Relations

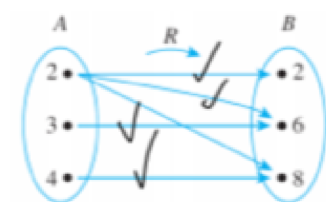
Relations are more general than functions. The key difference is that the same element in the Domain may be related to multiple elements in the Co-domain

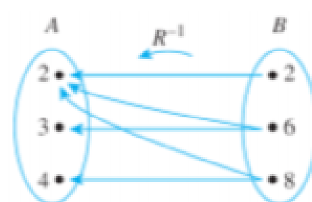
Inverse of a Relation

$\mathbf{R}^{-1} = (y, x) \in B \times A | (x, y) \in R.$

For all $x \in A$ and $y \in B$, $(y, x) \in R_{-1} \iff (x, y) \in R.$

What this means is visualized in the following diagram:





In this case, the second diagram is an inverse of the first.

Finite sets and directed graphs

A directed graph displays the relations inside a finite set. What we see is that from $A = 2, 3, 4, 6, 7, 9$ - we created the three sets $(4, 7)(2)(3, 6, 9)$

Example:
let $A = \{2, 3, 4, 6, 7, 9\}$ and a relation R on the set A be defined by the following directed graph:

Equivalence relations

An equivalence relation is only one if the three following conditions are fulfilled:

Reflexivity

R is reflexive if, and only if, for all $x \in A, xRx$.

What this means, is that x is related to itself. In a directed graph, this would be an arrow from a number showing back to the same number.

Symmetry

R is symmetric if, and only if, for all $x, y \in A$, if xRy then yRx.

What this means, is that two elements are related to each other symmetrically. In a directed graph, these two numbers are connected by two arrows pointing at each other.

Transitivity

R is transitive if, and only if, for all $x, y, z \in A$, if xRy amd yRz then xRz.

What this means, when we have three elements x, y, z, x is related to y and y is related to z which means that x is also related to z. In directed graph, we would have 3 elements in a triangular form all pointing at each other.

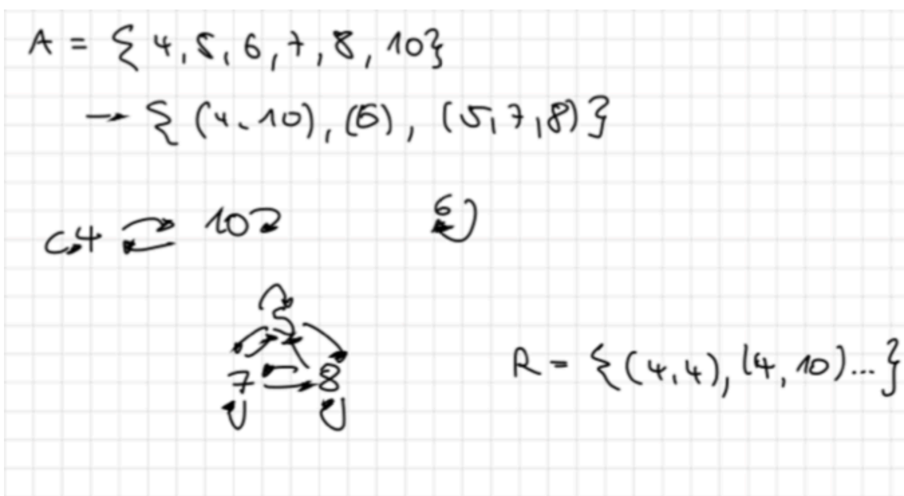
Example: Equivalence relation

From $A = 4, 5, 6, 7, 8, 10$ we arrived to the relation

$$R = (4,4), (4,10), (10,4), (10,10), (6,6), (5,5), (7,7), (8,8), (5,8), (8,7), (7,5)$$

Here we can also see the three conditions clearly.

- Reflexivity: $(4,4), (10,10), (6,6), (5,5), (7,7), (8,8)$
- Symmetry: $(4,10), (10,4)$
- Transitivity: $(5,8), (8,7), (7,5)$



Congruences

Let m and n be integers and let d be a postiive integer. We say that m is congruent to n modulo d and write

$m \equiv n$ (mod d)

if an only if, $d|(m-n)$

Examples

- $12 \equiv 7$ $(mod\ 5)$ -> $5|(12-7) = 5$ -> $5|5$ -> true
- $6 \equiv -8$ $(mod\ 4)$ -> $4|(6-(-8)) = 14$ -> $4|14$ -> false
- $3 \equiv 3$ $(mod\ 7)$ -> $ 3 |(3-3) = 0$ -> 7|3 -> true

Modular arithmetic

Theorem 8.4.1 Modular Equivalences

Let a, b, and n be any integers and suppose $n > 1$. The following statements are all equivalent:

1. $n \mid (a-b)$
2. $a \equiv b \pmod n$
3. $a = b + kn$ for some integer k
4. a and b have the same (nonnegative) remainder when divided by n
5. $a \ mod\ n = b\ mod\ n$

Inverse modulo n

The modular inverse of an integer a is an integer x such that: $a   b   1 (mod \ n) $

Example: Inverse of 3 modulo 7

- $3   0   0 (mod \ 7) $
- $3   1   3 (mod \ 7) $
- $3   2   6 (mod \ 7) $
- $3   3   2 (mod \ 7) $
- $3   4   5 (m o d \ 7) $
- **$3   5   1 (m o d \ 7)** $
- $3   6   4 (mod \ 7) $

Bezout's theorem and Euclidian Algorithm

$gcd(a,b) = \mathbf{s}a + \mathbf{t}b$

$$\gcd(35,27) = \gcd(27, 35 \ mod\ 27) = \gcd(27, 8) \quad \rightarrow 35 = 1 \cdot 27 + 8 \quad \rightarrow 8 = 35 - 27$$
$$= \gcd(8, 27 \ mod\ 8) = \gcd(8,3) \quad \rightarrow 27 = 3 \cdot 8 + 3 \quad \rightarrow 3 = 27 - 3 \cdot 8$$
$$= \gcd(3, 8 \ mod\ 3) = \gcd(3,2) \quad \rightarrow 8 = 2 \cdot 3 + 2 \quad \rightarrow 2 = 8 - 2 \cdot 3$$
$$= \gcd(2, 3 \ mod\ 2) = \gcd(2,1) = 1 \quad \rightarrow 3 = 1 \cdot 2 + 1 \quad \rightarrow 1 = 3 - 2$$

$$\rightarrow 1 = 3 - 2 = 3 - (8 - 2 \cdot 3) =$$
$$= 3 \cdot 3 - 8 = 3 \cdot (27 - 3 \cdot 8) - 8 = 3 \cdot 27 - 10 \cdot 8$$
$$= 3 \cdot 27 - 10 \cdot (35 - 27) = 13 \cdot 27 - 10 \cdot 35 \quad \boxed{\rightarrow 1 = 13 \cdot 27 - 10 \cdot 35}$$

Example: Ceasar cipher

An encryption system which uses the 26 letters of the alphabet but just by pushing them some places forward.

Example: $A = D$ because our steps we are using is 3, which would mean $B = E$, $C = F$ etc.

Formula for encryption: \$C = (M+3) \ mod \ 26 \$

Formula for decryption: \$C = (M-3) \ mod \ 26 \$

Very easy to hack once the factor is known.

RSA cryptography

In RSA, the plaintext M is converted into ciphertext C according to the following formula:

$C = M^e \ mod \ pq$

pq and e are the public keys and anyone can use them to encrypt their messages!

The plaintext M for a ciphertext C is then recovered as follows:

$M = C^d \ mod \ pq$. Where d is the private key; it is secret and only the recipient knows it.

When does the RSA Cipher work?

For the RSA to work, the following expression must all for all positive integers M < pq: $M = (M^e)^d \ mod \ pq$

This holds if:

- p and q are prime

- e and the product $(p-1)(q-1)$ are relatively prime (e.g., their greatest common divisor is 1)
- $ed \equiv 1$ (mod(p−1)(q−1)).
  - (i.e., d is the inverse of e modulo $(p-1)(q-1)$).