Weak Imposition of "No-Slip" Conditions in Finite Element Methods

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November 15, 2023

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- Consider the Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \qquad \text{in } \Omega \left(\subset \mathbb{R}^2\right) \\ \mathbf{u} = 0, & \qquad \text{on } \partial\Omega \end{aligned}$$

- The Stokes problem is a fundamental partial differential equation that describes fluid flow.
- It operates within a domain Ω in a two-dimensional space, $\mathbb{R}^2$, with specific boundary conditions where the fluid velocity $\mathbf{u}$ is zero on the boundary $\partial\Omega$.

Introduction

Motivation for Relaxed Methods

- The motivation for studying "relaxed" methods for imposing the "no-slip" condition in fluid flow problems is driven by their importance in high Reynolds number flow scenarios.
- At high Reynolds numbers, fluid behavior near boundaries plays a crucial role, leading to issues like vorticity and instability.
- The transition from adherence to free-stream velocities can be problematic.
- Consideration of three different approaches to impose the "no-slip" condition: Lagrange multiplier, penalty term, and "slip with friction."

Introduction

Method Formulation

- Define function spaces:

$$
\begin{aligned}
X &:= \left\{ \mathrm{v} \in \left(H^1(\Omega)\right)^2 : (\mathbf{v} \cdot \hat{n}, \lambda) = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\} \\
X_0 &:= \left\{ \mathrm{v} \in X : (\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\} \\
V &:= \left\{ \mathrm{v} \in X : (\nabla \cdot \mathrm{v}, q) = 0, \text{ for all } q \in L_0^2(\Omega) \right\} \\
V_0 &:= \left\{ \mathrm{v} \in V : (\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma = 0, \text{ for all } \lambda \in H^{-1/2}(\Gamma) \right\} \\
M &:= L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : (q, 1) = 0 \right\}, \text{ and} \\
L &:= H^{-1/2}(\Gamma).
\end{aligned}
$$

Introduction

Method Formulation

- The function spaces described are commonly used in the mathematical formulation of fluid dynamics problems, particularly when dealing with the Navier-Stokes equations or similar complex partial differential equations.
- These spaces are defined in the context of Sobolev spaces, which are a generalization of the classical notions of differentiation and integration to more irregular functions, suitable for dealing with partial differential equations.
- These spaces are essential in the weak formulation of fluid dynamics problems, where the equations are reformulated in a way that allows for solutions that are not necessarily smooth but still physically meaningful. This approach is particularly useful in numerical methods like finite element analysis.

Introduction

Method Formulation

- $H^1(\Omega)$:
  - This is a Sobolev space. The notation H^1 refers to functions whose first derivatives (in the sense of distributions) are square-integrable.
  - Ω typically represents a domain in $\mathbb{R}^n$ (often $n = 2$ or 3 for fluid dynamics problems).
  - $(H^1(\Omega))^2$ means vector-valued functions in this space, each component of which belongs to $H^1(\Omega)$.
- $H^{-1/2}(\Gamma)$:
  - This is a Sobolev space that consists of functions on a boundary Γ (which is part of the boundary of Ω) that have a certain regularity.
  - $H^{-1/2}$ indicates a space of functions that are "less regular" than L^2 functions. It's a kind of fractional Sobolev space.
- $L^2(\Omega)$ and $L^2_0(\Omega)$:
  - $L^2(\Omega)$ is the space of square-integrable functions on Ω. These are functions for which the integral of their square is finite.
  - $L^2_0(\Omega)$ typically refers to the subspace of $L^2(\Omega)$ consisting of functions with zero mean (i.e., $(q, 1) = 0$).

Introduction

Method Formulation

- X, X_0, V, V_0, M, L:
  - These are specific function spaces defined with respect to the above Sobolev spaces and are typically used to impose certain boundary conditions and other properties relevant to fluid dynamics problems.
  - For example, X is a space of vector-valued functions in $(H^1(\Omega))^2$ that satisfy a certain condition on the normal component $\mathbf{v} \cdot \hat{n}$ on the boundary Γ.
  - X_0, V, V_0 are further restrictions of X, each imposing additional conditions, like tangential boundary conditions or divergence-free conditions (important for incompressible flows).
  - M and L are also defined with specific roles in the formulation of fluid dynamics problems, often related to pressure spaces and boundary condition spaces, respectively.

- Stokes problem in (X_0, M):

$$a(\mathbf{u}, \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) - (p, \nabla \cdot \mathbf{u}) = (\mathbf{f}, \mathbf{v})$$

- To mathematically formulate the fluid flow problem, we define specific function spaces like X.
- The Stokes problem, when restricted to (X_0, M), is expressed through equations that model fluid behavior.

- The equation provided:

$$a(\mathbf{u}, \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

is a weak or variational formulation of the Stokes problem. Here's what each term represents:

  - $\mathbf{u}$ is the velocity field of the fluid.
  - p is the pressure field.
  - $\mathbf{f}$ represents external forces (like gravity).
  - $a(\mathbf{u}, \mathbf{v})$ is a bilinear form representing the viscous forces in the fluid, often related to the gradient of velocities.
  - $(q, \nabla \cdot \mathbf{u})$ and $(p, \nabla \cdot \mathbf{v})$ are terms involving the pressure and the divergence of the velocity field, ensuring the incompressibility condition.
  - $(\mathbf{f}, \mathbf{v})$ represents the work done by external forces.

Introduction

Method Formulation

- X_0 and M are the function spaces where the solution $\mathbf{u}$ (velocity) and p (pressure) are sought.
- X_0 typically includes vector-valued functions (for velocity) that satisfy certain boundary conditions.
- M is typically a space for scalar functions (like pressure) that are square-integrable with zero mean.
- The Stokes problem models the flow of viscous fluids at low speeds or in situations where inertial forces are negligible compared to viscous forces.
- This formulation is used in computational fluid dynamics, particularly in finite element methods, where the weak form is more suitable for numerical solutions.
- The equation balances the viscous forces, pressure gradients, and external forces in the fluid.

- In summary, the Stokes problem in the context of the function spaces (X_0, M) provides a mathematical framework to model and solve for the velocity and pressure fields in a slow-moving, viscous fluid under certain boundary conditions and external forces.

Introduction

Relaxed "No-Slip" Formulation

- $a(\mathbf{u}, \mathbf{v})$: Represents the internal forces in the fluid, typically viscous forces. It's a bilinear form involving the velocity fields $\mathbf{u}$ and $\mathbf{v}$.
- $-(p, \nabla \cdot \mathbf{v})$: Accounts for the pressure forces in the fluid. Here, p is the pressure, and $\nabla \cdot \mathbf{v}$ is the divergence of the test function $\mathbf{v}$.
- $(\mathbf{v} \cdot \hat{\tau}, \lambda)_\Gamma$: Relates to the boundary condition on the boundary Γ of the domain. $\hat{\tau}$ is the tangential vector, and λ is a Lagrange multiplier or an additional variable introduced to enforce the relaxed no-slip condition on the tangential component of velocity.
- $(\mathbf{f}, \mathbf{v})$: Represents the external forces acting on the fluid, like gravity or other external influences.
- $(q, \nabla \cdot \mathbf{u}) = 0$: Ensures that the fluid is incompressible (the divergence of $\mathbf{u}$ is zero).
- $(\mathbf{u} \cdot \hat{\tau}, \alpha)_\Gamma = 0$: This is the relaxed no-slip boundary condition, where $\mathbf{u} \cdot \hat{\tau}$ should be zero on the boundary Γ.
- X: Space for the velocity field.
- M: Space for the pressure field.
- L: Space for the Lagrange multiplier or additional variables related to the boundary conditions.

Introduction

Relaxed "No-Slip" Formulation

- The classic no-slip condition in fluid dynamics states that the velocity of the fluid at a solid boundary is equal to the velocity of the boundary (usually zero).
- The "relaxed" no-slip condition means that this constraint is somewhat loosened, often to account for slip at the boundary or for more complex boundary behaviors.
- This weak formulation provides a mathematical framework for numerically solving fluid dynamics problems where the no-slip condition is relaxed.
- It is particularly useful in computational fluid dynamics and is well-suited for numerical methods like finite element analysis, where handling complex boundary conditions is essential.

- In summary, this formulation allows for accurately modeling fluid flow with relaxed boundary conditions, ensuring that the velocity and pressure fields satisfy the desired constraints in a computationally efficient manner.

Introduction

Main Results

- **Theorem 3.1**: This theorem asserts the quasi-optimal convergence of the velocity $\mathbf{u}$ and pressure p as the mesh size h approaches zero. In other words, the numerical solution approaches the true solution efficiently.
- **Theorem 3.2**: This theorem states that the numerical approximation λ^h for the Lagrange multiplier is quasi-optimal under an additional discrete inf-sup condition. This condition ensures the stability of the numerical method.
- The suggested use of polynomial degree $(k, k-1, k-1)$ spaces indicates that using these specific polynomial degrees for the function spaces is optimal for the numerical solution.

- In conclusion, the paper presents a mathematically correct procedure for imposing the "no-slip" condition in the Stokes problem with fluid flow.
- Demonstrates that as the mesh size decreases, the numerical solution converges to the true solution, ensuring the accuracy of the approach.
- Concepts discussed have relevance beyond fluid flow problems, such as contact problems in simulations.

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Preliminaries

Introduction

- In this section, we gather some preliminary information related to the continuous problem.
- Notation: $\| \cdot \|$ and $\| \cdot \|_\Gamma$ represent $L(\Omega)$ and $L^2(\Gamma)$ norms, and $H^s(\Gamma)$ norm is denoted as $\| \cdot \|_{s,\Gamma}$.
- The aim is to verify an important inf-sup condition for the Stokes problem - to satisfy an inf-sup condition for the Stokes problem with a positive constant $\beta > 0$.
- The inf-sup condition ensures the stability of the numerical method.
- The condition involves various norms and inner products.
- This condition is critical for the success of the numerical approach.
- We can simplify the inf-sup condition using the Poincaré inequality and the trace lemma.
- These simplifications help us better understand and analyze the stability condition.

Preliminaries

Introduction

- The Poincaré Inequality provides a bound on the norm of a function in terms of the norm of its gradient. It is crucial in the analysis of PDEs, especially when proving the existence and uniqueness of solutions. The inequality states that, for a suitable function u and a bounded domain Ω, there exists a constant C such that:

$$\|u - u_{\text{mean}}\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

  where:
  - $\|u - u_{\text{mean}}\|_{L^2(\Omega)}$ is the L^2 norm of u minus its mean value over Ω.
  - $\|\nabla u\|_{L^2(\Omega)}$ is the L^2 norm of the gradient of u.
  - C is a constant that depends on the domain Ω but not on the function u.
- This inequality is particularly useful in establishing that a weak derivative (in the Sobolev space sense) is small if its gradient is small, providing a way to control the size of a function by its derivative.

Preliminaries

Introduction

- The Trace Lemma, also known as the Trace Theorem, is a result in functional analysis that concerns the restriction of a Sobolev space function to a lower-dimensional subset, such as the boundary of a domain. It guarantees that functions in a Sobolev space have well-defined and bounded traces (or restrictions) on the boundary of the domain. The theorem typically states that:
  - For a function u in a Sobolev space $H^1(\Omega)$, the trace of u on the boundary Γ of Ω exists and is in $L^2(\Gamma)$.
  - There exists a constant C such that:

$$\|u\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$$

  where:
  - $\|u\|_{L^2(\Gamma)}$ is the L^2 norm of u on the boundary Γ.
  - $\|u\|_{H^1(\Omega)}$ is the H^1 norm of u in the domain Ω.
- This theorem is essential in the analysis of boundary value problems and in the development of finite element methods, where the behavior of functions on the boundaries of domains is crucial.

Preliminaries
Useful Lemmas

- **Lemma 2.1**: This lemma provides a formal proof of the inf-sup condition (2.1).
- The proof involves constructing an auxiliary solution for $\hat{\lambda}$.
- The result is a positive constant $\beta > 0$ that ensures the stability of the numerical method.
- Another inf-sup condition (2.2) holds on the function spaces (X_0, M).
- Functions in X_0 vanish on Γ.
- We have a corresponding lemma (Lemma 2.2) that establishes this condition on (X, M).

Preliminaries

Useful Lemmas

- These conditions are important for stability and convergence.
- We define nontrivial, closed subspaces V and V_0 within X and X_0.
- These subspaces are crucial for setting up and solving the Stokes problem.
- They maintain the required constraints and relations.
- **Lemma 2.3**: This lemma introduces another important constant $\beta > 0$ related to λ and $\mathbf{v}$.
- The lemma is significant for the analysis of the numerical approach's stability.
- **Lemma 2.4**: This lemma provides a finite constant C related to the norms of gradients and function values.
- The result is useful for estimating the regularity of the solutions.

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Formulation of the FEM

Introduction

- Introduction to the finite element method with weakly imposed "no-slip" boundary conditions.
- Definition of finite element spaces X^h, M^h, and L^h.
- The FEM relies on approximating the solution using finite element spaces. Here, X^h, M^h, and L^h represent different spaces for approximating the velocity, pressure, and Lagrange multiplier, respectively.
- Assumptions on polygonal boundary Γ for well-defined spaces.
- The formulation requires certain assumptions about the geometry, such as having a polygonal boundary, to ensure that the finite element spaces are well-defined.
- Verification of the classical discrete inf-sup condition for pressure stability - crucial in FEM simulations.

Formulation of the FEM

Introduction

- Introduction of divergence-free spaces V^h and V_0^h.
- In simulations involving incompressible flows, divergence-free spaces V^h and V_0^h are introduced to handle the incompressibility constraint effectively.
- The Lagrange multiplier-FEM scheme defines the equations used to find the solution in FEM simulations.
- Equivalence between the original scheme and an alternative form.
- This slide discusses how the original scheme and an alternative form of the FEM formulation are equivalent, providing flexibility in solving the problem.
- Introduction of a discretely divergence-free space $\hat{V}^h$.
- The inclusion of discretely divergence-free spaces $\hat{V}^h$ further enhances the handling of incompressible flows in FEM simulations.

Formulation of the FEM

Error Analysis

- Error analysis for the method with respect to the continuous problem.
- Error analysis in FEM aims to understand how well the discrete solution approximates the continuous problem it represents.
- **Theorem 3.1: Convergence results under certain conditions.**
- *Explanation:* Theorem 3.1 provides convergence results, indicating that as the mesh is refined, the FEM solution approaches the true solution under certain conditions.
- **Error bounds for pressure and velocity.**
- *Explanation:* This part of the analysis establishes error bounds for both pressure and velocity, helping to quantify the accuracy of the FEM solution.

Formulation of the FEM

Analysis of the Lagrange Multiplier

- Introduction of conditions for bounding the error in the Lagrange multiplier.
- Conditions are introduced to bound the error associated with the Lagrange multiplier, a crucial component in FEM simulations of incompressible flows.
- Discussion of discrete versions of inf-sup conditions.
- The slide discusses discrete versions of the inf-sup conditions, which are essential for pressure stability in FEM simulations.
- **Theorem 3.2: Convergence results for the Lagrange multiplier.**
- *Explanation:* Theorem 3.2 provides convergence results specifically for the Lagrange multiplier, further validating its accuracy in the simulation.

Formulation of the FEM

Balancing Error Terms

- Balancing error terms in the error analysis.
- Balancing error terms is crucial in ensuring that the error analysis accurately reflects the behavior of the FEM method and that no individual error term dominates the analysis.
- Use of $H^{-1/2}(\Gamma)$ norm for estimating the Lagrange multiplier error.
- The choice of the $H^{-1/2}(\Gamma)$ norm is explained, which is used to estimate the error associated with the Lagrange multiplier.
- **Lemma 3.3: Estimations for different norms.**
- *Explanation:* Lemma 3.3 provides estimations for various norms, helping to quantify the errors in the simulation accurately.

Formulation of the FEM

Constructing Spaces

- **Condition 3.2** - inf-sup (or Ladyzhenskaya-Babuška-Brezzi, LBB) condition
- Crucial concept in numerical analysis, particularly in mixed finite element methods used to solve PDEs like the Navier-Stokes equations in fluid dynamics

$$\inf_{q \in M^h} \sup_{\mathbf{v} \in X^h} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \|q\|} \geq \beta > 0$$

  - M^h and X^h represent discrete function spaces used in the finite element method. M^h is typically associated with the pressure space, while X^h is associated with the velocity space.
  - $(q, \nabla \cdot \mathbf{v})$ represents the inner product of the function q from space M^h with the divergence of the function $\mathbf{v}$ from space X^h.
  - $\|\nabla \mathbf{v}\|$ and $\|q\|$ are the norms of the gradient of $\mathbf{v}$ and q, respectively.

Formulation of the FEM

Constructing Spaces

- The LBB condition is a mathematical criterion that ensures the stability of the mixed finite element method, guaranteeing convergence of both velocity and pressure approximations in a PDE.
- The expression $\inf_{q \in M^h} \sup_{\mathbf{v} \in X^h} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \|q\|}$ measures the coupling between pressure space M^h and velocity space X^h.
- The condition states that a positive constant β exists, ensuring the appropriate linkage of pressure and velocity, thus avoiding numerical instabilities.
- In designing a finite element method for fluid dynamics problems, satisfying the LBB condition is crucial to avoid inaccurate or unstable solutions, especially for incompressible flows.
- The LBB condition guides the selection of appropriate elements for the discretization of pressure and velocity fields.

- In summary, this condition is a fundamental stability criterion in mixed finite element methods, ensuring that the chosen discretization strategy for coupled variables (like velocity and pressure) will yield stable and accurate solutions.

Formulation of the FEM

Constructing Spaces

- **Condition 3.8** - Another form of the inf-sup condition, tailored to a specific scenario in numerical analysis
- This variant deals with boundary conditions and involves function spaces defined on the boundary of a domain.

$$\inf_{\alpha^h \in L^h} \sup_{\mathbf{v}^h \in X^h} \frac{\left(\alpha^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma}{\|\nabla \mathbf{v}^h\| \, \|\alpha^h\|_{1/2,\Gamma}} \geq \beta''' > 0, \quad \text{for some } \beta''' > 0$$

  - L^h and X^h represent discrete function spaces, with L^h associated with functions on the boundary Γ, and X^h representing a velocity space or similar vector field space.
  - α^h is an element of L^h, and $\mathbf{v}^h$ is an element of X^h.
  - $\left(\alpha^h, \mathbf{v}^h \cdot \hat{\tau}\right)_\Gamma$ is an inner product on the boundary Γ, involving α^h and the tangential component of $\mathbf{v}^h$.
  - $\left\|\nabla \mathbf{v}^h\right\|$ and $\left\|\alpha^h\right\|_{1/2,\Gamma}$ are the norms of $\mathbf{v}^h$ and α^h, respectively.

Formulation of the FEM

Constructing Spaces

- This is a stability condition focusing on the interaction between boundary values of functions and their behavior in the domain.
- It ensures that for each function α^h in L^h, there is a corresponding $\mathbf{v}^h$ in X^h such that their inner product to norms ratio is bounded below by a positive constant β'''.
- The condition is crucial for the well-posedness of boundary value problems in finite element methods, especially for problems with significant boundary conditions.
- This condition helps to ensure stability and convergence in finite element methods involving complex boundary conditions.
- It is particularly important for problems where accurate enforcement of boundary conditions is crucial, such as in fluid-structure interaction problems or in scenarios with complex boundary dynamics.

- In summary, this form of the inf-sup condition is a stability criterion that guarantees an appropriate discretization strategy in mixed finite element methods, especially when dealing with complex boundary conditions.
- It ensures a stable relationship between boundary values of functions

Formulation of the FEM

Constructing Spaces

- Constructing suitable finite element spaces that satisfy the conditions (3.2) and (3.8).
- Strategies for constructing finite element spaces that meet specific conditions (3.2) and (3.8) are discussed.
- Introduction of the MINI element and quadratic edge bubble functions.
- **Definition of the MINI element spaces Y^h and M^h.**
- *Explanation:* The MINI element spaces Y^h and M^h are defined, offering practical choices for FEM simulations.

Formulation of the FEM

Constructing Spaces

- The MINI element spaces are a type of finite element space used in the finite element method (FEM) for solving partial differential equations, particularly in the context of incompressible fluid flow problems.
  - These spaces are designed to provide stable and accurate approximations of both velocity and pressure fields in incompressible flow simulations.
  - In the context of MINI elements, there are typically two primary spaces:

    1. Velocity Space (Y^h): The Y^h space is used to approximate the velocity field in the FEM simulation. It typically consists of piecewise linear basis functions defined on triangular elements. These basis functions are chosen to ensure that the resulting velocity field is divergence-free, which is a fundamental requirement for incompressible flow problems.
    2. Pressure Space (M^h): The M^h space is used to approximate the pressure field. It usually consists of piecewise constant or piecewise linear basis functions defined on the same mesh as the velocity space. These basis functions are chosen to ensure that the pressure is approximated consistently with the velocity field to maintain stability.

Formulation of the FEM

Constructing Spaces

- The MINI element approach is known for its ability to satisfy the discrete inf-sup (LBB or Ladyzhenskaya-Babuska-Brezzi) condition, which is essential for pressure stability in incompressible flow simulations. This property makes MINI elements an attractive choice for solving problems involving fluid flow, particularly when dealing with the Navier-Stokes equations or related formulations.

- In summary, MINI element spaces are finite element spaces specifically tailored for incompressible flow simulations, offering stable and accurate approximations of both velocity and pressure fields.

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Penalty Methods in FEM

Introduction to Penalty Methods

- Penalty methods to replace the Lagrange multiplier imposition of the constraint $\mathbf{u} \cdot \hat{\tau}|_{\Gamma} = 0$.
- Introduces the concept of penalty functions and regularization.
  - Consider the following penalized functional:

$$J_{\epsilon}(\mathbf{v}) = \int_{\Omega} \frac{1}{2}|\nabla \mathbf{v}|^2 - \mathbf{f} \cdot \mathbf{v}\, dx + \frac{1}{2}\epsilon^{-1} \int_{\Gamma} (\mathbf{v} \cdot \hat{\tau})^2\, ds$$

  - where $\epsilon > 0$.
  - This functional is used to find $\mathbf{u}_{\epsilon} \in V$ that minimizes $J_{\epsilon}(\mathbf{v})$.

Penalty Methods in FEM

Introduction to Penalty Methods

- The given functional $J_\epsilon(\mathbf{v})$ represents a penalized form commonly used in variational problems, particularly in fluid dynamics or elasticity.

  1. **First Term:** $\int_\Omega \frac{1}{2}|\nabla \mathbf{v}|^2 \, dx$
     - ★ This term is an integral over the domain Ω.
     - ★ $|\nabla \mathbf{v}|^2$ represents the square of the gradient norm of the vector field $\mathbf{v}$, typically corresponding to kinetic or strain energy.
     - ★ It reflects the energy associated with the spatial variation of $\mathbf{v}$.
  2. **Second Term:** $-\int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx$
     - ★ This integral is over Ω.
     - ★ $\mathbf{f} \cdot \mathbf{v}$ is the dot product of a field $\mathbf{f}$ and the vector field $\mathbf{v}$.
     - ★ It accounts for the work done by the field $\mathbf{f}$ against $\mathbf{v}$.
  3. **Third Term:** $\frac{1}{2}\epsilon^{-1} \int_\Gamma (\mathbf{v} \cdot \hat{\tau})^2 \, ds$
     - ★ This integral is over the boundary Γ.
     - ★ $(\mathbf{v} \cdot \hat{\tau})^2$ is the square of the tangential component of $\mathbf{v}$ along the boundary.
     - ★ ϵ is a small positive parameter, making this a penalization term.
     - ★ It penalizes the tangential component of $\mathbf{v}$ at the boundary, enforcing a boundary condition.

Penalty Methods in FEM

Introduction to Penalty Methods

- The functional $J_\epsilon(\mathbf{v})$ is used in variational problems to find a vector field $\mathbf{v}$ that minimizes this functional.
- The penalization term enforces boundary conditions by penalizing certain behaviors at the boundary.
- The choice of ϵ is crucial for balancing boundary condition enforcement and numerical stability.

- In summary, $J_\epsilon(\mathbf{v})$ combines energy terms in the domain with a penalization term at the boundary. The goal is to find a vector field $\mathbf{v}$ that minimizes this functional while respecting boundary conditions enforced through the penalization term.

Penalty Methods in FEM

Penalty Method Equation

- The corresponding Euler-Lagrange equation for the penalty method is:

$$a(\mathbf{u}_\epsilon, \mathbf{v}) + \epsilon^{-1}(\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v} \cdot \hat{\tau})_\Gamma = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V$$

- This equation provides a way to find $\mathbf{u}_\epsilon$ that satisfies the penalty constraint.
- To facilitate error analysis, we can write the equation as follows:

$$\begin{aligned} a(\mathbf{u}_\epsilon, \mathbf{v}) + (\lambda_\epsilon, \mathbf{v} \cdot \hat{\tau})_\Gamma &= (\mathbf{f}, \mathbf{v}), \qquad \text{for all } \mathbf{v} \in V \\ (\mathbf{u}_\epsilon \cdot \hat{\tau}, \alpha)_\Gamma &= \epsilon(\lambda_\epsilon, \alpha)_\Gamma, \quad \text{for all } \alpha \in L \end{aligned}$$

- Here, $\lambda_\epsilon = \epsilon^{-1}\mathbf{u}_\epsilon \cdot \hat{\tau}$ is introduced as the Lagrange multiplier.

Penalty Methods in FEM

Penalty Method Equation

- The Euler-Lagrange equation for the penalty method arises from the functional $J_\epsilon(\mathbf{v})$ earlier.
  - Here, $a(\mathbf{u}_\epsilon, \mathbf{v})$ likely represents a bilinear form associated with the variational problem, including terms like diffusion or elasticity.
  - $\epsilon^{-1}(\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v} \cdot \hat{\tau})_\Gamma$ is the penalty term on the boundary Γ, enforcing a boundary condition by penalizing the tangential component of $\mathbf{u}_\epsilon$.
  - $(\mathbf{f}, \mathbf{v})$ is a linear form, representing external forces or influences.
  - The equation must hold for all $\mathbf{v} \in V$, a suitable function space.
  - λ_ϵ is introduced as a Lagrange multiplier, where $\lambda_\epsilon = \epsilon^{-1}\mathbf{u}_\epsilon \cdot \hat{\tau}$.
  - The first equation is similar to the original, but with the penalty term rewritten using the Lagrange multiplier.
  - The second equation relates the Lagrange multiplier to the boundary condition, encoding how it's enforced through the penalty method.
  - Useful for error analysis, studying how approximation error depends on ϵ and other properties.

Penalty Methods in FEM

Penalty Method Equation

- The goal is to find $\mathbf{u}_\epsilon$ that minimizes $J_\epsilon(\mathbf{v})$ while satisfying the imposed boundary conditions.
- The penalty method and Lagrange multipliers handle boundary conditions in variational problems.
- Adjusting ϵ controls the strength of the penalty and the adherence to the boundary condition.

- In summary, these equations provide a mathematical framework for solving variational problems with boundary conditions in a way that is amenable to both analytical treatment and numerical approximation.

Slip with Resistance

- Weakening the no-slip condition by introducing resistance to slippage leads to the following boundary value problem for $\epsilon > 0$:

$$\begin{aligned} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f}, \quad \nabla \cdot \mathbf{u}_\epsilon = 0, \quad \text{in } \Omega \\ \mathbf{u}_\epsilon \cdot \hat{n}_{|\Gamma} = 0, \\ \mathbf{u}_\epsilon \cdot \hat{\tau} + \epsilon(\nabla \mathbf{u}_\epsilon \cdot \hat{n}) \cdot \hat{\tau} = 0, \quad \text{on } \Gamma \end{aligned}$$

- The variational formulation of this problem with modified boundary conditions is:

$$\begin{aligned} (\nabla \mathbf{u}_\epsilon, \nabla \mathbf{v}) - (p_\epsilon, \nabla \cdot \mathbf{v}) + \frac{1}{\epsilon}(\mathbf{u}_\epsilon \cdot \hat{\tau}, \mathbf{v} \cdot \hat{\tau})_\Gamma = (\mathbf{f}, \mathbf{v}), \\ (q, \nabla \cdot \mathbf{u}_\epsilon) = 0 \end{aligned}$$

- This formulation is equivalent to the penalty formulation, and it accounts for slip with resistance.

Penalty Methods in FEM

Slip with Resistance

- This approach introduces a resistance to slippage at the boundary, which is more realistic in certain fluid flow scenarios.
  1. **Inside the Domain** Ω:
     - ★ $-\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f}$ represents the Navier-Stokes equations (without the time-dependent term), enforcing fluid dynamics and external forces.
     - ★ $\nabla \cdot \mathbf{u}_\epsilon = 0$ enforces incompressibility.
  2. **On the Boundary** Γ:
     - ★ $\mathbf{u}_\epsilon \cdot \hat{n}_{|\Gamma} = 0$ enforces zero normal velocity.
     - ★ $\mathbf{u}_\epsilon \cdot \hat{\tau} + \epsilon(\nabla \mathbf{u}_\epsilon \cdot \hat{n}) \cdot \hat{\tau} = 0$ allows for controlled slip in the tangential direction.

Penalty Methods in FEM

Slip with Resistance

- The variational formulation incorporates the modified boundary conditions and the equations governing fluid flow in a weak (variational) form.
- It is suitable for numerical methods like finite element analysis.
- The first equation captures the dynamics and the modified boundary conditions, while the second enforces incompressibility.
- This formulation accounts for slip with resistance at the boundary, offering a more realistic model in high Reynolds number scenarios.
- The parameter ϵ controls the degree of slip, approaching the traditional no-slip condition as it becomes smaller.
- Useful in numerical simulations where exact enforcement of no-slip can be challenging or less accurate.

Penalty Methods in FEM

Conclusion

- In conclusion, penalty methods offer a powerful approach for handling constraints and regularization in partial differential equations, particularly in the context of incompressible fluid flow problems.

Table of Contents

Introduction

- Overview of modifying the `no_slip_boundary_conditions` function in PINN.
- Introduction to three methods: Lagrange Multiplier, Penalty Method, and Slip with Resistance.
- Key changes required for each method in the context of enforcing boundary conditions.

Lagrange Multiplier Method

- Introduce additional variables (Lagrange multipliers) to enforce boundary conditions.
- Neural network architecture changes to incorporate these variables.
- Mechanism to tie boundary condition constraints with Lagrange multipliers.

Penalty Method

- Modify loss function to include a penalty term for boundary condition violations.
- The penalty term should be proportional to the square of velocity at boundary points.
- Scale the penalty term by a factor (ϵ) to control its impact.

Slip with Resistance

- Allow slip with linear resistance to slippage instead of enforcing strict no-slip condition.
- Modify the loss function to reflect the linear resistance relationship.
- Calculate tangential component of velocity and its gradient at boundary points.

Considerations and Implications

- Each method requires significant changes in network architecture and/or loss function.
- Choosing a method depends on the specific problem and computational resources.
- Modifications can affect the accuracy and convergence of the model.
- Importance of aligning network training with the principles of each method.