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Weak Imposition of "No-Slip" Conditions in Finite Element Methods

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Abstract—We begin the analysis of two schemes for imposing weakly the "no-slip" boundary conditions in the Navier-Stokes equations. The first is via a Lagrange multiplier constraining no tangential velocity on the boundary and the second is via penalization of tangential velocities along boundaries. Our analysis in this first step is for the Stokes problem although the motivation for relaxing this essential boundary condition comes from higher Reynolds number flow.

Keywords—No-slip, Navier-Stokes, Penalty, Lagrange multiplier.

1. INTRODUCTION

Consider the Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p = \mathbf{f}, &\qquad \text{in } \Omega \left(\subset \mathbb{R}^2\right), \\ \mathbf{u} = 0, &\qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ is composed of two separate conditions:

1.1 "no penetration": $\mathbf{u} \cdot \hat{n} = 0$, on Γ,
1.2 "no-slip": $\mathbf{u} \cdot \hat{\tau} = 0$, on Γ,

where $\hat{n}$ is the outward unit normal to Γ and $\hat{\tau}$ a unit tangent vector to Γ. This report studies algorithms for which the true no slip condition $\mathbf{u} \cdot \hat{\tau}_{|\Gamma} = 0$ is imposed weakly as a constraint and as a penalty term.

The study of "relaxed" methods of imposing the no-slip Condition 1.2 is motivated by, and a first step towards, the same idea for high Reynolds number flow problems. Indeed, the behavior of fluids near boundaries plays a key role in high Reynolds number flow. The sharp transition from the free-stream flow to the adherence condition generates large amounts of vorticity which (at sufficient levels) can cause detachment of boundary layers. For example, it is known, [1], that in $2d$ flow driven by a potential body force, *all* vorticity must be created at boundaries and then diffused into the fluid's interior. This situation becomes even more complex when approximated on a given computational mesh. If the transition from adherence to free-stream velocities is

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not resolvable on the given mesh, nonphysical vortex structures e.g., "wiggles" can form in the approximate solution and then be transported into the flow region's interior.

We therefore consider methods of imposing $\mathbf{u} = \mathbf{0}$ on Γ in such a way that the "no-penetration" condition $\mathbf{u} \cdot \hat{n} = 0$, appropriate for the Euler equations, is imposed strongly while the "no-slip" condition $\mathbf{u} \cdot \hat{\tau} = 0$ on Γ is relaxed. There are (at least) three natural ways of accomplishing this: imposing $\mathbf{u} \cdot \hat{\tau} = 0$ via

(i) a Lagrange multiplier implementation of $\mathbf{u} \cdot \hat{\tau}_{|_\Gamma} = 0$ as a constraint;
(ii) a penalty term imposing $\mathbf{u} \cdot \hat{\tau}_{|_\Gamma} = 0$ approximately;
(iii) replacing "no-slip" with "slip with friction" (see [1] for a precise statement of this boundary condition) with a friction coefficient which increases as $h \to 0$.

We show in Section 4 that (iii) is almost the same as the usual penalty function approach if linear resistance is used. Thus, we consider the first two possibilities.

There are many papers studying finite element methodology in which Dirichlet boundary conditions or other continuity conditions are imposed weakly. In this regard, the fundamental paper [2] of Babuska stands out. It is shown in [2] (using elliptic theory extensively), that Dirichlet boundary conditions can successfully be imposed using Lagrange multipliers in second order elliptic boundary value problems. *Inter-element* continuity can also be imposed weakly in various ways, for example, via hybrid and other formulations and mortar elements [3].

To formulate the methods we study, let $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ denote the usual $L^2(\Omega)$ and $L^2(\Gamma)$ inner products and associated duality pairings and introduce the spaces:

$$
\begin{aligned}
X &:= \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^2 : (\mathbf{v} \cdot \hat{n}, \lambda) = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\}, \\
X_0 &:= \left\{ \mathbf{v} \in X : (\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\}, \\
V &:= \left\{ \mathbf{v} \in X : (\nabla \cdot \mathbf{v}, q) = 0, \text{ for all } q \in L_0^2(\Omega) \right\}, \\
V_0 &:= \left\{ \mathbf{v} \in V : (\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\}, \\
M &:= L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : (q, 1) = 0 \right\}, \text{ and} \\
L &:= H^{-1/2}(\Gamma).
\end{aligned}
$$

For reference, we note that the most common formulation of the Stokes problem is in (X_0, M). Find $(\mathbf{u}, p) \in (X_0, M)$ satisfying

$$
a(\mathbf{u}, \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) - (p, \nabla \cdot \mathbf{u}) = (\mathbf{f}, \mathbf{v}), \tag{1.2}
$$

for all $(\mathbf{v}, q) \in (X_0, M)$, where $a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})$. Equation (1.2) is equivalent to the formulation of (1.1) as a constrained minimization problem.

Find $u \in V_0$ minimizing over V_0

$$
J(\mathbf{v}) := \int_\Omega \frac{1}{2} |\nabla \mathbf{v}|^2 - \mathbf{f} \cdot \mathbf{v} \, dx. \tag{1.3}
$$

If we proceed analogously to (1.3),(1.2) and impose $\mathbf{u} \cdot \hat{\tau}_{|_\Gamma} = 0$ as a constraint, we arrive at the following weak formulation, considered in this report.

Find $(\mathbf{u}, p, \lambda) \in (X, M, L)$ satisfying:

$$
\begin{gathered}
a(\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma = (\mathbf{f}, \mathbf{v}), \\
(q, \nabla \cdot \mathbf{u}) = 0, \qquad (\mathbf{u} \cdot \hat{\tau}, \alpha)_\Gamma = 0, \\
\text{for all } (\mathbf{v}, q, \alpha) \in (X, M, L),
\end{gathered} \tag{1.4}
$$

For future reference, we note that (1.4) has the following equivalent formulation in (V, L).

REMARK. It is possible to use different formulations of the viscous term in the numerical procedure, see [4]. With the one chosen here the Lagrange multiplier $-\lambda$ is almost the tangential component of the normal stress: $\lambda = -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$. If the stress formulation of the viscous terms in (1.2),(1.4) is used instead, $a(\mathbf{u}^h, \mathbf{v}^h) = (\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)$ is replaced by $a(\mathbf{u}^h, \mathbf{v}^h) = (\mathcal{D}(\mathbf{u}^h), \mathcal{D}(\mathbf{v}^h))$ where $\mathcal{D}(\mathbf{w}) := (\nabla \mathbf{w} + \nabla \mathbf{w}^t)/2$ is the deformation tensor and the corresponding λ is then $\lambda = -\mathcal{D}(\mathbf{u}) \cdot \hat{n} \cdot \hat{\tau}$.

The main result of this report is that this is a mathematically correct procedure. We show in Theorem 3.1 that with the Lagrange multiplier formulation, under only the assumption of div-stability of the velocity-pressure finite element spaces, that $\mathbf{u}^h \to \mathbf{u}$ and $p^h \to p$ quasi-optimally as $h \to 0$. We give an error estimate which suggests the use polynomial degree $(k, k-1, k-1)$ spaces are asymptotically optimal for (X^h, M^h, L^h). In Theorem 3.2, we show λ^h also provides a quasi-optimal approximation of $+\lambda$ provided an additional discrete inf-sup condition (relating X^h and L^h) holds. This condition is shown to hold in the limit as $h \to 0$ in Lemma 2.1 and it is shown that the discrete analog, (3.8), holds for a simple modification of the MINI element when L^h is taken to be L^2 piecewise constants.

Section 4 introduces some preliminary results for the usual penalty discretization associated with the Lagrange multiplier formulation of Section 3.

REMARK. The related idea of "regularization" of an approximation scheme by weakly imposing the no-penetration condition (using, e.g., penalty methods) also occurs in work on contact problems, e.g., [5]. The same idea can be considered here for boundaries in flow problems which allow small amounts of flow penetration into the surface.

2. PRELIMINARIES

We collect, in this section, some preliminary information concerning the continuous problem. Let $\|\cdot\|$ and $\|\cdot\|_\Gamma$ denote the usual $L(\Omega)$ and $L^2(\Gamma)$ norms. The $H^s(\Gamma)$ norm will be denoted $\|\cdot\|_{s,\Gamma}$. If we proceed by following the usual treatment of the Stokes problem, we are led to the following inf-sup condition to be verified: there is a $\beta > 0$ such that

$$\inf_{(q,\alpha)\in(M,L)} \sup_{\mathbf{v}\in X} \frac{(q, \nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \hat{\tau}, \alpha)_\Gamma}{\left(\|\nabla \mathbf{v}\|^2 + \|\mathbf{v} \cdot \hat{\tau}\|_{+1/2,\Gamma}^{1/2}\right)^2 \left(\|q\|^2 + \|\alpha\|_{-1/2,\Gamma}^2\right)^{1/2}} \geq \beta > 0. \tag{2.1}$$

Since $\mathbf{v} \cdot \hat{n} = 0$, on Γ, the Poincaré inequality holds in X [6], which, when combined with the trace lemma

$$\|\mathbf{v} \cdot \hat{\tau}\|_{1/2,\Gamma} \leq C\|\mathbf{v}\|_{1,\Omega} \leq C\|\nabla \mathbf{v}\|,$$

allows us to simplify (2.1) a bit. The next lemma contains a proof of (2.1).

LEMMA 2.1. *Let $X = \{\mathbf{w} \in H^1(\Omega)^2 : (\mathbf{w} \cdot \hat{n}, \lambda) = 0\}$, for all $\lambda \in L^2(\Gamma)$. Then, there is a $\beta > 0$ such that*

$$\inf_{\substack{\lambda\in H^{-1/2}(\Gamma)\\ p\in M}} \sup_{\mathbf{v}\in X} \frac{(p, \nabla \cdot \mathbf{v}) - (\lambda, \mathbf{v} \cdot \hat{\tau})_\Gamma}{\|\nabla \mathbf{v}\| \left(\|p\|^2 + \|\lambda\|_{-1/2,\Gamma}^2\right)^{1/2}} \geq \beta > 0. \tag{2.2}$$

PROOF. Let $p \in L_0^2(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma)$ be given. Then, there is a $\hat{\lambda} \in H^{+1/2}(\Gamma)$ such that $\|\hat{\lambda}\|_{1/2,\Gamma} = 1$ and $(\lambda, \hat{\lambda})_\Gamma \geq (1/2)\,\|\lambda\|_{-1/2,\Gamma}$. Let $\tilde{\mathbf{v}}$ be a solution of the following problem, where $\hat{p} = p/\|p\|$

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{v}} &= +p, && \text{in } \Omega, \\ \tilde{\mathbf{v}} \cdot \hat{n} &= 0, && \text{on } \Gamma = \partial\Omega, \\ \tilde{\mathbf{v}} \cdot \hat{\tau} &= -\hat{\lambda}, && \text{on } \Gamma. \end{aligned}$$

It is known (see [6, (3.2) Section III]) that such a $\tilde{\mathbf{v}}$ exists (although possibly nonuniquely) and satisfies:

$$\|\tilde{\mathbf{v}}\|_{1,\Omega} \le C(\Omega)\left[\|p\| + \left\|\hat{\lambda}\right\|_{1/2,\Gamma}\right] =: C'.$$

Thus, $\|\nabla(\tilde{\mathbf{v}}/C')\| \le 1$. Pick $\mathbf{v} = \tilde{\mathbf{v}}/C'$ in (2.2). This gives:

$$\begin{aligned}(p, \nabla\cdot\mathbf{v}) - (\lambda, \mathbf{v}\cdot\hat{\tau})_\Gamma &= (p,\hat{p}) + \left(\lambda,\hat{\lambda}\right)_\Gamma \\ &\ge \frac{1}{2}\left(\|p\| + \|\lambda\|_{-1/2,\Gamma}\right) \ge C''\left(\|p\|^2 + \|\lambda\|^2_{-1/2,\Gamma}\right)^{1/2}.\end{aligned}$$

Thus, for this choice of $\mathbf{v}$,

$$\frac{(p,\nabla\cdot\mathbf{v}) - (\lambda,\mathbf{v}\cdot\hat{\tau})_\Gamma}{\|\nabla\mathbf{v}\|} \ge C''\left(\|p\|^2 + \|\lambda\|^2_{1/2,\Gamma}\right)^{1/2},$$

and the inf-sup condition (2.1),(2.2) follows. ∎

REMARK. The idea of solving an auxiliary problem for $\hat{\lambda}$ (rather than λ) used in this proof occurs in a slightly different form in the work of Verfürth [7, Lemma 3.2, p. 22] and Liakos [8, Lemma 3.1] on problems with free-slip boundary conditions.

It is also known (see, e.g., [4,6,9], etc.) that the following inf-sup condition holds on (X_0, M) (note that functions in X_0 vanish on Γ): for some $\beta' > 0$

$$\inf_{q\in M}\sup_{\mathbf{v}\in X_0}\frac{(q,\nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|\,\|q\|} \ge \beta' > 0.$$

Since $X_0 \subset X$ the same must hold on (X, M) as recorded in this next lemma.

LEMMA 2.2.

$$\inf_{q\in M}\sup_{\mathbf{v}\in X}\frac{(q,\nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|\,\|q\|} \ge \beta' > 0$$

thus,

$$V := \{\mathbf{v}\in X : (q,\nabla\cdot\mathbf{v}) = 0,\ \forall\, q\in M\} \qquad \textit{and}$$
$$V_0 := \{v\in X_0 : (q,\nabla\cdot v) = 0,\ \forall\, q\in M\}$$

are nontrivial, closed subspaces of X and X_0, respectively, and

$$V_0 \subset V.$$ ∎

We record two other potentially useful results along this same line.

LEMMA 2.3. *There is a constant $\beta > 0$ such that*

$$\inf_{\lambda\in L}\sup_{\mathbf{v}\in V}\frac{(\lambda,\mathbf{v}\cdot\hat{\tau})_\Gamma}{\|\mathbf{v}\|_{1/2,\Gamma}\,\|\lambda\|_{-1/2,\Gamma}} \ge 1. \tag{2.3}$$

PROOF. Since the boundary value problem: given $\lambda\in H^{1/2}(\Gamma)$, find $\mathbf{v}\in V$ satisfying

$$\nabla\cdot\mathbf{v} = 0,\quad \text{in } \Omega, \qquad \mathbf{v}\cdot\hat{n} = 0,\quad \text{on } \Gamma, \qquad \mathbf{v}\cdot\hat{\tau} = \lambda,\quad \text{on } \Gamma,$$

has a solution $\mathbf{v}$ for every $\lambda\in H^{1/2}(\Gamma)$ (see [6, (3.25) Section III.3]) the supremum in the lemma is in fact taken over $\lambda\in H^{1/2}(\Gamma)$. Now $\sup_{\lambda\in H^{1/2}(\Gamma)}((\alpha,\lambda)_\Gamma/\|\lambda\|_{1/2,\Gamma})$ is the definition of $\|\alpha\|_{-1/2,\Gamma}$ so the lemma holds. ∎

LEMMA 2.4. *There is a finite constant $C = C(\Omega)$*

$$\sup_{0\ne\mathbf{v}\in V}\frac{\|\nabla\mathbf{v}\|}{\|\mathbf{v}\|_{1/2,\Gamma}} \le C(\Omega).$$

PROOF. Applying the regularity result of [4, (3.25)] gives that if $\nabla\cdot\mathbf{v} = 0$ in Ω, $\|\mathbf{v}\|_{W^{1,q}(\Omega)} \le C\|\mathbf{v}\|_{W^{1-1/q,q}(\Gamma)}$. If $q = 2$ this implies there is a $C = C(\Omega)$ with $\|\nabla\mathbf{v}\| \le C(\Omega)\|\mathbf{v}\|_{1/2,\Gamma}$ or

$$\sup_{0\ne\mathbf{v}\in V}\frac{\|\nabla\mathbf{v}\|}{\|\mathbf{v}\|_{1/2,\Gamma}} \le C(\Omega),$$

which is the desired result. ∎

3. FORMULATION OF THE FINITE ELEMENT METHOD

The finite element method with "no-slip" weakly imposed can now be formulated. Let $X^h \subset X, M^h \subset M$ and $L^h \subset L$ be finite element spaces. We shall assume Γ is polygonal so that, for example, $X_0^h = \{\mathbf{v} \in X^h : \mathbf{v} \cdot \hat{\tau} = 0 \text{ on } \Gamma\} = \{\mathbf{v} \in X^h : \mathbf{v} = 0 \text{ on } \Gamma\}$ is well defined. We shall assume the spaces (X^h, M^h) are chosen so that (X_0^h, M^h) verify the classical discrete inf-sup condition for pressure stability: for some $\beta > 0$:

$$\inf_{q \in M^h} \sup_{\mathbf{v} \in X_0^h} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \; \|q\|} \geq \beta > 0. \tag{3.1}$$

If the supremum is taken over a larger space X^h the quotient can only increase, as noted already in Lemma 2.2. Thus, (3.1) implies

$$\inf_{q \in M^h} \sup_{\mathbf{v} \in X^h} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \; \|q\|} \geq \beta > 0, \tag{3.2}$$

which implies that the discretely divergence-free spaces V^h and V_0^h

$$V^h := \left\{\mathbf{v}^h \in X^h : \left(q^h, \nabla \cdot \mathbf{v}^h\right) = 0, \ \forall\, q^h \in M^h\right\},$$
$$V_0^h := \left\{\mathbf{v}^h \in X_0^h : \left(q^h, \nabla \cdot \mathbf{v}^h\right) = 0, \ \forall\, q^h \in M^h\right\},$$

are well defined (and, in fact, satisfy $V_0^h \subset V^h$).

We can now formulate the Lagrange multiplier-finite element scheme: Find $\mathbf{u}^h \in X^h$, $p^h \in M^h$, and $\lambda^h \in L^h$ satisfying:

$$\begin{aligned} a\left(\mathbf{u}^h, \mathbf{v}\right) + \left(p^h, \nabla \cdot \mathbf{v}\right) + \left(\mathbf{v} \cdot \hat{\tau}, \lambda^h\right)_\Gamma &= (\mathbf{f}, \mathbf{v}), \\ \left(q, \nabla \cdot \mathbf{u}^h\right) = 0, \qquad \left(\mathbf{u}^h \cdot \hat{\tau}, \alpha\right)_\Gamma &= 0, \end{aligned} \tag{3.3}$$

for all $(\mathbf{v}, q, \alpha) \in (X^h, M^h, L^h)$. It follows from standard theory of saddle point problems that under (3.2),(3.3) is equivalent to the problem: find $\mathbf{u}^h \in V^h, \lambda^h \in L^h$ satisfying

$$\begin{aligned} a\left(\mathbf{u}^h, \mathbf{v}\right) + \left(\mathbf{v} \cdot \hat{\tau}, \lambda^h\right)_\Gamma &= (\mathbf{f}, \mathbf{v}), \\ \left(\mathbf{u}^h \cdot \hat{\tau}, \alpha\right)_\Gamma &= 0, \end{aligned} \tag{3.4}$$

for all $(\mathbf{v}, \alpha) \in (V^h, L^h)$.

Before proceeding, define a third discretely divergence free space $\hat{V}^h$ satisfying the discrete boundary condition weakly by:

$$\hat{V}^h := \left\{\mathbf{v} \in V^h : (\mathbf{v} \cdot \hat{\tau}, \alpha)_\Gamma = 0, \text{ for all } \alpha \in L^h\right\}.$$

Clearly,

$$V_0^h \subset \hat{V}^h \subset V^h$$

so that function in V_0 can be approximated in $\hat{V}^h$ at least as well as by functions in V_0^h.

REMARK. Imposing $\mathbf{v} \cdot \hat{n}_{|\Gamma} = 0$ in X^h.

There are practical issues which arise from the requirement $X^h \subset X$. If Γ is polygonal then this is implied by, for example, $\mathbf{v}^h(N) \cdot \hat{n}(N) = 0$ at every node N along a straight side and $\mathbf{v}^h(N) = 0$ at all corner nodes. If Γ^h is a polygonal approximation to a curved boundary Γ then each node is at a corner and this construction fails. For such problems it could well be preferable to impose both $\mathbf{u} \cdot \hat{n}_{|\Gamma} = 0$ and $\mathbf{u} \cdot \hat{\tau}_{|\Gamma} = 0$ weakly with different scalings. For example, it seems that the degree $\mathbf{u} \cdot \hat{\tau}_{|\Gamma} = 0$ is weakened should be scaled by the Reynolds number somehow. This, idea is currently under study.

LEMMA 3.1. *Suppose the classical discrete* inf-sup *condition (3.1) holds.*

Then, for any $u \in V_0$

$$\inf_{\mathbf{v}\in \hat{V}^h} \|\nabla(\mathbf{u}-\mathbf{v})\| \le C \inf_{\mathbf{v}\in V_0^h} \|\nabla(\mathbf{u}-\mathbf{v})\| \le C \inf_{\mathbf{v}\in X_0^h} \|\nabla(\mathbf{u}-\mathbf{v})\|.$$

PROOF. The first inequality follows since $V_0^h \subset \hat{V}^h$ (which is obvious). The second follows from standard theory of the discretized Stokes problem in, e.g., [9]. ∎

To analyze the error in method (3.3), we shall first derive the needed variational formulation of the continuous problem.

LEMMA 3.2. *Let $(\mathbf{u}, p)$ satisfy the Stokes problem and let $\mathbf{v}^h \in \hat{V}^h$ be given. If $a(\mathbf{u},\mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})$, then $(\mathbf{u}, p)$ satisfies:*

$$a\left(\mathbf{u}, \mathbf{v}^h\right) - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right) + \left(\lambda, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = \left(\mathbf{f}, \mathbf{v}^h\right),$$

for all $\mathbf{v}^h \in V^h$ and $q^h \in M^h$, where $\lambda := -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$. If $a(\mathbf{u},\mathbf{v}) = (\mathcal{D}(\mathbf{u}), (\mathbf{v}))$, then the same holds with λ replaced by $\lambda = -\mathcal{D}(\mathbf{u}), \cdot \hat{n} \cdot \hat{\tau}$.

PROOF. This follows by integration by parts, keeping careful track of all boundary integral terms. ∎

The next theorem (Theorem 3.1) presents the basic convergence result of the method. It is noteworthy that convergence of the fluid velocities and pressures follows only under the classical inf-sup condition (3.1); (3.2) for pressure stability. Thus, all div-stable elements used to approximate the Stokes problem may be used also when the no-slip condition is imposed weakly. On the other hand, optimal convergence of the Lagrange multiplier λ^h to $-(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$ will require a discrete analog of the inf-sup condition presented in Lemma 2.1 to hold.

THEOREM 3.1. *Suppose (3.1) and (3.2) hold. Let $(\mathbf{u}^h, p^h, \lambda^h)$ satisfy (3.3). Then, with $\lambda := -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$,*

$$\begin{aligned} \left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| + \left\|p - p^h\right\| \le C \Bigg\{ &\inf_{\mathbf{v}^h \in X_0^h} \left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\| \\ &+ \inf_{q^h \in M^h} \left\|p - q^h\right\| + \inf_{\alpha^h \in L^h} \left\|\lambda - \alpha^h\right\|_{-1/2,\Gamma} \Bigg\}. \end{aligned} \tag{3.5}$$

PROOF. Let $e = \mathbf{u} - \mathbf{u}^h$ denote the velocity error. Subtracting the equations for $\mathbf{u}$ (in Lemma 3.2) and $\mathbf{u}^h$ yields:

$$a\left(\mathbf{e}, \mathbf{v}^h\right) - \left(p - p^h, \nabla \cdot \mathbf{v}^h\right) + \left(\lambda - \lambda^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = 0, \tag{3.6}$$

where $\lambda = -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$. If $\mathbf{v} \in \hat{V}^h$ this simplifies to:

$$a\left(\mathbf{e}, \mathbf{v}^h\right) - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right) + \left(\lambda - \lambda^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = 0,$$

for any $\mathbf{v}^h \in \hat{V}^h$, $q^h \in M^h$. Further, by the definition of $\hat{V}^h$, $(-\lambda^h, \mathbf{v}^h \cdot \hat{\tau})_\Gamma = (\lambda - \alpha^h, \mathbf{v}^h \cdot \hat{\tau})_\Gamma$ for any $\alpha^h \in L^h$. Writing $\mathbf{e} = \mathbf{u} - \mathbf{u}^h = (\mathbf{u} - \mathbf{v}^h) - (\mathbf{u}^h - \mathbf{v}^h) = \eta - \phi^h$, where $\eta = \mathbf{u} - \mathbf{v}^h, \mathbf{v}^h \in \hat{V}^h$, and hence, $\phi^h := (\mathbf{u}^h - \mathbf{v}^h) \in \hat{V}^h$, we have:

$$a\left(\eta, \mathbf{v}^h\right) - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right) + \left(\lambda - \alpha^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = a\left(\phi^h, \mathbf{v}^h\right),$$

where $\lambda = -(\nabla \mathbf{v} \cdot \hat{n}) \cdot \tau$ and $\mathbf{v}^h \in \hat{V}^h$, $\alpha^h \in L^h$ an arbitrary. Thus, setting $\mathbf{v}^h = \phi^h \in \hat{V}^h$ gives

$$\left\|\nabla \phi^h\right\|^2 \le \|\nabla \eta\| \ \left\|\nabla \phi^h\right\| + \left\|p - q^h\right\| \ \left\|\nabla \cdot \phi^h\right\| + + \left\|\lambda - \alpha^h\right\|_{-1/2,\Gamma} \left\|\phi^h \cdot \hat{\tau}\right\|_{1/2,\Gamma}.$$

Since $\|\phi^h \cdot \hat{\tau}\|_{1/2,\Gamma} \leq C(\Omega)\|\nabla\phi^h\|$ and $\|\nabla \cdot \phi^h\| \leq \|\nabla\phi^h\|$ we have, by the triangle inequality,

$$\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| \leq C\left\{\inf_{\mathbf{v}^h\in\hat{V}} \left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\| + \inf_{q^h\in M^h} \left\|p-q^h\right\| + \inf_{\alpha^h\in L^h} \left\|\lambda-\alpha^h\right\|_{-1/2,\Gamma}\right\},$$

from which the result follows. The error bound for the pressure follows exactly as in, e.g., [4,9]. ∎

The error in the approximation λ^h of λ (essentially, the tangential component of the normal stresses), can be bounded by assuming a discrete version of (2.2) holds:

$$\inf_{\substack{\alpha^h\in L^h\\ q^h\in M^h}} \sup_{\mathbf{v}^h\in X^h} \frac{\left(q^h, \nabla\cdot\mathbf{v}^h\right) - \left(\alpha^h, \mathbf{v}^h\cdot\hat{\tau}\right)_\Gamma}{\left\|\nabla\mathbf{v}^h\right\| \left(\left\|q^h\right\|^2 + \left\|\alpha^h\right\|^2_{-1/2,\Gamma}\right)^{1/2}} \geq \tilde{\beta} > 0, \qquad \text{for some } \tilde{\beta} > 0. \tag{3.7}$$

It seems quite plausible that there are reasonable spaces for which this holds. However, in the discrete case (3.7) might also be stronger than assuming both the usual velocity-pressure inf-sup condition (3.2) *and* the following velocity-multiplier condition individually hold:

$$\inf_{\alpha^h\in L^h} \sup_{\mathbf{v}^h\in X^h} \frac{\left(\alpha^h, \mathbf{v}^h\cdot\hat{\tau}\right)_\Gamma}{\left\|\nabla\mathbf{v}^h\right\| \left\|\alpha^h\right\|_{1/2,\Gamma}} \geq \beta''' > 0, \qquad \text{for some } \beta''' > 0. \tag{3.8}$$

In the error bound for $\|\lambda - \lambda^h\|$ we shall therefore assume (3.8) instead of (3.7).

We shall show in Proposition 3.1 (later in this section) that the MINI element can be modified quite easily to satisfy (3.8). By an argument taking advantage of particular features of the MINI element, we shall also show in Proposition 3.2 that the modified MINI element satisfies (3.7) as well.

THEOREM 3.2. *Suppose (3.8) holds. Then, with $\lambda = -(\nabla\mathbf{u}\cdot\hat{n})\cdot\hat{\tau}$, we have*

$$\begin{aligned}\left\|\lambda-\lambda^h\right\|_{-1/2,\Gamma} \leq \left(\frac{1}{\beta'''}\right) \Big[&\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| \\ &+ \left\|p-p^h\right\| + \left(1+\frac{1}{\beta'''}\right) \inf_{\alpha^h\in L^h} \left\|\lambda-\alpha^h\right\|_{-1/2,\Gamma}\Big].\end{aligned} \tag{3.9}$$

PROOF. Condition (3.8) implies that for any $\alpha^h \in L^h$, $q^h \in M^h$

$$\beta''' \left\|\lambda^h - \alpha^h\right\|^2_{1/2,\Gamma} \leq \sup_{\mathbf{v}^h\in X^h} \frac{\left(\lambda^h-\alpha^h, \mathbf{v}^h\cdot\hat{\tau}\right)_\Gamma}{\left\|\nabla\mathbf{v}^h\right\|}.$$

The error equation (3.6) implies (after adding and subtracting terms)

$$\begin{aligned}&-\left(\lambda^h-\alpha^h, \mathbf{v}^h\cdot\hat{\tau}\right)_\Gamma = -a\left(\mathbf{e},\mathbf{v}^h\right) + \left(p-p^h, \nabla\cdot\mathbf{v}^h\right) - \left(\lambda-\alpha^h, \mathbf{v}^h\cdot\hat{\tau}\right)_\Gamma \\ &\leq \left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| \left\|\nabla\mathbf{v}^h\right\| + \left\|p-p^h\right\| \left\|\nabla\mathbf{v}^h\right\| + \left\|\lambda-\alpha^h\right\|_{-1/2,\Gamma} \left\|\mathbf{v}^h\cdot\hat{\tau}\right\|_{1/2,p} \\ &\leq \left(\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| + \left\|p-q^h\right\| + \left\|\lambda-\alpha^h\right\|_{-1/2,\Gamma}\right) \left\|\nabla\mathbf{v}^h\right\|.\end{aligned}$$

Thus,

$$\beta''' \left\|\lambda^h-\alpha^h\right\|_{-1/2,\Gamma} \leq \left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| + \left\|p-p^h\right\| + \left\|\lambda-\alpha^h\right\|_{1/2,\Gamma},$$

and the triangle inequality yields

$$\beta''' \left\|\lambda-\lambda^h\right\|_{-1/2,\Gamma} \leq \left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\| + \left\|p-p^h\right\| + (1+\beta''') \inf_{\alpha^h\in L^h} \left\|\lambda-\alpha^h\right\|_{-1/2,\Gamma},$$

which is the claimed result. ∎

REMARK. Bounds for $\|\lambda - \lambda^h\|_{-1/2,\Gamma}$ may also be given if a discrete version of Lemma 2.4 is assumed. However, the inf-sup condition (3.8) seems more natural and possibly easier verify than discrete versions of the condition in Lemma 2.4.

With the estimates (3.5) and (3.9), one can begin considering examples of spaces which balance the error terms on the right side of (3.5). To do this, the $H^{-1/2}(\Gamma)$ norm of $\lambda - \alpha^h$ must be estimated. This estimation is aided by the following fairly standard lemma.

LEMMA 3.3. *Let L^h consist of L^2 piecewise polynomials of degree k'. If $\mathbf{u} \in H^{k+1}(\Omega)$ for some $k \geq 0$, $\lambda = -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau}$, and $r = \min\{k'+1, k-1/2\}$,*

$$\inf_{\alpha^h \in L^h} \|\lambda - \alpha^h\|_{0,\Gamma} \leq Ch^r |\lambda|_{r,\Gamma} \leq Ch^r |\mathbf{u}|_{r+3/2,\Omega},$$

$$\inf_{\alpha^h \in L^h} \|\lambda - \alpha^h\|_{-1/2,\Gamma} \leq Ch^{r+1/2} |\lambda|_{r,\Gamma} \leq Ch^{r+1/2} |\mathbf{u}|_{r+3/2,\Omega}.$$

PROOF. If $u \in H^{\ell+1}(\Omega)$, then by the trace theorem, $\lambda \in H^{\ell-1/2}(\Gamma)$ and the first inequality follows from standard approximation theory in finite element spaces, e.g., [10].

The second inequality follows essentially from the definition of the $H^{-1/2}(\Gamma)$ norm. Indeed, if we choose α^h to be the best L^2 approximation of λ in L^h then $(\lambda - \alpha^h) \perp L^h$. Thus, for any s$g \in L^2(\Gamma)$

$$\left(\lambda - \alpha^h, g\right)_\Gamma = \left(\lambda - \alpha^h, g - g^h\right)_\Gamma,$$

holds for any $g^h \in L^h$. Thus,

$$\|\lambda - \alpha^h\|_{-1/2,\Gamma} = \sup_{g \in H^{1/2}(\Gamma)} \frac{\left(\lambda - \alpha^h, g\right)_\Gamma}{\|g\|_{1/2,\Gamma}}.$$

Now, by the L^2 error estimate,

$$\left(\lambda - \alpha^h, g\right)_\Gamma = \left(\lambda - \alpha^h, g - g^h\right)_\Gamma \leq \|\lambda - \alpha^h\|_\Gamma \inf_{g^h \in L^h} \|g - g^h\|_\Gamma \leq \|\lambda - \alpha^h\| \, Ch^{1/2} \|g\|_{1/2,\Gamma},$$

and hence,

$$\|\lambda - \alpha^h\|_{-1/2,\Gamma} \leq Ch^{1/2} \|\lambda - \alpha^h\|_\Gamma$$

and the second inequality follows. ∎

REMARK. Balancing error terms in (3.5). Suppose (X^h, M^h) contains C^0 of piecewise degree $(k, k-1)$ polynomials (a common case) and $\mathbf{u} \in (H^{k+1}(\Omega) \cap \overset{\circ}{H}{}^1(\Omega))^2$. Then, (3.5) implies

$$\|\nabla(\mathbf{u} - \mathbf{u}^h)\| \leq Ch^k(|\mathbf{u}|_{k+1,\Omega} + |p|_{k,\Omega}) + +C \inf_{\alpha^h \in L^h} \|\lambda - \alpha^h\|_{-1/2,\Gamma}.$$

If L^2 contains L^2 piecewise polynomials of degree k', then the last lemma implies

$$\inf_{\alpha^h \in L^h} \|\lambda - \alpha^h\|_{-1/2,\Gamma} \leq Ch^{r+1/2} |\mathbf{u}|_{r+3/2,\Omega},$$

$r = \min\{k'+1, k-1/2\}$. If we wish $r + 1/2 = k$ then, formally, $k' + 1 = k - 1/2$ or $k' = k - 3/2$ and, in this case,

$$\|\nabla(\mathbf{u} - \mathbf{u}^h)\| \leq Ch^k(|\mathbf{u}|_{k+1,\Omega} + |p|_{k,\Omega}).$$

In this example, we would choose L^h to be piecewise polynomials of degree $k - 1$.

It remains to be shown that it is possible to construct reasonable examples of spaces (X^h, M^h, L^h) which satisfy (3.2) *and* (3.8). We shall here begin this by showing how, beginning with a specific space (the "MINI" element) satisfying (3.2), condition (3.8) can be satisfied by adding quadratic edge bubble functions to X^h for the edges which lie on Γ.

Let $\Pi^h(\Omega)$ denote an edge-to-edge, locally quasi-uniform mesh on the polygonal domain Ω containing triangles (denoted by "T") and edges (denoted by "e"). The usual C^0, piecewise linear basis function associated with a vertex node (x_i, y_i) of $\Pi^h(\Omega)$ is denoted $\phi_i(x, y)$, and

$$b_T(x, y) := (\phi_i \phi_j \phi_k)(x, y), \qquad \text{for } (x_i, y_i), (x_j, y_j), (x_k, y_k) \text{ nodes of } T,$$

denotes the usual cubic bubble function on T. For an edge $e \subset \Gamma$ with endpoints $(x_i, y_i), (x_j, y_j)$

$$b_e(x, y) := (\phi_i \phi_j)(x, y), \qquad \text{for } e \subset \Gamma,$$

denotes the quadratic bubble associated with the midpoint of e. Define (Y^h, M^h) be the MINI element of [11] (adapted to the no penetration boundary condition), defined by:

$$\begin{aligned} M^h &:= \text{span } \{\phi_1 : \text{all vertice } (X_i, Y_i) \in \Pi^h(\Omega)\} \cap L_0^2(\Omega), \\ Y^h &:= \left[\text{span } \{\phi_1 : \text{vertices } (X_i, Y_i) \in \Pi^h(\Omega)\} \oplus \{b_T : \text{all } T \in \Pi^h(\Omega)\}\right]^2 \\ &\quad \cap \left\{\mathbf{w} \in H^1(\Omega)^2 : \mathbf{w} \cdot \hat{n} = 0, \text{ a.e. } (x,y) \in \Gamma\right\}. \end{aligned}$$

L^h will denote the piecewise constants on Γ:

$$L^h := \left\{\alpha^h(x,y) \in L^2(\Gamma) : \alpha^h \text{ is constant on each edge } e \in \Pi^h(\Omega), \text{ with } e \subset \Gamma\right\}.$$

X^h will denote Y^h augmented by quadratic edge bubbles $\psi_e(x,y)$ satisfy $\psi_e(x,y) \cdot \hat{n}_{|_e} = 0$, for $e \subset \Gamma$. Indeed, for an edge $e \subset \Gamma$, form $\psi_e(x,y) = (a(1,0)\, b_e(x,y) + b(0,1)\, b_e(x,y))$.

The coefficients a, b in $\psi_e(x,y)$ are chosen so that $\psi_e(x,y) \cdot \hat{n}_{|_e} = 0$. Define

$$X^h := Y^h \oplus \text{span } \left\{\psi_e : \text{all edges } e \in \Pi^h(\Omega) \text{ with } e \subset \Gamma\right\}.$$

PROPOSITION 3.1. *The finite element spaces satisfy the velocity-pressure* inf-sup *condition (3.2) and the velocity-stress multiplier* inf-sup *condition (3.8).*

PROOF. The MINI element (Y_0^h, M^h) satisfies (3.1) [11], thus it must also satisfy (3.2) since $Y_0^h \subset X^h$.

To show that (3.8) holds, we shall adapt an argument used by Verfürth [7,12] to balance the effects of boundary and interior norms in the (different) context of slip boundary conditions. First, note that by the assumption of local mesh quasi-uniformity a simple scaling argument gives, for $e \subset \partial T \cap \Gamma$:

$$\frac{1}{C} \leq \|\nabla \psi_e\|_{L^2(T)} \leq C. \tag{3.10}$$

Let $\chi^h \in L^h$ be given and define

$$\tilde{\mathbf{w}}(x,y) := \sum_{\substack{\text{all edges} \\ e \text{ on } \Gamma}} \chi^h(e) \alpha_e^h(x,y), \tag{3.11}$$

and consider $\int_T \alpha^h \tilde{\mathbf{w}}^h \cdot \hat{\tau}\, ds$. Using the fact that elements of L^h are piecewise constant gives:

$$\begin{aligned} \int_T \alpha^h \tilde{\mathbf{w}}^h \cdot \hat{\tau}\, ds &= \sum_{\substack{\text{all edges} \\ e\subset\Gamma}} \alpha^h(e)^2 \int_{\Gamma^h} \psi_e^h(x,y) \cdot \hat{\tau}\, ds \\ &\geq C \sum_{\substack{\text{all edges} \\ e\subset\Gamma}} h_e\, \alpha^h(e)^2 = C \left\|\alpha^h\right\|_\Gamma^2. \end{aligned}$$

Consider also $\|\nabla \tilde{\mathbf{w}}^h\|$. Using (3.11) gives:

$$\left\|\nabla \tilde{\mathbf{w}}^h\right\|^2 = \sum_{\substack{\text{all } T \text{ with an} \\ \text{edge } e\subset\Gamma}} \alpha^h(e)^2 \left\|\nabla \psi_e^h\right\|_T^2 \leq \text{using (3.10)} \leq Ch^{-1} \sum_{e\subset\Gamma} \alpha^h(e)^2 h \leq Ch^{-1} \left\|\alpha^h\right\|_\Gamma^2.$$

Thus, the following inf-sup condition holds (with the "wrong" norm on α^h)

$$\inf_{\alpha^h\in L^h} \sup_{\mathbf{w}^h\in \text{ span } \{\psi_e\}} \frac{\int_\Gamma \alpha^h \mathbf{w}^h \cdot \hat{\tau}\, ds}{\|\alpha^h\|_\Gamma \ \|\nabla \mathbf{w}^h\|} \geq Ch^{1/2} > 0. \tag{3.12}$$

Now, fix $\alpha^h \in L^h$ and let $\hat{\alpha} \in H^{1/2}(\Gamma)$ satisfy

$$\|\hat{\alpha}\|_{1/2,\Gamma} = 1, (\alpha^h, \hat{\alpha})_\Gamma \geq C \frac{1}{2} \left\|\alpha^h\right\|_{-1/2,\Gamma}.$$

Let $\hat{\mathbf{u}}$ be the solution of

$$\nabla \cdot \hat{\mathbf{u}} = 0, \quad \text{in } \Omega, \qquad \hat{\mathbf{u}} \cdot \hat{n} = 0, \quad \text{on } \Gamma, \qquad \hat{\mathbf{u}} \cdot \hat{\tau} = \hat{\alpha}, \quad \text{on } \Gamma.$$

Then $\hat{\mathbf{u}} \in X$ and (by [6])

$$\|\nabla \hat{\mathbf{u}}\| \leq C \|\hat{\chi}\|_{1/2,\Gamma} \leq C.$$

Let $I^h : X \to X^h$ be the quasi-interpolation operator [13] (which interpolates L^2 data via local averages), and define $\hat{\mathbf{u}}^h := I^h \mathbf{u}$ so that $\|\nabla \hat{\mathbf{u}}^h\| \leq C \|\nabla \hat{\mathbf{u}}\| \leq C$ (see [13]).

Consider,

$$\begin{aligned}
\int_\Gamma \alpha^h \hat{\mathbf{u}}^h \cdot \hat{\tau}\, ds &= \int_\Gamma \alpha^h \hat{\mathbf{u}} \cdot \hat{\tau}\, ds + \int_\Gamma \alpha^h \left(\hat{\mathbf{u}}^h - \hat{\mathbf{u}}\right) \cdot \hat{\tau}\, ds \\
&\geq \int_\Gamma \alpha^h \hat{\alpha}\, ds - \left\|\alpha^h\right\|_\Gamma \left\|\hat{\mathbf{u}}^h - \hat{\mathbf{u}}\right\|_\Gamma \\
&\geq \frac{1}{2} \left\|\alpha^h\right\|_{-1/2,\Gamma} - C h^{1/2} \left\|\alpha^h\right\|_\Gamma \|\nabla \mathbf{u}\| \\
&\geq \frac{1}{2} \left\|\alpha^h\right\|_{-1/2,\Gamma} - C h^{1/2} \left\|\alpha^h\right\|_\Gamma,
\end{aligned}$$

where the properties of I^h and $\hat{\mathbf{u}}_{|\Gamma}$ were used on the RHS. Since $\|\alpha^h\|_{-1/2,\Gamma} = 1$, it follows that

$$\int_\Gamma \alpha^h \hat{\mathbf{u}}^h \cdot \hat{\tau}\, ds \geq \frac{1}{2} - C h^{1/2} \left\|\alpha^h\right\|_\Gamma. \tag{3.13}$$

Combining this estimate with the previous lower bound for $\int_\Gamma \alpha^h \hat{\mathbf{u}}^h \cdot \hat{\tau}\, ds$ (i.e., (3.12) and (3.13)) gives:

$$\int_\Gamma \alpha^h \hat{\mathbf{u}}^h \cdot \hat{\tau}\, ds \geq \max\left\{\frac{1}{2} - C_1 h^{1/2} \left\|\alpha^h\right\|_\Gamma, C_2 h^{1/2} \left\|\alpha^h\right\|_\Gamma\right\}.$$

Note that since $h^{1/2}\|\alpha^h\|_\Gamma > 0$ it follows that

$$\max\left\{\frac{1}{2} - C_1 h^{1/2} \left\|\alpha^h\right\|_\Gamma, C_2 h^{1/2} \left\|\alpha^h\right\|_\Gamma\right\} \geq \min_{s>0} \max\left\{\frac{1}{2} - C_1, C_2 s\right\}.$$

A sketch of $y = (1/2) - C_1 s$ and $y = C_2 s$ reveals that there is a positive constant $\beta''' > 0$ such that $\min\limits_{s>0} \max\{(1/2) - C_1 s, C_2 s\} = \beta > 0$. Thus,

$$\inf_{\alpha^h \in L^h} \sup_{\mathbf{u}^h \in X^h} \frac{\int_\Gamma \alpha^h \mathbf{u}^h \cdot \hat{\tau}}{\|\alpha^h\|_{-1/2,\Gamma}\ \|\nabla \mathbf{u}^h\|} \geq \beta''' > 0,$$

which is the desired inf-sup condition. ∎

To prove the combined inf-sup condition (3.7) we will use the following lemma and sharpen the result of Proposition 3.1.

LEMMA 3.4. *Suppose (X_0^h, M^h) satisfies the* inf-sup *condition (3.1) and suppose, (V^h, L^h), where $V^h := \{\mathbf{v} \in X^h : (\nabla \cdot \mathbf{v}, q^h) = 0,\ \forall\, q^h \in M^h\}$, satisfies*

$$\inf_{\alpha^h \in L^h} \sup_{\mathbf{v}^h \in V^h} \frac{\left(\alpha^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma}{\|\nabla \mathbf{v}^h\|\ \|\alpha^h\|_{-1/2,\Gamma}} \geq C > 0. \tag{3.14}$$

Then, (X^h, M^h, L^h) satisfies the inf-sup *condition (3.7).*

PROOF. Let $\|q^h\|^2 + \|\alpha^h\|^2_{-1/2,\Gamma} = 1$. We shall construct a function $\mathbf{w}^h \in X^h$ with $\|\nabla \mathbf{w}^h\| \leq C < \infty$ and

$$(\nabla \cdot \mathbf{w}^h, q^h) - (\alpha^h, \mathbf{w}^h \cdot \hat{\tau})_\Gamma \geq C > 0,$$

thereby proving the result.

Since the (X_0^h, M^h) inf-sup condition (3.1) holds, we can find a $\hat{\mathbf{w}}^h$ with $\|\nabla \hat{\mathbf{w}}^h\| \leq C$ and $(q^h, \nabla \cdot \hat{\mathbf{w}}^h) \geq C_1 \|q^h\|$. Note that $\hat{\mathbf{w}}^h_{|\Gamma} = 0$. Similarly, by (3.13) we can construct a $\tilde{\mathbf{w}}^h$ with $\|\nabla \tilde{\mathbf{w}}^h\| \leq C$ and $(\alpha^h, \tilde{\mathbf{w}}^h \cdot \hat{\tau})_\Gamma \geq C_2 \|\alpha^h\|_{-1/2,\Gamma}$. Note also that $\nabla \cdot \tilde{\mathbf{w}}^h \perp M^h$ in $L^2(\Omega)$. Let $\mathbf{w}^h = \hat{\mathbf{w}}^h + \tilde{\mathbf{w}}^h$. Then, $\|\nabla \mathbf{w}\| \leq C$ and

$$(q^h, \nabla \cdot \mathbf{w}^h) - (\alpha^h, \mathbf{w}^h \cdot \hat{\tau})_\Gamma = (q^h, \nabla \cdot \hat{\mathbf{w}}^h) - (\alpha^h, \tilde{\mathbf{w}}^h \cdot \hat{\tau})_\Gamma \geq C_2 \|\alpha^h\|_{-1/2,\Gamma} + C_1 \|q^h\| > 0,$$

which proves the result. ∎

It remains to show that the MINI element satisfies the condition (3.14) in (V^h, L^h). To do this, we will begin with the analogous condition in (X^h, L^h) and use the internal bubbles $\psi_T(x, y)$ in an essential way.

PROPOSITION 3.2. *The space (X^h, M^h, L^h) satisfies (3.14), and hence, (3.7) as well.*

PROOF. Given $\alpha^h \in L^h$, $\|\alpha^h\|_{-1/2,\Gamma} = 1$ by Proposition 3.1 we can find $\mathbf{w}^h$ with

$$\|\nabla \mathbf{w}^h\| \leq C < \infty \qquad \text{and} \qquad (\alpha^h, \mathbf{w}^h \cdot \hat{\tau})_\Gamma \geq C > 0. \tag{3.15}$$

We shall show now (by a construction entirely analogous to the one used in [11] to verify div-stability of the MINI element) that this $\mathbf{w}^h$ can be modified using the bubbles $b_T(x, y)$ to lie in V^h and still satisfy (3.15). Let $q^h \in M^h$ then $\nabla q^h|_T$ is a constant vector on each triangle T. Since $\mathbf{v} \cdot \hat{n}|_\Gamma = 0$, integration by parts gives:

$$(q^h, \nabla \cdot \mathbf{v}^h) = -\sum_T \int_T \nabla q^h \cdot \mathbf{v}\, dx,$$

so that $(q^h, \nabla \cdot \mathbf{v}^h) = 0$ is implied by $\int_T \mathbf{v}_1^h\, dx = 0$ and $\int_T \mathbf{v}_2^h\, dx = 0$. For each triangle $T \in \Pi^h(\Omega)$ define a_1, a_2 by requiring $\int_T \mathbf{w}_j^h(x) - a_j b_T(x)\, dx = 0$, $j = 1, 2$. Thus,

$$\mathbf{v}^h(x) := \mathbf{w}^h(x) - \sum_{\text{all } T \in \Pi^h(\Omega)} [a_1(1,0) + a_2(0,1)] b_T(x)$$

satisfies $\int_T \mathbf{v}^h\, dx = 0$ for all T, and hence, $\nabla \cdot \mathbf{v}^h \perp M^h$ in $L^2(\Omega)$, or $\mathbf{v}^h \in V^h$. That $\|\nabla \mathbf{v}^h\| \leq C \|\nabla \mathbf{w}^h\| \leq C$, follows by a scaling argument entirely analogous to the scaling argument used to bound the bubble contributions in the proof in [11]. Note that $\mathbf{v} \cdot \hat{\tau}|_\Gamma = \mathbf{w}^h \cdot \hat{\tau}|_\Gamma$ so $\mathbf{v}^h \in V^h$ satisfies

$$\|\nabla \mathbf{v}^h\| \leq C \qquad \text{and} \qquad (\alpha^h, \mathbf{v}^h \cdot \hat{\tau})_\Gamma \geq C > 0,$$

and the result follows. ∎

4. PENALTY METHODS

In this section, we consider the replacement of the Lagrange multiplier imposition of the constraint $\mathbf{u} \cdot \hat{\tau}|_\Gamma = 0$ by a penalty function approach. To this end, let $\epsilon > 0$ be given and consider the following problem. Find $\mathbf{u}_\epsilon \in V$ minimizing over V

$$J_\epsilon(\mathbf{v}) := \int_\Omega \frac{1}{2} |\nabla \mathbf{v}|^2 - \mathbf{f} \cdot \mathbf{v}\, dx + \frac{1}{2} \epsilon^{-1} \int_\Gamma (\mathbf{v} \cdot \hat{\tau})^2\, ds. \tag{4.1}$$

The penalized functional $J_\epsilon(\cdot)$ is quadratic so its Euler-Lagrange equation is easily found to be the following. Find $\mathbf{u}_\epsilon \in V$ satisfying

$$a(\mathbf{u}_\epsilon, \mathbf{v}) + \epsilon^{-1} (\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v} \cdot \hat{\tau})_\Gamma = (\mathbf{f}, \mathbf{v}), \qquad \forall\, \mathbf{v} \in V.$$

To facilitate the error analysis it is helpful to write this as: find $\mathbf{u}_\epsilon \in V$, (where $\lambda_\epsilon := \epsilon^{-1}\mathbf{u}_\epsilon \cdot \hat{\tau}$) satisfying

$$\begin{aligned} a(\mathbf{u}_\epsilon, \mathbf{v}) + (\lambda_\epsilon, \mathbf{v} \cdot \hat{\tau})_\Gamma &= (\mathbf{f}, \mathbf{v}), && \text{for all } \mathbf{v} \in V, \\ (\mathbf{u}_\epsilon \cdot \hat{\tau}, \alpha)_\Gamma &= \epsilon(\lambda_\epsilon, \alpha)_\Gamma, && \text{for all } \alpha \in L. \end{aligned} \tag{4.2}$$

REMARK: SLIP WITH RESISTANCE. If the no slip condition $\mathbf{u} \cdot \hat{\tau}|_\Gamma = 0$ is weakened by replacing adherence with slip with linear resistance to slippage we have the following boundary value problem. For $\epsilon > 0$ find $(\mathbf{u}_\epsilon, p_\epsilon)$ satisfying

$$\begin{aligned} &-\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = f, \quad \nabla \cdot \mathbf{u}_\epsilon = 0, \qquad \text{in } \Omega, \\ &\mathbf{u}_\epsilon \cdot \hat{n}_{|_\Gamma} = 0, \quad \text{and} \quad \mathbf{u}_\epsilon \cdot \hat{\tau} + \epsilon(\nabla \mathbf{u}_\epsilon \cdot \hat{n}) \cdot \hat{\tau} = 0, \qquad \text{on } \Gamma. \end{aligned}$$

(Naturally, the requirement of frame invariance in the continuous problem suggests that $\nabla \mathbf{u}_\epsilon$ should be replaced by $(\nabla \mathbf{u}_\epsilon + \nabla \mathbf{u}_\epsilon^t)/2$. This modification is discussed below.) The variational formulation of this problem with modified boundary conditions is: find $(\mathbf{u}_\epsilon, p_\epsilon) \in (X, M)$ satisfying

$$(\nabla \mathbf{u}_\epsilon, \nabla \mathbf{v}) - (p_\epsilon, \nabla \cdot \mathbf{v}) + \frac{1}{\epsilon}\,(\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v} \cdot \hat{\tau})_\Gamma = (\mathbf{f}, \mathbf{v}), (q, \nabla \cdot \mathbf{u}_\epsilon) = 0,$$

for all $(\mathbf{v}, q) \in (X, M)$. This variational formulation is exactly the same as the penalty formulation (4.1) above.

The form of the resistance term in the boundary condition is related to the choice made for the viscous term. A more physically correct boundary condition would involve the tangential component of the stress vector, $\vec{\mathbf{t}} \cdot \hat{\tau} = (\mathcal{D}(\mathbf{u}_\epsilon) \cdot \hat{n}) \cdot \hat{\tau}$ rather than $(\nabla \mathbf{u}_\epsilon \cdot \hat{n}) \cdot \hat{\tau}$, where $\mathcal{D}(\mathbf{u}) = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$ is the deformation tensor. The variational formulation then becomes find: $(\mathbf{u}_\epsilon, p_\epsilon) \in (X, M)$ satisfying

$$(\mathcal{D}(\mathbf{u}_\epsilon), \mathcal{D}(\mathbf{v})) - (p_\epsilon, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}_\epsilon) + \frac{1}{\epsilon}\,(\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v}_\epsilon \cdot \hat{\tau})_\Gamma = (\mathbf{f}, \mathbf{v}),$$

for all $(\mathbf{v}, q) \in (X, M)$. Again, there is an equivalence between linear friction and penalty methods. Thus, to obtain a new algorithm from a slip with resistance boundary condition for approximating the no-slip condition, nonlinear resistance to slippage (as proposed by Serrin [1]) would be needed.

PROPOSITION 4.1. *Let $(\mathbf{u}, p, \lambda), (\lambda = -(\nabla \mathbf{u} \cdot \hat{n}) \cdot \hat{\tau})$ be the solution of the Stokes problem and let $(\mathbf{u}_\epsilon, \lambda_\epsilon), (\lambda_\epsilon = \epsilon^{-1}\, \mathbf{u}_\epsilon \cdot \hat{\tau}_{|_\Gamma})$ be the solution of (3.1) or (3.2). Then,*

$$\|\lambda - \lambda_\epsilon\|_{-1/2,\Gamma} + \|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\| \le \epsilon\, C\, \|\lambda\|_{1/2,\Gamma}.$$

PROOF. Subtracting (4.2) from the corresponding equations for $(\mathbf{u}, \lambda)$ gives:

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v}) + (\lambda - \lambda_\epsilon,\ \mathbf{v} \cdot \hat{\tau})_\Gamma &= 0, \\ ((\mathbf{u} - \mathbf{u}_\epsilon) \cdot \hat{\tau}, \alpha)_\Gamma + \epsilon(\lambda_\epsilon, \alpha)_\Gamma &= 0, \end{aligned} \tag{4.3}$$

for all $\mathbf{v} \in V$, $\alpha \in L$.

Applying the continuous inf-sup condition gives:

$$\begin{aligned} \beta' \|\lambda - \lambda_\epsilon\|_{-1/2,\Gamma} &\le \sup_{0 \ne \mathbf{v} \in V} \frac{(\lambda - \lambda_\epsilon, \mathbf{v} \cdot \hat{\tau})_\Gamma}{\|\mathbf{v}\|_{1/2,\Gamma}} \qquad \text{(by (3.3))} \\ &= \sup_{\mathbf{v} \in V} \frac{a(\mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v})}{\|\mathbf{v}\|_{1/2,\Gamma}} \\ &\le \left(\sup_{\mathbf{v} \in V} \frac{\|\nabla \mathbf{v}\|}{\|\mathbf{v}\|_{1/2,\Gamma}}\right) \|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\| \le C(\Omega)\|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\|, \end{aligned} \tag{4.4}$$

by Lemma 2.4. To proceed further, take $\mathbf{v} = \mathbf{u} - \mathbf{u}_\epsilon$ and $\alpha = \lambda - \lambda_\epsilon$ in (4.3). This gives

$$\|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\|^2 + (\lambda - \lambda_\epsilon, (\mathbf{u} - \mathbf{u}_\epsilon) \cdot \hat{\tau})_\Gamma = 0, \qquad \text{and}$$
$$((\mathbf{u} - \mathbf{u}_\epsilon) \cdot \hat{\tau}, \lambda - \lambda_\epsilon)_\Gamma + \epsilon(\lambda_\epsilon, \lambda - \lambda_\epsilon)_\Gamma = 0.$$

Thus,

$$\|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\|^2 = \epsilon(\lambda_\epsilon, \lambda - \lambda_\epsilon)_\Gamma = \epsilon(\lambda, \lambda - \lambda_\epsilon)_\Gamma - \epsilon(\lambda - \lambda_\epsilon, \lambda - \lambda_\epsilon)_\Gamma \le \epsilon(\lambda, \lambda - \lambda_\epsilon)_\Gamma$$
$$\le \epsilon\|\lambda\|_{1/2,\Gamma}\|\lambda - \lambda_\epsilon\|_{-1/2,\Gamma}.$$

Using (4.4) in the RHS of the above gives

$$\|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\|^2 \le \epsilon\|\lambda\|_{1/2,\Gamma} C(\Omega)\|\nabla(\mathbf{u} - \mathbf{u}_\epsilon)\|. \tag{4.5}$$

Equations (4.5) and (4.4) combine to prove the proposition. ∎

We can now pose the finite element-penalty method as the usual finite element approximation to the regularized problem (4.1),(4.2). We compute $(\mathbf{u}_\epsilon^h, p_\epsilon^h,) \in (X^h, M^h)$ satisfying

$$a\left(\mathbf{u}_\epsilon^h, \mathbf{v}^h\right) - \left(p^h, \nabla \cdot \mathbf{v}^h\right) + \epsilon^{-1}\left(\mathbf{u}_\epsilon^h \cdot \hat{\tau}, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = \left(\mathbf{f}, \mathbf{v}^h\right), \left(\nabla \cdot \mathbf{u}_\epsilon^h, q^h\right) = 0, \tag{4.6}$$

for all $(\mathbf{v}^h, q^h) \in (X^h, M^h)$. This is equivalent to finding $\mathbf{u}_\epsilon^h \in V^h$ satisfying

$$a\left(\mathbf{u}_\epsilon^h, \mathbf{v}^h\right) + \epsilon^{-1}\left(\mathbf{u}_\epsilon^h \cdot \hat{\tau}, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = \left(\mathbf{f}, \mathbf{v}^h\right), \tag{4.7}$$

for all $\mathbf{v}^h \in V^h$. For penalty methods, the discrete multiplier space L^h is not chosen but rather induced by X^h via

$$L^h := \left\{\mathbf{v}^h \cdot \hat{\tau} : \mathbf{v}^h \in X^h\right\}.$$

The true solution to (4.2) satisfies: for $\mathbf{v}^h \in V^h, \alpha^h \in L^h$ (where $\lambda_\epsilon = \epsilon^{-1}\mathbf{u}_\epsilon \cdot \tau$)

$$\begin{aligned} a\left(\mathbf{u}_\epsilon, \mathbf{v}^h\right) + \left(\lambda_\epsilon, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right) &= \left(\mathbf{f}, \mathbf{v}^h\right), \\ \left(\mathbf{u}_\epsilon \cdot \hat{\tau}, \alpha^h\right)_\Gamma &= \epsilon\left(\lambda_\epsilon, \alpha^h\right). \end{aligned} \tag{4.8}$$

PROPOSITION 4.2. *Define* $\lambda_\epsilon := \epsilon^{-1}\mathbf{u}_\epsilon \cdot \hat{\tau}$ *and* $\lambda_\epsilon^h := \epsilon^{-1}\mathbf{u}_\epsilon^h \cdot \hat{\tau}$. *Then,*

$$\begin{aligned} &\left\|\nabla\left(\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h\right)\right\|^2 + \left\|\lambda_\epsilon - \lambda_\epsilon^h\right\|^2 \\ &\quad \le C \inf_{q^h \in M^h} \left\|p - q^h\right\|^2 + \inf_{\mathbf{v}^h V^h} \left\{\left\|\nabla\left(\mathbf{u}_\epsilon - \mathbf{v}_\epsilon^h\right)\right\|^2 + \epsilon^{-1}\left\|\left(\mathbf{u}_\epsilon - \mathbf{v}^h\right) \cdot \hat{\tau}\right\|_\Gamma^2\right\}. \end{aligned}$$

PROOF. Subtracting (4.6) from (4.7) with $\mathbf{e} = \mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h$ gives

$$a\left(\mathbf{e}, \mathbf{v}^h\right) - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right) + \epsilon^{-1}\left(\mathbf{e} \cdot \hat{\tau}, \mathbf{v}^h \cdot \tau\right)_\Gamma = 0, \tag{4.9}$$

for all $\mathbf{v}^h \in V^h$. Write $\mathbf{e} = (\mathbf{u}_\epsilon - \mathbf{v}^h) - (\mathbf{u}_\epsilon^h - \mathbf{v}^h)$ where $\mathbf{v}^h \in V^h$. Define $\eta := \mathbf{u}_\epsilon - \mathbf{v}^h$ and $\phi^h := \mathbf{u}_\epsilon^h - \mathbf{v}^h$. Equation (4.8) becomes

$$a\left(\phi^h, \mathbf{v}^h\right) + \epsilon^{-1}\left(\phi^h \cdot \hat{\tau}, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma = a\left(\eta, \mathbf{v}^h\right) + \epsilon^{-1}\left(\eta \cdot \hat{\tau}, \mathbf{v}^h \cdot \hat{\tau}\right) - \left(p - q^h, \nabla \cdot \mathbf{v}^h\right).$$

Setting $\mathbf{v}^h = \phi^h$ gives:

$$\begin{aligned} \left\|\nabla\phi^h\right\|^2 + \epsilon^{-1}\left\|\phi^h \cdot \hat{\tau}\right\|_\Gamma^2 &\le \frac{1}{2}\|\nabla\eta\|^2 + \frac{1}{2}\left\|\phi^h\right\|^2 + \frac{1}{2}\epsilon^{-1}\|\eta \cdot \hat{\tau}\|_\Gamma^2 + \frac{1}{2}\epsilon^{-1}\left\|\phi^h \cdot \hat{\tau}\right\|_\Gamma \\ &\quad + \frac{1}{4}\left\|\nabla\phi^h\right\|^2 + \left\|p - q^h\right\|^2. \end{aligned}$$

Thus, by the triangle inequality

$$\begin{aligned} &\left\|\nabla\left(\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h\right)\right\|^2 + \left\|\left(\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h\right) \cdot \hat{\tau}\right\|_\Gamma \le C \\ &\qquad \le \inf_{\mathbf{v}^h \in V^h} \left\{\left\|\nabla\left(\mathbf{u}_\epsilon - \mathbf{v}^h\right)\right\|^2 + \epsilon^{-1}\left\|\left(\mathbf{u}_\epsilon - \mathbf{v}^h\right) \cdot \hat{\tau}\right\|_\Gamma^2\right\} \\ &\qquad\quad + C \inf_{q \in M^h} \left\|p - q^h\right\|^2, \qquad \lambda_\epsilon := \epsilon^{-1}\mathbf{u}_\epsilon \cdot \hat{\tau}, \quad \lambda_\epsilon^h := \epsilon^{-1}\mathbf{u}_\epsilon^h \cdot \hat{\tau}, \end{aligned} \tag{4.10}$$

which is a basic error estimate for $\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h$.

To estimate $\mathbf{u} - \mathbf{u}_\epsilon^h$ we combine Propositions 4.1. and 4.2.

REFERENCES

1. J. Serrin, Mathematical principles of classical fluid mechanics, In *Encyclopedia of Physics VIII/1*, (Edited by J. Flugge and C. Truesdall), pp. 125–263, Springer, Berlin, (1959).
2. I. Babuska, The finite element method with Lagrange multipliers, *Numer. Math.* **20**, 179–192, (1973).
3. C. Bernardi, Y. Mayday and A.T. Patera, Domain decomposition by the mortar element method, In *Asymptotic and Numerical Methods for P.D.E.'s with Critical Parameters*, (Edited by H.G. Kaper and M. Garbey), pp. 269–286, (1993).
4. M. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice and Algorithms*, Academic Press, Boston, MA, (1989).
5. J.T. Oden and J.A. Martins, Models and computational methods for dynamic friction phenomena, *Comp. Meth. in Appl. Mech. and Eng.* **52**, 527–634, (1988).
6. G.P. Galdi, *Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Vol. 1; *Springer Tracts in Natural Philosophy*, Vol. 38, Revised Edition, Springer, Berlin, (1998).
7. R. Verfürth, Finite element approximation of stationary Navier-Stokes equations with slip boundary conditions, Habilitationsschrift, Report 75, Univ. Bochum, (1986).
8. A Liakos, Discretization of the Navier-Stokes equations with slip boundary conditions, Technical Report, University of Pittsburgh, PA, (1998).
9. V. Girault and P.A. Raviart, *Finite Element Methods for the Navier-Stokes Equations*, Springer, Berlin, (1986).
10. P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, (1978).
11. D. Arnold, F. Brezzi and M. Fortin, A stable finite element for the Stokes equations, *Calcolo* **21**, 337–344, (1984).
12. R. Verfürth, Finite element approximation of incompressible Navier-Stokes equations with slip boundary conditions, *Numer. Math.* **50**, 697–721, (1987).
13. P. Clément, Approximation by finite element functions using local regularization, *RAIRO Anal. Numér.* **7** (R-3), 33–76, (1977).