Online appendix for the paper

Abstract Gringo

published in Theory and Practice of Logic Programming

MARTIN GEBSER
Aalto University, HIIT, Finland
University of Potsdam, Germany
gebser@cs.uni-potsdam.de

AMELIA HARRISON
Univeristy of Texas at Austin, USA
ameliaj@cs.utexas.edu

ROLAND KAMINSKI
University of Potsdam, Germany
kaminski@cs.uni-potsdam.de

VLADIMIR LIFSCHITZ
Univeristy of Texas at Austin, USA
vl@cs.utexas.edu

TORSTEN SCHAUB
University of Potsdam, Germany
INRIA Rennes, France
torsten@cs.uni-potsdam.de

Proofs of Theorems 1–3

Proof of Theorem 1

We will show that for any term t, the set of conjunctive terms of $\tau_t E$ is the union of the sets of conjunctive terms of $\tau_t E_{\leq}$ and $\tau_t E_{\geq}$. For any subset Δ of A,

	(22) is a conjunctive term of $\tau_t E$
iff	Δ does not justify E with respect to t
iff	$\widehat{\alpha}[\Delta] \neq t$
iff	$\widehat{\alpha}[\Delta] < t$ or $\widehat{\alpha}[\Delta] > t$
iff	Δ does not justify $E_{\geq}$ with respect to t or Δ does not justify $E_{\leq}$ with respect to t
iff	(22) is a conjunctive term of $\tau_t E_{\geq}$ or of $\tau_t E_{\leq}$
iff	(22) is a conjunctive term of $\tau_t E_{\leq} \wedge \tau_t E_{\geq}$.

Proof of Theorem 2
Since $[\overline{m}]$ is the singleton set $\{\overline{m}\}$, τE is $\tau_{\overline{m}} E$. Since E is monotone, the antecedent of (22) can be dropped (Section 5.1), so that $\tau_{\overline{m}} E$ is strongly equivalent to

$$\bigwedge_{\substack{\Delta \subseteq A \\ |[\Delta]| < m}} \bigvee_{(i,\mathbf{r}) \in A \setminus \Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{25}$$

To derive (25) from (23) in HT$^\infty$, assume (23). We will reason by cases, with one case corresponding to each disjunctive term

$$\bigwedge_{(i,\mathbf{r}) \in \Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}) \tag{26}$$

of (23). Let Δ' be a subset of A such that $|[\Delta']| < m$. We will show that the conjunctive term of (25) corresponding to Δ' can be derived from (26). Since

$$|[\Delta']| < m = |[\Delta]|, \tag{27}$$

there exists a pair $(i, \mathbf{r})$ that is an element of Δ but not an element of Δ'. Indeed, if $\Delta \subseteq \Delta'$ then $[\Delta] \subseteq [\Delta']$, which contradicts (27). Since $(i, \mathbf{r}) \in \Delta$, from (26) we can derive $\tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}})$. Since $(i, \mathbf{r}) \in A \setminus \Delta'$, we can further derive

$$\bigvee_{(i,\mathbf{r}) \in A \setminus \Delta'} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}).$$

It follows that each conjunctive term of (25) can be derived from (26).

We will prove by induction on m that (23) can be derived from (25) in HT$^\infty$. Base case: when $m = 0$ the disjunctive term of (23) corresponding to the empty Δ is $\top$. Inductive step: assume that (23) can be derived from (25), and assume

$$\bigwedge_{\substack{\Delta \subseteq A \\ |[\Delta]| < m+1}} \bigvee_{(i,\mathbf{r}) \in A \setminus \Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{28}$$

From (28) we can derive (25), and consequently (23). Now we reason by cases, with one case corresponding to each disjunctive term of (23). Assume

$$\bigwedge_{(i,\mathbf{r}) \in \Sigma} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}) \tag{29}$$

where Σ is a subset of A such that $|[\Sigma]| = m$. Consider the set

$$\Sigma' = \{(i, \mathbf{r}) : [(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}] \subseteq [\Sigma]\}.$$

By the definition of $[\Sigma]$, for any $(i, \mathbf{r}) \in \Sigma$, $[(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}] \subseteq [\Sigma]$. So $\Sigma \subseteq \Sigma'$. It follows that $[\Sigma] \subseteq [\Sigma']$. On the other hand,

$$[\Sigma'] = \bigcup_{(i,\mathbf{r}) \in \Sigma'} [(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}] = \bigcup_{(i,\mathbf{r})\,:\,[(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}] \subseteq [\Sigma]} [(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}] \subseteq [\Sigma].$$

Consequently $[\Sigma] = [\Sigma']$, and $|[\Sigma']| = |[\Sigma]| = m$. From (28),

$$\bigvee_{(i,\mathbf{r}) \in A \setminus \Sigma'} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{30}$$

Again, we reason by cases, with one case corresponding to each disjunctive term of (30).

Assume $\tau_\vee((\mathbf{L}_j)^{\mathbf{x}_j}_{\mathbf{s}})$, where $(j,\mathbf{s}) \in A \setminus \Sigma'$. Combining assumption (29) and $\tau_\vee((\mathbf{L}_j)^{\mathbf{x}_j}_{\mathbf{s}})$, we derive

$$\bigwedge_{(i,\mathbf{r})\in\Sigma\cup\{(j,\mathbf{s})\}} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{31}$$

Consider the set $[\Sigma \cup \{(j,\mathbf{s})\}]$, that is,

$$[\Sigma] \cup [(\mathbf{t}_j)^{\mathbf{x}_j}_{\mathbf{s}}]. \tag{32}$$

Recall that the cardinality of $[\Sigma]$ is m. Since $\mathbf{t}_j$ is interval-free, the cardinality of $[(\mathbf{t}_j)^{\mathbf{x}_j}_{\mathbf{s}}]$ is at most 1. Furthermore, since $(j,\mathbf{s}) \notin \Sigma'$ it follows that

$$[(\mathbf{t}_j)^{\mathbf{x}_j}_{\mathbf{s}}] \not\subseteq [\Sigma],$$

so that $[(\mathbf{t}_j)^{\mathbf{x}_j}_{\mathbf{s}}]$ is nonempty. Consequently, the set is a singleton, and therefore $[\Sigma]$ is disjoint from it. It follows that the cardinality of (32) is $m+1$. So from (31) we can derive

$$\bigvee_{\substack{\Delta\subseteq A\\ |[\Delta]|=m+1}} \bigwedge_{(i,\mathbf{r})\in\Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}).$$

Proof of Theorem 3

Since $[\overline{m}]$ is the singleton set $\{\overline{m}\}$, τE is $\tau_{\overline{m}} E$. Since the consequent of (22) can be replaced in this case by $\bot$, $\tau_{\overline{m}} E$ is strongly equivalent to

$$\bigwedge_{\substack{\Delta\subseteq A\\ |[\Delta]|>m}} \neg \bigwedge_{(i,\mathbf{r})\in\Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{33}$$

Every conjunctive term of (24) is a conjunctive term of (33). To derive (33) from (24), consider a set Δ such that $|[\Delta]| > m$. Let $f(i,\mathbf{r})$ stand for the set $[(\mathbf{t}_i)^{\mathbf{x}_i}_{\mathbf{r}}]$. Since each $\mathbf{t}_i$ is interval-free, this set is either empty or a singleton. Let $\mathbf{s}_1,\dots,\mathbf{s}_{m+1}$ be $m+1$ distinct elements of $[\Delta]$. Choose elements $(i_1,\mathbf{r}_1),\dots,(i_{m+1},\mathbf{r}_{m+1})$ of Δ such that each s_k belongs to $f(i_k,\mathbf{r}_k)$, and let Δ' be $\{(i_1,\mathbf{r}_1),\dots,(i_{m+1},\mathbf{r}_{m+1})\}$. The cardinality of $[\Delta']$ is at least $m+1$, because this set includes $\mathbf{s}_1,\dots,\mathbf{s}_{m+1}$. On the other hand, it is at most $m+1$, because this set is the union of $m+1$ sets of cardinality at most 1. Consequently, $|[\Delta']| = m+1$. From (24) we can conclude in HT$^\infty$ that

$$\neg \bigwedge_{(i,\mathbf{r})\in\Delta'} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}}). \tag{34}$$

Then the conjunctive term

$$\neg \bigwedge_{(i,\mathbf{r})\in\Delta} \tau_\vee((\mathbf{L}_i)^{\mathbf{x}_i}_{\mathbf{r}})$$

of (33) follows, because $\Delta' \subseteq \Delta$.

Correctness of the n-Queens Program

In this section, we prove the correctness of the program K, consisting of rules $R_1,\dots,R_7$ (Sections 2.3 and 3).

The n-queens problem involves placing n queens on an $n \times n$ chess board such that no two queens threaten each other. We will represent squares by pairs of integers (i,j) where

$1 \leq i, j \leq n$. Two squares (i_1, j_1) and (i_2, j_2) are said to be in the same row if $i_1 = i_2$; in the same column if $j_1 = j_2$; and in the same diagonal if $|i_1 - i_2| = |j_1 - j_2|$. A set Q of n squares is a *solution* to the n-queens problem if no two elements of Q are in the same row, in the same column, or in the same diagonal.

For any stable model I of K, by Q_I we denote the set of pairs (i, j) such that $q(\bar{i}, \bar{j}) \in I$.

Theorem 4
For each stable model I of K, Q_I is a solution to the n-queens problem. Furthermore, for each solution Q to the n-queens problem there is exactly one stable model I of K such that $Q_I = Q$.

Review: Supported Models and Constraints

We start by reviewing two familiar facts that will be useful in proving Theorem 4.

An *infinitary program* is a conjunction of (possibly infinitely many) infinitary formulas of the form $G \to A$, where A is an atom. We say that an interpretation I is *supported* by an infinitary program Π if each atom A from I is the consequent of a conjunctive term $G \to A$ of Π such that $I \models G$. Lifschitz and Yang (2013) give a condition, "tightness on an interpretation," under which the stable models of an infinitary program are identical to its supported models. Proposition 1 below gives a simpler condition of this kind that is sufficient for our purposes.

We say that an atom A *occurs nonnegated in a formula* F if

- F is A, or
- F is of the form $\mathcal{H}^\wedge$ or $\mathcal{H}^\vee$ and A occurs nonnegated in at least one element of $\mathcal{H}$, or
- F is of the form $G \to H$, where H is different from $\bot$, and A occurs nonnegated in G or in H.

It is clear, for instance, that no atom occurs nonnegated in a formula of the form $\neg F$.

The *positive dependency graph* of an infinitary program Π is the directed graph containing a vertex for each atom occuring in Π, and an edge from A to B for every conjunctive term $G \to A$ of Π and every atom B that occurs nonnegated in G. We say that an infinitary program Π is *extratight* if the positive dependency graph of Π contains no infinite paths.

The following fact is immediate from (Lifschitz and Yang 2013, Lemma 2).

Proposition 1
For any model I of an extratight infinitary program Π, I is stable iff I is supported by Π.

A *constraint* is an infinitary formula of the form $\neg F$ (which is shorthand for $F \to \bot$). The following theorem is a straightforward generalization of Proposition 4 from (Ferraris and Lifschitz 2005).

Proposition 2
Let $\mathcal{H}_1$ be a set of infinitary formulas and $\mathcal{H}_2$ be a set of constraints. A set I of atoms is a stable model of $\mathcal{H}_1 \cup \mathcal{H}_2$ iff I is a stable model of $\mathcal{H}_1$ and satisfies all formulas in $\mathcal{H}_2$.

Proof
Case 1: Every formula in $\mathcal{H}_1 \cup \mathcal{H}_2$ is satisfied by I. For each formula $\neg F$ in $\mathcal{H}_2$, I does not satisfy F. So the reduct of each formula in $\mathcal{H}_2$ w.r.t. I is $\neg\bot$. It follows that the set of reducts of all formulas in $\mathcal{H}_1 \cup \mathcal{H}_2$ is satisfied by the same interpretations as the set of reducts of all formulas in $\mathcal{H}_1$. Consequently, I is minimal among the sets satisfying the reducts of all formulas from $\mathcal{H}_1 \cup \mathcal{H}_2$ iff it is minimal among the sets satisfying the reducts of all formulas from $\mathcal{H}_1$. *Case 2:* Some formula F in $\mathcal{H}_1 \cup \mathcal{H}_2$ is not satisfied by I. Then I is not a stable model of $\mathcal{H}_1 \cup \mathcal{H}_2$. If $F \in \mathcal{H}_1$ then I is not a stable model of $\mathcal{H}_1$. Otherwise, it is not true that I satisfies all formulas in $\mathcal{H}_2$.

Proof of Theorem 4

To simplify notation, we will identify each set Q of squares with the set of atoms $q(\bar{i},\bar{j})$ where $(i,j) \in Q$. By D_n we denote the set of atoms of the forms $d1(\bar{i},\bar{j},\overline{i-j+n})$ and $d2(\bar{i},\bar{j},\overline{i+j-1})$ for all i,j from $\{1,\dots,n\}$. Recall that the rules of the program K are denoted by $R_1,\dots,R_7$.

Lemma 1
A set of atoms is a stable model of

$$\tau R_1 \cup \tau R_4 \cup \tau R_5 \tag{35}$$

iff it is of the form $Q \cup D_n$ where Q is a set of squares.

Proof
We can turn (35) into a strongly equivalent infinitary program as follows. The result of applying τ to R_1 is (21). Each conjunctive term in this formula is strongly equivalent to

$$\neg\neg q(\bar{i},\bar{j}) \to q(\bar{i},\bar{j}). \tag{36}$$

The set τR_4 is strongly equivalent to the set of formulas

$$\top \to d1(\bar{i},\bar{j},\overline{i-j+n}) \tag{37}$$

$(1 \le i,j \le n)$. (We take into account that $\tau(\bar{i} = \bar{1}..\bar{n})$ is equivalent to $\top$ if $1 \le i \le n$ and to $\bot$ otherwise, and similarly for j.) Similarly, τR_5 is strongly equivalent to the set of formulas

$$\top \to d2(\bar{i},\bar{j},\overline{i+j-1}) \tag{38}$$

$(1 \le i,j \le n)$. Consequently, (35) is strongly equivalent to the conjunction H of formulas (36)–(38). It is easy to check that H is an extratight infinitary program, so that by Proposition 1 its stable models are identical to its supported models. A set I of atoms is a model of H iff $D_n \subseteq I$. Furthermore, I is supported iff every element of I has the form $q(\bar{i},\bar{j})$ or is an element of D_n. Consequently, supported models of H are sets of the form $Q \cup D_n$ where Q is a set of squares.

Lemma 2
A set I of atoms is a stable model of τK iff it has the form $Q \cup D_n$, where Q is a solution to the n-queens problem.

Proof

Let $\mathcal{H}_1$ be (35) and $\mathcal{H}_2$ be

$$\tau R_2 \cup \tau R_3 \cup \tau R_6 \cup \tau R_7.$$

All formulas in $\mathcal{H}_2$ are constraints. Consequently, by Proposition 2, I is a stable model of τK iff it is a stable model of $\mathcal{H}_1$ and satisfies all formulas in $\mathcal{H}_2$. By Lemma 1, I is a stable model of $\mathcal{H}_1$ iff it is of the form $Q \cup D_n$, where Q is a set of squares. It remains to show that a set I of the form $Q \cup D_n$ satisfies all formulas in $\mathcal{H}_2$ iff Q is a solution to the n-queens problem. Specifically, we will show that for any set I of the form $Q \cup D_n$

(i) I satisfies τR_2 iff for all $i \in \{1, \dots, n\}$, I contains exactly one atom of the form $q(\bar{i}, \bar{j})$;
(ii) I satisfies τR_3 iff for all $j \in \{1, \dots, n\}$, I contains exactly one atom of the form $q(\bar{i}, \bar{j})$;
(iii) I satisfies $\tau R_6 \cup \tau R_7$ iff no two squares in I are in the same diagonal.

To prove (i), note first that τR_2 is equivalent to the set of formulas

$$\neg\neg\left(\tau(\mathit{count}\{Y : q(\bar{i}, Y)\} = \bar{1})\right)$$

$(1 \leq i \leq n)$. Let E be the aggregate atom above. Since $[\bar{1}]$ is a singleton set, τE is the same as $\tau_{\bar{1}}E$. By Theorem 1, this set is strongly equivalent to the set of formulas

$$\neg\neg\left(\tau_{\bar{1}}(\mathit{count}\{Y : q(\bar{i}, Y)\} \leq \bar{1}) \wedge \tau_{\bar{1}}(\mathit{count}\{Y : q(\bar{i}, Y)\} \geq \bar{1})\right). \tag{39}$$

Again note that the result of applying τ to first aggregate atom in (39) is the same as the result of applying $\tau_{\bar{1}}$. Then by Theorem 3 and the comment at the end of Section 5.3, τ applied to this aggregate atom is strongly equivalent to

$$\bigwedge_{\substack{\Delta \subseteq A \\ |\Delta| = 2}} \neg \bigwedge_{(1, r) \in \Delta} q(\bar{i}, r).$$

This formula can be written as

$$\bigwedge_{\substack{\Sigma \subseteq P \\ |\Sigma| = 2}} \neg \bigwedge_{r \in \Sigma} q(\bar{i}, r),$$

where P is the set of precomputed terms. It is easy to see that I satisfies this formula iff it contains at most one atom of the form $q(\bar{i}, r)$. On the other hand, by Theorem 2, the result of applying τ to the second aggregate atom in (39) is strongly equivalent to

$$\bigvee_{\substack{\Delta \subseteq A \\ |\Delta| = 1}} \bigwedge_{(1, r) \in \Delta} q(\bar{i}, r).$$

Similar reasoning shows that I satisfies this formula iff it contains at least one atom of the form $q(\bar{i}, r)$. Since $I = Q \cup D_n$, r in this atom is one of $\bar{1}, \dots, \bar{n}$.

Claim (ii) is proved in a similar way.

To prove (iii), note first that two squares $(\overline{i_1}, \overline{j_1}), (\overline{i_2}, \overline{j_2})$ are in the same diagonal iff there exists a $k \in \{1, \dots, 2n - 1\}$ such that

$$d1(\bar{i}_1, \bar{j}_1, \bar{k}), d1(\bar{i}_2, \bar{j}_2, \bar{k}) \in D_n \tag{40}$$

or

$$d2(\bar{i}_1, \bar{j}_1, \bar{k}), d2(\bar{i}_2, \bar{j}_2, \bar{k}) \in D_n. \tag{41}$$

We will show that a set I of the form $Q \cup D_n$ does not satisfy τR_6 iff there exists a k such that (40) holds for two distinct elements $q(\overline{i_1}, \overline{j_1}), q(\overline{i_2}, \overline{j_2}) \in Q$, and that it does not satisfy τR_7 iff there exists a k such that (41) holds for such two elements. The result of applying τ to R_6 is strongly equivalent to the set of formulas

$$\neg\tau(2 \leq \mathit{count}\{\overline{0}, q(X,Y) : q(X,Y), d1(X,Y,\overline{k})\}) \tag{42}$$

$(1 \leq k \leq 2n-1)$. Formula (42) is identical to

$$\neg\tau(\mathit{count}\{X, Y : q(X,Y), d1(X,Y,\overline{k})\} \geq 2).$$

In view of Theorem 2, it follows that it is strongly equivalent to

$$\neg \bigvee_{\substack{\Delta \subseteq A \\ |\Delta|=2}} \bigwedge_{(1,(r,s)) \in \Delta} (q(r,s) \wedge d1(r,s,\overline{k}))$$

$(1 \leq k \leq 2n-1)$. This formula can be written as

$$\neg \bigvee_{\substack{\Sigma \subseteq P \times P \\ |\Sigma|=2}} \bigwedge_{(r,s) \in \Sigma} (q(r,s) \wedge d1(r,s,\overline{k})). \tag{43}$$

For any set Q of squares,

$Q \cup D_n$ does not satisfy (43)
iff there exist two distinct pairs $(r_1, s_1), (r_2, s_2)$ from $P \times P$ such that $q(r_1, s_1), q(r_2, s_2) \in Q$ and $d1(r_1, s_1, \overline{k}), d1(r_2, s_2, \overline{k}) \in D_n$
iff there exist two distinct squares $(\bar{i}_1, \bar{j}_1), (\bar{i}_2, \bar{j}_2) \in Q$ such that (40) holds.

The claim about (41) is proved in a similar way.

Theorem 4 is immediate from the lemma.

References

FERRARIS, P. AND LIFSCHITZ, V. 2005. Mathematical foundations of answer set programming. In *We Will Show Them! Essays in Honour of Dov Gabbay*. King's College Publications, 615–664.

LIFSCHITZ, V. AND YANG, F. 2013. Lloyd-Topor completion and general stable models. *Theory and Practice of Logic Programming 13,* 4–5.