# Model Order Reduction and Proper Orthogonal Decomposition

Ameir

CNRS

March 19, 2024

# Table of Contents

# Introduction to Model Order Reduction

- Model Order Reduction (MOR) is a technique used in numerical simulations to reduce the complexity of mathematical models.
- Essential for managing computational costs while preserving important features and dynamics.
- Key in engineering, physics, and applied sciences, especially with large-scale systems.

# Motivation for Model Order Reduction

- **Complexity and Cost of High-fidelity Models:** High-fidelity models, essential for accurate simulations in engineering and physics, incorporate an immense amount of detail to replicate real-world phenomena accurately. Each additional detail, while improving model accuracy, also increases the computational resources required, including processing power and memory. For large-scale systems, such as those encountered in aerospace engineering, climate modeling, or structural analysis, the computational cost can become prohibitive, making simulations extremely slow and sometimes infeasible with available technology.

# Motivation for Model Order Reduction

- **Simplification through MOR:** Model Order Reduction (MOR) techniques address these challenges by simplifying the original, high-dimensional models. MOR strategically reduces the number of equations or variables, focusing on preserving the model's core dynamics and behavior. This simplification process reduces computational demands significantly, enabling faster simulations without a corresponding loss in essential accuracy. The key lies in identifying and retaining the most influential features of the system while discarding the redundant or less significant aspects.

# Motivation for Model Order Reduction

- **Enabling Real-time and Interactive Applications:** In many practical scenarios, such as control system design, interactive simulations for educational purposes, or real-time decision-making in engineering and environmental monitoring, the ability to perform quick simulations is crucial. Real-time simulation and analysis capabilities allow for immediate feedback and adjustments, which are vital in dynamic environments and systems where conditions change rapidly. MOR opens the door to these applications by making simulations sufficiently fast and responsive.

# Motivation for Model Order Reduction

- **Design Optimization:** In the design and optimization of engineering systems, multiple iterations of simulations are often required to explore the effects of various parameters and configurations. MOR significantly accelerates this iterative process by reducing simulation times, thus enabling a more exhaustive exploration of the design space within practical time frames. This can lead to more innovative designs and optimizations that might not be feasible to explore with full-scale simulations due to time or resource constraints.
- The fundamental idea behind MOR is to create a simpler, reduced model that approximates the behavior of the original, high-fidelity model but with significantly fewer degrees of freedom. This is achieved by identifying the system's most critical modes or states that capture the majority of its dynamics, and then expressing the system's behavior in terms of these modes or states.

# Table of Contents

# Techniques in Model Order Reduction

- **Proper Orthogonal Decomposition (POD):** A powerful technique for identifying the most energetically significant modes of a system. By analyzing the covariance of the data, POD extracts orthogonal modes that represent the system's dominant patterns of variation. These modes can then be used to construct a reduced-order model that captures the essential dynamics of the original system with far fewer degrees of freedom. This approach is particularly useful in fluid dynamics and structural analysis, where it can significantly reduce the complexity of simulations.
- **Krylov Subspace Methods:** These methods approximate the solution of large linear systems by projecting them onto a smaller subspace, generated by the action of the system's matrix on an initial vector. Techniques such as the Arnoldi and Lanczos algorithms are used to construct a basis for this subspace, enabling efficient computations and analysis. Krylov subspace methods are widely applied in solving sparse systems of linear equations and eigenvalue problems.

# Techniques in Model Order Reduction

- **Balanced Truncation:** This technique focuses on reducing the system while maintaining a balance between its controllability and observability. It involves computing singular value decompositions of controllability and observability Gramians, identifying states that contribute least to the system's input-output behavior, and truncating them. Balanced truncation is particularly favored in control theory for its ability to preserve the essential input-output characteristics of the system.
- **Moment Matching:** Moment matching approximates the original system by ensuring that the reduced-order model matches the first few moments (i.e., derivatives at a specific point) of the system's transfer function. This approach is especially useful in circuit simulation and control systems, where the goal is to replicate the input-output response of the system accurately. By matching moments, the technique ensures that the reduced model behaves similarly to the original system under specific conditions.

# Table of Contents

# Understanding Proper Orthogonal Decomposition (POD)

- **Principle of POD:** Proper Orthogonal Decomposition (POD) is a statistical method used to reduce the dimensionality of a system by identifying patterns in the data. It decomposes a dataset into a set of orthogonal modes, each representing a direction of maximum variance. This process is akin to finding the principal axes in the data space along which the data variability is maximized.
- **Projection onto Lower-dimensional Space:** Once the significant modes (also known as eigenfunctions or singular vectors) are identified, the high-dimensional system can be approximated by projecting it onto the subspace spanned by these modes. This projection effectively captures the most critical aspects of the system's dynamics, allowing for a simplified model that retains the essence of the original system's behavior.

# Understanding Proper Orthogonal Decomposition (POD)

- **Reduction of Variables:** The process significantly reduces the number of variables needed to describe the system. For example, a fluid dynamics simulation with millions of data points (representing velocity and pressure across a spatial domain) can be approximated using a much smaller set of POD modes. This reduction facilitates faster simulations and analyses while using substantially less computational resources.
- **Application and Benefits:** POD is widely used in various fields such as fluid dynamics, image processing, and data analysis. Its ability to extract dominant features from complex datasets makes it an invaluable tool for simplifying models, enhancing computational efficiency, and identifying underlying structures in the data.

# Understanding Proper Orthogonal Decomposition (POD)

- **Mathematical Foundation:** Mathematically, POD involves solving an eigenvalue problem derived from the covariance matrix of the dataset or directly applying singular value decomposition (SVD) to the data matrix. The resulting eigenvalues and eigenvectors (or singular values and singular vectors) provide a hierarchy of modes ordered by their energy content or variance explained.

# Step 1: Organize the Dataset

- **Data Preparation:** Begin by structuring your CFD simulation data. Each data point should represent a snapshot in time, with variables $X$, $Y$, $Z$ for spatial coordinates, and corresponding values for Velocity, Pressure, and Turbulent Viscosity.
- **Normalization:** Consider normalizing the data if the variables are on vastly different scales to ensure that no single variable dominates the POD due to its scale.

# Step 2: Identify Significant Modes

- **POD Application:** Apply POD to the dataset to extract significant modes. This involves computing the covariance matrix of the dataset and finding its eigenvalues and eigenvectors, or using Singular Value Decomposition (SVD) for a more direct approach.

# Step 2: Identify Significant Modes

- **Computing the Covariance Matrix:**
  1. The first step in applying POD is to compute the covariance matrix of the dataset. This matrix captures the variance and covariance of the variables across the dataset, essentially quantifying how much the variables change together.
  2. For a dataset with variables $X$, $Y$, $Z$, Velocity, Pressure, and Turbulent Viscosity, the covariance matrix provides a measure of the variability of these quantities with respect to each other, across all spatial points and time steps considered in the dataset.
  3. The covariance matrix,

$$\Sigma = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_p) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \cdots & \text{Cov}(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_p, X_1) & \text{Cov}(X_p, X_2) & \cdots & \text{Var}(X_p) \end{bmatrix}$$

  4. For each component, $\text{Cov}(X_i, X_j) = \frac{1}{n-1} \sum_{k=1}^{n} (X_{ik} - \bar{X}_i)(X_{jk} - \bar{X}_j)$

# Step 2: Identify Significant Modes

- **Eigenvalue Decomposition:**
  1. The next step involves eigenvalue decomposition of the covariance matrix. This process identifies the eigenvalues and their corresponding eigenvectors of the matrix, which represent the variance captured by each mode and the direction of these modes, respectively.
  2. The eigenvectors (modes) are orthogonal to each other, meaning they capture independent features or dynamics within the dataset. The eigenvalues indicate the amount of variance captured by each mode, providing a basis for selecting the most significant modes.
- **Using Singular Value Decomposition (SVD):**
  1. Alternatively, SVD can be applied directly to the data matrix $\mathbf{X}$ without explicitly computing the covariance matrix. SVD decomposes $\mathbf{X}$ into three matrices, $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices containing the left-singular vectors and right-singular vectors, and $\mathbf{\Sigma}$ is a diagonal matrix with singular values.
  2. The columns of $\mathbf{U}$ (the left-singular vectors) can be interpreted as the modes in POD, and the singular values in $\mathbf{\Sigma}$ represent the strength (or energy) of each mode. This method is numerically stable and efficient, especially for large datasets.

# Step 2: Identify Significant Modes

- **Mode Selection:**
  1. After identifying the modes and their corresponding strengths, select a subset of modes that capture a significant portion of the system's energy. This is typically done by choosing modes associated with the largest singular values (or eigenvalues), which effectively capture the most important dynamics of the fluid flow.
  2. **Mode Selection:** Select a number of leading modes that capture a substantial portion of the system's energy or variance, which represent the dominant flow structures and dynamics.

# Step 3: Construct the Reduced Model

- **Projection:** Project the original dataset onto the subspace spanned by the identified significant modes. This step reduces the dimensionality from potentially millions of points to just a few modes.
- **Model Formulation:** The reduced model is formulated in terms of these modes, significantly simplifying the representation of the fluid dynamics.

# Step 4: Analyze and Simulate

- **Simulation with Reduced Model:** Use the reduced model to perform fluid dynamics analyses and simulations. This should be much faster and less resource-intensive than using the full-scale model.
- **Interactive Exploration:** The efficiency of the reduced model allows for interactive exploration of different scenarios and parameters, facilitating deeper insights into the fluid dynamics.

# Step 5: Validate the Reduced Model

- **Comparison with Full Model:** Validate the reduced model by comparing its predictions with those from the original full-scale model. This could involve comparing summary statistics, flow patterns, or specific dynamic behaviors.
- **Adjustments:** Based on this comparison, adjustments to the model or the selection of modes may be necessary to improve accuracy and representation.

# Table of Contents

# Get Dataset

- Get dataset -
- Define dataset
  $\mathrm{X} : \mathrm{X} = [x_{ij}] = \{\vec{X}_i(\theta_j), \vec{V}_i(\theta_j) | \forall i \in [1, N], \forall j \in [1, n \times M]; i, j \in \mathbb{N}\}$
- In our example, $\mathrm{X} \in \mathbb{R}^{N \times 2M} = \mathbb{R}^{20 \times 12}$
- We take $N = 20$ (20 observations), $n = 2$ (2 angles), and $M = 6$ (6 variables)

# Dataset $X \in \mathbb{R}^{20\times 12}$

| Points:0 ($\theta = 0$) | Points:1 ($\theta = 0$) | Points:2 ($\theta = 0$) | Velocity:0 ($\theta = 0$) | Velocity:1 ($\theta = 0$) | Velocity:2 ($\theta = 0$) | Points:0 ($\theta = 135$) | Points:1 ($\theta = 135$) | Points:2 ($\theta = 135$) | Velocity:0 ($\theta = 135$) | Velocity:1 ($\theta = 135$) | Velocity:2 ($\theta = 135$) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -49.809 | -233.39 | 1.1732 | -0.11436 | -0.16909 | -0.012407 | -49.809 | -233.39 | 1.1732 | -0.28844 | -0.50923 | -0.063486 |
| 162.71 | -33.957 | 20.784 | 0.29694 | -0.11781 | -0.053469 | 162.71 | -33.957 | 20.784 | -0.14046 | -0.0409 | 0.23128 |
| -577.8 | 270.26 | 1.7004 | -0.28844 | -0.88723 | 0.0021901 | -577.8 | 270.26 | 1.7004 | -0.47265 | 0.69779 | -0.032282 |
| -409.59 | 635.01 | 27.039 | -0.45833 | -2.0245 | 0.032774 | -409.59 | 635.01 | 27.039 | -0.70382 | 0.099895 | 0.11435 |
| -45.52 | -226.9 | 15.933 | 0.016559 | -0.020085 | -0.038222 | -45.52 | -226.9 | 15.933 | -0.43829 | -0.32702 | -0.41848 |
| 187.5 | -50.85 | 49.567 | 0.26444 | -0.65064 | -0.23518 | 187.5 | -50.85 | 49.567 | -0.62553 | 0.79691 | 0.11876 |
| 400.15 | -712.43 | 23.415 | -0.28017 | -2.3111 | -0.042736 | 400.15 | -712.43 | 23.415 | -1.6547 | 1.63 | -0.00031295 |
| 251.73 | -610.99 | 219.74 | -0.031567 | -3.6066 | -0.28337 | 251.73 | -610.99 | 219.74 | -3.3551 | 3.285 | 0.044889 |
| -44.669 | -98.604 | 56.746 | 0.022383 | 0.17762 | -0.0021013 | -44.669 | -98.604 | 56.746 | -0.47375 | 0.42469 | 0.023411 |
| -142.85 | -160.2 | 7.2621 | -0.034768 | -0.082606 | -0.017066 | -142.85 | -160.2 | 7.2621 | -0.49406 | 0.12969 | 0.020166 |
| -56.055 | -108.65 | 49.824 | 0.030558 | 0.11737 | 0.0087919 | -56.055 | -108.65 | 49.824 | 0.016165 | 0.1787 | 0.12693 |
| 198.23 | -726.9 | 234.14 | 0.020346 | -3.7919 | -0.26086 | 198.23 | -726.9 | 234.14 | -3.4149 | 3.3526 | 0.034918 |
| -407.78 | 638.38 | 29.006 | -0.44665 | -2.0776 | 0.033642 | -407.78 | 638.38 | 29.006 | -0.73376 | 0.10373 | 0.12352 |
| 826.64 | -268.21 | 49.689 | 0.044228 | -3.489 | 0.0025637 | 826.64 | -268.21 | 49.689 | -2.2013 | 2.2569 | 0.022693 |
| 386.76 | 928.29 | 27.776 | 0.17297 | -2.6085 | 0.069452 | 386.76 | 928.29 | 27.776 | -1.9341 | 2.0256 | 0.0031161 |
| 234.02 | 68.108 | 216.36 | -0.32819 | -1.4283 | 0.14909 | 234.02 | 68.108 | 216.36 | -3.1677 | 3.2374 | 0.35045 |
| -14.569 | 15.468 | 153.18 | -0.22886 | -2.1073 | 0.7425 | -14.569 | 15.468 | 153.18 | 0.68777 | 0.26538 | 0.12476 |
| -6.6665 | 12.702 | 136.78 | -0.054038 | -0.52924 | -0.26808 | -6.6665 | 12.702 | 136.78 | 0.7487 | 0.04379 | 0.23887 |
| -474.07 | 435.17 | 31.053 | -0.96927 | -2.0113 | 0.061435 | -474.07 | 435.17 | 31.053 | -0.15149 | 0.072373 | -0.0096399 |
| 173.3 | 42.965 | 55.431 | -0.20574 | -1.1827 | 0.5018 | 173.3 | 42.965 | 55.431 | -0.81865 | 1.21 | 0.17383 |

# Standardize Dataset

- Define $f : \mathbb{R}^{20\times 12} \to \mathbb{R}^{20\times 12}$ such that $f(\mathrm{X}) = \mathrm{Z} = [z_{ij}]$, where $Z \in \mathbb{R}^{N\times M}$ is the standardized matrix, and each element $z_{ij}$ of $Z$ is given by:

$$z_{ij} = \frac{x_{ij} - \mu_j}{\sigma_j}$$

- With $\mu_j = \frac{1}{N}\sum_{i=1}^{N} x_{ij}$ is the mean of the $j$-th column of $X$,
- And $\sigma_j = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_{ij} - \mu_j)^2}$ is the standard deviation of the $j$-th column of $X$,
- $x_{ij}$ represents the original value in the $i$-th row and $j$-th column of $X$,
- $z_{ij}$ represents the standardized value corresponding to $x_{ij}$ in $Z$.
- This function $f$ thus transforms each feature in $X$ to have a mean of 0 and a standard deviation of 1 in $Z$, applied column-wise across the entire matrix.

# Standardized Dataset $Z \in \mathbb{R}^{20\times12}$

| Points:0 ($\theta = 0$) | Points:1 ($\theta = 0$) | Points:2 ($\theta = 0$) | Velocity:0 ($\theta = 0$) | Velocity:1 ($\theta = 0$) | Velocity:2 ($\theta = 0$) | Points:0 ($\theta = 135$) | Points:1 ($\theta = 135$) | Points:2 ($\theta = 135$) | Velocity:0 ($\theta = 135$) | Velocity:1 ($\theta = 135$) | Velocity:2 ($\theta = 135$) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.2422775230642614 | -0.5299456452302135 | -0.9263562703310546 | 0.05045384463351536 | 1.002311924281608 | -0.13555268487255895 | -0.2422775230642614 | -0.5299456452302135 | -0.9263562703310546 | 0.5732341608441163 | -1.1942511144808337 | -0.8349662911482649 |
| 0.4062579500808071 | -0.05844463045642818 | -0.6636690924848442 | 1.5079436024824224 | 1.0427534364237152 | -0.30979510525014087 | 0.4062579500808071 | -0.05844463045642818 | -0.6636690924848442 | 0.695752493701489 | -0.8100862791999293 | 1.135991818641859 |
| -1.8535258961571983 | 0.6607875137295223 | -0.9192944125710181 | -0.5664190390191414 | 0.435957253264974 7 | -0.07361137576697227 | -1.8535258961571983 | 0.6607875137295223 | -0.9192944125710181 | 0.4207196176919563 7 | -0.20414875187165366 | -0.6263201892983533 |
| -1.3402063720782595 | 1.5231322367128712 | -0.5798832042279629 | -1.1684441665682561 | -0.4609405336666635 | 0.05616829304082875 | -1.3402063720782595 | 1.5231322367128712 | -0.5798832042279629 | 0.2293250846020871 7 | -0.6945940237745349 | 0.3541329071883115 |
| -0.22918895853911686 | -0.5146019378904961 | -0.7286483640970153 | 0.5143806659508533 | 1.1198233813162162 | -0.24509601159914984 | -0.22918895853911686 | -0.5146019378904961 | -0.7286483640970153 | 0.4491675830722734 5 | -1.0447866884638508 | -3.2086399958049596 |
| 0.48190857133945336 | -0.09838318948413953 | -0.2781200562421884 | 1.392776043206737 | 0.6225418351234219 | -1.0808672510109605 | 0.48190857133945336 | -0.09838318948413953 | -0.2781200562421884 | 0.294144386395597 8 | -0.12284194156045591 | 0.3836248342760069 |
| 1.1308438117852433 | -1.662495655094349 | -0.6284267788151632 | -0.5371133247049901 | -0.6869650535560543 | -0.26425071145752027 | 1.1308438117852433 | -1.662495655094349 | -0.6284267788151632 | -0.5579450353291099 | 0.5605306432070555 | -0.412558566995286 |
| 0.677916615044288 | -1.4226704359046825 | 2.0013517203106885 | 0.3438405441754181 | -1.7086495110463045 | -1.2853566143638335 | 0.677916615044288 | -1.4226704359046825 | 2.0013517203106885 | -1.9657715875028936 | 1.9181050082699318 | -0.1103149231190327 |
| -0.22659199691382 | -0.21128355933018983 | -0.18195716386870311 | 0.5350186925730561 | 1.2757416566293076 | -0.09182149516710755 | -0.22659199691382 | -0.21128355933018983 | -0.18195716386870311 | 0.4198088853893269 | -0.42816902927266365 | -0.2539279527156357 |
| -0.5262069079987394 | -0.3569092908089625 | -0.8447952959094199 | 0.3324974254910652 | 1.0705167553287611 | -0.1553226774822505 | -0.5262069079987394 | -0.3569092908089625 | -0.8447952959094199 | 0.402993455328959 | -0.6701535837702 76 | -0.2756257030201788 |
| -0.26133818382995344 | -0.23503438890411912 | -0.27467753453443167 | 0.5639877632524016 | 1.228226034426461 | -0.04559730896923422 | -0.26133818382995344 | -0.23503438890411912 | -0.27467753453443167 | 0.8254283545190724 | -0.6299513362586557 | 0.4382536848116362 |
| 0.5146528700931958 | -1.6967057391013924 | 2.194240096153868 | 0.5278003443193771 | -1.8547846943523203 | -1.1898377189690799 | 0.5146528700931958 | -1.6967057391013924 | 2.194240096153868 | -2.0152823072276598 | 1.9735563841480286 | -0.17698619223662876 |
| -1.3346828696789796 | 1.5310996163330326 | -0.5535351878888286 | -1.1270547175731789 | -0.5028173724877611 | 0.05985156275218575 | -1.3346828696789796 | 1.5310996163330326 | -0.5535351878888286 | 0.2045366072014267 2 | -0.6914482245660659 | 0.4154526590678791 |
| 2.4323457666299406 | -0.6122673539465403 | -0.2764858630579614 | 0.6124290104923592 | -1.6159053256346074 | -0.07202604216770618 | 2.4323457666299406 | -0.6122673539465403 | -0.2764858630579614 | -1.0104961958902 66 | 1.0747683286801 17 | -0.25872887250567017 |
| 1.0899821006657082 | 2.2165070423511346 | -0.5700110699921003 | 1.0686413771468306 | -0.9215068965954986 | 0.2118076553370868 | 1.0899821006657082 | 2.2165070423511346 | -0.5700110699921003 | -0.7892710401969969 | 0.8850360322214808 | -0.3896301630871832 |
| 0.6238717379773003 | 0.18285821848475212 | 1.9560765320919424 | -0.7072778230563261 | 0.009247249857603246 | 0.5497434079558065 | 0.6238717379773003 | 0.18285821848475212 | 1.9560765320919424 | -1.8106159206731 08 | 1.8790593649001475 | 1.9328241464669438 |
| -0.13473706751142983 | 0.05840633029326893 | 1.1097778783079991 2 | -0.3552903257377462 | -0.5262400111504092 | 3.0678183842419315 | -0.13473706751142983 | 0.05840633029326893 | 1.1097778783079991 2 | 1.3814759618894994 | -0.5588488915676285 | 0.4237439411565179 |
| -0.11062133396982224 | 0.05186693206497024 | 0.8901003550363696 | 0.2642119218836048 | 0.7182828564582859 | -1.2204750545865886 | -0.11062133396982224 | 0.05186693206497024 | 0.8901003550363696 | 1.4319222520706032 | -0.7406162048934791 | 1.1867424888457054 |
| -1.5369773304791274 | 1.0506689894449002 | -0.5261155694616766 | -2.979019943070373 | -0.45053047203881974 | 0.17778833128645424 | -1.5369773304791274 | 1.0506689894449002 | -0.5261155694616766 | 0.6866203325214861 | -0.7171699522792 24 | -0.47492338431514985 |
| 0.438575016604771 | 0.12341494673706097 | -0.19957162319049354 | -0.2733618958776277 | 0.20293748741808593 | 2.04643241704881 | 0.438575016604771 | 0.12341494673706097 | -0.19957162319049354 | 0.1342529115921392 | 0.21601026053248876 | 0.7518513702609649 |

# Construct Correlation Matrix

- Our goal is to reduce the number of observations.
- Therefore, we construct the correlation matrix as $\mathrm{C}_R = \mathrm{Z} \cdot \mathrm{Z}^T \in \mathbb{R}^{20\times 20}; \mathrm{Z} = [z_{ij}], \mathrm{Z}^T = [z'_{jk}]$
- The subscript R in $\mathrm{C}_R$ makes it clear that this correlation is done row wise.
- The construction of the correlation matrix can be defined by bilinear function $g_R : \mathbb{R}^{20\times 12} \times \mathbb{R}^{12\times 20} \rightarrow \mathbb{R}^{20\times 20}$ where each element $(c_R)_{ik}$ of $\mathrm{C}_R$ is given by

$$(c_R)_{ik} = z_{ij} * z'_{jk}$$

# Row Correlation Matrix, $C_R \in \mathbb{R}^{20\times 20}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.87290459864389 | 2.675678642532029 | 3.327258014022279 | 0.24611755572497476 | 7.372363079686277 | 1.2218111717315918 | 1.053794957068926 | -7.374996732783923 | 2.9529911095876393 | 4.570725439303437 | 3.0099007007499994 | -7.551880784404862 | 0.07663384108097807 | -3.2431625344513626 | -3.903841897969999 | -9.116911652597098 | -1.9084481050084385 | -0.037340120239516 48 | 1.6265416258833914 | -0.8701389085416731 |
| 2.675678642532029 | 7.10582629124455 25 | -0.09317101323162649 | -1.6327399093089536 | 0.37426886536380827 | 4.596409366200743 | -0.8080498587075603 | -5.851206331687627 | 2.5980905891260386 | 2.911444415653912 5 | 3.9075073977081605 | -6.267535911836113 | -1.596692583719007 1 | -0.19191598233934593 | 0.2593593284886155 | -3.924312146820258 | -1.728995496060314 4 | 3.1924535839591046 | -5.171183105634569 | 0.5447638781530552 |
| 3.327258014022279 | -0.09317103231626449 | 10.56186778422532 | 8.520537498068053 | 4.135998508405838 | -1.9345031221012048 | -5.300371486960358 | -10.067464691509665 | 1.578260972176911 | 3.801013446835338 3 | 1.5839453213819428 | -10.344369360832516 | 8.37070223320099 2 | -10.846201393189599 | -1.3546592254471 28 | -7.659240827695216 | -1.2878595537934219 | -0.894223827321825 | 10.264115114032087 | -1.4616343711851743 |
| 0.24611755572497476 | -1.6327399093089536 | 8.520537498068053 | 11.14604581771377 | -1.5467854902399 | -2.9552712452872605 | -7.100701290340408 5 | -9.980567824357543 | -0.739824685095192 8 | 0.872595207089658 4 | -0.14255559828709213 | -10.817146088780985 | 11.109946493284472 | -9.108843610281278 | 2.7458304507656983 | -3.566678701496053 | 0.9370480339468137 | -0.023078777429657856 | 12.116405850384472 | -0.08033368528949444 |
| 7.372363079686277 | 0.37426886536380827 | 4.135998508405838 | -1.5467854902399 | 14.863907637898148 | 0.9936835463146605 | 1.615210172721880 7 | -5.717582936619223 | 3.763471242751258 | 5.013103175642719 | 2.0613751075605102 | -5.600487444060173 | -1.83358157702696 1 | -2.3053535055115508 | -2.513341719403684 | -12.791169062454449 | -3.2948385639957793 | -2.451297001676990 4 | 8.890985720192429 7 | -3.029950491181503 4 |
| 1.2218111717315918 | 4.596409366200743 | -1.9345031221012048 | -2.9552712452872605 | 0.9936835463146605 | 4.382968311020448 | 0.4852421830910822 5 | -0.2315877650766716 4 | 1.641671213082736 4 | 1.425513015514702 4 | 2.034871238764023 | -0.42727099683240 78 | -2.922731841227531 | 2.0149478493805635 | 1.1268669499493706 | -2.118183224729095 5 | -4.259446291368 72 | 2.4898212672337237 | -5.90933044755846 | -1.6555011777422333 |
| 1.053794957068926 | -0.8080498587075603 | -5.300371486960358 | -7.100701290340408 5 | 1.615210172721880 7 | 0.4852421830910822 5 | 10.501181474897587 | 7.294414488260794 | -1.090234508786216 6 | -0.3013417294118815 | -1.5934291758609271 | 7.656449284330826 | -7.151954381523308 | 9.957634559286218 | -3.0879510620810984 | -0.161143670331016 | -3.4109554116764182 | -3.557882131283439 | -5.034946767293603 | 0.03503499764029556 |
| -7.374996732783923 | -5.851206331687627 | -10.067464691509665 | -9.980567824357543 | -5.717582936619223 | -0.2315877650766716 4 | 7.294414488260794 | 25.22334686583483 | -3.930688147248183 4 | -6.641764462172 88 | -5.510370245935272 | 26.95500123643304 | -9.761237138079936 | 11.073911428072659 | -2.1486425757931964 | 14.139830074126422 | -2.9073380246721627 | -0.668783761918517 8 | -10.335266413857138 | -3.559007041058234 |
| 2.9529911095876393 | 2.5980905891260386 | 1.5782609721769 11 | -0.7398246850951928 | 3.763471124275 1258 | 1.6416712130827364 | -1.0902345087862166 | -3.930688147248183 4 | 2.6044267558706107 | 2.780685221426967 | 2.6955040500597964 | -3.935473566251795 | -0.814224165267736 3 | -3.288861480778475 | -2.4578103834602167 | -3.544396602754692 7 | -0.7989720125137 72 | 1.4909573236593852 | -1.0249846835186476 | -0.4805881225902831 6 |
| 4.570725439303437 | 2.911444415653912 5 | 3.801013446835383 | 0.8725952073896584 | 5.013103175642719 | 1.425513015514702 4 | -0.3013417294118815 | -6.641764462172 88 | 2.780685221426967 | 4.204078962345616 | 3.050356122483537 | -6.749081706768496 | 0.7593965073374 75 | -4.226844815430106 | -3.2406643719701063 | -6.924384421360537 | -2.1185021595 76835 | 0.3681268919234398 | 1.1425621142265 29 | -0.7018520284549667 |
| 3.009900700749994 | 3.907507397708160 5 | 1.5839453213819428 | -0.14255559828709213 | 2.061375107560510 2 | 2.034871238764023 | -1.5934291758609271 | -5.510370245935272 | 2.695504050597964 | 3.050356122483537 | 3.496910566621709 | -5.587294426401013 | -0.18749238972 10043 | -4.092176140457334 | -3.2170293130557237 | -3.730407029599674 7 | 0.1247653274795165 | 2.7999290279391 3 | -0.8327102855529022 | 0.1283997441137876 |
| -7.551880784404862 | -6.267535911836113 | -10.344369360832516 | -10.817146088780985 | -5.600484744060173 | -0.4272709968324078 | 7.656449284330826 | 26.95500123643304 | -3.935473566251795 | -6.749081706768496 | -5.587294426401013 | 29.038860037555732 | -10.582421955336422 | 10.977388242287757 | -3.4736368258482235 | 14.576534770656 84 | -2.2902872959825307 | -0.681735342991231 8 | -11.119503807545 99 | -3.776110757391452 7 |
| 0.07663384108097807 | -1.5966925837190071 | 8.370702233200992 | 11.109946493284472 | -1.833581577026961 | -2.9227318412275 31 | -7.151954381523308 | -9.761237138079936 | -0.814224166267736 3 | 0.755939650733747 5 | -0.18749238971200 43 | -10.582421955336422 | 11.083287842872531 | -9.000982089698393 | 2.8452127466643 65 | -3.312126364812769 5 | 1.0035821263689 4 | 0.034731814871837 13 | 11.936270001424147 | -0.052862263097940274 |
| -3.2431625344513626 | -0.19191598233934593 | -10.846201393189599 | -9.108843610281278 | -2.305335305115508 | 2.0149478493805635 | 9.957634559286218 | 11.073911428072659 | -3.288861480778475 | -4.226844815430106 | -4.092176140457334 | 10.977388242287757 | -9.000982089698393 | 17.969820564012284 | 6.881288767631291 | 4.590772911557556 | -3.035094107290448 | -4.554799289792854 | -10.923545645858 46 | 1.3519980724512257 |
| -3.903841897969999 | 0.2593593284886155 | -1.3546592254471 28 | 2.7458304507656983 | -2.513341719403684 | 1.1268669499493706 | -3.087951062081 0984 | -2.148642575791964 | -2.457810383460216 7 | -3.234066437197016 3 | -3.217029313055723 7 | -3.473636825848223 5 | 2.8452127466643 65 | 6.881288767631291 | 16.445835190448758 | 1.6317680381925486 | -2.294998802566892 6 | -3.912056054936686 | -1.815428157939581 5 | 1.4772809835587988 |
| -9.116911652597098 | -3.924312146820258 | -7.659240827695216 | -3.566678701496053 | -12.791169062454449 | -2.118183224790955 | -0.161143670331016 | 14.139830074126422 | -3.544396602754692 7 | -6.924384421360537 | -3.730407029599674 7 | 14.576534770656 84 | -3.312126364812769 5 | 4.590772911557556 | 1.6317680381925486 | 19.845325481217 78 | 3.9953928611603 96 | 0.821414661201288 4 | -4.899938624759136 | 2.7478535312973595 |
| -1.9084481050084385 | -1.728995496060314 4 | -1.2878595537934219 | 0.9370480339468137 | -3.2948385639957793 | -4.259446291368 72 | -3.410955411676418 2 | -2.907338024672162 7 | -0.798972012513772 | -2.118502159579835 | 0.12476532747935165 | -2.290287295982530 7 | 1.0035821263689 4 | -3.035094107290448 | -2.2949988025668926 | 3.3953928611603 96 | 14.721364896451758 | 0.690374007202568 3 | 2.3581829322692722 | 6.105025639629628 |
| -0.037340120239516 48 | 3.1924535839591046 | -0.8942238273218 25 | -0.023078777429657856 | -2.451297001676990 4 | 2.4898212672337237 | -3.557882131283439 | -0.668783761918517 8 | 1.490957323659385 2 | 0.368126891234398 | 2.7999290279391 3 | -0.681735342991231 8 | 0.034731814871837 13 | -4.554799289792854 | -3.912056054936686 | 0.821414661201288 4 | 0.690374007202568 3 | 7.6969807945868 39 | -0.864520651109751 | -1.9390724138768476 |
| 1.6265416258833914 | -5.171183105634569 | 10.264115114032087 | 12.116405850384472 | 0.8909857201924297 | -5.909330447558 46 | -0.534946767293603 | -10.335266413857138 | -1.024984683518647 6 | 1.144256211422652 9 | -0.832710285552902 2 | -11.119503807545 99 | 11.936270001424147 | -10.923545645858 46 | -1.815428157939581 5 | -4.899938624759136 | 2.3581829322692722 | 17.806483125077 4 | -0.211881990057421 53 | |
| -0.8701389085416731 | 0.5447638781530552 | -1.4616343711851743 | -0.08033368528949444 | -3.029950491181503 4 | -1.6555011777422333 | 0.03503499764029556 | -3.559007041058234 | -0.480588122592 8316 | -0.701852028549667 | 0.12839974411378766 | -3.776110757391452 7 | -0.052862263097940274 | 1.351998072451225 7 | 1.477280983557988 | 2.747853531297359 5 | 6.105025639629628 | -1.939072413876847 6 | -0.211881990057421 53 | 5.428577001456931 |

# Eigenvalue Decomposition of $\mathrm{C}_R$

- $\mathrm{C}_R$ and indeed, any correlation matrix is a real and symmetric matrix.
- Therefore, the decomposition must result in real valued eigenvalues and eigenvectors.
- Solve the characteristic equation to get the eigenvalues - $det\left(\mathrm{C}_R - \lambda I\right) = 0$
- Use the matrix of eigenvalues $\Lambda \in \mathbb{R}^{20\times 20}$ to determine the matrix of eigenvectors $V \in \mathbb{R}^{20\times 20}$
- Mathematica or Matlab is effective for this.

# Eigenvalue Decomposition of $C_R$

- Sort $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{20}), \in \mathbb{R}^{20\times 20}$ such that $\lambda_i > \lambda_j, \forall i < j$
- Sort the eigenvectors $V \in \mathbb{R}^{20\times 20}$ according to their position in $\Lambda$

# Sorted Eigenvalues and Sorted Eigenvectors $\Lambda, V \in \mathbb{R}^{20\times 20}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 104.33544793460241 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 45.25316341800748 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 28.644758524673954 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 24.726725619237783 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 18.543985544275134 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 9.102650911223483 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 5.580315312915288 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 3.185196442172311 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.6277562928916468 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.1968162656875224e-14 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 4.014445840899402e-16 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -9.692352108704102e-17 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -1.8908197794951261e-16 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -3.3863198278749 1e-16 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -5.307033533124646e-16 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -6.943845118625574e-16 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -1.280818511524653e-15 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -1.953768536666593e-15 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -2.914874713396433e-15 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -7.057239083296936e-15 |

Table: Sorted Eigenvalues $\Lambda \in \mathbb{R}^{20\times 20}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.14319551672159034 | 0.26179839635995494 | -0.10183951637481896 | 0.012594737555012359 | 0.09338330856696915 | -0.06918200208380862 | -0.04504644775046992 | 0.07797394972957264 | 0.3889296857580371 | -0.4780031224216076 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| -0.05982108765574215 | 0.22873195534302637 | 0.19542146804648586 | -0.2883894450807488 | -0.2076768459966512 | -0.14766742713541864 | 0.021243905529205433 | -0.23326625593850656 | 0.25353330339435987 | 0.5909831650528217 | 0.15603974236268654 | -0.31547497516305284 | -0.000362955058547904 | 0.10937664062286067 | 0.12330735201231 1 | 0.013276781703122 2 | 0.16612997423564102 | -0.11497642636356632 | -0.20319580065344117 | 0.23681338797868295 |
| -0.2650983034402287 4 | -0.07409883654642817 7 | -0.18814822523820304 | 0.17519596461532261 | -0.09364806046423688 | 0.062592045085277 95 | -0.2697132107352712 7 | -0.3582753176493896 | -0.5580762649711618 | 0.08017976292467131 | 0.05741450362059803 | 0.23255909653628865 | 0.18048262957470708 | 0.11892844010746584 | -0.02653495682315290 2 | 0.050123329919338 5 | 0.23910916230354562 | 0.09198248746168435 | -0.18469853563725633 | 0.34580678900569696 |
| -0.24808386309231775 | -0.28150900514130676 | 0.006819340146975952 | 0.15268230405501623 | -0.12912358747321137 | -0.08301833433776523 | 0.09529900849266552 | -0.14976244877255954 | 0.32588718484289214 | 0.07706105980065577 | -0.6083380095146 14 | 0.040159000281194 55 | 0.22951195458641294 | -0.11145207306923607 | 0.008216857093507416 | 0.30541398873054 97 | 0.29678254032558 68 | -0.06048526354545194 | 0.21223334163785776 | -0.04343059091059772 |
| -0.13412354358038417 | 0.39055479975140633 | -0.18688175746077137 | 0.25805307906374 86 | 0.17475732106776018 | 0.5161891955134156 | -0.05476159686878866 | 0.36143170195857655 | 0.1429467602970696 | 0.31182184040821 47 | -0.15954626873308445 | 0.016467050645638314 | -0.05898347271085712 | 0.16408162661246015 | -0.12272589519347 03 | -0.04528283174076 6 | 0.23712590591209 77 | 0.17936336089730 9 | -0.10219597080872714 | -0.08710784475193861 |
| 0.024346590820286357 | 0.21144524026566144 | 0.10876102889410047 | -0.10766160347112434 | -0.29513726837637 16 | -0.01726372821020817 8 | 0.07550614928929787 | -0.05156114614455765 | -0.15019614632480 35 | -0.10768279202064533 | -0.04220887433703602 4 | 0.04112234358372697 | -0.19538285347356 69 | 0.48273694735843 41 | -0.33434703747393446 | -0.11611792361232492 | 0.20716827924437 48 | -0.15104344231051686 | 0.5793706851768333 | 0.0018313405998883918 |
| 0.17563169527777464 | 0.22934920259521 1 | -0.03978453985540278 | 0.19398544235381596 | 0.29415905151596171 | -0.52030938599803759 | -0.00249398413119493 02 | -0.00542405537636109 3 | -0.16921456905583387 | 0.08413471189655017 | -0.24365059746919956 | -0.35522159324724476 | 0.35863945892393134 | 0.08216801524603642 | -0.33931205080569337 | -0.11706902139048433 | -0.10553625059107095 15 | 0.20902579923598288 | -0.00589691788619605 8 | 0.0037891223165778896 |
| 0.4582400647111323 | -0.08953146160050388 | -0.27108243940116 39 | 0.13260601034353 03 | -0.11581592364606696 | 0.10551727959473 57 | 0.06534193669045724 | -0.10787045262449677 | 0.04140435500689039 | 0.39652490135882895 | -0.13098132002965243 | 0.20904467758532885 | -0.16906482702403053 | -0.31128369955200303 | -0.15674319275624 87 | -0.15508426515116322 | -0.24765827188242412 | -0.02094318029445462 | 0.23776661193178497 | 0.37184536788886524 |
| -0.07237794082012114 | 0.15471511511904187 | -0.06226104394331967 | -0.13373100284815 14 | -0.05398760571315 18 | 0.11175336053420784 | -0.19368706591449197 | 0.11491153618069848 | -0.05997880465905467 | 0.025254518403149222 | 0.23774761118862417 | -0.13399555854747522 | 0.46867251379514785 | -0.41600031103914 28 | 0.24142772422574 75 | -0.21529596010071 14 | 0.18873840978479361 | 0.07614681238222215 | 0.5140049277155692 | -0.03544189006106151 4 |
| -0.13293551536657927 | 0.19476702881039323 | -0.0869576807737 | -0.03849347734491236 | -0.02365768063566752 2 | -0.07158697699568375 | -0.21758980949836893 | -0.14550496263215917 | 0.05251221904998465 | -0.11160552934383576 | -0.06891194015882 65 | 0.04832971089406963 | -0.11602428635261014 | -0.41855552502763605 | -0.40651925238044 24 | -0.21644344560004887 | 0.17728228395254947 | -0.50645489839970 27 | -0.24959391455359264 | -0.21709846601815538 |
| -0.09448985459360237 | 0.14219198808044417 | -0.03627165999984036 | -0.24255035750854317 | -0.07237213125952382 | -0.03597346427260083 | -0.08314588441609766 | -0.02807786072994782 | -0.11505744927478224 | -0.02848862513641 1986 | 0.05298492518191904 4 | -0.15681276081215853 | -0.35764320624398 38 | -0.37313614685801 35 | -0.31283333243980836 | 0.52735213965061 3 | 0.04921897756014587 6 | 0.44993406485327 19 | 0.07950947634224913 | -0.02485613428319533 |
| 0.4840265910241915 | -0.08597543955057538 | -0.33902708602027 14 | 0.11143429093936252 | -0.08750104978307564 | 0.16632735444004 93 | 0.062854943516495 43 | -0.27606137281409 8 | 0.06206148036176776 4 | -0.27935144507206 88 | 0.04284791050136639 | -0.43186334080235 32 | -0.07544292139235303 | 0.05681594248215875 | 0.13581992614419064 | -0.00779475329971292 | 0.44060118611090104 | 0.05662589291860344 | -0.11753350932511897 | -0.08174025499252 62 |
| -0.24327255979156373 | -0.28727030073839105 | 0.011966478809151648 | 0.14623250076347452 | -0.13616248545925388 | -0.08272088800897594 | 0.10176475259542384 | -0.17444567169176 57 | 0.35787595715773 69 | 0.00520169160946616 | 0.30259726820757105 | 0.08104043094300147 | -0.02042938144733811 | 0.02698039978201949 6 | -0.28180623433849 09 | -0.44279596706761787 | 0.05038240186883001 | 0.50454590884964 86 | -0.01748731439872189 6 | -0.10160099953256921 |
| 0.30304167174888996 | 0.16339766927221586 | 0.3884967958485783 | 0.24284952199206344 | 0.18465367398785412 | -0.21472770932630278 | 0.24564786365053637 | 0.05514025062536926 | -0.00277222365644119 8 | 0.02245305006550801 | 0.23527897392532174 | 0.43025479581088927 | 0.01065341055614587 8 | -0.17660848604194165 | 0.013416977252018 63 | 0.13377883810209262 | 0.48458076242993686 | 0.02719066932476507 7 | -0.03057660444085534 | 0.03338312897195204 5 |
| 0.009279852047937731 | -0.11852685709765191 | 0.6285701018687545 | 0.31223246171645586 | -0.21122185004021941 | 0.36537061433811129 | 0.01510594971102581 | 0.06171573617626962 | -0.16485880521079 92 | -0.10039156289717378 | -0.08507647556483136 | -0.37760291320192825 | 0.02552770159196515 | -0.18478099085053953 | -0.15914114335323126 | -0.08391992169107829 | -0.07675295977638363 | -0.02156971416387 7506 | -0.07602499783117816 | 0.21154042034245565 |
| 0.2911447205875365 | -0.39804727707769627 | 0.05030725865087144 | -0.23711063436421911 | -0.08408178425421008 | -0.0658981451593261 | -0.5294757052909312 | 0.44258605915559 65 | 0.12422863423477212 | 0.05246434225896036 | 0.03632624935162464 | 0.05324065756155084 | 0.14780896076920922 | -0.14707746164150895 | -0.26987615653487 91 | 0.07044323801923 68 | 0.20938991940138962 | -0.00523236552321303 2 | -0.10695136755182233 | 0.10714759842047 12 |
| -0.03184161834722066 | -0.23712662126919307 | 0.038233899762508 29 | -0.39920527810802 83 | 0.5845611534355446 | 0.3489478937849063 | 0.30155744432571 8 | -0.20651853075747606 | -0.01831003641531392 7 | 0.03342561495436046 | 0.07921047791987414 | -0.04785665711211 3406 | 0.17972872427519343 | 0.02412376543550442 | -0.31860060166880054 | 0.02510618412094793 | 0.08283832169510814 | -0.11045465427257879 | 0.07052028744939894 | 0.12567267995354 93 |
| -0.022475949346084 04 | 0.00074932428765752 6 | -0.12212355085011 37 | -0.37587555553263938 | -0.31629205922682546 | -0.04833664976553697 6 | 0.50647649290113 3 | 0.34329838919494654 | -0.25344109714837 55 | -0.06744575075358623 | -0.26468015426742 94 | 0.06962497289897 32 | 0.09879200057141545 | -0.11659683650774835 | 0.01246196973121547 | -0.24827766287770564 | 0.16186158709225 69 | 0.14827160190951916 | -0.27840228351555 91 | 0.10963832533499117 |
| -0.2845965770440508 | -0.31895979776785 92 | -0.22224451114562432 | 0.26653249902654 24 | 0.14716790898516546 | -0.25012420242232 31 | 0.18033991546159688 | 0.36316651152128 6 | -0.15702684583259205 | 0.15912508663248226 | 0.23752209926323 48 | -0.31175249943940 41 | -0.30874118307822 8 | -0.05955455674870537 | 0.01826682680040330 3 | 0.00397406608976756 | 0.18121326940474572 | -0.26482056987447 45 | 0.14669028539369695 | 0.14521315500047507 |
| -0.01339865701397269 | -0.08691902381738556 | 0.22804563903582086 | -0.18138113991012747 | 0.34789590262309544 | -0.08710435282129 15 | -0.27521865737098 68 | -0.08370384482626051 | -0.10049237554332 5 | 0.02135248849361639 | -0.37997375225986846 | -0.01040182220040166 92 | -0.43466058029295485 | -0.04002771979145729 | 0.31759395771664367 | -0.38741845726889 08 | 0.19732369250161 37 | 0.21789160828687845 | 0.08820768751404044 | 0.07254898848279265 |

Table: Sorted Eigenvalues $V \in \mathbb{R}^{20\times 20}$

# Deleting Eigenvalues

- Construct $\Lambda^{'} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_k)$
- $k$ is chosen such that it captures most of the variance present in $\mathbb{Z}$ as well as satisfying $1 < k < min(N, nM)$ (by a fundamental theorem in Linear Algebra).
- Here, we take $k = 12$ as the obvious choice.
- Similarly, construct $V^{'}$ by deleting the eigenvectors corresponding to the deleted eigenvalues.
- $\Lambda^{'} \in \mathbb{R}^{k \times k}$ and $V^{'} \in \mathbb{R}^{N \times k}$ where N=20 and k=12.

# Reduced Eigenvalues and Reduced Eigenvectors

$\Lambda' \in \mathbb{R}^{12\times 12}, V' \in \mathbb{R}^{20\times 12}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 104.33544793460241 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 45.25316341800748 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 28.644758524673954 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 24.726725619237783 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 18.543985544275134 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 9.102650911223483 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 5.580315312915288 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 3.18519644217231 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.6277562928916468 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.1968162656875224e-14 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 4.0144458408994026e-16 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -9.692352108704102e-17 |

Table: Reduced Eigenvalues $\Lambda \in \mathbb{R}^{12\times 12}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.14319551672159034 | 0.2617983963595494 | -0.10183951637481896 | 0.012594737555012359 | 0.09338330856696915 | -0.06918200208380862 | -0.04504644775046992 | 0.07797394972957264 | 0.3889296857580371 | -0.4780031224216076 | 0.0 | 0.0 |
| -0.059821087655748215 | 0.22873195533402637 | 0.19542146804648586 | -0.2883894450807488 | -0.207676845996512 | -0.14766742713541864 | 0.021243905529205433 | -0.23326625593850656 | 0.25353330339435987 | 0.5909831650528217 | 0.1560397423626854 | -0.31547497516305284 |
| -0.26509830340228774 | -0.07409883654628177 | -0.18814822523820304 | 0.1751956461532261 | -0.09364806046423688 | 0.06259204508527795 | -0.26971321073527127 | -0.35802753176493896 | -0.5580762649711618 | 0.08017976292467131 | 0.05741450362059803 | 0.23255909653628865 |
| -0.24808386309231775 | -0.2815090051430676 | 0.006819340146975952 | 0.15268230405501623 | -0.12912358747321137 | -0.08301833433776523 | 0.09529900849266552 | -0.14976244877255954 | 0.32588718484289214 | 0.07706105980065577 | -0.608338009154614 | 0.04015900028119455 |
| -0.13412354358038417 | 0.39055479975140633 | -0.18688175746077137 | 0.2580530790637486 | 0.17475732106776018 | 0.5161891955134156 | -0.05476159686878866 | 0.36143170195857655 | 0.1429467602970696 | 0.3118218404082147 | -0.15954626873308445 | 0.016467050645638314 |
| 0.0243465908202863 57 | 0.21144524026566144 | 0.10876102889410047 | -0.10766160347112434 | -0.2951372683763716 | -0.017263728210280178 | 0.07550614928929787 | -0.05156114614455765 | -0.1501961463248035 | -0.10768279200264533 | -0.042208874337036024 | 0.04112234358372697 |
| 0.17563169527777464 | 0.22934929295211 | -0.03978453985540278 | 0.19398544235381596 | 0.2941590531596171 | -0.5030938599803759 | -0.002493984319149302 | -0.005424055376361093 | -0.16921456905583387 | 0.08413471189655017 | -0.2436505974691956 | -0.3552315932472476 |
| 0.4582400647111323 | -0.08953146160050388 | -0.2710824394011639 | 0.1326060103435303 | -0.11581592364606696 | 0.10551727959547357 | 0.06534193669045724 | -0.10787045262449677 | 0.04140435500689039 | 0.39652490135882895 | -0.13098132002965243 | 0.20904467758532885 |
| -0.07237794082012114 | 0.15471511511904187 | -0.06226104394331967 | -0.1337310028481514 | -0.0539876051731518 | 0.11175336053420784 | -0.19368706591449197 | 0.11491153618069848 | -0.05997880465905467 | 0.025254518403149222 | 0.23774761118862417 | -0.13399555854747522 |
| -0.13293551536657927 | 0.19476702881039323 | -0.0869576807737 | -0.03849347734491236 | -0.023657680635667522 | -0.07158697699568375 | -0.21758980948936893 | -0.14550496263215917 | 0.05251221904998465 | -0.11160552934383576 | -0.0689119401588265 | 0.04832917089406963 |
| -0.09448985493640237 | 0.14219198080444417 | -0.03627165999984036 | -0.24255035750854317 | -0.0722721212952282 | -0.03597346427260083 | -0.08314588441609766 | -0.028077860729947782 | -0.11505744927478224 | -0.028488625136411986 | 0.052984925181919044 | -0.15681276081215853 |
| 0.48402659102419 15 | -0.08597543955057538 | -0.3390270860202714 | 0.11143429093936252 | -0.08750104978307564 | 0.1663273544400493 | 0.06285494351649543 | -0.276061372814098 | 0.062061480361767764 | -0.2793514450720688 | 0.04284791050136639 | -0.4318633408023532 |
| -0.24327255979156373 | -0.287270300738391 05 | 0.011966478809151648 | 0.14623250076347452 | -0.13616248545925388 | -0.08272088800897594 | 0.10176475259542384 | -0.1744456716917657 | 0.3578759571577369 | 0.005201691609406616 | 0.30259726820757105 | 0.08104043094300147 |
| 0.30304167174888996 | 0.16339766927221586 | 0.388496795848578 3 | 0.24284952199206344 | 0.18465367398785412 | -0.21472770932630278 | 0.24564786365053637 | 0.05514025062536926 | -0.002727223656441198 | 0.02245305006550801 | 0.23527897392532174 | 0.43025479581088927 |
| 0.009279652947937731 | -0.11852685709765191 | 0.6285701018687545 | 0.31223246171645586 | -0.2112218500421941 | 0.3653706143831129 | 0.01510594971102581 | 0.06171573617626962 | -0.1648588052107992 | -0.10039156289717378 | -0.08507647556483126 | -0.37760291201928525 |
| 0.2911447205875365 | -0.39804727707769627 | 0.05030725865087144 | -0.23711063436421911 | -0.08408178425421008 | -0.06589814519539261 | -0.5294757052909312 | 0.4425860591555965 | 0.12422863423477212 | 0.05244634225896036 | 0.036326249351623464 | 0.05324065756155084 |
| -0.031841618347222066 | -0.23712662126919307 | 0.03823389976250829 | -0.399205278108028 3 | 0.5845611534355446 | 0.3489478937849063 | 0.301557444325718 | -0.20651853075747606 | -0.018310036415313927 | 0.03342561495436046 | 0.07921047791987414 | -0.047856657112113406 |
| -0.022475949434608404 | 0.007493242876575 26 | -0.1221235508501137 | -0.3758755553263938 | -0.31629205922682546 | -0.048336649765536976 | 0.5064764929011 33 | 0.34329838919494654 | -0.2534410971483755 | -0.06744575075358623 | -0.2646801542674294 | 0.06962497289897 32 |
| -0.2845965770440508 | -0.3186959797678592 | -0.22224451114562432 | 0.2665324990265424 | 0.14716790898516546 | -0.2501242027422731 | 0.18033391546159688 | 0.3631665110521286 | -0.15702684583259205 | 0.15912508663248226 | 0.2375220992632348 | -0.3117524994394041 |
| -0.013398657013397269 | -0.08691902381738556 | 0.22804563903580286 | -0.18138113991012747 | 0.34789590262309544 | -0.0871043552812915 | -0.2752186573789868 | -0.08370384482629051 | -0.100492337554332 5 | 0.021352488493616 39 | -0.37973755252586846 | -0.010401822204016092 |

Table: Reduced Eigenvectors $\Lambda \in \mathbb{R}^{20\times 12}$

# Obtaining Reduced Dataset Row-Wise

- By projecting $(V')^T$ onto the original dataset $\mathrm{Z}$, we obtain the reduced dataset row-wise.
- Define bilinear function $h : \mathbb{R}^{12\times 20} \times \mathbb{R}^{20\times 12} \to \mathbb{R}^{12\times 12}$ such that $h\left((V')^T, \mathrm{Z}\right) = (V')^T\mathrm{Z} = \mathrm{W} \in \mathbb{R}^{12\times 12}$
- $\mathrm{W}$ is the new row reduced dataset. In comparison to $\mathrm{Z}$, it has 8 less rows but the same number of columns with the same column-names.

# Reduced Dataset Row-Wise $W \in \mathbb{R}^{12\times 12}$

| Points:0 (theta=0) | Points:1 (theta=0) | Points:2 (theta=0) | Velocity:0 (theta=0) | Velocity:1 (theta=0) | Velocity:2 (theta=0) | Points:0 (theta=135) | Points:1 (theta=135) | Points:2 (theta=135) | Velocity:0 (theta=135) | Velocity:1 (theta=135) | Velocity:2 (theta=135) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3.437988973407158 | -2.875153166611659 | 3.3832445233029427 | 1.610623050394167 | -2.74136388328951 | -1.1522803319122532 | 3.437988973407158 | -2.875153166611659 | 3.3832445233029427 | -3.6452385608110136 | 3.980154492006279 | 0.8389795232025505 |
| 1.423124325767138 | -2.330061615713827 | -1.9247407518415207 | 2.88910738339776 | 2.1399848951034985 | -1.5709445561093314 | 1.423124325767138 | -2.330061615713827 | -1.9247407518415207 | 0.9665959970626935 | -1.3573587310066606 | -2.18263495952959 |
| 2.300469807298714 | 2.0672507865589416 | -1.2958880215940494 | 1.5244965819914171 | -0.39675346377361 | 1.5059971159741263 | 2.300469807298714 | 2.0672507865589416 | -1.2958880215940494 | -0.12611851986711578 | 0.4131412969131234 6 | 1.1039069856189074 |
| -0.08594962290495341 | 0.5738320848741221 | -1.387975354952216 | -1.1666265218824303 | -2.1090549678137505 | -1.3512400286867186 | -0.08594962290495341 | 0.5738320848741221 | -1.387975354952216 | -1.8294744638932074 | 0.9010680945990733 | -2.89940559591530 44 |
| 0.5408282241629957 | -1.0855563331646887 | -0.19135769540390482 | -1.4716932001510166 | -0.6837242659191735 | 3.319716982892896 | 0.5408282241629957 | -1.0855563331646887 | -0.19135769540390482 | 0.5866539983374522 | -0.3014114423160430 5 | -1.2000009179307827 |
| -0.3148340861327329 | 0.65767092420173 | 0.9848902807217528 | 1.5178147863477727 | 0.2160232173264733 | 0.6983652164877412 | -0.3148340861327329 | 0.65767092420173 | 0.9848902807217528 | 0.07854349737891311 | 0.0086898756884 6576 | -1.8041462695248327 |
| 0.34790700991304674 | 0.09443771547694649 | 0.3283463072719048 | -0.026640723272553676 | -1.4064178580868674 | -0.6698540193683797 | 0.34790700991304674 | 0.09443771547694649 | 0.3283463072719048 | 1.3516398508378191 | -0.9178038432207628 | -0.0897645289912172 |
| 0.5848094117854946 | 0.2856821862417042 | 0.3549305761624681 5 | -0.950174500920428 7 | 0.8974154794548667 | -0.38864156495469 | 0.5848094117854946 | 0.2856821862417042 | 0.3549305761624681 5 | 0.08538180683832103 | -0.1383213817233308 6 | -0.4476170757789962 |
| 0.027263430970183362 | 0.00129177621537284 | 0.05305223115881089 | 0.025704772124744805 | -0.048496425712181206 | 0.05024753216908514 | 0.027263430970183362 | 0.00129177621537284 | 0.05305223115881089 | -0.48009684865693925 | -0.6171847227788423 | 0.0607442877465168 |
| 1.1449327691334164e-16 | -1.81891627014393e-16 | -3.5542523493829393e-16 | -2.7535088709745247e-15 | 2.7394051633896376e-15 | -3.3822894731466193e-16 | 1.1449327691334164e-16 | -1.81891627014393e-16 | -3.5542523493829393e-16 | 1.4453938286973778e-15 | 2.4986092454998115e-15 | -2.9137795683329616e-15 |
| -2.2891409265811 97e-16 | -1.9683694878705873e-16 | 1.6778559244976804e-16 | -4.3549211944859487e-16 | -1.9875467577473176e-16 | 1.8798781128996988e-16 | -2.2891409265811 97e-16 | -1.9683694878705873e-16 | 1.6778559244976804e-16 | -1.1104808954911545e-15 | -5.8342458362837 31e-16 | -4.1202059993299103e-16 |
| 2.6221152393624873e-16 | -1.9910971487933867e-16 | 3.7367378176008 58e-17 | 1.125986502615641e-15 | -5.842103943265712e-16 | 4.141225210715377e-16 | 2.6221152393624873e-16 | -1.9910971487933867e-16 | 3.7367378176008 58e-17 | -1.6636335237628148e-15 | -1.1097889623152853e-15 | 7.424432305890857e-16 |

# Column Correlation

- Having obtained $\mathbf{W} \in \mathbb{R}^{12\times 12}$ it seems that we are ready to train our neural network.
- Keeping it mind that our goal is to predict velocity at new angles not seen in the dataset, it is probably best to correlate the columns as well.
- However, this time, we do not have to reduce the number of columns and we can keep the maximum number of columns present such that the full variance is captured by the eigenvalues, small as they may be.
- We repeat the process described from Step 3 to Step 5 with a few key changes.

# Column Correlation

- We construct the correlation matrix as $\mathrm{C}_C = \mathrm{W}^T \cdot \mathrm{W} \in \mathbb{R}^{12\times 12}; \mathrm{W} = [w'_{jk}], \mathrm{W}^T = [w_{ij}]$
- The subscript C in $\mathrm{C}_C$ makes it clear that this correlation is done column wise.
- The construction of the correlation matrix can be defined by bilinear function $g_C : \mathbb{R}^{12\times 12} \times \mathbb{R}^{12\times 12} \to \mathbb{R}^{12\times 12}$ where each element $(c_C)_{ik}$ of $\mathrm{C}_C$ is given by

$$(c_C)_{ik} = w_{ij} * w'_{jk}$$

# Column Correlation Matrix, $C_C \in \mathbb{R}^{12\times12}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 19.999999999999993 | -9.088581217770104 | 5.940235106990982 | 11.418161016264317 | -7.514357771780367 | -1.49996778948842 | 19.999999999999993 | -9.088581217770104 | 5.940235106990982 | -10.489954061881173 | 12.042233349929832 | 2.1946170392122206 |
| -9.088581217770104 | 19.999999999999982 | -7.730034573127007 | -6.558619238143711 | 1.8728995443158716 | 5.992601051051302 | -9.088581217770104 | 19.999999999999982 | -7.730034573127007 | 6.484088679730555 | -6.703770197901935 | 3.271619071725043 |
| 5.940235106990982 | -7.730034573127007 | 20.0 | 0.9638920777909981 | -9.754405795273739 | -1.2535602826043506 | 5.940235106990982 | -7.730034573127007 | 20.0 | -11.076742826768315 | 13.975395341146964 | 7.900862067615191 |
| 11.418161016264317 | -6.558619238143711 | 0.9638920777909981 | 20.000000000000004 | 4.140608143468153 | -5.959438424719385 | 11.418161016264317 | -6.558619238143711 | 0.9638920777909981 | -2.0100967176547075 | 2.6643866024357803 | -0.432220677551209 |
| -7.514357771780367 | 1.8728995443158716 | -9.754405795273739 | 4.140608143468153 | 20.0 | 0.5213289040614937 | -7.514357771780367 | 1.8728995443158716 | -9.754405795273739 | 13.784719493126806 | -14.475521586391638 | -1.1413928249713878 |
| -1.49996778948842 | 5.992601051051302 | -1.2535602826043506 | -5.959438424719385 | 0.5213289040614937 | 19.999999999999996 | -1.49996778948842 | 5.992601051051302 | -1.2535602826043506 | 6.003662279305754 | -3.406280060860079 | 3.0358600965194507 |
| 19.999999999999993 | -9.088581217770104 | 5.940235106990982 | 11.418161016264317 | -7.514357771780367 | -1.49996778948842 | 19.999999999999993 | -9.088581217770104 | 5.940235106990982 | -10.489954061881173 | 12.042233349929832 | 2.1946170392122206 |
| -9.088581217770104 | 19.999999999999982 | -7.730034573127007 | -6.558619238143711 | 1.8728995443158716 | 5.992601051051302 | -9.088581217770104 | 19.999999999999982 | -7.730034573127007 | 6.484088679730555 | -6.703770197901935 | 3.271619071725043 |
| 5.940235106990982 | -7.730034573127007 | 20.0 | 0.9638920777909981 | -9.754405795273739 | -1.2535602826043506 | 5.940235106990982 | -7.730034573127007 | 20.0 | -11.076742826768315 | 13.975395341146964 | 7.900862067615191 |
| -10.489954061881173 | 6.484088679730555 | -11.076742826768315 | -2.0100967176547075 | 13.784719493126806 | 6.003662279305754 | -10.489954061881173 | 6.484088679730555 | -11.076742826768315 | 20.0 | -18.653399619727043 | -1.0372414462435116 |
| 12.042233349929832 | -6.703770197901935 | 13.975395341146964 | 2.6643866024357803 | -14.475521586391638 | -3.406280060860079 | 12.042233349929832 | -6.703770197901935 | 13.975395341146964 | -18.653399619727043 | 20.00000000000001 | 4.598221421175162 |
| 2.1946170392122206 | 3.271619071725043 | 7.900862067615191 | -0.432220677551209 | -1.1413928249713878 | 3.0358600965194507 | 2.1946170392122206 | 3.271619071725043 | 7.900862067615191 | -1.0372414462435116 | 4.598221421175162 | 19.999999999999996 |

# Eigenvalue Decomposition of $\mathrm{C}_C$

- Again, the decomposition must result in real valued eigenvalues and eigenvectors.
- Solve the characteristic equation to get the eigenvalues - $det\left(\mathrm{C}_C - \lambda I\right) = 0$
- Use the matrix of eigenvalues $\Lambda_C \in \mathbb{R}^{12\times 12}$ to determine the matrix of eigenvectors $V_C \in \mathbb{R}^{12\times 12}$

# Eigenvalue Decomposition of $C_C$

- Sort $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{12}), \in \mathbb{R}^{12\times12}$ such that $\lambda_i > \lambda_j, \forall i < j$
- Sort the eigenvectors $V_C \in \mathbb{R}^{12\times12}$ according to their position in $\Lambda$
- Construct $\Lambda_C^{'} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{k'})$
- Since we want all our columns to stay, we want to take $k^{'} = 12$.
- We simply use $\Lambda_C \in \mathbb{R}^{12\times12}$ and $V_C \in \mathbb{R}^{12\times12}$

# Sorted Eigenvalues and Sorted Eigenvectors
# $\Lambda_C, V_C \in \mathbb{R}^{12\times 12}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 104.3354479346024 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 45.25316341800758 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 28.64475852467392 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 24.726725619237776 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 18.543985544275124 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 9.102650911223483 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 5.580315312915283 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 3.1851964421723067 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.6277562928916579 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0658856570233323e-14 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.302915155099587e-15 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | -8.168869046324322e-16 |

Table: Sorted Eigenvalues $\Lambda \in \mathbb{R}^{12\times 12}$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.3365801789521992 | -0.21155260147517294 | 0.42982725961632084 | -0.01728465321951271 | -0.1255907945422605 | 0.10435128516568844 | 0.1472766204052892 | -0.3276772139823629 | 0.03441002366556592 | -0.544952479228502 | 0.45058494802057825 | 0.0 |
| -0.2814783801863367 | 0.34637212468137285 | 0.3862518593841857 | 0.11539886107757534 | 0.25208721792124233 | -0.21798404041810945 | 0.039977543360557574 | -0.16007188151480897 | 0.0016303909141223848 | -0.04692361666712166 | -0.056750988574851904 | -0.703262041841200 |
| 0.33122068043279074 | 0.2861197056679237 | -0.24212793200943872 | -0.2791248161042895 | 0.044436965257769474 | -0.32644040485877024 | 0.13899614862502638 | -0.19887276095469433 | 0.06695886998529824 | -0.44813499035623733 | -0.5419894186357049 | 0.07363762968310389 |
| 0.15768049249703175 | -0.4294763091548304 | 0.2848418988385949 | -0.23461109177549655 | 0.34175568151137026 | -0.5030774321307111 | -0.011277598832272516 | 0.532396584227492 | 0.03244279188844165 | -4.503110173757013e-16 | 3.23308419989998e-16 | 1.0567572559614897e-16 |
| -0.26838036815931476 | -0.3181165295819704 | -0.07413070736074524 | -0.42413547037736654 | 0.1587740246638197 | -0.07160057105170049 | -0.5953673341288739 | -0.502834937616012 7 | -0.06120884632152748 | 4.437924315677514e-16 | 3.1834376857364354e-17 | 4.047378137339695e-17 |
| -0.11280859924741647 | 0.23352661577128614 | 0.2813853984501271 | -0.2717372633269267 | -0.7709026173369629 | -0.23147210249904746 | -0.28356380678311227 | 0.21776151798461219 | 0.0634189722109988 | -2.0180317098835063e-16 | 1.398657759192876e-16 | -1.2928426232505807e-17 |
| 0.3365801789521989 | -0.21155260147517288 | 0.42982725961631985 | -0.01728465321951243 7 | -0.1255907945422603 8 | 0.10435128516568846 | 0.14727662040528824 | -0.32767721398236055 | 0.034410023665564365 | 0.5449524792285034 | -0.4505849480205798 | -2.0408043981633435e-16 |
| -0.2814783801863366 | 0.3463721246813729 | 0.3862518593841859 | 0.1153988610775754 | 0.2520872179212423 | -0.21798404041810968 | 0.03997754336055821 | -0.16007188151480917 | 0.0016303909141221701 | 0.04692361666712253 | 0.056750988574851405 | 0.7032620418412 |
| 0.3312206804327906 | 0.2861197056679235 | -0.24212793200943825 | -0.2791248161042892 | 0.04443696525776918 | -0.3264404048587701 | 0.1389961486250272 | -0.19887276095469383 | 0.06695886998529803 | 0.4481349903562382 | 0.5419894186357045 | -0.07363762968310403 |
| -0.3568699773650855 | -0.1436880067691029 | -0.023564394372557602 | -0.3679112323426416 | -0.13623242737862196 | -0.026033124283251215 | 0.572178609698766 1 | -0.047840667447776083 | -0.6059451556965026 | -2.9156476260229926e-16 | -1.0338586922305354e-16 | -5.898384528675288e-17 |
| 0.38965835013986194 | 0.20177630687656975 | 0.07719266339557743 | 0.18120672337951377 | 0.06999357805925406 | -0.002880246250208384 | -0.3885263716253456 | 0.07750348076463041 | -0.7789680227727597 | -8.408533348624302e-16 | 8.142591834433342e-16 | 1.3524384571593328e-16 |
| 0.08213635361863685 | 0.3244566166137629 | 0.20625757095114527 | -0.5830766741528113 | 0.2786634683639359 | 0.5979815723397471 | -0.03799928166265869 | 0.25080635394423595 | 0.07666725369934364 | -3.015520374822 17e-16 | 4.291061125657871e-17 | 1.4628576196843829e-16 |

Table: Sorted Eigenvalues $V \in \mathbb{R}^{12\times 12}$

# Obtaining Reduced Dataset Row-Wise and Column-Wise

- Now, we project the reduced dataset $\mathrm{W}$ onto the eigenvectors ($V_C$) to obtain the reduced dataset column-wise.
- Define bilinear function $h_C : \mathbb{R}^{12\times 12} \times \mathbb{R}^{12\times 12} \to \mathbb{R}^{12\times 12}$ such that $h_C(\mathrm{W}, V_C) = \mathrm{W}V_C = \mathrm{W}_C \in \mathbb{R}^{12\times 12}$
- $\mathrm{W}_C$ is the new column reduced dataset. In comparison to $\mathrm{W}$, it has the same number of rows and columns but different values.
- Now, we can use $\mathrm{W}_C$ for our training dataset.

# Reduced Datasets $\mathrm{W}, \mathrm{W}_C \in \mathbb{R}^{12\times 12}$

| Points:0 (theta=0) | Points:1 (theta=0) | Points:2 (theta=0) | Velocity:0 (theta=0) | Velocity:1 (theta=0) | Velocity:2 (theta=0) | Points:0 (theta=135) | Points:1 (theta=135) | Points:2 (theta=135) | Velocity:0 (theta=135) | Velocity:1 (theta=135) | Velocity:2 (theta=135) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3.437988973407158 | -2.875153166611659 | 3.3832445233029427 | 1.610623050394167 | -2.74136388328951 | -1.1522803319122532 | 3.437988973407158 | -2.875153166611659 | 3.3832445233029427 | -3.6452385608110136 | 3.980154492006279 | 0.8389795232025505 |
| 1.423124325767138 | -2.330061615713827 | -1.9247407518415207 | 2.88910738339776 | 2.1399848951034985 | -1.5709445561093314 | 1.423124325767138 | -2.330061615713827 | -1.9247407518415207 | 0.9665959970626935 | -1.3573587310066606 | -2.182634959529 59 |
| 2.300469807298714 | 2.0672507865589416 | -1.2958880215940494 | 1.5244965819914171 | -0.39675346377361 | 1.5059971159741263 | 2.300469807298714 | 2.0672507865589416 | -1.2958880215940494 | -0.12611851986711578 | 0.4131412969131 2346 | 1.1039069856189074 |
| -0.08594962290495341 | 0.5738320848741221 | -1.387975354952216 | -1.1666265218824303 | -2.1090549678137505 | -1.3512400286867186 | -0.08594962290495341 | 0.5738320848741221 | -1.387975354952216 | -1.8294744638932074 | 0.9010680945990733 | -2.8994055959153044 |
| 0.5408282241629957 | -1.0855563331646887 | -0.19135769540390482 | -1.4716932001510166 | -0.6837242659191735 | 3.319716982892896 | 0.5408282241629957 | -1.0855563331646887 | -0.19135769540390482 | 0.5866539983374522 | -0.30141144231604305 | -1.2000009179307827 |
| -0.3148340861327329 | 0.65767092420173 | 0.9848902807217528 | 1.5178147863477727 | 0.2160232173264733 | 0.6983652164877412 | -0.3148340861327329 | 0.65767092420173 | 0.9848902807217528 | 0.07854349737891311 | 0.00868987568846576 | -1.8041462695248327 |
| 0.34790700991304674 | 0.09443771547694649 | 0.3283463072719048 | -0.026640723272553676 | -1.4064178580868674 | -0.6698540193683797 | 0.34790700991304674 | 0.09443771547694649 | 0.3283463072719048 | 1.3516398508378191 | -0.9178038432207628 | -0.0897645289912172 |
| 0.5848094117854946 | 0.2856821862417042 | 0.35493057616246815 | -0.9501745009204287 | 0.8974154794548667 | -0.38864156495469 | 0.5848094117854946 | 0.2856821862417042 | 0.35493057616246815 | 0.08538180683832103 | -0.13832138172333086 | -0.4476170757789962 |
| 0.027263430970183362 | 0.00129177621537284 | 0.05305223115881089 | 0.025704772124744805 | -0.04849642571218 1206 | 0.05024753216908514 | 0.027263430970183362 | 0.00129177621537284 | 0.05305223115881089 | -0.48009684865693925 | -0.6171847227788423 | 0.0607442877465168 |
| 1.1449327691334164e-16 | -1.81891627014393e-16 | -3.55425234938293936-16 | -2.7535088709745247e-15 | 2.7394051633896376e-15 | -3.3822894731466193e-16 | 1.1449327691334164e-16 | -1.81891627014393e-16 | -3.55425234938293936-16 | 1.4453938286973778e-15 | 2.4986092454998115e-15 | -2.9137795683329616e-15 |
| -2.2891409265811 97e-16 | -1.9683694878705873e-16 | 1.6778559244976804e-16 | -4.3549211944859487e-16 | -1.9875467577473176e-16 | 1.8798781128996988e-16 | -2.2891409265811 97e-16 | -1.9683694878705873e-16 | 1.6778559244976804e-16 | -1.1104808954911545e-15 | -5.8342458362837 31e-16 | -4.1202059993299103e-16 |
| 2.6221152393624873e-16 | -1.9910971487933867e-16 | 3.7367378176008 58e-17 | 1.1259865026156 41e-15 | -5.8421039432657 12e-16 | 4.1412252107157 77e-16 | 2.6221152393624873e-16 | -1.9910971487933867e-16 | 3.7367378176008 58e-17 | -1.6636335237628148e-15 | -1.1097889623152853e-15 | 7.4244323058908 57e-16 |

Table: Reduced Dataset $\mathrm{W}$

| Points:0 (theta=0) | Points:1 (theta=0) | Points:2 (theta=0) | Velocity:0 (theta=0) | Velocity:1 (theta=0) | Velocity:2 (theta=0) | Points:0 (theta=135) | Points:1 (theta=135) | Points:2 (theta=135) | Velocity:0 (theta=135) | Velocity:1 (theta=135) | Velocity:2 (theta=135) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.21447247461183 | 3.586306986050404e-15 | -5.563313795445292e-15 | -1.376404249050 6875e-16 | 1.2888299264168004e-15 | -5.2137090857384 18e-16 | -5.3800875567303 03e-16 | 1.6513037552312 87e-15 | -1.4696203139951447e-17 | 1.4480693925819 19e-15 | -1.3626534267399486e-15 | 8.5271549213312 42e-16 |
| 4.4909934552477775e-15 | -6.727047154436158 | 2.8472946054653 71e-14 | -1.9677898085111 477e-15 | 5.3587463484766706e-15 | -2.4639485032396545e-15 | -2.7425227771932 066e-15 | 4.3072085501043 1e-15 | -3.4495850327304544e-16 | -3.0914387783546 44e-16 | -8.4208530669802 96e-16 | 8.7211730107300 98e-16 |
| 8.3693795610222391e-16 | 2.3626662064129994e-14 | 5.352079831679829 | -6.6880266653135625e-15 | -2.9582721984339 42e-15 | 1.8534228439378 35e-15 | 1.2518492042549414e-15 | -3.7205443147410 04e-15 | 2.4540866569723764e-16 | 2.1406595296072424e-15 | -2.9523714825234 37e-15 | -7.4907201935051 13e-16 |
| 3.0587542095297 05e-16 | -1.7581277820571272e-15 | 7.1308599592318 8e-15 | 4.972597472070085 | 1.2685553865948 44e-15 | -6.8154999284526 3e-15 | 2.6696150713570 536e-15 | -4.1210843749142 96e-16 | 4.9678012507854 8e-17 | -2.7163406032552 803e-16 | 6.0020919362457 29e-16 | -4.0488783894623 01e-16 |
| 1.8026995464959745e-15 | -7.203406395795962e-15 | -2.5889290285183 55e-15 | 1.4980821211514 31e-15 | -4.306272813498 37 | 5.7623483573389 64e-16 | 2.3801078336326 535e-15 | 4.3201168593424 96e-15 | -3.9323789329958 797e-16 | -1.7533947607613 037e-16 | -5.8014763330550 31e-16 | 9.4024186095625 34e-17 |
| -1.1437656781276188e-15 | 1.4439497958312 608e-15 | 2.2647849893198 366e-15 | -5.4859137178883 03e-15 | 3.8758985087488 5e-15 | -3.0170599780619 99 | -2.6015106247164 203e-16 | -2.2772087128334 918e-15 | 3.8727556112240 196e-16 | 8.2065556756295 98e-16 | 1.7806695773313 318e-16 | -4.8940403905213 32e-16 |
| -7.4934619312506 65e-17 | -1.0415844255351 769e-15 | 5.6540819226690 4e-16 | -2.3265635813099 65e-15 | 2.1027498058028 247e-15 | -7.4758850795734 35e-16 | 2.3622691025611 977 | 3.4141724995003 063e-15 | -9.5432721818828 53e-16 | 7.9987771192333 78e-16 | -1.7606976470836 083e-15 | -4.1515582765132 94e-16 |
| -5.1242650826094 81e-17 | -1.7044439444796 258e-15 | -3.6267370871137 15e-15 | -1.5551032399005 838e-15 | -1.6595245125265 913e-15 | 2.1660153909365 48e-15 | 3.2722297750148 404e-15 | -1.7847118653083 207 | -3.7468507111234 23e-15 | 2.5309701309917 87e-15 | -1.6451876002389 557e-15 | -4.6769161513076 68e-16 |
| 7.5463151878465 85e-16 | 1.6574109310874 999e-15 | -2.4590839314170 425e-15 | 7.3563651843313 6e-16 | 5.0704067366631 49e-16 | 1.3040543342463 604e-16 | 3.3638216911064 986e-16 | -4.0057851627537 44e-15 | 0.7923107300116 835 | 6.8815197396447 12e-16 | -5.0543892121614 8e-16 | -6.0154395163236 09e-17 |
| -9.6874192303423 62e-16 | -7.9462477764572 22e-16 | -1.3947065898261 13e-15 | 1.3484571833030 158e-15 | -1.2313779154138 94e-15 | -1.8459115122701 79e-16 | -1.6166447879128 434e-15 | -3.3988061230932 65e-15 | -3.3643264671111 533e-15 | 5.6345263101235 3e-31 | 9.9945965689553 44e-31 | -2.4965188062401 69e-31 |
| 1.6644867583749 55e-16 | 2.5883017992462 56e-16 | -5.9036167444344 38e-16 | 5.4728207155655 74e-16 | -3.5650622418560 987e-16 | -9.7492087201995 86e-17 | -3.5964668485541 835e-16 | -4.0105866197725 465e-17 | 1.1118038298863 08e-15 | 6.6235938212164 47e-31 | -1.2118138963264 285e-31 | -3.1508366500248 356e-32 |
| 8.2321828730735 32e-16 | -1.7251531136730 376e-16 | 6.4073653615186 3e-16 | -2.2672904841232 1e-16 | 1.6573350251120 877e-16 | -1.2879741795214 965e-17 | -2.5953108195600 063e-16 | 1.0402385669983 677e-15 | 2.0504343974221 09e-15 | 5.4550247740821 47e-31 | -6.1513239752121 71e-31 | 1.9090368421684 024e-31 |

Table: Reduced Dataset $\mathrm{W}_C$

# Reconstructing the original dataset

- Now, importantly, we need to know how we can reconstruct the original dataset.
- We first summarize the functions we took to get to the final reduced dataset $W_C \in \mathbb{R}^{12\times 12}$.
- $\mathcal{F}(\mathrm{X}) = h_C \circ g_{\lambda_C} \circ g_C \circ h_R \circ g_{\lambda_R} \circ g_R \circ f(\mathrm{X}) = W_C$
- I did not explicitly define $g_{\lambda_C} : \mathbb{R}^{20\times 20} \to \mathbb{R}^{20\times 12}, g_{\lambda_R} : \mathbb{R}^{12\times 12} \to \mathbb{R}^{12\times 12}$ but it is clear that they are the reduction process of the eigenvectors.
- This step is nuanced as there is a potential loss of information in this step.

# Reconstructing the original dataset

- If one did not (or could not) retain all the variance, one loses information in this step that would be impossible to recover. Therefore, any reconstruction is necessarily an approximation.
- This loss of information might seem "non-linear" when used colloquially but mathematically, the function is still a linear one.
- $\mathcal{F}$ is linear as all the individual components are linear and therefore $\mathcal{F}^{-1}$ necessarily exists.
- Whether or not $\mathcal{F}^{-1}$ gives a good reconstruction/approximation of the original dataset has to do with $g_{\lambda_C}, g_{\lambda_R}$ and how much variance one was able to keep.

# Reconstructing the original dataset

- It is somewhat trivial to see that $(h_C \circ g_{\lambda_C} \circ g_C)^{-1} = \mathcal{T}_C : \mathbb{R}^{12\times 5} \times \mathbb{R}^{12\times 5} \to \mathbb{R}^{12\times 5}$
- $\mathcal{T}_C(\mathrm{W_C}, V_C^{'}) = \mathrm{W_C}(V_C^{'})^T = \mathrm{W}_{\mathcal{R}}$ where the subscript $\mathcal{R}$ indicates that it is reconstructed.
- And similarly, we see that $(h_R \circ g_{\lambda_R} \circ g_R)^{-1} = \mathcal{T}_R : \mathbb{R}^{12\times 5} \times \mathbb{R}^{12\times 5} \to \mathbb{R}^{12\times 5}$
- $\mathcal{T}_R(V^{'}, \mathrm{W}) = (V^{'})\mathrm{W} = \mathrm{Z}_{\mathcal{R}}$ where the subscript $\mathcal{R}$ indicates that it is reconstructed.

# Reconstruction of Dataset $Z_{\mathcal{R}} \in \mathbb{R}^{20\times 12}$

| Points:0 ($\theta = 0$) | Points:1 ($\theta = 0$) | Points:2 ($\theta = 0$) | Velocity:0 ($\theta = 0$) | Velocity:1 ($\theta = 0$) | Velocity:2 ($\theta = 0$) | Points:0 ($\theta = 135$) | Points:1 ($\theta = 135$) | Points:2 ($\theta = 135$) | Velocity:0 ($\theta = 135$) | Velocity:1 ($\theta = 135$) | Velocity:2 ($\theta = 135$) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.2422775230642614 | -0.5299456452302135 | -0.9263562703310546 | 0.05045384463351536 | 1.002311924281608 | -0.13555268487255895 | -0.2422775230642614 | -0.5299456452302135 | -0.9263562703310546 | 0.5732341608441163 | -1.1942511144808337 | -0.8349662911482649 |
| 0.4062579500808071 | -0.05844463045642818 | -0.6636690924848442 | 1.5079436024824224 | 1.042753436423715 2 | -0.30979510525014087 | 0.4062579500808071 | -0.05844463045642818 | -0.6636690924848442 | 0.695752493701489 | -0.8100862791999293 | 1.135991818641859 |
| -1.8535258961571983 | 0.6607875137295223 | -0.9192944125710181 | -0.5664190390191414 | 0.4359572532649747 | -0.07361137576697227 | -1.8535258961571983 | 0.6607875137295223 | -0.9192944125710181 | 0.4207196176919563 7 | -0.20414875187165366 | -0.6263201892983533 |
| -1.3402063720782595 | 1.5231322367128712 | -0.5798832042279629 | -1.1684441665682561 | -0.4609405336666635 | 0.05616829304082875 | -1.3402063720782595 | 1.5231322367128712 | -0.5798832042279629 | 0.2293250846020871 7 | -0.6945940237745349 | 0.35413729071883115 |
| -0.22918895853911686 | -0.5146019378904961 | -0.7286483640970153 | 0.5143806659508533 | 1.1198233813162162 | -0.24509601159914984 | -0.22918895853911686 | -0.5146019378904961 | -0.7286483640970153 | 0.4491675830722734 5 | -1.0447866884638508 | -3.2086399958049596 |
| 0.48190857133945336 | -0.09838318948413953 | -0.2781200562421884 | 1.392776043206737 | 0.6225418351234219 | -1.0808672510109605 | 0.48190857133945336 | -0.09838318948413953 | -0.2781200562421884 | 0.2941443863955978 | -0.12284194156045591 | 0.3836248342760069 |
| 1.1308438117852433 | -1.662495655094349 | -0.6284267788151632 | -0.5371133247049901 | -0.6869650535560543 | -0.26425071145752027 | 1.1308438117852433 | -1.662495655094349 | -0.6284267788151632 | -0.557945035329109 9 | 0.5605306432070555 | -0.4125585669956286 |
| 0.677916615044288 | -1.4226704359046825 | 2.0013517203106885 | 0.3438045441754181 | -1.7086495110463045 | -1.2853566143638335 | 0.677916615044288 | -1.4226704359046825 | 2.0013517203106885 | -1.9657715875028936 | 1.9181050082699318 | -0.1103149231190327 |
| -0.226591996913 82 | -0.21128355933018983 | -0.18195716386870311 | 0.5350186925730561 | 1.2757416566293076 | -0.09182149516710755 | -0.226591996913 82 | -0.21128355933018983 | -0.18195716386870311 | 0.4198088853893269 | -0.42816902927266365 | -0.2539279527156357 |
| -0.5262069079987394 | -0.3569092908089625 | -0.8447952959094199 | 0.3324974254910652 | 1.0705167553287611 | -0.1553226774822505 | -0.5262069079987394 | -0.3569092908089625 | -0.8447952959094199 | 0.402993455328959 | -0.6701535837702 76 | -0.2756257030201788 |
| -0.26133818382995344 | -0.23503438890411912 | -0.27467753453443167 | 0.5639877632524016 | 1.228226034426461 | -0.04559730896923422 | -0.26133818382995344 | -0.23503438890411912 | -0.27467753453443167 | 0.8254283545190724 | -0.6299513362586557 | 0.4382536848116362 |
| 0.5146528700931958 | -1.6967057391013924 | 2.194240096153868 | 0.5278003443193771 | -1.8547846943523203 | -1.1898377189690799 | 0.5146528700931958 | -1.6967057391013924 | 2.194240096153868 | -2.0152823072276598 | 1.9735563841480286 | -0.1769861922366279 |
| -1.3346828696789796 | 1.5310996163330326 | -0.5535351878888286 | -1.1270547175731789 | -0.5028173724877611 | 0.059851562752185 75 | -1.3346828696789796 | 1.5310996163330326 | -0.5535351878888286 | 0.2045366072014267 2 | -0.6914482245660659 | 0.4154526590678791 |
| 2.4323457666299406 | -0.6122673539465403 | -0.2764858630579614 | 0.6124290104923592 | -1.6159053256346074 | -0.07202604216770618 | 2.4323457666299406 | -0.6122673539465403 | -0.2764858630579614 | -1.010496195890266 | 1.074768328680117 | -0.25872887250567017 |
| 1.0899821006657082 | 2.2165070423511346 | -0.5700110699921003 | 1.0686413771468306 | -0.9215068965954986 | 0.2118076553370868 | 1.0899821006657082 | 2.2165070423511346 | -0.5700110699921003 | -0.7892710401969969 | 0.8850360322214808 | -0.3896301630871832 |
| 0.6238717379773003 | 0.18285821848475212 | 1.9560765320919424 | -0.7072778230563261 | 0.009247249857603246 | 0.5497434079558065 | 0.6238717379773003 | 0.18285821848475212 | 1.9560765320919424 | -1.8106159206731 08 | 1.8790593649001475 | 1.9328241464669438 |
| -0.13473706751142983 | 0.05840633029326893 | 1.1097787830799912 | -0.3552903257377462 | -0.5262400111504092 | 3.0678183842419315 | -0.13473706751142983 | 0.05840633029326893 | 1.1097787830799912 | 1.3814759618894994 | -0.5588488915676285 | 0.4237439411565179 |
| -0.11062133396982224 | 0.05186693206497024 | 0.8901003550363696 | 0.2642119218836048 | 0.7182828564582859 | -1.2204750545865886 | -0.11062133396982224 | 0.05186693206497024 | 0.8901003550363696 | 1.4319225207060032 | -0.7406162048934791 | 1.1867424888457054 |
| -1.5369773304791274 | 1.0506689894449002 | -0.5261155694616766 | -2.979019943070373 | -0.4505304720388194 | 0.17778833128645424 | -1.5369773304791274 | 1.0506689894449002 | -0.5261155694616766 | 0.6866203325214861 | -0.7171699522792246 | -0.4749233843151498 5 |
| 0.438575016604771 | 0.12341494673706097 | -0.19957162319049354 | -0.2733618958776277 | 0.20293748741808593 | 2.04643241704881 | 0.438575016604771 | 0.12341494673706097 | -0.19957162319049354 | 0.1342529115921392 | 0.21601026053248876 | 0.7518513702609649 |

Table: Original Dataset $Z$

| Points:0 ($\theta = 0$) | Points:1 ($\theta = 0$) | Points:2 ($\theta = 0$) | Velocity:0 ($\theta = 0$) | Velocity:1 ($\theta = 0$) | Velocity:2 ($\theta = 0$) | Points:0 ($\theta = 135$) | Points:1 ($\theta = 135$) | Points:2 ($\theta = 135$) | Velocity:0 ($\theta = 135$) | Velocity:1 ($\theta = 135$) | Velocity:2 ($\theta = 135$) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.24227752306426067 | -0.5299456452302154 | -0.9263562703310542 | 0.050453844633514995 | 1.002311924281609 | -0.13555268487255948 | -0.24227752306426067 | -0.5299456452302154 | -0.9263562703310542 | 0.5732341608441149 | -1.1942511144808372 | -0.8349662911482659 |
| 0.4062579500808075 | -0.05844463045642779 | -0.6636690924848445 | 1.507943602482422 | 1.0427534364237163 | -0.3097951052501405 | 0.4062579500808075 | -0.05844463045642779 | -0.6636690924848445 | 0.6957524937014887 | -0.8100862791999314 | 1.1359918186418585 |
| -1.853525896157196 | 0.6607875137295197 | -0.9192944125710173 | -0.5664190390191409 | 0.4359572532649757 | -0.07361137576697314 | -1.853525896157196 | 0.6607875137295197 | -0.9192944125710173 | 0.4207196176919551 | -0.20414875187165543 | -0.6263201892983544 |
| -1.340206372078258 | 1.5231322367128697 | -0.5798832042279621 | -1.1684441665682552 | -0.4609405336666634 | 0.05616829304082877 | -1.340206372078258 | 1.5231322367128697 | -0.5798832042279621 | 0.2293250846020863 7 | -0.6945940237745344 | 0.3541372907188308 |
| -0.22918895853911803 | -0.5146019378904 95 | -0.7286483640970158 | 0.5143806659508523 | 1.1198233813162164 | -0.24509601159914932 | -0.22918895853911803 | -0.5146019378904 95 | -0.7286483640970158 | 0.4491675830722748 | -1.0447866884638517 | -3.2086399958046 |
| 0.48190857133945353 | -0.09838318948413907 | -0.27812005624218883 | 1.3927760432067369 | 0.6225418351234219 | -1.0808672510109598 | 0.48190857133945353 | -0.09838318948413907 | -0.27812005624218883 | 0.2941443863959587 | -0.12284194156045575 | 0.38362483427600674 |
| 1.1308438117852422 | -1.662495655094348 | -0.628426778815164 | -0.5371133247049913 | -0.6869650535560538 | -0.2642507114575201 | 1.1308438117852422 | -1.662495655094348 | -0.628426778815164 | -0.557945035329109 | 0.560530643207055 | -0.41255856699528565 |
| 0.6779166150442868 | -1.4226704359046822 | 2.001351720310688 | 0.3438405441754176 | -1.7086495110463047 | -1.2853566143638329 | 0.6779166150442868 | -1.4226704359046822 | 2.001351720310688 | -1.9657715875028932 | 1.9181050082699302 | -0.1103149231190326 5 |
| -0.22659199691382062 | -0.2112835593301898 | -0.18195716386870367 | 0.5350186925730558 | 1.2757416566293078 | -0.09182149516710704 | -0.22659199691382062 | -0.2112835593301898 | -0.18195716386870367 | 0.4198088853893271 5 | -0.42816902927266337 | -0.2539279527156354 5 |
| -0.5262069079987399 | -0.3569092908089619 4 | -0.8447952959094199 | 0.3324974254910656 | 1.0705167553287613 | -0.15532267748224965 | -0.5262069079987399 | -0.3569092908089619 4 | -0.8447952959094199 | 0.4029934553289586 4 | -0.6701535837702752 | -0.2756257030201786 |
| -0.26133818382995383 | -0.235034388904119 | -0.27467753453443 32 | 0.5639877632524015 | 1.2282260344264613 | -0.04559730896923402 | -0.26133818382995383 | -0.235034388904119 | -0.27467753453443 32 | 0.8254283545190728 | -0.6299513362586551 | 0.4382536848116357 |
| 0.5146528700931 95 | -1.6967057391013913 | 2.1942400961538686 | 0.5278003443193755 | -1.8547846943523207 | -1.1898377189690794 | 0.5146528700931 95 | -1.6967057391013913 | 2.1942400961538686 | -2.0152823072276598 | 1.9735563841480286 | -0.1769861922366279 |
| -1.3346828696 78979 | 1.5310996163330313 | -0.5535351878888278 | -1.1270547175731778 | -0.5028173724877615 | 0.059851562752185616 | -1.3346828696 78979 | 1.5310996163330313 | -0.5535351878888278 | 0.2045366072014262 | -0.6914482245660645 | 0.4154526590678789 |
| 2.4323457666299397 | -0.6122673539465391 | -0.2764858630579614 | 0.6124290104923585 | -1.615905325634607 | -0.07202604216770589 | 2.4323457666299397 | -0.6122673539465391 | -0.2764858630579614 | -1.0104961958902658 | 1.0747683286801166 | -0.2587288725056704 |
| 1.0899821006657098 | 2.216507042351134 | -0.5700110699921002 | 1.0686413771468322 | -0.9215068965954987 | 0.21180765533708626 | 1.0899821006657098 | 2.216507042351134 | -0.5700110699921002 | -0.7892710401969972 | 0.8850360322214803 | -0.3896301630871 83 |
| 0.6238717379773003 | 0.18285821848475114 | 1.9560765320919429 | -0.7072778230563264 | 0.009247249857603192 | 0.5497434079558057 | 0.6238717379773003 | 0.18285821848475114 | 1.9560765320919429 | -1.8106159206731094 | 1.8790593649001477 | 1.9328241464669427 |
| -0.13473706751142936 | 0.05840633029326831 | 1.1097787830799914 | -0.3552903257374575 | -0.5262400111504086 | 3.0678183842419307 | -0.13473706751142936 | 0.05840633029326831 | 1.1097787830799914 | 1.3814759618894985 | -0.5588488915676293 | 0.4237439411565171 |
| -0.11062133396982186 | 0.05186693206497005 5 | 0.89010035503637 | 0.26421192188360443 | 0.7182828564582857 | -1.220475054586589 | -0.11062133396982186 | 0.05186693206497005 5 | 0.89010035503637 | 1.4319222520706036 | -0.7406162048934803 | 1.186742488845705 |
| -1.5369773304791272 | 1.0506689894448982 | -0.5261155694616761 | -2.9790199430073725 | -0.4505304720388194 | 0.17778833128645327 | -1.5369773304791272 | 1.0506689894448982 | -0.5261155694616761 | 0.6866203325214856 | -0.7171699522792246 | -0.4749233843151503 |
| 0.4385750166047716 | 0.12341494673706059 | -0.1995716231904937 6 | -0.27336189587762677 | 0.20293748741808584 | 2.04643241704881 | 0.4385750166047716 | 0.12341494673706059 | -0.1995716231904937 6 | 0.1342529115921391 7 | 0.21601026053248853 | 0.7518513702609647 |

Table: Reconstructed Dataset $Z_{\mathcal{R}}$

# Reconstructing the original dataset

- In evaluating how good our approximation was, the best way is to compare $\mathrm{Z}_{\mathcal{R}}$ with $\mathrm{Z}$
- We can do this via the Mean Square Errors method, defined as $\mathrm{MSE}(\mathrm{Z}, \mathrm{Z}_{\mathcal{R}}) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (z_{ij} - z_{\mathcal{R}ij})^2$
- In our case, we obtain MSE $\sim 7E - 31 \approx 0$

# Table of Contents

# Conclusion and Future Outlook

- **Critical Role of MOR:** Model Order Reduction, especially techniques like Proper Orthogonal Decomposition (POD), plays a pivotal role in tackling the computational challenges posed by high-fidelity models in science and engineering. By distilling complex systems to their most influential dynamics, MOR enables simulations that were once impractical due to computational constraints, making it an indispensable tool in the researcher's toolkit.
- **Advancements in Computational Efficiency:** The application of MOR methodologies, such as POD, has led to significant advancements in computational efficiency. This not only accelerates the simulation process but also opens up new possibilities for real-time analysis and control, interactive simulation environments, and the ability to handle larger and more complex models than ever before.

# Conclusion and Future Outlook

- **Wide-ranging Applications:** The utility of MOR extends across numerous fields, including but not limited to fluid dynamics, structural analysis, electrical engineering, and control systems. Its versatility and effectiveness in simplifying complex problems while retaining essential characteristics make it a cornerstone technique in both theoretical research and practical applications.
- **Future Directions:** The ongoing development of MOR techniques promises to further broaden their applicability and enhance their effectiveness. Innovations in algorithmic efficiency, the integration of machine learning methods for automated mode selection, and the exploration of non-linear MOR methods are just a few areas where significant progress can be expected. As computational power continues to grow, the synergy between MOR techniques and high-performance computing will undoubtedly lead to new breakthroughs in simulation and modeling capabilities.

# Conclusion and Future Outlook

- **Final Thoughts:** In conclusion, the strategic simplification offered by MOR, particularly through POD, represents a fundamental advance in our ability to understand and predict complex systems. As we continue to push the boundaries of what is computationally feasible, MOR will remain at the forefront of this endeavor, enabling us to tackle the grand challenges of science and engineering with ever-greater sophistication and insight.

# Questions?

Questions?