December 10, 2018

## Narrow Resonance Model (homogeneous geometry)

We start with the Boltzmann Equation.

$$\Sigma_t(E)\phi(E) = \int_0^\infty \Sigma_s\left(E' \to E\right)\phi\left(E'\right)\mathrm{d}E' + \frac{\chi(E)}{k_{eff}}\int_0^\infty v\Sigma_f\left(E'\right)\phi\left(E'\right)\mathrm{d}E' \tag{1}$$

We're working in the resonance region, where scattering is the main form of neutrons slowing down, which allows us to get rid of our fission term, simplifying the Boltzmann Equation to

$$\Sigma_t(E)\phi(E) = \int_0^\infty \Sigma_s\left(E' \to E\right)\phi\left(E'\right)\mathrm{d}E'. \tag{2}$$

Since this method prompts us to consider one nuclide to be resonant, while all other "non-resonant" nuclides are considered to be constant for the time being, we separate the macroscopic cross section into its number density $N$ and microscopic cross section $\sigma$ components[1].

$$\left(\sum_k N_k\sigma_{t,k}(E)\right)\phi(E) = \sum_k \int_E^{E/\alpha_k} N_k\sigma_{s,k}\left(E'\right)\phi\left(E'\right)P(E' \to E)\mathrm{d}E' \tag{3}$$

Recalling that

$$P(E' \to E)dE' = \frac{1}{(1-\alpha_k)E'}dE', \tag{4}$$

we can further simplify the scattering kernel, bringing the equation to

$$\left(\sum_k N_k\sigma_{t,k}(E)\right)\phi(E) = \sum_k \frac{1}{1-\alpha_k}\int_E^{E/\alpha_k} \frac{1}{E'}N_k\sigma_{s,k}\left(E'\right)\phi\left(E'\right)\mathrm{d}E' \tag{5}$$

The potential scattering cross section is independent of neutron energy. For neutrons with energy in the resonance region that interact with non-resonant nuclides, this is the dominant reaction. Thus, we assume that $\sigma_{t,k}(E') = \sigma_{s,k} = \sigma_{potot,k}$. In other words, we neglect energy dependence and non-scattering reactions, such as absorption, for non-resonant nuclides.

$$\left(N_r\sigma_{t,r}(E) + \sum_{k\neq r} N_k\sigma_{pot,k}\right)\phi(E) = \frac{1}{1-\alpha_r}\int_E^{E/\alpha_r} \frac{1}{E'}N_r\sigma_{s,r}\left(E'\right)\phi\left(E'\right)dE' \tag{6}$$

$$+\sum_{k\neq r}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k} \frac{1}{E'}N_k\sigma_{pot,k}\phi\left(E'\right)dE' \tag{7}$$

We now need to simplify the latter integral, which represents non-resonant scattering. First, we remove all terms without energy dependence out of the integral, yielding

$$\sum_{k\neq r}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k} \frac{1}{E'}N_k\sigma_{pot,k}\phi\left(E'\right)\mathrm{d}E' = \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{1-\alpha_k}\int_E^{E/\alpha_k} \frac{1}{E'}\phi\left(E'\right)\mathrm{d}E'. \tag{8}$$

It is clear that without an adequate approximation for $\phi(E')$, we are unable to further simplify this integral. In invoking the Narrow Resonance (NR) approxiamtion, the resonance width of our resonant nuclide $r$is assumed to be narrow compared to the neutron slowing down width. In other words, neutrons are not able to stay in the resonance (the only neutrons that "see" the resonance are those that come from higher energies, away from the resonance peak. Additionally, we assume the standard $\phi(E) = 1/E$ shape. While it is true that the neutron spectrum surrounding the resonances is not a true $1/E$ shape, the error caused negligible due to the narrow resonance width.

[1] Note that this does not prevent us from considering multiple nuclides that have resonances. We simply focus on one resonant nuclide at a time.

$$\sum_{k\neq r}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k}\frac{1}{E'}N_k\sigma_{pot,k}\phi\left(E'\right)\mathrm{d}E' = \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{1-\alpha_k}\int_E^{E/\alpha_k}\frac{1}{(E')^2}\mathrm{d}E' \tag{9}$$

$$= \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{1-\alpha_k}\left(\frac{1}{E}-\frac{\alpha_k}{E}\right) \tag{10}$$

$$= \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{E} \tag{11}$$

$$\boxed{\sum_{k\neq r}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k}\frac{1}{E'}N_k\sigma_{pot,k}\phi\left(E'\right)\mathrm{d}E' = \sum_{k\neq r}N_k\sigma_{pot,k}\frac{1}{E}} \tag{12}$$

We now insert this approximation of the non-resonant nuclide moderation into our earlier equation, yielding,

$$\left(N_r\sigma_{t,r}(E)+\sum_{k\neq r}N_k\sigma_{pot,k}\right)\phi(E) = \frac{1}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}N_r\sigma_{s,r}\left(E'\right)\phi\left(E'\right)dE' + \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{E} \tag{13}$$

which obviously requires approximations to be made for the remaining integral, which represents neutron moderation via colliding with the resonant nuclide.

First, we assume that the resonant nuclide scattering cross section $\sigma_{s,r}(E')$ is adequarely represented using the potential scattering cross section $\sigma_{pot,r}$, which is not energy dependent. Similarly, we assume that the flux is $1/E$, as we assumed for the non-resonant nuclides. This results in

$$\frac{1}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}N_r\sigma_{s,r}\left(E'\right)\phi\left(E'\right)\mathrm{d}E' = \frac{N_r\sigma_{pot,r}}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}\phi\left(E'\right)\mathrm{d}E' \tag{14}$$

$$= \frac{N_r\sigma_{pot,r}}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}\frac{\mathrm{d}E'}{E'} \tag{15}$$

$$= N_r\sigma_{pot,r}\frac{1}{E} \tag{16}$$

$$\boxed{\frac{1}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}N_r\sigma_{s,r}\left(E'\right)\phi\left(E'\right)\mathrm{d}E' = N_r\sigma_{pot,r}\frac{1}{E}} \tag{17}$$

These NR approximations to the scattering behavior greatly simplifies the energy dependence of the neutron flux, as shown below.

$$\left(N_r\sigma_{t,r}(E)+\sum_{k\neq r}N_k\sigma_{pot,k}\right)\phi(E) = N_r\sigma_{pot,r}\frac{1}{E} + \sum_{k\neq r}\frac{N_k\sigma_{pot,k}}{E} \tag{18}$$

$$\phi(E) = \frac{N_r\sigma_{pot,r}+\sum\limits_{k\neq r}N_k\sigma_{pot,k}}{N_r\sigma_{t,r}(E)+\sum\limits_{k\neq r}N_k\sigma_{pot,k}}\frac{1}{E} \tag{19}$$

$$\boxed{\phi(E) = \frac{\sigma_{pot,r}+\sigma_0}{\sigma_{t,r}(E)+\sigma_0}\frac{1}{E}\text{ where }\sigma_0 = \frac{\sum\limits_{k\neq r}N_k\sigma_{pot,k}}{\Sigma_r}} \tag{20}$$

## Neutron Slowing Down in Heterogeneous Isolated System

Consider a neutron slowing down in a two-region heterogeneous problem, where $f, m$ represent fuel and moderator, respectively.

$$\Sigma_{t,f}(E)\phi_f(E)V_f = P_{f\to f}(E)V_f \int_0^\infty \Sigma_{s,f}\left(E' \to E\right)\phi_f\left(E'\right)dE' \tag{21}$$

$$+P_{m\to f}(E)V_m \int_0^\infty \Sigma_{s,m}\left(E' \to E\right)\phi_m\left(E'\right)dE' \tag{22}$$

We separate the macroscopic cross section for scattering to energy $E$ into its number density $N$, microscopic cross section $\sigma_s$, and probability of energy change $P(E' \to E)$, to rewrite the balance equation as

$$\Sigma_{t,f}(E)\phi_f(E)V_f = P_{f\to f}(E)V_f \int_0^\infty \sum_{k\in f} N_k\sigma_{s,k}P(E' \to E)\phi_f\left(E'\right)dE' \tag{23}$$

$$+P_{m\to f}(E)V_m \int_0^\infty \sum_{k\in m} N_k\sigma_{s,k}P\left(E' \to E\right)\phi_m\left(E'\right)dE' \tag{24}$$

Recalling the energy distribution of a single neutron scattering collision is

$$P(E' \to E) = \frac{1}{(1-\alpha)E} \text{ for } \alpha E \le E' \le E, \tag{25}$$

we can further simplify the balance equation to be

$$\Sigma_{t,f}(E)\phi_f(E)V_f = P_{f\to f}(E)V_f \sum_{k\in f}\int_E^{E/\alpha_k} \frac{N_k\sigma_{s,k}\left(E'\right)\phi_f\left(E'\right)}{\left(1-\alpha_k\right)E'}dE' \tag{26}$$

$$+P_{m\to f}(E)V_m \sum_{k\in m}\int_E^{E/\alpha_k} \frac{N_k\sigma_{s,k}\left(E'\right)\phi_m\left(E'\right)}{\left(1-\alpha_k\right)E'}dE'. \tag{27}$$

Recall Eq. 12 and Eq. 17, where we defined the neutron moderation from non-resonant and resonant nuclides, respectively, as

$$\sum_{k\neq r}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k}\frac{1}{E'}N_k\sigma_{s,k}\phi\left(E'\right)\mathrm{d}E' = \sum_{k\neq r}N_k\sigma_{pot,k}\frac{1}{E} \tag{12}$$

$$\frac{1}{1-\alpha_r}\int_E^{E/\alpha_r}\frac{1}{E'}N_r\sigma_{s,r}(E')\phi\left(E'\right)\mathrm{d}E' = N_r\sigma_{pot,r}\frac{1}{E} \tag{17}$$

which can be combined into

$$\sum_{k}\frac{1}{1-\alpha_k}\int_E^{E/\alpha_k}\frac{1}{E'}N_k\sigma_{pot,k}\phi\left(E'\right)\mathrm{d}E' = \sum_{k\neq r}\Sigma_{pot,k}\frac{1}{E} \tag{28}$$

and subsequently plugged into Eq. 27 to yield

$$\Sigma_{t,f}(E)\phi_f(E)V_f = \frac{1}{E}\Big(P_{f\to f}(E)V_f\Sigma_{pot,f} + P_{m\to f}(E)V_m\Sigma_{pot,m}\Big) \tag{29}$$

$$\phi_f(E) = \frac{P_{f\to f}(E)V_f\Sigma_{pot,f} + P_{m\to f}(E)V_m\Sigma_{pot,m}}{E\Sigma_{t,f}(E)V_f} \tag{30}$$

Note that while this result is derived using a two-region problem, it can be extended to solve for a flux in region $i \in N$, which is dependent on all $j \in N$ regions:

$$\phi_i(E) = \frac{1}{E}\Sigma_j\frac{P_{j\to i}(E)V_j\Sigma_{p,j}}{\Sigma_{t,i}(E)V_i} \tag{31}$$

## Tone's Method

We start with the result from Eq. 31, which represents the energy dependence of the neutron flux in region $i$ as it depends on the collision probabilities between itself and other $j$ regions.

$$\phi_i(E) = \frac{1}{E}\Sigma_j \frac{P_{j\to i}(E)V_j\Sigma_{p,j}}{\sum\limits_{t,i}(E)V_i} \tag{32}$$

The basis of Tone's method lies in the following approximation:

$$\frac{P_{j\to i}(E)}{\Sigma_{t,i}(E)} = \alpha_i(E)\frac{P_{j\to i,g}}{\Sigma_{t,i,g}} \tag{33}$$

Note what this is actually saying. We're approximating the collision probability and total cross section to be group constants. But to make it slightly better, we're adding in a fine energy term $\alpha_i(E)$, that is **only dependent on the region we're going into**. This is a major assumption, since in reality the actual collision probability is dependent on other regions (including the source region).

$$\phi_i(E) = \frac{1}{E}\alpha_i(E)\Sigma_j \frac{P_{j\to i,g}V_j\Sigma_{p,j}}{\Sigma_{t,i,g}V_i} \tag{34}$$

$$\tag{35}$$

Another tool that we're going to use is the reciprocity theorem

$$P_{j\to i}(E)V_j\Sigma_{t,j}(E) = P_{i\to j}(E)V_i\Sigma_{t,i}(E) \tag{36}$$

Finally, normalization requirement for collision probabilities is also considered.

$$\Sigma_j P_{i\to j}(E) = 1 \tag{37}$$

$$1 = \Sigma_j P_{i\to j}(E) \tag{38}$$

$$1 = \Sigma_j \frac{P_{j\to i}(E)V_j \sum\limits_{t,j}(E)}{V_i \sum\limits_{t,i}(E)} \tag{39}$$

$$1 = \Sigma_j \alpha_i(E)\frac{P_{j\to i,g}}{\sum\limits_{t,i,g}}\frac{V_j \sum\limits_{t,j}(E)}{V_i} \tag{40}$$

$$1 = \alpha_i(E)\frac{1}{\sum\limits_{t,i,g} V_i}\Sigma_j P_{j\to i,g}V_j\Sigma_{t,j}(E) \tag{41}$$

$$\alpha_i(E) = \frac{\sum\limits_{t,i,g} V_i}{\Sigma_j P_{j\to i,g}V_j\Sigma_{t,j}(E)} \tag{42}$$

$$\phi_i(E) = \frac{1}{E}\alpha_i(E)\Sigma_j \frac{P_{j\to i,g}V_j \sum\limits_{p,j}}{\sum\limits_{t,i,g} V_i} \tag{43}$$

$$= \frac{1}{E}\frac{\sum\limits_{t,i,g} V_i}{\Sigma_j P_{j-i,g}V_{t,j}(E)}\Sigma_j \frac{P_{j-i,g}V_j\Sigma_{p,j}}{\sum\limits_{t,i,g} V_i} \tag{44}$$

$$= \frac{1}{E} \frac{\Sigma_j P_{j \to i,g} V_j \Sigma_{p,j}}{\Sigma_j P_{j \to i,g} V_j \Sigma_{t,j}(E)} \tag{45}$$

$$\cong \frac{\Sigma_j P_{j \to i,g} V_j \cdot \left( N_{r,j} \sigma_{pot,r} + \sum_{k \neq r} N_{k,j} \sigma_{pot,k} \right)}{\Sigma_j P_{j \to i,g} V_j \cdot \left( N_{r,j} \sigma_{t,r}(E) + \sum_{k \neq r} N_{k,j} \sigma_{pot,k} \right)} \tag{46}$$

$$= \frac{1}{E} \frac{\sigma_{pot,r} \Sigma_j P_{j \to i,g} V_j N_{r,j} + \Sigma_j P_{j \to i,g} V_j \sum_{k \neq r} N_{k,j} \sigma_{pot,k}}{\sigma_{t,r}(E) \Sigma_j P_{j \to i,g} V_j N_{r,j} + \Sigma_j P_{j \to i,g} V_j \sum_{k \neq r} N_{k,j} \sigma_{pot,k}} \tag{47}$$

$$= \frac{1}{E} \frac{\sigma_{pot,r} + \left( \sum_j P_{j \to i,g} V_j \sum_{k \neq r} N_{k,j} \sigma_{pot,k} \right) / \left( \sum_j P_{j \to i,g} V_j N_{r,j} \right)}{\sigma_{t,r}(E) + \left( \sum_j P_{j \to i,g} V_j \sum_{k \neq r} N_{k,j} \sigma_{pot,k} \right) / \left( \Sigma_j P_{j \to i,g} V_j N_{r,j} \right)} \tag{48}$$

$$= \frac{1}{E} \frac{\sigma_{pot,r} + \sigma_0}{\sigma_{t,r}(E) + \sigma_0} \tag{49}$$

$$\sigma_0 = \frac{\Sigma_j \sum_{k \neq r} P_{j \to i,g} V_j N_{k,j} \sigma_{pot,k}}{\Sigma_j P_{j \to i,g} V_j N_{r,j}} \tag{50}$$