S_N Method

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The main equation that we want to solve is

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

where i is for the cell number, and n is for S_n, and deals with the Gauss-Legendre quadrature order that we want to use. Like, for S_2 commonly $\mu_1 = -\mu_2$ and $w_1 = w_2 = 1$.

There are three approaches that we're going to use that start from here. Step, Diamond-Difference, and Step-Characteristics.

Step

The main approximation here is that $\psi_i \approx \psi_{i-1/2}$ for $\mu > 0$ and $\psi_i \approx \psi_{i+1/2}$ for $\mu < 0$.

$\mu > 0$

$$\psi_i = \psi_{i-1/2}$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} = \Delta_i Q_i$$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$$\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i-1/2,n}}{\mu_n}$$

$$\psi_{i+\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i-1/2,n}}{\mu_n} + \psi_{i-\frac{1}{2},n}$$

$$\boxed{\psi_{i+\frac{1}{2},n} = \frac{\Delta_i Q_i}{\mu_n} + \psi_{i-1/2,n} \left(1 - \frac{\Delta_i \Sigma_{ti}}{\mu_n}\right)}$$

$\mu < 0$

$$\psi_i = \psi_{i+1/2}$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = \Delta_i Q_i$$

And since we're traveling from right to left, that means we're starting with $i + 1/2$ and we're trying to get to $i - 1/2$. So I'll solve for $i - 1/2$ in terms of $i + 1/2$.

$$\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i+1/2,n}}{\mu_n}$$

$$\psi_{i-\frac{1}{2},n} = \psi_{i+\frac{1}{2},n} - \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i+1/2,n}}{\mu_n}$$

$$\boxed{\psi_{i-\frac{1}{2},n} = -\frac{\Delta_i Q_i}{\mu_n} + \psi_{i+\frac{1}{2},n} \left(1 + \frac{\Delta_i \Sigma_{ti}}{\mu_n}\right)}$$

Diamond-Difference

The main approximation here is that

$$\psi_{i,n} = \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2}$$

$\mu > 0$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2} = \Delta_i Q_i$$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$$2\mu_n \psi_{i+\frac{1}{2},n} - 2\mu_n \psi_{i-\frac{1}{2},n} + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} + 2\mu_n) \psi_{i+\frac{1}{2},n} + (\Delta_i \Sigma_{ti} - 2\mu_n) \psi_{i-1/2,n} = 2\Delta_i Q_i$$

$$\psi_{i+\frac{1}{2},n} = \frac{2\Delta_i Q_i - (\Delta_i \Sigma_{ti} - 2\mu_n) \psi_{i-1/2,n}}{\Delta_i \Sigma_{ti} + 2\mu_n}$$

$$\boxed{\psi_{i+\frac{1}{2},n} = \frac{2\mu_n - \Delta_i \Sigma_{ti}}{2\mu_n + \Delta_i \Sigma_{ti}} \psi_{i-1/2,n} + \frac{2\Delta_i Q_i}{\Delta_i \Sigma_{ti} + 2\mu_n}}$$

$\mu < 0$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2} = \Delta_i Q_i$$

And since we're traveling from right to left, that means we're starting with $i + 1/2$ and we're trying to get to $i - 1/2$. So I'll solve for $i - 1/2$ in terms of $i + 1/2$.

$$2\mu_n \psi_{i+\frac{1}{2},n} - 2\mu_n \psi_{i-\frac{1}{2},n} + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} + 2\mu_n) \psi_{i+\frac{1}{2},n} + (\Delta_i \Sigma_{ti} - 2\mu_n) \psi_{i-1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} - 2\mu_n) \psi_{i-1/2,n} = -(\Delta_i \Sigma_{ti} + 2\mu_n) \psi_{i+\frac{1}{2},n} + 2\Delta_i Q_i$$

$$\psi_{i-1/2,n} = \frac{-(\Delta_i \Sigma_{ti} + 2\mu_n)}{(\Delta_i \Sigma_{ti} - 2\mu_n)} \psi_{i+\frac{1}{2},n} + \frac{2\Delta_i Q_i}{(\Delta_i \Sigma_{ti} - 2\mu_n)}$$

$$\boxed{\psi_{i-1/2,n} = \frac{2\mu_n + \Delta_i \Sigma_{ti}}{2\mu_n - \Delta_i \Sigma_{ti}} \psi_{i+\frac{1}{2},n} - \frac{2\Delta_i Q_i}{2\mu_n - \Delta_i \Sigma_{ti}}}$$

Step-Characteristic

The main approximation here is that

$$\psi_{i,n} = \psi_{i-1/2,n} e^{-\Sigma_{ti} \Delta_i / \mu_n} + \frac{Q_i}{\Sigma_{ti}} \left(1 - e^{-\Sigma_{ti} \Delta_i / \mu_n}\right) \text{ for } \mu > 0$$

$\mu > 0$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$\mu < 0$

And since we're traveling from right to left, that means we're starting with $i + 1/2$ and we're trying to get to $i - 1/2$. So I'll solve for $i - 1/2$ in terms of $i + 1/2$.