S_N Method

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The main equation that we want to solve is

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

where i is for the cell number, and n is for S_n, and deals with the Gauss-Legendre quadrature order that we want to use. Like, for S_2 commonly $\mu_1 = -\mu_2$ and $w_1 = w_2 = 1$.

There are three approaches that we're going to use that start from here. Step, Diamond-Difference, and Step-Characteristics.

Step

The main approximation here is that $\psi_i \approx \psi_{i-1/2}$ for $\mu > 0$ and $\psi_i \approx \psi_{i+1/2}$ for $\mu < 0$.

$\mu > 0$

$$\psi_i = \psi_{i-1/2}$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} = \Delta_i Q_i$$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$$\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i-1/2,n}}{\mu_n}$$

$$\psi_{i+\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i-1/2,n}}{\mu_n} + \psi_{i-\frac{1}{2},n}$$

$$\boxed{\psi_{i+\frac{1}{2},n} = \frac{\Delta_i Q_i}{\mu_n} + \psi_{i-1/2,n} \left(1 - \frac{\Delta_i \Sigma_{ti}}{\mu_n}\right)}$$

$\mu < 0$

$$\psi_i = \psi_{i+1/2}$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = \Delta_i Q_i$$

And since we're traveling from right to left, that means we're starting with $i + 1/2$ and we're trying to get to $i - 1/2$. So I'll solve for $i - 1/2$ in terms of $i + 1/2$.

$$\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n} = \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i+1/2,n}}{\mu_n}$$

$$\psi_{i-\frac{1}{2},n} = \psi_{i+\frac{1}{2},n} - \frac{\Delta_i Q_i - \Delta_i \Sigma_{ti} \psi_{i+1/2,n}}{\mu_n}$$

$$\boxed{\psi_{i-\frac{1}{2},n} = -\frac{\Delta_i Q_i}{\mu_n} + \psi_{i+\frac{1}{2},n} \left(1 + \frac{\Delta_i \Sigma_{ti}}{\mu_n}\right)}$$

Diamond-Difference

The main approximation here is that

$$\psi_{i,n} = \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2}$$

$\mu > 0$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2} = \Delta_i Q_i$$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$$2\mu_n \psi_{i+\frac{1}{2},n} - 2\mu_n \psi_{i-\frac{1}{2},n} + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} + 2\mu_n)\psi_{i+\frac{1}{2},n} + (\Delta_i \Sigma_{ti} - 2\mu_n)\psi_{i-1/2,n} = 2\Delta_i Q_i$$

$$\psi_{i+\frac{1}{2},n} = \frac{2\Delta_i Q_i - (\Delta_i \Sigma_{ti} - 2\mu_n)\psi_{i-1/2,n}}{\Delta_i \Sigma_{ti} + 2\mu_n}$$

$$\boxed{\psi_{i+\frac{1}{2},n} = \frac{2\mu_n - \Delta_i \Sigma_{ti}}{2\mu_n + \Delta_i \Sigma_{ti}} \psi_{i-1/2,n} + \frac{2\Delta_i Q_i}{\Delta_i \Sigma_{ti} + 2\mu_n}}$$

$\mu < 0$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \frac{\psi_{i-1/2,n} + \psi_{i+1/2,n}}{2} = \Delta_i Q_i$$

And since we're traveling from right to left, that means we're starting with $i + 1/2$ and we're trying to get to $i - 1/2$. So I'll solve for $i - 1/2$ in terms of $i + 1/2$.

$$2\mu_n \psi_{i+\frac{1}{2},n} - 2\mu_n \psi_{i-\frac{1}{2},n} + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} + \Delta_i \Sigma_{ti} \psi_{i+1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} + 2\mu_n)\psi_{i+\frac{1}{2},n} + (\Delta_i \Sigma_{ti} - 2\mu_n)\psi_{i-1/2,n} = 2\Delta_i Q_i$$

$$(\Delta_i \Sigma_{ti} - 2\mu_n)\psi_{i-1/2,n} = -(\Delta_i \Sigma_{ti} + 2\mu_n)\psi_{i+\frac{1}{2},n} + 2\Delta_i Q_i$$

$$\psi_{i-1/2,n} = \frac{-(\Delta_i \Sigma_{ti} + 2\mu_n)}{(\Delta_i \Sigma_{ti} - 2\mu_n)} \psi_{i+\frac{1}{2},n} + \frac{2\Delta_i Q_i}{(\Delta_i \Sigma_{ti} - 2\mu_n)}$$

$$\boxed{\psi_{i-1/2,n} = \frac{2\mu_n + \Delta_i \Sigma_{ti}}{2\mu_n - \Delta_i \Sigma_{ti}} \psi_{i+\frac{1}{2},n} - \frac{2\Delta_i Q_i}{2\mu_n - \Delta_i \Sigma_{ti}}}$$

Step-Characteristic

The main approximation here is that

$$\psi_{i,n} = \psi_{i-1/2,n} e^{-\Sigma_{ti}\Delta_i/\mu_n} + \frac{Q_i}{\Sigma_{ti}} \left(1 - e^{-\Sigma_{ti}\Delta_i/\mu_n}\right) \text{ for } \mu > 0$$

$\mu > 0$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \psi_{i,n} = \Delta_i Q_i$$

$$\mu_n \left(\psi_{i+\frac{1}{2},n} - \psi_{i-\frac{1}{2},n}\right) + \Delta_i \Sigma_{ti} \left(\psi_{i-1/2,n} e^{-\Sigma_{ti}\Delta_i/\mu_n} + \frac{Q_i}{\Sigma_{ti}}\left(1 - e^{-\Sigma_{ti}\Delta_i/\mu_n}\right)\right) = \Delta_i Q_i$$

$$\mu_n \psi_{i+\frac{1}{2},n} - \mu_n \psi_{i-\frac{1}{2},n} + \Delta_i \Sigma_{ti} \psi_{i-1/2,n} e^{-\Sigma_{ti}\Delta_i/\mu_n} + \Delta_i Q_i - \Delta_i Q_i e^{-\Sigma_{ti}\Delta_i/\mu_n} = \Delta_i Q_i$$

And since we're traveling from left to right, that means we're starting with $i - 1/2$ and we're trying to get to $i + 1/2$. So I'll solve for $i + 1/2$ in terms of $i - 1/2$.

$$\mu_n \psi_{i+\frac{1}{2},n} = \Delta_i Q_i + \mu_n \psi_{i-\frac{1}{2},n} - \Delta_i \Sigma_{ti} \psi_{i-1/2,n} e^{-\Sigma_{ti}\Delta_i/\mu_n} - \Delta_i Q_i + \Delta_i Q_i e^{-\Sigma_{ti}\Delta_i/\mu_n}$$

$$\psi_{i+\frac{1}{2},n} = \psi_{i-\frac{1}{2},n} - \frac{\Delta_i \Sigma_{ti}}{\mu_n} \psi_{i-1/2,n} e^{-\Sigma_{ti}\Delta_i/\mu_n} + \frac{\Delta_i Q_i}{\mu_n} e^{-\Sigma_{ti}\Delta_i/\mu_n}$$

$$\boxed{\psi_{i+\frac{1}{2},n} = \left(1 - \frac{\Delta_i \Sigma_{ti}}{\mu_n} e^{-\Sigma_{ti}\Delta_i/\mu_n}\right) \psi_{i-1/2,n} + \frac{\Delta_i Q_i}{\mu_n} e^{-\Sigma_{ti}\Delta_i/\mu_n}}$$

Linear Discontinuous

So the idea of Linear Discontinuous is that we alternate between "known" values, and artificial values that make solving this problem easier. So when I'm going from left to right, I start with a known source at $i = 1/2$. I use that to calculate an artificial value at $i = 1/2$. This artificial value gives me the real value at $i = 3/2$, which I then use to get the artificial values at $i = 3/2$, etc.

$\mu > 0$

$$\begin{array}{ccccccccccccc} \text{Real} & \rightarrow & \text{Fake} & \rightarrow & \text{Real} & \rightarrow & \text{Fake} & \rightarrow & \text{Real} & \rightarrow & \text{Fake} & \rightarrow & \text{Real} \\ \psi_0^R & \rightarrow & \psi_1^L & \rightarrow & \psi_1^R & \rightarrow & \psi_2^L & \rightarrow & \psi_2^R & \rightarrow & \psi_3^L & \rightarrow & \psi_3^R \end{array}$$

where ψ_0^R is the flux on the right side of the zero'th cell (or, in this indexing, the furthest left cell), and similarly, ψ_3^R is the real flux on the right side of the third cell (which here is the furthest right cell).

1. Compute source
2. Solve for $\psi_{i,m}^L$ knowing $\psi_{i-1,m}^R$

$$\psi_{i,m}^L = \frac{\left(\tilde{Q}^L + 2\mu_m \psi_{i-1,m}^R - \Sigma_t \Delta x_i \frac{\tilde{Q}^R}{3T_{2/3}} - \mu_m \frac{\tilde{Q}^R}{T_{2/3}}\right)}{\left(\mu_m + \mu_m \frac{s_{1/3}}{T_{2/3}} + \frac{2\Sigma_t \Delta x_i}{3} + \frac{\Sigma_t \Delta x_i S_{1/3}}{3T_{2/3}}\right)}$$

$$\tilde{Q}^L = \left(\frac{1}{3}Q^R + \frac{2}{3}Q^L\right)\Delta x$$

$$\tilde{Q}^R = \left(\frac{2}{3}Q^R + \frac{1}{3}Q^L\right)\Delta x$$

3. Solve for $\psi_{i,m}^R$ knowing $\psi_{i,m}^L$

$$\psi_{i,m}^R = \frac{\tilde{Q}^R + \psi_{i,m}^L\left(\mu_m - \Sigma_{t,i}\Delta x_i/3\right)}{\left(\mu_m + 2\Sigma_{t,i}\Delta x_i/3\right)}$$

$$\tilde{Q}^R = \left(\frac{2}{3}Q^R + \frac{1}{3}Q^L\right)\Delta x$$

4. Sweep form left to right