Unbalanced multisection in the stochastic block model

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Joint work with Alex Wein (MIT)

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This talk: a robust, statistically strong, poly-time algorithm to find communities.

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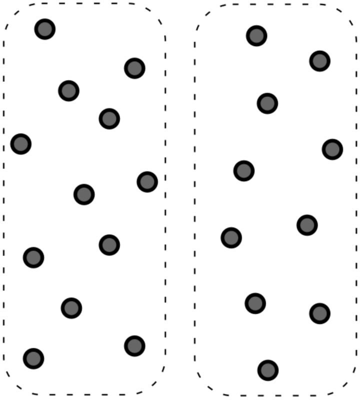

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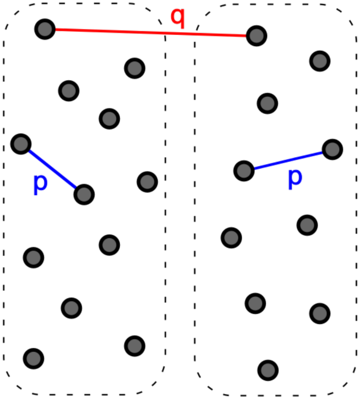

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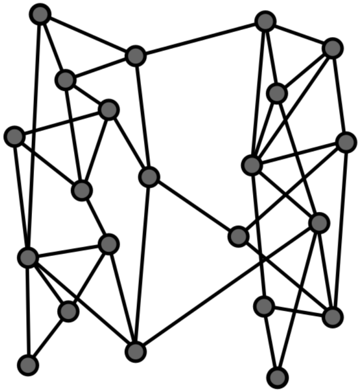

Goal: recover the planted communities (exactly or approximately)

Many algorithmic approaches

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- Belief propagation [DMKZ11, MNS14]
- Semidefinite programming [GW94, ABH14, HWX15, ABKK15]

Sharp thresholds

n — number of vertices
k — number of communities
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Denser regime

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Theorem [ABH14, MNS14, HWX15] With **two** equal-size communities, partial recovery is possible iff

$$(a - b)^2 > 2(a + b).$$

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- [RJM16]: planting just a few tetrahedra into a large SBM graph causes spectral clustering to fail.
- Q: How do we design algorithms that transfer to e.g. power-law graphs and many other models? Algorithms that are robust to model misspecification?

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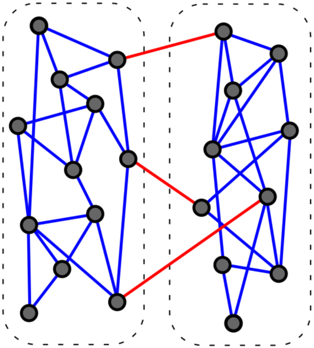

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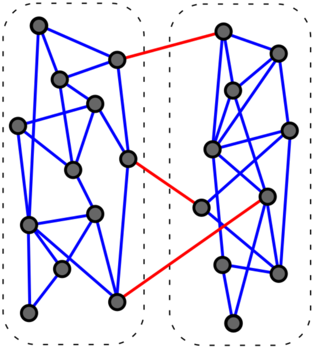

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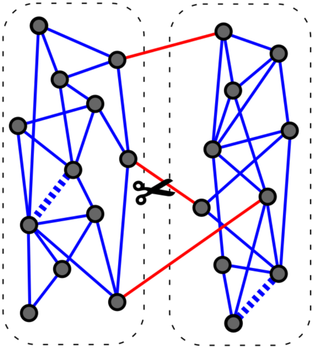

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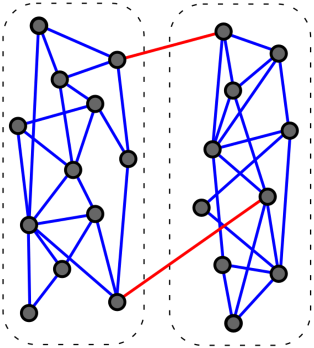

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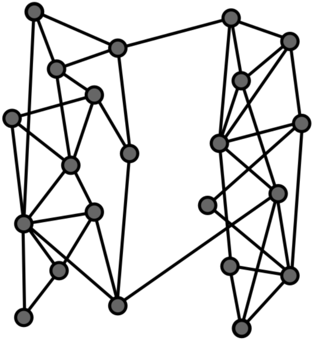

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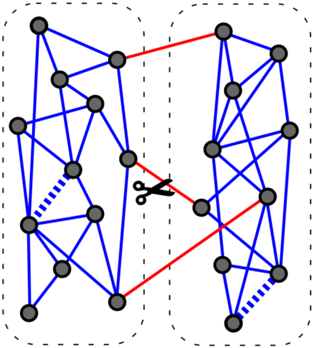
- Prevents algorithms from over-tuning to specific model statistics (degree distribution, spectrum, etc.)



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Reason: under a monotone change, the true SDP solution increases in value more than any other feasible solution.

If the true solution was optimal before the change, it remains optimal afterward.

Our convex program

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\text{s.t.} \quad & \forall u \quad X_{uu} = 1, \\
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- Allows k communities of different sizes.
- The hyperparameter ω is necessary for a robust algorithm.

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Theorem: our SDP achieves exact recovery w.h.p., and is robust to monotone changes, for all a, b, k and all community proportions for which the problem is statistically possible as $n \to \infty$.

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Depends on roughly the right choice of $\omega \approx \frac{a-b}{\log a - \log b} \frac{\log n}{n}$.

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Thanks for listening! Any questions?