Supplementary Material to "Translational and rotational nonGaussianities in homogeneous freely evolving granular gases"

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In this Supplementary Material we give some helpful expressions and integrals used to compute the collisional moments, both as functions of two-body averages, and from the SA. We also expose a comparison between the time evolution toward the HCS of the quantities θ, a_{20}, a_{02} and a_{11}, as predicted by the SA, and simulation results, as well as for μ_{20}^{H}, μ_{02}^{H}, and $\langle c_{12}\rangle^{\mathrm{H}}$. Finally, some details about HVT theoretical derivations and fitting are developed.

I. EVALUATION OF THE COLLISIONAL MOMENTS AS FUNCTIONS OF d_t AND d_r

A. Angular integrals

Some angular integrals are used in the computation of collisional moments. Here, we generalize the results for a d-dimensional Euclidean vector space,

$$\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^\ell = B_\ell c^\ell, \quad B_\ell \equiv \frac{\pi^{\frac{d_t-1}{2}}\Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+d_t}{2}\right)} \tag{1.1a}$$

$$\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\sigma}_i = B_{\ell+1}c^{\ell-1}c_i, \tag{1.1b}$$

$$\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\sigma}_i\widehat{\sigma}_j = B_{\ell+2}c^{\ell-2}c_ic_j + \frac{B_\ell - B_{\ell+2}}{d-1}c^l\delta_{ij}^\perp, \tag{1.1c}$$

$$\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k = B_{\ell+3}c^{\ell-3}c_ic_jc_k + 3\frac{B_{\ell+1} - B_{\ell+3}}{d-1}c^{\ell-1}c_{(i}\delta_{jk)}^\perp, \tag{1.1d}$$

$$\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k\widehat{\sigma}_m = B_{\ell+4}c^{\ell-4}c_ic_jc_kc_m + 6\frac{B_{\ell+2} - B_{\ell+4}}{d-1}c^{\ell-2}c_{(i}c_j\delta_{km)}^\perp + 3\frac{B_{\ell+4} - 2B_{\ell+2} + B_\ell}{d^2-1}c^\ell\delta_{(ij}^\perp\delta_{km)}^\perp, \tag{1.1e}$$

where $\delta_{ij}^\perp \equiv \delta_{ij} - \widehat{c}_i\widehat{c}_j$ and the notation with indices enclosed by parentheses means that one is totally symmetrizing the tensors over such indices, i.e.,

$$c_{(i}\delta_{jk)}^\perp = \frac{1}{3}\left(c_i\delta_{jk}^\perp + c_j\delta_{ik}^\perp + c_j\delta_{ij}^\perp\right), \tag{1.2a}$$

$$c_{(i}c_j\delta_{km)}^\perp = \frac{1}{6}\left(c_ic_j\delta_{km}^\perp + c_ic_k\delta_{jm}^\perp + c_ic_m\delta_{jk}^\perp + c_jc_k\delta_{im}^\perp + c_jc_m\delta_{ik}^\perp + c_kc_m\delta_{ij}^\perp\right), \tag{1.2b}$$

$$\delta_{(ij}^\perp\delta_{km)}^\perp = \frac{1}{3}\left(\delta_{ij}^\perp\delta_{km}^\perp + \delta_{ik}^\perp\delta_{jm}^\perp + \delta_{im}^\perp\delta_{jk}^\perp\right). \tag{1.2c}$$

B. Levi-Civita summations for HD and HS

As introduced in the main text, whereas we worked in a generalized framework, in which expressions are given in terms of the numbers of translational and rotational degrees of freedom of the problem, d_t and d_r, respectively,

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we took into account only two- and three-dimensional setups. In the case of HD, translational velocities are vectors of a two-dimensional Euclidean space $\mathfrak{C}$, whereas the space of the angular velocities is a one-dimensional Euclidean space $\mathfrak{W}$ orthogonal to the previous one, that is, $\mathfrak{W}=\mathfrak{C}^{\perp}$, such that the total space, $\mathfrak{E}^3=\mathfrak{C}\bigoplus\mathfrak{W}$. On the other hand, trivially for HS, $\mathfrak{C}=\mathfrak{W}=\mathfrak{E}^3$. Then, we wrote all relations using general vector notation for elements in the three-dimensional Euclidean space $\mathfrak{E}^3$.

Some vector cross products appear in the computation of the collisional moments, involving both translational and angular velocity variables. Then, it is convenient to express formally those vector products in terms of the three-dimensional Levi-Civita tensor in $\mathfrak{E}^3$, ε_{ijk}. For example, we face terms of the kind $(\mathbf{c}\times\mathbf{w})_i=\varepsilon_{ijk}c_j w_k$, where we are using Einstein's convention of summation over repeated indices.

Let us denote by $\bar{\delta}_{ij}$ the metric of our translational Euclidean space $\mathfrak{C}$ of dimension $d_t=2$ and 3 for HD and HS, respectively. Therefore, if δ_{ij} is the metric of the total embedding space $\mathfrak{E}^3$, then $\bar{\delta}_{ij}\delta_{jk}=\bar{\delta}_{ik}$.

During some computations we faced expressions of the kind $\varepsilon_{ijk}\varepsilon_{ilm}$. Thus, using the identity

$$\varepsilon_{ijk}\varepsilon_{lmn}=\delta_{il}(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})-\delta_{im}\delta_{jl}, \tag{1.3}$$

if the indices i and l are contracted by the metric in $\mathfrak{C}$, then

$$\bar{\delta}_{il}\varepsilon_{ijk}\varepsilon_{lmn}=d_t(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})-(\bar{\delta}_{jm}\delta_{kn}-\delta_{jn}\bar{\delta}_{km})-(\delta_{jm}\bar{\delta}_{kn}-\bar{\delta}_{jn}\delta_{km}), \tag{1.4}$$

where we have used that $\bar{\delta}_{ii}=d_t$.

Let us use this methodology in an example of an angular integral involving translational and rotational variables,

$$\begin{aligned}
\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}\,(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^p\left[\mathbf{c}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{w})\right]^2 =& c_i c_l\left[\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}\,(\mathbf{c}\cdot\widehat{\boldsymbol{\sigma}})^p\widehat{\sigma}_j\widehat{\sigma}_m\right]\varepsilon_{ijk}\varepsilon_{lmn}w_k w_n\\
=& c_i c_l\left[B_{p+2}c^{p-2}c_j c_m+\frac{B_p-B_{p+2}}{d_t-1}c^p\bar{\delta}^{\perp}_{jm}\right]\varepsilon_{ijk}\varepsilon_{lmn}w_k w_n\\
=& c_i c_l\left[B_{p+2}c^{p-2}c_j c_m+\frac{B_p-B_{p+2}}{d_t-1}c^p\left(\bar{\delta}_{jm}-\widehat{c}_j\widehat{c}_m\right)\right]\varepsilon_{ijk}\varepsilon_{lmn}w_k w_n\\
=& B_{p+2}c^{p-2}[\mathbf{c}\cdot(\mathbf{c}\times\mathbf{w})]^2+\frac{B_p-B_{p+2}}{d_t-1}\Big\{c^p\left[d_t(\delta_{il}\delta_{kn}-\delta_{in}\delta_{kl})\right.\\
&\left.-(\bar{\delta}_{il}\delta_{kn}-\delta_{in}\bar{\delta}_{kl})-(\delta_{il}\bar{\delta}_{kn}-\bar{\delta}_{in}\delta_{kl})\right]c_i c_l w_k w_n-c^{p-2}[\mathbf{c}\cdot(\mathbf{c}\times\mathbf{w})]^2\Big\}\\
=& \frac{B_p-B_{p+2}}{d_t-1}c^p\left[d_t(\delta_{il}\delta_{kn}-\delta_{in}\delta_{kl})-(\bar{\delta}_{il}\delta_{kn}-\delta_{in}\bar{\delta}_{kl})-(\delta_{il}\bar{\delta}_{kn}-\bar{\delta}_{in}\delta_{kl})\right]c_i c_l w_k w_n\\
=& \frac{B_p-B_{p+2}}{d_t-1}\left[c^{p+2}w^2-(4-d_t)c^p(\mathbf{c}\cdot\mathbf{w})^2\right].
\end{aligned} \tag{1.5}$$

For HD ($d_t=2$, $d_r=1$), we have $\mathbf{c}\perp\mathbf{w}$, so that the second term vanishes. Then, one can simply replace $(4-d_t)c^p(\mathbf{c}\cdot\mathbf{w})^2$ by $\frac{d_r-1}{2}c^p(\mathbf{c}\cdot\mathbf{w})^2$, which holds both for HD and HS. Following the same reasoning, one gets, for instance,

$$\begin{aligned}
\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}\,(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^p\left[\mathbf{C}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right]^2 =& \left(B_{p+2}-\frac{B_p-B_{p+2}}{d_t-1}\right)c_{12}^p[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})]^2\\
&+\frac{B_p-B_{p+2}}{d_t-1}\left[c_{12}^p C_{12}^2 W_{12}^2-\frac{d_r-1}{2}c_{12}^p(\mathbf{C}_{12}\cdot\mathbf{W}_{12})^2\right],
\end{aligned} \tag{1.6}$$

where we have called $C_{12}\equiv\frac{1}{2}(\mathbf{c}_1+\mathbf{c}_2)$.

C. Computations of collisional moments

1. Collisional impulse

In Sec. II B of the main paper we expressed the postcollisional velocities in terms of the (reduced) impulse $\mathbf{\Delta}_{12}$. It is then possible to derive the following results:

$$\mathbf{\Delta}_{12}^2 =\overline{\alpha}^2(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2+\overline{\beta}^2[c_{12}^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2+4\frac{\theta}{\kappa}(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})^2-4\sqrt{\frac{\theta}{\kappa}}\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})], \tag{1.7a}$$

$$\widehat{\boldsymbol{\sigma}}\cdot\mathbf{\Delta}_{12} =\overline{\alpha}(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}}), \tag{1.7b}$$

$$\begin{aligned}\mathbf{\Delta}_{12}^4 =&\overline{\alpha}^4(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^4+\overline{\beta}^4\Big\{c_{12}^4+(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^4-2c_{12}^2(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2+16\frac{\theta^2}{\kappa^2}\left[W_{12}^4+(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{W}_{12})^4-2W_{12}^2(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{W}_{12})^2\right]\\
&+16\frac{\theta}{\kappa}\left[\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right]^2-8\sqrt{\frac{\theta}{\kappa}}\left[\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right]\left[c_{12}^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2+\frac{4\theta}{\kappa}(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})^2\right]\\
&+8\frac{\theta}{\kappa}[W_{12}^2-(\mathbf{W}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2][c_{12}^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2]\Big\}+2\overline{\alpha}^2\overline{\beta}^2\Big\{c_{12}^2(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^4\\
&+4\frac{\theta}{\kappa}[W_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{W}_{12})^2](\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2\Big\},\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta}_{12}^2(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{\Delta}_{12})^2 =&\overline{\alpha}^4(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^4+\overline{\beta}^2\overline{\alpha}^2\Big\{c_{12}^2(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^4+4\frac{\theta}{\kappa}\left[W_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{W}_{12})^2\right](\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2\\
&-4\sqrt{\frac{\theta}{\kappa}}(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})\left[\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right]\Big\},\end{aligned} \tag{1.7c}$$

$$\widehat{\boldsymbol{\sigma}}\times\mathbf{\Delta}_{12} =\overline{\beta}\left\{(\widehat{\boldsymbol{\sigma}}\times\mathbf{c}_{12})+2\sqrt{\frac{\theta}{\kappa}}[\mathbf{W}_{12}-(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{W}_{12})\widehat{\boldsymbol{\sigma}}]\right\}, \tag{1.7d}$$

$$\mathbf{c}_{12}\cdot\mathbf{\Delta}_{12} =\overline{\alpha}(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2+\overline{\beta}\left[c_{12}^2-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})^2-2\sqrt{\frac{\theta}{\kappa}}\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right], \tag{1.7e}$$

$$\mathbf{C}_{12}\cdot\mathbf{\Delta}_{12} =\overline{\alpha}(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{C}_{12}\cdot\widehat{\boldsymbol{\sigma}})+\overline{\beta}\left[\mathbf{c}_{12}\cdot\mathbf{C}_{12}-(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{C}_{12}\cdot\widehat{\boldsymbol{\sigma}})-2\sqrt{\frac{\theta}{\kappa}}\mathbf{C}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{W}_{12})\right], \tag{1.7f}$$

where use has been made the following vector relations,

$$\widehat{\boldsymbol{\sigma}}\times(\widehat{\boldsymbol{\sigma}}\times\mathbf{A})=(\widehat{\boldsymbol{\sigma}}\times\mathbf{A})\widehat{\boldsymbol{\sigma}}-\mathbf{A},\qquad(\widehat{\boldsymbol{\sigma}}\times\mathbf{A})\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{B})=\mathbf{A}\cdot\mathbf{B}-(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{A})(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{B}). \tag{1.8}$$

2. Collisional changes

As seen in the main text, the (reduced) collisional moments can be written as

$$\mu_{pq}^{(r)}=-\frac{1}{2}\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_1\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_2\int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}\,(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})\phi(\widetilde{\mathbf{\Gamma}}_1)\phi(\widetilde{\mathbf{\Gamma}}_2)(\mathcal{B}_{12,\widehat{\boldsymbol{\sigma}}}-1)\left[\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}}_1)+\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}}_2)\right],\quad \psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}})\equiv c^pw^q(\mathbf{c}\cdot\mathbf{w})^r. \tag{1.9}$$

Using Eqs. (1.7a), we have obtained the collisional changes associated with the second- and fourth-order collisional moments. They are displayed in Table I.

3. Collisional moments in terms of two-body averages

Once the collisional changes displayed in Table I are inserted into Eq. (1.9) and the angular integrals are performed (see Sec. I A), the collisional moments can be expressed in terms of two-body averages. The results are listed in Table II, where, as in the main text, we have simplified the notation as $\mu_{pq}^{(0)}\to\mu_{pq}$. We do not include $\mu_{00}^{(2)}$ because it is meaningful only for HS and is already known.[1]

TABLE I. Collisional changes of the quantities $\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}})$ with $p+q+2r=2$ and 4.

(p,q,r)	$\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}})$	$(\mathcal{B}_{12,\widehat{\boldsymbol{\sigma}}}-1)\left[\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}}_1)+\psi_{pq}^{(r)}(\widetilde{\mathbf{\Gamma}}_2)\right]$
$(2,0,0)$	c^2	$2\left[\boldsymbol{\Delta}_{12}^2-(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\right]$
$(0,2,0)$	w^2	$\frac{2}{\kappa\theta}\left[\boldsymbol{\Delta}_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^2\right]-\frac{4}{\sqrt{\kappa\theta}}\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})$
$(4,0,0)$	c^4	$2\boldsymbol{\Delta}_{12}^4+2\left[(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})^2+4(\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})^2\right]+\boldsymbol{\Delta}_{12}^2(c_{12}^2+4C_{12}^2)-4\boldsymbol{\Delta}_{12}^2(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})-8(\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})(\mathbf{c}_{12}\cdot\mathbf{C}_{12})$ $-(c_{12}^2+4C_{12}^2)(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})$
$(0,4,0)$	w^4	$\frac{2}{\kappa^2\theta^2}\left[\boldsymbol{\Delta}_{12}^4+(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^4-2\boldsymbol{\Delta}_{12}^2(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^2\right]+\frac{2}{\kappa\theta}\Big\{4[\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2+[\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2+\left(2W_{12}^2+\frac{w_{12}^2}{2}\right)$ $\times\left[\boldsymbol{\Delta}_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^2\right]\Big\}-\frac{4}{\sqrt{\kappa\theta}}\left[\left(2W_{12}^2+\frac{w_{12}^2}{2}\right)\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})+(\mathbf{W}_{12}\cdot\mathbf{w}_{12})\,\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\right]$ $-\frac{8}{\kappa\theta\sqrt{\kappa\theta}}\left[\boldsymbol{\Delta}_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^2\right]\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})$
$(2,2,0)$	c^2w^2	$\frac{1}{\kappa\theta}\left[\boldsymbol{\Delta}_{12}^2-(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\Delta}_{12})^2\right]\left\{\frac{1}{2}(4C_{12}^2+c_{12}^2)+2[\boldsymbol{\Delta}_{12}^2-(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})]\right\}+\frac{2}{\sqrt{\kappa\theta}}\Big\{\mathbf{C}_{12}\cdot(2\boldsymbol{\Delta}_{12}-\mathbf{c}_{12})\left[\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\right]$ $-\frac{1}{2}\left[4C_{12}^2+c_{12}^2+4(\boldsymbol{\Delta}_{12}^2-\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\right]\left[\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\right]\Big\}+\frac{1}{2}\left(4W_{12}^2+w_{12}^2\right)\left[\boldsymbol{\Delta}_{12}^2-(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\right]$ $-4\left(\mathbf{W}_{12}\cdot\mathbf{w}_{12}\right)\left(\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12}\right)$
$(0,0,2)$	$(\mathbf{c}\cdot\mathbf{w})^2$	$2(\mathbf{W}_{12}\cdot\boldsymbol{\Delta}_{12})^2+\frac{1}{2}(\mathbf{w}_{12}\cdot\boldsymbol{\Delta}_{12})^2+\frac{2}{\kappa\theta}\left\{[\mathbf{C}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2+\frac{1}{4}[\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2\right\}-2\left(\mathbf{C}_{12}\cdot\mathbf{w}_{12}+\mathbf{W}_{12}\cdot\mathbf{c}_{12}\right)$ $\times\left(\mathbf{W}_{12}\cdot\boldsymbol{\Delta}_{12}\right)-2\left(\mathbf{C}_{12}\cdot\mathbf{W}_{12}+\frac{1}{4}\mathbf{c}_{12}\cdot\mathbf{w}_{12}\right)\left(\mathbf{w}_{12}\cdot\boldsymbol{\Delta}_{12}\right)+\frac{2}{\sqrt{\kappa\theta}}\Big\{\left(\mathbf{W}_{12}\cdot\boldsymbol{\Delta}_{12}\right)[\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]+(\mathbf{w}_{12}\cdot\boldsymbol{\Delta}_{12})$ $\times[\mathbf{C}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]-\left[2(\mathbf{C}_{12}\cdot\mathbf{W}_{12})+\frac{1}{2}(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\right]\mathbf{C}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})-\frac{1}{2}\left[(\mathbf{C}_{12}\cdot\mathbf{w}_{12})+(\mathbf{W}_{12}\cdot\mathbf{c}_{12})\right]\mathbf{c}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\Big\}$

Upon derivation of the results of Table II, we have needed to take into account the following relations:

$$\begin{aligned}\frac{\langle\boldsymbol{\Delta}_{12}^4\rangle}{2B_5}=&\frac{1}{2}\left(\overline{\alpha}^4+\overline{\beta}^4\frac{d_t^2-1}{8}+\overline{\alpha}^2\overline{\beta}^2\frac{d_t-1}{2}\right)\langle c_{12}^5\rangle+\frac{\theta^2}{\kappa^2}\overline{\beta}^4\left[15\langle c_{12}W_{12}^4\rangle-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right.\\&\left.-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right]+\frac{\theta\overline{\beta}^2}{2\kappa}\left[\overline{\beta}^2(d_t+1)-\overline{\alpha}^2\right]\left[5\langle c_{12}^3W_{12}^2\rangle-3\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right],\end{aligned}\tag{1.10a}$$

$$\begin{aligned}\frac{\langle\boldsymbol{\Delta}_{12}^2(4C_{12}^2+c_{12}^2)\rangle}{2B_5}=&\frac{d_t+3}{8}\left\{\left(\overline{\alpha}^2+\overline{\beta}^2\frac{d_t-1}{2}\right)\langle c_{12}^3(c_{12}^2+4C_{12}^2)\rangle+2\overline{\beta}^2\frac{\theta}{\kappa}\left[3\langle c_{12}W_{12}^2(c_{12}^2+4C_{12}^2)\rangle\right.\right.\\&\left.\left.-\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2(c_{12}^2+4C_{12}^2)\rangle\right]\right\},\end{aligned}\tag{1.10b}$$

$$\frac{\langle(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})^2\rangle}{2B_5}=\frac{1}{2}\left(\overline{\alpha}^2+\overline{\beta}^2\frac{d_t^2-1}{8}+\overline{\alpha}\overline{\beta}\frac{d_t-1}{2}\right)\langle c_{12}^5\rangle+\overline{\beta}^2\frac{\theta}{\kappa}\frac{d_t+3}{4}\left[\langle c_{12}^3W_{12}^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right],\tag{1.10c}$$

$$\begin{aligned}\frac{\langle(\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})(4C_{12}^2+c_{12}^2)\rangle}{2B_5}=&\frac{d_t+3}{8}\left(\overline{\alpha}+\overline{\beta}\frac{d_t-1}{2}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(4C_{12}^2+c_{12}^2)\rangle\\&-\overline{\beta}\sqrt{\frac{\theta}{\kappa}}\frac{B_2}{B_5}\langle(4C_{12}^2+c_{12}^2)\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle,\end{aligned}\tag{1.10d}$$

$$\begin{aligned}\frac{\langle\boldsymbol{\Delta}_{12}^2(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\rangle}{2B_5}=&\frac{1}{2}\left[\overline{\alpha}^3+\frac{d_t-1}{4}\left(\overline{\alpha}\overline{\beta}^2+\overline{\alpha}^2\overline{\beta}\right)+\frac{d_t^2-1}{8}\overline{\beta}^3\right]\langle c_{12}^5\rangle\\&+\frac{1}{2}\frac{\theta}{\kappa}\overline{\beta}^2\left\{\left[5\overline{\alpha}+\left(\frac{3d_t-1}{2}+(d_t+3)\right)\overline{\beta}\right]\langle c_{12}^3W_{12}^2\rangle-\left[3\overline{\alpha}+\left(\frac{d_t-3}{2}+(d_t+3)\right)\overline{\beta}\right]\right.\\&\left.\times\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right\},\end{aligned}\tag{1.10e}$$

TABLE II. Collisional moments μ_{pq} with $p+q=2$ and 4 in terms of two-body averages.

(p,q)	$-\mu_{pq}/B_5$
$(2,0)$	$\frac{d_t+3}{4}\left\{\left[\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t-1}{2}\overline{\beta}(\overline{\beta}-1)\right]\langle c_{12}^3\rangle+2\overline{\beta}^2\frac{\theta}{\kappa}\left[3\langle c_{12}W_{12}^2\rangle-\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]\right\}$
$(0,2)$	$\frac{d_t+3}{4}\frac{\overline{\beta}}{\kappa}\left\{-2\left(1-\frac{\overline{\beta}}{\kappa}\right)\left[3\langle c_{12}W_{12}^2\rangle-\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]+\frac{\overline{\beta}}{\theta}\frac{d_t-1}{2}\langle c_{12}^3\rangle\right\}$
$(4,0)$	$\left\{\overline{\alpha}^3(\overline{\alpha}-2)+\frac{d_t^2-1}{8}\overline{\beta}^3(\overline{\beta}-2)+\frac{d_t-1}{2}\overline{\alpha}\overline{\beta}(\overline{\alpha}\overline{\beta}-\overline{\alpha}-\overline{\beta}+1)+\frac{d_t+3}{8}\left[\overline{\alpha}(2\overline{\alpha}-1)+\frac{d_t-1}{2}\overline{\beta}(2\overline{\beta}-1)\right]+\overline{\alpha}^2+\frac{d_t^2-1}{8}\overline{\beta}^2\right\}\langle c_{12}^5\rangle$
	$+\left\{(\overline{\alpha}-\overline{\beta})^2+\frac{d_t+3}{2}\left[\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t-1}{2}\overline{\beta}(\overline{\beta}-1)\right]\right\}\langle c_{12}^3C_{12}^2\rangle+\left\{3\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t-1}{2}\overline{\beta}\left[(d_t+3)\overline{\beta}-3\right]+d_t\left[2\overline{\beta}(\overline{\alpha}-\overline{\beta})-\overline{\alpha}-\overline{\beta}\frac{d_t-1}{2}\right]\right\}\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^2\rangle+2\frac{\theta}{\kappa}\overline{\beta}^2$
	$\left\{\frac{3d_t-1}{2}\langle c_{12}^3W_{12}^2\rangle+\overline{\alpha}(\overline{\alpha}-1)\left(5\langle c_{12}^3W_{12}^2\rangle-3\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)+(d_t+3)\overline{\beta}(\overline{\beta}-1)\left(\langle c_{12}^3W_{12}^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)+(d_t+3)\left[\langle c_{12}^{-1}[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})]^2\rangle\right.\right.$
	$\left.+\langle c_{12}C_{12}^2W_{12}^2\rangle-\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]+3\left(\langle c_{12}^3W_{12}^2\rangle+4\langle c_{12}C_{12}^2W_{12}^2\rangle\right)-\left(\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle+4\langle c_{12}^{-1}C_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)+\frac{\theta}{\kappa}\overline{\beta}^2\left[15\langle c_{12}W_{12}^4\rangle\right.$
	$\left.\left.-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right]\right\}-\frac{16}{3}\frac{B_4}{B_5}\sqrt{\frac{\theta}{\kappa}}\overline{\beta}\left[(d_t-2)\overline{\beta}+4\overline{\alpha}+(d_t+2)\right]\langle(\mathbf{C}_{12}\cdot\mathbf{c}_{12})[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})]\rangle$
$(0,4)$	$\frac{d_t^2-1}{8}\frac{\overline{\beta}^4}{\kappa^2\theta^2}\langle c_{12}^5\rangle+\frac{2}{\kappa\theta}\overline{\beta}^2\left\{\left[\frac{(d_t+3)(d_t+1)}{4}-\frac{\overline{\beta}}{\kappa}\frac{3d_t-1}{2}+\frac{\overline{\beta}^2}{\kappa^2}\left(\frac{3d_t-1}{2}+d_t+3\right)\right]\langle c_{12}^3W_{12}^2\rangle-(d_t+3)\left(\frac{1}{4}+2\frac{\overline{\beta}}{\kappa}+\frac{\overline{\beta}^2}{\kappa^2}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle+\frac{(d_t+3)}{32}\right.$
	$\times\left(\langle c_{12}^3w_{12}^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{w}_{12})^2\rangle\right)\Big\}+2\frac{\overline{\beta}}{\kappa}\left\{\left(\frac{\overline{\beta}}{\kappa}-\frac{1}{2}\right)\left[\frac{3(d_t+3)}{4}\left(4\langle c_{12}W_{12}^4\rangle+\langle c_{12}w_{12}^2W_{12}^2\rangle\right)-\frac{d_t+3}{4}\left(4\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle+\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)\right]\right.$
	$+\frac{\overline{\beta}}{\kappa}\left[\frac{11-d_t(d_t-4)}{2}\langle c_{12}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})^2\rangle-(7-d_t)\langle c_{12}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle-\frac{1}{2}\langle c_{12}^{-2}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle+\frac{d_t-2}{2}\left(\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right.\right.$
	$\left.\left.\left.+\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{w}_{12})^2\rangle+\langle c_{12}w_{12}^2W_{12}^2\rangle\right)\right]+\left(\frac{\overline{\beta}}{\kappa}-4\frac{\overline{\beta}^2}{\kappa^2}+\frac{\overline{\beta}^3}{\kappa^3}\right)\left(15\langle c_{12}W_{12}^4\rangle-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right)\right\}$
$(2,2)$	$\frac{d_t+3}{16}\left[\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t-1}{2}\overline{\beta}(\overline{\beta}-1)\right]\left(4\langle c_{12}^3W_{12}^2\rangle+\langle c_{12}^3w_{12}^2\rangle\right)-\frac{d_t+3}{4}[\overline{\alpha}+(d_t-1)\overline{\beta}]\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{w}_{12}\cdot\mathbf{W}_{12})\rangle+\frac{\overline{\beta}^2}{\kappa\theta}\frac{d_t-1}{4}\left\{\left[\frac{d_t+3}{8}+\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t+1}{2}\overline{\beta}(\overline{\beta}-1)\right]\langle c_{12}^5\rangle\right.$
	$\left.+\frac{d_t+3}{2}\langle c_{12}^3C_{12}^2\rangle\right\}+\frac{\overline{\beta}^2}{\kappa^2}\left\{\frac{d_t+3}{2}\left(3\langle c_{12}C_{12}^2W_{12}^2\rangle-\langle c_{12}^{-1}C_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)+\langle c_{12}^3W_{12}^2\rangle\left[\frac{3(d_t+3)}{8}+5\overline{\alpha}(\overline{\alpha}-1)+5\frac{d_t+1}{2}\overline{\beta}(\overline{\beta}-1)+(d_t+3)\overline{\beta}^2\right]-\left[\frac{d_t+3}{8}+3\overline{\alpha}(\overline{\alpha}-1)\right.\right.$
	$\left.\left.+2\frac{d_t+1}{2}\overline{\beta}(\overline{\beta}+1)+(d_t+3)\overline{\beta}^2\right]\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right\}+2\frac{\theta}{\kappa}\overline{\beta}\left\{\overline{\beta}\left[\frac{\overline{\beta}}{\kappa}\left(\frac{\overline{\beta}}{\kappa}-1\right)\right]\left(15\langle c_{12}W_{12}^4\rangle-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right)+\frac{d_t+3}{8}\left[4\left(3\langle c_{12}W_{12}^4\rangle\right.\right.\right.$
	$\left.\left.\left.-\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)+3\langle c_{12}w_{12}W_{12}^2\rangle-\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]\right\}-\frac{\overline{\beta}}{\kappa}\left\{\left[\frac{3(d_t+3)}{8}+5\overline{\alpha}(\overline{\alpha}-1)+\frac{3d_t-1}{2}\overline{\beta}(\overline{\beta}-1)+\frac{d_t+3}{2}\overline{\beta}(2\overline{\beta}-1)\right]\langle c_{12}^3W_{12}^2\rangle\right.$
	$-\left[\frac{d_t+3}{8}+3\overline{\alpha}(\overline{\alpha}-1)+\frac{d_t+3}{2}\overline{\beta}(2\overline{\beta}-1)\right]\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle+\frac{d_t+3}{2}\left[3\langle c_{12}C_{12}^2W_{12}^2\rangle-\langle c_{12}^{-1}C_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]+\left[\frac{3(d_t+3)}{4}-5\overline{\alpha}-(2d_t+1)\overline{\beta}\right]\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{w}_{12}\cdot\mathbf{W}_{12})\rangle$
	$\left.-\left[\frac{d_t+3}{4}-\overline{\alpha}-\frac{d_t+1}{2}\overline{\beta}\right]\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle+\left(\overline{\alpha}+\frac{d_t+1}{2}\overline{\beta}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{w}_{12}\cdot\mathbf{C}_{12})\rangle+\left(\overline{\alpha}+\overline{\beta}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{w}_{12})(\mathbf{C}_{12}\cdot\mathbf{W}_{12})\rangle\right\}-\frac{d_t+3}{4}\left[\overline{\alpha}+\frac{d_t-1}{2}\overline{\beta}\right]$
	$\times\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{w}_{12}\cdot\mathbf{W}_{12})\rangle-\frac{B_4}{3B_5}\overline{\beta}\sqrt{\frac{\theta}{\kappa}}\left\{4(d_t+2)\langle(\mathbf{w}_{12}\cdot\mathbf{W}_{12})\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle-\frac{1}{\theta}\left[(\overline{\alpha}+\overline{\beta})\langle c_{12}^2\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{w}_{12})\rangle+8\frac{\overline{\beta}}{\kappa}\left(4\langle(\mathbf{w}_{12}\cdot\mathbf{W}_{12})\mathbf{c}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})\rangle\right.\right.\right.$
	$\left.\left.\left.-\langle(\mathbf{c}_{12}\cdot\mathbf{W}_{12})\mathbf{w}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})\rangle+6\langle c_{12}^{-2}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\mathbf{c}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})\rangle\right)\right]\right\}$

$$
\begin{aligned}
\frac{\langle (\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})^2\rangle}{2B_5} =& \frac{1}{8}(\overline{\alpha}-\overline{\beta})^2\langle c_{12}^3C_{12}^2\rangle+\frac{1}{8}\left(3\overline{\alpha}^2+\frac{d_t^2-3}{2}\overline{\beta}^2+2d_t\overline{\alpha}\overline{\beta}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^2\rangle+\frac{\theta}{\kappa}\overline{\beta}^2\frac{d_t+3}{4}\\
&\times\left[\langle c_{12}^{-1}[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})]^2\rangle+\langle c_{12}C_{12}^2W_{12}^2\rangle-\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]+\frac{2B_4}{B_5}\sqrt{\frac{\theta}{\kappa}}\overline{\beta}\\
&\times\left[\left(1-\frac{B_2}{B_4}+\frac{B_2/B_4-1}{d_t-1}\right)\overline{\beta}-2\overline{\alpha}\left(1+\frac{B_2/B_4-1}{d_t-1}\right)\right]\langle(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle,
\end{aligned}
\tag{1.10f}
$$

$$
\frac{\langle (c_{12}^2+4C_{12}^2)(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\rangle}{2B_5}=\frac{d_t+3}{8}\left(\overline{\alpha}+\overline{\beta}\frac{d_t-1}{2}\right)\left(\langle c_{12}^5\rangle+4\langle c_{12}^3C_{12}^2\rangle\right),
\tag{1.10g}
$$

$$
\frac{\langle (\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\rangle}{2B_5}=\frac{d_t+3}{8}\left(\overline{\alpha}+\overline{\beta}\frac{d_t-1}{2}\right)\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^2\rangle-2\sqrt{\frac{\theta}{\kappa}}\frac{B_2}{B_3}\overline{\beta}\langle(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle,
\tag{1.10h}
$$

$$
\begin{aligned}
\frac{\langle (\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})^2(4C_{12}^2+c_{12}^2)\rangle}{2B_5} =& \overline{\beta}^2\frac{d_t+3}{4}\Big\{\frac{d_t-1}{4}(4\langle c_{12}^3C_{12}^2\rangle+\langle c_{12}^5\rangle)+\frac{\theta}{\kappa}\left[12\langle c_{12}W_{12}^2C_{12}^2\rangle+3\langle c_{12}^3W_{12}^2\rangle\right.\\
&\left.-4\langle C_{12}^2c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle)\right]\Big\},
\end{aligned}
\tag{1.11a}
$$

$$
\begin{aligned}
\frac{\langle (\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})^2[\boldsymbol{\Delta}_{12}^2-(\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})]\rangle}{2B_5} =& \frac{\overline{\beta}^2}{2}\Big\{\overline{\alpha}(\overline{\alpha}-1)\left[\frac{d_t-1}{4}\langle c_{12}^5\rangle+\frac{\theta}{\kappa}\left(5\langle c_{12}^3W_{12}^2\rangle-3\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right)\right]\\
&+\overline{\beta}(\overline{\beta}-1)\left(\frac{d_t^2-1}{8}\langle c_{12}^5\rangle+\frac{\theta}{\kappa}\frac{3d_t-1}{2}\langle c_{12}^3W_{12}^2\rangle\right)\\
&+\overline{\beta}(2\overline{\beta}-1)\frac{\theta}{\kappa}\frac{d_t+3}{2}\left[\langle c_{12}^3W_{12}^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]+\overline{\beta}^2\frac{\theta}{\kappa}\left[\frac{3d_t-1}{2}\langle c_{12}^3W_{12}^2\rangle\right.\\
&\left.+\frac{2\theta}{\kappa}\left(15\langle c_{12}W_{12}^4\rangle-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right)\right]\Big\},
\end{aligned}
\tag{1.11b}
$$

$$
\frac{\langle (\mathbf{C}_{12}\cdot\mathbf{c}_{12})[\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]\rangle}{2B_5}=\overline{\beta}\sqrt{\frac{\theta}{\kappa}}\frac{d_t+3}{8}\left[3\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{W}_{12}\cdot\mathbf{w}_{12})\rangle-\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle\right],
\tag{1.11c}
$$

$$
\begin{aligned}
\frac{\langle (\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})[\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]\rangle}{2B_5} =& \frac{1}{2}\overline{\alpha}\overline{\beta}\Big\{\frac{B_2-B_4}{B_5(d_t-1)}\langle c_{12}^2\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{w}_{12})\rangle+2\sqrt{\frac{\theta}{\kappa}}\Big[\frac{5}{4}\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{c}_{12})(\mathbf{W}_{12}\cdot\mathbf{w}_{12})\rangle\\
&-\frac{1}{4}\Big(\langle c_{12}^{-1}(\mathbf{C}_{12}\cdot\mathbf{c}_{12})(\mathbf{W}_{12}\cdot\mathbf{c}_{12})(\mathbf{w}_{12}\cdot\mathbf{c}_{12})\rangle+\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{w}_{12})(\mathbf{W}_{12}\cdot\mathbf{c}_{12})\rangle\\
&+\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{w}_{12})(\mathbf{W}_{12}\cdot\mathbf{C}_{12})\rangle\Big)\Big]\Big\}+\frac{1}{2}\overline{\beta}^2\Big\{\frac{B_4-B_2}{B_5(d_t-1)}\langle c_{12}^2[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{w}_{12})]\rangle\\
&+\sqrt{\frac{\theta}{\kappa}}\Big[\frac{2d_t+1}{2}\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{c}_{12})(\mathbf{W}_{12}\cdot\mathbf{w}_{12})\rangle+\frac{1}{2}\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle\\
&-\frac{d_t+1}{4}\left(\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle+\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{w}_{12})(\mathbf{W}_{12}\cdot\mathbf{c}_{12})\rangle\right)\Big]\\
&-\frac{4}{3}\frac{B_4}{B_5}\frac{\theta}{\kappa}\Big[4\langle(\mathbf{W}_{12}\cdot\mathbf{w}_{12})[\mathbf{c}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})]\rangle-\Big(\langle(\mathbf{c}_{12}\cdot\mathbf{W}_{12})[\mathbf{w}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})]\rangle\\
&+2\langle c_{12}^{-2}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})[\mathbf{c}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})]\rangle\Big)\Big]\Big\},
\end{aligned}
\tag{1.11d}
$$

$$\frac{\langle\!\langle (4C_{12}^2+\mathbf{c}_{12}^2)\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\rangle\!\rangle}{2B_5} = \overline{\beta}\sqrt{\frac{\theta}{\kappa}}\frac{d_t+3}{8}\left[3\left(4\langle\!\langle c_{12}C_{12}^2W_{12}^2\rangle\!\rangle+\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle\right)-4\langle\!\langle c_{12}^{-1}C_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle \right. \\ \left. +\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right] \tag{1.11e}$$

$$\frac{\langle\!\langle (\boldsymbol{\Delta}_{12}^2-\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})\rangle\!\rangle}{2B_5} = \frac{\overline{\beta}}{4}\sqrt{\frac{\theta}{\kappa}}\left\{\overline{\alpha}(\overline{\alpha}-1)\left[5\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle-3\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right]+\overline{\beta}(\overline{\beta}-1)\frac{3d_t-1}{2}\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle \right. \\ \left. +\overline{\beta}(2\overline{\beta}-1)\frac{d_t+3}{2}\left[\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle-\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right]\right\} \\ +\frac{1}{2}\overline{\beta}^3\left(\frac{\theta}{\kappa}\right)^{3/2}\left[15\langle\!\langle c_{12}W_{12}^4\rangle\!\rangle-2d_t\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle-\langle\!\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\!\rangle\right], \tag{1.11f}$$

$$\frac{\langle\!\langle (4W_{12}^2+w_{12}^2)(\boldsymbol{\Delta}_{12}^2-\mathbf{c}_{12}\cdot\boldsymbol{\Delta}_{12})\rangle\!\rangle}{2B_5} = \frac{d_t+3}{8}\left\{\left[\overline{\alpha}(\overline{\alpha}-1)+\overline{\beta}(\overline{\beta}-1)\frac{d_t-1}{2}\right]\left(\langle\!\langle c_{12}^3w_{12}^2\rangle\!\rangle+4\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle\right)+\right. \\ +2\frac{\theta}{\kappa}\overline{\beta}^2\left(3\langle\!\langle c_{12}W_{12}^2w_{12}^2\rangle\!\rangle+12\langle\!\langle c_{12}W_{12}^4\rangle\!\rangle-4\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle \right. \\ \left.\left. -\langle\!\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right)\right\}, \tag{1.11g}$$

$$\frac{\langle\!\langle (\mathbf{W}_{12}\cdot\mathbf{w}_{12})(\mathbf{C}_{12}\cdot\boldsymbol{\Delta}_{12})\rangle\!\rangle}{2B_5} = \overline{\alpha}\frac{d_t+3}{16}\langle\!\langle c_{12}(\mathbf{W}_{12}\cdot\mathbf{w}_{12})(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\rangle\!\rangle+\frac{\overline{\beta}}{2}\left[\frac{(d_t-1)(d_t+3)}{8}\langle\!\langle c_{12}(\mathbf{W}_{12}\cdot\mathbf{w}_{12})(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\rangle\!\rangle \right. \\ \left. -2\frac{B_2}{B_5}\sqrt{\frac{\theta}{\kappa}}\langle\!\langle (\mathbf{W}_{12}\cdot\mathbf{w}_{12})\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle\!\rangle\right], \tag{1.11h}$$

$$\frac{\langle\!\langle (\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})^4\rangle\!\rangle}{2B_5} = \overline{\beta}^4\frac{d_t^2-1}{16}\langle\!\langle c_{12}^5\rangle\!\rangle+\frac{\theta^2}{\kappa^2}\overline{\beta}^4\left[15\langle\!\langle c_{12}W_{12}^4\rangle\!\rangle-2d_t\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle-\langle\!\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\!\rangle\right] \\ +\frac{\theta}{\kappa}\overline{\beta}^4\frac{d_t+1}{2}\left[5\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle-3\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right], \tag{1.12a}$$

$$\frac{\langle\!\langle [\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2\rangle\!\rangle}{2B_5} = \frac{\overline{\beta}^2}{2}\left\{\frac{d_t+3}{8}\left[\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle-\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle\right]+\frac{\theta}{2\kappa}\left[15\langle\!\langle c_{12}W_{12}^4\rangle\!\rangle-2d_t\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle \right.\right. \\ \left.\left. -\langle\!\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\!\rangle\right]\right\}, \tag{1.12b}$$

$$\frac{\langle\!\langle [\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]^2\rangle\!\rangle}{2B_5} = \frac{\overline{\beta}^2}{2}\left\{\frac{(d_t+3)}{8}\left[\langle\!\langle c_{12}^3w_{12}^2\rangle\!\rangle-\langle\!\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{w}_{12})^2\rangle\!\rangle\right]+\frac{\theta}{\kappa}\left[\frac{11-d_t(d_t-4)}{2}\langle\!\langle c_{12}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle \right.\right. \\ -(7-d_t)\langle\!\langle c_{12}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle\!\rangle-\frac{1}{2}\langle\!\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2(\mathbf{c}_{12}\cdot\mathbf{w}_{12})^2\rangle\!\rangle \\ \left.\left. +\frac{d_t-2}{2}\left(\langle\!\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle+\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{w}_{12})^2\rangle\!\rangle+\langle\!\langle c_{12}w_{12}^2W_{12}^2\rangle\!\rangle\right)\right]\right\}, \tag{1.12c}$$

$$\frac{\langle\!\langle (4W_{12}^2+w_{12}^2)(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})^2\rangle\!\rangle}{2B_5} = \overline{\beta}^2\frac{d_t+3}{16}\left\{(d_t-1)\left[4\langle\!\langle c_{12}^3W_{12}^2\rangle\!\rangle+\langle\!\langle c_{12}^3w_{12}^2\rangle\!\rangle\right]+4\frac{\theta}{\kappa}\left[12\langle\!\langle c_{12}W_{12}^4\rangle\!\rangle+3\langle\!\langle c_{12}w_{12}^2W_{12}^2\rangle\!\rangle \right.\right. \\ \left.\left. -4\langle\!\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle-\langle\!\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\!\rangle)\right]\right\}, \tag{1.12d}$$

$$\frac{\langle(4W_{12}^2+w_{12}^2)[\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]\rangle}{2B_5} = \overline{\beta}\sqrt{\frac{\theta}{\kappa}}\frac{d_t+3}{8}\left[12\langle c_{12}W_{12}^4\rangle+3\langle c_{12}w_{12}^2W_{12}^2\rangle-4\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right.$$
$$\left.-\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right], \tag{1.12e}$$

$$\frac{\langle(\mathbf{w}_{12}\cdot\mathbf{W}_{12})[\mathbf{w}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]\rangle}{2B_5} = \overline{\beta}\sqrt{\frac{\theta}{\kappa}}\frac{d_t+3}{8}\left[3\langle c_{12}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-1}(\mathbf{w}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle\right], \tag{1.12f}$$

$$\frac{\langle(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})^2[\mathbf{W}_{12}\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\Delta}_{12})]\rangle}{2B_5} = \overline{\beta}^3\sqrt{\frac{\theta}{\kappa}}\left\{\frac{3d_t-1}{4}\langle c_{12}^3W_{12}^2\rangle-\frac{d_t-3}{4}\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right.$$
$$+\frac{d_t+3}{2}\left[\langle c_{12}^3W_{12}^2\rangle-\langle c_{12}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle\right]$$
$$\left.+\frac{\theta}{\kappa}\left[15\langle c_{12}W_{12}^4\rangle-2d_t\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^4\rangle\right]\right\}. \tag{1.12g}$$

Equations (1.10), (1.11), and (1.12) are related to the evaluation of μ_{40}, μ_{22}, and μ_{04}, respectively.

D. Useful integrals and changes of variable for two-body averages in the Sonine approximation

In this subsection we summarize the most common integral expressions appearing in the two-body averages of the collisional moments appearing in Table II, in the SA.

1. *Maxwellian-type integrals I, J_1, and J_2*

Let us start by introducing the integrals[2]

$$I(\epsilon,p,d)\equiv\int \mathrm{d}\mathbf{x}_1\int \mathrm{d}\mathbf{x}_2\, x_{12}^p e^{-\epsilon x_1^2-x_2^2}, \quad \epsilon>0, \tag{1.13}$$

d being the dimension of the vector space where $\mathbf{x}$ resides. It is convenient to transform our general variables $\mathbf{x}_1$ and $\mathbf{x}_2$ (in analogy to $\mathbf{c}_1$ or $\mathbf{w}_1$, and $\mathbf{c}_2$ or $\mathbf{w}_2$, respectively) into relative and center-of-mass-like variables of the form

$$\mathbf{x}_{12}=\mathbf{x}_1-\mathbf{x}_2, \qquad \mathbf{X}_{12}=\frac{1}{2}(\epsilon\mathbf{x}_1+\mathbf{x}_2). \tag{1.14}$$

Therefore,

$$\mathbf{x}_1=\frac{\mathbf{x}_{12}+2\mathbf{X}_{12}}{1+\epsilon}, \qquad \mathbf{x}_2=\frac{2\mathbf{X}_{12}-\epsilon\mathbf{x}_{12}}{1+\epsilon}, \tag{1.15}$$

with associated Jacobian of the transformation

$$\left|\frac{\partial(\mathbf{x}_1,\mathbf{x}_2)}{\partial(\mathbf{x}_{12},\mathbf{X}_{12})}\right| = 2^d(1+\epsilon)^{-d}, \tag{1.16}$$

Note that the original center-of-mass variable is obtained by setting $\varepsilon=1$. Using this change and d-spherical coordinates, Eq. (1.13) reads

$$I(\epsilon,p,d)=(1+\epsilon)^{-d}\Omega_d^2\int_0^\infty \mathrm{d}x_{12}\, x_{12}^{d+p-1}e^{-\frac{\epsilon}{1+\epsilon}x_{12}^2}\int_0^\infty \mathrm{d}X_{12}\, X_{12}^{d-1}e^{-\frac{4}{1+\epsilon}X_{12}^2} \tag{1.17}$$
$$=\pi^d\epsilon^{-d/2}\left(\frac{1+\epsilon}{\epsilon}\right)^{p/2}\frac{\Gamma\left(\frac{d+p}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},$$

where $\Omega_d=2\pi^{d/2}/\Gamma\left(\frac{d}{2}\right)$.

Analogously, one can obtain

$$\int d\mathbf{x}_1 \int d\mathbf{x}_2\, X_{12}^p e^{-\epsilon x_1^2 - x_2^2} = \frac{1}{2^p} I(\epsilon, p, d). \tag{1.18}$$

Since Eq. (1.17) applies to any $\epsilon > 0$, we can derive with respect to ϵ and then take $\epsilon = 1$ to get

$$\int d\mathbf{x}_1 \int d\mathbf{x}_2\, x_{12}^p x_1^{2q} e^{-x_1^2 - x_2^2} = (-1)^q \left[\frac{\partial^q I(\epsilon, p, d)}{\partial \epsilon^q}\right]_{\epsilon=1}, \tag{1.19a}$$

$$\int d\mathbf{x}_1 \int d\mathbf{x}_2\, X_{12}^p x_1^{2q} e^{-x_1^2 - x_2^2} = (-1)^q \frac{1}{2^p} \left[\frac{\partial^q I(\epsilon, p, d)}{\partial \epsilon^q}\right]_{\epsilon=1}. \tag{1.19b}$$

Similar steps lead to

$$J_1(p, q, d) \equiv \int d\mathbf{x}_1 \int d\mathbf{x}_2\, x_{12}^p X_{12}^q e^{-x_1^2 - x_2^2} = \frac{2^{\frac{p-q}{2}} \pi^d \Gamma\left(\frac{d+p}{2}\right)\left(\frac{d+q}{2}\right)}{\left[\Gamma\left(\frac{d}{2}\right)\right]^2}, \tag{1.20a}$$

$$\begin{aligned} J_2(p, q, r, d) \equiv & \int d\mathbf{x}_1 \int d\mathbf{x}_2\, x_{12}^p X_{12}^q (\mathbf{x}_{12} \cdot \mathbf{X}_{12})^r e^{-x_1^2 - x_2^2} \\ = & \frac{1 + (-1)^r}{2} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+r}{2}\right)} J_1(p + r, q + r, d). \end{aligned} \tag{1.20b}$$

Note that $J_2(p, q, 0, d) = J_1(p, q, d)$ and $J_2(p, q, r, d) = 0$ if $r =$ odd.

Let us suppose that the vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{E}^{d_1}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathfrak{E}^{d_2}$ are all embedded in the same d-Euclidean space, $\mathfrak{E}^d$. Then, the following identity holds,

$$\begin{aligned} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \int d\mathbf{y}_1 \int d\mathbf{y}_2\, & x_{12}^p X_{12}^q (\mathbf{x}_{12} \cdot \mathbf{X}_{12})^r y_{12}^{p'} Y_{12}^{q'} (\mathbf{y}_{12} \cdot \mathbf{Y}_{12})^{r'} (\mathbf{x}_{12} \cdot \mathbf{Y}_{12})^k e^{-x_1^2 - x_2^2 - y_1^2 - y_2^2} \\ & = \frac{1 + (-1)^k}{2} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1+k}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+k}{2}\right)} J_2(p + k, q, r, d_1) J_2(p', q' + k, r', d_2), \end{aligned} \tag{1.21}$$

unless $\mathfrak{E}^{d_1} \perp \mathfrak{E}^{d_2}$, in which case the integral with $k > 0$ vanishes because $\mathbf{x}_{12} \cdot \mathbf{Y}_{12} = 0$.

2. *Sonine integral* I_S

The SA of the VDF implies the action of the Sonine polynomials into the integrals involved in the two-body averages. The Sonine polynomial of degree r of a scalar variable x in a d-dimensional problem is given by

$$S_r(x) = \sum_{k=0}^{r} \frac{(-1)^k \Gamma\left(\frac{d}{2} + r\right)}{\Gamma\left(\frac{d}{2} + k\right)(r - k)! k!} x^k. \tag{1.22}$$

The first three Sonine polynomials are

$$S_0(x) = 1, \quad S_1(x) = -x + \frac{d}{2}, \quad S_2(x) = \frac{1}{2}x^2 - \frac{d+2}{2}x + \frac{d(d+2)}{8}. \tag{1.23}$$

Let us define the following integral where Sonine polynomials are involved,

$$\begin{aligned} I_\mathrm{S}(p, q, r, d) \equiv & \int d\mathbf{x}_1 \int d\mathbf{x}_2\, x_{12}^p x_1^{2q} e^{-x_1^2 - x_2^2} S_r(x_1^2) \\ = & (-1)^q \sum_{k=0}^{r} \frac{\Gamma\left(\frac{d}{2} + r\right)}{\Gamma\left(\frac{d}{2} + k\right)(r - k)! k!} \left[\frac{\partial^{(q+k)} I(\epsilon, p)}{\partial \epsilon^{q+k}}\right]_{\epsilon=1}, \end{aligned} \tag{1.24}$$

where in the second step we have used Eq. (1.19a).

3. Two-body averages in the Sonine approximation

Within the SA described in the main text, we can obtain

$$\begin{aligned}\langle c_{12}^p\rangle_\mathrm{S} =&\pi^{-d_t-d_r}\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_1\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_2\, c_{12}^p e^{-(c_1^2+c_2^2+w_1^2+w_2^2)}\left[1+2a_{20}S_2(c_1^2)+2a_{02}S_2(w_1^2)+2a_{11}S_1(c_1^2)S_1(w_1^2)\right.\\ &\left.+2a_{00}^{(1)}P_2(\widehat{\mathbf{c}}_1\cdot\widehat{\mathbf{w}}_1)\right]\\ =&\pi^{-d_t}\left[I(1,p,d_t)+2a_{20}I_\mathrm{S}(p,0,2,d_t)\right],\end{aligned} \tag{1.25}$$

where $p=$ even and in the second step we have taken into account the orthogonality relations of the Sonine polynomials. Analogously,

$$\begin{aligned}\langle c_{12}W_{12}^2\rangle_\mathrm{S} =&\frac{\pi^{-d_t-d_r}}{4}\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_1\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_2\, c_{12}W_{12}^2 e^{-(c_1^2+c_2^2+w_1^2+w_2^2)}\left[1+2a_{20}S_2(c_1^2)+2a_{02}S_2(w_1^2)+2a_{11}S_1(c_1^2)S_1(w_1^2)\right]\\ =&\frac{\pi^{-d_t-d_r}}{4}\left[I(1,1,d_t)I(1,2,d_r)+2a_{20}I_\mathrm{S}(1,0,2,d_t)I(1,2,d_r)+2a_{02}I(1,1,d_t)I_\mathrm{S}(2,0,2,d_r)\right.\\ &\left.+2a_{11}I_\mathrm{S}(1,0,1,d_t)I_\mathrm{S}(2,0,1,d_r)\right],\end{aligned} \tag{1.26a}$$

$$\begin{aligned}K_1(p,q,r,s,d_t,d_r)\equiv&\langle c_{12}^pC_{12}^q(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^rW_{12}^s\rangle_\mathrm{S}\\ =&\frac{\pi^{-d_t-d_r}}{2^s}\left\{I_\mathrm{S}(s,0,0,d_r)\left[J_2(p,q,r,d_t)+a_{20}\Big(J_2(p,q+4,r,d_t)+J_2(p,q,r+2,d_t)\right.\right.\\ &+\frac{1}{16}J_2(p+4,q,r,d_t)+\frac{1}{2}J_2(p+2,q+2,r,d_t)-\frac{d_t+2}{4}\left(4J_2(p,q+2,r,d_t)+J_2(p+2,q,r,d_t)\right)\\ &\left.+\frac{d_t(d_t+2)}{4}J_2(p,q,r,d_t)\Big)\right]+2a_{02}J_2(p,q,r,d_t)I_\mathrm{S}(s,0,2,d_r)+a_{11}I_\mathrm{S}(s,0,1,d_r)\Big[d_tJ_2(p,q,r,d_t)\\ &\left.-2J_2(p,q+2,r,d_t)-\frac{1}{2}J_2(p+2,q,r,d_t)\Big]\right\},\end{aligned} \tag{1.26b}$$

$$\langle c_{12}^pC_{12}^q(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^rW_{12}^s(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^t\rangle_\mathrm{S} =\frac{d_r-1}{2}\left[\frac{(1+(-1)^t)\Gamma\left(\frac{d_t}{2}\right)\Gamma\left(\frac{t+1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d_t+t}{2}\right)}K_1(p+t,q,r,s+t,d_t,d_r)\right. \tag{1.26c}$$

$$\left.+2a_{00}^{(1)}F(p,q,r,s,t,d_t,d_r)\right]. \tag{1.26d}$$

In Eq. (1.26c),

$$\begin{aligned}F(p,q,r,s,t,d_t,d_r)\equiv&\pi^{-d_t-d_r}\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_1\int \mathrm{d}\widetilde{\mathbf{\Gamma}}_2\, c_{12}^pC_{12}^qW_{12}^s(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^r(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^te^{-c_1^2-c_2^2-w_1^2-w_2^2}P_2(\widehat{\mathbf{c}}_1\cdot\widehat{\mathbf{w}}_1)\\ =&\pi^{-d_t-d_r}\frac{1+(-1)^t}{2^{s+t+5}}\frac{t}{(1+t)(3+t)}J_2(p+t+2,q,r,3)I(1,s+t+2,3),\end{aligned} \tag{1.27}$$

where we have taken into account that the function F is meaningful only for HS.

Furthermore, we have also faced vector products in the averages, for instance,

$$\begin{aligned}\langle c_{12}^{-1}[\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})]^2\rangle =&\langle c_{12}C_{12}^2W_{12}^2\rangle-\langle C_{12}^2c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})^2\rangle-\langle c_{12}^{-1}W_{12}^2(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^2\rangle\\ &+2\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{C}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})\rangle,\end{aligned} \tag{1.28}$$

where we have used the identity $[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2=a^2b^2c^2-a^2(\mathbf{b}\cdot\mathbf{c})^2-b^2(\mathbf{c}\cdot\mathbf{a})^2-c^2(\mathbf{a}\cdot\mathbf{b})^2+2(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a})$. From parity arguments, one can prove that

$$\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{C}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})\rangle=\langle c_{12}^{-3}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle, \tag{1.29a}$$

$$\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})^2\rangle=\langle C_{12}^2c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle, \tag{1.29b}$$

$$\langle(\mathbf{c}_{12}\cdot\mathbf{C}_{12})\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle=\langle(4C_{12}^2+c_{12}^2)\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{W}_{12})\rangle=0. \tag{1.29c}$$

Some of these quantities are similar to those found in the smooth case, [3] but here they are more complex due to the introduction of the angular velocities.

In the computation of μ_{22} from the SA, one needs to deal with the generalized quantity

$$\begin{aligned}K_2(p,q,r,s,t,u,d_t,d_r)\equiv&\langle c_{12}^p C_{12}^q(\mathbf{c}_{12}\cdot\mathbf{C}_{12})^r w_{12}^s W_{12}^t(\mathbf{w}_{12}\cdot\mathbf{W}_{12})^u\rangle_{\mathrm{S}}\\
=&\pi^{-d_t-d_r}\Bigg\{J_2(s,t,u,d_r)\Bigg[J_2(p,q,r,d_t)+a_{20}\Bigg(J_2(p,q+4,r,d_t)+J_2(p,q,r+2,d_t)\\
&+\frac{1}{16}J_2(p+4,q,r,d_t)+\frac{1}{2}J_2(p+2,q+2,r,d_t)-\frac{d_t+2}{4}\left(4J_2(p,q+2,r,d_t)+J_2(p+2,q,r,d_t)\right)\\
&+\frac{d_t(d_t+2)}{4}J_2(p,q,r,d_t)\Bigg)\Bigg]+a_{02}J_2(p,q,r,d_t)\Bigg(J_2(s,t+4,u,d_r)+J_2(s,t,u+2,d_r)\\
&+\frac{1}{16}J_2(s+4,t,u,d_r)+\frac{1}{2}J_2(s+2,t+2,u,d_r)-\frac{d_t+2}{4}\left(4J_2(s,t+2,u,d_r)+J_2(s+2,t,u,d_r)\right)\\
&+\frac{d_t(d_t+2)}{4}J_2(s,t,u,d_r)\Bigg)+2a_{11}\Bigg[\Bigg(J_2(p,q+2,r,d_t)+\frac{1}{4}J_2(p+2,q,r,d_t)\\
&-\frac{d_t}{2}J_2(p,q,r,d_t)\Bigg)\Bigg(J_2(s,t+2,u,d_r)+\frac{1}{4}J_2(s+2,t,u,d_r)-\frac{d_r}{2}J_2(s,t,u,d_r)\Bigg)\\
&+J_2(p,q,r+1,d_t)J_2(s,t,u+1,d_r)\Bigg]\Bigg\}.\end{aligned} \tag{1.30}$$

In particular,

$$\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{c}_{12})(\mathbf{W}_{12}\cdot\mathbf{w}_{12})\rangle_{\mathrm{S}}=K_2(1,0,1,0,0,1,d_t,d_r). \tag{1.31}$$

Moreover,

$$\begin{aligned}\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\cdot\mathbf{w}_{12})\rangle_{\mathrm{S}}=&\langle c_{12}(\mathbf{C}_{12}\cdot\mathbf{w}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})\rangle_{\mathrm{S}}\\
=&\frac{d_r-1}{54}\pi^{-6}\left[2a_{11}+5a_{00}^{(1)}\right]J_2(3,2,0,3)J_2(2,2,0,3),\end{aligned} \tag{1.32a}$$

$$\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{C}_{12})(\mathbf{c}_{12}\cdot\mathbf{W}_{12})(\mathbf{c}_{12}\dot{\mathbf{w}}_{12})\rangle_{\mathrm{S}}=\frac{d_r-1}{27}\pi^{-6}\left[a_{11}+a_{00}^{(1)}\right]J_2(3,2,0,3)J_2(2,2,0,3), \tag{1.32b}$$

$$\langle c_{12}^{-1}w_{12}^2(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle_{\mathrm{S}}=\frac{d_r-1}{2}\left[\frac{1}{3}K_2(1,0,0,2,2,0,3,3)+a_{00}^{(1)}\frac{\pi^{-6}}{15}J_2(3,0,0,3)J_2(2,4,0,3)\right]. \tag{1.32c}$$

From symmetry arguments, the averages involving a power of $\mathbf{c}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})$, $\mathbf{w}_{12}\cdot(\mathbf{C}_{12}\times\mathbf{W}_{12})$, $\mathbf{C}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{w}_{12})$, or $\mathbf{W}_{12}\cdot(\mathbf{c}_{12}\times\mathbf{w}_{12})$ vanish.

II. HIGH-VELOCITY TAIL OF THE MARGINAL DISTRIBUTION $\phi_{cw}(c^2w^2)$

In this section we present an alternative justification for the HVT of $\phi_{cw}(c^2w^2)$ given in the main text.

Assuming $c\gg 1$ and $w\gg 1$ in the stationary version of the BE for the reduced VDF $\phi(\widetilde{\mathbf{\Gamma}})$, we get

$$c\frac{\partial\phi^{\mathrm{H}}}{\partial c}+w\frac{\partial\phi^{\mathrm{H}}}{\partial w}\approx-\gamma_c c\phi^{\mathrm{H}}. \tag{2.1}$$

Here, we have (i) neglected the collisional gain term versus the loss term, (ii) taken $c_{12}\to c_1$, (iii) ignored the dependence on the angle $\cos^{-1}(\widehat{\mathbf{c}}\cdot\widehat{\mathbf{w}})$ (which exists for HS only), (iv) neglected ϕ^{H} versus $c\phi^{\mathrm{H}}$, and (v) taken into account that $\mu_{20}^{\mathrm{H}}/d_t=\mu_{02}^{\mathrm{H}}/d_r$ and $\gamma_c=d_tB_1/\mu_{20}^{\mathrm{H}}$. The general solution of this linear first-order partial differential equation can be obtained from the method of characteristics as

$$\phi^{\mathrm{H}}(\widetilde{\mathbf{\Gamma}})\approx e^{-\gamma_c c}G\left(\frac{w}{c}\right), \tag{2.2}$$

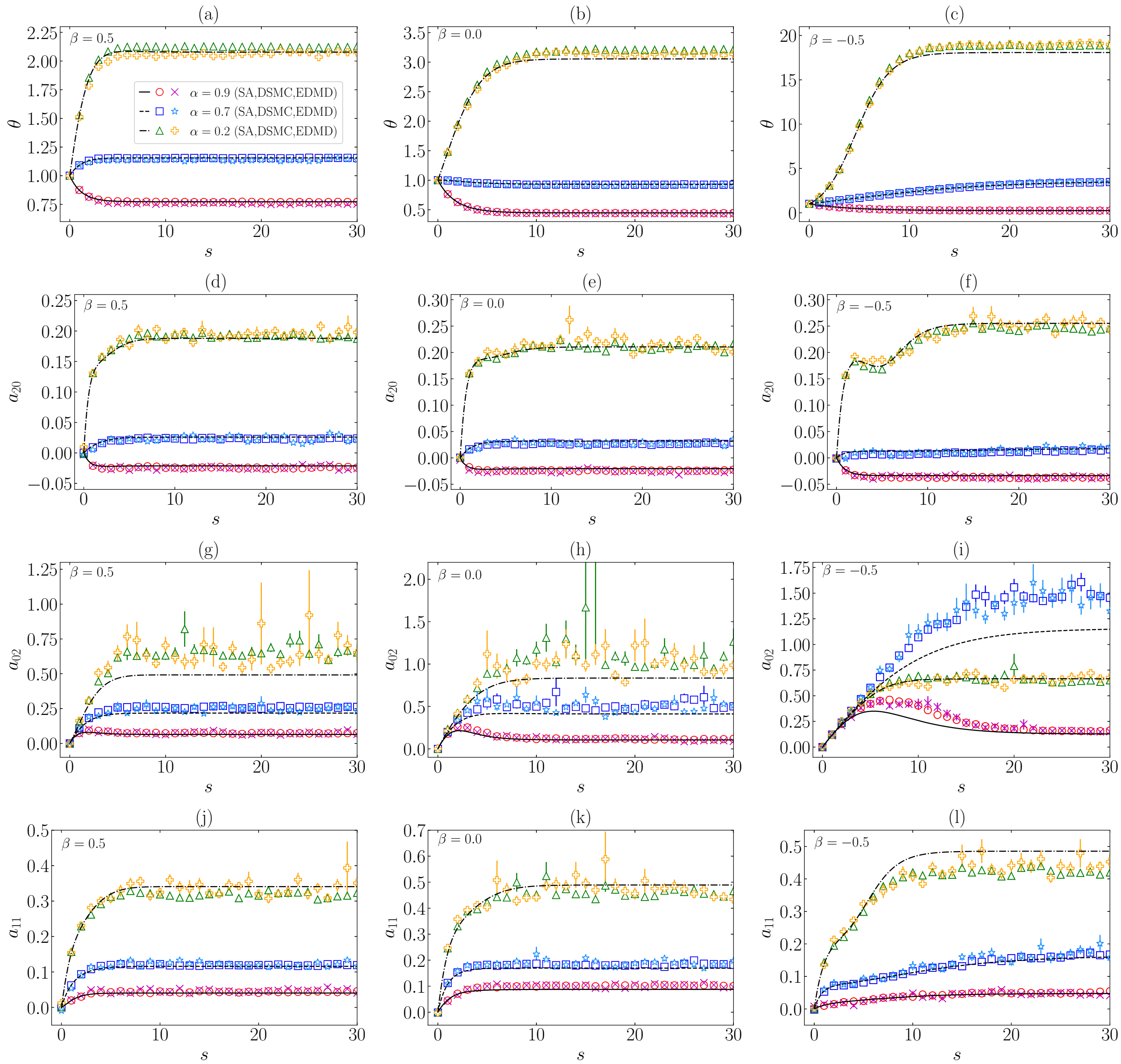

FIG. 1. Plots of (a–c) the temperature ratio $\theta(s)$, (d–f) the cumulant $a_{20}(s)$, (g–i) the cumulant $a_{02}(s)$, and (j)–(l) the cumulant $a_{11}(s)$, for uniform HD ($\kappa = \frac{1}{2}$), as functions of the number of collisions per particle s. The left (a, d, g, j), middle (b, e, h, k), and right (c, f, i, l) panels correspond to $\beta = 0.5$, 0, and -0.5, respectively. In each panel, three values of α are considered: 0.9 (DSMC: $\circ$; EDMD: $\times$), 0.7 (DSMC: $\square$; EDMD: $\star$), and 0.2 (DSMC: $\triangle$; EDMD: $+$). The lines are theoretical predictions from the SA.

where $G(y)$ is an unknown function. Now we take the liberty of assuming that the HVT of the marginal distributions $\phi_{\mathbf{w}}(\mathbf{w})$ and $\phi_{c^2w^2}(x)$ can be obtained from Eq. (2.2), i.e.,

$$\phi_{\mathbf{w}}(\mathbf{w}) \approx \Omega_{d_t} \int_0^\infty \mathrm{d}c\, c^{d_t-1} e^{-\gamma_c c} G\left(\frac{w}{c}\right), \tag{2.3a}$$

$$\phi_{cw}(x) \approx \frac{\Omega_{d_t}\Omega_{d_r}}{2} x^{\frac{d_r}{2}-1} \int_0^\infty \mathrm{d}c\, c^{d_t-d_r-1} e^{-\gamma_c c} G\left(\frac{\sqrt{x}}{c^2}\right). \tag{2.3b}$$

Consistency of Eq. (2.3a) with the HVT $\phi_{\mathbf{w}}(\mathbf{w}) \sim w^{-\gamma_w}$ implies that $G(y) \sim y^{-\gamma_w}$ for large y. Insertion of this asymptotic form of $G(y)$ into Eq. (2.3b) finally yields

$$\phi_{cw}^{\mathrm{H}}(x) \sim x^{-\gamma_{cw}}, \quad \gamma_{cw} = 1 + \frac{\gamma_w - d_r}{2}, \tag{2.4}$$

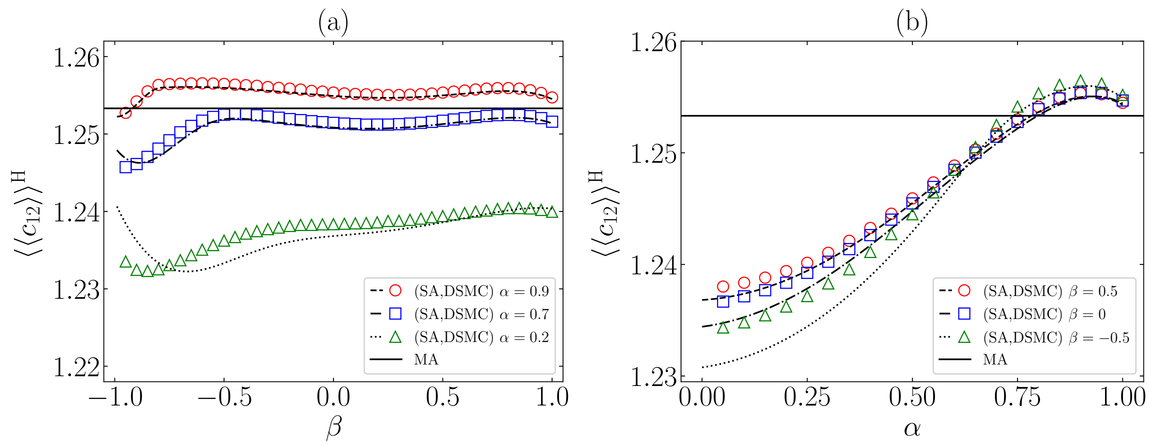

FIG. 2. Two-body average $\langle c_{12}\rangle^{\rm H}$ from MA and SA (lines) and DSMC simulation outcomes (symbols) for uniform HD ($\kappa = \frac{1}{2}$) as a function of (a) the coefficient of tangential restitution β (at $\alpha = 0.9$, 0.7, and 0.2), (b) the coefficient of normal restitution α (at $\beta = 0.5$, 0, and -0.5).

in agreement with the result in the main text.

III. TRANSIENT STATES

In the main text, we focused on the comparison between the theoretical predictions and the simulation data for the HCS. As a complement, here we provide a comparison for the temporal evolution toward the HCS.

Figure 1 shows the evolution of $\theta(s)$, $a_{20}(s)$, $a_{02}(s)$, and $a_{11}(s)$, starting from a Maxwellian and equipartioned initial state, so that $\theta(0) = 1$ and $a_{20}(0) = a_{02}(0) = a_{11}(0) = 0$. We observe that the SA theoretical predictions agree very well with simulations, except close to the HCS values for the cases in which $a_{02}^{\rm H}$, $a_{11}^{\rm H} \sim \mathcal{O}(1)$.

IV. COMPUTATION OF $\langle c_{12}\rangle^{\rm H}$, $\mu_{20}^{\rm H}$, AND $\mu_{02}^{\rm H}$ FROM DSMC. COMPARISON WITH THE MA AND SA

An important point of our work is the exact expression—in the framework of the BE—of the relevant collisional moments in terms of two-body averages, as displayed in Table II. It is then interesting to compute the HCS collisional moments $\mu_{20}^{\rm H}$ and $\mu_{02}^{\rm H}$ from DSMC and compare the results with the MA and SA predictions.

Before starting with the collisional moments, let us first consider the simple two-body average $\langle c_{12}\rangle$. It can be computed in simulations as

$$\langle c_{12}\rangle = \frac{1}{N'}\sum_{ij}^{N'} c_{ij}, \tag{4.1}$$

with, in principle, $N' = N(N-1)/2$ being the total number of pairs. Since we had $N = 10^4$ particles, this would imply $N' \simeq 5\times 10^7$ pairs. Instead, in order to accelerate the computation, we took a random sample of $N' = 10^5$ pairs, which represent a 2% of the total number of pairs. The results for $\langle c_{12}\rangle^{\rm H}$ are displayed in Fig. 2. While the MA value, $\langle c_{12}\rangle_{\rm M} = \sqrt{\pi/2} \simeq 1.253$, is independent of α and β, the dependence of $\langle c_{12}\rangle^{\rm H}$ on both coefficients of restitution is well predicted by the SA, at least semi-quantitatively.

Now we turn to the collisional moments μ_{20} and μ_{02}, whose expressions as linear combinations of the three two-body averages $\langle c_{12}^3\rangle$, $\langle c_{12}W_{12}^2\rangle$, and $\langle c_{12}^{-1}(\mathbf{c}_{12}\cdot\mathbf{W}_{12})^2\rangle$ are displayed in Table II. Those two-body averages are evaluated by the DSMC method by sums over pairs analogous to Eq. (4.1), again with $N' = 10^5$. From Figs. 3(a–d), we infer that both MA and, especially, SA provide good estimates of the two first collisional moments $\mu_{20}^{\rm H}$ and $\mu_{02}^{\rm H}$. Moreover, as Figs. 3(e, f) show, the HCS condition $\mu_{20}^{\rm H}/2\mu_{02}^{\rm H} = 1$ is very accurately fulfilled by the DSMC data.

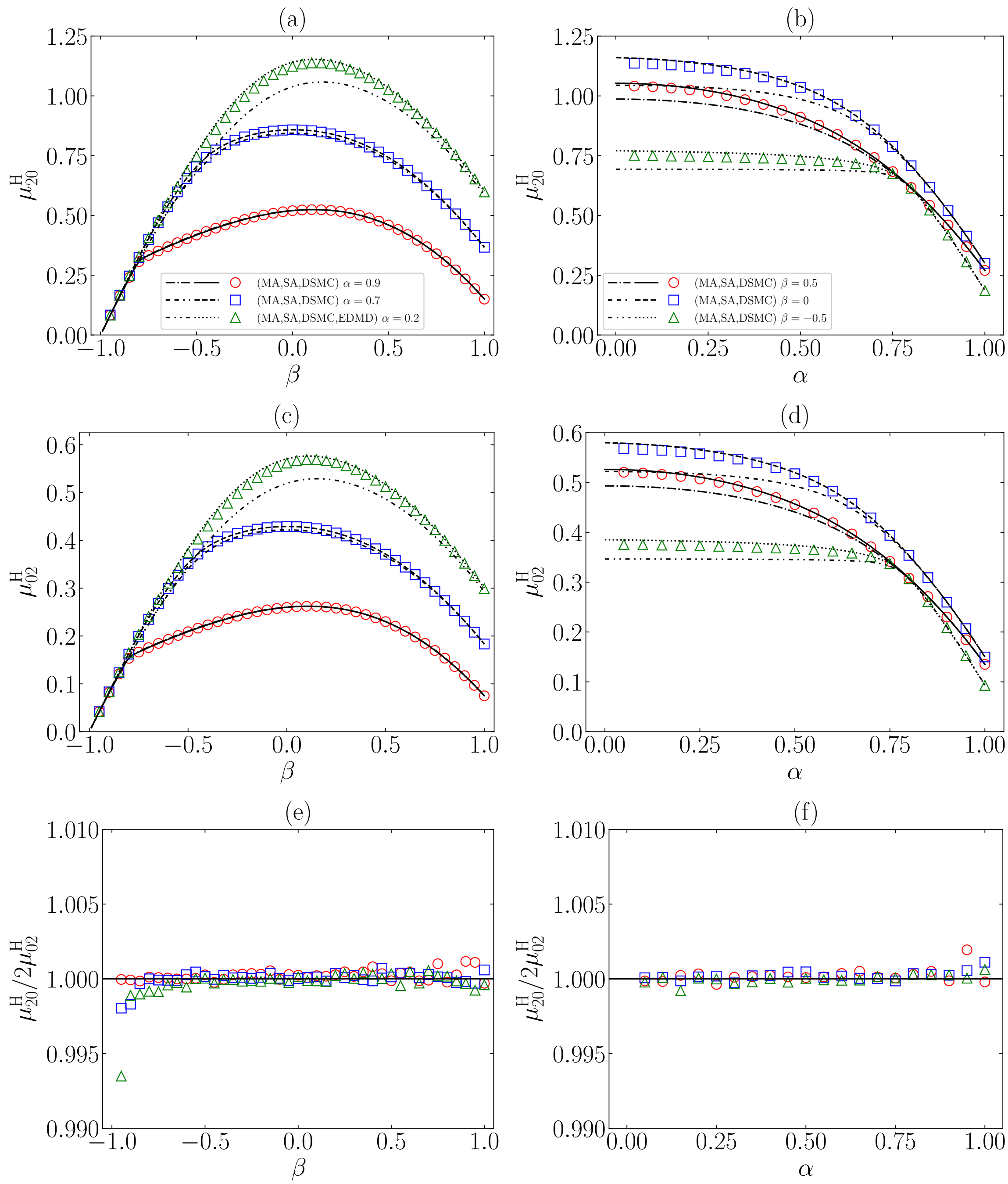

FIG. 3. Plots of (a, b) the collisional moment μ_{20}^{H}, (c, d) the collisional moment μ_{02}^{H}, and (e, f) the ratio $\mu_{20}^{\mathrm{H}}/2\mu_{02}^{\mathrm{H}}$ for uniform HD ($\kappa = \frac{1}{2}$). The quantities are plotted versus (a, c, e) the coefficient of tangential restitution β (at $\alpha = 0.9$, 0.7, and 0.2) and (b, d, f) the coefficient of normal restitution α (at $\beta = 0.5$, 0, and -0.5). Symbols represent DSMC values and lines in panels (a–d) correspond to the SA predictions. The thick black lines in panels (e, f) represent the HCS condition $\mu_{20}^{\mathrm{H}}/2\mu_{02}^{\mathrm{H}} = 1$.

V. SOME TECHNICAL DETAILS ABOUT THE HIGH-VELOCITY FITTING

A. Fitting of the exponents γ_c, γ_w, and γ_{cw}

To get the HVT exponents γ_c, γ_w, and γ_{cw} from simulations (see Fig. 9 of the main text), we fitted the data according to some conditions. First of all, we defined threshold values for the velocity variables $x = c, w, c^2w^2$, beyond which the velocities were considered high enough as to observe the asymptotic behavior. Those thresholds are defined as $x_{\rm thres} = \max\{\widetilde{x}, x_*^{\rm M}\}$, where the values of $\widetilde{x}$ are

$$\widetilde{c} = \frac{5/2}{\sqrt{\frac{\overline{\alpha}^2}{2\alpha^2} + \frac{\overline{\beta}^2}{2\beta^2}}}, \quad \widetilde{w} = \frac{5}{2}\frac{\kappa|\beta|}{\overline{\beta}}, \quad \widetilde{c^2 w^2} = \widetilde{c}^2\widetilde{w}^2. \tag{5.1}$$

This ensures that $c_2'' \gg 1$ and $w_2'' \gg 1$ in Eqs. (B2) and (B4) of the main text. In what concerns to the values of $x_*^{\rm M}$, they were determined under the condition of fulfilling a continuous and differentiable match between the MA VDF and the HVT behavior, as derived in Section V B.

For each histogram, we firstly chose the range of points comprised between $x_{\rm thres}$ and $x_{\rm max}$, where $x_{\rm max}$ represents the maximum value of x in our dataset for a given system. If $x_{\rm max} < x_{\rm thres}$, we concluded that there were not enough data to get a trusted fitting. On the other hand, if $x_{\rm max} > x_{\rm thres}$, we proceeded to choose the proper subrange of data to be fitted from a minimization of $|\chi^2/\text{d.o.f} - 1|$,[4] where χ^2 is the *chi-square* statistic (computed assuming diagonal covariance) and d.o.f is the number of degrees of freedom of the fit. If the number of points in the subrange were larger than 5, we fitted them to the desirable form, getting the slope, as well as the standard deviation of the chosen points in the dataset with respect to the fitting parameters. Finally, we computed Pearson's coefficient of determination, R^2, concluding that the fit was trustable if $R^2 \geq 0.9$, discarding the results otherwise.

B. Matching points $x_*^{\rm M}$

In previous works for the smooth case,[5] a merged HCS VDF was built from a match of the thermal part (as described by the SA) and the asymptotic HVT, the matching point c_* being determined by imposing continuity of the VDF and of its first derivative. In our work, we used this same method to compute the matching points x_* for the marginal distributions $\phi_{\mathbf{c}}^{\rm H}$, $\phi_{\mathbf{w}}^{\rm H}$, and $\phi_{cw}^{\rm H}$.

Although we present below the derivation of x_* from the SA, we actually considered in the fitting the matching points $x_*^{\rm M}$ provided by the MA. This is due to the appearance of bimodal thermal regions of the VDF (especially for $\phi_{\mathbf{w}}^{\rm H}$) in the SA, not observed in simulations, and even unphysical negative values in a small range of values, as previously reported for HS.[1] We exclusively show below the results for HD ($d_t = 2$, $d_r = 1$).

1. *Matching of* $\phi_{\mathbf{c}}^{\rm H}(\mathbf{c})$*:* $c_*^{\rm M}$

We construct a merged distribution $\phi_{\mathbf{c}}^{\rm H}(\mathbf{c})$, such that it coincides with that of the SA for $c < c_*$ and with its asymptotic HVT for $c > c_*$, i.e.,

$$\phi_{\mathbf{c}}^{\rm H}(\mathbf{c}) = \mathcal{A}_c^{\rm th} e^{-c^2}\left[1 + a_{20}^{\rm H} S_2(c^2)\right]\Theta(c_* - c) + \mathcal{A}_c e^{-\gamma_c c}\Theta(c - c_*). \tag{5.2}$$

Imposing the continuity of $\phi_{\mathbf{c}}^{\rm H}(\mathbf{c})$ and its first derivative at the matching point c_* yields the following 5th-degree polynomial equation:

$$c_* = \frac{\gamma_c}{2} - \frac{a_{20}^{\rm H} c_*(c_* - 2)}{1 + a_{20}^{\rm H} S_2(c_*^2)}. \tag{5.3}$$

As said above, however, we take the MA for the thermal part ($c < c_*$), i.e., $c_*^{\rm M} = \gamma_c/2$.

2. *Matching of* $\phi_{\mathbf{w}}^{\rm H}(\mathbf{w})$*:* $w_*^{\rm M}$

In this case, we have

$$\phi_{\mathbf{w}}^{\rm H}(\mathbf{w}) = \mathcal{A}_w^{\rm th} e^{-w^2}\left[1 + a_{02}^{\rm H} S_2(w^2)\right]\Theta(w_* - w) + \mathcal{A}_w w^{-\gamma_w}\Theta(w - w_*). \tag{5.4}$$

Again, we assume continuity of the function and its first derivative at w^*, which gives a cubic equation for w_*^2,

$$w_*^2 = \frac{\gamma_w}{2} - \frac{a_{02}^{\rm H} w_*^2\left(3 - 2w_*^2\right)}{1 + a_{02}^{\rm H} S_2(w_*^2)}. \tag{5.5}$$

Thus, $w_*^{\rm M} = \sqrt{\gamma_w/2}$.

3. *Matching of* $\phi^{\rm H}_{c^2w^2}(x)$*:* $x_*^{\rm M}$

The merged function is now

$$\phi^{\rm H}_{cw}(x) = \mathcal{A}^{\rm th}_{cw}x^{-\frac{1}{2}}e^{-2\sqrt{x}}\left(1+\frac{16a^{\rm H}_{20}+4a^{\rm H}_{11}+6a^{\rm H}_{02}}{8}-\sqrt{x}\frac{5a^{\rm H}_{20}+6a^{\rm H}_{11}+5a^{\rm H}_{02}}{4}+x\frac{a^{\rm H}_{20}+2a^{\rm H}_{11}+a^{\rm H}_{02}}{2}\right)\Theta(x_*-x)$$
$$+\mathcal{A}_{cw}x^{-\gamma_{cw}}\Theta(x-x_*), \tag{5.6}$$

where we have used $K_{\frac{1}{2}}(2\sqrt{x})=\sqrt{\pi}e^{-2\sqrt{x}}/2x^{1/4}$. From the continuity conditions one gets the cubic equation

$$x_*^{3/2}\left(a^{\rm H}_{20}+2a^{\rm H}_{11}+a^{\rm H}_{02}\right)-x_*\left[\left(a^{\rm H}_{20}+a^{\rm H}_{02}\right)(3+\gamma_{cw})+a^{\rm H}_{11}(2+\gamma_{cw})\right]$$
$$+x_*^{1/2}\left[2+a^{\rm H}_{20}\left(4+\frac{5\gamma_{cw}}{2}\right)+a^{\rm H}_{11}(1+3\gamma_{cw})+\frac{a^{\rm H}_{02}}{2}(3+5\gamma_{cw})\right]$$
$$-(2\gamma_{cw}-1)\left[1+2a^{\rm H}_{20}+\frac{1}{2}a^{\rm H}_{11}+\frac{3}{4}a^{\rm H}_{02}\right]=0. \tag{5.7}$$

$$(2\gamma_{cw}-1)\left(1+2a^{\rm H}_{20}+\frac{1}{2}a^{\rm H}_{11}+\frac{3}{4}a^{\rm H}_{02}\right)=x_*^{3/2}\left(a^{\rm H}_{20}+2a^{\rm H}_{11}+a^{\rm H}_{02}\right)-x_*\left[\left(a^{\rm H}_{20}+a^{\rm H}_{02}\right)(3+\gamma_{cw})+a^{\rm H}_{11}(2+\gamma_{cw})\right]$$
$$+x_*^{1/2}\left[2+a^{\rm H}_{20}\left(4+\frac{5\gamma_{cw}}{2}\right)+a^{\rm H}_{11}(1+3\gamma_{cw})+\frac{a^{\rm H}_{02}}{2}(3+5\gamma_{cw})\right]. \tag{5.8}$$

In the MA, we simply get $x_*^{\rm M}=\left(\gamma_{cw}-\frac{1}{2}\right)^2$.

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