Solid Mechanics 2 Tutorial Sheets

Tutorial Sheets and Answers for DE2's Enjoyment

Tutorial Sheet 6: 3D Dynamics Answers

Topics covered are:

- Euler equations
- Constructing inertia matrices in 3D

Tips

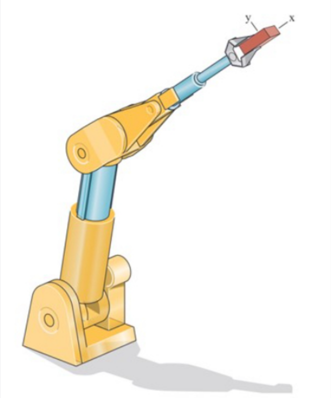
- Surprise, the equations are getting longer! I'd always have the Euler equation pulled up for reference so you can easily see what you need to calculate.
- The inertia matrices can get confusing if you need to construct them yourself. Be clear with directions and do more practice.
- Make sure you are clear with matrix manipulation.

Question 1

A robotic manipulator moves a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{kgm}^2$$

At the present instant, the angular velocity and angular acceleration of the casting are ω =1.2i+0.8j−0.4k rad/s and α=0.26i−0.07j+0.13k rad/s ^{2}. What moment is exerted about the center of mass of the casting by the manipulator?



Answer

The Euler equation is

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We are given all information required, so just sub in! Remember $d\omega/dt$ is α

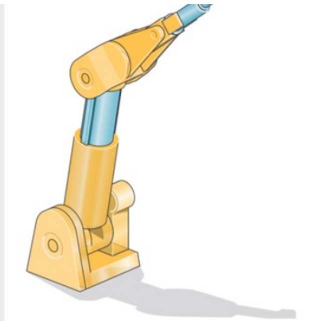
$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \begin{bmatrix} 0.26 \\ -0.07 \\ 0.13 \end{bmatrix} + \begin{bmatrix} 0 & 0.4 & 0.8 \\ -0.4 & 0 & -1.2 \\ -0.8 & 1.2 & 0 \end{bmatrix} \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \begin{bmatrix} 1.2 \\ 0.8 \\ -0.4 \end{bmatrix}$$
$$= 0.0135i + 0.0086j + 0.01k \text{ Nm}$$

Question 2

A robotic manipulator holds a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{kgm}^2$$

At the present instant, the casting is stationary. If the manipulator exerts a moment $\sum M$= 0.042i+0.036j+0.066k Nm about the center of mass, what is the angular acceleration of the casting at that instant?



Answer

In the Euler equation, because the casting is stationary, the ω terms are zero, so we are left with the matrix version of $M = I\alpha$

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

Then sub in values

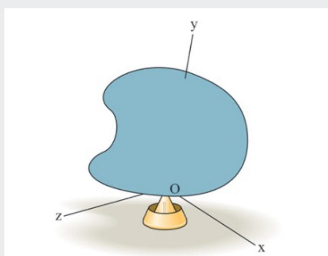
$$\begin{bmatrix} 0.042 \\ 0.036 \\ 0.066 \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$
$$= 1.43i + 0.987j + 1.65k \text{ rad/s}^2$$

Question 3

The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{kgm}^2$$

At the present instant, the rigid body's angular velocity is ω=6i+6j−4k rad/s and its angular acceleration is zero. What total moment about *O* is being exerted on the rigid body?



Answer

The angular acceleration is zero so the equation is simplified to

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Then sub in values

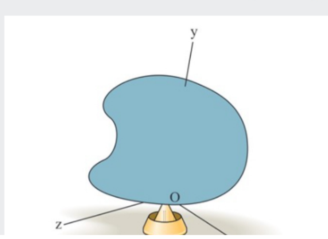
$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ -4 & 0 & -6 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ -4 \end{bmatrix}$$
$$= -76i + 36j - 60k \text{ Nm}$$

Question 4

The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{kgm}^2$$

At the present instant, the rigid body's angular velocity is ω=6i+6j−4k rad/s. The total moment about O due to the forces and couples acting on the rigid body is zero. What is its angular acceleration?



Answer

We have to use the full Euler equation - sub in the values!

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}4 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 5\end{bmatrix}\begin{bmatrix}d\omega_x/dt\\d\omega_y/dt\\d\omega_z/dt\end{bmatrix} + \begin{bmatrix}0 & 4 & 6\\-4 & 0 & -6\\-6 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & -2 & 0\\-2 & 3 & 1\\0 & 1 & 5\end{bmatrix}\begin{bmatrix}6\\6\\-4\end{bmatrix}$$
$$\alpha = 16.2i - 5.6j + 13.1k \,\mathrm{rad/s^2}$$

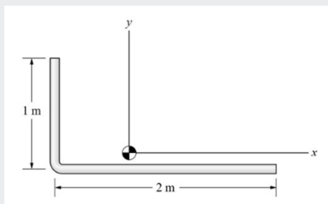
Question 5

In terms of the coordinate system shown, the inertia matrix of the 6-kg slender bar is

$$\begin{bmatrix}I_{xx} & -I_{xy} & -I_{xz}\\-I_{yx} & I_{yy} & -I_{yz}\\-I_{zx} & -I_{zy} & I_{zz}\end{bmatrix} = \begin{bmatrix}0.5 & 0.667 & 0\\0.667 & 2.667 & 0\\0 & 0 & 3.167\end{bmatrix}\mathrm{kgm^2}$$

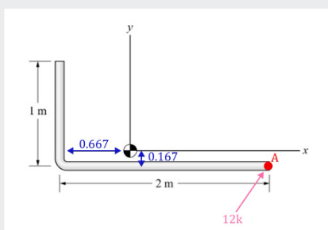
The bar is stationary relative to an inertial reference frame when the force F=12k N is applied at the right end of the bar. No other forces or couples act on the bar. Determine

(a) The bar's angular acceleration relative to the inertial reference frame.
(b) The acceleration of the right end of the bar relative to the inertial reference frame at the instant the force is applied.



Answer

(a) Taking the individual bar CoM distances from the reference frame labelled



The CoM can be found

$$\frac{x_1m_1 + x_2m_2}{m_1+m_2}, \frac{y_1m_1+y_2m_2}{m_1+m_2}$$
$$= \frac{(0)(\frac{1}{3}.6) + (1)(\frac{2}{3}.6)}{(\frac{1}{3}.6)+(\frac{2}{3}.6)}, \frac{(0.5)(\frac{1}{3}.6)+(0)(\frac{2}{3}.6)}{(\frac{1}{3}.6)+(\frac{2}{3}.6)}$$
$$= (0.667, 0.167)\ \mathrm{m}$$

Remembering the distance 'r' is taken from the right side where the force is applied, calculating moments

$$M = C + r \times F = I\alpha$$
$$0 + (1.33i - 0.167j) \times 12k = I\alpha$$
$$\begin{bmatrix}i & j & k\\1.33 & -0.167 & 0\\0 & 0 & 12\end{bmatrix} = \begin{bmatrix}0.5 & 0.667 & 0\\0.667 & 2.667 & 0\\0 & 0 & 3.167\end{bmatrix}\alpha$$
$$\begin{bmatrix}-2\\-16\\0\end{bmatrix} = \begin{bmatrix}0.5 & 0.667 & 0\\0.667 & 2.667 & 0\\0 & 0 & 3.167\end{bmatrix}\begin{bmatrix}d\omega_x/dt\\d\omega_y/dt\\d\omega_z/dt\end{bmatrix}$$
$$\alpha = 6.01i - 7.5j \,\mathrm{rad/s^2}$$

(b) We can work out the linear acceleration of the CoM with Newton's second law

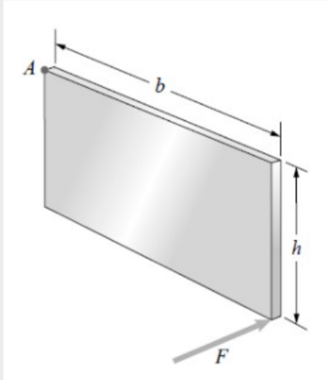
$$F = ma_O$$
$$12k = 6a_O$$
$$a_O = 2k \,\mathrm{m/s^2}$$

Then use the acceleration equation to solve acceleration of the right end (denoted A) from the CoM

$$a_A = a_O + \alpha \times r_{A/O} + \omega \times (\omega \times r_{A/O})$$
$$a_A = 2k + \begin{vmatrix}i & j & k\\6.01 & -7.5 & 0\\1.33 & -0.167 & 0\end{vmatrix} + 0$$
$$a_A = 11k \,\mathrm{m/s^2}$$

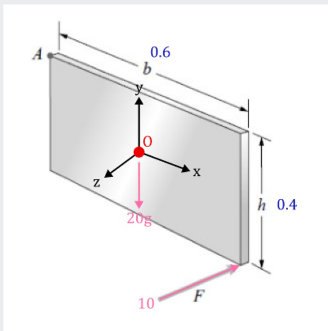
Question 6

The dimensions of the 20 kg thin plate are h=0.4 m and b=0.6 m. The plate is stationary relative to an inertial reference frame when the force F=10 N is applied in the direction perpendicular to the plate. No other forces or couples act on the plate. At the instant F is applied, what is the magnitude of the acceleration of point A relative to the inertial reference frame?



Answer

First we need to construct the inertia matrix. Referring to the formula sheet (thin rectangular plate) we construct



$$\begin{bmatrix} \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{12}mh^2 & 0 \\ 0 & 0 & \frac{1}{12}m(b^2+h^2) \end{bmatrix} = \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.867 \end{bmatrix} \mathrm{kgm}^2$$

The moment can be found with

$$\sum M = C + r \times F$$

$$\sum M = 0 + \begin{vmatrix} i & j & k \\ 0.3 & -0.2 & 0 \\ 0 & 0 & -10 \end{vmatrix}$$

$$= 2i + 3j$$

We know that ω is zero so the Euler equation is simplified and we can find the angular acceleration

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.867 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

$$\alpha = 7.6i + 5j \ \mathrm{m/s}^2$$

Then to find the acceleration of the CoM we use Newton's second law

$$F = ma_O$$

$$-10 = 20a_O$$

$$a_O = -0.5k$$

Then the acceleration of point A

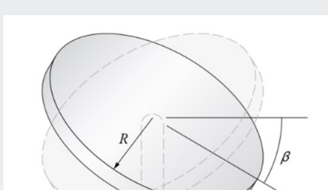
$$a_A = a_O + \alpha \times r_{A/O} + \omega \times (\omega \times r_{A/O})$$

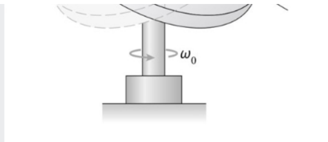
$$a_A = -0.5k + \begin{vmatrix} i & j & k \\ 7.6 & 5 & 0 \\ -0.3 & 0.2 & 0 \end{vmatrix} + 0$$

$$a_A = 2.5k \ \mathrm{m/s}^2$$

Question 7

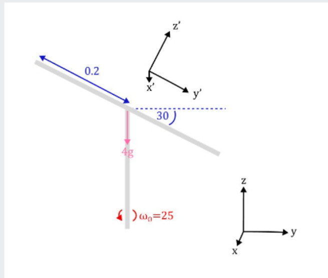
The thin circular disk of radius R=0.2 m and mass m=4 kg is rigidly attached to the vertical shaft. The plane of the disk is slanted at an angle β=30° relative to the horizontal. The shaft rotates with constant angular velocity ω_O=25 rad/s. Determine the magnitude of the couple exerted on the disk by the shaft.





Answer

Let's draw out a free body diagram the information in the question, considering its body-fixed reference frame



Construct the inertia matrix of the thin circular disk according to the body-fixed reference frame - note the reference frame is tiled by β in order for construction using the formulae provided.

$$\begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \text{kgm}^2$$

The disk's angular velocity needs to be written in terms of the 'new' body-fixed reference frame (which has been tilted by β)

$$\Omega = \omega_O \sin(\beta) j + \omega_O \cos(\beta) k$$
$$\Omega = 25\sin(30) j + 25\cos(30) k$$
$$\Omega = 12.5k + 21.65k$$

The angular acceleration is zero (constant velocity) so the Euler equation simplifies to

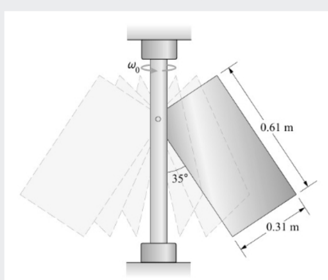
$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Then subbing in values

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -21.65 & 12.5 \\ 21.65 & 0 & 0 \\ -12.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \begin{bmatrix} 0 \\ 12.5 \\ 21.65 \end{bmatrix}$$
$$M = 12.8i \text{ Nm}$$

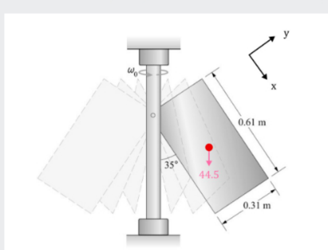
Question 8

The vertical shaft rotates with constant angular velocity ω_O. The 35◦ angle between the edge of the 44.5 N thin rectangular plate pinned to the shaft and the shaft remains constant. Determine ω_O.



Answer

Draw a free body diagram with the origin at the pinned joint and labelling angles and axes



We need to calculate a lot of things before we can sub into the Euler equation. First construct the inertia matrix

$$\begin{bmatrix} \frac{1}{3}mh^2 & \frac{1}{3}mbh & 0 \\ \frac{1}{3}mbh & \frac{1}{3}mb^2 & 0 \end{bmatrix} = \begin{bmatrix} 0.14 & -0.21 & 0 \\ -0.21 & 0.56 & 0 \end{bmatrix} \text{kgm}^2$$

$$\begin{bmatrix} 0 & 0 & \frac{1}{3}m(b^2+h^2) \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0.7 \end{bmatrix}$$

The weight force in terms of the reference frame is

$$F = W(\cos(\beta)i - \sin(\beta)j)$$
$$F = 44.5(\cos(35)i - \sin(35)j)$$
$$F = 36.4i - 25.5j \text{ N}$$

The moment about the CoM is

$$\sum M = C + r \times F$$
$$\sum M = 0 + \begin{vmatrix} i & j & k \\ 0.305 & 0.155 & 0 \\ 36.4 & -25.5 & 0 \end{vmatrix}$$
$$= -13.4k \text{ Nm}$$

The plate rotates with angular velocity

$$\omega = -\omega_O \cos(\beta)i + \omega_O \sin(\beta)j$$
$$\omega = -0.82\omega_O i + 0.57\omega_O j$$

As the shaft and plate are attached,

$$\omega = \Omega$$

We can then use the Euler equation with ω_O factored out, taking into account angular acceleration is zero

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \\ -13.4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.57 \\ 0 & 0 & 0.82 \\ -0.57 & -0.82 & 0 \end{bmatrix} \begin{bmatrix} 0.14 & -0.21 & 0 \\ -0.21 & 0.56 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} -0.82 \\ 0.57 \\ 0 \end{bmatrix} \omega_O^2$$
$$\begin{bmatrix} 0 \\ 0 \\ -13.4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.27 \end{bmatrix} \omega_O^2$$

Then finally

$$-13.4 = -0.27\omega_O^2$$
$$\omega_O = 7.04 \text{ rad/s}$$