Derivation of RANS

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Mean-Flow Equations

Reynolds Equations

- The equations that govern the mean velocity field, $\langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle$, are those derived by Reynolds (1894)
- The decomposition of the velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ into its mean $\langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle$ and the fluctuation given by $\boldsymbol{u}(\boldsymbol{x}, t) \equiv \boldsymbol{U}(\boldsymbol{x}, t) - \langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle$ is referred to as the Reynolds decomposition given by

$$\boldsymbol{U}(\boldsymbol{x}, t) = \langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle + \boldsymbol{u}(\boldsymbol{x}, t) \tag{1}$$

- It follows from the continuity equation given by

$$\nabla \cdot \boldsymbol{U} = \nabla \cdot (\langle \boldsymbol{U} \rangle + \boldsymbol{u}) = 0 \tag{2}$$

- that both $\langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle$ and $\boldsymbol{u}(\boldsymbol{x}, t)$ are solenoidal

Mean-Flow Equations

Reynolds Equations

- The mean of this equation is simply

$$\nabla \cdot \langle \boldsymbol{U} \rangle = 0 \tag{3}$$

- and then by subtraction we obtain

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4}$$

- The operations of taking the mean and differentiation commute
- For the momentum, the covariant derivative reads

$$\frac{\mathrm{D}U_j}{\mathrm{D}t} = \frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_i}\left(U_i U_j\right) \tag{5}$$

- The mean of the momentum is then

$$\left\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \right\rangle = \frac{\partial \left\langle U_j \right\rangle}{\partial t} + \frac{\partial}{\partial x_i}\left\langle U_i U_j \right\rangle \tag{6}$$

Mean-Flow Equations

Reynolds Equations

- Then, substituting the Reynolds decomposition for U_i and U_j, the nonlinear term becomes

$$
\begin{aligned}
\langle U_i U_j \rangle &= \langle (\langle U_i \rangle + u_i)(\langle U_j \rangle + u_j) \rangle \\
&= \langle \langle U_i \rangle \langle U_j \rangle + u_i \langle U_j \rangle + u_j \langle U_i \rangle + u_i u_j \rangle \\
&= \langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle .
\end{aligned}
\tag{7}
$$

- The velocity covariances $\langle u_i u_j \rangle$ are called Reynolds stresses.
- Thus, from the previous two equations along with $\partial \langle U_i \rangle / \partial x_i = 0$ in the second line, we obtain

$$
\begin{aligned}
\left\langle \frac{\mathrm{D} U_j}{\mathrm{D} t} \right\rangle &= \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial}{\partial x_i} \left(\langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle \right) \\
&= \frac{\partial \langle U_j \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle ,
\end{aligned}
\tag{8}
$$

Mean-Flow Equations

Reynolds Equations

- We define,

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t} \equiv \frac{\partial}{\partial t} + \langle \boldsymbol{U} \rangle \cdot \nabla \tag{9}$$

- Then equation 8 becomes

$$\left\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \right\rangle = \frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t} \langle U_j \rangle + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle \tag{10}$$

- Notice that $\langle \mathrm{D}U_j / \mathrm{D}t \rangle$ does not equal $\overline{\mathrm{D}} \langle U_j \rangle / \overline{\mathrm{D}}t$.

Mean-Flow Equations

Reynolds Equations

- Taking the mean of the momentum equation, we arrive at the Reynolds equations

$$\frac{\overline{\mathrm{D}} \langle U_j \rangle}{\overline{\mathrm{D}} t} = \nu \nabla^2 \langle U_j \rangle - \frac{\partial \langle u_i u_j \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} \tag{11}$$

- Upon expansion we get

$$\frac{\partial \langle U_j \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} = \nu \nabla^2 \langle U_j \rangle - \frac{\partial \langle u_i u_j \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} \tag{12}$$

Mean-Flow Equations

Reynolds Stresses

- The Reynolds equations can be rewritten

$$\rho \frac{\overline{\mathrm{D}} \langle U_j \rangle}{\overline{\mathrm{D}} t} = \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \langle p \rangle \delta_{ij} - \rho \langle u_i u_j \rangle \right] \tag{13}$$

- This is the general form of a momentum conservation equation, with the term in square brackets representing the sum of three stresses: the viscous stress, the isotropic stress $-\langle p \rangle \delta_{ij}$ from the mean pressure field, and the apparent stress arising from the fluctuating velocity field, $-\rho \langle u_i u_j \rangle$.

Mean-Flow Equations

Reynolds Stresses

- The Reynolds stresses are the components of a second-order tensor, which is obviously symmetric, i.e., $\langle u_i u_j \rangle = \langle u_j u_i \rangle$.
- The diagonal components $\left(\left\langle u_1^2 \right\rangle = \langle u_1 u_1 \rangle, \left\langle u_2^2 \right\rangle, \text{ and } \left\langle u_3^2 \right\rangle\right)$ are normal stresses, while the off-diagonal components (e.g., $\langle u_1 u_2 \rangle$) are shear stresses.
- The turbulent kinetic energy $k(x,t)$ is defined to be half the trace of the Reynolds stress tensor: $k \equiv \frac{1}{2}\langle \boldsymbol{u} \cdot \boldsymbol{u} \rangle = \frac{1}{2} \langle u_i u_i \rangle$
- It is the mean kinetic energy per unit mass in the fluctuating velocity field.
- In the principal axes of the Reynolds stress tensor, the shear stresses are zero, and the normal stresses are the eigenvalues, which are non-negative (i.e., $\left\langle u_1^2 \right\rangle \geq 0$).
- Thus the Reynolds stress tensor is symmetric positive semidefinite. In general, all eigenvalues are strictly positive; but, in special or extreme circumstances, one or more of the eigenvalues can be zero.

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The Closure Problem

- For a general statistically three-dimensional flow, there are four independent equations governing the mean velocity field; namely three components of the Reynolds equations together with either the mean continuity equation.
- However, these four equations contain more than four unknowns. In addition to $\langle \boldsymbol{U} \rangle$ and $\langle p \rangle$ (four quantities), there are also the Reynolds stresses.
- This is a manifestation of the closure problem.
- In general, the evolution equations (obtained from the Navier-Stokes equations) for a set of statistics contain additional statistics to those in the set considered.
- Consequently, in the absence of separate information to determine the additional statistics, the set of equations cannot be solved.
- Such a set of equations - with more unknowns than equations - is said to be unclosed.
- The Reynolds equations are unclosed: they cannot be solved unless the Reynolds stresses are somehow determined.

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Anisotropy

- The distinction between shear stresses and normal stresses is dependent on the choice of coordinate system.
- An intrinsic distinction can be made between isotropic and anisotropic stresses.
- The isotropic stress is $\frac{2}{3}k\delta_{ij}$, and then the deviatoric anisotropic part is

$$a_{ij} \equiv \langle u_i u_j \rangle - \frac{2}{3}k\delta_{ij} \tag{14}$$

- The normalized anisotropy tensor is defined by

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\langle u_i u_j \rangle}{\langle u_\ell u_\ell \rangle} - \frac{1}{3}\delta_{ij} \tag{15}$$

- In terms of these anisotropy tensors, the Reynolds stress tensor is

$$\begin{aligned} \langle u_i u_j \rangle &= \frac{2}{3}k\delta_{ij} + a_{ij} \\ &= 2k\left(\frac{1}{3}\delta_{ij} + b_{ij}\right) \end{aligned} \tag{16}$$

Anisotropy

- It is only the anisotropic component a_{ij} that is effective in transporting momentum as the isotropic component $\left(\frac{2}{3}k\right)$ can be absorbed in a modified mean pressure like so

$$\rho\frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{\partial \langle p \rangle}{\partial x_j} = \rho\frac{\partial a_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\langle p \rangle + \frac{2}{3}\rho k\right) \tag{17}$$

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Turbulent-Viscosity Hypotheses

- The turbulent-viscosity hypothesis - introduced by Boussinesq in 1877 - is mathematically analogous to the stress-rate-of-strain relation for a Newtonian fluid.
- According to the hypothesis, the deviatoric Reynolds stress $\left(-\rho \langle u_i u_j \rangle + \frac{2}{3}\rho k \delta_{ij}\right)$ is proportional to the mean rate of strain,

$$
\begin{aligned}
-\rho \langle u_i u_j \rangle + \frac{2}{3}\rho k \delta_{ij} &= \rho \nu_{\mathrm{T}} \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) \\
&= 2\rho \nu_{\mathrm{T}} \bar{S}_{ij}
\end{aligned}
\tag{18}
$$

- The positive scalar coefficient v_{T} is the turbulent viscosity (also called the eddy viscosity).

Turbulent-Viscosity Hypotheses

- The mean of momentum equation incorporating the turbulent-viscosity hypothesis then reads

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t}\langle U_j\rangle = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\langle p\rangle\right) + \nu\nabla^2\langle U_j\rangle + \frac{\partial}{\partial x_i}\left[-\frac{2}{3}k + \nu_{\mathrm{T}}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right)\right] \tag{19}$$

- Incorporating the gradient diffusion hypothesis, we arrive at

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t}\langle U_j\rangle = \frac{\partial}{\partial x_i}\left[v_{\mathrm{eff}}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\langle p\rangle + \frac{2}{3}\rho k\right) \tag{20}$$

- where

$$v_{\mathrm{eff}}(\boldsymbol{x}, t) = v + v_{\mathrm{T}}(\boldsymbol{x}, t) \tag{21}$$

Turbulent-Viscosity Hypotheses

- Finally, upon expansion, we arrive at

$$\frac{\partial \langle U_j \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} = \frac{\partial}{\partial x_i} \left[v_{\text{eff}} \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) \right] - \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\langle p \rangle + \frac{2}{3} \rho k \right) \tag{22}$$