Notes on inverse Higgs Constraints

Ameen Ismail

August 29, 2019

1 Formalism

This is mostly adapted from Low and Manohar's paper. Consider a theory with a global symmetry G which is broken down to a subgroup H. If G is an internal symmetry (that is, its generators T^A have no spacetime dependence), then Goldstone's theorem states that our theory will have one massless boson for each broken symmetry generator. However, as we will see, if G is a spacetime symmetry then some of these degrees of freedom may be redundant, and so the number of Goldstone boson is less than the number of broken generators.

We denote the unbroken generators by $T^\alpha(x)$ and the broken generators by $T^a(x)$. The unbroken generators annihilate the vacuum $\langle\phi\rangle$: $T^\alpha\langle\phi\rangle = 0$. We can parametrize fluctuations of the vacuum in terms of the Goldstone modes $c_a(x)$:

$$\phi = e^{c_a(x)T^a(x)}\langle\phi\rangle \approx (1 + c_a(x)T^a(x))\langle\phi\rangle . \tag{1}$$

So small fluctuations about the vacuum are given by $\delta\phi = c_a T^a\langle\phi\rangle$. Suppose there exists a nontrivial solution to

$$c_a(x)T^a(x)\langle\phi\rangle = 0. \tag{2}$$

This indicates a redundancy in our counting of the Goldstone modes, and thus we can eliminate one of the modes. We can eliminate a degree of freedom for every nontrivial solution to the above equation, and so the true number of Goldstone bosons is the number of broken generators minus the number of independent solutions to that equation. For an intuitive example of this, think of a string breaking (2+1)D Poincaré symmetry down to (1+1)D Poincaré. A local rotation of the string can be undone by a local translation.

The situation illustrated by the above example occurs whenever the commutator of the unbroken translation generators P_μ and a broken generator T^a is proportional to another broken generator. Generally speaking, we can write

$$[P_\mu, T^a] = f_{\mu ab}T^b + f_{\mu a\beta}T^\beta. \tag{3}$$

Now we can act on the equation we're trying to solve with P_μ:

$$0 = P_\mu c_a T^a\langle\phi\rangle = [P_\mu, c_a T^a]\langle\phi\rangle = -((\partial_\mu c_a)T^a - f_{\mu ab}c_a T^b)\langle\phi\rangle = -(\partial_\mu c_a - f_{\mu ba}c_b)T^a\langle\phi\rangle . \tag{4}$$

This is easy to interpret intuitively if we consider the case where there's only one nonzero $f_{\mu ab}$. Then the c_b mode is redundant and can be written in terms of the derivative of c_a, reducing the number of Goldstone bosons by one.

The above equation provides us with a constraint on the c_a. We will show that in the CCWZ formalism, we can implement such constraints by setting components of the Maurer-Cartan form to zero. We parametrize an element of the coset space G/H as

$$\tilde{g} = e^{x^\mu P_\mu}e^{\zeta^a(x)T_a}. \tag{5}$$

We must include even the unbroken translation generators in $\tilde{g}$ because they induce nonlinear transformations of the coordinates, and the coset space is the space of all nonlinear symmetry transformations. The Maurer-Cartan form is given by $\Omega = \tilde{g}^{-1}\,\mathrm{d}\tilde{g}$. It can be resolved into components:

$$\Omega = \Omega^\mu P_\mu + \Omega^a T_a + \Omega^\alpha T_\alpha. \tag{6}$$

The Maurer-Cartan form is determined by the algebra. In particular, at linear order we find

$$\Omega^a = (\partial_\mu \zeta^a - f_{\mu b a}\zeta^b)\,\mathrm{d}x^\mu\,. \tag{7}$$

Comparing to our constraint equation from before, we see we can implement the constraint by insisting

$$\Omega^a T_a \langle \phi \rangle = 0. \tag{8}$$

So the constraint (which is often called an inverse Higgs constraint) is applied by setting the component of the Maurer-Cartan form along T^a to 0.

2 Application to Abelian gauge theory

This section is mostly adapted from Goon, Joyce, and Trodden's paper. Consider a $U(1)$ gauge symmetry. We parametrize gauge transformations as $e^{\alpha(x)Q}$, where Q is the $U(1)$ charge and α is an arbitrary function. Under this symmetry, a gauge field A_μ transforms as $A_\mu \to A_\mu + \partial_\mu \alpha$. The symmetry is nonlinearly realized by the gauge field, except for global transformations (i.e. when α is constant). Now, we can expand $\alpha(x)$ in a power series:

$$\alpha(x)Q = \alpha_0 Q + \sum_{k=1}^{\infty} \alpha_{\mu_1\ldots\mu_k} x^{\mu_1}\ldots x^{\mu_k} Q \tag{9}$$

where the $\alpha_{\mu_1\ldots\mu_k}$ are arbitrary constants. We then define generators $Q^{\mu_1\cdots\mu_k}$ by

$$Q^{\mu_1\cdots\mu_k} = x^{\mu_1}\ldots x^{\mu_k} Q \tag{10}$$

so that we can rewrite $\alpha(x)$ as $\alpha(x)Q = \alpha_0 Q + \sum_{k=1}^{\infty} \alpha_{\mu_1\ldots\mu_k} Q^{\mu_1\cdots\mu_k}$. We have now rephrased the gauge symmetry as the combination of an infinite number of global symmetries. Q induces global transformations and is linearly realized by gauge fields, while the $Q^{\mu_1\cdots\mu_k}$ induce local transformations and are nonlinearly realized. So we consider the symmetry breaking pattern in which the $U(1)$ gauge symmetry is broken down to the global $U(1)$; all generators are broken except for Q.

Proceeding in the CCWZ formalism, we parametrize an element of the coset space as

$$\tilde{g} = e^{x^\mu P_\mu} e^{\Phi_\mu(x)Q^\mu} e^{\Phi_{\mu\nu}(x)Q^{\mu\nu}} e^{\Phi_{\mu\nu\rho}(x)Q^{\mu\nu\rho}} \ldots. \tag{11}$$

Note that the fields $\Phi_{\mu_1\ldots\mu_k}$ are totally symmetric. To compute the Maurer-Cartan form $\Omega = \tilde{g}^{-1}\,\mathrm{d}\tilde{g}$ we need to work out the algebra. The only nontrivial commutators are

$$[P_\mu, Q^{\nu_1\ldots\nu_k}] = -k\delta_\mu^{(\nu_1} Q^{\nu_2\ldots\nu_k)}. \tag{12}$$

We expand the Maurer-Cartan form as $\Omega = \Omega_P^\mu P_\mu + \Omega^Q Q + \Omega_\mu^Q Q^\mu + \ldots$. Computing the first few components yields

$$\Omega_P^\mu = \mathrm{d}x^\mu \tag{13}$$

$$\Omega^Q = \mathrm{d}x^\nu\,\Phi_\nu \tag{14}$$

$$\Omega_\mu^Q = \mathrm{d}x^\nu\,(\partial_\nu \Phi_\mu + 2\Phi_{\mu\nu}). \tag{15}$$

The commutation relations imply we can eliminate $\Phi_{\mu\nu}$ in favour of Φ_μ, by setting the component of the Maurer-Cartan form along Q^μ to zero. However, we cannot simply set Ω_μ^Q to zero, because $\Phi_{\mu\nu}$ is symmetric in its indices, while $\partial_\mu \Phi_\nu$ is not. (This is related to the symmetrizing in the commutation relation.) Instead we define $\Omega_{\mu\nu}$ by $\Omega_\mu^Q = \mathrm{d}x^\nu\,\Omega_{\mu\nu}$. Then we can split $\Omega_{\mu\nu}$ into symmetric and antisymmetric pieces:

$$\Omega_{(\mu\nu)} = \partial_{(\mu}\Phi_{\nu)} + 2\Phi_{\mu\nu} \tag{16}$$

$$\Omega_{[\mu\nu]} = \partial_{[\mu}\Phi_{\nu]} = \frac{1}{2}F_{\mu\nu}. \tag{17}$$

To apply the inverse Higgs constraint, we set the symmetric piece to zero. Then $\Phi_{\mu\nu} = -\frac{1}{2}\partial_{(\mu}\Phi_{\nu)}$, and $\Omega^Q_\mu = \frac{1}{2}\,\mathrm{d}x^\nu\,F_{\mu\nu}$. Going back to the commutation relations, we see we can eliminate all the fields with more indices by the same procedure. After applying these constraints, the higher-order components of the Maurer-Cartan form $\Omega^Q_{\mu\nu\rho\ldots}$ reduce to covariant derivatives of $F_{\mu\nu}$.

Finally we'll construct the effective action from Ω^μ_P and Ω^Q_μ, the lowest-order components of the Maurer-Cartan form along nonlinearly realized generators. (Since Q is linearly realized, we do not use Ω^Q to construct the action.) Our Lagrangian should be a scalar, so first we contract upper and lower indices to build the 2-form

$$F = \Omega^\mu_P \wedge \Omega^Q_\mu = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{18}$$

The simplest possible action we can construct is then

$$S = \int F \wedge \star F. \tag{19}$$

But this is just the familiar Abelian Yang-Mills action, since $F \wedge \star F \sim \mathrm{d}^d x F_{\mu\nu}F^{\mu\nu}$. We could, of course, construct more complicated terms using more components of the Maurer-Cartan form, but these would all give actions built from the covariant derivative and $F_{\mu\nu}$.