

Introduction to Topological Groups

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To the memory of Ivan Prodanov

Abstract

These notes provide a brief introduction to topological groups with a special emphasis on Pontryagin-van Kampen's duality theorem for locally compact abelian groups. We give a completely self-contained elementary proof of the theorem following the line from [36]. According to the classical tradition, the structure theory of the locally compact abelian groups is built parallelly.

1 Introduction

Let \mathcal{L} denote the category of locally compact abelian groups and continuous homomorphisms and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle group. For $G \in \mathcal{L}$ denote by \widehat{G} the group of continuous homomorphisms (*characters*) $G \rightarrow \mathbb{T}$ equipped with the compact-open topology¹. Then the assignment

$$G \mapsto \widehat{G}$$

is a contravariant endofunctor $\widehat{\cdot} : \mathcal{L} \rightarrow \mathcal{L}$. The celebrated Pontryagin-van Kampen duality theorem ([82]) says that this functor is, up to natural equivalence, an involution i.e., $\widehat{\widehat{G}} \cong G$ (see Theorem 7.36 for more detail). Moreover, this functor sends compact groups to discrete ones and viceversa, i.e., it defines a duality between the subcategory \mathcal{C} of compact abelian groups and the subcategory \mathcal{D} of discrete abelian groups. This allows for a very efficient and fruitful tool for the study of compact abelian groups, reducing all problems to the related problems in the category of discrete groups. The reader is advised to give a look at the Mackey's beautiful survey [75] for the connection of charactres and Pontryagin-van Kampen duality to number theory, physics and elsewhere. This duality inspired a huge amount of related research also in category theory, a brief comment on a specific categorical aspect (uniqueness and representability) can be found in §8.1 of the Appendix.

The aim of these notes is to provide a self-contained proof of this remarkable duality theorem, providing all necessary steps, including basic background on topological groups and the structure theory of locally compact abelian groups. Peter-Weyl's theorem asserting that the continuous characters of the compact abelian groups separate the points of the groups (see Theorem 6.4) is certainly the most important tool in proving the duality theorem. The usual proof of Peter-Weyl's theorem involves Haar integration in order to produce sufficiently many finite-dimensional unitary representations. In the case of abelian groups the irreducible ones turn out to be one-dimensional, i.e., characters. We prefer here a different approach. Namely, Peter-Weyl's theorem can be obtained as an immediate corollary of a theorem of Følner (Theorem 5.12) whose elementary proof uses nothing beyond elementary properties of the finite abelian groups, a local version of the Stone-Weierstraß approximation theorem proved in §2 and the Stone-Čech compactification of discrete spaces. As another application of Følner's theorem we describe the precompact groups (i.e., the subgroups of the compact groups) as having a topology generated by continuous characters. As a third application of Følner's theorem one can obtain the existence of the Haar integral on locally compact abelian groups for free (see [36, §2.4, Theorem 2.4.5]).

The notes are organized as follows. In Section 2 we recall basic results and notions on abelian groups and general topology, which will be used in the rest of the paper. Section 3 contains background on topological groups, starting from scratch. Various ways of introducing a group topology are considered (§3.2), of which the prominent one is by means of characters (§3.2.3). In §3.6 we recall the construction of Protasov and Zelenyuk [88] of topologies arising from a given sequence that is required to be convergent to 0. Connectedness and related properties in topological groups are discussed in §3.5. In §3.7 the Markov's problems on the existence of non-discrete Hausdorff group topologies is discussed. In §3.7.1 we introduce two topologies, the Markov topology and the Zariski topology, that allow for an easier understanding of Markov's problems. In §3.7.2 we

¹having as a base of the neighborhoods of 0 the family of sets $W(K, U) = \{\chi \in \widehat{G} : \chi(K) \subseteq U\}$, where $K \subseteq G$ is compact and U is an open neighborhood of 0 in \mathbb{T} .

describe the Markov topology of the infinite permutation groups, while §7.3.3 contains the first two examples of non-topologizable groups, given by Shelah and Ol'shanskii, respectively. The problems arising in extension of group topologies are the topic of §7.3.4. Several cardinal invariants (weight, character and density character) are introduced in §3.8, whereas §3.9 discusses completeness and completions. Further general information on topological groups can be found in the monographs or surveys [3, 26, 27, 28, 36, 70, 79, 82].

Section 4 is dedicated to specific properties of the (locally) compact groups used essentially in these notes. The most important property we recall in §4.1 is the open mapping theorem. In §4.2 we give an internal description of the precompact groups using the notion of a big (large) set of the group. In §4.3 we recall (with complete proofs) the structure of the closed subgroups of \mathbb{R}^n as well as the description of the closure of an arbitrary subgroup of \mathbb{R}^n . These groups play an important role in the whole theory of locally compact abelian groups. In §5 we expose the proof of Følner's theorem (see Theorem 5.12). The proof, follows the line of [36]. An important ingredient of the proof is also the crucial idea, due to Prodanov, that consists in elimination of all discontinuous characters in the uniform approximation of continuous functions via linear combinations of characters obtained by means of Stone-Weierstraß approximation theorem (Prodanov's lemma 5.10).

In Section 6 we give various applications of Følner's theorem. The main one is an immediate proof of Peter-Weyl's theorem. In this chapter we give also several other applications of Følner's theorem and Peter-Weyl's theorem: a description of the precompact group topologies of the abelian groups (§6.1) and the structure of the compactly generated locally compact abelian groups (§6.3). Here we consider also a precompact version of Protasov and Zelenyuk's construction [88] of topologies making a fixed sequence converge to 0.

Section 7 is dedicated to Pontryagin-van Kampen duality. In §§7.1-7.3 we construct all tools for proving the duality theorem 7.36. More specifically, §§7.1 and 7.2 contain various properties of the dual groups that allow for an easier computation of the dual in many cases. Using further the properties of the dual, we see in §7.3 that many specific groups G satisfy the duality theorem, i.e., $G \cong \widehat{\widehat{G}}$. In §7.4 we stress the fact that the isomorphism $G \cong \widehat{\widehat{G}}$ is *natural* by studying in detail the natural transformation $\omega_G : G \rightarrow \widehat{\widehat{G}}$ connecting the group with its bidual. It is shown in several steps that ω_G is an isomorphism, considering larger and larger classes of locally compact abelian groups G where the duality theorem holds (elementary locally compact abelian groups, compact abelian groups, discrete abelian groups, compactly generated locally compact abelian groups). The last step uses the fact that the duality functor is exact, this permits us to use all previous steps in the general case.

In the Appendix we dedicate some time to several topics that are not discussed in the main body of the notes: uniqueness of the duality, dualities for non-abelian or non-locally compact-groups, some connection to the topological properties of compact group and dynamical systems.

A large number of exercises is given in the text to ease the understanding of the basic properties of group topologies and the various aspects of the duality theorem.

These notes are born out of two courses in the framework of the PhD programs at the Department of Mathematics at Milan University and the Department of Geometry and Topology at the Complutense University of Madrid held in April/May 2007. Among the participants there were various groups, interested in different fields. To partially satisfy the interest of the audience I included various parts that can be eventually skipped, at least during the first reading. For example, the reader who is not interested in non-abelian groups can skip §§3.2.4, the entire §3.7 and take all groups abelian in §§3 and 4 (conversely, the reader interested in non-abelian groups or rings may dedicate more time to §§3.2.4, 3.7 and consider the non-abelian case also in the first half of §4.2, see the footnote at the beginning of §4.2). For the category theorists §§4.3, 5.1-5.3, 6.2-6.3 may have less interest, compared to §§3.1-3.9, 4.2, 6.1, 7.1-7.4 and 8.1-8.2. Finally, those interested to get as fast as possible to the proof of the duality theorem can skip §§3.2.3, 3.2.4 and 3.6-3.9 (in particular, the route §§5-7 is possible for the reader with sufficient knowledge of topological groups).

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This notes are dedicated to the memory of Ivan Prodanov whose original contributions to Pontryagin-van Kampen duality can hardly be overestimated. The line adopted here follows his approach from [84] and [36].

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Contents

1	Introduction	i
2	Background on topological spaces and abstract groups	1
2.1	Background on abelian groups	1
2.2	Background on topological spaces	3
3	General properties of topological groups	5
3.1	Definition of a topological group	5
3.2	Examples of group topologies	6
3.2.1	Linear topologies	7
3.2.2	Topologies generated by characters	7
3.2.3	Pseudonorms and pseudometrics in a group	8
3.2.4	Permutation groups	8
3.3	Subgroups and quotients of topological groups	9
3.4	Separation axioms	11
3.5	Connectedness in topological groups	12
3.6	Group topologies determined by sequences	14
3.7	Markov's problems	15
3.7.1	The Zariski topology and the Markov topology	15
3.7.2	The Markov topology of the symmetric group	16
3.7.3	Existence of Hausdorff group topologies	18
3.7.4	Extension of group topologies	19
3.8	Cardinal invariants of topological groups	20
3.9	Completeness and completion	22
4	Compactness and local compactness in topological groups	23
4.1	Specific properties of (local) compactness	23
4.2	Subgroups of the compact groups	27
4.3	Subgroups of \mathbb{R}^n	29
5	Følner's theorem	32
5.1	Fourier theory for finite abelian groups	32
5.2	Bogoliouboff and Følner Lemmas	34
5.3	Prodanov's lemma and proof of Følner's theorem	37
6	Peter-Weyl's theorem and other applications of Følner's theorem	40
6.1	Precompact group topologies on abelian groups	40
6.2	Precompact group topologies determined by sequences	41
6.3	On the structure of compactly generated locally compact abelian groups	43
7	Pontryagin-van Kampen duality	46
7.1	The dual group	46
7.2	Computation of some dual groups	48
7.3	Some general properties of the dual	50
7.4	The natural transformation ω	53
8	Appendix	58
8.1	Uniqueness of Pontryagin-van Kampen duality	58
8.2	Non-abelian or non-locally compact groups	59
8.3	Relations to the topological theory of topological groups	59
8.4	Relations to dynamical systems	59
	Index	61
	Bibliography	61

Notation and terminology

We denote by \mathbb{P} , \mathbb{N} and \mathbb{N}_+ respectively the set of primes, the set of natural numbers and the set of positive integers. The symbol \mathfrak{c} stands for the cardinality of the continuum. The symbols \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} will denote the integers, the rationals, the reals and the complex numbers, respectively.

The quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a compact divisible abelian group, topologically isomorphic to the unitary circle \mathbb{S} (i.e., the subgroup of all $z \in \mathbb{C}$ with $|z| = 1$). For \mathbb{S} we use the multiplicative notation, while for \mathbb{T} we use the additive notation.

For an abelian group G we denote by $\text{Hom}(G, \mathbb{T})$ the group of all homomorphisms from G to \mathbb{T} written additively. The multiplicative form $G^* = \text{Hom}(G, \mathbb{S}) \cong \text{Hom}(G, \mathbb{T})$ will be used when necessary (e.g., concerning easier computation in \mathbb{C} , etc.). We call the elements of $\text{Hom}(G, \mathbb{T}) \cong \text{Hom}(G, \mathbb{S})$ *characters*.

For a topological group G we denote by $c(G)$ the connected component of the identity 1 in G . If $c(G)$ is trivial, the group G is said to be *totally disconnected*. If M is a subset of G then $\langle M \rangle$ is the smallest subgroup of G containing M and \overline{M} is the closure of M in G . The symbol $w(G)$ stands for the weight of G . Moreover \widetilde{G} stands for the completion of a Hausdorff topological abelian group G (see §3.9).

2 Background on topological spaces and abstract groups

2.1 Background on abelian groups

Generally a group G will be written multiplicatively and the neutral element will be denoted by e_G or simply e or 1 when there is no danger of confusion. For a subset A, A_1, A_2, \dots, A_n of a group G we write

$$A^{-1} = \{a^{-1} : a \in A\}, \quad \text{and} \quad A_1 A_2 \dots A_n = \{a_1 \dots a_n : a_i \in A_i, i = 1, 2, \dots, n\} \quad (*)$$

and we write A^n for $A_1 A_2 \dots A_n$ if all $A_i = A$. Moreover, for $A \subseteq G$ we denote by $c_G(A)$ the *centralizer* of A , i.e., the subgroup $\{x \in G : xa = ax \text{ for every } a \in A\}$.

We use additive notation for abelian groups, consequently 0 will denote the neutral element in such a case. Clearly, the counterpart of (*) will be $-A, A_1 + A_2 + \dots + A_n$ and nA .

A standard reference for abelian groups is the monograph [46]. We give here only those facts or definitions that appear very frequently in the sequel.

For $m \in \mathbb{N}_+$, we use \mathbb{Z}_m or $\mathbb{Z}(m)$ for the finite cyclic group of order m . Let G be an abelian group. The subgroup of torsion elements of G is $t(G)$ and for $m \in \mathbb{N}_+$

$$G[m] = \{x \in G : mx = 0\} \quad \text{and} \quad mG = \{mx : x \in G\}.$$

For a family $\{G_i : i \in I\}$ of groups we denote by $\prod_{i \in I} G_i$ the direct product G of the groups G_i . The underlying set of G is the Cartesian product $\prod_{i \in I} G_i$ and the operation is defined coordinatewise. The direct sum $\bigoplus_{i \in I} G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of all elements of finite support. If all G_i are isomorphic to the same group G and $|I| = \alpha$, we write $\bigoplus_\alpha G$ (or $G^{(\alpha)}$, or $\bigoplus_I G$) for the direct sum $\bigoplus_{i \in I} G_i$.

A subset X of an abelian group G is independent, if $\sum_{i=1}^n k_i x_i = 0$ with $k_i \in \mathbb{Z}$ and distinct elements x_i of X , $i = 1, 2, \dots, n$, imply $k_1 = k_2 = \dots = k_n = 0$. The maximum size of an independent subset of G is called *free-rank* of G and denoted by $r_0(G)$. An abelian group G is *free*, if G has an independent set of generators X . In such a case $G \cong \bigoplus_{|X|} \mathbb{Z}$.

For an abelian group G and a prime number p the subgroup $G[p]$ is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$. We denote by $r_p(G)$ its dimension over $\mathbb{Z}/p\mathbb{Z}$ and call it *p-rank* of G .

Let us start with the structure theorem for finitely generated abelian groups.

Theorem 2.1. *If G is a finitely generated abelian group, then G is a finite direct product of cyclic groups. Moreover, if G has m generators, then every subgroup of G is finitely generated as well and has at most m generators.*

Definition 2.2. An abelian group G is

- (a) *torsion* if $t(G) = G$;
- (b) *torsion-free* if $t(G) = 0$;
- (c) *bounded* if $mG = 0$ for some $m > 0$;
- (d) *divisible* if $G = mG$ for every $m > 0$;

(e) *reduced* if the only divisible subgroup of G is the trivial one.

Example 2.3. (a) The groups \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} are torsion-free. The class of torsion-free groups is stable under taking direct products and subgroups.

(b) The groups \mathbb{Z}_m , \mathbb{Q}/\mathbb{Z} are torsion. The class of torsion groups is stable under taking direct sums, subgroups and quotients.

(c) Let $m_1, m_2, \dots, m_k > 1$ be naturals and let $\alpha_1, \alpha_2, \dots, \alpha_k$ be cardinal numbers. Then the group $\bigoplus_{i=1}^k \mathbb{Z}_{m_i}^{(\alpha_i)}$ is bounded. According to a theorem of Prüfer every bounded abelian group has this form [46]. This generalizes the Frobenius-Stickelberger theorem about the structure of the finite abelian groups (see Theorem 2.1).

Example 2.4. (a) The groups \mathbb{Q} , \mathbb{R} , \mathbb{C} , and \mathbb{T} are divisible.

(b) For $p \in \mathbb{P}$ we denote by $\mathbb{Z}(p^\infty)$ the *Prüfer group*, namely the p -primary component of the torsion group \mathbb{Q}/\mathbb{Z} (so that $\mathbb{Z}(p^\infty)$ has generators $c_n = 1/p^n + \mathbb{Z}$, $n \in \mathbb{N}$). The group $\mathbb{Z}(p^\infty)$ is divisible.

(c) The class of divisible groups is stable under taking direct products, direct sums and quotients. In particular, every abelian group has a maximal divisible subgroup $d(G)$.

(d) [46] Every divisible group G has the form $(\bigoplus_{r_0(G)} \mathbb{Q}) \oplus (\bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(G))})$.

If X is a set, a set Y of functions of X to a set Z *separates the points* of X if for every $x, y \in X$ with $x \neq y$, there exists $f \in Y$ such that $f(x) \neq f(y)$. Now we see that the characters separate the points of a discrete abelian groups.

Theorem 2.5. *Let G be an abelian group, H a subgroup of G and D a divisible abelian group. Then for every homomorphism $f : H \rightarrow D$ there exists a homomorphism $\bar{f} : G \rightarrow D$ such that $\bar{f}|_H = f$.*

If $a \in G \setminus H$ and D contains elements of arbitrary finite order, then \bar{f} can be chosen such that $\bar{f}(a) \neq 0$.

Proof. Let H' be a subgroup of G such that $H' \supseteq H$ and suppose that $g : H' \rightarrow D$ is such that $g|_H = f$. We prove that for every $x \in G$, defining $N = H' + \langle x \rangle$, there exists $\bar{g} : N \rightarrow D$ such that $\bar{g}|_{H'} = g$. There are two cases.

If $\langle x \rangle \cap H' = \{0\}$, then take any $y \in D$ and define $\bar{g}(h + kx) = g(h) + ky$ for every $h \in H'$ and $k \in \mathbb{Z}$. Then \bar{g} is a homomorphism. This definition is correct because every element of N can be represented in a unique way as $h + kx$, where $h \in H'$ and $k \in \mathbb{Z}$.

If $C = \langle x \rangle \cap H' \neq \{0\}$, then C is cyclic, being a subgroup of a cyclic group. So $C = \langle lx \rangle$ for some $l \in \mathbb{Z}$. In particular, $lx \in H'$ and we can consider the element $a = g(lx) \in D$. Since D is divisible, there exists $y \in D$ such that $ly = a$. Now define $\bar{g} : N \rightarrow D$ putting $\bar{g}(h + ky) = g(h) + ky$ for every $h + kx \in N$, where $h \in H'$ and $k \in \mathbb{Z}$. To see that this definition is correct, suppose that $h + kx = h' + k'x$ for $h, h' \in H'$ and $k, k' \in \mathbb{Z}$. Then $h - h' = k'x - kx = (k' - k)x \in C$. So $k - k' = sl$ for some $s \in \mathbb{Z}$. Since $g : H' \rightarrow D$ is a homomorphism and $lx \in H'$, we have

$$g(h) - g(h') = g(h - h') = g(s(lx)) = sg(lx) = sa = sy = (k' - k)y = k'y - ky.$$

Thus, from $g(h) - g(h') = k'y - ky$ we conclude that $g(h) + ky = g(h') + k'y$. Therefore \bar{g} is correctly defined. Moreover \bar{g} is a homomorphism and extends g .

Let \mathcal{M} be the family of all subgroups H_i of G such that $H \leq H_i$ and of all homomorphisms $f_i : H_i \rightarrow D$ that extend $f : H \rightarrow D$. For $(H_i, f_i), (H_j, f_j) \in \mathcal{M}$ put $(H_i, f_i) \leq (H_j, f_j)$ if $H_i \leq H_j$ and f_j extends f_i . In this way (\mathcal{M}, \leq) is partially ordered. Let $\{(H_i, f_i)\}_{i \in I}$ a totally ordered subset of (\mathcal{M}, \leq) . Then $H_0 = \bigcup_{i \in I} H_i$ is a subgroup of G and $f_0 : H_0 \rightarrow D$ defined by $f_0(x) = f_i(x)$ whenever $x \in H_i$, is a homomorphism that extends f_i for every $i \in I$. This proves that (\mathcal{M}, \leq) is inductive and so we can apply Zorn's lemma to find a maximal element (H_{\max}, f_{\max}) of (\mathcal{M}, \leq) . It is easy to see that $H_{\max} = G$.

Suppose now that D contains elements of arbitrary finite order. If $a \in G \setminus H$, we can extend f to $H + \langle a \rangle$ defining it as in the first part of the proof. If $\langle a \rangle \cap H = \{0\}$ then $\bar{f}(h + ka) = f(h) + ky$ for every $k \in \mathbb{Z}$, where $y \in D \setminus \{0\}$. If $\langle a \rangle \cap H \neq \{0\}$, since D contains elements of arbitrary order, we can choose $y \in D$ such that $\bar{f}(h + ka) = f(h) + ky$ with $y \neq 0$. In both cases $\bar{f}(a) = y \neq 0$. \square

Corollary 2.6. *Let G be an abelian group and H a subgroup of G . If $\chi \in \text{Hom}(H, \mathbb{T})$ and $a \in G \setminus H$, then χ can be extended to $\bar{\chi} \in \text{Hom}(G, \mathbb{T})$, with $\bar{\chi}(a) \neq 0$.*

Corollary 2.7. *If G is an abelian group, then $\text{Hom}(G, \mathbb{T})$ separates the points of G .*

Corollary 2.8. *If G is an abelian group and D a divisible subgroup of G , then there exists a subgroup B of G such that $G = D \times B$.*

Proof. Consider the homomorphism $f : D \rightarrow G$ defined by $f(x) = x$ for every $x \in D$. By Theorem 2.5 we can extend f to $\bar{f} : G \rightarrow G$. Then put $B = \ker \bar{f}$ and observe that $G = D + B$ and $D \cap B = \{0\}$; consequently $G \cong D \times B$. \square

Corollary 2.9. *Every abelian group G can be written as $G = d(G) \times R$, where R is a reduced subgroup of G .*

Proof. By Corollary 2.8 there exists a subgroup R of G such that $G = d(G) \times R$. To conclude that R is reduced it suffices to apply the definition of $d(G)$. \square

The ring of endomorphisms of the group $\mathbb{Z}(p^\infty)$ will be denoted by \mathbb{J}_p , it is isomorphic to the inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$, known also as the ring of *p-adic integers*. The field of quotients of \mathbb{J}_p (i.e., the field of *p-adic numbers*) will be denoted by \mathbb{Q}_p . Sometimes we shall consider only the underlying groups of these rings (and speak of "the group *p-adic integers*", or "the group *p-adic numbers*").

2.2 Background on topological spaces

We assume the reader is familiar with the basic definitions and notions related to topological spaces. For the sake of completeness we recall here some frequently used properties related to compactness.

Definition 2.10. *A topological space X is*

- compact if for every open cover of X there exists a finite subcover;
- Lindelöf if for every open cover of X there exists a countable subcover;
- locally compact if every point of X has compact neighborhood in X ;
- σ -compact if X is the union of countably many compact subsets;
- of first category, if $X = \bigcup_{n=1}^{\infty} A_n$ and every A_n is a closed subset of X with empty interior;
- of second category, if X is not of first category;
- connected if for every proper open subset of X with open complement is empty.

Here we recall properties of maps:

Definition 2.11. For a map $f : (X, \tau) \rightarrow (Y, \tau')$ between topological spaces and a point $x \in X$ we say:

- f is *continuous* at x if for every neighborhood U of $f(x)$ in Y there exists a neighborhood V of x in X such that $f(V) \subseteq U$,
- f is *open* in $x \in X$ if for every neighborhood V of x in X there exists a neighborhood U of $f(x)$ in Y such that $f(V) \supseteq U$,
- f is continuous (resp., open) if f is continuous (resp., open) at every point $x \in X$.
- f is *closed* if the subset $f(A)$ of Y is closed for every closed subset $A \subseteq X$.

Some basic properties relating spaces to continuous maps are collected in the next lemma:

Lemma 2.12. • *If $f : X \rightarrow Y$ is a continuous surjective map, then Y is compact (resp., Lindelöf, σ -compact, connected) whenever X has the same property.*

- *If X is a closed subspace of a space Y , then X is compact (resp., Lindelöf, σ -compact, locally compact) whenever Y has the same property.*
- *If $X = \prod_{i \in I} X_i$, then X is compact (resp., connected) iff every space X_i has the same property. If I is finite, the same holds for local compactness and σ -compactness.*

A partially ordered set (A, \leq) is *directed* if for every $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. A subset B of A is *cofinal*, if for every $\alpha \in A$ there exists $\beta \in B$ with $\beta \geq \alpha$.

A *net* in a topological space X is a map from a directed set A to X . We write x_α for the image of $\alpha \in A$ so that the net can be written in the form $N = \{x_\alpha\}_{\alpha \in A}$. A *subnet* of a net N is $S = \{x_\beta\}_{\beta \in B}$ such that B is a cofinal subset of A .

A net $\{x_\alpha\}_{\alpha \in A}$ in X *converges* to $x \in X$ if for every neighborhood U of x in X there exists $\beta \in A$ such that $\alpha \in A$ and $\alpha \geq \beta$ implies $\alpha \in U$.

Lemma 2.13. *Let X be a topological space.*

- (a) *If Z is a subset of X , then $x \in \overline{Z}$ if and only if there exists a net in Z converging to x .*
- (b) *X is compact if and only if every net in X has a convergent subnet.*
- (c) *A function $f : X \rightarrow Y$ (where Y is a topological space) is continuous if and only if $f(x_\alpha) \rightarrow f(x)$ in Y for every net $\{x_\alpha\}_{\alpha \in A}$ in X with $x_\alpha \rightarrow x$.*
- (d) *The space X is Hausdorff if and only if every net in X converges to at most one point in X .*

Let us recall that the *connected component* of a point x in a topological space X is the largest connected subset of X containing x . It is always a closed subset of X . The space X is called *totally disconnected* if all connected components are singletons.

In a topological space X the *quasi-component* of a point $x \in X$ is the intersection of all clopen sets of X containing x .

Lemma 2.14. (Shura-Bura) *In a compact space X the quasi-components and the connected components coincide.*

A topological space X *zero-dimensional* if X has a base of clopen sets. Zero-dimensional T_2 spaces are totally disconnected (as every point is an intersection of clopen sets).

Theorem 2.15. (Vedenissov) *Every totally disconnected locally compact space is zero-dimensional.*

By βX we denote the *Čech-Stone compactification* of a topological Tychonov space X , that is the compact space βX together with the dense immersion $i : X \rightarrow \beta X$, such that for every function $f : X \rightarrow [0, 1]$ there exists $f^\beta : \beta X \rightarrow [0, 1]$ which extends f (this is equivalent to ask that every function of X to a compact space Y can be extended to βX). Here βX will be used only for a discrete space X .

Theorem 2.16 (Baire category theorem). *A Hausdorff locally compact space X is of second category.*

Proof. Suppose that $X = \bigcup_{n=1}^{\infty} A_n$ and assume that every A_n is closed with empty interior. Then the sets $D_n = G \setminus A_n$ are open and dense in X . To get a contradiction, we show that $\bigcap_{n=1}^{\infty} D_n$ is dense, in particular non-empty (so $G \neq \bigcup_{n=1}^{\infty} A_n$, a contradiction).

We use the fact that a Hausdorff locally compact space is regular. Pick an arbitrary open set $V \neq \emptyset$. Then there exists an open set $U_0 \neq \emptyset$ with $\overline{U_0}$ compact and $\overline{U_0} \subseteq V$. Since D_1 is dense, $U_0 \cap D_1 \neq \emptyset$. Pick $x_1 \in U_0 \cap D_1$ and an open set $U_1 \ni x_1$ in X with $\overline{U_1}$ compact and $\overline{U_1} \subseteq U_0 \cap D_1$. Proceeding in this way, for every $n \in \mathbb{N}_+$ we can find an open set $U_n \neq \emptyset$ in G with $\overline{U_n}$ compact and $\overline{U_n} \subseteq U_{n-1} \cap D_n$. By the compactness of every $\overline{U_n}$ there exists a point $x \in \bigcap_{n=1}^{\infty} \overline{U_n}$. Obviously, $x \in V \cap \bigcap_{n=1}^{\infty} D_n$. \square

Lemma 2.17. *If G is a locally compact σ -compact space, then G is a Lindelöff space.*

Proof. Let $G = \bigcup_{\alpha \in I} U_\alpha$. Since G is σ -compact, $G = \bigcup_{n=1}^{\infty} K_n$ where each K_n is a compact subset of G . Thus for every $n \in \mathbb{N}_+$ there exists a finite subset F_n of I such that $K_n \subseteq \bigcup_{\alpha \in F_n} U_\alpha$. Now $I_0 = \bigcup_{n=1}^{\infty} F_n$ is a countable subset of I and $K_n \subseteq \bigcup_{\alpha \in I_0} U_\alpha$ for every $n \in \mathbb{N}_+$ yields $G = \bigcup_{\alpha \in I_0} U_\alpha$. \square

Let X be a topological space. Let $\mathcal{C}(X, \mathbb{C})$ be the \mathbb{C} -algebra of all continuous complex valued functions on X . If $f \in \mathcal{C}(X, \mathbb{C})$ let

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

Theorem 2.18 (Stone-Weierstraß theorem). *Let X be a compact topological space. A \mathbb{C} -subalgebra \mathcal{A} of $\mathcal{C}(X, \mathbb{C})$ containing all constants and closed under conjugation is dense in $\mathcal{C}(X, \mathbb{C})$ for the norm $\|\cdot\|_\infty$ if and only if \mathcal{A} separates the points of X .*

We shall need in the sequel the following local form of Stone-Weierstraß theorem.

Corollary 2.19. *Let X be a compact topological space and $f \in \mathcal{C}(X, \mathbb{C})$. Then f can be uniformly approximated by a \mathbb{C} -subalgebra \mathcal{A} of $\mathcal{C}(X, \mathbb{C})$ containing all constants and closed under the complex conjugation if and only if \mathcal{A} separates the points of X separated by $f \in \mathcal{C}(X, \mathbb{C})$.*

Proof. Denote by $G : X \rightarrow \mathbb{C}^{\mathcal{A}}$ the diagonal map of the family $\{g : g \in \mathcal{A}\}$. Then $Y = G(X)$ is a compact subspace of $\mathbb{C}^{\mathcal{A}}$ and by the compactness of X , its subspace topology coincides with the quotient topology of the map $G : X \rightarrow Y$. The equivalence relation \sim in X determined by this quotient is as follows: $x \sim y$ for $x, y \in X$ by if and only if $G(x) = G(y)$ (if and only if $g(x) = g(y)$ for every $g \in \mathcal{A}$). Clearly, every continuous function $h : X \rightarrow \mathbb{C}$, such that $h(x) = h(y)$ for every pair x, y with $x \sim y$, can be factorized as $h = \bar{h} \circ q$, where $\bar{h} \in \mathcal{C}(Y, \mathbb{C})$. In particular, this holds true for all $g \in \mathcal{A}$ and for f (for the latter case this follows from our hypothesis). The \mathbb{C} -subalgebra $\bar{\mathcal{A}} \subseteq \mathcal{C}(Y, \mathbb{C})$ is closed under the complex conjugation and contains all constants. It is easy to see that it separates the points of Y . Hence we can apply Stone - Weierstraß theorem 2.18 to Y and $\bar{\mathcal{A}}$ to deduce that we can uniformly approximate the function \bar{f} by functions of $\bar{\mathcal{A}}$. This produces uniform approximation of the function f by functions of \mathcal{A} . \square

3 General properties of topological groups

3.1 Definition of a topological group

Let us start with the following fundamental concept:

Definition 3.1. Let G be a group.

- A topology τ on G is said to be a *group topology* if the map $f : G \times G \rightarrow G$ defined by $f(x, y) = xy^{-1}$ is continuous.
- A *topological group* is a pair (G, τ) of a group G and a group topology τ on G .

If τ is Hausdorff (resp., compact, locally compact, connected, etc.), then the topological group (G, τ) is called Hausdorff (resp., compact, locally compact, connected, etc.). Analogously, if G is cyclic (resp., abelian, nilpotent, etc.) the topological group (G, τ) is called cyclic (resp. abelian, nilpotent, etc.). Obviously, a topology τ on a group G is a group topology iff the maps

$$\mu : G \times G \rightarrow G \text{ and } \iota : G \rightarrow G$$

defined by $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$ are continuous when $G \times G$ carries the product topology.

Here are some examples, starting with two trivial ones: for every group G the discrete topology and the indiscrete topology on G are group topologies. Non-trivial examples of a topological group are provided by the additive group \mathbb{R} of the reals and by the multiplicative group \mathbb{S} of the complex numbers z with $|z| = 1$, equipped both with their usual topology. This extends to all powers \mathbb{R}^n and \mathbb{S}^n . These are abelian topological groups. For every n the linear group $GL_n(\mathbb{R})$ equipped with the topology induced by \mathbb{R}^{n^2} is a non-abelian topological group. The groups \mathbb{R}^n and $GL_n(\mathbb{R})$ are locally compact, while \mathbb{S} is compact.

Example 3.2. For every prime p the group \mathbb{J}_p of p -adic integers carries the topology induced by $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)$, when we consider it as the inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. The same topology can be obtained also when we consider \mathbb{J}_p as the ring of all endomorphisms of the group $\mathbb{Z}(p^\infty)$. Now \mathbb{J}_p embeds into the product $\mathbb{Z}(p^\infty)^{\mathbb{Z}(p^\infty)}$ carrying the product topology, while $\mathbb{Z}(p^\infty)$ is discrete. We leave to the reader the verification that this is a compact group topology on \mathbb{J}_p . Basic open neighborhoods of 0 in this topology are the subgroups $p^n\mathbb{J}_p$ of $(\mathbb{J}_p, +)$ (actually, these are ideals of the ring \mathbb{J}_p) for $n \in \mathbb{N}$. The field \mathbb{Q}_p becomes a locally compact group by declaring \mathbb{J}_p open in \mathbb{Q}_p (i.e., an element $x \in \mathbb{Q}_p$ has as typical neighborhoods the cosets $x + p^n\mathbb{J}_p$, $n \in \mathbb{N}$).

Other examples of group topologies will be given in §3.2.

If G is a topological group written multiplicatively and $a \in G$, then the *translations* $x \mapsto ax$ and $x \mapsto xa$ as well as the *internal automorphism* $x \mapsto axa^{-1}$ are homeomorphisms. Consequently, the group G is discrete iff the point 1 is isolated, i.e., the singleton $\{1\}$ is open. In the sequel aM will denote the image of a subset $M \subseteq G$ under the (left) translation $x \mapsto ax$, i.e., $aM := \{am : m \in M\}$. This notation will be extended also to families of subsets of G , in particular, for every filter \mathcal{F} we denote by $a\mathcal{F}$ the filter $\{aF : F \in \mathcal{F}\}$.

Making use of the homeomorphisms $x \mapsto ax$ one can prove:

Exercise 3.3. *Let $f : G \rightarrow H$ be a homomorphism between topological groups. Prove that f is continuous (resp., open) iff f is continuous (resp., open) at $1 \in G$.*

For a topological group G and $g \in G$ we denote by $\mathcal{V}_{G,\tau}(g)$ the filter of all neighborhoods of the element g of G . When no confusion is possible, we shall write briefly also $\mathcal{V}_G(g)$, $\mathcal{V}_\tau(g)$ or even $\mathcal{V}(g)$. Among these filters the filter $\mathcal{V}_{G,\tau}(1)$, obtained for the neutral element $g = 1$, plays a central role. It is useful to note that for every $a \in G$ the filter $\mathcal{V}_G(a)$ coincides with $a\mathcal{V}_G(1) = \mathcal{V}_G(1)a$. More precisely, we have the following:

Theorem 3.4. *Let G be a group and let $\mathcal{V}(1)$ be the filter of all neighborhoods of 1 in some group topology τ on G . Then:*

- (a) *for every $U \in \mathcal{V}(1)$ there exists $V \in \mathcal{V}(1)$ with $V \cdot V \subseteq U$;*
- (b) *for every $U \in \mathcal{V}(1)$ there exists $V \in \mathcal{V}(1)$ with $V^{-1} \subseteq U$;*
- (c) *for every $U \in \mathcal{V}(1)$ and for every $a \in G$ there exists $V \in \mathcal{V}(1)$ with $aVa^{-1} \subseteq U$.*

Conversely, if \mathcal{V} is a filter on G satisfying (a), (b) and (c), then there exists a unique group topology τ on G such that \mathcal{V} coincides with the filter of all τ -neighborhoods of 1 in G .

Proof. To prove (a) it suffices to apply the definition of the continuity of the multiplication $\mu : G \times G \rightarrow G$ at $(1, 1) \in G \times G$. Analogously, for (b) use the continuity of the map $\iota : G \rightarrow G$ at $1 \in G$. For item (c) use the continuity of the internal automorphism $x \mapsto axa^{-1}$ at $1 \in G$.

Let \mathcal{V} be a filter on G satisfying all conditions (a), (b) and (c). Let us see first that every $U \in \mathcal{V}$ contains 1. In fact, take $W \in \mathcal{V}$ with $W \cdot W \subseteq U$ and choose $V \in \mathcal{V}(1)$ with $V \subseteq W$ and $V^{-1} \subseteq W$. Then $1 \in V \cdot V^{-1} \subseteq U$.

Now define a topology τ on G whose open sets O are defined by the following property:

$$\tau := \{O \subseteq G : (\forall a \in O)(\exists U \in \mathcal{V}) \text{ such that } aU \subseteq O\}.$$

It is easy to see that τ is a topology on G . Let us see now that for every $g \in G$ the filter $g\mathcal{V}$ coincides with the filter $\mathcal{V}_{(G,\tau)}(g)$ of all τ -neighborhoods of g in (G, τ) . The inclusion $g\mathcal{V} \supseteq \mathcal{V}_{(G,\tau)}(g)$ is obvious. Assume $U \in \mathcal{V}$. To see that $gU \in \mathcal{V}_{(G,\tau)}(g)$ we have to find a τ -open $O \subseteq gU$ that contains g . Let $O := \{h \in gU : (\exists W \in \mathcal{V}) hW \subseteq gU\}$. Obviously $g \in O$. To see that $O \in \tau$ pick $x \in O$. Then there exists $W \in \mathcal{V}$ with $xW \subseteq gU$. Let $V \in \mathcal{V}$ with $V \cdot V \subseteq W$, then $xV \subseteq O$ since $xvV \subseteq gU$ for every $v \in V$.

We have seen that τ is a topology on G such that the τ -neighborhoods of any $x \in G$ are given by the filter $x\mathcal{V}$. It remains to see that τ is a group topology. To this end we have to prove that the map $(x, y) \mapsto xy^{-1}$ is continuous. Fix x, y and pick a $U \in \mathcal{V}$. By (c) there exists a $W \in \mathcal{V}$ with $Wy^{-1} \subseteq y^{-1}U$. Now choose $V \in \mathcal{V}$ with $V \cdot V^{-1} \subseteq W$. Then $O = xV \times yV$ is a neighborhood of (x, y) in $G \times G$ and $f(O) \subseteq xV \cdot V^{-1}y^{-1} \subseteq xWy^{-1} \subseteq xy^{-1}U$. \square

In the above theorem one can take instead of a filter \mathcal{V} also a *filter base*, i.e., a family \mathcal{V} with the property

$$(\forall U \in \mathcal{V})(\forall V \in \mathcal{V})(\exists W \in \mathcal{V}) W \subseteq U \cap V$$

beyond the proprieties (a)–(c).

A neighborhood $U \in \mathcal{V}(1)$ is *symmetric*, if $U = U^{-1}$. Obviously, for every $U \in \mathcal{V}(1)$ the intersection $U \cap U^{-1} \in \mathcal{V}(1)$ is a symmetric neighborhood, hence every neighborhood of 1 contains a symmetric one.

Let $\{\tau_i : i \in I\}$ be a family of group topologies on a group G . Then their supremum $\tau = \sup_{i \in I} \tau_i$ is a group topology on G with a base of neighborhoods of 1 formed by the family of all finite intersection $U_1 \cap U_2 \cap \dots \cap U_n$, where $U_k \in \mathcal{V}_{\tau_{i_k}}(1)$ for $k = 1, 2, \dots, n$ and the n -tuple i_1, i_2, \dots, i_n runs over all finite subsets of I .

Exercise 3.5. *If (a_n) is a sequence in G such that $a_n \rightarrow 1$ for every member τ_i of a family $\{\tau_i : i \in I\}$ of group topologies on a group G , then $a_n \rightarrow 1$ also for the supremum $\sup_{i \in I} \tau_i$.*

3.2 Examples of group topologies

Now we give several series of examples of group topologies, introducing them by means of the filter $\mathcal{V}(1)$ of neighborhoods of 1 as explained above. However, in all cases we avoid the treat the whole filter $\mathcal{V}(1)$ and we prefer to deal with an essential part of it, namely a base. Let us recall the precise definition of a base of neighborhoods.

Definition 3.6. Let G be a topological group. A family $\mathcal{B} \subseteq \mathcal{V}(1)$ is said to be a *base of neighborhoods of 1* (or briefly, a *base at 1*) if for every $U \in \mathcal{V}(1)$ there exists a $V \in \mathcal{B}$ contained in U (such a family will necessarily be a filterbase).

3.2.1 Linear topologies

Let $\mathcal{V} = \{N_i : i \in I\}$ be a filter base consisting of normal subgroups of a group G . Then \mathcal{V} satisfies (a)–(c), hence generates a group topology on G having as basic neighborhoods of a point $g \in G$ the family of cosets $\{gN_i : i \in I\}$. Group topologies of this type will be called *linear topologies*. Let us see now various examples of linear topologies.

Example 3.7. Let G be a group and let p be a prime:

- the *pro-finite* topology, with $\{N_i : i \in I\}$ all normal subgroups of finite index of G ;
- the *pro- p -finite* topology, with $\{N_i : i \in I\}$ all normal subgroups of G of finite index that is a power of p ;
- the *p -adic* topology, with $I = \mathbb{N}$ and for $n \in \mathbb{N}$, N_n is the subgroup (necessarily normal) of G generated by all powers $\{g^{p^n} : g \in G\}$.
- the *natural* topology (or *\mathbb{Z} -topology*), with $I = \mathbb{N}$ and for $n \in \mathbb{N}$, N_n is the subgroup (necessarily normal) of G generated by all powers $\{g^n : g \in G\}$.
- the *pro-countable* topology, with $\{N_i : i \in I\}$ all normal subgroups of at most countable index $[G : N_i]$.

The next simple construction belongs to Taimanov. Now neighborhoods of 1 are subgroups, that are not necessarily normal.

Exercise 3.8. Let G be a group with trivial center. Then G can be considered as a subgroup of $\text{Aut}(G)$ making use of the internal automorphisms. Identify $\text{Aut}(G)$ with a subgroup of the power G^G and equip $\text{Aut}(G)$ with the group topology τ induced by the product topology of G^G , where G carries the discrete topology. Prove that:

- the filter of all τ -neighborhoods of 1 has as base the family of centralizers $\{c_G(F)\}$, where F runs over all finite subsets of G ;
- τ is Hausdorff;
- τ is discrete iff there exists a finite subset of G with trivial centralizer.

3.2.2 Topologies generated by characters

Let G be an abelian group. A *character* of G is a homomorphism $\chi : G \rightarrow \mathbb{S}$. For characters χ_i , $i = 1, \dots, n$, of G and $\delta > 0$ let

$$U_G(\chi_1, \dots, \chi_n; \delta) := \{x \in G : |\text{Arg}(\chi_i(x))| < \delta, i = 1, \dots, n\}, \quad (1)$$

where the argument $\text{Arg}(z)$ of a complex number z is taken in $(-\pi, \pi]$.

Exercise 3.9. Let G be an abelian group and let H be a family of characters of G . Then the family

$$\{U_G(\chi_1, \dots, \chi_n; \delta) : \delta > 0, \chi_i \in H, i = 1, \dots, n\}$$

is a filter base satisfying the conditions (a)–(c) of Theorem 3.4, hence it gives rise to a group topology \mathcal{T}_H on G (this is the initial topology of the family H , i.e., the coarsest topology that makes continuous all the characters of H).

We refer to the group topology \mathcal{T}_H as *topology generated by the characters* of H . The topology \mathcal{T}_{G^*} , generated by all characters of G , is called *Bohr topology* of G .

For an abelian group G some of the linear topologies on G are also generated by appropriate families of characters.

Exercise 3.10. Let G be an abelian group.

1. Prove that the profinite topology of G is contained in the Bohr topology of G . Give an example of a group G where these two topologies differ.
2. Let H be the family of all characters χ of G such that the subgroup $\chi(G)$ is finite. Prove that the topology \mathcal{T}_H coincides with the pro-finite topology on G .
3. Let H be the family of all characters χ of G such that the subgroup $\chi(G)$ is finite and contained in the subgroup $\mathbb{Z}(p^\infty)$ of \mathbb{T} . Prove that the topology \mathcal{T}_H coincides with the pro- p -finite topology on G .

This exercise suggests to call a character $\chi : G \rightarrow \mathbb{T}$ *torsion* if there exists $n > 0$ such that χ vanishes on the subgroup $nG := \{nx : x \in G\}$. (Equivalently, the character $n \cdot \chi$ coincides with the trivial character, where the character $n \cdot \chi : G \rightarrow \mathbb{T}$ is defined by $(n \cdot \chi)(x) := n\chi(x)$.)

Exercise 3.11. *Let G be an abelian group. Prove that:*

1. *if H is a family of characters of G , then the topology T_H is contained in the pro-finite topology of G iff every character of H is torsion.*
2. *if G is bounded, then the Bohr topology of G coincides with the profinite topology of G .*
3. *if the Bohr topology of G coincides with the profinite topology of G , then G is bounded.*

3.2.3 Pseudonorms and pseudometrics in a group

According to Markov a *pseudonorm* in an abelian group G is a map $\nu : G \rightarrow \mathbb{R}_+$ such that for every $x, y \in G$:

- (1) $\nu(1) = 0$;
- (2) $\nu(x^{-1}) = \nu(x)$;
- (3) $\nu(xy) \leq \nu(x) + \nu(y)$.

The norms defined in a real vector space are obviously pseudonorms (with the additional property, in additive notation, $\nu(0) = 0$ iff $x = 0$).

Every pseudonorm ν generates a pseudometric d_ν on G defined by $d_\nu(x, y) := \nu(x^{-1}y)$. This pseudometric is *left invariant* in the sense that $d_\nu(ax, ay) = d_\nu(x, y)$ for every $a, x, y \in G$. Denote by τ_ν the topology induced on G by this pseudometric. A base of $\mathcal{V}_{\tau_\nu}(1)$ is given by the open balls $\{B_{1/n}(1) : n \in \mathbb{N}_+\}$.

In order to build metrics inducing the topology of a given topological group (G, τ) we need the following lemma (for a proof see [67, 8.2], [79]). We say that a pseudometric d on G is *continuous* if the map $d : G \times G \rightarrow \mathbb{R}_+$ is continuous. This is equivalent to have the topology induced by the metric d coarser than the topology τ (i.e., every open set with respect to the metric d is τ -open).

Lemma 3.12. *Let G be a topological group and let*

$$U_0 \supseteq U_1 \supseteq \dots \supseteq U_n \supseteq \dots \quad (2)$$

be symmetric neighborhoods of 1 with $U_n^3 \subseteq U_{n-1}$ for every $n \in \mathbb{N}$. Then there exists a continuous left invariant pseudometric d on G such that $U_n \subseteq B_{1/n}(1) \subseteq U_{n-1}$ for every n .

Exercise 3.13. *Prove that in the previous lemma $H = \bigcap_{n=1}^\infty U_n$ is a closed subgroup of G with the property $H = \{x \in G : d(x, 1) = 0\}$. In particular, d is a metric iff $H = \{1\}$.*

If the chain (2) has also the property $xU_nx^{-1} \subseteq U_{n-1}$ for every $x \in G$ and for every n , the subgroup H is normal and d defines a metric on the quotient group letting $\tilde{d}(xH, yH) := d(x, y)$. The metric \tilde{d} induces the quotient topology on G/H .

3.2.4 Permutation groups

Let X be an infinite set and let G briefly denotes the group $S(X)$ of all permutations of X . A very natural topology on G is defined by taking as filter of neighborhoods of $1 = id_X$ the family of all subgroups of G of the form

$$S_F = \{f \in G : (\forall x \in F) f(x) = x\},$$

where F is a finite subset of X .

This topology can be described also as the topology induced by the natural embedding of G into the Cartesian power X^X equipped with the product topology, where X has the discrete topology.

This topology is also the point-wise convergence topology on G . Namely, if $(f_i)_{i \in I}$ is a net in G , then f_i converges to $f \in G$ precisely when for every $x \in X$ there exists an $i_0 \in I$ such that for all $i \geq i_0$ in I one has $f_i(x) = f(x)$.

Exercise 3.14. *If $S_\omega(X)$ denotes the subset of all permutations of finite support in $S(X)$ prove that S_ω is a dense normal subgroup of G .*

Exercise 3.15. *Prove that $S(X)$ has no proper closed normal subgroups.*

3.3 Subgroups and quotients of topological groups

Let G be a topological group and let H be a subgroup of G . Then H becomes a topological group when endowed with the topology induced by G . Sometimes we refer to this situation by saying H is a topological subgroup of G .

Let G and H be topological groups and let $f : G \rightarrow H$ be a continuous homomorphism. If f is simultaneously an isomorphism and a homeomorphism, then f is called a *topological isomorphism*. If $f : G \rightarrow f(G) \subseteq H$ is a topological isomorphism, where $f(G)$ carries the topology induced by H , then f is called *topological group embedding*, or shortly *embedding*.

Proposition 3.16. *Let G be a topological group and let H be a subgroup of G . Then:*

- (a) *H is open in G iff H has a non-empty interior;*
- (b) *if H is open, then H is also closed;*
- (c) *if H is discrete and G is T_1 , then H is closed.*

Proof. (a) Let $\emptyset \neq V \subseteq H$ be an open set and let $h_0 \in V$. Then $1 \in h_0^{-1}V \subseteq H = h_0^{-1}H$. Now $U = h_0^{-1}V$ is open, contains 1 and $h \in hU \subseteq H$ for every $h \in H$. Therefore H is open.

(b) If H is open then every coset gH is open and consequently the complement $G \setminus H$ is open. So H is closed.

(c) Since H is discrete there exists $U \in \mathcal{V}(1)$ with $U \cap H = \{1\}$. Choose $V \in \mathcal{V}(1)$ with $V^{-1} \cdot V \subseteq U$. Then $|xV \cap H| \leq 1$ for every $x \in G$, as $h_1 = xv_1 \in xV \cap H$ and $h_2 = xv_2 \in xV \cap H$ give $h_1^{-1}h_2 \in V^{-1} \cdot V \cap H = \{1\}$, hence $h_1 = h_2$. Therefore, if $x \notin H$ one can find a neighborhood $W \subseteq xV$ of x with $W \cap H = \emptyset$, i.e., $x \notin \overline{H}$. Indeed, if $xV \cap H = \emptyset$, just take $W = xV$. In case $xV \cap H = \{h\}$ for some $h \in H$, one has $h \neq x$ as $x \notin H$. Then $W = xV \setminus \{x\}$ is the desired neighborhood of x . \square

Exercise 3.17. *Let H be a discrete non-trivial group and let $G = H \times N$, where N is an indiscrete non-trivial group. Prove that $H \times \{1\}$ is a discrete non-closed subgroup of G .*

Let us see now how the closure \overline{H} of a subset H of a topological group G can be computed.

Lemma 3.18. *Let H be a subset of G . Then with $\mathcal{V} = \mathcal{V}(1)$ one has*

- (a) $\overline{H} = \bigcap_{U \in \mathcal{V}} UH = \bigcap_{U \in \mathcal{V}} HU = \bigcap_{U, V \in \mathcal{V}} UHV$;
- (b) *if H is a subgroup of G , then \overline{H} is a subgroup of G ; if H a normal subgroup, then also \overline{H} is normal subgroup;*
- (c) $N = \{1\}$ *is a closed normal subgroup.*

Proof. (a) For $x \in G$ one has $x \notin \overline{H}$ iff there exists $U \in \mathcal{V}$ such that $xU \cap H = \emptyset = Ux \cap H$. Pick a symmetric U , i.e., $U = U^{-1}$. Then the latter property is equivalent to $x \notin UH \cup HU$. This proves $\overline{H} = \bigcap_{U \in \mathcal{V}} UH = \bigcap_{U \in \mathcal{V}} HU$. To prove the last equality in (a) note that the already established equalities yield

$$\bigcap_{U, V \in \mathcal{V}} UHV = \bigcap_{U \in \mathcal{V}} \left(\bigcap_{V \in \mathcal{V}} UHV \right) = \bigcap_{U \in \mathcal{V}} \overline{UH} \subseteq \bigcap_{U \in \mathcal{V}} U^2H = \bigcap_{W \in \mathcal{V}} WH = \overline{H}.$$

(b) Let $x, y \in \overline{H}$. According to (a), to verify $xy \in \overline{H}$ it suffices to see that $xy \in UHU$ for every $U \in \mathcal{V}$. This follows from $x \in UH$ and $y \in HU$ for every $U \in \mathcal{V}$. If H is normal, then for every $a \in G$ and for $U \in \mathcal{V}$ there exists a symmetric $V \in \mathcal{V}$ with $aV \subseteq Ua$ and $Va^{-1} \subseteq a^{-1}U$. Now for every $x \in \overline{H}$ one has $x \in VHV^{-1}$, hence $axa^{-1} \in aVHV^{-1}a^{-1} \subseteq UaHa^{-1}U \subseteq UHU$. This proves $axa^{-1} \in \overline{H}$ according to (a).

(c) follows from (b) with $H = \{1\}$. \square

Exercise 3.19. *Prove that:*

- *the subgroup $H \times \{1\}$ from Exercise 3.17 of G is dense.*
- *for every infinite set X and every group topology on the permutation group $S(X)$ the subgroups $S_x = \{f \in S(X) : f(x) = x\}$, $x \in X$, are either closed or dense. (Hint. Prove that S_x is a maximal subgroup of $S(X)$, see Fact 3.56.)*

Exercise 3.20. *Prove that every proper closed subgroup of \mathbb{R} is cyclic.*

(Hint. If H is a proper closed non-trivial subgroup of \mathbb{R} prove that the set $\{h \in H : h > 0\}$ has a greatest lower bound h_0 and conclude that $H = \langle h_0 \rangle$.)

Let G be a topological group and H a normal subgroup of G . Consider the quotient G/H with the quotient topology, namely the finest topology on G/H that makes the canonical projection $q : G \rightarrow G/H$ continuous. Since we have a group topology on G , the quotient topology consists of all sets $q(U)$, where U runs over the family of all open sets of G (as $q^{-1}(q(U))$ is open in G in such a case). In particular, the canonical projection q is open.

The next theorem is due to Frobenius.

Theorem 3.21. *If G and H are topological groups, $f : G \rightarrow H$ is a continuous surjective homomorphism and $q : G \rightarrow G/\ker f$ is the canonical homomorphism, then the unique homomorphism $f_1 : G/\ker f \rightarrow H$, such that $f = f_1 \circ q$, is a continuous isomorphism. Moreover, f_1 is a topological isomorphism iff f is open.*

Proof. Follows immediately from the definitions of quotient topology and open map. \square

Independently on its simplicity, this theorem is very important since it produces topological isomorphisms. Openness of the map f is its main ingredient, so from now on we shall be interested in providing conditions that ensure openness (see also §4.1).

Lemma 3.22. *Let X, Y be topological spaces and let $\varphi : X \rightarrow Y$ be a continuous open map. Then for every subspace P of Y with $P \cap \varphi(X) \neq \emptyset$ the restriction $\psi : H_1 \rightarrow P$ of the map φ to the subspace $H_1 = \varphi^{-1}(P)$ is open.*

Proof. To see that ψ is open choose a point $x \in H_1$ and a neighborhood U of x in H_1 . Then there exists a neighborhood W of x in X such that $U = H_1 \cap W$. To see that $\psi(U)$ is a neighborhood of $\psi(x)$ in P it suffices to note that if $\varphi(w) \in P$ for $w \in W$, then $w \in H_1$, hence $w \in H_1 \cap W = U$. Therefore $\varphi(W) \cap P \subseteq \varphi(U) = \psi(U)$. \square

We shall apply this lemma when $X = G$ and $Y = H$ are topological group and $\varphi = q : G \rightarrow H$ is a continuous open homomorphism. Then the restriction $q^{-1}(P) \rightarrow P$ of q is open for every subgroup P of H . Nevertheless, even in the particular case when q is surjective, the restriction $H_1 \rightarrow \varphi(H_1)$ of q to an arbitrary closed subgroup H_1 of G need not be open.

In the next theorem we see some isomorphisms related the quotient groups.

Teoema 3.23. *Let G be a topological group, let N be a normal closed subgroup of G and let $p : G \rightarrow G/N$ be the canonical homomorphism.*

- (a) *If H is a subgroup of G , then the homomorphism $i : HN/N \rightarrow p(H)$, defined by $i(xN) = p(x)$, is a topological isomorphism.*
- (b) *If H is a closed normal subgroup of G with $N \subseteq H$, then $p(H) = H/N$ is a closed normal subgroup of G/N and the map $j : G/H \rightarrow (G/N)/(H/N)$, defined by $j(xH) = (xN).(H/N)$, is a topological isomorphism.*

(Both in (a) and (b) the quotient groups are equipped with the quotient topology.)

Proof. (a) As $HN = p^{-1}(p(H))$ we can apply Lemma 3.22 and conclude that p' is an open map. Now Theorem 3.21 applies to the restriction $p' : HN \rightarrow p(H)$ of p .

(b) Since $H = HN$, item (a) implies that the induced topology of $p(H)$ coincides with the quotient topology of H/N . Hence we can identify H/N with the topological subgroup $p(H)$ of G/N . Since $H = HN$, the set $(G/N) \setminus p(HN) = p(G \setminus HN)$ is open, hence $p(H)$ is closed. Finally note that the composition $f : G \rightarrow (G/N)/(H/N)$ of p with the canonical homomorphism $G/N \rightarrow (G/N)/(H/N)$ is open, being the latter open. Applying to the open homomorphism f with $\ker f = H$ Theorem 3.21 we can conclude that j is a topological isomorphism. \square

Exercise 3.24. *Let G be an abelian group equipped with its Bohr topology and let H be a subgroup of G . Prove that:*

- H is closed in G ;
- the topological subgroup topology of H coincides with its Bohr topology;
- the quotient topology of G/H coincides with the Bohr topology of G/H .
- * G has no convergent sequences [36, §3.4].

Exercise 3.25. Let H be a discrete subgroup of a topological group G . Prove that:

- $H \cap \overline{\{1\}} = \{1\}$;
- \overline{H} is isomorphic to the semi-direct product of H and $\overline{\{1\}}$, carrying the product topology, where H is discrete and $\{1\}$ is indiscrete.

3.4 Separation axioms

Lemma 3.18 easily implies that every topological group is regular, hence:

Proposition 3.26. For a topological group G the following are equivalent:

- (a) G is Hausdorff;
- (b) G is T_0 .
- (c) G is T_3 (where T_3 stands for "regular and T_1 ").
- (d) $\overline{\{1\}} = \{1\}$.

A topological group G is *monothetic* if there exists $x \in G$ with $\langle x \rangle$ dense in G .

Exercise 3.27. Prove that:

- a Hausdorff monothetic group is necessarily abelian.
- \mathbb{T} is monothetic.

Is \mathbb{T}^2 monothetic? What about $\mathbb{T}^{\mathbb{N}}$?

Now we relate properties of the quotient G/H to those of the subgroup H of G .

Lemma 3.28. Let G be a topological group and let H be a normal subgroup of G . Then:

- (1) the quotient G/H is discrete if and only if H is open;
- (2) the quotient G/H is Hausdorff if and only if H is closed.

Let us see now that every T_0 topological group is also a Tychonov space.

Theorem 3.29. Every Hausdorff topological group is a Tychonov space.

Proof. Let F be a closed set with $1 \notin F$. Then we can find a chain (2) of open neighborhoods of 1 as in Lemma 3.12 such that $F \cap U_0 = \emptyset$. Let d be the pseudometric defined in Lemma 3.12 and let $f_F(x) = d(x, F)$ be the distance function from F . This function is continuous in the topology induced by the pseudometric. By the continuity of d it will be continuous also with respect to the topology of G . It suffices to note now that $f_F(F) = 0$, while $f_F(1) = 1$. This proves that the space G is Tychonov, as the pseudometric is left invariant, so the same argument provides separation of a generic point $a \in G$ from a closed set F that does not contain a . \square

Let G be an abelian group and let H be a family of characters of G . Then the characters of H separate the points of G iff for every $x \in G$, $x \neq 0$, there exists a character $\chi \in H$ with $\chi(x) \neq 1$.

Exercise 3.30. Let G be an abelian group and let H be a family of characters of G . Prove that the topology \mathcal{T}_H is Hausdorff iff the characters of H separate the points of G .

Proposition 3.31. Let G be an infinite abelian group and let $H = \text{Hom}(G, \mathbb{S})$. Then the following holds true:

- (a) the characters of H separate the points of G ,
- (b) the Bohr topology \mathcal{T}_H is Hausdorff and non-discrete.

Proof. (a) This is Corollary 2.7.

(b) According to Exercise 3.30 item (a) implies that the topology \mathcal{T}_H is Hausdorff. Suppose, for a contradiction, that \mathcal{T}_H is discrete. Then there exist $\chi_i \in H, i = 1, \dots, n$ and $\delta > 0$ such that $U(\chi_1, \dots, \chi_n; \delta) = \{0\}$. In particular, $H = \bigcap_{i=1}^n \ker \chi_i = \{0\}$. Hence the diagonal homomorphism $f = \chi_1 \times \dots \times \chi_n : G \rightarrow \mathbb{S}^n$ is injective and $f(G) \cong G$ is an infinite discrete subgroup of \mathbb{S}^n . According to Proposition 3.16 $f(G)$ is closed in \mathbb{S}^n and consequently, compact. The compact discrete spaces are finite, a contradiction. \square

Most often the topological groups in the sequel will be assumed to be Hausdorff.

Example 3.32. Contrary to what we proved in Theorem 3.29 Hausdorff topological groups need not be normal as topological spaces (see Exercise 3.37). A nice “uniform” counter-example to this was given by Trigos: for every uncountable group G the topological group $G^\#$ is not normal as a topological space (countable groups are ruled out since every countable Hausdorff topological group is normal, being a regular Lindelöf space).

Theorem 3.33. (Birkhoff-Kakutani) *A topological group is metrizable iff it has a countable base of neighborhoods of 1.*

Proof. The necessity is obvious as every point x in a metric space has a countable base of neighborhoods. Suppose now that G has countable base of neighborhoods of 1. Then one can build a chain (2) of neighborhoods of 1 as in Lemma 3.12 that form a base of $\mathcal{V}(1)$, in particular, $\bigcap_{n=1}^{\infty} U_n = \{1\}$. Then the pseudometric produced by the lemma is a metric that induces the topology of the group G because of the inclusions $U_n \subseteq B_{1/n} \subseteq U_{n-1}$. \square

Exercise 3.34. *Prove that subgroups and quotients of metrizable topological groups are metrizable.*

Exercise 3.35. *Prove that every topological abelian group admits a continuous isomorphism into a product of metrizable abelian groups.*

[Hint. For $x \in G$, $x \neq 0$ choose an open neighborhood U of 0 with $x \in U$. Build a sequence $\{U_n\}$ of symmetric open neighborhoods of 0 with $U_0 \subseteq U$ and $U_n + U_n \subseteq U_{n-1}$. Then $H_U = \bigcap_{n=1}^{\infty} U_n$ is a closed subgroup of G . Let τ_U be the group topology on the quotient G/H_U having as a local base at 0 the family $\{f_U(U_n)\}$, where $f_U : G \rightarrow G/H_U$ is the canonical homomorphism. Show that $(G/H, \tau_U)$ is metrizable. Now take the product of all groups $(G/H, \tau_U)$. To conclude observe that the diagonal map of the family f_U into the product of all groups $(G/H, \tau_U)$ is continuous and injective.]

Exercise 3.36. *Let G be a Hausdorff topological group. Prove that the centralizer of an element $g \in G$ is a closed subgroup. In particular, the center $Z(G)$ is a closed subgroup of G .*

Exercise 3.37.* *The group $\mathbb{Z}^{\mathbb{N}_1}$ equipped with the Tychonov topology (where \mathbb{Z} is discrete) is not a normal space [67].*

Furstenberg used the natural topology ν of \mathbb{Z} (see Example 3.7) to find a new proof of the infinitude of prime numbers.

Exercise 3.38. *Prove that there are infinitely many primes in \mathbb{Z} using the natural topology ν of \mathbb{Z} .*

(Hint. If p_1, p_2, \dots, p_n were the only primes, then consider the union of the open subgroups $p_1\mathbb{Z}, \dots, p_n\mathbb{Z}$ and use the fact that every integer $n \neq 0, \pm 1$ has a prime divisor, so belongs to $\bigcup_{i=1}^n p_i\mathbb{Z}$.)

3.5 Connectedness in topological groups

For a topological group G we denote by $c(G)$ the connected component of 1 and we call it briefly *connected component of G* .

Before proving some basic facts about the connected component, we need an elementary property of the connected sets in a topological groups.

Lemma 3.39. *Let G be a topological group.*

(a) *If C_1, C_2, \dots, C_n are connected sets in G , then also $C_1 C_2 \dots C_n$ is connected.*

(b) *If C is a connected set in G , then the set C^{-1} as well as the subgroup generated by C are connected.*

Proof. (a) Let us consider the case $n = 2$, the general case easily follows from this one by induction. The subset $C_1 \times C_2$ of $G \times G$ is connected. Now the map $\mu : G \times G \rightarrow G$ defined $\mu(x, y) = xy$ is continuous and $\mu(C_1 \times C_2) = C_1 C_2$.

(a) For the first part it suffices to note that C^{-1} is a continuous image of C under the continuous map $x \mapsto x^{-1}$.

To prove the second assertion consider the set $C_1 = CC^{-1}$. It is connected by the previous lemma and obviously $1 \in C_1$. Moreover, $C_1^2 \supseteq C \cup C^{-1}$. It remains to note now that the subgroup generated by C_1 coincides with the subgroup generated by C . Since the former is the union of all sets C_1^n , $n \in \mathbb{N}$ and each set C_1^n is connected by item (a), we are done. \square

Proposition 3.40. *The connected component $c(G)$ of a topological group G is a closed normal subgroup of G . The connected component of an element $x \in G$ is simply the coset $xc(G) = c(G)x$.*

Proof. To prove that $c(G)$ is stable under multiplication it suffices to note that $c(G)c(G)$ is still connected (applying item (a) of the above lemma) and contains 1, so must be contained in the connected component $c(G)$. Similarly, an application of item (b) implies that $c(G)$ is stable also w.r.t. the operation $x \mapsto x^{-1}$, so $c(G)$ is a subgroup of G . Moreover, for every $a \in G$ the image $ac(G)a^{-1}$ under the conjugation is connected and contains 1, so must be contained in the connected component $c(G)$. So $c(G)$ is stable also under conjugation. Therefore $c(G)$ is a normal subgroup. The fact that $c(G)$ is closed is well known.

To prove the last assertion it suffices to recall that the maps $y \mapsto xy$ and $y \mapsto yx$ are homeomorphisms. \square

Our next aim is to see that the quotient $G/c(G)$ is totally disconnected. We need first to see that connectedness and total connectedness are properties stable under extension:

Proposition 3.41. *Let G be a topological group and let N be a closed normal subgroup of G .*

- (a) *If both N and G/N are connected, then also G is connected.*
- (b) *If both N and G/N are totally disconnected, then also G is totally disconnected.*

Proof. Let $q : G \rightarrow G/N$ be the canonical homomorphism.

(a) Let $A \neq \emptyset$ be a clopen set of G . As every coset aN is connected, one has either $aN \subseteq A$ or $aN \cap A = \emptyset$. Hence, $A = q^{-1}(q(A))$. This implies that $q(A)$ is a non-empty clopen set of the connected group G/N . Thus $q(A) = G/N$. Consequently $A = G$.

(b) Assume C is a connected set in G . Then $q(C)$ is a connected set of G/N , so by our hypothesis, $q(C)$ is a singleton. This means that C is contained in some coset xN . Since xN is totally disconnected as well, we conclude that C is a singleton. This proves that G is totally disconnected. \square

Lemma 3.42. *If G is a topological group, then the group $G/c(G)$ is totally disconnected.*

Proof. Let $q : G \rightarrow G/c(G)$ be the canonical homomorphism and let H be the inverse image of $c(G/c(G))$ under q . Now apply Proposition 3.41 to the group H and the quotient group $H/c(G) \cong c(G/c(G))$ to conclude that H is connected. Since it contains $c(G)$, we have $H = c(G)$. Hence $G/c(G)$ is totally disconnected. \square

For a topological group G denote by $Q(G)$ the quasi-component of the neutral element 1 of G (i.e., the intersection of all clopen sets of G containing 1) and call it *quasi-component* of G .

Proposition 3.43. *For a topological group G the quasi-component $Q(G)$ is a closed normal subgroup of G . The quasi-component of $x \in G$ coincides with the coset $xQ(G) = Q(G)x$.*

Proof. Let $x, y \in Q(G)$. To prove that $xy \in Q(G)$ we need to verify that $xy \in O$ for every clopen set O containing 1. Let O be such a set, then $x, y \in O$. Obviously Oy^{-1} is a clopen set containing 1, hence $x \in Oy^{-1}$. This implies $xy \in O$. Hence $Q(G)$ is stable under multiplication. For every clopen set O containing 1 the set O^{-1} has the same propriety, hence $Q(G)$ is stable also w.r.t. the operation $a \mapsto a^{-1}$. This implies that $Q(G)$ is a subgroup. Moreover, for every $a \in G$ and for every clopen set O containing 1 also its image aOa^{-1} under the conjugation is a clopen set containing 1. So $Q(G)$ is stable also under conjugation. Therefore $Q(G)$ is a normal subgroup. Finally, as an intersection of clopen sets, $Q(G)$ is closed. \square

Remark 3.44. It follows from Lemma 2.14 that $c(G) = Q(G)$ for every compact topological group G . Actually, this remains true also in the case of locally compact groups G (cf. 4.22).

In the next remark we discuss zero-dimensionality.

Remark 3.45. (a) It follows immediately from Proposition 3.16 that every linear group topology is zero-dimensional; in particular, totally disconnected.

- (b) Every countable Hausdorff topological group is zero-dimensional (this is true for topological spaces as well).

We shall see in the sequel that for locally compact abelian groups or compact groups the implication from item (a) can be inverted (see Theorem 4.18). On the other hand, the next example shows that local connectedness is essential.

Example 3.46. The group \mathbb{Q}/\mathbb{Z} is zero-dimensional but has no proper open subgroups.

3.6 Group topologies determined by sequences

Let G be an abelian group and let (a_n) be a sequence in G . The question of the existence of a Hausdorff group topology that makes the sequence (a_n) converge to 0 is not only a mere curiosity. Indeed, assume that some Hausdorff group topology τ makes the sequence (p_n) of all primes converge to zero. Then $p_n \rightarrow 0$ would yield $p_n - p_{n+1} \rightarrow 0$ in τ , so this sequence cannot contain infinitely many entries equal to 2. This would provide a very easy negative solution to the celebrated problem of the infinitude of twin primes (actually this argument would show that the shortest distance between two consecutive primes converges to ∞).

Definition 3.47. [89] A sequence $A = \{a_n\}_n$ in an abelian group G is called a T -sequence if there exists a Hausdorff group topology on G such that $a_n \rightarrow 0$.²

Let (a_n) be a T -sequence in an abelian group G . Hence the family $\{\tau_i : i \in I\}$ of Hausdorff group topologies on the group G such that $a_n \rightarrow 0$ in τ_i is non-empty. Let $\tau = \sup_{i \in I} \tau_i$, then by Exercise 3.5 $a_n \rightarrow 0$ in τ as well. Clearly, this is the finest group topology in which a_n converges to 0. This is why we denote it by τ_A or $\tau_{(a_n)}$.³

Before discussing the topology $\tau_{(a_n)}$ and how T -sequences can be described in general we consider a couple of examples:

Example 3.48. (a) Let us see that the sequences (n^2) and (n^3) are not a T -sequence in \mathbb{Z} . Indeed, suppose for a contradiction that some Hausdorff group topology τ on \mathbb{Z} makes n^2 converge to 0. Then $(n+1)^2$ converges to 0 as well. Taking the difference we conclude that $2n+1$ converges to 0 as well. Since obviously also $2n+3$ converges to 0, we conclude, after subtraction, that the constant sequence 2 converges to 0. This is a contradiction, since τ is Hausdorff. We leave the case (n^3) as an exercise to the reader.

(b) A similar argument proves that the sequence $P_d(n)$, where $P_d(x) \in \mathbb{Z}[x]$ is a fixed polynomial with $\deg P_d = d > 0$, is not a T -sequence in \mathbb{Z} .

Protasov and Zelenyuk [88] established a number of nice properties of the finest group topology $\tau_{(a_n)}$ on G that makes (a_n) converge to 0.

For an abelian group G and subsets $A_1, \dots, A_n \dots$ of G we denote by $\pm A_1 \pm \dots \pm A_n$ the set of all sums $g = g_1 + \dots + g_n$, where $g_i \in \{0\} \cup A_i \cup -A_i$ for every $i = 1, \dots, n$. Let

$$\pm A_1 \pm \dots \pm A_n \pm \dots = \bigcup_{n=1}^{\infty} \pm A_1 \pm \dots \pm A_n.$$

If $A = \{a_n\}_n$ is a sequence in G , for $m \in \mathbb{N}$ denote by A_m the “tail” $\{a_m, a_{m+1}, \dots\}$. For $k \in \mathbb{N}$ let $A(k, m) = \pm A_m \pm \dots \pm A_m$ (k times).

Remark 3.49. The existence of a finest group topology τ_A on an abelian group G that makes an arbitrary given sequence $A = \{a_n\}_n$ in G converge to 0 is easy to prove as far as we are not interested on imposing the Hausdorff axiom. Indeed, as a_n converges to 0 in the indiscrete topology, τ_A is simply the supremum of all group topologies τ on G such that a_n converges to 0 in τ . This gives no idea on how this topology looks like. One can easily describe it as follows.

Let m_1, \dots, m_n, \dots be a sequence of natural numbers. Denote by $A(m_1, \dots, m_n, \dots)$ the set

$$\pm A_{m_1} \pm \dots \pm A_{m_n} \pm \dots$$

and by \mathcal{B}_A the family of all sets $A(m_1, \dots, m_n, \dots)$ when m_1, \dots, m_n, \dots vary in $\mathbb{N}^{\mathbb{N}}$. Then \mathcal{B}_A is a filter base, satisfying the axioms of group topology. The group topology τ defined in this way satisfies the required conditions. Indeed, obviously $a_n \rightarrow 0$ in (G, τ) and τ contains any other group topology with this property. Consequently, $\tau = \tau_A$.

Note that

$$A(k, m) \subseteq A(m_1, \dots, m_n, \dots), \quad (1)$$

for every $k \in \mathbb{N}$, where $m = \max\{m_1, \dots, m_k\}$. The sets $A(k, m)$, for $k, m \in \mathbb{N}$, form a filter base, but the filter they generate need not be the filter of neighborhoods of 0 in a group topology. The utility of this family becomes clear now.

²We shall see below that the sequence (p_n) of all primes is not a T -sequence in the group \mathbb{Z} (see Exercise 4.32). So the above mentioned possibility to resolve the problem of the infinitude of twin primes does not work.

³To simplify things we consider only sequences without repetition, hence the convergence to zero $a_n \rightarrow 0$ depends only on the set $A = \{a_n\}_n$, it does not depend on the enumeration of the sequence.

Theorem 3.50. *A sequence $A = \{a_n\}_n$ in an abelian group G is a T -sequence iff*

$$\bigcap_{m=1}^{\infty} A(k, m) = 0 \text{ for every } k \in \mathbb{N}. \quad (2)$$

Proof. Obviously the sequence $A = \{a_n\}_n$ is a T -sequence iff the topology τ_A is Hausdorff. Clearly, τ_A is Hausdorff iff $\bigcap_{m_1, \dots, m_n, \dots}^{\infty} A(m_1, \dots, m_n, \dots) = 0$. If τ_A is Hausdorff, then (2) holds by (1). It remains to see that (2) implies $\bigcap_{m_1, \dots, m_n, \dots}^{\infty} A(m_1, \dots, m_n, \dots) = 0$. First of all note that $A(m_1, \dots, m_n, \dots) \supseteq A(m_1^*, \dots, m_n^*, \dots)$, where $m_n^* = \max\{m_1, \dots, m_n\}$. Moreover, the sequence (m_n^*) is increasing. Hence

$$\bigcap_{m_1, \dots, m_n, \dots}^{\infty} A(m_1, \dots, m_n, \dots) = \bigcap_{m_1^*, \dots, m_n^*, \dots}^{\infty} A(m_1^*, \dots, m_n^*, \dots),$$

where the second intersection is taken only over the increasing sequences (m_n^*) . Obviously, for every increasing sequence (m_n^*) one has

$$A(m_1^*, \dots, m_n^*, \dots) \subseteq \bigcup_{k=1}^{\infty} A(k, m_1^*).$$

This yields

$$\bigcap_{m_1^*, \dots, m_n^*, \dots}^{\infty} A(m_1^*, \dots, m_n^*, \dots) \subseteq \bigcap_{m_1^*=1}^{\infty} \bigcup_{k=1}^{\infty} A(k, m_1^*) = \bigcup_{k=1}^{\infty} \bigcap_{m_1^*=1}^{\infty} A(k, m_1^*) = 0.$$

□

Since every infinite abelian group G admits a non-discrete metrizable group topology, there exist non-trivial (i.e., having all members non-zero) T -sequences.

A notion similar to T -sequence, but defined with respect to only topologies induced by characters, will be given in §6.2. From many points of view it turns out to be easier to deal with than T -sequence. In particular, we shall see easy sufficient condition for a sequence of integers to be a T -sequence.

We give without proof the following technical lemma that will be useful in §6.2.

Lemma 3.51. [89] *For every T -sequence $A = \{a_n\}$ in \mathbb{Z} there exists a sequence $\{b_n\}$ in \mathbb{Z} such that for every choice of the sequence (e_n) , where $e_n \in \{0, 1\}$, the sequence q_n defined by $q_{2n} = b_n + e_n$ and $q_{2n-1} = a_n$, is a T -sequence.*

Exercise 3.52. (a)* *Prove that there exists a T -sequence (a_n) in \mathbb{Z} with $\lim_n \frac{a_{n+1}}{a_n} = 1$ [89] (see also Example 6.12).*

(b)* *Every sequence (a_n) in \mathbb{Z} with $\lim_n \frac{a_{n+1}}{a_n} = +\infty$ is a T -sequence [89, 7] (see Theorem 6.11).*

(c)* *Every sequence (a_n) in \mathbb{Z} such that $\lim_n \frac{a_{n+1}}{a_n} \in \mathbb{R}$ is transcendental is a T -sequence [89].*

3.7 Markov's problems

3.7.1 The Zariski topology and the Markov topology

Let G be a Hausdorff topological group, $a \in G$ and $n \in \mathbb{N}$. Then the set $\{x \in G : x^n = a\}$ is obviously closed in G . This simple fact motivated the following notions due to Markov [76].

A subset S of a group G is called:

(a) *elementary algebraic* if there exist an integer $n > 0$, $a_1, \dots, a_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$S = \{x \in G : x^{\varepsilon_1} a_1 x^{\varepsilon_2} a_2 \dots a_{n-1} x^{\varepsilon_n} = a_n\},$$

(b) *algebraic* if S is an intersection of finite unions of elementary algebraic subsets,

(c) *unconditionally closed* if S is closed in every Hausdorff group topology of G .

Since the family of all finite unions of elementary algebraic subsets is closed under finite unions and contains all finite sets, it is a base of closed sets of some T_1 topology \mathfrak{Z}_G on G , called the *Zariski topology*⁴. Clearly, the \mathfrak{Z}_G -closed sets are precisely the algebraic sets in G .

Analogously, the family of all unconditionally closed subsets of G coincides with the family of closed subsets of a T_1 topology \mathfrak{M}_G on G , namely the infimum (taken in the lattice of all topologies on G) of all Hausdorff group topologies on G . We call \mathfrak{M}_G the *Markov topology* of G . Note that (G, \mathfrak{Z}_G) and (G, \mathfrak{M}_G) are quasi-topological groups, i.e., the inversion and translations are continuous. Nevertheless, when G is abelian (G, \mathfrak{Z}_G) and (G, \mathfrak{M}_G) are not group topologies unless they are discrete.

Since an elementary algebraic set of G must be closed in every Hausdorff group topology on G , one always has $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$. In 1944 Markov [76] asked if the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ holds for every group G . He himself showed that the answer is positive in case G is countable [76]. Moreover, in the same manuscript Markov attributes to Perel'man the fact that $\mathfrak{Z}_G = \mathfrak{M}_G$ for every Abelian group G (a proof has never appeared in print until [37]). An example of a group G with $\mathfrak{Z}_G \neq \mathfrak{M}_G$ was given by Gerchard Hesse [66].

Exercise 3.53. *Show that if (G, \cdot) is an abelian group, then every elementary algebraic set of G has the form $\{x \in G : x^n = a\}$, $a \in G$.*

3.7.2 The Markov topology of the symmetric group

Let X be an infinite set. In the sequel we denote by τ_X the pointwise convergence topology of the infinite symmetric group $S(X)$ defined in §3.2.4. It turns out that the Markov topology of $S(X)$ coincides with τ_X :

Theorem 3.54. *Then Markov topology on $S(X)$ coincides with the topology τ_X of pointwise convergence of $S(X)$.*

This theorem follows immediately from the following old result due to Gaughan.

Theorem 3.55. ([36]) *Every Hausdorff group topology of the infinite permutation group $S(X)$ contains the topology τ_X .*

The proof of this theorem follows more or less the line of the proof exposed in [36, §7.1] with several simplifications. The final stage of the proof is preceded by a number of claims (and their corollaries) and two facts about *purely algebraic* properties of the group $S(X)$ (3.56 and 3.59). The claims and their corollaries are given with complete proofs. To give an idea about the proofs of the two algebraic facts, we prove the first one; the proof of the second one can be found in [36, Lemmas 7.1.4, 7.1.8] (actually, only a fragment of the proof of [36, Lemmas 7.1.8] is needed for the proof of item (b) of Fact 3.59).

We say for a subset A of $S(X)$ that A is m -transitive for some positive integer m if for every $Y \subseteq X$ of size at most m and every injection $f : Y \rightarrow X$ there exists $a \in A$ that extends f .⁵ The leading idea is that a transitive subset A of $S(X)$ is placed “generically” in $S(X)$, whereas a non-transitive one is a subset of some subgroup of $S(X)$ that is a direct product $S(Y) \times S(X \setminus Y)$. (Here and in the sequel, for a subset Y of X we tacitly identify the group $S(Y)$ with the subgroup of $S(X)$ consisting of all permutations of $S(X)$ that are identical on $X \setminus Y$.)

The first fact concerns the stabilizers $S_x = S_{\{x\}} = \{f \in S(X) : f(x) = x\}$ of points $x \in X$. They constitute a prebase of the filter of neighborhoods of id_X in τ_X .

Fact 3.56. *For every $x \in X$ the subgroup S_x of $S(X)$ is maximal.*

Proof. Assume H is a subgroup of $S(X)$ properly containing S_x . To show that $H = S(X)$ take any $f \in S(X)$. If $y = f(x)$ coincides with x , then $f \in S_x \subseteq H$ and we are done. Assume $y \neq x$. Get $h \in H \setminus S_x$. Then $z = h(x) \neq x$, so $x \notin \{z, y\}$. There exists $g \in S(X)$ such that $g(x) = x, g(y) = z$ and $g(z) = y$. Then $g \in S_x \subseteq H$ and $f(x) = g(h(x)) = y$, so $h^{-1}g^{-1}f(x) = x$ and $h^{-1}g^{-1}f \in S_x \cap H \subseteq H$. So $f \in ghH = H$. \square

Claim 3.57. *Let T be a Hausdorff group topology on $S(X)$. If a subgroups of $S(X)$ of the form S_x is T -closed, then it is also T -open.*

Proof. As S_x is T -closed, for every fixed $y \neq x$ the set $V_y = \{f \in S(X) : f(x) \neq y\}$ is T -open and contains 1. So there exists a symmetric neighborhood W of 1 in T such that $W \cdot W \subseteq V_y$. By the definition of V_y this gives $Wx \cap Wy = \emptyset$. Then either $|X \setminus Wx| = |X|$ or $|X \setminus Wy| = |X|$. Suppose this occurs with x , i.e., $|X \setminus Wx| = |X|$.

⁴Some authors call it also the *verbal topology* [20], we prefer here *Zariski topology* coined by most authors [10].

⁵Note that a countable subset H of $S(X)$ cannot be transitive unless X itself is countable.

Then one can find a permutation $f \in S(X)$ that sends $Wx \setminus \{x\}$ to the complement of Wx and $f(x) = x$. Such an f satisfies:

$$fWf^{-1} \cap W \subseteq S_x$$

as $fWf^{-1}(x)$ meets Wx precisely in the singleton $\{x\}$ by the choice of f . This proves that S_x is T -open.

Analogous argument works for S_y when $|X \setminus Wy| = |X|$. \square

Corollary 3.58. *If T be a Hausdorff group topology on $S(X)$ that does not contain τ_X , then all subgroups of $S(X)$ of the form S_x are T -dense.*

Proof. Since the subgroups S_x of $S(X)$ form a prebase of the filter of neighborhoods of id_X in $S(X)$, our hypothesis implies that some subgroup S_x is not T -open. By Claim 3.57 S_x is not T -closed either. By Fact 3.56 S_x is T -dense. Since all subgroups of the form S_y are conjugated, this implies that stabilizers S_y are T -dense. \square

This was the first step in the proof. The next step will be establishing that $S_{x,y}$ are never dense in any Hausdorff group topology on $S(X)$ (Corollary 3.62).

In the sequel we need the subgroup $\tilde{S}_{x,y} := S_{x,y} \times S(\{x,y\})$ of $S(X)$ that contains $S_{x,y}$ as a subgroup of index 2. Note that $\tilde{S}_{x,y}$ is precisely the subgroup of all permutations in $S(X)$ that leave the doubleton $\{x,y\}$ set-wise invariant.

Fact 3.59. *For any doubleton x,y in X the following holds true:*

- (a) *the subgroup $\tilde{S}_{x,y}$ of $S(X)$ is maximal;*
- (b) *every proper subgroup of $S(X)$ properly containing $S_{x,y}$ coincides with one of the subgroups S_x, S_y or $\tilde{S}_{x,y}$.*

Claim 3.60. *Let T be a Hausdorff group topology on $S(X)$, then there exists a T -nbd of 1 that is not 2-transitive.*

Proof. Assume for a contradiction that all T -neighborhoods of id_X that are 2-transitive. Fix distinct $u, v, w \in X$. We show now that the 3-cycle $(u, v, w) \in V$ for every arbitrarily fixed T -neighborhood of id_X . Indeed, choose a symmetric T -neighborhood W of id_X such that $W^2 \subseteq V$. Let f be the transposition (uv) . Then $U = fWf \cap W \in T$ is a neighborhood of 1 and $fUf = U$. Since U is 2-transitive there exists $g \in U$ such that $g(u) = u$ and $g(v) = w$. Then $(u, v, w) = gfg^{-1}f \in W \cdot (fUf) \subseteq W^2 \subseteq V$. \square

Claim 3.61. *Let T be a group topology on $S(X)$. Then*

- (a) *every T -nbd V of id_X in $S(X)$ is transitive iff every stabilizer S_x is T -dense;*
- (b) *every T -nbd V of id_X in $S(X)$ is m -transitive iff every stabilizer S_F with $|F| \leq m$ is T -dense.*

Proof. Assume that some (hence all) S_z is T -dense in $S(X)$. To prove that V is transitive consider a pair $x, y \in X$. Let $t = (xy)$. By the T -density of S_x the T -nbd $t^{-1}V$ of t^{-1} meets S_x , i.e., for some $v \in V$ one has $t^{-1}v \in S_x$. Then $v \in tS_x$ obviously satisfies $vx = y$.

A similar argument proves that transitivity of each T -nbd of 1 entails that every stabilizer S_x is T -dense.

(b) The proof in the case $m > 1$ is similar. \square

What we really need further on (in particular, in the next corollary) is that the density of the stabilizers $S_{x,y}$ imply that every T -nbd V of id_X in $S(X)$ is 2-transitive.

Corollary 3.62. *Let T be a Hausdorff group topology on $S(X)$. Then $S_{x,y}$ is T -dense for no pair x, y in X .*

Proof. Follows from claims 3.60 and 3.61 \square

Proof of Theorem 3.55. Assume for a contradiction that T is a Hausdorff group topology on $S(X)$ that does not contain τ_X . Then by corollaries 3.58 and 3.62 all subgroups of the form S_x are T -dense and no subgroup of the form $S_{x,y}$ is T -dense. Now fix a pair $x, y \in X$ and let $G_{x,y}$ denote the T -closure of $S_{x,y}$. Then $G_{x,y}$ is a proper subgroup of $S(X)$ containing $S_{x,y}$. Since S_x is dense, $G_{x,y}$ cannot contain S_x , so $S_x \cap G_{x,y}$ is a proper subgroup of S_x containing $S_{x,y}$. By Claim 3.56 applied to $S_x = S(X \setminus \{x\})$ and its subgroup $S_{x,y}$ (the stabilizer of y in S_x), we conclude that $S_{x,y}$ is a maximal subgroup of S_x . Therefore, $S_x \cap G_{x,y} = S_{x,y}$. This shows that $S_{x,y}$ is a T -closed subgroup of S_x . By Claim 3.57 applied to $S_x = S(X \setminus \{x\})$ and its subgroup $S_{x,y}$, we conclude that $S_{x,y}$ is a T -open subgroup of S_x . Since S_x is dense in $S(X)$, we can claim that $G_{x,y}$ is a T -open subgroup of $S(X)$. Since S_x is a proper dense subgroup of $S(X)$, it is clear that S_x cannot contain $G_{x,y}$. Analogously, S_y

cannot contain $G_{x,y}$ either. So $G_{x,y} \neq S_{x,y}$ is a proper subgroup of $S(X)$ containing $S_{x,y}$ that does not coincide with S_x or S_y . Therefore $G_{x,y} = \tilde{S}_{x,y}$ by Fact 3.59. This proves that $\tilde{S}_{x,y}$ is T -open. Since all subgroups of the form $\tilde{S}_{x,y}$ are pairwise conjugated, we can claim that all subgroups $\tilde{S}_{x,y}$ is T -open.

Now we can see that the stabilizers S_F with $|F| > 2$ are T -open, as

$$S_F = \bigcap \{\tilde{S}_{x,y} : x, y \in F, x \neq y\}.$$

This proves that all basic neighborhoods S_F of 1 in τ_X are T -open. In particular, also the subgroups S_x are T -open, contrary to our hypothesis.

3.7.3 Existence of Hausdorff group topologies

According to Proposition 3.31 every infinite abelian group admits a non-discrete Hausdorff group topology, for example the Bohr topology. This gives immediately the following

Corollary 3.63. *Every group with infinite center admits a non-discrete Hausdorff group topology.*

Proof. The center $Z(G)$ of the group G has a non-discrete Hausdorff group topology τ by the above remark. Now consider the family \mathcal{B} of all sets of the form aU , where $a \in G$ and U is a non-empty τ -subset of $Z(G)$. It is easy to see that it is a base of a non-discrete Hausdorff group topology on G . \square

In 1946 Markov set the problem of the existence of a (countably) infinite group G that admits no Hausdorff group topology beyond the discrete one. Let us call such a group a *Markov group*. Obviously, G is a Markov group precisely when \mathfrak{M}_G is discrete. A Markov group must have finite center by Corollary 3.63.

According to Proposition 3.26, the closure of the neutral element of every topological group is always a normal subgroup of G . Therefore, a simple topological group is either Hausdorff, or indiscrete. So a simple Markov group G admits only two group topologies, the discrete and the indiscrete ones.

The equality $\mathfrak{Z}_G = \mathfrak{M}_G$ established by Markov in the countable case was intended to help in finding a countably infinite Markov group G . Indeed, a countable group G is Markov precisely when \mathfrak{Z}_G is discrete. Nevertheless, Markov failed in building a countable group G with discrete Zariski topology; this was done much later, in 1980, by Ol'shanskii [78] who made use of the so called *Adian groups* $A = A(m, n)$ (constructed by Adian to negatively resolve the famous 1902 Burnside problem on finitely generated groups of finite exponent). Let us sketch here Ol'shanskii's elegant short proof.

Example 3.64. [78] Let m and n be odd integers ≥ 665 , and let $A = A(m, n)$ be Adian's group having the following properties

- (a) A is generated by n -elements;
- (b) A is torsion-free;
- (c) the center C of A is infinite cyclic.
- (d) the quotient A/C is infinite, of exponent m , i.e., $y^m \in C$ for every $y \in A$.⁶

By (a) the group A is countable. Denote by C^m the subgroup $\{c^m : c \in C\}$ of A . Let us see that (b), (c) and (d) jointly imply that the Zariski topology of the infinite quotient $G = A/C^m$ is discrete (so G is a countably infinite Markov group). Let d be a generator of C . Then for every $x \in A \setminus C$ one has $x^m \in C \setminus C^m$. Indeed, if $x^m = d^{ms}$, then $(xd^{-s})^m = 1$ for some $s \in \mathbb{Z}$, so $xd^{-s} = 1$ and $x \in C$ by (b). Hence

$$\text{for every } u \in G \setminus \{1\} \text{ there exists } a \in C \setminus C^m, \text{ such that either } u = a \text{ or } u^m = a. \quad (3)$$

As $|C/C^m| = m$, every $u \in G \setminus \{1\}$ is a solution of some of the $2(m-1)$ equations in (3). Thus, $G \setminus \{e\}$ is closed in the Zariski topology \mathfrak{Z}_G of G . Therefore, \mathfrak{Z}_G is discrete.

Now we recall an example, due to Shelah [92], of an uncountable group which is non-topologizable. It appeared about a year or two earlier than the ZFC-example of Ol'shanskii exposed above.

Example 3.65. [92] Under the assumption of CH there exists a group G of size ω_1 satisfying the following conditions (a) (with $m = 10000$) and (b) (with $n = 2$):

- (a) there exists $m \in \mathbb{N}$ such that $A^m = G$ for every subset A of G with $|A| = |G|$;

⁶i.e., the finitely generated infinite quotient A/C negatively resolves Burnside's problem.

- (b) for every subgroup H of G with $|H| < |G|$ there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in G$ such that the intersection $\bigcap_{i=1}^n x_i^{-1} H x_i$ is finite.

Let us see that G is a Markov group (i.e., \mathfrak{M}_G is discrete)⁷. Assume \mathcal{T} be a Hausdorff group topology on G . There exists a \mathcal{T} -neighbourhood V of e_G with $V \neq G$. Choose a \mathcal{T} -neighbourhood W of e_G with $W^m \subseteq V$. Now $V \neq G$ and (a) yield $|W| < |G|$. Let $H = \langle W \rangle$. Then $|H| = |W| \cdot \omega < |G|$. By (b) the intersection $O = \bigcap_{i=1}^n x_i^{-1} H x_i$ is finite for some $n \in \mathbb{N}$ and elements $x_1, \dots, x_n \in G$. Since each $x_i^{-1} H x_i$ is a \mathcal{T} -neighbourhood of e_G , this proves that $e_G \in O \in \mathcal{T}$. Since \mathcal{T} is Hausdorff, it follows that $\{e_G\}$ is \mathcal{T} -open, and therefore \mathcal{T} is discrete.

One can see that even the weaker form of (a) (with m depending on $A \in [G]^{|G|}$), yields that every proper subgroup of G has size $< |G|$. In the case $|G| = \omega_1$, the groups with this property are known as *Kurosh groups* (in particular, this is a *Jonsson semigroup* of size ω_1 , i.e., an uncountable semigroup whose proper subsemigroups are countable).

Finally, this remarkable construction from [92] furnished also the first consistent example to a third open problem. Namely, a closer look at the above argument shows that the group G is simple. As G has no maximal subgroups, it shows also that taking Frattini subgroup⁸ “does not commute” with taking finite direct products (indeed, $\text{Fratt}(G) = G$, while $\text{Fratt}(G \times G) = \Delta_G$ the “diagonal” subgroup of $G \times G$).

3.7.4 Extension of group topologies

The problem of the existence of (Hausdorff non-discrete) group topologies can be considered also as a problem of extension of (Hausdorff non-discrete) group topologies.

The theory of extension of topological spaces is well understood. If a subset Y of a set X carries a topology τ , then it is easy to extend τ to a topology τ^* on X such that (Y, τ) is a subspace of (X, τ^*) . The easiest way to do it is to consider $X = Y \cup (X \setminus Y)$ as a partition of the new space (X, τ^*) into clopen sets and define the topology of $X \setminus Y$ arbitrarily. Usually, one prefers to define the extension topology τ^* on X in such a way to have Y dense in X . In such a case the extensions of a given space (Y, τ) can be described by means of appropriate families of open filters of Y (i.e., filters on Y having a base of τ -open sets).

The counterpart of this problem for groups and group topologies is much more complicated because of the presence of group structure. Indeed, let H be a subgroup of a group G and assume that τ is a group topology of H . Now one has to build a group topology τ^* on G such that (H, τ) is a topological subgroup of (G, τ^*) . The first idea to extend τ is to imitate the first case of extension considered above by declaring the subgroup H a τ^* -open topological subgroup of the new topological group (G, τ^*) . Let us note that this would immediately determine the topology τ^* in a unique way. Indeed, every coset gH of H must carry the topology transported from H to gH by the translation $x \mapsto gx$, i.e., the τ^* -open subsets of gH must have the form gU , where U is an open subset of (H, τ) . In other words, the family $\{gU : \emptyset \neq U \in \tau\}$ is a base of τ^* . This idea has worked in the proof of Corollary 3.63 where H was the center of G . Indeed, this idea works in the following more general case.

Lemma 3.66. *Let H be a subgroup of a group G such that $G = Hc_G(G)$. Then for every group topology τ on H the above described topology τ^* is a group topology of G such that (H, τ) is a topological subgroup of (G, τ^*) .*

Proof. The first two axioms on the neighborhood base are easy to check. For the third one pick a basic τ^* -neighborhood U of 1 in G . Since H is τ^* -open, we can assume wlog that $U \subseteq H$, so U is a τ -neighborhood of 1. Let $x \in G$. We have to produce a τ^* -neighborhood V of 1 in G such that $x^{-1}Vx \subseteq U$. By our hypothesis there exist $h \in H, z \in c_G(G)$, such that $x = hz$. Since τ is a group topology on H there exist $V \in \mathcal{V}_{H, \tau}(1)$ such that $h^{-1}Vh \subseteq U$. Then

$$x^{-1}Vx = z^{-1}h^{-1}Vhz \subseteq z^{-1}Uz = U$$

as $z \in c_G(G)$. This proves that τ^* is a group topology of G . □

Clearly, the condition $G = Hc_G(G)$ is satisfied when H is a central subgroup of G . It is satisfied also when H is a direct summand of G . On the other hand, subgroups H satisfying $G = Hc_G(G)$ are normal.

Two questions are in order here:

- is the condition $G = Hc_G(G)$ really necessary for the extension problems;
- is it possible to define the extension τ^* in a different way in order to have *always* the possibility to extend a group topology?

⁷Hesse [66] showed that the use of CH in Shelah's construction of a Markov group of size ω_1 can be avoided.

⁸the Frattini subgroup of a group G is the intersection of all maximal subgroups of G .

Our next theorem shows that the difficulty of the extension problem are not hidden in the special features of the extension τ^* .

Theorem 3.67. *Let H be a normal subgroup of the group G and let τ be a group topology on H . Then the following are equivalent:*

- (a) *the extension τ^* is a group topology on G ;*
- (b) *τ can be extended to a group topology of G ;*
- (c) *for every $x \in G$ the automorphism of H induced by the conjugation by x is τ -continuous.*

Proof. The implication (a) \rightarrow (b) is obvious, while the implication (b) \rightarrow (c) follows from the fact that the conjugations are continuous in any topological group. To prove the implication (c) \rightarrow (a) assume now that all automorphisms of N induced by the conjugation by elements of G are τ -continuous. Take the filter of all neighborhoods of 1 in (H, τ^*) as a base of neighborhoods of 1 in the group topology τ^* of G . This works since the only axiom to check is to find for every $x \in G$ and every τ^* -nbd U of 1 a τ^* -neighborhood V of 1 such that $V^x := x^{-1}Vx \subseteq U$. Since we can choose U, V contained in H , this immediately follows from our assumption of τ -continuity of the restrictions to H of the conjugations in G . \square

Now we give an example showing that the extension problem cannot be resolved for certain triples G, H, τ of a group G , its subgroup H and a group topology τ on H .

Example 3.68. In order to produce an example when the extension is not possible we need to produce a triple G, H, τ such that at least some conjugation by an element of G is not τ -continuous when considered as an automorphism of H . The best tool to face this issue is the use of semi-direct products.

Let us recall that for groups K, H and a group homomorphism $\theta : K \rightarrow \text{Aut}(H)$ one defines the semi-direct product $G = H \rtimes_{\theta} K$, where we shall identify H with the subgroup $H \times \{1\}$ of G . In such a case, the conjugation in G by an element k of K restricted to H is precisely the automorphism $\theta(k)$ of H . Now consider a group topology τ on H . According to Theorem 3.67 τ can be extended to a group topology of G iff for every $k \in K$ the automorphism $\theta(k)$ of H is τ -continuous. (Indeed, every element $x \in G$ has the form $x = hk$, where $h \in H$ and $k \in K$; hence it remains to note that the conjugation by x is composition of the (continuous) conjugation by h and the conjugation by k .)

In order to produce the required example of a triple G, H, τ such that τ cannot be extended to G it suffices to find a group K and a group homomorphism $\theta : K \rightarrow \text{Aut}(H)$ such that at least one of the automorphisms $\theta(k)$ of H is τ -discontinuous. Of course, one can simplify the construction by taking the cyclic group $K_1 = \langle k \rangle$ instead of the whole group K , where $k \in K$ is chosen such that the automorphisms $\theta(k)$ of H is τ -discontinuous. A further simplification can be arranged by taking k in such a way that the automorphism $f = \theta(k)$ of H is also an involution, i.e., $f^2 = \text{id}_H$. Then H will be an index two subgroup of G .

Here is an example of a topological abelian group (H, τ) admitting a τ -discontinuous involution f . Then the triple G, H, τ such that τ cannot be extended to G is obtained by simply taking $G = H \rtimes \langle f \rangle$, where the involution f acts on H . Take as (H, τ) the torus group \mathbb{T} with the usual topology. Then \mathbb{T} is algebraically isomorphic to $(\mathbb{Q}/\mathbb{Z}) \oplus_{\mathbb{C}} \bigoplus \mathbb{Q}$, so \mathbb{T} has $2^{\mathfrak{c}}$ many involutions. Of these only the involutions $\pm \text{id}_{\mathbb{T}}$ of \mathbb{T} are continuous.

Let us conclude now with a series of examples when the extension problem has always a positive solution.

Example 3.69. Let p be a prime number. If the group of p -adic integers $N = \mathbb{Z}_p$ is a normal subgroup of some group G , then the p -adic topology of N can be extended to a group topology on G . Indeed, it suffices to note that if $\xi : N \rightarrow N$ is an automorphism of N , then $\xi(p^n N) = p^n N$. Since the subgroups $p^n N$ define the topology of N , this proves that every automorphism of N is continuous. Now Theorem 3.67 applies.

Clearly, the p -adic integers can be replaced by any topological group N such that every automorphism of N is continuous (e.g., products of the form $\prod_p \mathbb{Z}_p^{k_p} \times F_p$, where $k_p < \omega$ and F_p is a finite abelian p -group).

3.8 Cardinal invariants of topological groups

Here we shall be interested in measuring the minimum size of a base (of neighborhoods of 1) in a topological group H , as well as other cardinal functions related to H .

It is important to relate the bases (of neighborhoods of 1) in H to those of a subgroup G of H .

Exercise 3.70. *If G is a subgroup of a topological group H and if \mathcal{B} is a base (of neighborhoods of 1) in H then a base (of neighborhoods of 1) in G is given by $\{U \cap G : U \in \mathcal{B}\}$.*

Now we consider the case when G is a *dense* subgroup of H .

Lemma 3.71. *If G is a dense subgroup of a topological group H and \mathcal{B} is a base of neighborhoods of 1 in G , then $\{\overline{U}^H : U \in \mathcal{B}\}$ is a base of neighborhoods of 1 in H .*

Proof. Since the topological group H is regular, the closed neighborhoods form a base at 1 in H . Hence for a neighborhood $V \ni 1$ in H one can find another neighborhood $V_0 \ni 1$ such that $\overline{V_0} \subseteq V$. Since $G \cap V_0$ is a neighborhood of 1 in G , there exists $U \in \mathcal{B}$ such that $U \subseteq G \cap V_0$. There exists also an open neighborhood W of 1 in H such that $U = W \cap G$. Obviously, one can choose $W \subseteq V_0$. Hence $\overline{U}^H = \overline{W}$ as G is dense in H and W is open in H . Thus $\overline{U}^H = \overline{W} \subseteq \overline{V_0} \subseteq V$ is a neighborhood of 1 in H . \square

Lemma 3.72. *Let G be a dense subgroup of a topological group H and let \mathcal{B} be a base of neighborhoods of 1 in H . Then $\{gU : U \in \mathcal{B}, g \in G\}$ is a base of the topology of H .*

Proof. Let $x \in H$ and let $x \in O$ be an open set. Then there exists a $U \in \mathcal{U}$ symmetric with $xU^2 \subseteq O$. Pick a $g \in G \cap xU$. Then $x \in gU \subseteq O$. \square

For a topological group G set $d(G) = \min\{|X| : X \text{ is dense in } G\}$,

$$w(G) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } G\} \text{ and } \chi(G) = \min\{|\mathcal{B}| : \mathcal{B} \text{ a base of neighborhoods of 1 in } G\}.$$

Lemma 3.73. *Let H be a subgroup of a topological group G . Then:*

- (a) $w(H) \leq w(G)$ and $\chi(H) \leq \chi(G)$;
- (b) if H is dense in G , then $w(G) = w(H)$ and $\chi(G) = \chi(H)$.

Lemma 3.74. $w(G) = \chi(G) \cdot d(G)$ for every topological group G .

Proof. The inequality $w(G) \geq \chi(G)$ is obvious. To see that $w(G) \geq d(G)$ choose a base \mathcal{B} of size $w(G)$ and for every $U \in \mathcal{B}$ pick a point $d_U \in U$. Then the set $D = \{d_U : U \in \mathcal{B}\}$ is dense in G and $|D| \leq w(G)$. This proves the inequality $w(G) \geq \chi(G) \cdot d(G)$.

The inequality $w(G) \leq \chi(G) \cdot d(G)$ follows from the previous lemma. \square

The *cardinal invariants* of the topological groups are cardinal numbers, say $\rho(G)$, associated to every topological group G such that if G is topologically isomorphic to the topological group H , then $\rho(G) = \rho(H)$. For example, the size $|G|$ is the simplest cardinal invariant of a topological group, it does not depend on the topology of G . Other cardinal invariants are the *weight* $w(G)$, the *character* $\chi(G)$ and the *density character* $d(G)$ defined above. Beyond the equality $w(G) = \chi(G) \cdot d(G)$ proved in Lemma 3.74, one has also the following inequalities:

Lemma 3.75. *Let G be a topological group. Then:*

- (a) $d(G) \leq w(G) \leq 2^{d(G)}$;
- (b) $|G| \leq 2^{w(G)}$ if G is Hausdorff.

Proof. (a) $d(G) \leq w(G)$ has already been proved in Lemma 3.74 (a). To prove $w(G) \leq 2^{d(G)}$ note that G is regular, hence every open base \mathcal{B} on G contains a base \mathcal{B}_r of the same size consisting of regular open sets⁹. Let \mathcal{B} be a base of G of regular open sets and let D be a dense subgroup of G of size $d(G)$. If $U, V \in \mathcal{B}$, with $U \cap D = V \cap D$, then $\overline{U} = \overline{U \cap D} = \overline{V \cap D} = \overline{V}$. Being U and V regular open, the equality $\overline{U} = \overline{V}$ implies $U = V$. Hence the map $U \mapsto U \cap D$ from \mathcal{B} to the power set $P(D)$ is injective. Therefore $w(G) \leq 2^{d(G)}$.

(b) To every point $x \in G$ assign the set $O_x = \{U \in \mathcal{B} : x \in U\}$. Then the axiom T_2 guarantees that map $x \mapsto O_x$ from G to the power set $P(\mathcal{B})$ is injective. Therefore, $|G| \leq 2^{w(G)}$. \square

Remark 3.76. Two observations related to item (b) of the above lemma are in order here.

- The equality in item (b) can be attained (see Theorem 4.46).
- One cannot remove Hausdorffness in item (b) (any large indiscrete group provides a counter-example). This dependence on separation axioms is due to that the presence of the size of the group in (b). We see in the next exercise that the Hausdorff axiom is not relevant as far as the other cardinal invariants are involved.

⁹an open set is said to be *regular open* if it coincides with the interior of its closure.

Exercise 3.77. Let G be a topological group. Prove that:

- $w(G) = w(G/\overline{\{1\}})$, $\chi(G) = \chi(G/\overline{\{1\}})$ and $d(G) = d(G/\overline{\{1\}})$;
- $d(U) = D(G)$ for every non-empty open set U , if G is Lindelöf;
- $w(G) = \chi(G)$ if locally compact and σ -compact.

3.9 Completeness and completion

A net $\{g_\alpha\}_{\alpha \in A}$ in a topological group G is a *Cauchy net* if for every neighborhood U of 1 in G there exists $\alpha_0 \in A$ such that $g_\alpha^{-1}g_\beta \in U$ and $g_\beta g_\alpha^{-1} \in U$ for every $\alpha, \beta > \alpha_0$.

Exercise 3.78. Let G be a dense subgroup of a topological group H . If (g_α) is a net in G that converges to some element $h \in H$, then (g_α) is a Cauchy net.

By the previous exercise, the convergent nets are Cauchy nets. A topological group G is *complete* (in the sense of Raïkov) if every Cauchy net in G converges in G . We omit the tedious proof of the next theorem.

Theorem 3.79. For every topological Hausdorff group G there exists a complete topological group \tilde{G} and a topological embedding $i : G \rightarrow \tilde{G}$ such that $i(G)$ is dense in \tilde{G} . Moreover, if $f : G \rightarrow H$ is a continuous homomorphism and H is a complete topological group, then there is a unique continuous homomorphism $\tilde{f} : \tilde{G} \rightarrow H$ with $f = \tilde{f} \circ i$.

Therefore every Hausdorff topological abelian group has a unique, up to topological isomorphisms, (Raïkov-)completion (\tilde{G}, i) and we can assume that G is a dense subgroup of \tilde{G} .

Definition 3.80. A net $\{g_\alpha\}_{\alpha \in A}$ in G is a left [resp., right] Cauchy net if for every neighborhood U of 1 in G there exists $\alpha_0 \in A$ such that $g_\alpha^{-1}g_\beta \in U$ [resp., $g_\beta g_\alpha^{-1} \in U$] for every $\alpha, \beta > \alpha_0$.

Lemma 3.81. Let G be a Hausdorff topological group. Every left (resp., right) Cauchy net in G with a convergent subnet is convergent.

Proof. Let $\{g_\alpha\}_{\alpha \in A}$ be a left Cauchy net in G and let $\{g_\beta\}_{\beta \in B}$ be a subnet convergent to $x \in G$, where B is a cofinal subset of A . Let U be a neighborhood of 1 in G and V a symmetric neighborhood of 1 in G such that $VV \subseteq U$. Since $g_\beta \rightarrow x$, there exists $\beta_0 \in B$ such that $g_\beta \in xV$ for every $\beta > \beta_0$. On the other hand, there exists $\alpha_0 \in A$ such that $\alpha_0 \geq \beta_0$ and $g_\alpha^{-1}g_\gamma \in V$ for every $\alpha, \gamma > \alpha_0$. With $\gamma = \beta_0$ we have $g_\alpha \in xVV \subseteq xU$ for every $\alpha > \alpha_0$, that is $g_\alpha \rightarrow x$. \square

A topological group G is *complete in the sense of Weil* if every left Cauchy net converges in G .

Every Weil-complete group is also complete, but the converse does not hold in general. It is possible to define the Weil-completion of a Hausdorff topological group in analogy with the Raïkov-completion.

Exercise 3.82. Prove that if a Hausdorff topological group G admits a Weil-completion, then in G the left Cauchy and the right Cauchy nets coincide.

Exercise 3.83. Let X be an infinite set and let $G = S(X)$ equipped with the topology described in §3.2.4. Prove that:

- a net $\{f_\alpha\}_{\alpha \in A}$ in G is left Cauchy iff there exists $f \in X^X$ so that $f_\alpha \rightarrow f$ in X^X , prove that such an f must necessarily be injective;
- a net $\{f_\alpha\}_{\alpha \in A}$ in G is right Cauchy iff there exists $g \in X^X$ so that $f_\alpha^{-1} \rightarrow g$ in X^X ;
- the group $S(X)$ admits no Weil-completion. (Hint. Build a left Cauchy net in $S(X)$ that is not right Cauchy and use items (a) and (b), as well as the previous exercise.)
- $S(X)$ is Raïkov-complete. (Hint. Use items (a) and (b).)

Exercise 3.84. (a) Let G be a linearly topologized group and let $\{N_i : i \in I\}$ be its system of neighborhoods of 1 consisting of open normal subgroups. Then the completion of G is isomorphic to the inverse limit $\varprojlim G/N_i$ of the discrete quotients G/N_i .

- Show that the completion in (a) is compact iff all N_i have finite index in G .

- (c) Let p be a prime number. Prove that the completion of \mathbb{Z} equipped with the p -adic topology (see Example 3.7) is the compact group \mathbb{J}_p of p -adic integers.
- (d) Prove that the completion of \mathbb{Z} equipped with the natural topology (see Example 3.7) is isomorphic to $\prod_p \mathbb{J}_p$.

Exercise 3.85. Let p be a prime number. Prove that:

- (a) \mathbb{Z} admits a finest group topology τ such that p^n converges to 0 in τ (this is $\tau_{(p^n)}$ in the notation of §3.4);
- (b)* [89, 88] (\mathbb{Z}, τ) is complete;
- (c) conclude that τ is not metrizable.

Exercise 3.86. Let G be a Hausdorff topologized group. Call a filter \mathcal{F} on G Cauchy if for every $U \in \mathcal{V}_G(1)$ there exists $g \in G$ such that $gU \in \mathcal{F}$. Prove that:

- (a) a filter \mathcal{F} on G Cauchy iff for every $U \in \mathcal{V}_G(1)$ there exists $g \in G$ such that $Ug \in \mathcal{F}$.
- (b) if \mathcal{F} is a Cauchy filter on G and $x_F \in F$ for every $F \in \mathcal{F}$, then the net $\{x_F : F \in \mathcal{F}\}$ is a Cauchy net (here \mathcal{F} is considered as a directed partially ordered set w.r.t. inclusion);
- (c) if $\{x_i : i \in I\}$ is a Cauchy net in G and $F_i = \{x_j : j \in I, j \geq i\}$, then the family $\{F_i : i \in I\}$ is a filter base of a Cauchy filter on G ;
- (d) G is complete iff every Cauchy filter in G converges.

4 Compactness and local compactness in topological groups

Clearly, a topological group G is locally compact if there exists a compact neighborhood of e_G in G (compare with Definition 2.10). We shall assume without explicitly mentioning it, that all locally compact groups are Hausdorff.

As an immediate consequence of Tychonov's theorem of compactness of products we obtain the following the first example of a compact abelian group (it will become clear with the duality theorem that this is the most general one).

Remark 4.1. Let us see that for every abelian group G the group $G^* = \text{Hom}(G, \mathbb{S})$ is closed in the product \mathbb{S}^G , hence G^* is compact. Consider the projections $\pi_x : \mathbb{S}^G \rightarrow \mathbb{T}$ for every $x \in G$ and the following equalities

$$\begin{aligned} G^* &= \bigcap_{h, g \in G} \{f \in \mathbb{S}^G : f(h+g) = f(h)f(g)\} = \bigcap_{h, g \in G} \{f \in \mathbb{S}^G : \pi_{h+g}(f) = \pi_h(f)\pi_g(f)\} \\ &= \bigcap_{h, g \in G} \{f \in \mathbb{S}^G : (\pi_{h+g}^{-1}\pi_h\pi_g)(f) = 1\} = \bigcap_{h, g \in G} \ker(\pi_{h+g}^{-1}\pi_h\pi_g). \end{aligned}$$

Since π_x is continuous for every $x \in G$ and $\{1\}$ is closed in \mathbb{S} , then all $\ker(\pi_{h+g}^{-1}\pi_h\pi_g)$ are closed; so $\text{Hom}(G, \mathbb{S})$ is closed too.

The next lemma contains a well known useful fact – the existence of a “diagonal subnet”.

Lemma 4.2. Let G be an abelian group and let $N = \{\chi_\alpha\}_\alpha$ be a net in G^* . Then there exist $\chi \in G^*$ and a subnet $S = \{\chi_{\alpha_\beta}\}_\beta$ of N such that $\chi_{\alpha_\beta}(x) \rightarrow \chi(x)$ for every $x \in G$.

Proof. By Tychonov's theorem, the group \mathbb{S}^G endowed with the product topology is compact. Then N has a convergent (to χ) subnet S . Therefore $\chi_{\alpha_\beta}(x) \rightarrow \chi(x)$ for every $x \in G$ and $\chi \in G^*$, because G^* is closed in \mathbb{S}^G by 4.1. \square

4.1 Specific properties of (local) compactness

Here we shall see the impact of local compactness in various directions (the open mapping theorem, properties related to connectedness, etc.).

Lemma 4.3. Let G be a topological group and let C and K be closed subsets of G :

- (a) if K is compact, then both CK and KC are closed;
- (b) if both C and K are compact, then CK and KC are compact;
- (c) if K is contained in an open subset U of G , then there exists an open neighborhood V of 1 such that $KV \subseteq U$.

Proof. (a) Let $\{x_\alpha\}_{\alpha \in A}$ be a net in CK such that $x_\alpha \rightarrow x_0 \in G$. It is sufficient to show that $x_0 \in CK$. For every $\alpha \in A$ we have $x_\alpha = y_\alpha z_\alpha$, where $y_\alpha \in C$ and $z_\alpha \in K$. Since K is compact, then there exist $z_0 \in K$ and a subnet $\{z_{\alpha_\beta}\}_{\beta \in B}$ such that $z_{\alpha_\beta} \rightarrow z_0$. Thus $(x_{\alpha_\beta}, z_{\alpha_\beta})_{\beta \in B}$ is a net in $G \times G$ which converges to (x_0, z_0) . Therefore $y_{\alpha_\beta} = x_{\alpha_\beta} z_{\alpha_\beta}^{-1}$ converges to $x_0 z_0^{-1}$ because the function $(x, y) \mapsto xy^{-1}$ is continuous. Since $y_{\alpha_\beta} \in C$ for every $\beta \in B$ and C is closed, $x_0 z_0^{-1} \in C$. Now $x_0 = (x_0 z_0^{-1}) z_0 \in CK$. Analogously it is possible to prove that KC is closed.

(b) The product $C \times K$ is compact by the Tychonov theorem and the function $(x, y) \mapsto xy$ is continuous and maps $C \times K$ onto CK . Thus CK is compact.

(c) Let $C = G \setminus U$. Then C is a closed subset of G disjoint with K . Therefore, for the compact subset K^{-1} of G one has $1 \notin K^{-1}C$. By (a) $K^{-1}C$ is closed, so there exists a symmetric neighborhood V of 1 that misses $K^{-1}C$. Then KV misses C and consequently KV is contained in U . \square

Compactness of K cannot be omitted in item (a). Indeed, $K = \mathbb{Z}$ and $C = \langle \sqrt{2} \rangle$ are closed subgroups of $G = \mathbb{R}$ but the subgroup $K + C$ of \mathbb{R} is dense (see Exercice 3.20 or Proposition 4.45).

Lemma 4.4. *Let G be a topological group and K a compact subgroup of G . Then the canonical projection $\pi : G \rightarrow G/K$ is closed.*

Proof. Let C be a closed subset of G . Then CK is closed by Lemma 4.3 and so $U = G \setminus CK$ is open. For every $x \notin CK$, that is $\pi(x) \notin \pi(C)$, $\pi(U)$ is an open neighborhood of $\pi(x)$ such that $\pi(U) \cap \pi(C)$ is empty. So $\pi(C)$ is closed. \square

Lemma 4.5. *Let G be a topological group and let H be a closed subgroup of G .*

- (1) If G is compact, then G/H is compact.
- (2) If H and G/H are compact, then G is compact.

Proof. (1) is obvious.

(2) Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a family of closed sets of G with the finite intersection property. If $\pi : G \rightarrow G/H$ is the canonical projection, $\pi(\mathcal{F})$ is a family of closed subsets with the finite intersection property in G/H by Lemma 4.4. By the compactness of G/H there exists $\pi(x) \in \pi(F_\alpha)$ for every $\alpha \in A$. So $x \in \bigcap_{\alpha \in A} F_\alpha H$. Let $x = f_\alpha h_\alpha$ with $h_\alpha \in H$ and $f_\alpha \in F_\alpha$. It is not restrictive to assume that \mathcal{F} is closed for finite intersections. Define a partial order on A by $\alpha \leq \alpha'$ if $F_\alpha \supseteq F_{\alpha'}$. Then (A, \leq) is a right-filtered partially ordered set and so $\{f_\alpha\}_{\alpha \in A}$ is a net in G . By the compactness of H we can assume wlog that h_α converges to $h \in H$ (otherwise pass to a convergent subnet). But then $f_\alpha = x h_\alpha^{-1} \rightarrow x h^{-1}$. Since f_α is contained definitively in F_α , also the limit $x h^{-1} \in F_\alpha$. So the intersection of all F_α is not empty. \square

Lemma 4.6. *Let G be a locally compact group, H be a closed subgroup of G and $\pi : G \rightarrow G/H$ be the canonical projection. Then:*

- (a) G/H is locally compact too;
- (b) If C is a compact subset of G/H , then there exists a compact subset K of G such that $\pi(K) = C$.

Proof. Let U be an open neighborhood of 1 in G with compact closure. Consider the open neighborhood $\pi(U)$ of 1 in G/H . Then $\pi(\overline{U}) \subseteq \pi(U)$ by the continuity of π . Now $\pi(\overline{U})$ is compact in G/H , which is Hausdorff, and so $\pi(\overline{U})$ is closed. Since $\pi(U)$ is dense in $\pi(\overline{U})$, we have $\overline{\pi(U)} = \pi(\overline{U}) = \pi(U)$. So G/H is locally compact.

(b) Let U be an open neighborhood of 1 in G with compact closure. Then $\{\pi(sU) : s \in G\}$ is an open covering of G/H . Since C is compact, a finite subfamily $\{\pi(s_i U) : i = 1, \dots, m\}$ covers C . Then we can take $K = (s_1 U \cup \dots \cup s_m U) \cap \pi^{-1}(C)$. \square

Lemma 4.7. *A locally compact group is Weil-complete.*

Proof. Let U be a neighborhood of 1 in G with compact closure and let $\{g_\alpha\}_{\alpha \in A}$ be a left Cauchy net in G . Then there exists $\alpha_0 \in A$ such that $g_\alpha^{-1}g_\beta \in U$ for every $\alpha, \beta \geq \alpha_0$. In particular, $g_\beta \in g_{\alpha_0}U$ for every $\beta \geq \alpha_0$. By the compactness of $g_{\alpha_0}\overline{U}$, we can conclude that there exists a convergent subnet $\{g_\beta\}_{\beta \in B}$ (for some cofinal $B \subseteq A$) such that $g_\beta \rightarrow g \in G$. Then also g_α converges to g by Lemma 3.81. \square

Lemma 4.8. *A locally compact countable group is discrete.*

Proof. By the Baire category theorem 2.16 G is of second category. Since $G = \{g_1, \dots, g_n, \dots\} = \bigcup_{n=1}^\infty \{g_n\}$, there exists $n \in \mathbb{N}_+$ such that $\text{Int } \{g_n\}$ is not empty and so $\{g_n\}$ is open. \square

Now we prove the open mapping theorem for topological groups.

Theorem 4.9 (Open mapping theorem). *Let G and H be locally compact topological groups and let f be a continuous homomorphism of G onto H . If G is σ -compact, then f is open.*

Proof. Let U be an open neighborhood of 1 in G . There exists an open symmetric neighborhood V of 1 in G such that $\overline{VV} \subseteq U$ and \overline{V} is compact. Since $G = \bigcup_{x \in G} xV$ and G is Lindelöf by Lemma 2.17, we have $G = \bigcup_{n=1}^\infty x_n V$. Therefore $H = \bigcup_{n=1}^\infty h(x_n \overline{V})$, because h is surjective. Put $y_n = h(x_n)$, hence $H = \bigcup_{n=1}^\infty y_n h(\overline{V})$ where each $h(\overline{V})$ is compact and so closed in H . Since H is locally compact, Theorem 2.16 yields that there exists $n \in \mathbb{N}_+$ such that $\text{Int } h(\overline{V})$ is not empty. So there exists a non-empty open subset W of H such that $W \subseteq h(\overline{V})$. If $w \in W$, then $w \in h(\overline{V})$ and so $w = h(v)$ for some $v \in \overline{V} = \overline{V}^{-1}$. Hence

$$1 \in w^{-1}W \subseteq w^{-1}h(\overline{V}) = h(v^{-1})h(\overline{V}) \subseteq h(\overline{V}\overline{V}) \subseteq h(U)$$

and this implies that $h(U)$ is an open neighborhood of 1 in H . \square

The following immediate corollary is frequently used:

Corollary 4.10. *If $f : G \rightarrow H$ is a continuous surjective homomorphism of Hausdorff topological groups and G is compact, then f is open.*

Now we introduce a special class of σ -compact groups that will play an essential role in determining the structure of the locally compact abelian groups.

Definition 4.11. *A group G is compactly generated if there exists a compact subset K of G which generates G , that is $G = \langle K \rangle = \bigcup_{n=1}^\infty (K \cup K^{-1})^n$.*

Lemma 4.12. *If G is a compactly generated group then G is σ -compact.*

Proof. By the definition $G = \bigcup_{n=1}^\infty (K \cup K^{-1})^n$, where every $(K \cup K^{-1})^n$ is compact, since K is compact. \square

It should be emphasized that while σ -compactness is a purely topological property, being compactly generated involves essentially the algebraic structure of the group.

Exercise 4.13. (a) Give examples of σ -compact groups that are not compactly generated.

(b) Show that every connected locally compact group is compactly generated.

Lemma 4.14. *Let G be a locally compact group.*

(a) *If K a compact subset of G and U an open subset of G such that $K \subseteq U$, then there exists an open neighborhood V of 1 in G such that $(KV) \cup (VK) \subseteq U$ and $\overline{(KV) \cup (VK)}$ is compact.*

(b) *If G is compactly generated, then there exists an open neighborhood U of 1 in G such that \overline{U} is compact and U generates G .*

Proof. (a) By Lemma 4.3 (c) there exists an open neighborhood V of 1 in G such that $(KV) \cup (VK) \subseteq U$. Since G is locally compact, we can choose V with compact closure. Thus $K\overline{V}$ is compact by Lemma 4.3. Since $KV \subseteq K\overline{V}$, then $\overline{KV} \subseteq K\overline{V}$ and so \overline{KV} is compact. Analogously \overline{VK} is compact, so $\overline{(KV) \cup (VK)} = \overline{KV} \cup \overline{VK}$ is compact.

(b) Let K be a compact subset of G such that K generates G . So $K \cup \{1\}$ is compact and by (a) there exists an open neighborhood U of 1 in G such that $U \supseteq K \cup \{1\}$ and \overline{U} is compact. \square

In the case of first countable topological groups Fujita and Shakhmatov [49] have described the precise relationship between σ -compactness and the property of being compactly generated.

Theorem 4.15. *A metrizable topological group G is compactly generated if and only if G is σ -compact and, for every open subgroup H of G , there exists a finite set $F \subseteq G$ such that $F \cup H$ algebraically generates G [49].*

This gives the following:

Corollary 4.16. *A σ -compact metrizable group G is compactly generated in each of the following cases (for the definition of total boundedness see Definition 4.25):*

- G has no open subgroups
- the completion \tilde{G} is connected;
- G is totally bounded.

Moreover,

Theorem 4.17. *A countable metrizable group is compactly generated iff it is algebraically generated by a sequence (possibly eventually constant) converging to its neutral element.*

Examples showing that the various conditions above cannot be omitted can be found in [49].

The question when will a topological group contain a compactly generated dense subgroup is considered in [50].

Now we see that linearity and total disconnectedness of group topologies coincide for compact groups and for locally compact abelian groups.

Theorem 4.18. *Every locally compact totally disconnected group has a base of neighborhoods of 1 consisting of open subgroups. In particular, a locally compact totally disconnected group that is either abelian or compact has linear topology.*

This can be derived from the following more precise result:

Theorem 4.19. *Let G be a locally compact topological group and let $C = c(G)$. Then :*

- (a) C coincides with the intersection of all open subgroups of G ;
- (b) if G is totally disconnected, then every neighborhood of 1 contains an open subgroup of G .

If G is compact, then the open subgroups in items (a) and (b) can be chosen normal.

Proof. (a) follows from (b) as G/C is totally disconnected hence the neutral element of G/C is intersection of open (resp. open normal) subgroups of G/C . Now the intersection of the inverse images, w.r.t. the canonical homomorphism $G \rightarrow G/C$, of these subgroups coincides with C .

(b) Let G be a locally compact totally disconnected group. By Vedenissov's Theorem G has a base \mathcal{O} of clopen symmetric compact neighborhoods of 1. Let $U \in \mathcal{O}$. The $U = \overline{U} = \bigcap_{V \in \mathcal{O}} UV$. Then every set $U \cdot V$ is compact by Lemma 4.3, hence closed. According to Lemma 2.14 there exist $V_1, \dots, V_n \in \mathcal{O}$ such that $U = \bigcap_{k=1}^n UV_k$. Then for $V := U \cap \bigcap_{k=1}^n V_k$ one has $UV = U$. This implies also $VV \subseteq U$, $VVV \subseteq U$ etc. Since V is symmetric, the subgroup $H = \langle V \rangle$ is contained in U as well. From $V \subseteq H$ one can deduce that H is open (cf. 3.16). In case G is compact, note that the heart $H_G = \bigcap_{x \in G} x^{-1}Hx$ of H is an open normal subgroup as the number of all conjugates $x^{-1}Hx$ of H is finite (being equal to $[G : N_G(H)] \leq [G : H] < \infty$). Hence H_G is an open normal subgroup of G contained in H , hence also in U . \square

Corollary 4.20. *The quotient of a locally compact totally disconnected group is totally disconnected.*

Proof. Let G be a locally compact totally disconnected group and let N be a closed normal subgroup of G . It follows from the above theorem that G has a linear topology. This yields that the quotient G/N has a linear topology too. Thus G/N is totally disconnected. \square

Corollary 4.21. *The continuous homomorphic images of compact totally disconnected groups are totally disconnected.*

Proof. Follows from the above corollary and the open mapping theorem. \square

According to Example 3.46 none of the items (a) and (b) of the theorem remain true without the hypothesis "locally compact".

Corollary 4.22. *Let G be a locally compact group. Then $Q(G) = c(G)$.*

Proof. By item (a) of the above theorem $C(G)$ is an intersection of open subgroups, that are clopen being open subgroups (cf. Proposizione 3.16). Hence $c(G)$ contains $Q(G)$ which in turn coincides with the intersection of all clopen sets of G containing 1. The inclusion $C(G) \subseteq Q(G)$ is always true. \square

4.2 Subgroups of the compact groups

For a subset E of an abelian group G we set $E_{(2)} = E - E$, $E_{(4)} = E - E + E - E$, $E_{(6)} = E - E + E - E + E - E$ and so on.

A subset X of an abelian group $(G, +)$ is *big*¹⁰ if there exists a finite subset F of G such that $G = X + F$. Obviously, every non-empty set of a finite group is big; on the other hand, every big set in an infinite group is necessarily infinite.

Exercise 4.23. Let B be an infinite subset of \mathbb{Z} . Show that B is big iff the following two conditions hold:

- (a) B is unbounded from above and from below;
- (b) if $B = \{b_n\}_{n=-\infty}^{\infty}$ is a one-to-one monotone enumeration of B then the differences $b_{n+1} - b_n$ are bounded.

Exercise 4.24. (a) Assume B_ν is a big set of the group G_ν for $\nu = 1, 2, \dots, n$. Prove that $B_1 \times \dots \times B_n$ is a big set of $G_1 \times \dots \times G_n$.

- (b) if $f : G \rightarrow H$ is a surjective group homomorphism and B is a big subset of H , then $f^{-1}(B)$ is a big subset of G .

Definition 4.25. A topological group G is *totally bounded* if every open non-empty subset U of G is big. A Hausdorff totally bounded group will be called *precompact*.

Clearly, compact groups are precompact.

Note that if f in item (b) of 4.24 is not surjective, then the property may fail. The next proposition gives an easy remedy to this.

Proposition 4.26. Let A be an abelian group and let B be a big subset of A . Then $(B - B) \cap H$ is big with respect to H for every subgroup H of A .

If $a \in A$ then there exists a sufficiently large positive integer n such that $na \in B - B$.

Proof. There exists a finite subset F of A such that $B + F = A$. For every $f \in F$, if $(B + f) \cap H$ is not empty, choose $a_f \in (B + f) \cap H$, and if $(B + f) \cap H$ is empty, choose an arbitrary $a_f \in H$. On the other hand, for every $x \in H$ there exists $f \in F$ such that $x \in B + f$; since $a_f \in B + f$, we have $x - a_f \in B - B$ and so $H \subseteq (B - B) \cap H + \{a_f : f \in F\}$, that is $(B - B) \cap H$ is big in H .

For the last assertion it suffices to take $H = \langle a \rangle$. If H is finite, then there is nothing to prove as $0 \in B - B$. Otherwise $H \cong \mathbb{Z}$ so the first item of Exercise 4.23 applies. \square

Combining this proposition with item (b) of 4.24 we get:

Corollary 4.27. For every group homomorphism $f : G \rightarrow H$ and every big subset B of H , the subset $f^{-1}(B - B)$ of G is a big.

Here comes the most important consequence of the above proposition.

Corollary 4.28. Subgroups of precompact groups are precompact. In particular, all subgroups of compact groups are precompact.

One can show that the precompact groups are precisely the subgroups of the compact groups. This requires two steps as the next theorem shows:

Theorem 4.29. (a) A group having a dense precompact subgroup is necessarily precompact.

- (b) The compact groups are precisely the complete precompact groups.

Proof. (a) Indeed, assume that H is a dense precompact subgroup of a group G . Then for every $U \in \mathcal{V}_G(0)$ choose an open $V \in \mathcal{V}_G(0)$ with $V + V \subseteq U$. By the precompactness of H there exists a finite set $F \subseteq H$ such that $H = F + V \cap H$. Then

$$G = V + H \subseteq V + F + V \cap H \subseteq F + V + V \subseteq F + U.$$

¹⁰Some authors use also the terminology *large*, *relatively dense*, or *syndetically dense*. This notion can be given for non-abelian groups as well, but then both versions, *left large* and *right large*, do not coincide. This creates some technical difficulties that we prefer to avoid since the second part of this section is relevant only for abelian groups. The first half, including the characterization 4.29, remains valid in the non-abelian case as well (since, fortunately, the “left” and “right” versions of total boundedness coincide).

(b) Compact groups are complete and precompact. To prove the other implication take a complete precompact group G . To prove that G is compact it suffices to prove that every ultrafilter on G converges. Assume \mathcal{U} is such an ultrafilter. We show first that it is a Cauchy filter. Indeed, if $U \in \mathcal{V}_G(0)$, then U is a big set of G so there exists $g_1, g_2, \dots, g_n \in G$ such that $G = \bigcup_{i=1}^n g_i + U$. Since \mathcal{U} is an ultrafilter, $g_i + U \in \mathcal{U}$ for some i . Hence \mathcal{U} is a Cauchy filter. According to Exercise 3.86 \mathcal{U} converges. \square

In this way we have described the precompact groups internally (as the Hausdorff topological groups having big non-empty open sets), or externally (as the subgroups of the compact groups).

Now we adopt a different approach to describe the precompact groups, based on the use of characters. Our first aim will be to see that the topologies induced by characters are always totally bounded.

Proposition 4.30. *If A is an abelian group, $\delta > 0$ and $\chi_1, \dots, \chi_s \in A^*$ ($s \in \mathbb{N}_+$), then $U(\chi_1, \dots, \chi_s; \delta)$ is big in A . Moreover for every $a \in A$ there exists a sufficiently large positive integer n such that $na \in U(\chi_1, \dots, \chi_s; \delta)$.*

Proof. Define $h : A \rightarrow \mathbb{T}^s$ such that $h(x) = (\chi_1(x), \dots, \chi_s(x))$ and

$$B = \left\{ (z_1, \dots, z_s) \in \mathbb{S}^s : |\text{Arg } z_i| < \frac{\delta}{2} \text{ for } i = 1, \dots, s \right\} = \left\{ z \in \mathbb{S} : |\text{Arg } z| < \frac{\delta}{2} \right\}^s.$$

Then B is big in \mathbb{S}^s and by Proposition 4.26 the set $(B - B) \cap h(A)$ is big with respect to $h(A)$. Since

$$B - B \subseteq C = \{(z_1, \dots, z_s) \in \mathbb{S}^s : \|\text{Arg } z_i\| < \delta \text{ for } i = 1, \dots, s\},$$

we have that $C \cap h(A)$ is big with respect to $h(A)$. Therefore $U(\chi_1, \dots, \chi_s; \delta) = h^{-1}(C)$ is big in A .

The second statement follows from Proposition 4.26, since

$$U\left(\chi_1, \dots, \chi_s; \frac{\delta}{2}\right) - U\left(\chi_1, \dots, \chi_s; \frac{\delta}{2}\right) \subseteq U(\chi_1, \dots, \chi_s; \delta).$$

\square

Corollary 4.31. *For an abelian group G all topologies of the form \mathcal{T}_H , where $H \leq G^*$, are totally bounded. Moreover, \mathcal{T}_H is precompact iff H separates the points of G .*

It requires a considerable effort to prove that, conversely, every totally bounded group topology has the form \mathcal{T}_H for some H (see Remark 6.3).

It follows easily from Corollary 4.31 and Proposition 4.30 that for every neighborhood E of 0 in the Bohr topology (namely, a set E containing a subset of the form $U(\chi_1, \dots, \chi_n; \varepsilon)$ with characters $\chi_i : G \rightarrow \mathbb{S}$, $i = 1, 2, \dots, n$, and $\varepsilon > 0$) there exists a big set B of G such that $B_{(8)} \subseteq E$ (just take $B = U(\chi_1, \dots, \chi_n; \varepsilon/8)$). Surprisingly, the converse is also true. Namely, we shall obtain as a corollary of Følner's lemma that every set E satisfying $B_{(8)} \subseteq E$ for some big set B of G must be a neighborhood of 0 in the Bohr topology of G (see Corollary 5.8).

Exercise 4.32. *If G is a countably infinite Hausdorff abelian group, then for every compact set K in G the set $K_{(2n)}$ is big for no $n \in \mathbb{N}$.*

(Hint. By Lemma 4.3 every set $K_{(2n)}$ is compact. So if $K_{(2n)}$ were big for some n , then G itself would be compact. Now Lemma 4.8 applies.)

Exercise 4.33. *Call a subset S of an infinite abelian group G small if there exist (necessarily distinct) elements $g_1, g_2, \dots, g_n, \dots$ of G such that $(g_n + S) \cap (g_m + S) = \emptyset$ whenever $m \neq n$.*

- (a) *Show that a subset S of G such that $S - S$ is not big is necessarily small.*
- (b) *Show that every finite subset is small.*
- (c) *Show that the group \mathbb{Z} is not a finite union of small sets.*
- (d) ** Show that no infinite abelian group G is a finite union of small sets.*
- (e) *If $S = (a_n)$ is a one-to-one T -sequence in an abelian group G , then for every $n \in \mathbb{N}$ the set $S_{(2n)}$ is small in G .*
- (f) *Show that the sequence (p_n) of prime numbers in \mathbb{Z} is not a T -sequence.*

(Hint. (d) Use a finitely additive invariant (Banach) measure on G . For (e) consider the (countable) subgroup generated by S and note that if $a_n \rightarrow 0$ in some Hausdorff group topology τ on G , then the set $S \cup \{0\}$ would be compact in τ , so item (a) and Exercise 4.32 apply. For (f) use (e) and the fact that there exists a constant m such that every integer number is a sum of at most m^{11} prime numbers.)

Exercise 4.34. *Show that for an infinite abelian group G and a subgroup H of G the following are equivalent:*

- (a) H has infinite index;
- (b) H is not big;
- (c) H is small.

Exercise 4.35. * *Every infinite abelian group has a small set of generators.*

This can be extended to arbitrary groups [30]. One can find in the literature also different (weaker) forms of smallness ([4, 11]).

Exercise 4.36. (a) *If $f : G \rightarrow H$ is a continuous surjective homomorphism of topological groups, then H is totally bounded whenever G is totally bounded.*

(b) *Prove that a topological group G is totally bounded iff $G/\overline{\{1\}}$ is totally bounded.*

(c) *If $\{G_i : i \in I\}$ is a family of topological groups, then $\prod_i G_i$ is totally bounded iff each G_i is totally bounded.*

(d) *Prove that every topological abelian group G admits a “universal” totally bounded continuous surjective homomorphic image $q : G \rightarrow q(G)$ (i.e., every continuous homomorphism $G \rightarrow P$, where P is a totally bounded group, factors through q^{12}).*

4.3 Subgroups of \mathbb{R}^n

Our main goal here is to prove that every closed subgroup of \mathbb{R}^n is topologically isomorphic to $\mathbb{R}^s \times \mathbb{Z}^m$, with $s, m \in \mathbb{N}$ and $s + m \leq n$. More precisely:

Theorem 4.37. *Let $n \in \mathbb{N}_+$ and let H be a closed subgroup of \mathbb{R}^n . Then there exist closed subgroups V and D of \mathbb{R}^n such that $H = V + D \cong V \times D$, $V \cong \mathbb{R}^s$, $D \cong \mathbb{Z}^m$ and $s + m \leq n$.*

The proof is split in several steps. Before starting it, we note the following curious dichotomy hidden in this theorem:

- the closed connected subgroups of \mathbb{R}^n are always isomorphic to some \mathbb{R}^s with $s \leq n$;
- the totally disconnected closed subgroups D of \mathbb{R}^n must be free and have free-rank $r_0(D) \leq n$; in particular they are discrete.

In the general case, for every closed subgroup H of \mathbb{R}^n the connected component $c(H)$ is open in H and isomorphic to \mathbb{R}^s for some $s \leq n$. Therefore, by the divisibility of \mathbb{R}^s one can write $H = c(H) \times D$ for some discrete subgroup D of H (see Corollary 2.8). Necessarily $r_0(D) \leq n - s$ as $c(H) \cong \mathbb{R}^s$ contains a discrete subgroup D_1 of rank s , so that $D_1 \times D$ will be a discrete subgroup of \mathbb{R}^n .

It is not hard to see that every discrete subgroup of \mathbb{R} is cyclic (Exercise 3.20). The first part of the proof consists in appropriately extending this property to discrete subgroups of \mathbb{R}^n (see Proposition 4.39). The first step is to see that the free-rank $r_0(H)$ of a discrete subgroup H of \mathbb{R}^n coincides with the dimension of the subspace of \mathbb{R}^n generated by H .

Lemma 4.38. *Let H be a discrete subgroup of \mathbb{R}^n . If the elements v_1, \dots, v_m of H are \mathbb{Q} -linearly independent, then they are also \mathbb{R} -linearly independent.*

¹¹use the fact that according to the positive solution of the ternary Goldbach’s conjecture there exists a constant $C > 0$ such that every odd integer $\geq C$ is a sum of three primes (see [96] for further details on Goldbach’s conjecture).

¹²in other words, the subcategory of all totally bounded groups forms an epireflective subcategory of the category of all topological groups.

Proof. Let $V \cong \mathbb{R}^k$ be the subspace of \mathbb{R}^n generated by H . We can assume wlog that $V = \mathbb{R}^n$, i.e., $k = n$. Hence we have to prove that the free-rank $m = r_0(H)$ of H coincides with n . Obviously $m \geq n$. We need to prove that $m \leq n$. Let us fix n \mathbb{R} -linearly independent vectors v_1, \dots, v_n in H . It is enough to see that for every $h \in H$ the vectors v_1, \dots, v_n, h are not \mathbb{Q} -linearly independent. This would imply $m \leq n$. Let us note first that we can assume wlog that $H \supseteq \mathbb{Z}^n$. Indeed, as v_1, \dots, v_n are \mathbb{R} -linearly independent, there exists a linear isomorphism $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\alpha(v_i) = e_i$ for $i = 1, 2, \dots, n$, where e_1, \dots, e_n is the canonical base of \mathbb{R}^n . Clearly, $\alpha(H)$ is still a discrete subgroup of \mathbb{R}^n and the vectors v_1, \dots, v_n, h are \mathbb{Q} -linearly independent iff the vectors $e_1 = \alpha(v_1), \dots, e_n = \alpha(v_n), \alpha(h)$ are. The latter fact is equivalent to $\alpha(h) \notin \mathbb{Q}^n$. Therefore, arguing for a contradiction, assume for simplicity that $H \supseteq \mathbb{Z}^n$ and there exists $h = (h_1, \dots, h_n) \in H$ such that

$$h \notin \mathbb{Q}^n. \quad (4)$$

By the discreteness of H there exists an $\varepsilon > 0$ with $\max\{|h_i| : i = 1, 2, \dots, n\} \geq \varepsilon$ for every $0 \neq h = (h_1, \dots, h_n) \in H$. Represent the cube $C = [0, 1]^n$ as a finite union $\bigcup_i C_i$ of cubes C_i of diameter $< \varepsilon$ (e.g., take them with faces parallel to the coordinate axes, although their precise position is completely irrelevant). For a real number r denote by $\{r\}$ the unique number $0 \leq x < 1$ such that $r - x \in \mathbb{Z}$. Then $(\{mv_1\}, \dots, \{mv_n\}) \neq (\{lh_1\}, \dots, \{lh_n\})$ for every positive $l \neq m$, since otherwise, $(m-l)h \in \mathbb{Z}^n$ with $m-l \neq 0$ in contradiction with (4). Among the infinitely many points $a_m = (\{mh_1\}, \dots, \{mh_n\}) \in C$ there exist two $a_m \neq a_l$ belonging to the same cube C_i . Hence, $|\{mh_j\} - \{lh_j\}| < \varepsilon$ for every $j = 1, 2, \dots, n$. So there exists a $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$, such that $0 \neq (m-l)h - z \in H$ and $|(m-l)h_j - z_j| < \varepsilon$ for every $j = 1, 2, \dots, n$, this contradicts the choice of ε . \square

The aim of the next step is to see that the discrete subgroups of \mathbb{R}^n are free.

Proposition 4.39. *If H is a discrete subgroup of \mathbb{R}^n , then H is free and $r(H) \leq n$.*

Proof. In fact, let $m = r(H)$. By the definition of $r(H)$ there exist m \mathbb{Q} -linearly independent vectors v_1, \dots, v_m of H . By the previous lemma the vectors v_1, \dots, v_m are also \mathbb{R} -linearly independent. Hence, $m \leq n$. Let $V \cong \mathbb{R}^m$ be the subspace of \mathbb{R}^n generated by v_1, \dots, v_m . Obviously, $H \subseteq V$, since H is contained in the \mathbb{Q} -subspace of \mathbb{R}^n generated by the free subgroup $F = \langle v_1, \dots, v_m \rangle$ of H . Since H is a discrete subgroup of V too, we can argue with V in place of \mathbb{R}^n . So, we can assume wlog that $m = n$ and $V = \mathbb{R}^n$. It suffices to see that H/F is finite. Then H will be finitely generated and torsion-free, hence H must be free.

Since the vectors v_1, \dots, v_n are linearly independent on \mathbb{R} we can assume wlog that $H \supseteq \mathbb{Z}^n$. In fact, let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear isomorphism with $\alpha(v_i) = e_i$ for $i = 1, 2, \dots, n$, where e_1, \dots, e_n is the canonical base of \mathbb{R}^n . Then $\alpha(H)$ is still a discrete subgroup of \mathbb{R}^n , $\mathbb{Z}^n = \alpha(F) \subseteq \alpha(H)$ and H/F is finite iff $\alpha(H)/\alpha(F) \cong H/F$ is finite.

In the sequel we assume $H \supseteq \mathbb{Z}^n = F$ for the sake of simplicity. To check that H/F is finite consider the canonical homomorphism $q : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$. According to Theorem 3.23, q sends the closed subgroup H onto a closed (hence compact) subgroup $q(H)$ of \mathbb{T}^n ; moreover $H = q^{-1}(q(H))$, hence the restriction of q to H is open and $q(H)$ is discrete. Thus $q(H) \cong H/F$ is both compact and discrete, so $q(H)$ must be finite. \square

Now we are in position to prove Theorem 4.37. We advise the reader to review the warming exercise 3.20.

Proof of Theorem 4.37. Let $H \neq 0$ a closed subgroup of \mathbb{R}^n . If H discrete, then H is free and generates a linear subspace of \mathbb{R}^n of dimension $r(H) \leq n$ by Proposition 4.39, so the assertion is true with $s = 0$.

In case H is not discrete we argue by induction on n . The case $n = 1$ is Exercise 3.20. Let $n > 1$ and assume the theorem is true for $n - 1$. Consider the subset

$$M = \{u \in \mathbb{R}^n : \|u\| = 1 \text{ and } \exists \lambda \in (0, 1) \text{ with } \lambda u \in H\}$$

of the unitary sphere S in \mathbb{R}^n . For $u \in S$ let $N_u = \{r \in \mathbb{R} : ru \in H\}$. Then N_u is a closed subgroup of \mathbb{R} and $H \cap \mathbb{R}u = N_u u$. Our aim will be to find some $u \in S$ such that the whole line $\mathbb{R}u$ is contained in H . This will allow us to use our inductive hypothesis. Since the proper closed subgroups of \mathbb{R} are cyclic (see Exercise 3.20), it suffices to find some $u \in S$ such that N_u is not cyclic.

Case 1. If $M = \{u_1, \dots, u_n\}$ is finite, then there exists an index i such that $\lambda u_i \in H$ for infinitely many $\lambda \in (0, 1)$. Then the closed subgroup N_{u_i} cannot be cyclic, so H contains to line $\mathbb{R}u_i$ and we are done.

Case 2. Assume M is infinite. By the assumption H is not discrete there exists a sequence $u_n \in M$ such that the corresponding λ_n , with $\lambda_n u_n \in H$, converge to 0. By the compactness of S there exists a limit point $u_0 \in S$ for the sequence $u_n \in M$. We can assume wlog that $u_n \rightarrow u_0$. Let $\varepsilon > 0$ and let Δ_ε be the open interval $(\varepsilon, 2\varepsilon)$. As $\lambda_n \rightarrow 0$, there exists n_0 such that $\lambda_n < \varepsilon$ for every $n \geq n_0$. Hence for every $n \geq n_0$ there exists an

appropriate $k_n \in \mathbb{N}$ with $\eta_n = k_n \lambda_n \in \Delta_\varepsilon$. Taking again a subsequence we can assume wlog that there exists some $\xi_\varepsilon \in \overline{\Delta_\varepsilon}$ such that $\eta_n \rightarrow \xi_\varepsilon$. Hence $\xi_\varepsilon u_0 = \lim_n k_n \lambda_n u_0 \in H$. This argument shows that N_{u_0} contains $\xi_\varepsilon \in \overline{\Delta_\varepsilon}$ with arbitrarily small ε . Therefore, N_{u_0} cannot be cyclic. Hence H contains the line $\mathbb{R}u_0$.

We proved in all cases that our assumption of non-discreteness of H yields the existence of a line $L \cong \mathbb{R}$ as a subgroup of H . Let $L' \cong \mathbb{R}^{n-1}$ be a subspace of \mathbb{R}^n complementing L . Then $\mathbb{R}^n = L \times L'$ and the projection $\mathbb{R}^n \rightarrow L'$ sends H to a closed subgroup H_1 of L' as $L \leq H$ (cf. 3.23 (b)). Moreover, $H = L \times H_1$ in view of $L \leq H$ again. Now proceed by induction with the subgroup H_1 of $L' \cong \mathbb{R}^{n-1}$. This proves Theorem 4.37.

The next corollary describes the quotients of \mathbb{R}^n .

Corollary 4.40. *A quotient of \mathbb{R}^n is isomorphic to $\mathbb{R}^k \times \mathbb{T}^m$, where $k + m \leq n$. In particular, a compact quotient of \mathbb{R}^n is isomorphic to \mathbb{T}^m for some $m \leq n$.*

Proof. Let H be a closed subgroup of \mathbb{R}^n . Then $H = V + D$, where V, D are as in Theorem 4.37. If $s = \dim V$ and $m = r_0(D)$, then $s + m \leq n$. Let V_1 be the linear subspace of \mathbb{R}^n spanned by D . Pick a complementing subspace V_2 for the subspace $V + V_1$ and let $k = n - (s + m)$. Then $\mathbb{R}^n = V + V_1 + V_2$ is a factorization in direct product. Therefore $\mathbb{R}^n/H \cong (V_1/D) \times V_2$. Since $\dim V_1 = r_0(D) = m$, one has $V_1/D \cong \mathbb{T}^m$. Therefore, $\mathbb{R}^n/H \cong \mathbb{T}^m \times \mathbb{R}^k$. \square

Let us denote by $(x|y)$ the usual scalar product in \mathbb{R}^n . Recall that every base v_1, \dots, v_n di \mathbb{R}^n admits a dual base v'_1, \dots, v'_n defined by the relations $(v_i|v'_j) = \delta_{ij}$. For a subgroup H of \mathbb{R}^n define the *associated subgroup* H^\dagger setting

$$H^\dagger := \{u \in \mathbb{R}^n : (\forall x \in H)(x|u) \in \mathbb{Z}\}.$$

Then obviously $(\mathbb{Z}^n)^\dagger = \mathbb{Z}^n$.

Lemma 4.41. *Let H be a subgroup di \mathbb{R}^n . Then:*

1. H^\dagger is a closed subgroup of \mathbb{R}^n and the correspondence $H \mapsto H^\dagger$ is decreasing;
2. $(\overline{H})^\dagger = H^\dagger$.

Proof. The map $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $(x, y) \mapsto (x|y)$ is continuous.

(a) Let $q : \mathbb{R} \rightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the canonical homomorphism. Then for every fixed $a \in \mathbb{R}^n$ the assignment $x \mapsto (a|x) \mapsto f((a|x))$ is a continuous homomorphism $\chi_a : \mathbb{R}^n \rightarrow \mathbb{T}$. Hence the set $\chi_h^{-1}(0) = \{u \in \mathbb{R}^n : (\forall h \in H)(h|u) \in \mathbb{Z}\}$ is closed in \mathbb{R}^n . Therefore $H^\dagger = \bigcap_{h \in H} \chi_h^{-1}(0)$ is closed. The same equality proves that the correspondence $H \mapsto H^\dagger$ is decreasing.

(b) From the second part of (a) one has $(\overline{H})^\dagger \subseteq H^\dagger$. Suppose that $u \in H^\dagger$ e $x \in \overline{H}$. By the continuity of the map $\chi_x(u) = \chi_u(x)$, as a function of x , one can deduce that $\chi_x(u) = 0$, being $\chi_u(h) = 0$ for every $h \in H$. \square

We study in the sequel the subgroup H^\dagger associated to a closed subgroup H of \mathbb{R}^n . According to Theorem 4.37 there exist a base v_1, \dots, v_n of \mathbb{R}^n and $k \leq n$, such that for some $0 \leq s \leq k$ $H = V \oplus L$ where V is the linear subspace generated by v_1, \dots, v_s and $L = \langle v_{s+1}, \dots, v_k \rangle$. Let v'_1, \dots, v'_n be the dual base of v_1, \dots, v_n .

Lemma 4.42. *In the above notation the subgroup H^\dagger coincides with $\langle v'_{s+1}, \dots, v'_k \rangle + W$, where W is the linear subspace generated by v'_{k+1}, \dots, v'_n .*

Proof. Let V' be the linear subspace generated by v'_1, \dots, v'_s , V'' the linear subspace generated by v'_{s+1}, \dots, v'_k and $L' = \langle v'_{s+1}, \dots, v'_k \rangle$. Then $L^\dagger = V' + L' + W$, while $V^\dagger = V'' + W$. Hence $H^\dagger \leq L^\dagger \cap V^\dagger = L' + W$. On the other hand, obviously $L' + W \leq H^\dagger$. \square

Corollario 4.43. $\overline{H} = (H^\dagger)^\dagger$ for every subgroup H of \mathbb{R}^n .

Proof. If H is closed of the form $V + L$ in the notation of the previous lemma, then $H^\dagger = L' + W$ with v'_1, \dots, v'_n , L' and W defined as above. Now $H^\dagger = L' + W$ is a closed subgroup of \mathbb{R}^n by Lemma 4.41 and v_1, \dots, v_n is a dual base of v'_1, \dots, v'_n . Therefore, $H = V + L$ coincides with $(H^\dagger)^\dagger$. \square

Lemma 4.44. *Let V be a hyperplain in \mathbb{R}^n determined by the equation $\sum_{i=1}^n a_i x_i = 0$ such that there exists at least one coefficient $a_i = 1$. Then the subgroup $H = V + \mathbb{Z}^n$ of \mathbb{R}^n is not dense iff all the coefficients a_i are rational.*

Proof. We can assume wlog that $i = n$. Suppose that H is not dense in \mathbb{R}^n . Then $H^\dagger \neq 0$ by Corollary 4.43. Let $0 \neq z \in H^\dagger$. Since $\mathbb{Z}^n \leq H$, one has $H^\dagger \leq \mathbb{Z}^n = (\mathbb{Z}^n)^\dagger$, so $z \in \mathbb{Z}^n$. If $j < n$, then $a_j \in \mathbb{Q}$ as $v = (0, \dots, 0, 1, 0, \dots, 0, -a_j) \in V$ \square

The next proposition is a particular case of the well-known Kronecker's theorem.

Proposition 4.45. *Let $v_1, \dots, v_n \in \mathbb{R}$. Then for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ the subgroup $\langle v \rangle + \mathbb{Z}^n$ of \mathbb{R}^n is dense iff $v_0 = 1, v_1, \dots, v_n \in \mathbb{R}$ are linearly independent as elements of the vector space \mathbb{R} over \mathbb{Q} .*

Proof. Assume $v_0 = 1, v_1, \dots, v_n \in \mathbb{R}$ are linearly independent and let $H = \langle v \rangle + \mathbb{Z}^n$. Then $H^\dagger \subseteq \mathbb{Z}^n = (\mathbb{Z}^n)^\dagger$. It is easy to see now that some $z \in \mathbb{Z}^n$ belongs to $(\langle v \rangle)^\dagger$ iff $z = 0$. This proves that $H^\dagger = 0$. Consequently H is dense in \mathbb{R}^n by Corollary 4.43. If $\sum_{i=0}^n k_i v_i = 0$ is a non-trivial linear combination with $k_i \in \mathbb{Z}$, then the vector $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ is non-zero and obviously $k \in H^\dagger$. Thus $H^\dagger \neq 0$, hence H is not dense. \square

Theorem 4.46. \mathbb{T}^c is monothetic.

Proof. Let B be a Hamel base of \mathbb{R} on \mathbb{Q} that contains 1 and let $B_0 = B \setminus \{1\}$. Applying the previous proposition one can see that the element $x = (x_b)_{b \in B_0} \in \mathbb{T}^{B_0}$, defined by $x_b = b + \mathbb{Z} \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, is a generator of the group \mathbb{T}^{B_0} . To conclude note that $|B_0| = |\mathbb{R}| = c$. \square

Corollary 4.47. *Let C be a closed subgroup of \mathbb{T}^n . Then C is isomorphic to $\mathbb{T}^s \times F$ where $s \leq n$ and F is a finite abelian group.*

Proof. Let $q : \mathbb{R}^n \rightarrow \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the canonical projection. If C is a closed subgroup of \mathbb{T}^n , then $H = q^{-1}(C)$ is a closed subgroup of \mathbb{R}^n that contains $\mathbb{Z}^n = \ker q$. Hence H is a direct product $H = V + D$ with $V \cong \mathbb{R}^s$ and $D \cong \mathbb{Z}^m$, where s and m satisfy $s + m = n$ as $\mathbb{Z}^n \leq H$. Since the restriction of q to H is open by Theorem 3.23, we conclude that the restriction of q to V is open as far as V is open in H . Hence $q|_V : V \rightarrow q(V)$ is an open surjective homomorphism and the subgroup $q(V)$ is open in C . Since $q(V)$ is also divisible, we can write $C = q(V) \times F$, where the subgroup F of C must be discrete. Since C is compact, this implies that F is finite. On the other hand, as a compact quotient of $V \cong \mathbb{R}^s$ the group $q(V)$ is isomorphic to \mathbb{T}^s by Corollary 4.40. Therefore, $C \cong \mathbb{T}^s \times F$. \square

Exercise 4.48. Determine for which of the following possible choices of the vector $v \in \mathbb{R}^4$

$$(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}), (\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}), (\log 2, \log 3, \log 5, \log 6),$$

$$(\log 2, \log 3, \log 5, \log 7), (\log 3, \log 5, \log 7, \log 9) \text{ and } (\log 5, \log 7, \log 9, \log 11)$$

the subgroup $\langle v \rangle + \mathbb{Z}^4$ of \mathbb{R}^4 is dense.

Exercise 4.49. (a) Prove that a subgroup H of \mathbb{T} is dense iff H is infinite.

(b) Determine the minimal (w.r.t. inclusion) dense subgroups \mathbb{T} .

(c)* Determine the minimal (w.r.t. inclusion) dense subgroups \mathbb{T}^2 .

5 Følner's theorem

This section is entirely dedicated to Følner's theorem.

5.1 Fourier theory for finite abelian groups

In the sequel G will be a finite abelian group, so $G^* \cong G$, so in particular $|G^*| = |G|$.

Here we recall some well known properties of the scalar product in finite-dimensional complex spaces $V = \mathbb{C}^n$. Since our space will be "spanned" by a finite abelian group G of size n (i.e., $V = \mathbb{C}^G$), we have also an action of G on V . We normalize the scalar product in a such way to let the vector $(1, 1, \dots, 1)$ (i. e., the constant function 1) to have norm 1. The reader familiar with Haar integration may easily recognize in this the Haar integral on G .

Define the scalar product by

$$(f, g) = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}.$$

Let us see first that the elements of the subset G^* of V are pairwise orthogonal and have norm 1:

Proposition 5.1. *Let G be an abelian finite group and $\varphi, \chi \in G^*$, $x, y \in G$. Then:*

$$(a) \frac{1}{|G|} \sum_{x \in G} \varphi(x) \overline{\chi(x)} = \begin{cases} 1 & \text{if } \varphi = \chi; \\ 0 & \text{if } \varphi \neq \chi; \end{cases}$$

$$(b) \frac{1}{|G^*|} \sum_{\chi \in G^*} \chi(x) \overline{\chi(y)} = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$$

Proof. (a) If $\varphi = \chi$ then $\chi(x) \overline{\chi(x)} = \chi(x) \chi(x)^{-1} = 1$.

If $\varphi \neq \chi$ there exists $z \in G$ such that $\varphi(z) \neq \chi(z)$. Therefore the following equalities

$$\sum_{x \in G} \varphi(x) \overline{\chi(x)} = \frac{\varphi(z)}{\chi(z)} \sum_{x \in G} \varphi(x - z) \overline{\chi(x - z)} = \frac{\varphi(z)}{\chi(z)} \sum_{x \in G} \varphi(x) \overline{\chi(x)}$$

imply that $\sum_{x \in G} \varphi(x) \overline{\chi(x)} = 0$.

(b) If $x = y$ then $\chi(x) \overline{\chi(x)} = \chi(x) \chi(x)^{-1} = 1$.

If $x \neq y$, by Corollary 2.7 there exists $\chi_0 \in G^*$ such that $\chi_0(x) \neq \chi_0(y)$. Now we can proceed as before, that is

$$\sum_{\chi \in G^*} \chi(x) \overline{\chi(y)} = \frac{\chi_0(x)}{\chi_0(y)} \sum_{\chi \in G^*} (\chi \chi_0)(x) \overline{(\chi \chi_0)(y)} = \frac{\chi_0(x)}{\chi_0(y)} \sum_{\chi \in G^*} \chi(x) \overline{\chi(y)}$$

yields $\sum_{\chi \in G^*} \chi(x) \overline{\chi(y)} = 0$. \square

If G is a finite abelian group and f is a complex valued function on G , then for every $\chi \in G^*$ we can define

$$c_\chi = (f, \chi) = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{\chi(x)},$$

that is the *Fourier coefficient* of f corresponding to χ .

For complex valued functions f, g on a finite abelian group G define the *convolution* $f * g$ by $(f * g)(x) = \frac{1}{|G|} \sum_{y \in G} \overline{f(y)} g(x + y)$.

Proposition 5.2. *Let G be an abelian finite group and f a complex valued function on G with Fourier coefficients c_χ where $\chi \in G^*$. Then:*

- (a) $f(x) = \sum_{\chi \in G^*} c_\chi \chi(x)$ for every $x \in G$;
- (b) if $\{a_\chi\}_{\chi \in G^*}$ is such that $f(x) = \sum_{\chi \in G^*} a_\chi \chi(x)$, then $a_\chi = c_\chi$ for every $\chi \in G^*$;
- (c) $\frac{1}{|G|} \sum_{x \in G} |f(x)|^2 = \sum_{\chi \in G^*} |c_\chi|^2$;
- (d) if g is an other complex valued function on G with Fourier coefficients $(d_\chi)_{\chi \in G^*}$, then $f * g$ has Fourier coefficients $(\overline{c_\chi} d_\chi)_{\chi \in G^*}$.

Proof. (a) The definition of the coefficients c_χ yields

$$\sum_{\chi \in G^*} c_\chi \chi(x) = \sum_{\chi \in G^*} \frac{1}{|G|} \sum_{y \in G} f(y) \overline{\chi(y)} \chi(x).$$

Computing $\sum_{\chi \in G^*} \overline{\chi(y)} \chi(x)$ with Proposition 5.1(b) we get $\sum_{\chi \in G^*} c_\chi \chi(x) = \frac{|G^*|}{|G|} f(x)$ for every $x \in G$. Now $|G| = |G^*|$ gives $f(x) = \sum_{\chi \in G^*} c_\chi \chi(x)$ for every $x \in G$.

(b) By Proposition 5.1 the definition of the coefficients c_χ and the relation $f(x) = \sum_{\chi \in G^*} a_\chi \chi(x)$

$$c_\chi = \frac{1}{|G|} \sum_{\varphi \in G^*} a_\varphi \sum_{x \in G} \varphi(x) \overline{\chi(x)} = a_\chi.$$

(d) By item (a) $g(x) = \sum_{\varphi \in G^*} d_\varphi \varphi(x)$ for every $x \in G$. Therefore

$$\begin{aligned} \sum_{y \in G} \overline{f(y)} g(x + y) &= \sum_{y \in G} \left(\sum_{\chi \in G^*} \overline{c_\chi} \overline{\chi(y)} \right) \left(\sum_{\varphi \in G^*} d_\varphi \varphi(x + y) \right) = \\ &= \sum_{\chi \in G^*} \sum_{\varphi \in G^*} \overline{c_\chi} d_\varphi \varphi(x) \sum_{y \in G} \overline{\chi(y)} \varphi(y) = |G| \sum_{\chi \in G^*} \overline{c_\chi} d_\chi \chi(x). \end{aligned}$$

(c) It is sufficient to apply (d) with $g = f$ and let $x = 0$. \square

Corollary 5.3. *Let G be a finite abelian group, E be a non-empty subset of G and let f be the characteristic function of E . Then for the convolution $g = f * f$ one has*

- (a) $g(x) > 0$ iff $x \in E_{(2)}$;
- (b) $g(x) = \sum_{\chi \in G^*} |c_\chi|^2 \chi(x)$.

Proof. (a) $g(x) > 0$ if and only if there exists $y \in E$ with $x + y \in E$, that is $x \in E - E = E_{(2)}$.
 (b) follows obviously from Proposition 5.2(d). □

5.2 Bogoliouboff and Følner Lemmas

Lemma 5.4 (Bogoliouboff lemma). *If F is a finite abelian group and E is a non-empty subset of F , then there exist $\chi_1, \dots, \chi_m \in F^*$, where $m = \left\lceil \left(\frac{|F|}{|E|} \right)^2 \right\rceil$, such that $U(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq E_{(4)}$.*

Proof. Let f be the characteristic function of E . By Proposition 5.2(a) we have

$$f(x) = \sum_{\chi \in F^*} c_\chi \chi(x), \text{ with } c_\chi = \frac{1}{|F|} \sum_{x \in F} f(x) \overline{\chi(x)}. \quad (1)$$

Define $g = f * f$ and $h = g * g$. The functions f and $g = f * f$ have real values and by Corollary 5.3

$$g(x) = \sum_{\chi \in F^*} |c_\chi|^2 \chi(x) \text{ and } h(x) = \sum_{\chi \in F^*} |c_\chi|^4 \chi(x) \text{ for } x \in F. \quad (2)$$

Moreover, $g(x) > 0$ if and only if $x \in E - E = E_{(2)}$. Analogously $h(x) > 0$ if and only if $x \in E_{(4)}$.

By Proposition 5.2(c) $\sum_{\chi \in F^*} |c_\chi|^2 = \frac{|E|}{|F|}$. Set $a = \frac{|E|}{|F|}$ and order the Fourier coefficients of f so that

$$|c_{\chi_0}| \geq |c_{\chi_1}| \geq \dots \geq |c_{\chi_k}| \geq \dots$$

(note that they are finitely many). Thus $\chi_0 = 1$ and $c_{\chi_0} = a$ by (1). Then $\sum_{i=0}^k |c_{\chi_i}|^2 \leq \sum_{\chi \in F^*} |c_\chi|^2 = a$ for every $k \geq 0$. Consequently $(k+1)|c_{\chi_k}|^2 \leq a$, so

$$|c_{\chi_k}|^4 \leq \frac{a^2}{(k+1)^2}. \quad (3)$$

Now let $m = \left\lceil \frac{1}{a^2} \right\rceil$. We are going to show now that with these $\chi_1, \dots, \chi_m \in F^*$ one has

$$h(x) > 0 \text{ for every } x \in U(\chi_1, \dots, \chi_m; \frac{\pi}{2}). \quad (4)$$

Clearly $\operatorname{Re} \chi_k(x) \geq 0$ for $k = 1, 2, \dots, m$ whenever $x \in U(\chi_1, \dots, \chi_m; \frac{\pi}{2})$ thus

$$|a^4 + \sum_{k=0}^m |c_{\chi_k}|^4 \chi_k(x)| \geq \operatorname{Re}(a^4 + \sum_{k=1}^m |c_{\chi_k}|^4 \chi_k(x)) \geq a^4. \quad (5)$$

On the other hand, (3) yields

$$\sum_{k \geq m+1} |c_{\chi_k}|^4 \leq \sum_{k \geq m+1} \frac{a^2}{(k+1)^2} < a^2 \sum_{k \geq m+1} \frac{1}{k(k+1)} \leq \frac{a^2}{m+1}. \quad (6)$$

Since h has real values, (2), (5) and (6) give

$$h(x) = |h(x)| = |a^4 + |c_{\chi_1}|^4 \chi_1(x) + \dots| \geq \left| a^4 + \sum_{k=1}^m |c_{\chi_k}|^4 \chi_k(x) \right| - \sum_{k \geq m+1} |c_{\chi_k}|^4 \geq a^4 - \frac{a^2}{m+1} \geq a^2 \left(a^2 - \frac{1}{m+1} \right) > 0.$$

This proves (4). Therefore $U(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq E_{(4)}$. □

Let us note that the estimate for the number m of characters is certainly non-optimal when E is too small. For example, when E is just the singleton $\{0\}$, the upper bound given by the lemma is just $|F|^2$, while one can certainly find at most $m = |F| - 1$ characters χ_1, \dots, χ_m (namely, all non-trivial $\chi_i \in F^*$) such that $U(\chi_1, \dots, \chi_m; \frac{\pi}{2}) = \{0\}$. For certain groups (e.g., $F = \mathbb{Z}_2^k$) one can find even a much smaller number (say $m = \log_2 |F|$). Nevertheless, in the cases relevant for the proof of Følner's theorem, namely when the subset E is relatively large with respect to F , this estimate seems more reasonable.

The next lemma will be needed in the following proofs.

Lemma 5.5. *Let A be an abelian group and $\{A_n\}_{n=1}^\infty$ be a sequence of finite subsets of A such that*

$$\lim_{n \rightarrow \infty} \frac{|(A_n - a) \cap A_n|}{|A_n|} = 1$$

for every $a \in A$. If k is a positive integer and V is a subset of A such that k translates of V cover A , then for every $\varepsilon > 0$ there exists $N > 0$ such that

$$|V \cap A_n| > \left(\frac{1}{k} - \varepsilon\right) |A_n|$$

for every $n \geq N$.

Proof. Let $a_1, \dots, a_k \in A$ be such that $\bigcup_{i=1}^k (a_i + V) = A$. If $\varepsilon > 0$, then there exists $N_1 > 0$ such that for every $n \geq N_1$

$$|(A_n - a_i) \cap A_n| > (1 - \varepsilon) |A_n|$$

and consequently,

$$|(A_n - a_i) \setminus A_n| < \varepsilon |A_n| \quad (7)$$

for every $i = 1, \dots, k$. Since $A_n = \bigcup_{i=1}^k (a_i + V) \cap A_n$, for every n there exists $i_n \in \{1, \dots, k\}$ such that

$$\frac{1}{k} |A_n| \leq |(a_{i_n} + V) \cap A_n| = |V \cap (A_n - a_{i_n})|.$$

Since $V \cap (A_n - a_{i_n}) \subseteq (V \cap A_n) \cup ((A_n - a_{i_n}) \setminus A_n)$, (7) yields

$$\frac{1}{k} |A_n| \leq |V \cap (A_n - a_{i_n})| \leq |V \cap A_n| + |(A_n - a_{i_n}) \setminus A_n| < |V \cap A_n| + \varepsilon |A_n|.$$

□

Lemma 5.6 (Bogoliouboff-Følner lemma). *Let A be a finitely generated abelian group and let $r = r_0(A)$. If k is a positive integer and V is a subset of A such that k translates of V cover A , then there exist $\rho_1, \dots, \rho_s \in A^*$, where $s = 3^{2r} k^2$, such that $U_A(\rho_1, \dots, \rho_s; \frac{\pi}{2}) \subseteq V_{(4)}$.*

Proof. By Theorem 2.1 we have $A \cong \mathbb{Z}^r \times F$, where F is a finite abelian group; so we can identify A with the group $\mathbb{Z}^r \times F$. Define $A_n = (-n, n]^r \times F$, let $a = (a_1, \dots, a_r; f) \in \mathbb{Z}^r \times F$. Then $J_{ni} = (-n, n] \cap (-n - a_i, n - a_i]$ satisfies $|J_{ni}| \geq 2n - |a_i|$. In particular, $J_{ni} \neq \emptyset$ for every $n > n_0 = \max\{|a_i| : i = 1, 2, \dots, r\}$. As $(A_n - a) \cap A_n = \prod_{i=1}^r J_{ni} \times F$, we have

$$|(A_n - a) \cap A_n| \geq |F| \cdot \prod_{i=1}^r (2n - |a_i|)$$

or all $n > n_0$. Since $|A_n| = |F|(2n)^r$, we can apply Lemma 5.5. Thus for every $\varepsilon > 0$ we have

$$|V \cap A_n| > \left(\frac{1}{k} - \varepsilon\right) |A_n|. \quad (8)$$

for every sufficiently large n . For n with (8) define $G = A/(6n\mathbb{Z}^r)$ and $E = q(V \cap A_n)$ where q is the canonical projection of A onto G . Observe that $q|_{A_n}$ is injective, as $(A_n - A_n) \cap \ker q = \{0\}$. Then (8) gives

$$|E| = |V \cap A_n| > \left(\frac{1}{k} - \varepsilon\right) |A_n| = \left(\frac{1}{k} - \varepsilon\right) (2n)^r |F|$$

and so

$$\frac{|G|}{|E|} \leq \frac{(6n)^r |F|}{\left(\frac{1}{k} - \varepsilon\right) (2n)^r |F|} = \frac{3^r k}{1 - k\varepsilon}.$$

Fix $\varepsilon > 0$ sufficiently small to have $\left[\frac{3^{2r} k^2}{(1 - k\varepsilon)^2}\right] = 3^{2r} k^2$ and pick sufficiently large n to have (8). Now apply the Bogoliouboff Lemma 5.4 to find $s = 3^{2r} k^2$ characters $\xi_{1n}, \dots, \xi_{sn} \in G^*$ such that $U_G(\xi_{1n}, \dots, \xi_{sn}; \frac{\pi}{2}) \subseteq E_{(4)}$. For $j = 1, \dots, s$ define $\varrho_{jn} = \xi_{jn} \circ \pi \in A^*$. If $a \in A_n \cap U_A(\varrho_{1n}, \dots, \varrho_{sn}; \frac{\pi}{2})$ then $q(a) \in U_A(\xi_{1n}, \dots, \xi_{sn}; \frac{\pi}{2}) \subseteq E_{(4)}$ and so there exist $b_1, b_2, b_3, b_4 \in V \cap A_n$ and $c = (c_i) \in 6n\mathbb{Z}^r$ such that $a = b_1 - b_2 + b_3 - b_4 + c$. Now

$$c = a - b_1 + b_2 - b_3 + b_4 \in (A_n)_{(4)} + A_n$$

implies $|c_i| \leq 5n$ for each i . So $c = 0$ as $6n$ divides c_i for each i . Thus $a \in V_{(4)}$ and so

$$A_n \cap U_A \left(\varrho_{1n}, \dots, \varrho_{sn}; \frac{\pi}{2} \right) \subseteq V_{(4)} \quad (9)$$

for all n satisfying (8).

By Lemma 4.2 there exist $\varrho_1, \dots, \varrho_s \in A^*$ and a subsequence $\{n_l\}_l$ of $\{n\}_{n \in \mathbb{N}_+}$ such that $\varrho_i(a) = \lim_l \varrho_{in_l}(a)$ for every $i = 1, \dots, s$ and $a \in A$. We are going to prove now that

$$U_A \left(\varrho_1, \dots, \varrho_s; \frac{\pi}{2} \right) \subseteq V_{(4)}. \quad (10)$$

Take $a \in U_A(\varrho_1, \dots, \varrho_s; \frac{\pi}{2})$. Since $A = \bigcup_{l=k}^{\infty} A_{n_l}$ for every $k \in \mathbb{N}_+$, there exists n_0 satisfying (8) and $a \in A_{n_0}$. As $\varrho_i(a) = \lim_l \varrho_{in_l}(a)$ for every $i = 1, \dots, s$, we can pick l to have $n_l \geq n_0$ and $|\text{Arg}(\varrho_{in_l}(a))| < \pi/2$ for every $i = 1, \dots, s$, i.e., $a \in U_A(\varrho_{1n_l}, \dots, \varrho_{sn_l}; \frac{\pi}{2}) \cap A_{n_l}$. Now (9), applied to n_l , yields $a \in V_{(4)}$. This proves (10). \square

Our next aim is to eliminate the dependence of the number m of characters on the free rank of the group A in Bogoliouboff - Følner's lemma. The price to pay for this is taking $V_{(8)}$ instead of $V_{(4)}$.

Lemma 5.7 (Følner lemma). *Let A be an abelian group. If k is a positive integer and V be a subset of A such that k translates of V cover A , then there exist $\chi_1, \dots, \chi_m \in A^*$, where $m = k^2$, such that $U_A(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq V_{(8)}$.*

Proof. We consider first the case when A is finitely generated. Let $r = r_0(A)$. By Lemma 5.6 there exist $\varrho_1, \dots, \varrho_s \in A^*$, where $s = 3^{2r}k^2$, such that

$$U_A \left(\varrho_1, \dots, \varrho_s; \frac{\pi}{2} \right) \subseteq V_{(4)}.$$

Since it is finitely generated, we can identify A with $\mathbb{Z}^r \times F$, where F is a finite abelian group. For $t \in \{1, \dots, r\}$ define a monomorphism $i_t : \mathbb{Z} \hookrightarrow A$ by letting

$$i_t(n) = (\underbrace{0, \dots, 0}_t, n, 0, \dots, 0; 0) \in A.$$

Then each $\kappa_{jt} = \varrho_j \circ i_t$, where $j \in \{1, \dots, s\}, t \in \{1, \dots, r\}$, is a character of \mathbb{Z} . By Proposition 4.30 the subset

$$L = U_{\mathbb{Z}} \left(\{\kappa_{jt} : j = 1, \dots, s, t = 1, \dots, r\}; \frac{\pi}{8r} \right)$$

of \mathbb{Z} is infinite. Let $L^0 = \bigcup_{t=1}^r i_t(L)$, i.e., this is the set of all elements of A of the form $\pm i_t(n)$ with $n \in L$ and $t \in \{1, \dots, r\}$. Then obviously $L^0 = -L^0 \subseteq U_A(\varrho_1, \dots, \varrho_s; \frac{\pi}{8r})$, therefore,

$$L_{(4r)}^0 \subseteq U_A \left(\varrho_1, \dots, \varrho_s; \frac{\pi}{2} \right) \subseteq V_{(4)}. \quad (\lambda)$$

Define $A_n = (-n, n]^r \times F$ and pick $\varepsilon > 0$ such that $\left[\left(\frac{k}{1-k\varepsilon} \right)^2 \right] = k^2$. As in Lemma 5.6 A_n satisfies the hypotheses of Lemma 5.5 and so $|V \cap A_n| > (\frac{1}{k} - \varepsilon)|A_n|$ for sufficiently large n . Moreover, we choose this sufficiently large n from L . Let $G_n = A/(2n\mathbb{Z}^r) \cong \mathbb{Z}_{2n}^r \times F$ and $E = q(A_n \cap V)$ where q is the canonical projection $A \rightarrow G_n$. Then $q \upharpoonright_{A_n}$ is injective as $(A_n - A_n) \cap \ker q = 0$. So q induces a bijection between A_n and G_n on one hand, and between $V \cap A_n$ and E . Thus $|A_n| = |G_n| = (2n)^r |F|$, $|E| > (\frac{1}{k} - \varepsilon)|A_n|$ and so

$$\left(\frac{|G_n|}{|E|} \right)^2 \leq \left(\frac{k}{\varepsilon k - 1} \right)^2 \leq k^2.$$

To the finite group G_n apply the Bogoliouboff Lemma 5.4 to get $\xi_{1n}, \dots, \xi_{mn} \in G_n^*$, where $m = k^2$, such that

$$U_{G_n} \left(\xi_{1n}, \dots, \xi_{mn}; \frac{\pi}{2} \right) \subseteq E_{(4)}.$$

Let $\chi_{jn} = \xi_{jn} \circ q \in A^*$. If $a \in A_n \cap U_A(\chi_{1n}, \dots, \chi_{mn}; \frac{\pi}{2})$, then $q(a) \in U_{G_n}(\xi_{1n}, \dots, \xi_{mn}; \frac{\pi}{2}) \subseteq E_{(4)}$. Therefore there exist $b_1, b_2, b_3, b_4 \in A_n \cap V$ and $c = (c_i) \in 2n\mathbb{Z}^r$ such that $a = b_1 - b_2 + b_3 - b_4 + c$. Since $2n$ divides c_i for every i and $|c_i| \leq 5n$, we conclude that $c_i \in \{0, \pm 2n \pm 4n\}$ for $i = 1, 2, \dots, r$. This means that c can be written as a sum of at most $4r$ elements of L^0 . This gives $c \in L_{(4r)}^0 \subseteq V_{(4)}$ by (λ) , consequently $a \in V_{(8)}$. Therefore

$$A_n \cap U_A \left(\chi_{1n}, \dots, \chi_{mn}; \frac{\pi}{2} \right) \subseteq V_{(8)}$$

for $n \in L$ sufficiently large n . By Lemma 4.2 there exist $\chi_1, \dots, \chi_m \in A^*$ and a subsequence $\{n_l\}_l$ of $\{n\}_{n \in \mathbb{N}_+}$ such that $\chi_j(a) = \lim_l \chi_{jn_l}(a)$ for every $j = 1, \dots, m$ and for every $a \in A$. Being $A = \bigcup \{A_n : l > k, n_l \in L\}$ for every $k \in \mathbb{N}_+$ we can conclude as above that $U_A(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq V_{(8)}$.

Consider now the general case. Let $g_1, \dots, g_k \in A$ be such that $A = \bigcup_{i=1}^k (g_i + V)$. Suppose that G is a finitely generated subgroup of A containing g_1, \dots, g_k . Then $G = \bigcup_{i=1}^k (g_i + V \cap G)$ and so k translates of $V \cap G$ cover G . By the above argument and by Theorem 2.5 there exist $\varphi_{1G}, \dots, \varphi_{mG} \in G^*$, where $m = k^2$, such that

$$U_G\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) \subseteq (V \cap G)_{(8)} \subseteq V_{(8)}.$$

By Corollary 2.6 we can extend each φ_{iG} to a character of A , so that we assume from now on $\varphi_{1G}, \dots, \varphi_{mG} \in A^*$ and

$$G \cap U_A\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) = U_G\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) \subseteq V_{(8)}. \quad (11)$$

Let \mathcal{G} be the family of all finitely generated subgroups G of A containing g_1, \dots, g_k . It is a directed set under inclusion. So we get m nets $\{\varphi_{jG}\}_{G \in \mathcal{G}}$ in A^* for $j = 1, \dots, m$. By Lemma 4.2 there exist subnets $\{\varphi_{jG_\beta}\}_\beta$ and $\chi_1, \dots, \chi_m \in A^*$ such that

$$\varphi_j(x) = \lim_\beta \varphi_{jG_\beta}(x) \text{ for every } x \in A \text{ and } j = 1, \dots, m. \quad (12)$$

From (11) and (12) we conclude as before that $U_A(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq V_{(8)}$. \square

As a corollary of Følner's lemma we obtain the following description of the neighborhoods of 0 in the Bohr topology of A .

Corollary 5.8. *For a subset E of an abelian group A the following are equivalent:*

- (a) *E contains $V_{(8)}$ for some big subset V of A ;*
- (b) *for every $n \in \mathbb{N}_+$ E contains $V_{(2n)}$ for some big subset V of A ;*
- (c) *E is a neighborhood of 0 in the Bohr topology of A .*

Proof. The implication (c) \Rightarrow (b) follows from Følner's lemma. The implication (c) \Rightarrow (b) follows from Corollary 4.31 and Proposition 4.30. \square

It follows from results of Følner [45] obtained by less elementary tools, that (b) can be replaced by the weaker assumption $V_{(4)} \subseteq E$ (see also Ellis and Keynes [43] or Cotlar and Ricabarra [24] for further improvements). Nevertheless the following old problems concerning the group \mathbb{Z} is still open (see Cotlar and Ricabarra [24], Ellis and Keynes [43], Følner [45], Glasner [54], Pestov [80, Question 1025] or Veech [94]):

Question 5.9. Does there exist a big set $V \subseteq \mathbb{Z}$ such that $V - V$ is not a neighborhood of 0 in the Bohr topology of G ?

It is known that every infinite abelian group G admits a big set with empty interior with respect to the Bohr topology [4] (more precisely, these authors prove that every totally bounded group has a big subset with empty interior).

5.3 Prodanov's lemma and proof of Følner's theorem

Let C be a set in a real or complex vector space. Then C is said to be *convex* if, for all $x, y \in C$ and all $t \in [0, 1]$, the point $(1 - t)x + ty \in C$.

The next lemma, due to Prodanov [84], allows us to eliminate the discontinuous characters in uniform approximations of continuous functions via linear combinations of characters.

Lemma 5.10 (Prodanov's lemma). *Let G be a topological abelian group, let U be an open subset of G , f a complex valued continuous function on G and M a convex closed subset of \mathbb{C} . Let $k \in \mathbb{N}_+$ and $\chi_1, \dots, \chi_k \in G'$. Suppose that $c_1, \dots, c_k \in \mathbb{C}$ are such that $\sum_{j=1}^k c_j \chi_j(x) - f(x) \in M$ for every $x \in U$. If $\chi_{m_1}, \dots, \chi_{m_s}$, with $m_1 < \dots < m_s, s \in \mathbb{N}, \{m_1, \dots, m_s\} \subseteq \{1, \dots, k\}$, are the continuous among χ_1, \dots, χ_k , then $\sum_{i=1}^s c_{m_i} \chi_{m_i}(x) - f(x) \in M$ for every $x \in U$.*

Proof. Let $\chi_k \in G^*$ be discontinuous. Then it is discontinuous at 0. Consequently there exists a net $\{x_\gamma\}_\gamma$ in G such that $\lim_\gamma x_\gamma = 0$ and there exist $y_j = \lim_\gamma \chi_j(x_\gamma)$ for all $j = 1, \dots, k$, but $y_k \neq 1$. Notice that always $|y_j| = 1$. Moreover, $y_j = 1$ when χ_j is continuous because $x_\gamma \rightarrow 0$, so $y_j = \lim \chi_j(x_\gamma) = 1$.

Consider $\sum_{j=1}^k c_j \chi_j(x + tx_\gamma) - f(x + tx_\gamma)$, where $t \in \mathbb{Z}$. Since $\lim_\gamma x_\gamma = 0$, we have $x + tx_\gamma \in U$ for every $x \in U$ and for every sufficiently large γ . Thus $\sum_{j=1}^k c_j \chi_j(x) \chi_j(x_\gamma)^t - f(x + tx_\gamma) \in M$ and so passing to the limit $\sum_{j=1}^k c_j \chi_j(x) y_j^t - f(x) \in M$, because f is continuous and M is closed.

Take an arbitrary $n \in \mathbb{N}$. By the convexity of M and the relation above for $t = 0, \dots, n$, we obtain

$$\frac{1}{n+1} \sum_{t=0}^n \left(\sum_{j=1}^k c_j \chi_j(x) y_j^t - f(x) \right) \in M.$$

Note that $\sum_{t=0}^n y_k^t = \frac{y_k^{n+1} - 1}{y_k - 1}$ because $y_k \neq 1$. Hence we get

$$\sum_{j=1}^{k-1} c_{jn} \chi_j(x) + \frac{c_k}{1+n} \frac{1 - y_k^{n+1}}{1 - y_k} \chi_k(x) - f(x) \in M$$

for every $x \in U$, where $c_{jn} = \frac{\sum_{t=0}^n c_j y_j^t}{n+1}$. Now for every $j = 1, 2, \dots, k-1$

- $|c_{jn}| \leq |c_j| \frac{\sum_{t=0}^n |y_j|^t}{n+1} = |c_j|$ (because $|y_j| = 1$), and
- if $y_j = 1$ then $c_{jn} = c_j$.

By the boundedness of the sequences $\{c_{jn}\}_{n=1}^\infty$ for $j = 1, \dots, k-1$, there exists a subsequence $\{n_m\}_{m=1}^\infty$ such that all limits $c'_j = \lim_m c_{jn_m}$ exist for $j = 1, \dots, k-1$. On the other hand, $|y_k| = 1$, so

$$\lim_n \frac{c_k}{n+1} \frac{1 - y_k^{n+1}}{1 - y_k} = 0.$$

Taking the limit for $m \rightarrow \infty$ in

$$\sum_{j=1}^{k-1} c_{jn_m} \chi_j(x) + \frac{c_k}{1+n_m} \frac{1 - y_k^{n_m+1}}{1 - y_k} \chi_k(x) - f(x) \in M$$

gives

$$\sum_{j=1}^{k-1} c'_j \chi_j(x) - f(x) \in M \quad \text{for } x \in U; \tag{13}$$

moreover $c'_j = c_j$ for every $j = 1, \dots, k-1$ such that χ_j is continuous.

The condition (13) is obtained by the hypothesis, removing the discontinuous character χ_k in such a way that the coefficients of the continuous characters remain the same. Iterating this procedure, we can remove all discontinuous characters among χ_1, \dots, χ_k . \square

Now we give an (apparently) topology-free form of the local version of the Stone-Weierstraß theorem 2.19.

Proposition 5.11. *Let G be an abelian group and H be a group of characters of G . If X is a subset of G and f is a complex valued bounded function on X then the following conditions are equivalent:*

- f can be uniformly approximated on X by a linear combination of elements of H with complex coefficients;*
- for every $\varepsilon > 0$ there exist $\delta > 0$ and $\chi_1, \dots, \chi_m \in H$ such that $x - y \in U_G(\chi_1, \dots, \chi_m; \delta)$ yields $|f(x) - f(y)| < \varepsilon$ for every $x, y \in X$.*

Proof. (a) \Rightarrow (b) Let $\varepsilon > 0$. By (a) there exist $c_1, \dots, c_m \in \mathbb{C}$ and $\chi_1, \dots, \chi_m \in H$ such that $\|\sum_{i=1}^m c_i \chi_i - f\|_\infty < \frac{\varepsilon}{4}$, that is $|\sum_{i=1}^m c_i \chi_i(x) - f(x)| < \frac{\varepsilon}{4}$ for every $x \in X$.

On the other hand note that $|\sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y)| \leq \sum_{i=1}^m |c_i| \cdot |\chi_i(x) - \chi_i(y)|$ and that $|\chi_i(x) - \chi_i(y) - 1| = |\chi_i(x) \chi_i(y)^{-1} - 1| = |\chi_i(x) - \chi_i(y)|$. If we take

$$\delta = \frac{\varepsilon}{2m \max_{i=1, \dots, m} |c_i|}$$

then $x - y \in U(\chi_1, \dots, \chi_m; \delta)$ implies $\sum_{i=1}^m |c_i| \cdot |\chi_i(x) - \chi_i(y)| < \frac{\varepsilon}{2}$ and so also $|\sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y)| < \frac{\varepsilon}{2}$. Consequently,

$$|f(x) - f(y)| \leq \left| f(x) - \sum_{i=1}^m c_i \chi_i(x) \right| + \left| \sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y) \right| + \left| \sum_{i=1}^m c_i \chi_i(y) - f(y) \right| < \varepsilon.$$

(b) \Rightarrow (a) Let βX be the Čech-Stone compactification of X endowed with the discrete topology. If $F : X \rightarrow \mathbb{C}$ is bounded, there exists a unique continuous extension F^β of F to βX . Let \mathcal{S} be the collection of all continuous functions g on βX such that $g = \sum_{j=1}^n c_j \chi_j^\beta$ with $\chi_j \in H$, $c_j \in \mathbb{C}$ and $n \in \mathbb{N}_+$. Then \mathcal{S} is a subalgebra of $\mathcal{C}(\beta X, \mathbb{C})$ closed under conjugation and contains all constants. In fact in \mathcal{S} we have $\chi_k^\beta \chi_j^\beta = (\chi_k \chi_j)^\beta$ by definition and $\overline{\chi^\beta} = (\overline{\chi})^\beta$ because $\chi \overline{\chi} = 1$ and so $(\chi \overline{\chi})^\beta = \chi^\beta (\overline{\chi})^\beta = 1$, that is $(\overline{\chi})^\beta = (\chi^{-1})^\beta = \overline{\chi^\beta}$.

Now we will see that \mathcal{S} separates the points of βX separated by f^β , to apply the local Stone-Weierstraß Theorem 2.19. Let $x, y \in \beta X$ and $f^\beta(x) \neq f^\beta(y)$. Consider two nets $\{x_i\}_i$ and $\{y_i\}_i$ in X such that $x_i \rightarrow x$ and $y_i \rightarrow y$. Since f^β is continuous, we have $f^\beta(x) = \lim f(x_i)$ and $f^\beta(y) = \lim f(y_i)$. Along with $f^\beta(x) \neq f^\beta(y)$ this implies that there exists $\varepsilon > 0$ such that $|f(x_i) - f(y_i)| \geq \varepsilon$ for every sufficiently large i . By the hypothesis there exist $\delta > 0$ and $\chi_1, \dots, \chi_k \in H$ such that for every $u, v \in X$ if $u - v \in U_G(\chi_1, \dots, \chi_k; \delta)$ then $|f(u) - f(v)| < \varepsilon$. Assume $\chi_j^\beta(x) = \chi_j^\beta(y)$ holds true for every $j = 1, \dots, k$. Then $x_i - y_i \in U_G(\chi_1, \dots, \chi_k; \delta)$ for every sufficiently large i , this contradicts (a). So each pair of points of βX separated by f^β is also separated by \mathcal{S} . Since βX is compact, one can apply the local version of the Stone-Weierstraß Theorem 2.19 to \mathcal{S} and f^β and so f^β can be uniformly approximated by \mathcal{S} . To conclude note that if $g = \sum c_j \chi_j^\beta$ on βX then $g \upharpoonright_X = \sum c_j \chi_j$. \square

The reader familiar with uniform spaces will note that item (b) is nothing else but uniform continuity of f w.r.t. the uniformity on X induced by the uniformity of the whole group G determined by the topology \mathcal{T}_H .

Theorem 5.12 (Følner theorem). *Let G be a topological abelian group. If k is a positive integer and E is a subset of G such that k translates of E cover G , then for every neighborhood U of 0 in G there exist $\chi_1, \dots, \chi_m \in \widehat{G}$, where $m = k^2$, and $\delta > 0$ such that $U_G(\chi_1, \dots, \chi_m; \delta) \subseteq U - U + E_{(8)}$.*

Proof. By Følner's lemma 5.7 there exist $\varphi_1, \dots, \varphi_m \in G^*$ such that $U_G(\varphi_1, \dots, \varphi_m; \frac{\pi}{2}) \subseteq E_{(8)}$, where the characters φ_j can be discontinuous. Our aim will be to replace these characters by continuous ones "enlarging" $E_{(8)}$ to $U - U + E_{(8)}$.

It follows from Lemma 3.18 that $C := \overline{E_{(8)} + U} \subseteq E_{(8)} + U - U$. Consider the open set $X = U \cup (G \setminus C)$ and the function $f : X \rightarrow \mathbb{C}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in G \setminus C \end{cases}$$

Then f is continuous as $X = U \cup (G \setminus C)$ is a clopen partition of X .

Let H be the group generated by $\varphi_1, \dots, \varphi_m$. Take $x, y \in X$ with $x - y \in U_G(\varphi_1, \dots, \varphi_m; \frac{\pi}{2}) \subseteq E_{(8)}$. So if $y \in U$ then $x \in E_{(8)} + U$ and consequently $x \notin G \setminus \overline{E_{(8)} + U}$, that is $x \in U$. In the same way it can be showed that $x \in U$ yields $y \in U$. This gives $f(x) = f(y)$ by the definition of f . So by Proposition 5.11 one can uniformly approximate f on X by characters of H . Hence one can find a finite number of m -uples $\vec{j} = (j_1, \dots, j_m)$ of integers and $c_{\vec{j}} \in \mathbb{C}$ such that

$$\left| \sum_{\vec{j}} c_{\vec{j}} \varphi_1^{j_1}(x) \cdots \varphi_m^{j_m}(x) - f(x) \right| \leq \frac{1}{3} \quad (13)$$

holds for every $x \in X$. Since X is open and f is continuous, we can apply Lemma 5.10 to the convex closed set $M = \{z \in \mathbb{C} : |z| \leq \frac{1}{3}\}$ and this permits us to assume that all products $\varphi_1^{j_1} \cdots \varphi_m^{j_m}$ are continuous. Letting $x = 0$ in (13) one gets $|\sum_{\vec{j}} c_{\vec{j}}| \leq \frac{1}{3}$, and consequently,

$$\frac{2}{3} \leq \left| \sum_{\vec{j}} c_{\vec{j}} - 1 \right|. \quad (14)$$

Let now Φ be the subgroup of H consisting of all *continuous* characters of H , i.e., $\Phi = H \cap \widehat{G}$. By Theorem 2.1 there exist $\chi_1, \dots, \chi_m \in \Phi$ that generate Φ . Choose $\varepsilon > 0$ with $\varepsilon \sum_{\vec{j}} |c_{\vec{j}}| < \frac{1}{3}$. By the continuity of $\chi_1, \dots, \chi_m \in \Phi$ there exists $\delta > 0$ such that $x \in U_G(\chi_1, \dots, \chi_m; \delta)$ implies $|\varphi_1^{j_1}(x) \cdots \varphi_m^{j_m}(x) - 1| \leq \varepsilon$ for all summands $\varphi_1^{j_1} \cdots \varphi_m^{j_m}$ in (13).

To prove

$$U_G(\chi_1, \dots, \chi_m; \delta) \subseteq U - U + E_{(8)}$$

assume for a contradiction that some $z \in U_G(\chi_1, \dots, \chi_m; \delta)$ and $z \notin U - U + E_{(8)}$. Since $C = \overline{E_{(8)} + U} \subseteq E_{(8)} + U - U$, then $z \in G \setminus C \subseteq X$. Thus, by the definition of f , (13), (14) and $|\varphi_1^{j_1}(z) \cdots \varphi_m^{j_m}(z) - 1| \leq \varepsilon$, we have

$$\frac{2}{3} \leq \left| \sum_{\vec{j}} c_{\vec{j}} - 1 \right| \leq \left| \sum_{\vec{j}} c_{\vec{j}} (1 - \varphi_1^{j_1}(z) \cdots \varphi_m^{j_m}(z)) \right| + \left| \sum_{\vec{j}} c_{\vec{j}} \varphi_1^{j_1}(z) \cdots \varphi_m^{j_m}(z) - f(z) \right| \leq \varepsilon \sum_{\vec{j}} |c_{\vec{j}}| + \frac{1}{3}.$$

These inequalities together give $\frac{2}{3} \leq \varepsilon \sum_{\vec{j}} |c_{\vec{j}}| + \frac{1}{3}$. This contradicts the choice of ε . \square

6 Peter-Weyl's theorem and other applications of Følner's theorem

In this section we prove Peter-Weyl's theorem using Følner's theorem.

6.1 Precompact group topologies on abelian groups

Let us recall here that for an abelian group G and a subgroup H of G^* , the group topology \mathcal{T}_H generated by H is the coarsest group topology on G that makes every character from H continuous. We recall its description and properties in the next proposition:

Proposition 6.1. *Let G be an abelian group and let H be a group of characters of G . A base of the neighborhoods of 0 in (G, \mathcal{T}_H) is given by the sets $U(\chi_1, \dots, \chi_m; \delta)$, where $\chi_1, \dots, \chi_m \in H$ and $\delta > 0$. Moreover (G, \mathcal{T}_H) is a Hausdorff if and only if H separates the points of G .*

Now we can characterize the precompact topologies on abelian groups.

Theorem 6.2. *Let (G, τ) be an abelian group. The following conditions are equivalent:*

- (a) τ is precompact;
- (b) τ is Hausdorff on G and the neighborhoods of 0 in G are big subsets;
- (c) there exists a group H of continuous characters of G that separates the points of G and such that $\tau = \mathcal{T}_H$.

Proof. (a) \Rightarrow (b) is the definition of precompact topology.

(b) \Rightarrow (c) If $H = \widehat{(G, \tau)}$ then $\mathcal{T}_H \subseteq \tau$. Let U and V be open neighborhoods of 0 in (G, τ) such that $V_{(10)} \subseteq U$. Then V is big and by Følner's Theorem 5.12 there exist continuous characters χ_1, \dots, χ_m of G such that $U_G(\chi_1, \dots, \chi_m; \delta) \subseteq V_{(10)} \subseteq U$ for some $\delta > 0$. Thus $U \in \mathcal{T}_H$ and $\tau \subseteq \mathcal{T}_H$.

(c) \Rightarrow (a) Even if this implication is contained in Corollary 4.31, we give a direct proof here. Let $i : G \rightarrow \mathbb{S}^H$ be defined by $i(g) = i_g : H \rightarrow \mathbb{S}$ (if $g \in G$) with $i_g(\chi) = \chi(g)$ for every $\chi \in H$. Since H separates the points of G , the function i is injective. The product \mathbb{S}^H endowed with the product topology is compact and so i is a topological immersion by Proposition 6.1. The closure of $i(G)$ in \mathbb{S}^H is compact and \tilde{G} is isomorphic to it, hence \tilde{G} is compact. \square

Remark 6.3. The above theorem essentially belongs to Comfort and Ross [23]. It can be given in the following simpler “Hausdorff-free” version: τ is totally bounded iff $\tau = \mathcal{T}_H$ for some group H of continuous characters of G .

Corollary 6.4 (Peter-Weyl's theorem). *If G is a compact abelian group, then \hat{G} separates the points of G .*

Proof. Let τ be the topology of G . By Theorem 6.2 there exists a group H of continuous characters of G (i.e., $H \subseteq \hat{G}$) such that $\tau = \mathcal{T}_H$. Since $\tau \supseteq \mathcal{T}_{\hat{G}}$ and $H \subseteq \hat{G}$ we conclude that $H = \hat{G}$ separates the points of G . \square

The next theorem will allow us to sharpen this property (see Corollary 6.6).

Theorem 6.5. *Let G be an abelian group. Let \mathcal{H} be the set of all groups of characters of G separating the points of G and \mathcal{P} be the set of all precompact group topologies on G . Then the function $T : \mathcal{H} \rightarrow \mathcal{P}$ which associates to $H \in \mathcal{H}$ the topology $\mathcal{T}_H \in \mathcal{P}$ is an order preserving bijection (if $H_1, H_2 \in \mathcal{H}$ then $\mathcal{T}_{H_1} \subseteq \mathcal{T}_{H_2}$ if and only if $H_1 \subseteq H_2$).*

Proof. The equivalence (a) \Leftrightarrow (c) of Theorem 6.2 yields that $\mathcal{T}_H \in \mathcal{P}$ for every $H \in \mathcal{H}$ and that T is surjective.

Let $H \in \mathcal{H}$ and suppose that $\chi \in (\widehat{G, \mathcal{T}_H})$. To show that $\chi \in H$ let $\varepsilon > 0$. Since χ is continuous in 0, by Proposition 6.1 there exist $\chi_1, \dots, \chi_m \in H$ and $\delta > 0$ such that $|\chi(x) - 1| < \varepsilon$ for $x \in U(\chi_1, \dots, \chi_m; \delta)$. Therefore for every $x, y \in G$ with $x - y \in U(\chi_1, \dots, \chi_m; \delta)$ we get $|\chi(x - y) - 1| < \varepsilon$, that is $|\chi(x) - \chi(y)| < \varepsilon$. Apply now Proposition 5.11 to find $\chi_1, \dots, \chi_m \in H$ and $c_1, \dots, c_m \in \mathbb{C}$ such that $|\sum_{j=1}^m c_j \chi_j(x) - \chi(x)| \leq \frac{1}{2}$ for every $x \in G$. This yields $|\sum_{j=1}^m c_j \chi_j(x) \chi^{-1}(x) - 1| \leq \frac{1}{2}$.

Suppose now that $\chi \notin H$. Then each $\chi_j \chi^{-1}$ in the previous condition is non-constant. Equip G with the indiscrete topology. Then each character $\chi_j \chi^{-1}$ is discontinuous. Applying Lemma 5.10 we get the inequality $1 \leq \frac{1}{2}$, which is a contradiction. Therefore $\chi \in H$ and so $H = (\widehat{G, \mathcal{T}_H})$ for every $H \in \mathcal{H}$.

If $H_1, H_2 \in \mathcal{H}$ and $\mathcal{T}_{H_1} = \mathcal{T}_{H_2}$ then $H_1 = H_2$, so T is a bijection.

The last statement of the theorem is obvious. \square

As a corollary of Theorem 6.5 we obtain the following important fact that completes Corollary 6.4. It will be essentially used in the proof of the duality theorem.

Corollary 6.6. *If (G, τ) is a compact abelian group and $H \leq \widehat{G}$ separates the points of G , then $H = \widehat{G}$.*

Proof. By Theorem 6.2 it holds $\tau = \mathcal{T}_{\widehat{G}}$. Since $\mathcal{T}_H \subseteq \mathcal{T}_{\widehat{G}}$ by Theorem 6.5 and \mathcal{T}_H is Hausdorff, then $\mathcal{T}_H = \mathcal{T}_{\widehat{G}}$. Now again Theorem 6.5 yields $H = \widehat{G}$. \square

Definition 6.7. An abelian topological group is *elementary compact* if it is topologically isomorphic to $\mathbb{T}^s \times F$, where n is a positive integer and F is a finite abelian group.

Proposition 6.8. *Let G be a compact abelian group and let U be an open neighborhood of 0 in G . Then there exists a closed subgroup C of G such that $C \subseteq U$ and G/C is an elementary compact abelian group. In particular, G is an inverse limit of elementary compact abelian groups.*

Proof. By the Peter-Weyl Theorem 6.4 $\bigcap_{\chi \in \widehat{G}} \ker \chi = \{0\}$ and each $\ker \chi$ is a closed subgroup of G . By the compactness of G there exists a finite subset F of \widehat{G} such that $C = \bigcap_{\chi \in F} \ker \chi \subseteq U$. Define now $g = \prod_{\chi \in F} \chi : G \rightarrow \mathbb{T}^F$. Thus $\ker g = C$ and G/C is topologically isomorphic to the closed subgroup $g(G)$ of \mathbb{T}^F by the compactness of G . So G/C is elementary compact abelian by Lemma 4.47.

To prove the last statement, fix for every open neighborhood U_i of 0 in G a closed subgroup C_i of G with $C_i \subseteq U_i$ and such that G/C_i is elementary compact abelian. Note that for C_i and C_j obtained in this way the subgroup $C_i \cap C_j$ has the same property as $G/C_i \cap C_j$ is isomorphic to a closed subgroup of the product $G/C_i \times G/C_j$ which is again an elementary compact abelian group. Enlarging the family (C_i) with all finite intersections we obtain an inverse system of elementary compact abelian groups G/C_i where the connecting homomorphisms $G/C_i \rightarrow G/C_j$, when $C_i \leq C_j$, are simply the induced homomorphisms. Then the inverse limit G' of this inverse system is a compact abelian group together with a continuous homomorphism $f : G \rightarrow G'$ induced by the projections $p_i : G \rightarrow G/C_i$. Assume $x \in G$ is non-zero. Pick an open neighborhood U of 0. By the first part of the proof, there exists $C_i \subseteq U$, hence $x \notin C_i$. Therefore, $p_i(x) \neq 0$, so $f(x) \neq 0$ as well. This proves that f is injective. To check surjectivity of f take an element $x' = (x_i + C_i)$ of the inverse limit G' . Then the family of closed cosets $x_i + C_i$ in G has the finite intersection property, so has a non-empty intersection. For every element x of that intersection one has $f(x) = x'$. Finally, the continuous isomorphism $f : G \rightarrow G'$ must be open by the compactness of G . \square

For a topological abelian group G we say that G has *no small subgroups*, or shortly, G is *NSS*, if there exists a neighborhood U of 0 such that U contain no non-trivial subgroups of G . It follows immediately from the above proposition that the compact abelian group G has no small subgroups precisely when G is an elementary compact abelian group.

6.2 Precompact group topologies determined by sequences

Large and lacunary sets (mainly in \mathbb{Z} or elsewhere) are largely studied in number theory, harmonic analysis and dynamical systems ([43], [24], [80], [52], [53], [54], [55], [59]).

Let us consider a specific problem. For a strictly increasing sequence $\underline{u} = (u_n)_{n \geq 1}$ of integers, the interest in the distribution of the multiples $\{u_n \alpha : n \in \mathbb{N}\}$ of a non-torsion element α of the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has roots in number theory (Weyl's theorem of uniform distribution modulo 1) and in ergodic theory (Sturmian sequences and Hartman sets [99]). According to Weyl's theorem, the set $\{u_n \alpha : n \in \mathbb{N}\}$ will be uniformly dense in \mathbb{T} for almost all $\alpha \in \mathbb{T}$. One can consider the subset $t_{\underline{u}}(\mathbb{T})$ of all elements $\alpha \in \mathbb{T}$ such that $\lim_n u_n \alpha = 0$ in \mathbb{T} . Clearly

it will have measure zero. Moreover, it is a subgroup of \mathbb{T} as well as a Borel set, so it is either countable or has size \mathfrak{c} . It was observed by Armacost [3] that when $u_n = p^n$ for all n and some prime p , then $t_{\underline{u}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$. He posed the question of describing the subgroup $t_{\underline{u}}(\mathbb{T})$ for the sequence $u_n = n!$, this was done by Borel [19] (see also [36] and [31] for the more general problem concerning sequences \underline{u} with $u_{n-1} | u_n$ for every n).

Another motivation for the study of the subgroups of the form $t_{\underline{u}}(\mathbb{T})$ come from the fact that they lead to the description of precompact group topologies on \mathbb{Z} that make the sequence u_n converge to 0 in \mathbb{Z} (see the comment after proposition 6.9). Let us start by an easy to prove general fact:

Proposition 6.9. [7] *A sequence $A = \{a_n\}_n$ in a precompact abelian group G converges to 0 in G iff $\chi(a_n) \rightarrow 0$ in \mathbb{T} for every continuous character of G .*

In the case of $G = \mathbb{Z}$ the characters of G are simply elements of \mathbb{T} , i.e., a precompact group topology on \mathbb{Z} has the form \mathcal{T}_H for some subgroup H of \mathbb{T} . Thus the above proposition for $G = \mathbb{Z}$ can be reformulated as: a sequence $A = \{a_n\}_n$ in $(\mathbb{Z}, \mathcal{T}_H)$ converges to 0 iff $a_n x \rightarrow 0$ for every $x \in H$, i.e., simply $H \subseteq t_{\underline{a}}(\mathbb{T})$.

Now we can discuss a counterpart of the notion of T -sequences (introduced in §3.5), defined with respect to topologies induced by characters, i.e., precompact topologies.

Definition 6.10. [7, 9] A sequence $A = \{a_n\}_n$ in an abelian group G is called a *TB-sequence* if there exists a precompact group topology on G such that $a_n \rightarrow 0$.

Clearly, every TB-sequence is a T -sequence (see Example 6.12 for a T -sequence in \mathbb{Z} that is not a TB-sequence). The advantage of TB-sequences over the T -sequences is in the easier way of determining sufficient condition for a sequence to be a TB-sequence [7, 9]. For example, a sequence (a_n) in \mathbb{Z} is a TB-sequence iff the subgroup $t_{\underline{a}}(\mathbb{T})$ of \mathbb{T} is infinite.

Egglestone [42] proved that the asymptotic behavior of the sequence of ratios $q_n = \frac{u_{n+1}}{u_n}$ may have an impact on the size of the subgroup $t_{\underline{u}}(\mathbb{T})$ in the following remarkable dichotomy:

Theorem 6.11. *Let (a_n) be a sequence in \mathbb{Z} .*

- *If $\lim_n \frac{a_{n+1}}{a_n} = +\infty$, then (a_n) is a TB-sequence and $|t_{\underline{a}}(\mathbb{T})| = \mathfrak{c}$.*
- *If $\frac{a_{n+1}}{a_n}$ is bounded, then $t_{\underline{a}}(\mathbb{T})$ is countable.*

Example 6.12. [9] *There exists a TB-sequence (a_n) in \mathbb{Z} with $\lim_n \frac{a_{n+1}}{a_n} = 1$.*

Here is an example of a T -sequence in \mathbb{Z} that is not a TB-sequence.

Example 6.13. For every TB-sequence $A = \{a_n\}$ in \mathbb{Z} such that $t_{\underline{a}}(\mathbb{T})$ is countable, there exists a sequence $\{c_n\}$ in \mathbb{Z} such that the sequence q_n defined by $q_{2n} = c_n$ and $q_{2n-1} = a_n$, is a T -sequence, but not a TB-sequence.

Proof. Let $\{z_1, \dots, z_n, \dots\}$ be an enumeration of $t_{\underline{a}}(\mathbb{T})$.

According to Lemma 3.51 there exists a sequence b_n in \mathbb{Z} such that for every choice of the sequence (e_n) , where $e_n \in \{0, 1\}$, the sequence q_n defined by $q_{2n} = b_n + e_n$ and $q_{2n-1} = a_n$, is a T -sequence. Now we define the sequence q_n with $q_{2m-1} = a_m$ and $q_{2m} = b_m$ when m is not a prime power. Let p_1, \dots, p_n, \dots be all prime numbers enumerated one-to-one. Now fix k and define $e_k \in \{0, 1\}$ depending on $\lim_n b_{p_k^n} z_k$ as follows:

- if $\lim_n b_{p_k^n} z_k = 0$, let $e_k = 1$,
- if $\lim_n b_{p_k^n} z_k \neq 0$ (in particular, if the limit does not exist) let $e_k = 0$.

Now let $q_{2p_k^n} = b_{p_k^n} + e_k$ for $n \in \mathbb{N}$. Hence for every $k \in \mathbb{N}$

$$\lim_n q_{2p_k^n} z_k = 0 \implies e_k = 1. \quad (*)$$

To see that (q_n) is not a TB-sequence assume that $\chi : \mathbb{Z} \rightarrow \mathbb{T}$ a character such that $\chi(q_n) \rightarrow 0$ in \mathbb{T} . Then $x = \chi(1) \in \mathbb{T}$ satisfies $q_n x \rightarrow 0$, so $x \in t_{\underline{q}}(\mathbb{T}) \subseteq t_{\underline{a}}(\mathbb{T})$. So there exists $k \in \mathbb{N}$ with $x = z_k$. By (*) $e_k = 1$. Hence $q_{2p_k^n} = b_{p_k^n} + 1$ and $\lim_n b_{p_k^n} z_k = 0$, so $x \in t_{\underline{q}}(\mathbb{T})$ yields $0 = \lim_n q_{2p_k^n} x = 0 + x$, i.e., $x = 0$. This proves that every character $\chi : \mathbb{Z} \rightarrow \mathbb{T}$ such that $\chi(q_n) \rightarrow 0$ in \mathbb{T} is trivial. In particular, (q_n) not a TB-sequence. \square

Let us note that the above proof gives much more. Since $q_n \rightarrow 0$ in $\tau_{(q_n)}$, it shows that every $\tau_{(q_n)}$ -continuous character of \mathbb{Z} is trivial, i.e., $(\widehat{\mathbb{Z}}, \tau_{(q_n)}) = 0$.

The information accumulated on the properties of the subgroups $t_{\underline{u}}(\mathbb{T})$ of \mathbb{T} motivated the problem of describing those subgroups H of \mathbb{T} that can be characterized as $H = t_{\underline{u}}(\mathbb{T})$ for some sequence \underline{u} . As already

mentioned, such an H can be only countable or can have size \mathfrak{c} being of measure zero. A measure zero subgroup H of \mathbb{T} of size \mathfrak{c} that is not even contained in any proper subgroup of \mathbb{T} of the form $t_{\underline{u}}(\mathbb{T})$ was built in [7] (under the assumption of Martin Axiom) and in later in [61, 62] (in ZFC). Much earlier Borel [19] had already resolved in the positive the remaining part of the problem showing that every countable subgroup of \mathbb{T} can be characterized (in the above sense). Unaware of his result, Larcher [74], and later Kraaikamp and Liardet [71], proved that some cyclic subgroups of \mathbb{T} are characterizable (see also [16, 15, 12, 14, 13] for related results). The paper [9] describes the algebraic structure of the subgroup $t_{\underline{u}}(\mathbb{T})$ when the sequence $\underline{u} := (u_n)$ verifies a linear recurrence relation of order $\leq k$,

$$u_n = a_n^{(1)}u_{n-1} + a_n^{(2)}u_{n-2} + \dots + a_n^{(k)}u_{n-k}$$

for every $n > k$ with $a_n^{(i)} \in \mathbb{Z}$ for $i = 1, \dots, k$.

Three proofs of Borel's theorem of characterizability of the countable subgroups of \mathbb{T} were given in [13]. These author mentioned that the theorem can be extended to compact abelian groups in place of \mathbb{T} , without giving any precise formulation. There is a natural way to extend the definition of $t_{\underline{u}}(\mathbb{T})$ to an arbitrary topological abelian group G by letting $t_{\underline{u}}(G) = \{x \in G : \lim_n u_n x = 0 \text{ in } G\}$. Actually, for the sequence $u_n = p^n$ (resp., $u_n = n!$) an element x satisfying $\lim_n u_n x = 0$ has been called *topologically p -torsion* (resp., *topologically torsion*) by Braconnier and Vilenkin in the forties of the last century and these notions played a prominent role in the development of the theory of locally compact abelian groups. One can easily reduce the computation of $t_{\underline{u}}(G)$ for an arbitrary locally compact abelian group to that of $t_{\underline{u}}(\mathbb{T})$ [26]. Independently on their relevance in other questions, the subgroups $t_{\underline{u}}(G)$ turned out to be of no help in the characterization of countable subgroups of the compact abelian groups. Indeed, a much weaker condition, turned out to characterize the circle group \mathbb{T} in the class of all locally compact abelian groups:

Theorem 6.14. [31] *In a locally compact abelian group G every cyclic subgroup of the group G is an intersection of subgroups of the form $t_{\underline{u}}(G)$ iff $G \cong \mathbb{T}$.*

Actually, one can remove the “abelian” restraint in the theorem remembering that in the non-abelian case $t_{\underline{u}}(G)$ is just a subset of G , not a subgroup in general [31].

The above theorem suggested to use in [35] a different approach to the problem, replacing the sequence of integers u_n (characters of $\mathbb{T}!$) by a sequence u_n in the Pontryagin-van Kampen dual \widehat{G} . Then the subgroup $s_{\underline{u}}(G) = \{x \in G : \lim_n u_n(x) = 0 \text{ in } \mathbb{T}\}$ of G really can be used for such a characterization of all countable subgroups of the compact metrizable groups (see [35, 33, 17] for major detail).

6.3 On the structure of compactly generated locally compact abelian groups

From now on all groups are Hausdorff; quotients are taken for closed subgroups and so they are still Hausdorff.

An abelian topological group is *elementary locally compact* if it is topologically isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^s \times F$, where n, m, s are positive integers and F is a finite abelian group. Observe that the class of elementary locally compact abelian groups is closed under taking quotient, closed subgroups and finite products (see Theorem 4.37 and Corollary 4.47).

Lemma 6.15. *Let G be a locally compact monothetic group. Then G is either compact or is discrete.*

Proof. If G is finite, then G is both compact and discrete. So we can suppose without loss of generality that $\langle x \rangle \cong \mathbb{Z}$ is infinite and so also that \mathbb{Z} is a subgroup of G .

If G induces the discrete topology on \mathbb{Z} , then \mathbb{Z} is closed and so $G = \mathbb{Z}$ is discrete.

Suppose now that G induces on \mathbb{Z} a non-discrete topology. Our aim is to show that it is totally bounded. Then the density of \mathbb{Z} in G yields that $G = \widetilde{\mathbb{Z}} = \overline{\mathbb{Z}}$ is compact, as G is locally compact and so complete (see Lemma 4.7).

Every open subset of G has no maximal element. Indeed, if U is an open subset of \mathbb{Z} that contains 0 and it has a maximal element, then $-U$ is an open subset of \mathbb{Z} that contains 0 and it has a minimal element and $U \cap -U$ is an open finite neighborhood of 0 in \mathbb{Z} ; thus \mathbb{Z} is discrete against the assumption. Consequently every open subset of \mathbb{Z} contains positive elements.

Let U be a compact neighborhood of 0 in G and V a symmetric neighborhood of 0 in G such that $V + V \subseteq U$. There exist $g_1, \dots, g_m \in G$ such that $U \subseteq \bigcup_{i=1}^m (g_i + V)$. Let $n_1, \dots, n_m \in \mathbb{Z}$ be positive integers such that $n_i \in g_i + V$ for every $i = 1, \dots, m$. Equivalently $g_i \in n_i - V = n_i + V$. Thus

$$U \subseteq \bigcup_{i=1}^m (g_i + V) \subseteq \bigcup_{i=1}^m (n_i + V + V) \subseteq \bigcup_{i=1}^m (n_i + U)$$

implies

$$U \cap \mathbb{Z} \subseteq \bigcup_{i=1}^m (n_i + U \cap \mathbb{Z}). \quad (1)$$

We show that $U \cap \mathbb{Z}$ is big with respect to \mathbb{Z} . Let $t \in \mathbb{Z}$; since $U \cap \mathbb{Z}$ has no maximal element, then there exists $s \in U \cap \mathbb{Z}$ such that $s \geq t$. Define $s_t = \min\{s \in U \cap \mathbb{Z} : s \geq t\}$. By (1) $s_t = n_i + u_t$ for some $i \leq m$ and $u_t \in U \cap \mathbb{Z}$. Since $n_i > 0$, then $u_t < s_t$ and so $u_t < t \leq s_t$. Now put $N = \max\{n_1, \dots, n_m\}$ and $F = \{1, \dots, N\}$. Hence $U \cap \mathbb{Z} + F = \mathbb{Z}$. This proves that the topology induced on \mathbb{Z} by G is totally bounded. \square

Corollary 6.16. *Let G be a locally compact abelian group and $x \in G$. Then $\overline{\langle x \rangle}$ is either compact or discrete.*

Proposition 6.17. *Let G be a compactly generated locally compact abelian group. Then there exists a discrete subgroup H of G such that $H \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and G/H is compact.*

Proof. Suppose first that there exist $g_1, \dots, g_m \in G$ such that $G = \overline{\langle g_1, \dots, g_m \rangle}$. We proceed by induction. For $m = 1$ apply Lemma 6.15: if G is infinite and discrete take $H = G$ and if G is compact $H = \{0\}$. Suppose now that the property holds for $m \geq 1$ and $G = \overline{\langle g_1, \dots, g_{m+1} \rangle}$. If every $\overline{\langle g_i \rangle}$ is compact, then so is G and $H = \{0\}$. If $\langle g_{m+1} \rangle$ is discrete, consider the canonical projection $\pi : G \rightarrow G_1 = G/\langle g_{m+1} \rangle$. Since G_1 has a dense subgroup generated by m elements, by the inductive hypothesis there exists a discrete subgroup H_1 of G_1 such that $H_1 \cong \mathbb{Z}^n$ and G_1/H_1 is compact. Therefore $H = \pi^{-1}(H_1)$ is a closed countable subgroup of G . Thus H is locally compact and countable, hence discrete by Lemma 4.8.

Since H is finitely generated, it is isomorphic to $H_2 \times F$, where $H_2 \cong \mathbb{Z}^s$ for some $s \in \mathbb{N}$ and F is a finite abelian group (see Theorem 2.1). Now G/H is isomorphic to G_1/H_1 and H/H_2 is finite, so G/H_2 is compact thanks to Lemma 4.5.

Now consider the general case. There exists a compact subset K of G that generates G . By Lemma 4.14 we can assume wlog that $K = \overline{U}$, where U is a symmetric neighborhood of 0 in G with compact closure. We show now that there exists a finite subset F of G such that

$$K + K \subseteq K + \langle F \rangle. \quad (2)$$

In fact, pick a symmetric neighborhood V of 0 in G such that $V + V \subseteq U$. For the compact set K satisfying $K \subseteq \bigcup_{x \in K} (x + V)$ there exists a finite subset F of K such that $K \subseteq \bigcup_{x \in F} (x + V) = F + V$. Then

$$K + K \subseteq F + F + V + V \subseteq \langle F \rangle + U \subseteq \langle F \rangle + K.$$

gives (2). An easy inductive argument shows that $\langle K \rangle = G$ and (2) imply $G = \langle K \rangle \subseteq K + \langle F \rangle$.

Let $G_1 = \langle F \rangle$. By $G = \langle F \rangle + K$ the quotient $\pi(K) = G/G_1$ is compact. By the first part of the proof there exists a discrete subgroup H of the locally compact subgroup G_1 of G , such that $H \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and G_1/H is compact. Since G_1/H is a compact subgroup of G/H such that $(G/H)/(G_1/H) \cong G/G_1$ is compact, we conclude that also G/H is compact. \square

Proposition 6.18. *Let G be a compactly generated locally compact abelian group. Then there exists a compact subgroup K of G such that G/K is elementary locally compact abelian.*

Proof. By Proposition 6.17 there exists a discrete subgroup H of G such that the quotient G/H is compact. Consider the canonical projection π of G onto G/H . Let U be a compact symmetric neighborhood of 0 in G such that $(U + U + U) \cap H = \{0\}$. So $\pi(U)$ is a neighborhood of 0 in G/H and applying Lemma 6.8 we find a closed subgroup $L \supseteq H$ of G such that the closed subgroup L/H of G/H satisfies

$$L/H \subseteq \pi(U) \text{ and } (G/H)/(L/H) = G/L \cong \mathbb{T}^t \times F, \quad (4)$$

where F is a finite abelian group and $t \in \mathbb{N}$, i.e., G/L is elementary compact abelian.

The set $K = L \cap U$ is compact being closed in the compact neighborhood U . Let us see now that K is a subgroup of G . To this end take $x, y \in K$. Then $x - y \in L$ and $\pi(x - y) \in C \subseteq \pi(U)$. Thus $\pi(x - y) = \pi(u)$ for some $u \in U$. As $\pi(x - y - u) = 0$ in G/H , one has $x - y - u \in (U + U + U) \cap H = \{0\}$. Hence $x - y = u \in L \cap U = K$.

Now take $x \in L$; consequently $\pi(x) \in C \subseteq \pi(U)$ so $\pi(x) = \pi(u)$ for some $u \in U$. Clearly, $u \in L \cap U = K$, hence $\pi(L) = \pi(K)$. Thus $L = K + H$ and $K \cap H = \{0\}$ yields that the canonical projection $l : G \rightarrow G/K$ restricted to H is a continuous isomorphism of H onto $l(H) = l(L)$. Let us see now that $l(H)$ is discrete. The compact set K is contained in the open set $W_1 = G \setminus (H \setminus \{0\}) = G \setminus H \cup \{0\}$ (H is discrete). By Lemma 4.3 (c) there exists an open neighborhood V of 0 in G such that $K + V \subseteq W_1$. This implies that $(K + V) \cap H = \{0\}$ and so $(K + V) \cap (K + H) = K$, that gives $l(V) \cap l(H) = \{0\}$ in G/K . Thus

$$l(L) = l(H) \cong H \cong \mathbb{Z}^s$$

is discrete in G/K .

Observe that (4) yields the following isomorphisms:

$$(G/K)/l(L) = (G/K)/(L/K) \cong G/L \cong \mathbb{T}^t \times F.$$

Denote by ϱ the composition $G/K \rightarrow G/L \rightarrow \mathbb{T}^t \times F$. Let W be a compact neighborhood of 0 in G/K such that $W + W \subseteq l(V)$ and $\varrho(W) \subseteq \mathbb{T}^t \times \{0\}$. Then $\varrho|_W$ is injective because $l(V) \cap l(L) = \{0\}$. In particular, ϱ is a local homeomorphism.

Consider now the canonical projection $q : \mathbb{R}^t \rightarrow \mathbb{T}^t$. Our aim is to lift it to a continuous homomorphism $f : \mathbb{R}^t \rightarrow G/K$ such that $\varrho \circ f = q$. The existence of such a lifting is immediate from the facts that both q and ϱ are covering homomorphisms and \mathbb{R}^t simply connected. In particular, $D = f(\mathbb{R}^t)$ is an open subgroup of G/K as has a non-empty interior (as q and ϱ are local homeomorphisms). Since \mathbb{R}^t is divisible, by Lemma 2.8 $G/K = D \times B$ where B is a discrete subgroup of G/K because $D \cap B = \{0\}$ and D is open. Moreover B is compactly generated as it is a quotient of G . Since it is also discrete, B is finitely generated. Then $f : \mathbb{R}^t \rightarrow D$ is open by Theorem 4.9 and so D is isomorphic to a quotient of \mathbb{R}^t , which is elementary locally compact abelian.

For the reader who is not familiar with covering maps we provide now a self-contained proof.

Take an open neighborhood U of 0 in \mathbb{R}^t such that $U - U \cap \ker q = \{0\}$ and let $U_0 = U \cap q^{-1}(\varrho(W))$. Then U_0 is an open neighborhood of 0 in \mathbb{R}^t such that $q|_{U_0}$ is one-to-one from U_0 to $\varrho(W)$. Pick a symmetric neighborhood U_1 of 0 in \mathbb{R}^t such that $U_1 + U_1 \subseteq U_0$. Define a map $f : \mathbb{R}^t \rightarrow G/K$ as follows: $f|_{U_0}$ is simply the composition $\varrho^{-1} \circ q$. So f maps U_0 onto the open subset $\varrho^{-1}(q(U_0))$ of G/K . If $x \in \mathbb{R}^t$ there exists $n \in \mathbb{N}_+$ such that $\frac{1}{n}x \in U_0$. We put $f(x) = nf(\frac{1}{n}x)$ and we note that this definition does not depend on n . Moreover, $f(x_1 + x_2) = f(x_1) + f(x_2)$ for every $x_1, x_2 \in U_1$.

We can prove now that f is a homomorphism. First of all we note that for every $x \in \mathbb{R}^t$ $f|_{\langle x \rangle}$ is a homomorphism, i.e., $f(kx) = kf(x)$ for every $k \in \mathbb{Z}$. Now take $x, y \in \mathbb{R}^t$. There exists an integer $n > 0$ such that $\frac{1}{n}x, \frac{1}{n}y \in U_1$ and so $\frac{1}{n}x + \frac{1}{n}y \in U_0$. By the previous step

$$f(x + y) = nf\left(\frac{1}{n}(x + y)\right) = nf\left(\frac{1}{n}x + \frac{1}{n}y\right) = nf\left(\frac{1}{n}x\right) + nf\left(\frac{1}{n}y\right) = f(x) + f(y),$$

for all $x, y \in \mathbb{R}^t$.

So f is continuous and also a local homeomorphism on \mathbb{R}^t because it is the composition of local homeomorphisms: restricted to the open subset U_0 , f is the composition of q and ϱ^{-1} (note that both $\varrho|_W$ and $q|_{U_0}$ are continuous and open). \square

To prove the Pontryagin-van Kampen duality theorem in the general case (for $G \in \mathcal{L}$), we need Theorem 6.19, which generalizes the Peter-Weyl Theorem 6.4.

Theorem 6.19. *If G is a locally compact abelian group, then \widehat{G} separates the points of G .*

Proof. Let V be a compact neighborhood of 0 in G . Take $x \in G \setminus \{0\}$. Then $G_1 = \langle V \cup \{x\} \rangle$ is an open (it has non-void interior) compactly generated subgroup of G . In particular G_1 is locally compact. By Proposition 6.17 there exists a discrete subgroup H of G_1 such that $H \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}$ and G_1/H is compact. Thus $\bigcap_{n \in \mathbb{N}_+} nH = \{0\}$ and so there exists $n \in \mathbb{N}_+$ such that $x \notin nH$. Since H/nH is finite, the quotient $G_2 = G_1/nH$ is compact by Lemma 4.5. Consider the canonical projection $\pi : G_1 \rightarrow G_2$ and note that $\pi(x) = y \neq 0$ in G_2 . By the Peter-Weyl Theorem 6.4 there exists $\xi \in \widehat{G}$ such that $\xi(y) \neq 0$. Consequently $\chi = \xi \circ \pi \in \widehat{G_1}$ and $\chi(x) \neq 0$. By Theorem 2.5 there exists $\bar{\chi} \in \widehat{G}$ such that $\bar{\chi}|_{G_1} = \chi$. \square

It follows from Theorem 6.19 and Remark 7.24 that ω_G is a continuous monomorphism for every locally compact abelian group G .

Corollary 6.20. *Let G be a locally compact abelian group and K a compact subgroup of G . Then for every $\chi \in \widehat{K}$ there exists $\xi \in \widehat{G}$ such that $\xi|_K = \chi$.*

Proof. Define $H = \{\chi \in \widehat{K} : \text{there exists } \xi \in \widehat{G} \text{ with } \xi|_K = \chi\}$. By Theorem 6.19 the continuous characters of G separate the points of G . Therefore H separate the points of K . Now apply Corollary 6.6 to conclude that $H = \widehat{K}$. \square

Here is another corollary of Theorem 6.19:

Corollary 6.21. *A σ -compact and locally compact abelian group is totally disconnected iff for every continuous character χ of G the image $\chi(G)$ is a proper subgroup of \mathbb{T} .*

Proof. Assume that G is a locally compact abelian group such that $\chi(G)$ is a proper subgroup of \mathbb{T} for every continuous character χ of G . According to Theorem 6.19 the diagonal homomorphism $f : G \rightarrow \prod \{\chi(G) : \chi \in \widehat{G}\}$ of all $\chi \in \widehat{G}$ is injective. Since the proper subgroups of \mathbb{T} are totally disconnected, the whole product will be totally disconnected, so also G will be totally disconnected. Now assume that G is σ -compact, locally compact and totally disconnected. Consider $\chi \in \widehat{G}$ and assume for a contradiction that $\chi(G) = \mathbb{T}$. Then $\chi : G \rightarrow \mathbb{T}$ will be an open map by the open mapping theorem, so \mathbb{T} will be a quotient of G . As total disconnectedness is inherited by quotients of locally compact groups (see Corollary 4.20), we conclude that \mathbb{T} must be totally disconnected, a contradiction. \square

Corollary 6.22. *A compact abelian group is totally disconnected iff every continuous character of G is torsion.*

Proof. For a compact abelian group G the image $\chi(G)$ under a continuous character χ of G is a compact, hence closed subgroup of \mathbb{T} . Hence $\chi(G)$ is a proper subgroup of \mathbb{T} precisely when it is finite. This means that the character χ is torsion. \square

Compactness plays an essential role here. We shall see examples of totally disconnected σ -compact and locally compact abelian groups G such that no continuous character of G is torsion (e.g., $G = \mathbb{Q}_p$).

7 Pontryagin-van Kampen duality

7.1 The dual group

In the sequel we shall write the circle additively as $(\mathbb{T}, +)$ and we denote by $q_0 : \mathbb{R} \rightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z}$ the canonical projection. For every $k \in \mathbb{N}_+$ let $\Lambda_k = q_0((-\frac{1}{3k}, \frac{1}{3k}))$. Then $\{\Lambda_k : k \in \mathbb{N}_+\}$ is a base of the neighborhoods of 0 in \mathbb{T} , because $\{(-\frac{1}{3k}, \frac{1}{3k}) : k \in \mathbb{N}_+\}$ is a base of the neighborhoods of 0 in \mathbb{R} .

For every abelian group $G^* = \text{Hom}(G, \mathbb{T})$. For a subset K of G and a subset U of \mathbb{T} let

$$W_{G^*}(K, U) = \{\chi \in G^* : \chi(K) \subseteq U\}.$$

For any subgroup H of G^* we abbreviate $H \cap W(K, U)$ to $W_H(K, U)$. When there is no danger of confusion we shall write only $W(K, U)$ in place of $W_{G^*}(K, U)$. The group G^* will be considered only with one topology, namely the induced from \mathbb{T}^G compact topology (see Remark 4.1).

If G is a topological abelian group, \widehat{G} will denote the subgroup of G^* consisting of continuous characters.

The group \widehat{G} will carry the *compact open topology* that has as basic neighborhoods of 0 the sets $W_{\widehat{G}}(K, U)$, where K is a compact subset of G and U is neighborhood of 0 in \mathbb{T} . We shall see below that when $U \subseteq \Lambda_1$, then $W_{\widehat{G}}(K, U)$ coincides with $W_{G^*}(K, U)$ in case K is a neighborhood of 0 in G . Therefore we shall use mainly the notation $W(K, U)$ when the group G is clear from the context.

Let us start with an easy example.

Example 7.1. Let G be an abelian topological group.

- (1) If G is compact, then \widehat{G} is discrete.
- (2) If G is discrete, then \widehat{G} is compact.

Indeed, to prove (1) it is sufficient to note that $W_{\widehat{G}}(G, \Lambda_1) = \{0\}$ as Λ_1 contains no subgroup of \mathbb{T} beyond 0.

(2) Suppose that G is discrete. Then $\widehat{G} = \text{Hom}(G, \mathbb{T})$ is a subgroup of the compact group \mathbb{T}^G . The compact-open topology of \widehat{G} coincides with the topology inherited from \mathbb{T}^G : let F be a finite subset of G and U an open neighborhood of 0 in \mathbb{T} , then

$$\begin{aligned} \bigcap_{x \in F} \pi_x^{-1}(U) \cap \text{Hom}(G, \mathbb{T}) &= \{\chi \in \text{Hom}(G, \mathbb{T}) : \pi_x \in U \text{ for every } x \in F\} \\ &= \{\chi \in \text{Hom}(G, \mathbb{T}) : \chi(x) \in U \text{ for every } x \in F\} = W(F, U). \end{aligned}$$

Moreover $\text{Hom}(G, \mathbb{T})$ is closed in the compact product \mathbb{T}^G by Remark 4.1 and we can conclude that \widehat{G} is compact.

Now we prove that the dual group is always a topological group. If the group G is locally compact, then its dual is locally compact too. This is the first step of the Pontryagin-van Kampen duality theorem.

Theorem 7.2. *For an abelian topological group G the following assertions hold true:*

- (a) *if $x \in \mathbb{T}$ and $k \in \mathbb{N}_+$, then $x \in \Lambda_k$ if and only if $x, 2x, \dots, kx \in \Lambda_1$;*
- (b) *$\chi \in \text{Hom}(G, \mathbb{T})$ is continuous if and only if $\chi^{-1}(\Lambda_1)$ is a neighborhood of 0 in G ;*
- (c) *$\{W_{\widehat{G}}(K, \Lambda_1) : K \text{ compact} \subseteq G\}$ is a base of the neighborhoods of 0 in \widehat{G} , in particular \widehat{G} is a topological group.*
- (d) *$W_{\widehat{G}}(A, \Lambda_s) + W_{\widehat{G}}(A, \Lambda_s) \subseteq W_{\widehat{G}}(A, \Lambda_{s-1})$ and $W_{\widehat{G}}(\overline{A}, \Lambda_s) + W_{\widehat{G}}(\overline{A}, \Lambda_s) \subseteq W_{\widehat{G}}(\overline{A}, \Lambda_{s-1})$ for every $A \subseteq G$ and $s > 1$.*
- (e) *if F is a closed subset of \mathbb{T} , then for every $K \subseteq G$ the subset $W_{G^*}(K, F)$ of G^* is closed (hence, compact);*
- (f) *if U is neighborhood of 0 in G , then*
 - (f₁) *$W_{\widehat{G}}(\overline{U}, V) = W_{G^*}(\overline{U}, V)$ for every neighborhood of 0 $V \subseteq \Lambda_1$ in \mathbb{T} ;*
 - (f₂) *$W(\overline{U}, \Lambda_4)$ has compact closure;*
 - (f₃) *if U has compact closure, then $W(\overline{U}, \Lambda_4)$ is a neighborhood of 0 in \widehat{G} with compact closure, so \widehat{G} is locally compact.*

Proof. (a) Note that for $s \in \mathbb{N}$, $sx \in \Lambda_1$ if and only if $x \in A_{s,t} = \Lambda_s + \pi_{\mathbb{T}}(\frac{t}{s})$ for some integer t with $0 \leq t \leq s$. On the other hand, $A_{s,0} = \Lambda_s$ and $\Lambda_s \cap A_{s+1,t}$ is non-empty if and only if $t = 0$. Hence, if $x \in \Lambda_s$ and $(s+1)x \in \Lambda_1$, then $x \in \Lambda_{s+1}$ and this holds in particular for $1 \leq s < k$. This proves that $sx \in \Lambda_1$ for $s = 1, \dots, k$ if and only if $x \in \Lambda_k$.

(b) Suppose that $\chi^{-1}(\Lambda_1)$ is a neighborhood of 0 in G . So there exists an open neighborhood U of 0 in G such that $U \subseteq \chi^{-1}(\Lambda_1)$. Moreover, there exists an other neighborhood V of 0 in G with $\underbrace{V + \dots + V}_k \subseteq U$ where

$k \in \mathbb{N}_+$. Now $s\chi(y) \in \Lambda_1$ for every $y \in V$ and $s = 1, \dots, k$. By item (a) $\chi(y) \in \Lambda_k$ and so $\chi(V) \subseteq \Lambda_k$.

(c) Let $k \in \mathbb{N}_+$ and K be a compact subset of G . Define $L = \underbrace{K + \dots + K}_k$, which is a compact subset of

G because it is a continuous image of the compact subset K^k of G^k . Take $\chi \in W(L, \Lambda_1)$. For every $x \in K$ we have $s\chi(x) \in \Lambda_1$ for $s = 1, \dots, k$ and so $\chi(x) \in \Lambda_k$ by item (a). Hence $W(L, \Lambda_1) \subseteq W(K, \Lambda_k)$.

(d) obvious.

(e) If $\pi_x : \mathbb{T}^G \rightarrow \mathbb{T}$ is the projection defined by the evaluation at x , for $x \in G$, then obviously

$$W_{G^*}(K, F) = \bigcap_{x \in K} \{\chi \in G^* : \chi(x) \in F\} = \bigcap_{x \in K} \pi_x^{-1}(F)$$

is closed as each $\pi_x^{-1}(F)$ is closed in G^* .

(f₁) follows immediately from item (c).

(f₂) To prove that the closure of $W_0 = W(\overline{U}, \Lambda_4)$ is compact it is sufficient to note that $W_0 \subseteq W_1 := W(\overline{U}, \overline{\Lambda_4})$ and prove that W_1 is compact. Let τ_s denote the subspace topology of W_1 in \widehat{G} . We prove in the sequel that (W_1, τ_s) is compact.

Consider on the set W_1 also the weaker topology τ induced from G^* and consequently from \mathbb{T}^G . By (e) (W_1, τ) is compact.

It remains to show that both topologies τ_s and τ of W_1 coincide. Since τ_s is finer than τ , it suffices to show that if $\alpha \in W_1$ and K is a compact subset of G , then $(\alpha + W(K, \Lambda_1)) \cap W_1$ is also a neighborhood of α in (W_1, τ) .

Since $\bigcup\{a + U : a \in K\} \supseteq K$ and K is compact, $K \subseteq F + U$, where F is a finite subset of K . We prove now that

$$(\alpha + W(F, \Lambda_2)) \cap W_1 \subseteq (\alpha + W(K, \Lambda_1)) \cap W_1. \quad (*)$$

Let $\xi \in W(F, \Lambda_2)$, so that $\alpha + \xi' \in W_1 = W(\overline{U}, \overline{\Lambda_4})$. As $\alpha \in W_1$ as well, we deduce from items (c) and (d) that $\xi = (\alpha + \xi') - \alpha \in W_1 - W_1$. Hence $\xi(\overline{U}) \subseteq \overline{\Lambda_2}$ and consequently

$$\xi(K) \subseteq \xi(F + U) \subseteq \Lambda_2 + \overline{\Lambda_2} \subseteq \Lambda_1.$$

This proves $\xi \in W(K, \Lambda_1)$ and (*).

(f₃) Follows obviously from (f₂) and the definition of the compact open topology. \square

The above proof shows another relevant fact. The neighborhood $W(\overline{U}, \Lambda_4)$ of 0 in the dual group \widehat{G} carries the same topology in \widehat{G} and G^* , nevertheless the inclusion map $j : \widehat{G} \hookrightarrow G^*$ need not be an embedding:

Corollary 7.3. *For a locally compact abelian group G the following are equivalent:*

- (a) *the inclusion map $j : \widehat{G} \hookrightarrow G^*$ is an embedding;*
- (b) *G is discrete;*
- (c) *$\widehat{G} = G^*$ is compact.*

Proof. Since G^* is compact, j can be an embedding iff \widehat{G} itself is compact. According to Example 7.1 this occurs precisely when G is discrete. In that case $\widehat{G} = G^*$ is compact. \square

Actually, it can be proved, once the duality theorem is available, that $j : \widehat{G} \hookrightarrow G^*$ need not be even a local homeomorphism. (If j is a local homeomorphism, then the topological subgroup $j(\widehat{G})$ of G^* will be locally compact, hence closed in G^* . This would yield that $j(\widehat{G})$ is compact. On the other hand, the topology of $j(\widehat{G})$ is precisely the initial topology of all projections p_x restricted to \widehat{G} . By the Pontryagin duality theorem, these projections form the group of all continuous characters of \widehat{G} . So this topology coincides with $\mathcal{T}_{\widehat{G}}$. By a general theorem of Glicksberg, a locally compact abelian groups H and $(H, \mathcal{T}_{\widehat{H}})$ have the same compact sets. In particular, compactness of $(H, \mathcal{T}_{\widehat{H}})$ yields compactness of H . This proves that if $j : \widehat{G} \hookrightarrow G^*$ is a local homeomorphism, then \widehat{G} is compact and consequently G is discrete.)

7.2 Computation of some dual groups

In the next proposition we show, roughly speaking, that the projective order between continuous surjective open homomorphisms with the same domain corresponds to the order by inclusion of their kernels.

Proposition 7.4. *Let G, H_1 and H_2 be topological abelian groups and let $\chi_i : G \rightarrow H_i$, $i = 1, 2$, be continuous surjective open homomorphisms. Then there exists a continuous homomorphism $\iota : H_1 \rightarrow H_2$ such that $\chi_2 = \iota \circ \chi_1$ iff $\ker \chi_1 \leq \ker \chi_2$. If $\ker \chi_1 = \ker \chi_2$ then ι will be a topological isomorphism.*

Proof. The necessity is obvious. So assume that $\ker \chi_1 \leq \ker \chi_2$ holds. By the homomorphism theorem applied to χ_i there exists a topological isomorphism $j_i : G/\ker \chi_i \rightarrow H_i$ such that $\chi_i = j_i \circ q_i$, where $q_i : G \rightarrow G/\ker \chi_i$ is the canonical homomorphism for $i = 1, 2$. As $\ker \chi_1 \leq \ker \chi_2$ we get a continuous homomorphism t that makes commutative the following diagram

$$\begin{array}{ccccc}
 & & G & & \\
 \chi_1 \swarrow & & \searrow \chi_2 & & \\
 H_1 & \xleftarrow{j_1} & G/\ker \chi_1 & \xrightarrow{t} & G/\ker \chi_2 & \xrightarrow{j_2} & H_2 \\
 & \searrow \iota & & & & & \\
 & & & & & &
 \end{array}$$

Obviously $\iota = j_2 \circ t \circ j_1^{-1}$ works. If $\ker \chi_1 = \ker \chi_2$, then t is a topological isomorphism, hence ι will be a topological isomorphism as well. \square

In the sequel we denote by $k \cdot id_G$ the endomorphism of an abelian group G obtained by the map $x \mapsto kx$, for a fixed $k \in \mathbb{Z}$. The next lemma will be used for the computation of the dual groups in Example 7.7.

Lemma 7.5. *Every continuous homomorphism $\chi : \mathbb{T} \rightarrow \mathbb{T}$ has the form $k \cdot id_{\mathbb{T}}$, for some $k \in \mathbb{Z}$. In particular, the only topological isomorphisms $\chi : \mathbb{T} \rightarrow \mathbb{T}$ are $\pm id_{\mathbb{T}}$.*

Proof. We prove first that the only topological isomorphisms $\chi : \mathbb{T} \rightarrow \mathbb{T}$ are $\pm id_{\mathbb{T}}$. The proof will exploit the fact that the arcs are the only connected sets of \mathbb{T} . Hence χ sends any arc of \mathbb{T} to an arc, sending end points to end points. Denote by φ the canonical homomorphism $\mathbb{R} \rightarrow \mathbb{T}$ and for $n \in \mathbb{N}$ let $c_n = \varphi(1/2^n)$ be the generators of the Prüfer subgroup $\mathbb{Z}(2^\infty)$ of \mathbb{T} . Then, c_1 is the only element of \mathbb{T} of order 2, hence $g(c_1) = c_1$. Therefore, the arc $A_1 = \varphi([0, 1/2])$ either goes onto itself, or goes onto its symmetric image $-A_1$. Let us consider the first case. Clearly, either $g(c_2) = c_2$ or $g(c_2) = -c_2$ as $o(g(c_2)) = 4$ and being $\pm c_2$ the only elements of order 4 of \mathbb{T} . By our assumption $g(A_1) = A_1$ we have $g(c_2) = c_2$ since c_2 is the only element of order 4 on the arc A_1 . Now the arc $A_2 = [0, c_2]$ goes onto itself, hence for c_3 we must have $g(c_3) = c_3$ as the only element of order 8 on the

arc A_2 , etc. We see in the same way that $g(c_n) = c_n$. Hence g is identical on the whole subgroup $\mathbb{Z}(2^\infty)$. As this subgroup is dense in \mathbb{T} , we conclude that g coincides with $id_{\mathbb{T}}$. In the case $g(A_1) = -A_1$ we replace g by $-g$ and the previous proof gives $-g = id_{\mathbb{T}}$, i.e., $g = -id_{\mathbb{T}}$.

For $k \in \mathbb{N}_+$ let $\pi_k = k \cdot id_{\mathbb{T}}$. Then $\ker \pi_k = \mathbb{Z}_k$ and π_k is surjective. Let now $\chi : \mathbb{T} \rightarrow \mathbb{T}$ be a non-trivial continuous homomorphism. Then $\ker \chi$ is a closed proper subgroup of \mathbb{T} , hence $\ker \chi = \mathbb{Z}_k$ for some $k \in \mathbb{N}_+$. Moreover, $\chi(\mathbb{T})$ is a connected non-trivial subgroup of \mathbb{T} , hence $\chi(\mathbb{T}) = \mathbb{T}$. By Proposition 7.4 $\chi = \pm \pi_k$. \square

Obviously, $\chi = \pm \xi$ for characters $\chi, \xi : G \rightarrow \mathbb{T}$ implies $\ker \chi = \ker \xi$ and $\chi(G) = \xi(G)$. More generally, if $\chi = k \cdot \xi$ for some $k \in \mathbb{Z}$, then $\ker \chi \geq \ker \xi$ and $\chi(G) \leq \xi(G)$. Now we see that this implication can be (partially) inverted under appropriate hypotheses.

Corollary 7.6. *Let G be a σ -compact locally compact abelian group and let $\chi, \xi : G \rightarrow \mathbb{T}$ be continuous characters such that $\ker \chi \geq \ker \xi$ and $\chi(G) \leq \xi(G)$.*

- (a) *If $\chi(G) = \xi(G) = \mathbb{T}$ then $\chi = k \cdot \xi$ for some $k \in \mathbb{Z}$; moreover, $\ker \chi = \ker \xi$ iff $\chi = \pm \xi$.*
- (b) *If G is compact and $|\xi(G)| = m$ for some $m \in \mathbb{N}_+$, then $\chi = k\xi$ for some $k \in \mathbb{Z}$; moreover, $\ker \chi = \ker \xi$ iff $\chi(G) = \xi(G)$, in such a case k must be coprime to m .*
- (c) *If $\ker \xi = \ker \chi$ is open and $H = \chi(G) = \xi(G)$, then $\chi = \iota \circ \xi$, where $\iota : H \rightarrow H$ is an arbitrary automorphism of the subgroup H of \mathbb{T} equipped with the discrete topology.*

Proof. (a) As $\chi(G) = \xi(G) = \mathbb{T}$ and G is σ -compact, we can apply Lemma 7.4 and observe that the only ι given by the lemma can be $k \cdot id_{\mathbb{T}}$ for some $k \in \mathbb{Z}$ in view of the previous lemma. The same lemma yields $k = \pm 1$ when $\ker \chi = \ker \xi$.

(b) If G is compact and $|\xi(G)| = m$ for some $m \in \mathbb{N}_+$, $\xi(G)$ is a cyclic subgroup of \mathbb{T} of order m . Note that \mathbb{T} has a unique such cyclic subgroup. By Proposition 7.4 there exists a homomorphism $\iota : \xi(G) \rightarrow \chi(G)$ such that $\chi = \iota \circ \xi$. The hypothesis $\chi(G) \leq \xi(G)$ implies that there such a ι must be the multiplication by some $k \in \mathbb{Z}$. In case $\chi(G) = \xi(G)$ this k is coprime to m .

(c) Obvious. \square

Example 7.7. Let p be a prime. Then $\widehat{\mathbb{Z}(p^\infty)} \cong \mathbb{J}_p$, $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$, $\widehat{\mathbb{T}} \cong \mathbb{Z}$, $\widehat{\mathbb{Z}} \cong \mathbb{T}$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$.

Proof. The first isomorphism $\widehat{\mathbb{Z}(p^\infty)} = \mathbb{J}_p$ follows from our definition $\mathbb{J}_p = \text{End}(\mathbb{Z}(p^\infty)) = \text{Hom}(\mathbb{Z}(p^\infty), \mathbb{T}) = \widehat{\mathbb{Z}(p^\infty)}$.

To verify the isomorphism $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$ consider first the quotient homomorphism $\eta_n : \mathbb{J}_p \rightarrow \mathbb{J}_p/p^n\mathbb{J}_p \cong \mathbb{Z}_{p^n} \leq \mathbb{T}$. With this identifications we consider $\eta_n \in \widehat{\mathbb{J}_p}$. It is easy to see that under this identification $p\eta_n = \eta_{n-1}$. Therefore, the subgroup H of $\widehat{\mathbb{J}_p}$ generated by the characters η_n is isomorphic to $\mathbb{Z}(p^\infty)$. Let us see that $H = \widehat{\mathbb{J}_p}$. Indeed, take any non-trivial character $\chi : \mathbb{J}_p \rightarrow \mathbb{T}$. Then $N = \ker \chi$ is a closed proper subgroup of \mathbb{J}_p . Moreover, $N \neq 0$ as \mathbb{J}_p is not isomorphic to a subgroup of \mathbb{T} by Exercise 4.49. Thus $N = p^n\mathbb{J}_p$ for some $n \in \mathbb{N}_+$. Since $N = \ker \eta_n$, we conclude with (b) of Corollary 7.6 that $\chi = k\eta_n$ for some $k \in \mathbb{Z}$. This proves that $\chi \in H$ and consequently $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$.

The isomorphism $g : \widehat{\mathbb{Z}} \rightarrow \mathbb{T}$ is obtained by setting $g(\chi) := \chi(1)$ for every $\chi : \mathbb{Z} \rightarrow \mathbb{T}$. It is easy to check that this isomorphism is topological.

According to 7.5 every $\chi \in \widehat{\mathbb{T}}$ has the form $\chi = k \cdot id_{\mathbb{T}}$ for some $k \in \mathbb{Z}$. This gives a homomorphism $\widehat{\mathbb{T}} \rightarrow \mathbb{Z}$ assigning $\chi \mapsto k$. It is obviously injective and surjective. This proves $\widehat{\mathbb{T}} \cong \mathbb{Z}$ since both groups are discrete.

To prove $\widehat{\mathbb{R}} \cong \mathbb{R}$ consider the character $\chi_1 : \mathbb{R} \rightarrow \mathbb{T}$ obtained simply by the canonical map $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. For every non-zero $r \in \mathbb{R}$ consider the map $\rho_r : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\rho_r(x) = rx$. Then its composition $\chi_r = \chi_1 \circ \rho_r$ with χ_1 gives a continuous character of \mathbb{R} that is surjective and $\ker \chi_r = \langle 1/r \rangle$. Now consider any continuous non-trivial character $\chi \in \widehat{\mathbb{R}}$. Then χ is surjective and $N = \ker \chi$ is a proper closed subgroup of \mathbb{R} . Hence N is cyclic by Exercise 3.20. Let $N = \langle 1/r \rangle$. Then $\ker \chi = \ker \chi_r$, so that Corollary 7.6 yields $\chi = \pm \chi_r$. The assignment $\chi \mapsto \pm r$ defines a homomorphism $\widehat{\mathbb{R}} \rightarrow \mathbb{R}$ that is obviously injective and surjective. Its continuity immediately follows from the definition of the compact-open topology of $\widehat{\mathbb{R}}$. As \mathbb{R} is σ -compact, this isomorphism is also open by the open mapping theorem. \square

Exercise 7.8. *Let G be an abelian group and p be a prime. Prove that*

- (a) $\chi \in p\widehat{G}$ iff $\chi(G[p]) = 0$.
- (b) $p\chi = 0$ in \widehat{G} iff $\chi(pG) = 0$.

Conclude that

(i) a discrete abelian group G is divisible (resp., torsion-free) iff \widehat{G} is torsion-free (resp., divisible).

(ii) the groups $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{Q}_p}$ are torsion-free and divisible.

Exercise 7.9. Let G be a totally disconnected locally compact abelian group. Prove that $\ker \chi$ is an open subgroup of G for every $\chi \in \widehat{G}$.

(Hint. Use the fact that by the continuity of χ and the total disconnectedness of G there exists an open subgroup O of G such that $\chi(O) \subseteq \Lambda_1$.)

Exercise 7.10. Let p be a prime. Prove that $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$, where \mathbb{Q}_p denotes the field of all p -adic numbers.

(Hint. Fix $N = \{\chi \in \widehat{\mathbb{Q}_p} : \ker \chi \geq \mathbb{J}_p\}$. By the compactness of \mathbb{J}_p , conclude that N is an open subgroup of $\widehat{\mathbb{Q}_p}$ topologically isomorphic to \mathbb{J}_p using Exercise 7.9 and Corollary 7.6 (c). For every $n \in \mathbb{N}_+$ let $\xi_n : \mathbb{Q}_p \rightarrow \mathbb{Q}_p/p^n \mathbb{J}_p$ be the canonical homomorphism. As $\mathbb{Q}_p/p^n \mathbb{J}_p \cong \mathbb{Z}(p^\infty) \leq \mathbb{T}$, we can consider $\xi_n \in \widehat{\mathbb{Q}_p}$. Show that $p\xi_{n+1} = \xi_n$ for $n \in \mathbb{N}_+$ and $p\xi_1 \in N$. The subgroup of $\widehat{\mathbb{Q}_p}$ generated by N and (ξ_n) is isomorphic to \mathbb{Q}_p . Using Corollary 7.6 (c) and Exercise 7.9 deduce that it coincides with the whole group $\widehat{\mathbb{Q}_p}$.)

Exercise 7.11. Let H be a subgroup of \mathbb{R}^n . Prove that every $\chi \in \widehat{H}$ extends to a continuous character of \mathbb{R}^n .

7.3 Some general properties of the dual

We prove next that the dual group of a finite product of abelian topological groups is the product of the dual groups of each group.

Lemma 7.12. If G and H are topological abelian groups, then $\widehat{G \times H}$ is isomorphic to $\widehat{G} \times \widehat{H}$.

Proof. Define $\Phi : \widehat{G} \times \widehat{H} \rightarrow \widehat{G \times H}$ by $\Phi(\chi_1, \chi_2)(x_1, x_2) = \chi_1(x_1) + \chi_2(x_2)$ for every $(\chi_1, \chi_2) \in \widehat{G} \times \widehat{H}$ and $(x_1, x_2) \in G \times H$. Then Φ is a homomorphism, in fact $\Phi(\chi_1 + \psi_1, \chi_2 + \psi_2)(x_1, x_2) = (\chi_1 + \psi_1)(x_1) + (\chi_2 + \psi_2)(x_2) = \chi_1(x_1) + \psi_1(x_1) + \chi_2(x_2) + \psi_2(x_2) = \Phi(\chi_1, \chi_2)(x_1, x_2) + \Phi(\psi_1, \psi_2)(x_1, x_2)$.

Moreover Φ is injective, because

$$\begin{aligned} \ker \Phi &= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \Phi(\chi, \psi) = 0\} \\ &= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \Phi(\chi, \psi)(x, y) = 0 \text{ for every } (x, y) \in G \times H\} \\ &= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \chi(x) + \psi(y) = 0 \text{ for every } (x, y) \in G \times H\} \\ &= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \chi(x) = 0 \text{ and } \psi(y) = 0 \text{ for every } (x, y) \in G \times H\} \\ &= \{(0, 0)\}. \end{aligned}$$

To prove that Φ is surjective, take $\psi \in \widehat{G \times H}$ and note that $\psi(x_1, x_2) = \psi(x_1, 0) + \psi(0, x_2)$. Now define $\psi_1(x_1) = \psi(x_1, 0)$ for every $x_1 \in G$ and $\psi_2(x_2) = \psi(0, x_2)$ for every $x_2 \in H$. Hence $\psi_1 \in \widehat{G}$, $\psi_2 \in \widehat{H}$ and $\psi = \Phi(\psi_1, \psi_2)$.

Now we show that Φ is continuous. Let $W(K, U)$ be an open neighborhood of 0 in $\widehat{G \times H}$ (K is a compact subset of $G \times H$ and U is an open neighborhood of 0 in \mathbb{T}). Since the projections π_G and π_H of $G \times H$ onto G and H are continuous, $K_G = \pi_G(K)$ and $K_H = \pi_H(K)$ are compact in G and in H respectively. Taking an open symmetric neighborhood V of 0 in \mathbb{T} , it follows $\Phi(W(K_G, V) \times W(K_H, V)) \subseteq W(K, U)$.

It remains to prove that Φ is open. Consider two open neighborhoods $W(K_G, U_G)$ of 0 in \widehat{G} and $W(K_H, U_H)$ of 0 in \widehat{H} , where $K_G \subseteq G$ and $K_H \subseteq H$ are compact and U_G, U_H are open neighborhoods of 0 in \mathbb{T} . Then $K = (K_G \cup \{0\}) \times (K_H \cup \{0\})$ is a compact subset of $G \times H$ and $U = U_G \cap U_H$ is an open neighborhood of 0 in \mathbb{T} . Thus $W(K, U) \subseteq \Phi(W(K_G, U_G) \times W(K_H, U_H))$, because if $\chi \in W(K, U)$ then $\chi = \Phi(\chi_1, \chi_2)$, where $\chi_1(x_1) = \chi(x_1, 0) \in U \subseteq U_G$ for every $x_1 \in G$ and $\chi_2(x_2) = \chi(0, x_2) \in U \subseteq U_H$ for every $x_2 \in H$. \square

It follows from Proposition 7.7 that the groups \mathbb{T} , \mathbb{Z} , $\mathbb{Z}(p^\infty)$, \mathbb{J}_p e \mathbb{R} satisfy $\widehat{\widehat{G}} \cong G$, namely the Pontryagin-van Kampen duality theorem. Using the next theorem this property extends to all finite direct products of these groups.

Call a topological abelian group G *autodual*, if G satisfies $\widehat{\widehat{G}} \cong G$. We have seen already that \mathbb{R} and \mathbb{Q}_p are autodual. By Lemma 7.12 finite direct products of autodual groups are autodual. Now using this observation and Lemma 7.12 we provide a large supply of groups for which the Pontryagin-van Kampen duality holds true.

Proposition 7.13. *Let P_1, P_2 and P_3 be finite sets of primes, $m, n, k, k_p \in \mathbb{N}$ ($p \in P_3$) and $n_p, m_p \in \mathbb{N}_+$ ($p \in P_1 \cup P_2$). Then every group of the form*

$$G = \mathbb{T}^n \times \mathbb{Z}^m \times \mathbb{R}^k \times F \times \prod_{p \in P_1} \mathbb{Z}(p^\infty)^{n_p} \times \prod_{p \in P_2} \mathbb{J}_p^{m_p} \times \prod_{j \in P_3} \mathbb{Q}_p^{k_p},$$

where F is a finite abelian group, satisfies $\widehat{\widehat{G}} \cong G$.

Moreover, such a group is autodual iff $n = m$, $P_1 = P_2$ and $n_p = m_p$ for all $p \in P_1 = P_2$. In particular, $\widehat{\widehat{G}} \cong G$ holds true for all elementary locally compact abelian groups.

Proof. Let us start by proving $\widehat{\widehat{F}} = F^* \cong F$. Recall that F has the form $F \cong \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_m}$. So applying Theorem 7.14 we are left with the proof of the isomorphism $\mathbb{Z}_n^* \cong \mathbb{Z}_n$ for every $n \in \mathbb{N}_+$. The elements x of \mathbb{T} satisfying $nx = 0$ are precisely those of the unique cyclic subgroup of order n of \mathbb{T} , we shall denote that subgroup by \mathbb{Z}_n . Therefore, the group $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$ of all homomorphisms $\mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is isomorphic to \mathbb{Z}_n .

It follows easily from Lemma 7.12 that if $\widehat{\widehat{G_i}} \cong G_i$ (resp., $\widehat{G_i} \cong G_i$) for a finite family $\{G_i\}_{i=1}^n$ of topological abelian groups, then also $G = \prod_{i=1}^n G_i$ satisfies $\widehat{\widehat{G}} \cong G$ (resp., $\widehat{G} \cong G$). Therefore, it suffices to verify that the groups \mathbb{T} , \mathbb{Z} , $\mathbb{Z}(p^\infty)$, and \mathbb{J}_p satisfy $\widehat{\widehat{G}} \cong G$, while $\widehat{\mathbb{R}} \cong \mathbb{R}$, $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$ were already checked.

It follows from Proposition 7.7 that $\widehat{\widehat{\mathbb{Z}}} \cong \mathbb{T}$ and $\widehat{\widehat{\mathbb{T}}} \cong \mathbb{Z}$, hence $\mathbb{Z} \cong \widehat{\widehat{\mathbb{Z}}}$ and $\mathbb{T} \cong \widehat{\widehat{\mathbb{T}}}$. Analogously, $\widehat{\mathbb{Z}(p^\infty)} \cong \mathbb{J}_p$ and $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$ yield $\mathbb{Z}(p^\infty) \cong \widehat{\widehat{\mathbb{Z}(p^\infty)}}$ and $\mathbb{J}_p \cong \widehat{\widehat{\mathbb{J}_p}}$. \square

The problem of characterizing all autodual locally compact abelian groups is still open [47, 48].

Theorem 7.14. *Let $\{D_i\}_{i \in I}$ be a family of discrete abelian groups and let $\{G_i\}_{i \in I}$ be a family of compact abelian groups. Then*

$$\widehat{\bigoplus_{i \in I} D_i} \cong \prod_{i \in I} \widehat{D_i} \quad \text{and} \quad \widehat{\prod_{i \in I} G_i} \cong \bigoplus_{i \in I} \widehat{G_i}. \quad (5)$$

Proof. Let $\chi : \bigoplus_{i \in I} D_i \rightarrow \mathbb{T}$ be a character and let $\chi_i : D_i \rightarrow \mathbb{T}$ be its restriction to D_i . Then $\chi \mapsto (\chi_i) \in \prod_{i \in I} \widehat{D_i}$ is the first isomorphism in (5).

Let $\chi : \prod_{i \in I} G_i \rightarrow \mathbb{T}$ be a continuous character. Pick a neighborhood U of 0 containing no non-trivial subgroups of \mathbb{T} . Then there exists a neighborhood V of 0 in $G = \prod_{i \in I} G_i$ with $\chi(V) \subseteq U$. By the definition of the Tychonov topology there exists a finite subset $F \subseteq I$ such that V contains the subproduct $B = \prod_{i \in I \setminus F} G_i$. Being $\chi(B)$ a subgroup of \mathbb{T} , we conclude that $\chi(B) = 0$ by the choice of U . Hence χ factorizes through the projection $p : G \rightarrow \prod_{i \in F} G_i = G/B$; so there exists a character $\chi' : \prod_{i \in F} G_i \rightarrow \mathbb{T}$ such that $\chi = \chi' \circ p$. Obviously, $\chi' \in \bigoplus_{i \in I} \widehat{G_i}$. Then $\chi \mapsto \chi'$ is the second isomorphism in (5). \square

In order to extend the isomorphism (5) to the general case of locally compact abelian groups one has to consider a specific topology on the direct sum.

Algebraic properties of the dual group \widehat{G} of a compact abelian group G can be described in terms of topological properties of the group G . We saw in Corollary 6.22 that \widehat{G} is torsion precisely when G is totally disconnected. Here is the counterpart of this property in the connected case:

Proposition 7.15. *Let G be a topological abelian group.*

- (a) *If G is connected, then the dual group \widehat{G} is torsion-free.*
- (b) *If G is compact, then the dual group \widehat{G} is torsion-free iff G is connected.*

Proof. (a) Since for every non-zero continuous character $\chi : G \rightarrow \mathbb{T}$ the image $\chi(G)$ is a non-trivial connected subgroup of \mathbb{T} , we deduce that $\chi(G) = \mathbb{T}$ for every non-zero $\chi \in \widehat{G}$. Hence \widehat{G} is torsion-free.

(b) If the group G is compact and disconnected, then by Theorem 4.19 there exists a proper open subgroup N of G . Take any non-zero character ξ of the finite group G/N . Then $m\xi = 0$ for some positive integer m . Now the composition χ of ξ and the canonical homomorphism $G \rightarrow G/N$ satisfies $m\chi = 0$ as well. So \widehat{G} has a non-zero torsion character. This proves the implication left open by item (a). \square

Let G and H be abelian topological groups. If $f : G \rightarrow H$ is a continuous homomorphism, define $\widehat{f} : \widehat{H} \rightarrow \widehat{G}$ putting $\widehat{f}(\chi) = \chi \circ f$ for every $\chi \in \widehat{H}$.

Lemma 7.16. *If $f : G \rightarrow H$ is a continuous homomorphism of topological abelian group, then $\widehat{f}(\chi) = \chi \circ f$ is a continuous homomorphism as well.*

- (a) *If $f(G)$ is dense in H , then \widehat{f} is injective.*
- (b) *If f is injective and $f(G)$ is either open or dense in H , then \widehat{f} is surjective.*
- (c) *if f is a surjective homomorphism, such that every compact subset of H is covered by some compact subset of G , then \widehat{f} is an embedding.*
- (d) *if f is a quotient homomorphism and G is locally compact, then \widehat{f} is an embedding.*
- (e) *If f is a topological isomorphism, then \widehat{f} is a topological isomorphism too.*

Proof. Assume K is a compact subset of G and U a neighborhood of 0 in \mathbb{T} . Then $f(K)$ is a compact set in H , so $W = W_{\widehat{G}}(f(K), U)$ is a neighborhood of 0 in \widehat{H} and $\widehat{f}(W) \subseteq W(K, U)$. This proves the continuity of \widehat{f} .

- (a) If $\widehat{f}(\chi) = 0$, then $\chi \circ f = 0$. By the density of $f(G)$ in H this yields $\chi = 0$.
- (b) Let $\chi \in \widehat{G}$. If $f(G)$ is open in H , then any extension $\xi : H \rightarrow \mathbb{T}$ of χ will be continuous on $f(G)$. There exists at least one such extension ξ by Corollary 2.6. Hence $\xi \in \widehat{H}$ and $\chi = \widehat{f}(\xi)$. Now consider the case when $f(G)$ is dense in H . Then $\widehat{H} = \widehat{G}$ and the characters of H can be extended to characters of G (see Theorem 3.79).
- (c) Assume L is a compact subset of G/H and U a neighborhood of 0 in \mathbb{T} . Let K be a compact set in G such that $f(K) = L$. Then $\widehat{f}(W_{\widehat{H}}(L, U)) = \text{Im } \widehat{f} \cap W_{\widehat{G}}(K, U)$, so \widehat{f} is an embedding.
- (d) Follows from (c) and Lemma 4.6.
- (e) Obvious.

□

Exercise 7.17. *Prove that $\widehat{\mathbb{Q}/\mathbb{Z}} \cong \prod_p \mathbb{J}_p$.*

(Hint. Use the isomorphism $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$, Example 7.7 and Theorem 7.14.)

Now we shall see that the group \mathbb{Q} satisfies the duality theorem (see item (b) below).

Example 7.18. Let K denote the compact group $\widehat{\mathbb{Q}}$. Then:

- (a) K contains a closed subgroup H isomorphic to $\widehat{\mathbb{Q}/\mathbb{Z}}$ such that $K/H \cong \mathbb{T}$;
- (ii) $\widehat{K} \cong \mathbb{Q}$.

(a) Denote by H the subgroup of all $\chi \in K$ such that $\chi(\mathbb{Z}) = 0$. To prove that H is a closed subgroup of K such that K/H is isomorphic to \mathbb{T} . To this end consider the continuous map $\rho : K \rightarrow \widehat{\mathbb{Z}}$ obtained by the restriction to \mathbb{Z} of every $\chi \in K$. Obviously, $\ker \rho = H$, so $\mathbb{T} \cong \widehat{\mathbb{Z}} \cong K/H$. To see that $H \cong \widehat{\mathbb{Q}/\mathbb{Z}}$ note that the characters of \mathbb{Q}/\mathbb{Z} correspond precisely to those characters of \mathbb{Q} that vanish on \mathbb{Z} , i.e., precisely H .

(b) By Exercise 7.8 K is a divisible torsion-free group, every non-zero $r \in \mathbb{Q}$ defines a continuous automorphism λ_r of K by setting $\lambda_r(x) = rx$ for every $x \in K$. Then the composition $\rho \circ \lambda_r : K \rightarrow \mathbb{T}$ defines a character $\chi_r \in \widehat{K}$ with $\ker \chi_r = r^{-1}H$. For the sake of completeness let $\chi_0 = 0$. By Exercise 7.17 $\widehat{\mathbb{Q}/\mathbb{Z}} \cong \prod_p \mathbb{J}_p$ is totally disconnected, so by Corollary 6.21 H has no surjective characters $\chi : H \rightarrow \mathbb{T}$. Now let $\chi \in \widehat{K}$ be non-zero. Then $\chi(K)$ will be a non-zero closed divisible subgroup of \mathbb{T} , hence $\chi(K) = \mathbb{T}$. On the other hand, $N = \ker \chi$ is a proper closed subgroup of K such that $N + H \neq \mathbb{T}$, as $\chi(H)$ is a proper closed subgroup of \mathbb{T} by the previous argument. Hence, $\chi(H)$ is finite, say of order m . Then $N + H$ contains N is a finite-index subgroup, more precisely $[H : (N \cap H)] = [(N + H) : N] = m$. Then $mH \leq N$. Consider the character $\chi_{m^{-1}}$ of K having $\ker \chi_{m^{-1}} = mH \leq N$. By Corollary there exists $k \in \mathbb{Z}$ such that $\chi = k\chi_{m^{-1}} = \chi_r$, where $r = km^{-1} \in \mathbb{Q}$. This shows that $\widehat{K} = \{\chi_r : r \in \mathbb{Q}\} \cong \mathbb{Q}$.

The compact group $\widehat{\mathbb{Q}}$ is closely related to the adèle rings of the field \mathbb{Q} , more detail can be found in [34, 38, 75, 97].

Exercise 7.19. *Prove that a discrete abelian group G satisfies $\widehat{\widehat{G}} \cong G$ whenever*

- (a) G is divisible;
- (b) G is free;
- (c) G is of finite exponent;
- (d) G is torsion and every primary component of G is of finite exponent.

(Hint. (a) Use Examples 7.7 and 7.18 (b) and the fact that every divisible group is a direct sum of copies of \mathbb{Q} and the groups $\mathbb{Z}(p^\infty)$.

(c) and (d) Use that fact that every abelian group of finite exponent is a direct sum of cyclic subgroups (i.e., Prüfer's theorem, see (d) of Example 2.3).

Exercise 7.20. Prove that every torsion compact abelian group G is bounded. More precisely, there exists natural numbers m_1, \dots, m_n and cardinals $\alpha_1, \dots, \alpha_n$ such that $G \cong \prod_{i=1}^n \mathbb{Z}(m_i)^{\alpha_i}$.

(Hint. Use the Baire category theorem for the union $G = \bigcup_{n=1}^\infty G[n!]$ of closed subgroups. Conclude that $G[n!]$ is open for some n , so must have finite index by the compactness of G . This yields $mG = 0$ for some m . Show that this yields also $m\widehat{G} = 0$. Now apply Prüfer's theorem to \widehat{G} and the fact that $G \cong \widehat{\widehat{G}}$.)

7.4 The natural transformation ω

Let G be a topological abelian group. Define $\omega_G : G \rightarrow \widehat{\widehat{G}}$ such that $\omega_G(x)(\chi) = \chi(x)$, for every $x \in G$ and for every $\chi \in \widehat{G}$. We show now that $\omega_G(x) \in \widehat{\widehat{G}}$.

Proposition 7.21. If G is a topological abelian group. Then $\omega_G(x) \in \widehat{\widehat{G}}$ and $\omega_G : G \rightarrow \widehat{\widehat{G}}$ is a homomorphism. If G is locally compact, then the homomorphism ω_G is a continuous.

Proof. In fact,

$$\omega_G(x)(\chi + \psi) = (\chi + \psi)(x) = \chi(x) + \psi(x) = \omega_G(x)(\chi) + \omega_G(x)(\psi),$$

for every $\chi, \psi \in \widehat{G}$. Moreover, if U is an open neighborhood of 0 in \mathbb{T} , then $\omega_G(x)(W(\{x\}, U)) \subseteq U$. This proves that $\omega_G(x)$ is a character of \widehat{G} , i.e., $\omega_G(x) \in \widehat{\widehat{G}}$. For every $x, y \in G$ and for every $\chi \in \widehat{G}$ we have $\omega_G(x + y)(\chi) = (\chi)(x + y) = \chi(x) + \chi(y) = \omega_G(x)(\chi) + \omega_G(y)(\chi)$ and so ω_G is a homomorphism.

Now assume G is locally compact. To prove that ω_G is continuous, pick an open neighborhood A of 0 in \mathbb{T} and a compact subset K of \widehat{G} . Then $W(K, A)$ is an open neighborhood of 0 in $\widehat{\widehat{G}}$. Let U be an open neighborhood of 0 in G with compact closure. Take an open symmetric neighborhood B of 0 in \mathbb{T} with $B + B \subseteq A$. Thus $W(\overline{U}, B)$ is an open neighborhood of 0 in \widehat{G} . Since K is compact, there exist finitely many characters χ_1, \dots, χ_m of G such that $K \subseteq (\chi_1 + W(\overline{U}, B)) \cup \dots \cup (\chi_m + W(\overline{U}, B))$. For every $i = 1, \dots, m$ there is an open neighborhood V_i of 0 in G such that $\chi_i(V_i) \subseteq B$. Define $V = U \cap V_1 \cap \dots \cap V_m \subseteq U$ and note that $\chi_i(V) \subseteq B$ for every $i = 1, \dots, m$. Thus $\omega_G(V) \subseteq W(K, A)$. Indeed, if $x \in V$ and $\chi \in K$, then $\chi_i(x) \in B$ for every $i = 1, \dots, m$ and there exists $i_0 \in \{1, \dots, m\}$ such that $\chi \in \chi_{i_0} + W(\overline{U}, B)$; so $\chi(x) = \chi_{i_0}(x) + \psi(x)$ with $\psi \in W(\overline{U}, B)$ and then $\omega_G(x)(\chi) = \chi(x) \in B + B \subseteq A$. \square

In this chapter we shall have a precise approach, by saying that a group G satisfies the Pontryagin-van Kampen duality theorem when ω_G is a topological isomorphism.

Lemma 7.22. If the topological abelian groups G_i satisfy Pontryagin-van Kampen duality theorem for $i = 1, 2, \dots, n$, then also $G = \prod_{i=1}^n G_i$ satisfies Pontryagin-van Kampen duality theorem.

Proof. Apply Lemma 7.12 twice to obtain an isomorphism $j : \prod_{i=1}^n \widehat{\widehat{G_i}} \rightarrow \widehat{\widehat{G}}$. It remains to verify that the product $\pi : G \rightarrow \prod_{i=1}^n \widehat{\widehat{G_i}}$ of the isomorphisms $\omega_{G_i} : G_i \rightarrow \prod_{i=1}^n \widehat{\widehat{G_i}}$ given by our hypothesis composed with the isomorphism j gives precisely ω_G . \square

Consider two categories \mathcal{A} and \mathcal{B} . A covariant [contravariant] functor $F : \mathcal{A} \rightarrow \mathcal{B}$ assigns to each object $A \in \mathcal{A}$ an object $FA \in \mathcal{B}$ and to each arrow $f : A \rightarrow A'$ in \mathcal{A} an arrow $Ff : FA \rightarrow FA'$ [$Ff : FA' \rightarrow FA$] such that $Fid_A = id_{FA}$ and $F(g \circ f) = Fg \circ Ff$ [$F(g \circ f) = Ff \circ Fg$] for every arrow $f : A \rightarrow A'$ and $g : A' \rightarrow A''$ in \mathcal{A} .

Let $F, F' : \mathcal{A} \rightarrow \mathcal{B}$ be covariant functors. A *natural transformation* γ from F to F' assigns to each $A \in \mathcal{A}$ an arrow $\gamma_A : FA \rightarrow F'A$ such that for every arrow $f : A \rightarrow A'$ in \mathcal{A} the following diagram is commutative

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \gamma_A \downarrow & & \downarrow \gamma_{A'} \\ F'A & \xrightarrow{F'f} & F'A' \end{array}$$

A *natural equivalence* is a natural transformation γ such that each γ_A is an isomorphism.

If \mathcal{H} denote the category of all Hausdorff abelian topological groups, the *Pontryagin-van Kampen duality functor*, defined by

$$G \mapsto \widehat{G} \text{ and } f \mapsto \widehat{f}$$

for objects G and morphisms f of \mathcal{H} , is a contravariant functor $\widehat{} : \mathcal{H} \rightarrow \mathcal{H}$. Let \mathcal{L} be the full subcategory of \mathcal{H} having as objects all locally compact abelian groups. According to Proposition 7.2, the functor $\widehat{}$ sends \mathcal{L} to itself, i.e., defines a functor $\widehat{} : \mathcal{L} \rightarrow \mathcal{L}$. The Pontryagin-van Kampen duality theorem states that ω is a natural equivalence from $id_{\mathcal{L}}$ to $\widehat{} : \mathcal{L} \rightarrow \mathcal{L}$. We start by proving that ω is a natural transformation.

Proposition 7.23. ω is a natural transformation from $id_{\mathcal{L}}$ to $\widehat{} : \mathcal{L} \rightarrow \mathcal{L}$.

Proof. By Proposition 7.21 ω_G is continuous for every $G \in \mathcal{L}$. Moreover for every continuous homomorphism $f : G \rightarrow H$ the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \omega_G \downarrow & & \downarrow \omega_H \\ \widehat{\widehat{G}} & \xrightarrow{\widehat{f}} & \widehat{\widehat{H}} \end{array}$$

In fact, if $x \in G$ and $\xi \in \widehat{H}$, then $\omega_H(f(x))(\xi) = \xi(f(x))$. On the other hand,

$$(\widehat{f}(\omega_G(x)))(\xi) = (\omega_G(x) \circ \widehat{f})(\xi) = \omega_G(x)(\widehat{f}(\xi)) = \omega_G(x)(\xi \circ f) = \xi(f(x)).$$

Hence $\omega_H(f(x)) = \widehat{f}(\omega_G(x))$ for every $x \in G$. □

Remark 7.24. Note that ω_G is a monomorphism if and only if \widehat{G} separates the points of G . Moreover, $\omega_G(G)$ is a subgroup of $\widehat{\widehat{G}}$ that separates the points of \widehat{G} .

Now we can prove the Pontryagin-van Kampen duality theorem in the case when G is either compact or discrete.

Theorem 7.25. *If the abelian topological group G is either compact or discrete, then ω_G is a topological isomorphism.*

Proof. If G is discrete, then \widehat{G} separates the points of G by Corollary 2.7 and if G is compact, then \widehat{G} separates the points of G by the Peter-Weyl Theorem 6.4. Therefore ω_G is injective by Remark 7.24. If G is discrete, then \widehat{G} is compact and $\omega_G(G) = \widehat{\widehat{G}}$ by Corollary 6.6. Since $\widehat{\widehat{G}}$ is discrete, ω_G is a topological isomorphism.

Let now G be compact. Then ω_G is open thanks to Theorem 4.9. Suppose that $\omega_G(G)$ is a proper subgroup of $\widehat{\widehat{G}}$. By the compactness of G , $\widehat{\widehat{G}}$ is compact, hence closed in $\widehat{\widehat{G}}$. By the Peter-Weyl Theorem 6.4 applied to $\widehat{\widehat{G}}/\omega_G(G)$, there exists $\xi \in \widehat{\widehat{G}} \setminus \{0\}$ such that $\xi(\omega_G(G)) = \{0\}$. Since \widehat{G} is discrete, $\omega_{\widehat{G}}$ is a topological isomorphism and so there exists $\chi \in \widehat{G}$ such that $\omega_{\widehat{G}}(\chi) = \xi$. Thus for every $x \in G$ we have $0 = \xi(\omega_G(x)) = \omega_{\widehat{G}}(\chi)(\omega_G(x)) = \omega_G(x)(\chi) = \chi(x)$. It follows that $\chi \equiv 0$ and so that also $\xi \equiv 0$, a contradiction. □

Our next step is to prove the Pontryagin-van Kampen duality theorem when G is elementary locally compact abelian:

Theorem 7.26. *If G is an elementary locally compact abelian group, then ω_G is a topological isomorphism of G onto $\widehat{\widehat{G}}$.*

Proof. According to Lemma 7.22 and Theorem 7.25 it suffices to prove that $\omega_{\mathbb{R}}$ is a topological isomorphism. Of course, by the fact that $\widehat{\mathbb{R}}$ is topologically isomorphic to \mathbb{R} , one concludes immediately that also \mathbb{R} and $\widehat{\widehat{\mathbb{R}}}$ are topologically isomorphic. A more careful analysis of the dual $\widehat{\mathbb{R}}$ shows the crucial role of the (\mathbb{Z}) -bilinear map $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{T}$ defined by $\lambda(x, y) = \chi_1(xy)$, where $\chi_1 : \mathbb{R} \rightarrow \mathbb{T}$ is the character determined by the canonical quotient map $\mathbb{R} \rightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Indeed, for every $y \in \mathbb{R}$ the map $\chi_y : \mathbb{R} \rightarrow \mathbb{T}$ defined by $x \mapsto \lambda(x, y)$ is an element of $\widehat{\mathbb{R}}$. Hence the second copy $\{0\} \times \mathbb{R}$ of \mathbb{R} in $\mathbb{R} \times \mathbb{R}$ can be identified with $\widehat{\mathbb{R}}$. On the other hand, every element $x \in \mathbb{R}$ gives a continuous character $\mathbb{R} \rightarrow \mathbb{T}$ defined by $y \mapsto \lambda(x, y)$, so can be considered as the element $\omega_{\mathbb{R}}(x)$ of $\widehat{\widehat{\mathbb{R}}}$. We have seen that every $\xi \in \widehat{\widehat{\mathbb{R}}}$ has this form. This means that $\omega_{\mathbb{R}}$ is surjective. Since continuity of $\omega_{\mathbb{R}}$, as well as local compactness of $\widehat{\mathbb{R}}$ are already established, $\omega_{\mathbb{R}}$ is a topological isomorphism by the open mapping theorem. \square

For a subset X of G the *annihilator* of X in \widehat{G} is $A_{\widehat{G}}(X) = \{\chi \in \widehat{G} : \chi(A) = \{0\}\}$ and for a subset Y of \widehat{G} the *annihilator* of Y in G is $A_G(Y) = \{x \in G : \chi(x) = 0 \text{ for every } x \in Y\}$. When no confusion is possible we shall omit the subscripts \widehat{G} and G .

The next lemma will help us in computing the dual of a subgroup and a quotient group.

Lemma 7.27. *Let G be a locally compact abelian group. If M is a subset of G , then $A_{\widehat{G}}(M)$ is a closed subgroup of \widehat{G} .*

Proof. It suffices to note that

$$A_{\widehat{G}}(M) = \bigcap_{x \in M} \{\chi \in \widehat{G} : \chi(x) = 0\} = \bigcap \{\ker \omega_G(x) : x \in M\},$$

where each $\ker \omega(x)$ is a closed subgroup of \widehat{G} . \square

Call a continuous homomorphism $f : G \rightarrow H$ of topological groups *proper* if $f : G \rightarrow f(G)$ is open, whenever $f(G)$ carries the topology inherited from H . In particular, a surjective continuous homomorphism is proper iff it is open.

A short sequence $0 \rightarrow G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \rightarrow 0$ in \mathcal{L} , where f and h are continuous homomorphisms, is *exact* if f is injective, h is surjective and $\text{im } f = \ker h$. It is *proper* if f and h are proper.

Lemma 7.28. *Let G be a locally compact abelian group, H a subgroup of G and $i : H \rightarrow G$ the canonical inclusion of H in G . Then*

- (a) $\widehat{i} : \widehat{G} \rightarrow \widehat{H}$ is surjective if H is dense or open or compact;
- (b) \widehat{i} is injective if and only if H is dense in G ;
- (c) if H is closed and $\pi : G \rightarrow G/H$ is the canonical projection, then the sequence

$$0 \rightarrow \widehat{G/H} \xrightarrow{\widehat{\pi}} \widehat{G} \xrightarrow{\widehat{i}} \widehat{H}$$

is exact, $\widehat{\pi}$ is proper and $\text{im } \widehat{\pi} = A_{\widehat{G}}(H)$. If H is open or compact, then \widehat{i} is open and surjective.

Proof. (a) Note that \widehat{i} is surjective if and only if for every $\chi \in \widehat{H}$ there exists $\xi \in \widehat{G}$ such that $\xi|_H = \chi$. If H is compact apply Corollary 6.20. Otherwise Lemma 7.16 applies.

(b) If H is dense, then \widehat{i} is injective by Lemma 7.16. Conversely, assume that \overline{H} is a proper subgroup of G and let $q : G \rightarrow G/\overline{H}$ be the canonical projection. By Theorem 6.19 there exists $\chi \in \widehat{G/\overline{H}}$ not identically zero. Then $\xi = \chi \circ q \in \widehat{G}$ is non-zero and satisfies $\xi(H) = \{0\}$, i.e., $\widehat{i}(\xi) = 0$. This implies that \widehat{i} is not injective.

(c) According to Lemma 7.16 $\widehat{\pi}$ is a monomorphism, since π is surjective. We have that $\widehat{i} \circ \widehat{\pi} = \widehat{\pi} \circ i = 0$. If $\xi \in \ker \widehat{i} = \{\chi \in \widehat{G} : \chi(H) = \{0\}\}$, then $\xi(H) = \{0\}$. So there exists $\xi_1 \in \widehat{G/H}$ such that $\xi = \xi_1 \circ \pi$ (i.e. $\xi = \widehat{\pi}(\xi_1)$) and we can conclude that $\ker \widehat{i} = \text{im } \widehat{\pi}$. So the sequence is exact and $\text{im } \widehat{\pi} = \ker \widehat{i} = A_{\widehat{G}}(H)$.

To show that $\widehat{\pi}$ is proper it suffices to apply Lemma 7.16.

If H is open or compact, (a) implies that \widehat{i} is surjective. It remains to show that \widehat{i} is open. If H is compact then \widehat{H} is discrete by Example 7.1(2), so \widehat{i} is obviously open. If H is open, let K be a compact neighborhood of 0 in G such that $K \subseteq H$. Then $W = W_{\widehat{G}}(K, \overline{\Lambda_4})$ is a compact neighborhood of 0 in \widehat{G} . Since \widehat{i} is surjective, $V = \widehat{i}(W) = W_{\widehat{H}}(K, \overline{\Lambda_4})$ is a neighborhood of 0 in \widehat{H} . Now $M = \langle W \rangle$ and $M_1 = \langle V \rangle$ are open compactly generated subgroups respectively of \widehat{G} and \widehat{H} , and $\widehat{i}(M) = M_1$. Since M is σ -compact by Lemma 4.12, we can apply Theorem 4.9 to the continuous surjective homomorphism $\widehat{i}|_M : M \rightarrow M_1$ and so also \widehat{i} is open. \square

The lemma gives these immediate corollaries:

Corollary 7.29. *Let G be a locally compact abelian group and let H be a closed subgroup of G . Then $\widehat{G/H} \cong A_{\widehat{G}}(H)$. Moreover, if H is open or compact, then $\widehat{H} \cong \widehat{G}/A_{\widehat{G}}(H)$.*

The next corollary says that the duality functor preserves proper exactness for some sequences.

Corollary 7.30. *If the sequence $0 \rightarrow G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \rightarrow 0$ in \mathcal{L} is proper exact, with G_1 compact or G_2 discrete, then $0 \rightarrow \widehat{G_2} \xrightarrow{\widehat{h}} \widehat{G} \xrightarrow{\widehat{f}} \widehat{G_1} \rightarrow 0$ is proper exact with the same property.*

Now we can prove the Pontryagin-van Kampen duality theorem, namely ω is a natural equivalence from $id_{\mathcal{L}}$ to $\widehat{\cdot}: \mathcal{L} \rightarrow \mathcal{L}$.

Theorem 7.31. *If G is a locally compact abelian group, then ω_G is a topological isomorphism of G onto $\widehat{\widehat{G}}$.*

Proof. We know by Proposition 7.23 that ω is a natural transformation from $id_{\mathcal{L}}$ to $\widehat{\cdot}: \mathcal{L} \rightarrow \mathcal{L}$. Our plan is to chase the given locally compact abelian group G into an appropriately chosen proper exact sequence

$$0 \rightarrow G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \rightarrow 0$$

in \mathcal{L} , with G_1 compact or G_2 discrete, such that G_1 and G_2 satisfy the duality theorem. By Corollary 7.30 the sequences

$$0 \rightarrow \widehat{G_2} \xrightarrow{\widehat{h}} \widehat{G} \xrightarrow{\widehat{f}} \widehat{G_1} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \widehat{\widehat{G_1}} \xrightarrow{\widehat{\widehat{f}}} \widehat{\widehat{G}} \xrightarrow{\widehat{\widehat{h}}} \widehat{\widehat{G_2}} \rightarrow 0$$

are proper exact. According to Proposition 7.23 the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_1 & \xrightarrow{f} & G & \xrightarrow{h} & G_2 \longrightarrow 0 \\ & & \omega_{G_1} \downarrow & & \downarrow \omega_G & & \downarrow \omega_{G_2} \\ 0 & \longrightarrow & \widehat{\widehat{G_1}} & \xrightarrow{\widehat{\widehat{f}}} & \widehat{\widehat{G}} & \xrightarrow{\widehat{\widehat{h}}} & \widehat{\widehat{G_2}} \longrightarrow 0 \end{array}$$

According to Theorem 6.19, ω_{G_1} , ω_G , ω_{G_2} are injective. Moreover, ω_{G_1} and ω_{G_2} are surjective by our choice of G_1 and G_2 . Then ω_G must be surjective too. (Indeed, if $x \in \ker \widehat{h}$, then there exists $y \in \omega_G(G)$ with $\widehat{h}(x) = \widehat{h}(y)$, because $\widehat{h}(\omega_G(G)) = \widehat{\widehat{G_2}}$. Now $y - x \in \ker \widehat{h} \subseteq \omega_G(G)$ and so $x \in y + \omega_G(G) = \omega_G(G)$.)

If G is locally compact abelian and compactly generated, by Proposition 6.18 we can choose G_1 compact and G_2 elementary locally compact abelian. Then G_1 and G_2 satisfy the duality theorem by Theorems 7.25 and 7.26, hence ω_G is surjective. Since ω_G is a continuous isomorphism and G is σ -compact, we conclude with Theorem 4.9 that ω_G is a topological isomorphism.

In the general case of locally compact abelian group G , we can take an open compactly generated subgroup G_1 of G . This will produce a proper exact sequence $0 \rightarrow G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \rightarrow 0$ with G_1 compactly generated and $G_2 \cong G/G_1$ discrete. By the previous case ω_{G_1} is a topological isomorphism and ω_{G_2} is an isomorphism thanks to Theorem 7.25. Therefore ω_G is a continuous isomorphism.

Moreover $\omega_G \upharpoonright_{f(G_1)}: f(G_1) \rightarrow \widehat{\widehat{f(G_1)}}$ is a topological isomorphism (as ω_{G_1} , $f: G_1 \rightarrow f(G_1)$ and $\widehat{f}: \widehat{G_1} \rightarrow \widehat{\widehat{f(G_1)}}$ are topological isomorphisms) and $f(G_1)$ and $\widehat{\widehat{f(G_1)}}$ are open subgroups respectively of G and $\widehat{\widehat{G}}$. Thus ω_G is a topological isomorphism. \square

Our last aim is to prove that the annihilators define an inclusion-inverting bijection between the family of all closed subgroups of a locally compact group G and the family of all closed subgroups of \widehat{G} . We use that fact that one can identify G and $\widehat{\widehat{G}}$ by the topological isomorphism ω_G . In more precise terms:

Exercise 7.32. *Let G be a locally compact abelian group and Y be a subset of \widehat{G} . Then $A_{\widehat{\widehat{G}}}(Y) = \omega_G(A_G(Y))$.*

Lemma 7.33. *Let G be a locally compact abelian group and H a closed subgroup of G . If $a \in G \setminus H$ then there exists $\chi \in A(H)$ such that $\chi(a) \neq 0$.*

Proof. Let $\rho: \widehat{G/H} \rightarrow A(H)$ be the topological isomorphism of Corollary 7.29. By Theorem 6.19 there exists $\psi \in \widehat{G/H}$ such that $\psi(a + H) \neq 0$. Therefore $\chi = \rho(\psi) \in A(H)$ and $\chi(a) = \rho(\psi)(a) = \psi(a + H) \neq 0$. \square

Corollary 7.34. *If G is a locally compact abelian group and H a closed subgroup of G , then*

$$H = A_G(A_{\widehat{G}}(H)) = \omega_G^{-1}(A_{\widehat{G}}(A_{\widehat{G}}(H))).$$

Proof. The first equality follows immediately from the above lemma.

The last equality follows from the equality $H = A_G(A_{\widehat{G}}(H))$ and Exercise 7.32. \square

By Lemma 7.29 the equality $H = A_G(A_{\widehat{G}}(H))$ holds if and only if H is a closed subgroup of G .

Proposition 7.35. *Let G be a locally compact abelian group and H a closed subgroup of G . Then $\widehat{H} \cong \widehat{G}/A(H)$.*

Proof. Since $H = \omega_G^{-1}(A_{\widehat{G}}(A_{\widehat{G}}(H)))$ by Lemma 7.34 we have a topological isomorphism ϕ from H to $\widehat{G}/A(H)$ given by $\phi(h)(\alpha + A(H)) = \alpha(h)$ for every $h \in H$ and $\alpha \in \widehat{G}$. This gives rise to another topological isomorphism $\widehat{\phi} : \widehat{\widehat{G}/A(H)} \rightarrow \widehat{H}$. By Pontryagin's duality theorem 7.31 $\omega_{\widehat{G}/A(H)}$ is a topological isomorphism from $\widehat{G}/A(H)$ to $\widehat{\widehat{G}/A(H)}$. The composition gives the desired isomorphism. \square

Finally, let us resume for reader's benefit some of the most relevant points of Pontryagin-van Kampen duality theorem established so far:

Theorem 7.36. *Let G be a locally compact abelian group. Then \widehat{G} is a locally compact abelian group and:*

- (a) *the correspondence $H \mapsto A_{\widehat{G}}(H)$, $N \mapsto A_G(N)$, where H is a closed subgroup of G and N is a closed subgroup of \widehat{G} , defines an order-inverting bijection between the family of all closed subgroups of G and the family of all closed subgroups of \widehat{G} ;*
- (b) *for every closed subgroup H of G the dual group \widehat{H} is isomorphic to $\widehat{G}/A(H)$, while $A(H)$ is isomorphic to the dual \widehat{G}/\widehat{H} ;*
- (c) *$\omega_G : G \rightarrow \widehat{\widehat{G}}$ is a topological isomorphism;*
- (d) *G is compact (resp., discrete) if and only if \widehat{G} is discrete (resp., compact);*

Proof. The first sentence is proved in Theorem 7.2. (a) is Corollary 7.34 while (b) is Proposition 7.35. (c) is Theorem 7.31. To prove (d) apply Theorem 7.31 and Lemma 7.1. \square

Using the full power of the duality theorem one can prove the following structure theorem on compactly generated locally compact abelian groups.

Theorem 7.37. *Let G be a locally compact compactly generated abelian group. Prove that $G \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$, where $n, m \in \mathbb{N}$ and K is a compact abelian group.*

Proof. According to Theorem 6.18 there exists a compact subgroup K of G such that G/K is an elementary locally compact abelian group. Taking a bigger compact subgroup one can get the quotient G/K to be of the form $\mathbb{R}^n \times \mathbb{Z}^m$ for some $n, m \in \mathbb{N}$. Now the dual group \widehat{G} has an open subgroup $A(K) \cong \widehat{G/K} \cong \mathbb{R}^n \times \mathbb{T}^m$. Since this subgroup is divisible, one has $\widehat{G} \cong \mathbb{R}^n \times \mathbb{T}^m \times D$, where $D \cong \widehat{G}/A(K)$ is discrete and $D \cong \widehat{K}$. Taking duals gives $G \cong \widehat{\widehat{G}} \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$. \square

Making sharp use of the annihilators one can prove the structure theorem on locally compact abelian groups (see [67, 36] for a proof).

Theorem 7.38. *Let G be a locally compact abelian group. Then $G \cong \mathbb{R}^n \times G_0$, where G_0 is a closed subgroup of G containing an open compact subgroup K .*

As a corollary one can prove:

Corollary 7.39. *Every locally compact abelian group is isomorphic to a subgroup of a group of the form $\mathbb{R}^n \times D \times K$, where $n \in \mathbb{N}$, D is a discrete abelian group and K is a compact abelian group.*

Exercise 7.40. *Let G be a locally compact abelian group. Prove that for $\chi_1, \dots, \chi_n \in \widehat{G}$ and $\delta > 0$ one has*

$$U_G(\chi_1, \dots, \chi_n; \delta) = \omega_G^{-1}(W_{\widehat{G}}(\{\chi_1, \dots, \chi_n\}, U)),$$

where U is the neighborhood of 0 in $\mathbb{T} \cong \mathbb{S}$ determined by $|\text{Arg} z| < \delta$.

Exercise 7.41. Let G be a compact connected abelian group. Prove that $t(G)$ is dense in G iff \widehat{G} is reduced. Deduce that every compact connected abelian group G has the form $G \cong G_1 \times \mathbb{Q}^\alpha$ for some cardinal α , where the compact subgroup G_1 coincides with the closure of the subgroup $t(G)$ of G .

(Hint. Note first that \widehat{G} is torsion-free. Deduce that \widehat{G} is reduced iff $\bigcap_{n=1}^\infty n\widehat{G} = 0$. Show that this equality is equivalent to density of $t(G) = \bigcup_{n=1}^\infty G[n]$ in G . To prove the second assertion consider the torsion-free dual \widehat{G} and its decomposition $\widehat{G} = d(\widehat{G}) \times R$, where R is a reduced subgroup of \widehat{G} . Now apply the first part and the isomorphism $G \cong \widehat{\widehat{G}}$.)

Exercise 7.42. Give example of a reduced abelian group G such that $\bigcap_{n=1}^\infty nG \neq 0$.

(Hint. Fix a prime number p and take an appropriate quotient of the group $\bigoplus_{n=1}^\infty \mathbb{Z}(p^n)$.)

8 Appendix

8.1 Uniqueness of Pontryagin-van Kampen duality

For topological abelian groups G, H denote by $\text{Chom}(G, H)$ the group of all continuous homomorphisms $G \rightarrow H$ equipped with the compact-open topology. It was pointed out already by Pontryagin that the group \mathbb{T} is the unique locally compact group L with the property $\text{Chom}(\text{Chom}(\mathbb{T}, L), L) \cong \mathbb{T}$ (note that this is much weaker than asking $\text{Chom}(-, L)$ to define a duality of \mathcal{L}). Much later Roeder [91] proved that Pontryagin-van Kampen duality is the unique functorial duality of \mathcal{L} , i.e., the unique involutive contravariant endofunctor $\mathcal{L} \rightarrow \mathcal{L}$. Several years later Prodanov [85] rediscovered this result in the following much more general setting. Let R be a locally compact commutative ring and \mathcal{L}_R be the category of locally compact topological R -modules. A *functorial duality* $\# : \mathcal{L}_R \rightarrow \mathcal{L}_R$ is a contravariant functor such that $\# \cdot \#$ is naturally equivalent to the identity of \mathcal{L}_R and for each morphism $f : M \rightarrow N$ in \mathcal{L}_R and $r \in R$ $(rf)^\# = rf^\#$ (where, as usual, rf is the morphism $M \rightarrow N$ defined by $(rf)(x) = rf(x)$). It is easy to see that the restriction of the Pontryagin-van Kampen duality functor on \mathcal{L}_R is a functorial duality, since the Pontryagin-van Kampen dual \widehat{M} of an $M \in \mathcal{L}_R$ has a natural structure of an R -module. So *there is always a functorial duality in \mathcal{L}_R* . This stimulated Prodanov to raise the question *how many* functorial dualities can carry \mathcal{L}_R and extend this question to other well known dualities and adjunctions, such as Stone duality¹³, the spectrum of a commutative rings [86], etc. at his Seminar on dualities (Sofia University, 1979/83). Uniqueness of the functorial duality was obtained by L. Stoyanov [93] in the case of a compact commutative ring R . In 1988 Gregorio [56] extended this result to the general case of compact (not necessarily commutative) ring R (here left and right R -modules should be distinguished, so that the dualities are no more *endofunctors*). Later Gregorio jointly with Orsatti [58] offered another approach to this phenomenon.

Surprisingly the case of a discrete ring R turned out to be more complicated. For each functorial duality $\# : \mathcal{L}_R \rightarrow \mathcal{L}_R$ the module $T = R^\#$ (the *torus* of the duality $\#$) is compact and for every $X \in \mathcal{L}_R$ the module $\Delta_T(X) := \text{Chom}_R(X, T)$ of all continuous R -module homomorphisms $X \rightarrow T$, equipped with the compact-open topology, is algebraically isomorphic to $X^\#$. The duality $\#$ is called *continuous* if for each X this isomorphism is also topological, otherwise $\#$ is *discontinuous*. Clearly, continuous dualities are classified by their tori, which in turn can be classified by means of the Picard group $\text{Pic}(R)$ of R . In particular, the unique continuous functorial duality on \mathcal{L}_R is the Pontryagin-van Kampen duality if and only if $\text{Pic}(R) = 0$ ([29, Theorem 5.17]). Prodanov [85] (see also [36, §3.4]) proved that every functorial duality on $\mathcal{L} = \mathcal{L}_{\mathbb{Z}}$ is continuous, which in view of $\text{Pic}(\mathbb{Z}) = 0$ gives another proof of Roeder's theorem of uniqueness. Continuous dualities were studied in the non-commutative context by Gregorio [57]. While the Picard group provides a good tool to measure the failure of uniqueness for continuous dualities, there is still no efficient way to capture it for discontinuous ones. The first example of a discontinuous duality was given in [29, Theorem 11.1]. Discontinuous dualities of $\mathcal{L}_{\mathbb{Q}}$ and its subcategories are discussed in [34]. It was conjectured by Prodanov that in case R is an algebraic number ring uniqueness of dualities is available if and only if R is a principal ideal domain. This conjecture was proved to be true for real algebraic number rings, but Prodanov's conjecture was shown to fail in case R is an order in an imaginary quadratic number field [25].

We will not touch other well-known dualities for module categories such as Morita duality (see [76]) or more general setting of dualities of (representable dualities, adjunctions rather than involutions, etc. [40], [41] and [83]).

¹³his conjecture that the Stone duality is the unique functorial duality between compact totally disconnected Hausdorff spaces and Boolean algebras was proved to be true by Dimov [39].

8.2 Non-abelian or non-locally compact groups

The Pontryagin-van Kampen duality theorem was extended to some non-locally compact abelian topological groups (e.g., infinite powers of the reals, the underlying additive groups of certain linear topological spaces, etc.). A characterization of the abelian topological groups admitting duality were proposed by Venkatamaran [95] and Kye [73], but they contained flaws. These gaps were removed in the recent paper of Hernández [63]. An important class of abelian groups (nuclear groups) were introduced and studied in the monograph [6] (see also [5]) in relation to the duality theorem. Further reference can be found also in [21, 51, 65].

We do not discuss here non-commutative versions of duality for locally compact groups. The difficulties arise already in the compact case – there is no appropriate (or at least, comfortable) structure on the set of irreducible unitary representations of a compact non-abelian group. The reader is referred to [67] for a historical panorama of this trend (Tanaka-Klein duality, etc.). In the locally compact case one should see the pioneering paper of H. Chu [22], as well as the monograph of Heyer [68] (see also [69]). In the recent survey of Galindo, Hernández, and Wu [53] the reader can find the last achievements in this field (see also [64]).

8.3 Relations to the topological theory of topological groups

The Pontryagin-van Kampen dual of a compact abelian group K carries a lot of useful information about the topology of H . For example,

- $w(K) = |\widehat{K}|$,
- $d(K) = \log |\widehat{K}| = \min\{\kappa : 2^\kappa \geq |\widehat{K}|\}$,
- K is connected iff \widehat{K} is torsion-free,
- K is totally connected iff \widehat{K} is torsion,
- $c(K) = A(t(\widehat{K}))$, where $t(\widehat{K})$ is the torsion subgroup of \widehat{K} ,
- $\dim K = r_0(\widehat{K})$,
- $H^1(K, \mathbb{Z}) \cong \widehat{K}$ if K is connected (here $H^1(K, \mathbb{Z})$ denotes the first cohomology group),
- for two compact connected abelian groups K_1 and K_2 the following are equivalent: (i) K_1 and K_2 are homotopically equivalent as topological spaces; (ii) K_1 and K_2 are homeomorphic as topological spaces; (iii) $\widehat{K}_1 \cong \widehat{K}_2$; (iv) $K_1 \cong K_2$ as topological groups.

The first equality can be generalized to $w(K) = w(\widehat{K})$ for all locally compact abelian groups K .

The Pontryagin-van Kampen duality can be used to easily build the *Bohr compactification* bG of a locally compact abelian group G (this is the reflection of G into the subcategory of compact abelian groups). In the case when G is discrete, bG is simply the completion of $G^\#$, the group G equipped with its Bohr topology. One can prove that $bG \cong \widehat{\widehat{G}_d}$, where \widehat{G}_d denotes the group \widehat{G} equipped with the discrete topology. For a comment on the non-abelian case see [28, 53].

Many nice properties of $\mathbb{Z}^\#$ can be found in Kunen and Rudin [72]. For a fast growing sequence (a_n) in $\mathbb{Z}^\#$ the range is a closed discrete set of $\mathbb{Z}^\#$ (see [53] for further properties of the lacunary sets in $\mathbb{Z}^\#$), whereas for a polynomial function $n \mapsto a_n = P(n)$ the range has no isolated points [72, 44, Theorem 5.4]. Moreover, the range $P(\mathbb{Z})$ is closed when $P(x) = x^k$ is a monomial. For quadratic polynomials $P(x) = ax^2 + bx + c$, $(a, b, c, \in \mathbb{Z}, a \neq 0)$ the situation is already more complicated: the range $P(\mathbb{Z})$ is closed iff there is at most one prime that divides a , but does not divide b [72, 44, Theorem 5.6]. This leaves open the general question [26, Problem 954].

Problem 8.1. Characterize the polynomials $P(x) \in \mathbb{Z}[x]$ such that $P(\mathbb{Z})$ is closed in $\mathbb{Z}^\#$.

8.4 Relations to dynamical systems

Among the known facts relating the dynamical systems with the topic of these notes let us mention just two.

- A compact group G admits ergodic translations $T_a(x) = ax$ iff G is monothetic. The ergodic rotations T_a of G are precisely those defined by a topological generator a of G .
- A continuous surjective endomorphism $T : K \rightarrow K$ of a compact abelian group is ergodic iff the dual $\widehat{T} : \widehat{K} \rightarrow \widehat{K}$ has no periodic points except $x = 0$.

The Pontryagin-van Kampen duality has an important impact also on the computation of the entropy of endomorphisms of (topological) abelian groups. Adler, Konheim, and McAndrew introduced the notion of topological entropy of continuous self-maps of compact topological spaces in the pioneering paper [1]. In 1975 Weiss [98] developed the definition of entropy for endomorphisms of abelian groups briefly sketched in [1]. He

called it “algebraic entropy”, and gave detailed proofs of its basic properties. His main result was that the topological entropy of a continuous endomorphism ϕ of a profinite abelian group coincides with the algebraic entropy of the adjoint map $\widehat{\phi}$ of ϕ (note that pro-finite abelian groups are precisely the Pontryagin duals of the torsion abelian groups).

In 1979 Peters [82] extended Weiss’s definition of entropy for automorphisms of a discrete abelian group G . He generalized Weiss’s main result to metrizable compact abelian groups, relating again the topological entropy of a continuous automorphism of such a group G to the entropy of the adjoint automorphism of the dual group \widehat{G} . The definition of entropy of automorphisms given by Peters is easily adaptable to endomorphisms of Abelian groups, but it remains unclear whether his theorem can be extended to the computation of the topological entropy of a continuous endomorphism of compact abelian groups.

References

- [1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, TAMS 114 (1965), 309–319.
- [2] D. Alcaraz, D. Dikranjan, M. Sanchis, *Infinitude of Bowen's entropy for group endomorphisms*, Preprint.
- [3] D. L. Armacost, *The structure of locally compact Abelian groups*, Monographs and Textbooks in Pure and Applied Mathematics, **68**, Marcel Dekker, Inc., New York, 1981.
- [4] G. Artico, V. Malykhin and U. Marconi, *Some large and small sets in topological groups*, Math. Pannon. 12 (2001), no. 2, 157–165.
- [5] L. Aussenhofer, *Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups*, Dissertationes Math. (Rozprawy Mat.) 384 (1999), 113 pp.
- [6] W. Banaszczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Mathematics **1466**, Springer-Verlag, Berlin (1991).
- [7] G. Barbieri, D. Dikranjan, C. Milan, H. Weber, *Answer to Raczkowski's quests on convergent sequences*, Top. Appl. **1321** (2003), 89–101.
- [8] G. Barbieri, D. Dikranjan, C. Milan, H. Weber, *Convergent sequences in precompact group topologies*, Appl. Gen. Topol. 6 (2005), no. 2, 149–169.
- [9] G. Barbieri, D. Dikranjan, C. Milan and H. Weber, *Topological torsion related to some recursive sequences of integers*, Math. Nachrichten, to appear.
- [10] G. Baumslag, A. Myasnikov and V. Remeslennikov, *Algebraic geometry over groups. I. Algebraic sets and ideal theory*, J. Algebra 219 (1999), no. 1, 16–79..
- [11] A. Bella and V. Malykhin, *On certain subsets of a group*, Questions Answers Gen. Topology, 17:183–197, 1999.
- [12] A. B  r  , *Characterizing sets for subgroups of compact groups*, J. Number Theory, to appear.
- [13] A. B  r  , J.-M. Deshouillers and V. S  s, *Good approximation and characterization of subgroups of \mathbf{R}/\mathbf{Z}* , Studia Sci. Math. Hungar. **38** (2001), 97–113.
- [14] A. B  r   and V. S  s, *Strong characterizing sequences in simultaneous Diophantine approximation*, J. Number Theory **99** (2003), no. 2, 405–414.
- [15] M. Beigl    , *Filters in number theory and combinatorics*, PhD Dissertation, Technische Universit  t Wien, Wien, October 2004.
- [16] M. Beigl    , *Strong characterizing sequences of countable subgroups*, J. Number Theory, submitted.
- [17] M. Beigl    , C. Steineder and R. Winkler, *Sequences and filters of characters characterizing subgroups of compact abelian groups*, Topology Appl. 153 (2006), no. 11, 1682–1695.
- [18] J. P. Borel, *Sous-groupes de \mathbf{R} li  s    r  partition modulo 1 de suites* Ann. Fac. Sci. Toulouse Math. (5) 5 (1983), no. 3-4, 217–235.
- [19] J. P. Borel, *Sur certains sous-groupes de \mathbf{R} li  s    la suite des factorielles.*, Colloq. Math. 62 (1991), no. 1, 21–30.
- [20] R. Bryant, *The verbal topology of a group*, J. Algebra 48 (1977), no. 2, 340–346.
- [21] M.J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math. (Basel) 70 (1998), no. 1, 22–28.
- [22] H. Chu, *Compactification and duality of topological groups*, Trans. Amer. Math. Soc. 123 (1966), 310–324.
- [23] W.W. Comfort and Ross, *Topologies induced by groups of characters*, Fund. Math. 55 (1964) 283–291.
- [24] M. Cotlar and R. Ricabarra, *On the existence of characters in topological groups*, Amer. J. Math. 76 (1954), 375–388.

- [25] D. Dikranjan, *Uniqueness of dualities*, in: A. Facchini and C. Menini ed., Abelian Groups and Modules, Proc. of the Padova Conference, Padova, Italy, June 23-July 1, 1994, Mathematics and its Applications, vol. 343, Kluwer Academic Publishers, Dordrecht-Boston-London, The Netherlands, 1995, pp. 123-133.
- [26] D. Dikranjan, *Topologically torsion elements of topological groups*, Topology Proc., **26**, 2001/2002, pp. 505–532.
- [27] D. Dikranjan, *Topological groups*, in Klaas Pieter Hart, Jun-iti Nagata and Jerry E. Vaughan editors, *Encyclopedia of general topology*, Edited by Elsevier Science Publishers, B.V., Amsterdam, 2004. x+526 pp.
- [28] D. Dikranjan, *Closure operators in topological groups related to von Neumann’s kernel*, Topology and its Applications, 153 no. 11, (2006), 1930–1955 .
- [29] D. Dikranjan and A. Orsatti, *On an unpublished manuscript of Ivan Prodanov concerning locally compact modules and their dualities*, Comm. Algebra, 17 (1989) 2739-2771.
- [30] D. Dikranjan and I. Protasov, *Every infinite group can be generated by a P -small subset*, Appl. Gen. Topol., Volume 7, No. 2 (2006) 265-268.
- [31] D. Dikranjan and R. Di Santo, *On Armacost’s quest on topologically torsion elements*, Communications Algebra, **32** (2004), 133–146.
- [32] D. Dikranjan and R. Di Santo, *A characterization of the circle group via uniqueness of roots*, Topology and its Applications (2007), to appear.
- [33] D. Dikranjan and K. Kunen, *Characterizing Subgroups of Compact abelian Groups*, J. Pure Appl. Algebra, 208 (2006) 285–291.
- [34] D. Dikranjan and C. Milan, *Dualities of locally compact modules over the rationals*, J. Algebra, 256 (2002) pp. 433–466.
- [35] D. Dikranjan, C. Milan and A. Tonolo *A characterization of the MAP abelian groups*, J. Pure Appl. Algebra, **197** (2005) 23–41.
- [36] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Pure and Applied Mathematics, Vol. **130**, Marcel Dekker Inc., New York-Basel, 1989.
- [37] D. Dikranjan and D. Shakhmatov, *Reflection principle characterizing groups in which unconditionally closed sets are algebraic*, J. Group Theory, to appear.
- [38] D. Dikranjan and Umberto Zannier, *On a property of the adele ring of the rationals*, Rings, Modules, Algebras and Abelian Groups, Proceedings of the Algebra Conference – Venezia, “Lecture Notes in Pure and Applied Mathematics”, vol. 236, Marcel Dekker 2004, pp. 133-137.
- [39] G. Dimov, *An axiomatic characterization of Stone duality*, Serdika **10** (1984) 165–173.
- [40] G. Dimov and W. Tholen, *Characterization of representable dualities*, Proceedings of the International Conference on Categorical Topology, Prague, 1988 (World Scientific, Singapore 1989) 336–357.
- [41] G. Dimov and W. Tholen, *Group of dualities*, Trans. Amer. Math. Soc. **336**, n.2 (1993), 901–913.
- [42] H.G. Eggleston, *Sets of fractional dimensions which occur in some problems of number theory*, Proc. London Math. Soc. (2) 54, (1952) 42–93.
- [43] R. Ellis and H. B. Keynes, *Bohr compactifications and a result of Følner*, Israel J. Math. 12 (1972), 314–330.
- [44] R. Engelking, *General Topology*, Heldermann Verlag, Berlin 1989.
- [45] E. Følner, *Generalization of a theorem of Bogoljuboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups*, Math. Scand. 2 (1954), 518.
- [46] L. Fuchs, *Infinite Abelian groups*, vol. I, Academic Press New York and London, 1973.
- [47] L. Fuchs and K. -H. Hofmann, *Extensions of compact abelian groups by discrete ones and their duality theory. I*, J. Algebra 196 (1997), no. 2, 578–594.

- [48] L. Fuchs and K. -H. Hofmann, *Extensions of compact abelian groups by discrete ones and their duality theory. II*, in: D. Dikranjan and L. Salce editors, *Abelian groups, module theory, and topology* (Padua, 1997), 205–225, Lecture Notes in Pure and Appl. Math., 201, Dekker, New York, 1998.
- [49] H. Fujita and D. Shakhmatov, *A characterization of compactly generated metric groups*, Proc. Amer. Math. Soc. 131 (2003), no. 3, 953–961.
- [50] H. Fujita and D. Shakhmatov, *Topological groups with dense compactly generated subgroups*, Appl. Gen. Topol. 3 (2002), no. 1, 85–89.
- [51] J. Galindo and S. Hernández, *Pontryagin-van Kampen reflexivity for free abelian topological groups*, Forum Math. 11 (1999), no. 4, 399–415.
- [52] J. Galindo and S. Hernández, *Interpolation sets and the Bohr topology of locally compact groups*, Adv. Math. 188 (2004), no. 1, 51–68.
- [53] J. Galindo, S. Hernández, and T. S. Wu, *Recent results and open questions relating Chu duality and Bohr compactifications of locally compact groups*, Open problems in topology II (E. Pearl, ed.), Elsevier Science.
- [54] E. Glasner, *On minimal actions of Polish groups*, Topology Appl. 85 (1998), 119–125.
- [55] C. C. Graham and K. E. Hare, *ε -Kronecker and I_0 -sets in abelian groups. III. Interpolation by measures on small sets*, Studia Math. 171 (2005), no. 1, 1532.
- [56] E. Gregorio, *Dualities over compact rings*, Rend. Sem. Mat. Univ. Padova, **80** (1988), 151–174.
- [57] E. Gregorio, *Tori and continuous dualities*, Rend. Accad. Naz. dei XL, Memorie di Mat., **107**, vol. XIII, fasc. 13, (1989) 211–221.
- [58] E. Gregorio and A. Orsatti, *Uniqueness and existence of dualities over compact rings*, Tsukuba J. Math. **18**, n. 1 (1994) 39–61.
- [59] D. Grow, *Sidon sets and I_0 -sets*, Colloq. Math. 53 (1987), no. 2, 269270.
- [60] K. E. Hare and L. T. Ramsey, *I_0 -sets in non-abelian groups*, Math. Proc. Cambridge Philos. Soc. 135 (2003), no. 1, 81–98.
- [61] J. Hart and K. Kunen, *Limits in function spaces*, 151 (2005), no. 1-3, 157–168.
- [62] J. Hart and K. Kunen, *Limits in compact abelian groups*, Topology Appl. 153 (2006), no. 7, 991–1002.
- [63] S. Hernández, *Pontryagin duality for topological abelian groups*, Math. Z. Math. Z. 238 (2001), no. 3, 493–503.
- [64] S. Hernández, *Extension of continuous functions on product spaces, Bohr compactification and almost periodic functions*, Proceedings of the 18th Summer Conference on Topology and its Applications. Topology Proc. 28 (2004), no. 2, 527–540.
- [65] S. Hernández and V. Uspenskij, *Pontryagin duality for spaces of continuous functions*, J. Math. Anal. Appl. 242 (2000), no. 2, 135–144.
- [66] G. Hesse, *Zur Topologisierbarkeit von Gruppen*, Dissertation, Univ. Hannover, Hannover, 1979.
- [67] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Volume I, Springer-Verlag, Berlin-Göttingen-Heidelberg 1979.
- [68] H. Heyer, *Dualität lokalkompakter Gruppen*, Lecture Notes in Math., vol. 150, Springer, Berlin, 1970.
- [69] H. Heyer, *Groups with Chu duality*, Probability and information theory, II, Lecture Notes in Math., vol. 296, Springer, Berlin, 1973, pp. 181215.
- [70] K. H. Hofmann and S. Morris, *The structure of compact groups. A primer for the student—a handbook for the expert*, de Gruyter Studies in Mathematics, 25. Walter de Gruyter & Co., Berlin, 1998. xviii+835 pp.
- [71] C. Kraaikamp and P. Liardet, *Good approximations and continued fractions*, Proc. Amer. Math. Soc. 112 (1991), no. 2, 303–309.

- [72] K. Kunen and W. Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 1, 117–137.
- [73] S.-H. Kye, *Pontryagin duality in real linear topological spaces*, Chinese J. Math. **12** (1984), no. 2, 129–136.
- [74] G. Larcher, *A convergence problem connected with the continued fractions*, Proc. Amer. Math. Soc., **103** (1988), no. 3, 718–722.
- [75] G. W. Mackey, *Harmonic analysis as the exploitation of symmetry—a historical survey*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), no. 1, part 1, 543–698.
- [76] A. A. Markov, *On unconditionally closed sets*, In: Topology and Topological Algebra, Translations Series 1, vol. 8, pages 273–304. American Math. Society, 1962. Russian original in: Comptes Rendus Dokl. AN SSSR (N.S.) **44** (1944), 180–181.
- [77] K. Morita, *Duality for modules and its application to the theory of rings with minimum condition*, Sci. Report Tokyo Kyoiku Daigaku, **6** (1958), 83–142.
- [78] A. Yu. Olshanskii, *A remark on a countable nontopologized group*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., No. 3, p. 103 (1980);
- [79] A. Orsatti, *Introduzione ai groups astratti e topologici*, Quaderni dell’Unione Matematica Italiana, Pitagora Ed., Bologna, 1979.
- [80] V. Pestov, *Forty-plus annotated questions about large topological groups*, in: Open Problems in Topology II, Elliott Pearl Editor, Elsevier B.V., March, 2007, pp. 439 – 450.
- [81] J. Peters, *Entropy on discrete Abelian groups*, Adv. Math. **33** (1979), 1–13.
- [82] L. Pontryagin, *Topological groups*, Transl. Russian, Arlen Brown Gordon and Breach Science Publishers, Inc., New York-London-Paris 1966.
- [83] H. Porst and W. Tholen, *Concrete dualities*, Category theory at work (Bremen, 1990), Res. Exp. Math. **18**, (Heldermann, Berlin, 1991), 111–136.
- [84] Iv. Prodanov, *Elementary proof of the Peter-Weyl theorem*, (Russian) C. R. Acad. Bulgare Sci. **34** (1981), no. 3, 315–318.
- [85] Iv. Prodanov, *An axiomatic characterization of the Pontryagin duality*, Preprint, 1979.
- [86] Iv. Prodanov, *An abstract approach to the algebraic notion of spectrum*, (Russian) *Topology (Moscow, 1979)*. Trudy Mat. Inst. Steklov, **154** (1983), 200–208, Amer. Math. Soc. Transl. *Proceedings of the Steklov Institute of Mathematics* 1984, Issue
- [87] I. V. Protasov and T. Banach, *Ball Structures and Colorings of Graphs and Groups*, Matem. Stud. Monogr. Series, Vol 11, Lviv, 2003.
- [88] I. V. Protasov and E. G. Zelenyuk, *Topologies on abelian groups*, Math. USSR Izvestiya **37** (1991), 445–460. Russian original: Izvestia Akad. Nauk SSSR **54** (1990), 1090–1107.
- [89] I. Protasov, E. Zelenyuk, *Topologies on Groups Determined by Sequences*, Math. Stud. Monogr. Ser., **4**, 1999.
- [90] D. W. Roeder, *A characterization of unitary duality*, Trans. Amer. Math. Soc. **148** (1970), 129–135.
- [91] D. W. Roeder, *Functorial characterization of Pontryagin duality*, Trans. Amer. Math. Soc. **154** (1971), 151–175.
- [92] S. Shelah, *On a problem of Kurosh, Jonsson groups, and applications*, in Word Problems II, Ed. by S. I. Adian, W. W. Boone, and G. Higman (North-Holland, Amsterdam, 1980), pp. 373–394.
- [93] L. Stoyanov, *Dualities over compact commutative rings*, Rend. Accad. Naz. delle Sci. detta dei XL, Memorie di Mat. **101**, Vol. VII, fasc. 10 (1983), 155–176.
- [94] W. A. Veech, *The equicontinuous structure relation for minimal Abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732.

- [95] R. Venkataraman, *A characterization of Pontryagin duality*, Math. Z. 149 (1976), no. 2, 109–119.
- [96] Yuan Wang (editor), *The Goldbach conjecture*, Second edition, Series in Pure Mathematics, 4. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.
- [97] A. Weil, Basic number theory, *Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen*, Band 144, Springer-Verlag, Berlin Heidelberg New York, 1970.
- [98] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. system Theory, 8 (1974/75), no. 3, 243–248.
- [99] R. Winkler, *Ergodic group rotations, Hartman sets and Kronecker sequences*, Dedicated to Edmund Hlawka on the occasion of his 85th birthday. Monatsh. Math. **135** (2002), no. 4, 333–343.