newton's third law of motion. This can provide a significant thrust, enabling the bottle to reach fairly long distances (or heights) depending on the volume of water, air pressure and launching angle.

The low cost, simplicity and safety (a combustion process is not required to provide the propulsion thrust like in model rocketry) of water rockets make them an attractive tool to be presented at various educational levels. They can be shown to small children at elementary school with the purpose of introducing them to the amazing world of physics and having some fun. A water rocket can be used at secondary school to explain and provide a real hands-on application of Newton's third law. Besides, the students can carry out simple field measurements to check the influence of several parameters on the distance reached by the rocket and discuss the trends observed. Water rockets can be also an exciting problem for undergraduate students if a challenging goal is proposed [6, 7]. This can be, for example, to hit a specific target or to reach a maximum distance (or height) with limited air pressure. Within the scope of a physics or fluid mechanics course, this practice can be complemented with a physical model for the rocket that must be solved by the students to analytically estimate the required parameters.

A physical model for a water rocket can be set up from mass, momentum and energy conservation principles, in addition to some thermodynamic relation for the expansion of the air in the rocket, which usually led to a set of ordinary differential equations. A precise solution of these equations is difficult to obtain, though some simplifications can be made. For instance, one can suppose a constant exit velocity of the jet of water using an average air pressure, or neglect the effect of aerodynamic drag thus reducing the flight of the rocket to a simple parabolic throw. This kind of simplifications can be imposed to predict trends and also to investigate the influence of some parameters, but satisfactory calculations are obtained only if physical models as realistic as possible are used. However, the real equations are so complicated that certain simplifications are still made even in the scientific bibliography available. This includes overriding the thrust provided by the remaining air once water is expelled, or to assume quasi-steady flow and neglect dissipative effects. These simplifications are assumed in the models reported in [8, 9] and turn the general expression of energy conservation into the simple Bernoulli's equation [10–12]. Although the models that include these assumptions have been proven to provide relatively reasonable predictions, they fail to describe precisely the complex physics of the flow that takes place during the propulsion of the water rocket.

This paper gives a detailed theoretical analysis and proposal of a physical model for water rockets. In contrast to previous investigations reported in the scientific literature, the present analysis includes the contribution of the air to the total thrust on the rocket; also, the intrinsic unsteadiness of the flow and the presence of dissipative effects are taken into account in the energy conservation law. The predictions obtained from the physical model are checked by means of a series of experimental tests. Once validated, the model is applied to investigate the temporal evolution of some significant variables during the propulsion and flight of the water rocket. Although relatively complex and thus more suitable for graduate students, the model can also be used at undergraduate level if certain simplifications are assumed, as will be explained in the following sections.

2. The experience with the students

The physical model reported in the following sections is used in a field practice that we have been carrying out with our undergraduate students of fluid mechanics during the past 10 years. The main objective is to design a water rocket (and a proper launching system) performing the calculations needed to meet a predetermined target. This is usually to reach a specific distance or to obtain the maximum range with a limited air pressure of 3 bar (for safety reasons). The students are split into groups of a maximum of four members to encourage teamwork. Each team has to build its own rocket and launching system and perform the necessary calculations to estimate the distance reached from a set of input parameters. Basically, they can vary three parameters for a specific water rocket: the volume of water, the pressure of the air and the launching angle. Once the distance is estimated, the students have to carry out field tests to check the validity of their predictions.

A system of ordinary differential equations (deduced later) is given to the students and must be solved to estimate the distance reached by the water rocket. The students can solve the system by programming the equations in a mathematical framework like Matlab® to get a precise solution; this is what we have done to obtain the results presented in this paper. Additionally, we also accept rough solutions obtained from first-order Taylor approximations of time derivatives and a simple spreadsheet. Besides the specific competences of the subject (control volume theory, hydrodynamics and aerodynamics), this practice improves several generic competences of Tuning's project [13] like the 'ability to apply knowledge in practice', 'problem solving and decision making', 'planning and time management', 'teamwork' and 'ability to work autonomously' to cite just some of them.

The work done is evaluated in a public contest. Each group has to hand over a previous report including the geometric data of the water rocket, initial hypothesis, simplifications, method of solving the equations and calculations performed. The students have up to three attempts to show that their water rockets meet the proposed target using the volume of water, air pressure and launching angle estimated in the calculations. Competitive factors are introduced by assigning a final score to each group. This score results from evaluating the design of the rocket and launching system and also the degree of attainment of the target. The challenge and the spectacular nature of this practice motivate the students. The launching of the rocket is very impressive because the water discharge occurs in an explosive way (about tenths of a second) and the distances covered can reach up to a hundred meters. Some of this can be seen in figure 1, which presents three frames of a movie obtained with a high-speed camera. The jet of water at the nozzle is clearly illustrated in the figure.

The use of water rockets as an educational activity can be appreciated just by typing the words 'water rocket' in any internet search engine. Lots of web pages devoted to this subject are found, some of them even including simple applets that can be used to calculate the flight trajectory of the rocket from an initial set of input parameters. Also, some well-known internet video portals contain enough material uploaded by students on their own showing their most glorious feats.

3. Theoretical basis

The study of the motion of a water rocket is divided into three stages. At the very beginning, the rocket contains pressurized air plus an initial volume of water, and hence the weight W of the system can be significant (see figure 2). When the bottle is opened, a jet of water is pushed through the nozzle with a velocity that depends on the air pressure. This jet of fluid provides a propulsion thrust that increases the rocket momentum. The aerodynamic drag F_D

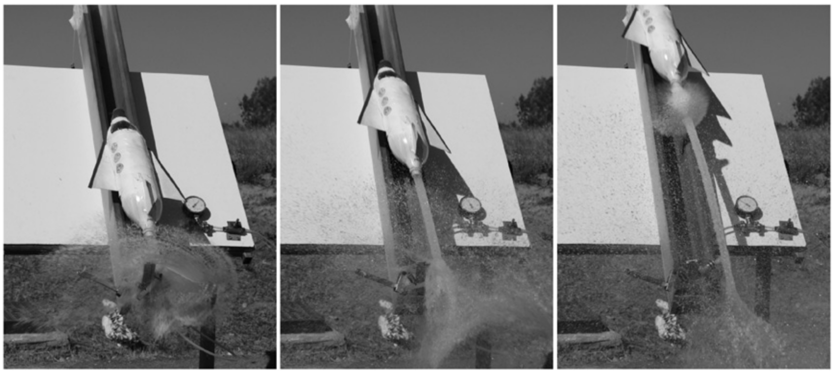

Figure 1. Three frames showing propulsion of a water rocket obtained with a high-speed camera.

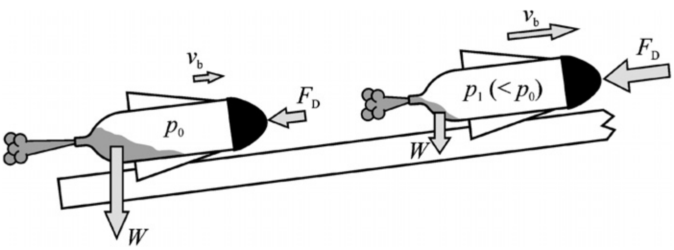

Figure 2. Water propulsion.

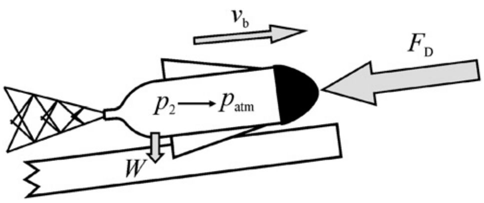

Figure 3. Air propulsion.

(proportional to the squared velocity) in the first instance is still small because the magnitude of the rocket velocity v_b is small too. The weight of the system decreases as water is expelled; also, the air pressure decreases thus reducing the water exit velocity and the propulsion thrust. In contrast, the velocity (and hence the drag force) is increased. The water propulsion lasts until the entire volume of water is out of the bottle.

The air that remains inside the rocket at the end of the previous stage is still pressurized (figure 3). This air causes an additional thrust on the rocket when being expelled, although this effect is usually neglected in simple theoretical analysis. However, the increment in rocket momentum can be significant depending on the magnitude of the remaining pressure, since the current weight of the system is small. Hence, the velocity of the water rocket still increases (though smoother) during this air propulsion stage because the weight is small and the air velocity at the nozzle is high (at the very beginning it is near sound velocity). At the end of this air propulsion stage, the rocket is empty (or, actually, full of air at atmospheric pressure)

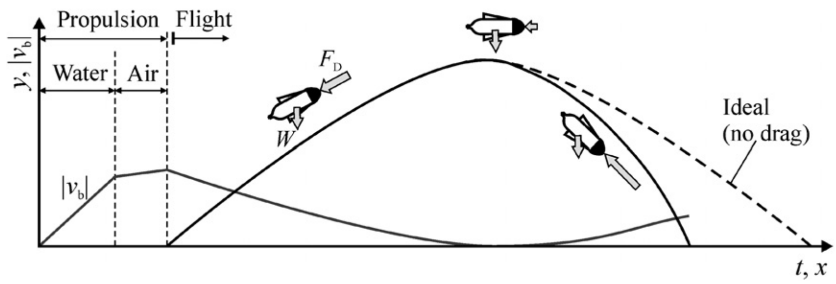

Figure 4. Evolution of velocity modulus and trajectory during the propulsion and flight of the water rocket.

and has attained its maximum velocity; in consequence, there is also a maximum drag on the rocket.

Some of the effects indicated previously are shown in figure 4. At the end of the propulsion stage, the rocket is launched with an initial velocity and angle, and follows a typical ballistic trajectory. The drag on the rocket during the ascent decreases continuously as velocity decreases until reaching a maximum height. Then, the drag force increases again due to the increase in fall velocity (see figure 4). The effect of the drag causes the rocket to reach a smaller distance than that corresponding to an ideal flight. The drag always opposes the motion of the rocket; in contrast, the gravity opposes rocket motion during the ascent while favouring it in the descent. Terminal velocity can be reached (i.e. drag equals gravity) if the fall distance is large enough.

This section has exposed the physics of a water rocket during its propulsion and free flight from a basic point of view. The deduction of a precise physical model for the rocket will be described in the following section.

4. Mathematical model

A mathematical model for a water rocket can be deduced by the application of the basic conservation laws of mass, momentum and energy. These laws can be imposed to a suitable control volume using the Reynolds transport theorem in a classical fluid mechanics approach.

4.1. Water propulsion

We can consider a control volume (CV) like the one depicted in figure 5. This volume contains the water rocket and intersects it at the nozzle (control surface (CS)). The initial volume of water in the rocket is V_{w0} and p_{a0} is the absolute pressure of air before the launching. The mass of water is m_w and m_a is the mass of pressurized air, which remains unaltered during this stage. The mass of the empty rocket (the plastic chassis) is m_b, the water density is ρ_w and θ is the launching angle. The absolute reference system (x, y) will be used to define velocities.

The mass conservation in the CV can be expressed in terms of the Reynolds transport theorem [10–12]:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho\,\mathrm{d}V + \int_{\mathrm{CS}} \rho(v_r \cdot \mathrm{d}S). \tag{1}$$

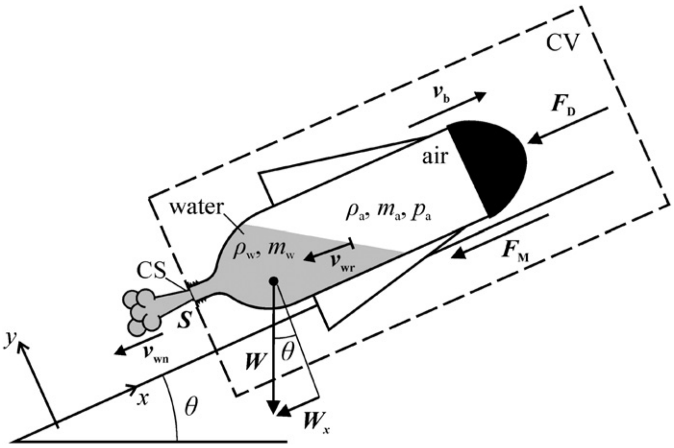

Figure 5. Schematic of the water rocket and CV used for the calculations.

As indicated previously, the mass of air (and obviously, the mass of plastic) remains constant. Hence

$$\frac{\mathrm{d}m_\mathrm{w}}{\mathrm{d}t} = -\rho_\mathrm{w} v_\mathrm{wn} S, \tag{2}$$

where v_wn is the relative velocity of the water jet with respect to the nozzle and S is the section of the nozzle.

The general expression of momentum conservation in the CV using the Reynolds transport equation is

$$\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\int_\mathrm{CV} \rho v \,\mathrm{d}V + \int_\mathrm{CS} \rho v (v_r \cdot \mathrm{d}S), \tag{3}$$

which can be applied along the x-direction (see figure 5):

$$\sum F_x = (m_\mathrm{b} + m_\mathrm{a})\frac{\mathrm{d}v_\mathrm{b}}{\mathrm{d}t} + m_\mathrm{w}\frac{\mathrm{d}v_\mathrm{w}}{\mathrm{d}t} + v_\mathrm{w}\frac{\mathrm{d}m_\mathrm{w}}{\mathrm{d}t} + \rho_\mathrm{w}(v_\mathrm{b} - v_\mathrm{wn})v_\mathrm{wn}S. \tag{4}$$

The velocity v_b in this equation is the absolute velocity of the water rocket. The variable v_w is the absolute velocity of the water inside the rocket. As the water is discharged quickly through the nozzle, this velocity cannot be strictly considered equal to v_b. The velocity v_w can be expressed in terms of v_b and of the relative velocity of the water in the rocket v_wr:

$$v_\mathrm{w} = v_\mathrm{b} - v_\mathrm{wr}, \tag{5}$$

and the relative velocity at the nozzle can be related to the relative velocity of the water in the rocket through the section ratio:

$$v_\mathrm{wr} = v_\mathrm{wn}\frac{S}{S_\mathrm{b}}, \tag{6}$$

where S_b is the cross-sectional area of the rocket.

The surface forces on the CV along the x-direction are due to the effect of the drag $\boldsymbol{F}_\mathbf{D}$ (proportional to v_b^2) and the mechanical friction $\boldsymbol{F}_\mathbf{M}$ between the rocket and the launching ramp. The only body forces here are due to the x-component of weight $\boldsymbol{W}$. These forces can be expressed as

$$F_\mathrm{D} = 0.5\rho_\mathrm{atm} C_\mathrm{D} S_\mathrm{b} v_\mathrm{b}^2; \tag{7}$$

$$F_{\mathrm{M}} = \mu(m_{\mathrm{b}} + m_{\mathrm{a}} + m_{\mathrm{w}})g\cos\theta; \tag{8}$$

$$W_x = (m_{\mathrm{b}} + m_{\mathrm{a}} + m_{\mathrm{w}})g\sin\theta. \tag{9}$$

In the above equations, ρ_{atm} is the density of the atmospheric air, C_{D} is the drag coefficient of the water rocket and μ is the friction coefficient.

The conservation of energy in the CV leads to the following relation:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho e\,\mathrm{d}V + \int_{\mathrm{CS}} \rho\left(e + \frac{p}{\rho}\right)(\mathbf{v}_r \cdot \mathrm{d}\boldsymbol{S}), \tag{10}$$

where $e = u + v^2/2 + gh$ is the specific energy, u is the specific internal energy, g is the gravity, h is the vertical position and p is the absolute pressure. The expulsion of the water through the nozzle is fast enough to be considered as adiabatic, so that $\mathrm{d}Q = 0$. Additionally, the time derivative of the work W on the CV can be written as follows:

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = -v_{\mathrm{b}}(F_{\mathrm{D}} + F_{\mathrm{M}}). \tag{11}$$

The volume integral can be evaluated by considering separately the contribution of the chassis of the rocket (b), the water (w) and the pressurized air (a). This yields

$$\int_{\mathrm{CV}} \rho e\,\mathrm{d}V = m_{\mathrm{b}}e_{\mathrm{b}} + m_{\mathrm{a}}e_{\mathrm{a}} + m_{\mathrm{w}}e_{\mathrm{w}}. \tag{12}$$

The mass of the rocket chassis and its specific internal energy remains constant; also, the mass of pressurized air and the specific internal energy of the water do not change during the present stage. The time derivative of h can be expressed as a function of the rocket velocity v_{b} and of the launching angle θ using the following expression:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v_{\mathrm{b}}\sin\theta. \tag{13}$$

On the other hand, the time derivative of the specific internal energy of the pressurized air can be related to absolute pressure and density through the perfect gas law $p_{\mathrm{a}} = \rho_{\mathrm{a}}RT_{\mathrm{a}}$:

$$\frac{\mathrm{d}u_{\mathrm{a}}}{\mathrm{d}t} = \frac{\mathrm{d}(c_v T_{\mathrm{a}})}{\mathrm{d}t} = \frac{c_v}{R}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_{\mathrm{a}}}{\rho_{\mathrm{a}}}\right), \tag{14}$$

where c_v is the specific heat at constant volume, T_{a} is the absolute temperature, R is the specific gas constant, p_{a} is the absolute pressure of the air and ρ_{a} is its density. The expansion of the air is fast enough to be considered as adiabatic (with exponent γ) and is related to the initial values of pressure p_{a0} and density ρ_{a0} with $R = c_v(\gamma - 1)$, and

$$\frac{p_{\mathrm{a0}}}{\rho_{\mathrm{a0}}^{\gamma}} = \frac{p_{\mathrm{a}}}{\rho_{\mathrm{a}}^{\gamma}}. \tag{15}$$

Also, the mass of air during this stage is constant:

$$m_{\mathrm{a}} = \rho_{\mathrm{a0}}(V - V_{\mathrm{w0}}) = \rho_{\mathrm{a}}(V - V_{\mathrm{w}}), \tag{16}$$

where V is the volume of the rocket and V_w is the volume of water.

The surface integral can be evaluated using the following expression:

$$\begin{aligned}\int_{\mathrm{CS}} \rho\left(e + \frac{p}{\rho}\right)(\mathbf{v}_r \cdot \mathrm{d}\boldsymbol{S}) &= \left(e_{\mathrm{wn}} + \frac{p_{\mathrm{wn}}}{\rho_{\mathrm{w}}}\right)\rho_{\mathrm{w}}v_{\mathrm{wn}}S \\ &= \left[u_{\mathrm{w}} + \frac{(v_{\mathrm{b}} - v_{\mathrm{wn}})^2}{2} + gh + \frac{p_{\mathrm{atm}}}{\rho_{\mathrm{w}}}\right]\rho_{\mathrm{w}}v_{\mathrm{wn}}S,\end{aligned} \tag{17}$$

where p_{wn} is the absolute pressure of the water at the nozzle (i.e. the atmospheric pressure).

After some maths, the conservation laws of mass, momentum and energy, in addition to the equation for an adiabatic expansion of the air, lead to the following set of four ordinary differential equations:

$$\frac{\mathrm{d}m_{\mathrm{w}}}{\mathrm{d}t} = -\rho_{\mathrm{w}} v_{\mathrm{wn}} S; \tag{18}$$

$$\frac{\mathrm{d}v_{\mathrm{b}}}{\mathrm{d}t} = -\frac{1}{m_{\mathrm{b}}+m_{\mathrm{a}}}(F_{\mathrm{D}}+F_{\mathrm{M}}+W_x) + \frac{1}{m_{\mathrm{b}}+m_{\mathrm{a}}}\left[S_{\mathrm{b}}(p_{\mathrm{a}}-p_{\mathrm{atm}}) - \frac{(S_{\mathrm{b}}-S)^2}{2S_{\mathrm{b}}}\rho_{\mathrm{w}} v_{\mathrm{wn}}^2\right]; \tag{19}$$

$$\frac{\mathrm{d}v_{\mathrm{wn}}}{\mathrm{d}t} = \frac{S_{\mathrm{b}}^2(p_{\mathrm{a}}-p_{\mathrm{atm}})}{S}\left(\frac{1}{m_{\mathrm{w}}}+\frac{1}{m_{\mathrm{b}}+m_{\mathrm{a}}}\right) - \frac{v_{\mathrm{wn}}^2\rho_{\mathrm{w}}}{2S}\left[\frac{S_{\mathrm{b}}^2-S^2}{m_{\mathrm{w}}}+\frac{(S_{\mathrm{b}}-S)^2}{m_{\mathrm{b}}+m_{\mathrm{a}}}\right]$$
$$- \frac{S_{\mathrm{b}}}{S(m_{\mathrm{b}}+m_{\mathrm{a}})}(F_{\mathrm{D}}+F_{\mathrm{M}}+W_x); \tag{20}$$

$$\frac{\mathrm{d}p_{\mathrm{a}}}{\mathrm{d}t} = \frac{-\gamma\, p_{\mathrm{a}}^{(\gamma+1)/\gamma}}{(V-V_{\mathrm{w0}})p_{\mathrm{a0}}^{1/\gamma}} v_{\mathrm{wn}} S. \tag{21}$$

The above equations can be used to obtain the time evolution of the mass of water m_w, the pressure of the air p_{a}, the water velocity at the nozzle v_{wn} and the velocity of the rocket v_{b}. The initial conditions for the system are the volume of water V_{w0}, the air pressure p_{a0}, the launching angle θ and some physical variables of the rocket: the mass of the empty bottle m_{b}, the cross-sectional diameter D (to calculate S_{b}), the nozzle diameter d (to calculate S) and the drag coefficient C_{D}. This system of equations is valid until the mass of water m_w inside the rocket equals zero.

4.2. Air propulsion

The air in the water rocket at the end of the previous stage is still pressurized. This air is expelled through the nozzle causing an additional increase in the rocket momentum that sometimes cannot be neglected: for example, a rocket with air pressurized at 2 bars can reach a distance of about 10 m, as discussed later.

The mathematical equations that describe the motion of the rocket at the present stage can be deduced following a similar procedure to that exposed in the previous section. The same CV shown in figure 5 will be used (see figure 6). The absolute pressure and the air density at the nozzle are p_{an} and ρ_{an}, respectively, as indicated.

The integral expression of continuity equation (1) can be applied to the CV shown in figure 6 and yields

$$\frac{\mathrm{d}m_{\mathrm{a}}}{\mathrm{d}t} = V\frac{\mathrm{d}\rho_{\mathrm{a}}}{\mathrm{d}t} = -\rho_{\mathrm{an}} S v_{\mathrm{an}}, \tag{22}$$

where v_{an} is the velocity of the air at the nozzle. The momentum conservation (3) on the CV leads to the following expression:

$$-F_{\mathrm{D}} - W_x - F_{\mathrm{M}} = \frac{\mathrm{d}}{\mathrm{d}t}(m_{\mathrm{b}}v_{\mathrm{b}} + m_{\mathrm{a}}v_{\mathrm{a}}) + \rho_{\mathrm{a}}(v_{\mathrm{b}} - v_{\mathrm{an}})v_{\mathrm{an}} S. \tag{23}$$

On the other hand, the integral equation of energy (10) can be expressed as

$$-v_{\mathrm{b}}(F_{\mathrm{D}}+F_{\mathrm{M}}) = \frac{\mathrm{d}}{\mathrm{d}t}(m_{\mathrm{b}}e_{\mathrm{b}} + m_{\mathrm{a}}e_{\mathrm{a}}) + \left(e_{\mathrm{an}} + \frac{p_{\mathrm{an}}}{\rho_{\mathrm{an}}}\right)\rho_{\mathrm{an}} v_{\mathrm{an}} S. \tag{24}$$

The absolute velocity of the air in the rocket v_{a} is assumed as equal to the velocity of the rocket v_{b}. This is because the decompression of the air takes place in the whole volume of

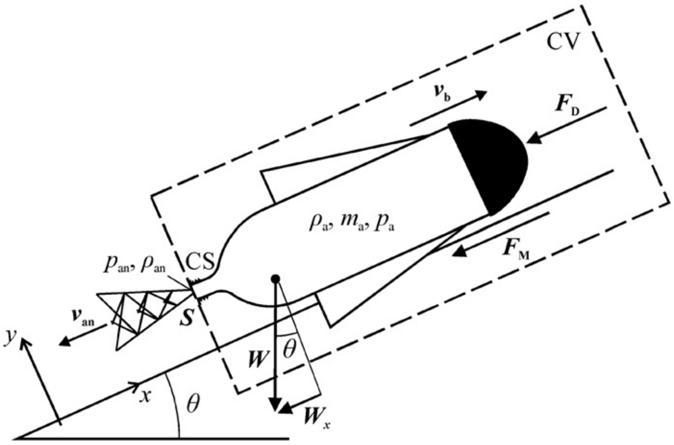

Figure 6. Schematic of the rocket during the air propulsion stage.

the bottle; it is not a simple displacement of the gas–liquid interface as in the previous stage. If someone disregards this warning, he must note that the relative velocity of the pressurized air changes with position in the rocket as well as with time (this is not a threat but a fact...). Equations (23) and (24) can be expanded and combined, and led to

$$m_{\mathrm{a}} \frac{\mathrm{d}u_{\mathrm{a}}}{\mathrm{d}t} + \left(u_{\mathrm{an}} - u_{\mathrm{a}} + \frac{v_{\mathrm{an}}^2}{2} + \frac{p_{\mathrm{an}}}{\rho_{\mathrm{an}}} \right) \rho_{\mathrm{an}} v_{\mathrm{an}} S. \tag{25}$$

The absolute pressure of the air at the nozzle p_{an} is not equal to p_{atm} in the present case because the flow is compressible. If an adiabatic expansion for the air is supposed

$$m_{\mathrm{a}} \frac{\mathrm{d}u_{\mathrm{a}}}{\mathrm{d}t} = \rho_{\mathrm{a}} V \frac{c_v}{R} \frac{\mathrm{d}T_{\mathrm{a}}}{\mathrm{d}t} = -\rho_{\mathrm{an}} v_{\mathrm{an}} S \frac{p_{\mathrm{a}}}{\rho_{\mathrm{a}}}, \tag{26}$$

and the difference of specific internal energy can be expressed as

$$u_{\mathrm{an}} - u_{\mathrm{a}} = c_v (T_{\mathrm{an}} - T_{\mathrm{a}}) = \frac{1}{\gamma - 1} \frac{p_{\mathrm{a0}}^{1/\gamma}}{\rho_0} \left(p_{\mathrm{an}}^{(\gamma-1)/\gamma} - p_{\mathrm{a}}^{(\gamma-1)/\gamma} \right). \tag{27}$$

Basically, this expression leads to the equations for compressible flow through a convergent nozzle. After some algebra, the three conservation laws plus the equation for an adiabatic expansion of the air lead to one algebraic equation and three ordinary differential equations:

$$v_{\mathrm{an}}^2 = \frac{2\gamma}{\gamma - 1} \frac{p_{\mathrm{a0}}^{1/\gamma}}{\rho_0} \left(p_{\mathrm{a}}^{(\gamma-1)/\gamma} - p_{\mathrm{an}}^{(\gamma-1)/\gamma} \right); \tag{28}$$

$$\frac{\mathrm{d}v_{\mathrm{b}}}{\mathrm{d}t} = \frac{S}{m_{\mathrm{b}} + m_{\mathrm{a}} + m_{\mathrm{w}}} \frac{2\gamma}{\gamma - 1} \left(p_{\mathrm{a}}^{(\gamma-1)/\gamma} - p_{\mathrm{an}}^{(\gamma-1)/\gamma} \right) p_{\mathrm{an}}^{1/\gamma} - \frac{1}{m_{\mathrm{b}} + m_{\mathrm{a}} + m_{\mathrm{w}}} (F_{\mathrm{D}} + F_{\mathrm{M}} + W_x); \tag{29}$$

$$\frac{\mathrm{d}m_{\mathrm{a}}}{\mathrm{d}t} = -S p_{\mathrm{an}}^{1/\gamma} \left[\frac{2\gamma}{\gamma - 1} \frac{\rho_{\mathrm{a0}}}{p_{\mathrm{a0}}^{1/\gamma}} \left(p_{\mathrm{a}}^{(\gamma-1)/\gamma} - p_{\mathrm{an}}^{(\gamma-1)/\gamma} \right) \right]^{1/2}; \tag{30}$$

$$\frac{\mathrm{d}p_{\mathrm{a}}}{\mathrm{d}t} = -\frac{S}{V} \left(\frac{2\gamma}{\gamma - 1} \right)^{1/2} \left(\frac{p_{\mathrm{a0}}^{1/\gamma}}{\rho_0} \right)^{1/2} p_{\mathrm{a}}^{(\gamma-1)/\gamma} p_{\mathrm{an}}^{1/\gamma} \left(p_{\mathrm{a}}^{(\gamma-1)/\gamma} - p_{\mathrm{an}}^{(\gamma-1)/\gamma} \right)^{1/2}. \tag{31}$$

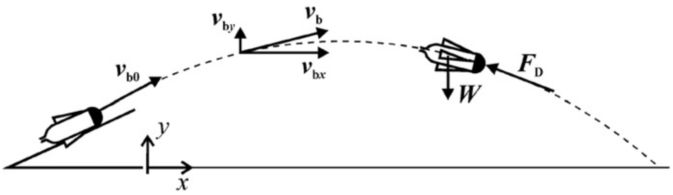

Figure 7. Typical ballistic trajectory of the water rocket.

This system of four equations describes the physics of the water rocket during the air propulsion stage. The unknown variables are the velocity of the rocket v_b, the velocity of the air at the nozzle v_{an}, the mass of air m_a and the absolute pressure of the air in the bottle p_a. The initial conditions are the last values of p_a and v_b calculated in the previous stage. The system is valid until the pressure in the rocket equals atmospheric conditions, that is, for $p_a \geqslant p_{atm}$.

4.3. Ballistic flight

The rocket is launched from the ramp at the beginning of the present stage with a high initial velocity v_{b0}. This velocity results from the increase in momentum due to water and air expulsion, as explained previously.

The rocket follows a typical ballistic trajectory (see figure 7). The differential equations that describe the motion of a body in ballistic flight are well known from physics textbooks, and thus they are simply written here as

$$\frac{\mathrm{d}v_{bx}}{\mathrm{d}t} = -\frac{\rho_{atm} C_D A}{2 m_b} v_{bx} \left(v_{bx}^2 + v_{by}^2\right)^{1/2}; \tag{32}$$

$$\frac{\mathrm{d}v_{by}}{\mathrm{d}t} = -g - \frac{\rho_{atm} C_D A}{2 m_b} v_{by} \left(v_{bx}^2 + v_{by}^2\right)^{1/2}, \tag{33}$$

where v_{bx} and v_{by} are the x, y components of the rocket velocity. The position of the water rocket can be obtained from integration of the previous equations. It must be noted that the effect of the drag force is important though it is usually neglected in many introductory physics textbooks. The above equations describe the physics of the rocket during the flight and are valid until landing (i.e. for $y \geqslant 0$).

5. Experimental validation

The validity of the mathematical model deduced in the previous section was checked by a series of field tests that were carried out using the handcrafted launching ramp shown in figure 8. The ramp is long enough to assure that at least the propulsion takes place on it; otherwise, the weight of the water would cause a premature fall of the rocket (a typical length of about 1.5–2 m is adequate). The launching angle of the ramp can be regulated with a precision of $\pm 1^\circ$ using a slider and a protractor with a plumb line. The pressurized air is introduced in the rocket through a specially designed cap (see detail in the figure) using a hand pump. It is more convenient to introduce the air through the nozzle rather than using (for instance) a bicycle valve at the head of the rocket, as can be usually found in several designs. This weakens the bottle and makes it more prone to explode when introducing the pressurized air; though spectacular, obviously this can be dangerous. The relative pressure of the air in the bottle is measured with a manometer (precision of ± 0.1 bar).

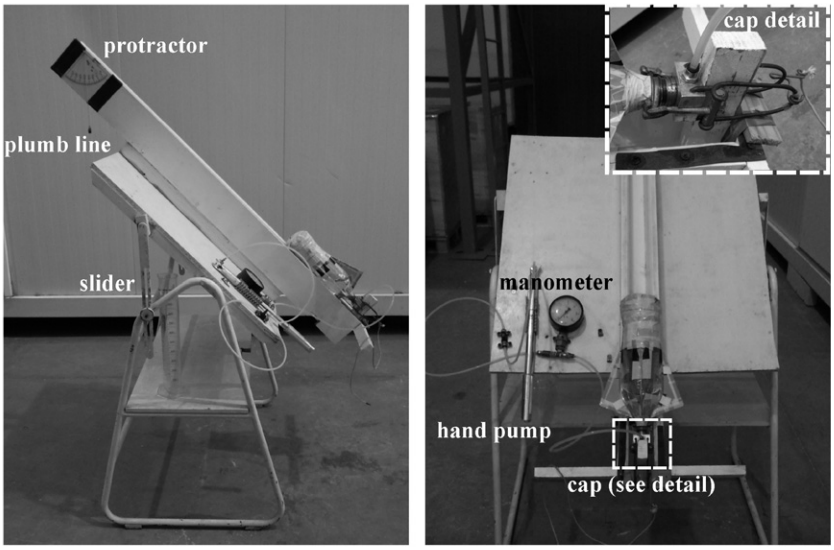

Figure 8. Several details of the launching ramp used in the field tests.

A plastic bottle for carbonated drinks of volume $V = 2$ l was used in the experimental tests. This type of bottle stands higher pressures than the plastic bottles used typically for still water. The bottle has a mass $m_b = 133.2 \pm 0.1$ g, a cross-sectional diameter $D = 106.0 \pm 0.1$ mm, and a nozzle diameter $d = 21.5 \pm 0.1$ mm. The magnitude of the drag coefficient for a body of similar shape ranges typically between 0.2 and 0.4 [10]. Based on previous experiences [14], the drag coefficient of this particular water rocket was estimated to be $C_D = 0.36$. The volume of water poured in the bottle was measured with a precision of $\pm 5 \times 10^{-3}$ l using a burette. The distance reached by the water rocket was obtained with a tape measure with an estimated precision of ± 0.1 m. The magnitude of atmospheric pressure and temperature at test conditions was 1020 ± 1 mbar and 22 ± 1 °C respectively.

The experiments were conducted for different values of the relative pressure of the air $p_{r0} = p_{a0} - p_{atm}$, volume of water V_{w0} and launching angle θ. One of these variables was modified in each test series while keeping the other two constant. The horizontal distance x reached in the tests was compared with that predicted from the model. The mathematical equations deduced previously were solved in Matlab® as successive systems of ordinary differential equations.

The distance reached by the water rocket is presented in figure 9 as a function of the relative pressure of the air. The data shown in the figure were obtained with a constant volume of water of 0.5 l and launching angle of 45°. As expected, the horizontal distance increases continuously with p_{r0} due to the higher velocities obtained at the nozzle and the corresponding increase in rocket momentum. The comparison between the test data and the predictions from the mathematical model presents quite a good agreement, showing relative differences that are below 6%.

Figure 10 presents the magnitude of x as a function of the initial volume of water V_{w0} with a constant relative pressure of 2 bars and launching angle of 45°. As observed, the distance reached by the rocket increases fast for volumes of water below 0.3 l. Subsequently, it progresses smoother until $V_{w0} = 0.5 - 0.6$ l (where maximum distances are reached), and then declines slowly due to the more pronounced effect of the weight of water and the inertial

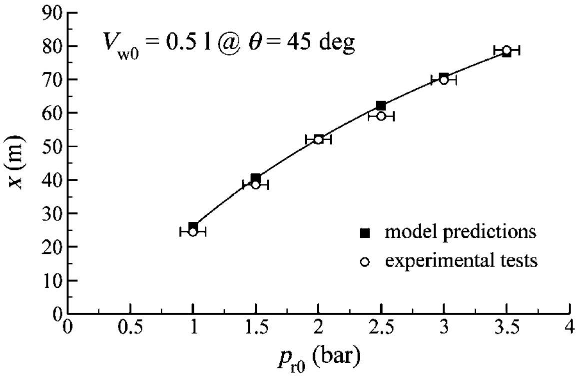

Figure 9. Horizontal distance reached by the water rocket as a function of the relative pressure of the air (V_{w0}, θ = constant).

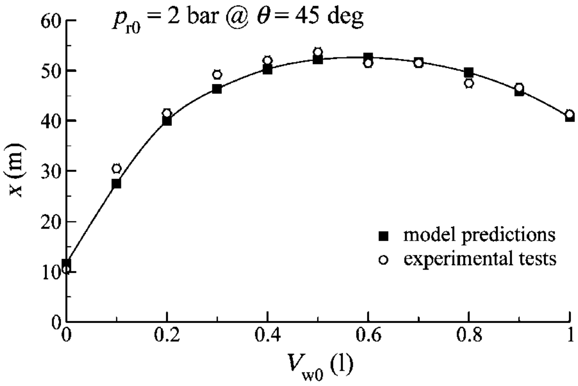

Figure 10. Horizontal distance reached by the water rocket as a function of the volume of water (p_{r0}, θ = constant).

forces. It is noted in figure 10 that there is a good agreement between the predictions obtained from the model and the experimental tests: the model predicts reasonably the tendency of the data and their magnitude. The relative difference between the predictions and the field tests also remains below 6%. Finally, the contribution of the air thrust to the bottle momentum can be appreciated in the first point of figure 10: a mass of pressurized air at 2 bars without any water ($V_{w0} = 0$ in the figure) can propel the rocket to a distance of about 10 m.

The horizontal distance x for several launching angles θ is presented in figure 11 with a constant volume of water of 0.5 l and a relative pressure of 2 bars. As observed, the distance x seems to increase with θ until a value of this angle of around 40°, where a maximum magnitude of x is obtained. From this point on, the distance reached by the rocket decreases as the angle θ is increased. Additionally, it can be seen in figure 11 that the predictions from the model are in good agreement with the experimental tests. The model predicts a similar tendency to that experimentally observed and also shows a good quantitative agreement, with typical differences below 4%.

In summary, it can be concluded that the proposed model properly simulates the propulsion and flight of the bottle. The model will be used in the following section to show some details of water rocket physics.

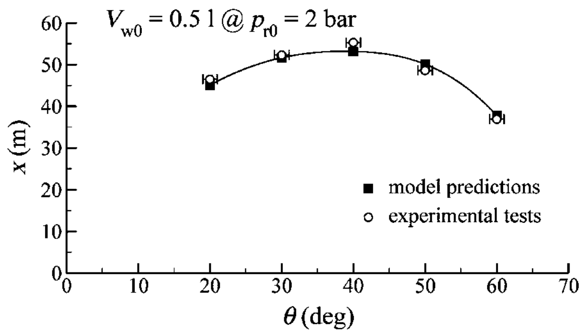

Figure 11. Horizontal distance reached by the water rocket as a function of the launching angle (V_{w0}, p_{r0} = constant).

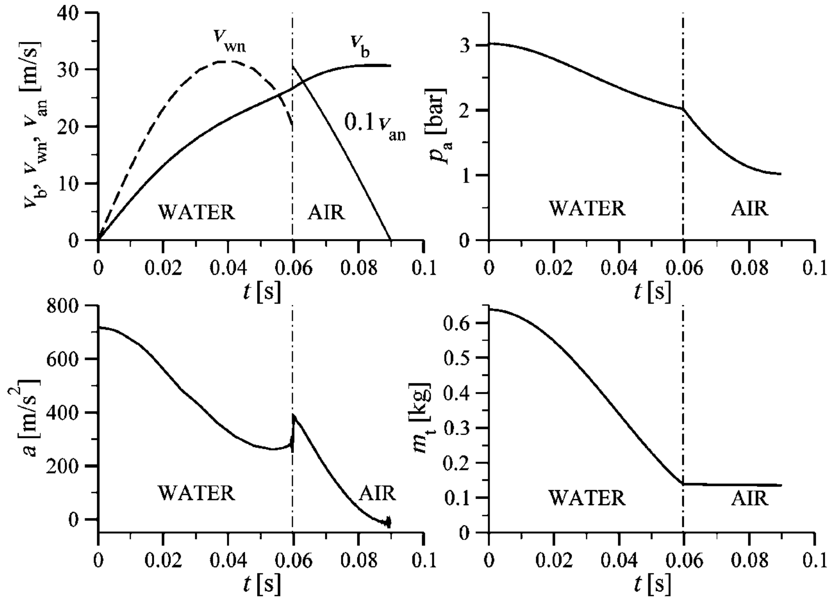

Figure 12. Temporal evolution of several significant variables during the propulsion stage of the rocket. Results obtained for $V_{w0} = 0.5$ l, $p_{r0} = 2$ bar, $\theta = 45°$.

6. Results and discussion

The mathematical model deduced in previous sections was used to investigate the physics of the water rocket in the propulsion and flight stage. Some significant results for the former stage are presented in figure 12. These results were obtained for an initial volume of water $V_{w0} = 0.5$ l, relative air pressure $p_{r0} = 2$ bar and launching angle $\theta = 45°$. The vertical line in each figure is the limit between the water and air propulsion.

As observed, the velocity v_b of the rocket increases continuously during the water propulsion stage. The initial acceleration a is very high (slightly above 70 g) so that the rocket can reach a velocity of about 26 m s^{-1} in only 60 ms. The absolute pressure p_a

decreases gradually from 3 to 2 bar (almost following a linear trend) as water is pushed out through the nozzle by the confined air; at the same time, the total mass m_t inside the CV (i.e. plastic + water + air, see figure 5) diminishes. It can also be observed that the relative velocity v_{wn} of the exit jet of water starts from a zero value and then increases to a maximum that is slightly above 30 m s^{-1}; subsequently, this relative velocity decreases towards the end of the water propulsion stage. It seems surprising that the maximum acceleration is obtained at $t = 0$, that is, for a minimum velocity v_{wn} of the exit jet of water, but this is explained if equation (4) is considered. Leaving aside the external forces F_x on the rocket (they are several orders of magnitude smaller than the terms in the right-hand side), it can be deduced that the acceleration of the rocket (first term) is due to, on the one hand, the momentum of the outgoing flow (last two terms) and, on the other hand, the change in momentum of the mass of water in the rocket (second term). At the very beginning (just when the bottle is opened) all the velocities equal zero and the pressure of the air in the rocket generates a non-balanced force that leads to an increment of the water exit velocity through the nozzle and, at the same time, of the velocity of the confined water according to the section ratio. If it were considered, to simplify the equations, that the section of the nozzle is very small when compared to the cross-section of the bottle, it can be assumed that the change in pressure within the rocket is very slow and also that the relative velocity of the confined water is insignificant. These assumptions (imposed in most simple models) lead to a quasi-steady solution that neglects the inertia of the confined water and, therefore, the velocity through the nozzle can be calculated simply as $v_{wn} = (p_{ra}/\rho_w)^{0.5}$ [8, 9].

The section of the nozzle however, though relatively small, is not insignificant compared to the rocket cross-sectional area (for example, this relation is 4% for the bottle used in the investigation), and there is a transient state of about 40 ms (see evolution of v_{wn} in figure 12) during which a significant part of the thrust is obtained by the change in momentum of the water in the bottle (that diminishes continuously), in addition to the thrust caused by the outgoing flow through the nozzle, which increases its velocity from $t = 0$ to the quasi-steady state ($t = 40$ ms). From this time on, the thrust is obtained almost exclusively from the momentum of the exit flow whose velocity decreases as the pressure of the confined air also decreases. At the end of this stage, the rocket has gained a velocity of 26.7 m s^{-1}.

The exit jet of air provides an additional increment in the acceleration at the beginning of the second propulsion stage. This causes the rocket to gain an extra velocity and reach 30.7 m s^{-1} at the end of the stage. As appreciated, the thrust provided by the air alone is not negligible at all for the present case. The initial pressure of the confined air at the beginning of the stage (about 2 bar) is high enough to induce sonic conditions in the flow through the nozzle. This is observed in the magnitude of v_{an} presented in figure 12 (note that the velocity v_{an} is divided by 10 to fit the vertical scale). The initial velocity of 306 m s^{-1} corresponds to sound velocity under the pressure and temperature conditions in the nozzle (choking is reached at this region during the first instance of the stage). As the pressure of the confined air decreases exponentially, the exit velocity of the fluid through the nozzle falls below sound velocity and the choking condition disappears. The total mass in the CV remains almost unaltered during the air propulsion stage; there is only a small decrement as the pressurized air leaves the rocket.

To see the effect of the amount of water, the temporal evolution of the velocity and acceleration of the water rocket is presented in figure 13 as a function of the initial volume of water V_{w0}. The results shown were obtained with an air relative pressure $p_{r0} = 2$ bar and launching angle $\theta = 45°$. As explained in the previous sections, the propulsion of the rocket during the exit of the water is obtained in two stages and is due to (i) the change in momentum of the confined water (that decreases quickly), and (ii) the velocity of the outgoing flow, which

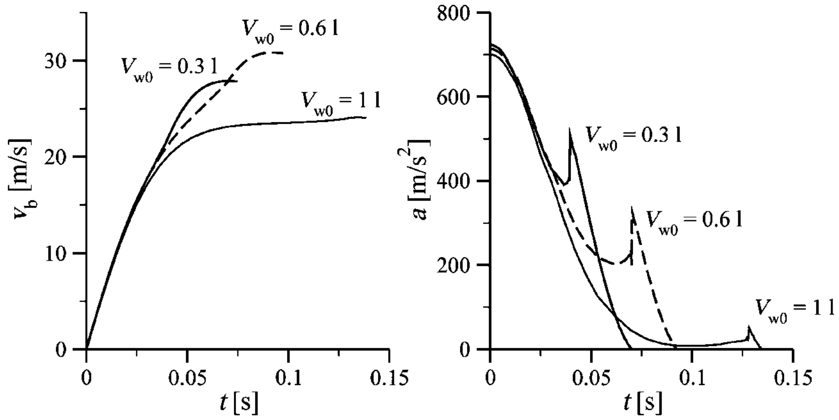

Figure 13. Effect of the initial volume of water on the velocity and acceleration of the water rocket ($p_{r0} = 2$ bar, $\theta = 45°$).

increases fast until reaching a maximum magnitude and then decreases towards the end of the water propulsion cycle. While the first stage is relatively independent of the volume of water (both the magnitude of the acceleration and the slope of the curve during the first instance do not change significantly with V_{w0}, as observed in figure 13), this volume has a strong influence during the second stage. If the initial volume of water is increased, this will result in a displacement of the quasi-steady state towards a higher t, near the minimum value of the acceleration in the right-hand side of figure 13. Once this state is reached, the exit jet of fluid goes on for a longer time when increasing its initial volume, thus providing propulsion thrust during a larger time interval. In contrast, the outgoing flow has to propel a bigger mass of fluid so that the magnitude of the thrust is decreased and, additionally, the pressure of the confined air (and hence the velocity of the flow through the nozzle) is lower due to a larger expansion inside the bottle: the initial volume and the mass of air is smaller for increasing volumes of water. An inverse reasoning can be made for decreasing volumes of water: a small volume of fluid originates a brief jet with a higher exit velocity (the pressure at the beginning of the quasi-steady state is high due to a smaller expansion of the air) that provides a higher thrust also because the mass of the system is smaller. There is an optimum volume of fluid to obtain a balance between the velocity and time length of the exit jet of fluid, and the mass of water that is to be propelled, to get a maximum rocket velocity. According to the results presented in figure 13, this optimum volume seems to be around 0.6 l for the volume and pressure tested.

The effect of the launching angle on the trajectory and distance reached by the rocket is shown in figure 14 (left-hand side). It is observed that a high launching angle θ results in a more vertical trajectory and thus in increasing heights for a specific air pressure and water volume. It is known from basic ballistics that the maximum distance x is attained with a launching angle of 45° if drag is neglected. In the present case, however, the inclusion of the drag on the rocket causes this angle to deviate slightly from this theoretical value, as will be shown shortly. Also, it can be seen in figure 14 that the rocket trajectory is not vertically symmetrical as it should be for a basic ballistic flight. More to the point, the descending path falls faster due to the combined effect of gravity and drag.

One of the challenges that can be proposed to the students is to reach a maximum distance with a specific air pressure. Basically, they have to determine the optimum volume of water

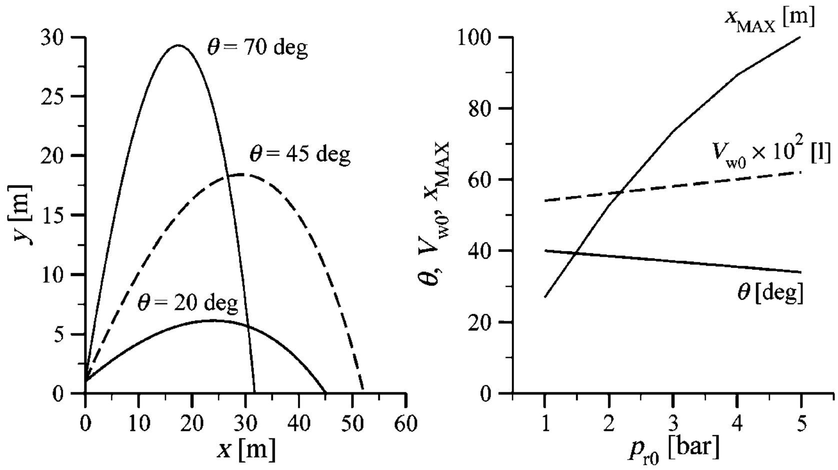

Figure 14. Effect of several significant parameters on the distance reached. The results of the left figure were obtained with $V_{w0} = 0.5$ l and $p_{r0} = 2$ bar.

and the launching angle and to estimate the distance from the equations. Figure 14 (right-hand side) shows the maximum distance that can be reached (using the water rocket of the present investigation) as a function of the relative air pressure (an upper limit of 5 bar was set for safety reasons). This figure also shows the volume of water V_{w0} and the launching angle θ that must be used to obtain that x_{MAX} for each pressure value. As shown, the distance x_{MAX} increases as pressure is increased (as previously observed in figure 9, even without the optimum values of θ and V_{w0}). It is seen that the optimum volume of water increases with p_{r0} but, for the pressure interval considered in the investigation, lies between 0.5 and 0.6 l. In contrast, the optimum launching angle decreases as pressure is increased, as known when drag is included in ballistic calculations. This decrease goes from an initial value of 40° to a magnitude of 30° at 5 bar. Obviously, the quantitative results of the figure will change if a water rocket of different geometry is used; nonetheless, the trends observed will be essentially the same.

7. Conclusions

This paper has presented a mathematical model that was developed to investigate the physics of water rockets. The model includes some new aspects of flow (when compared to other models reported previously in the literature) like the inertial effects of the water in the rocket and also the exit of the remaining air once the entire volume of water is expelled. The numerical predictions were validated by a series of field tests under variation of several significant parameters. A good agreement between the predictions and the experimental measurements was found, showing maximum relative errors in the range 4–6%. Once validated, the model was used to study the flow physics during the propulsion of the water rocket.

It was observed that the thrust on the rocket is not only obtained by the outgoing jet of water but by the combined effect of (i) the change in momentum of the water in the rocket, and (ii) the momentum of the exit jet through the nozzle. The former effect is more notorious during the first instance of the propulsion, when the acceleration gets its maximum magnitude; whereas, the importance of the latter effect increases until reaching a quasi-steady state. Also, it was seen that the additional propulsion obtained by the exit of the remaining air is not negligible; for the tests reported, an increment in the velocity of the rocket of about 15% was observed (choking is reached at the nozzle during the initial instants of the air propulsion). An investigation of the most significant variables (initial water volume, air pressure, and launching angle) on the distance reached revealed that an initial volume of water between 0.5 and 0.6 l provides an optimum balance between the velocity (and time length) of the outgoing jet of fluid, and the propelled mass, to reach a maximum distance with a specific initial pressure. The optimum launching angle is a bit different from the theoretical value of 45° for a ballistic flight in the absence of drag; the predictions from the mathematical model revealed that this angle is in the range 40–30°, decreasing as pressure increases.

Although the model reported is more suitable for graduate projects, a simplified version (for example, neglecting the inertia of the confined water or the air thrust) can be used at an undergraduate level to provide the students with an insight into the basis of fluid dynamics, CV theory, aerodynamic drag, ballistics, etc. Even without solving the equations, the field tests are not only spectacular but also provide a meaningful experience because they show the effect of jet propulsion and the importance of fluid mechanics in real-life situations. If the tests are designed and conducted by the students on their own, it can help them to develop several generic competences like the capacity for applying knowledge in practice, teamwork, or the ability to work autonomously. The personal experience of the authors is that this practice is very well received by the students because they feel it very different from traditional and well-guided laboratory sessions. (Warning: this mathematical model is not a toy and must be used under adult supervision.)

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