Marginal likelihood for DAG-Gaussian

$$P(\underset{\sim}{x}_1,\dots,\underset{\sim}{x}_d \mid \mathcal{G}) = \prod_{j=1}^{d} P(\underset{\sim}{x}_j \mid X_{A_j}, \mathcal{G})$$

To define A_j, Let $\gamma_{kj} = 1$ if x_k is a parent of x_j & $\gamma_{kj} = 0$ otherwise. Then

$$A_j = \{k ; \gamma_{kj} = 1\}$$

Note $P(\underset{\sim}{x}_j \mid X_{A_j}, \mathcal{G})$ is referred to as the marginal likelihood of $\underset{\sim}{x}_j$

Let $\underset{\sim}{\beta}_j$ $(\beta_0, \underset{\sim}{\beta}_{A_j})$ be the set of regression co-efficients, & let σ_j^2 be the variance of the error term e_j in

$$\underset{\sim}{x}_j = \beta_0 + X_{A_j}\underset{\sim}{\beta}_{A_j} + \underset{\sim}{e}_j ; \quad e_j \sim N(0, \sigma_j^2)$$

then the marginal likelihood

$$P(\underset{\sim}{x}_j \mid X_{A_j}) = \int_0^\infty \int_{\mathbb{R}^{|A_j|+1}} P(\underset{\sim}{x}_j \mid \underset{\sim}{\beta}_j, \sigma_j^2, X_{A_j})\, P(\beta_j \mid \sigma_j^2)\, P(\sigma_j^2)\, d\beta_j\, d\sigma_j^2$$

where $|A_j|$ is the length of A_j

$$P(\underset{\sim}{x}_j \mid \underset{\sim}{\beta}_j, \sigma_j^2, X_{A_j}) = \frac{1}{(2\pi\sigma_j^2)^{n/2}} \exp -\frac{1}{2}\frac{(\underset{\sim}{x}_j - X_{A_j}\beta_j)'(\underset{\sim}{x}_j - X_{A_j}\beta_j)}{\sigma_j^2}$$

Assume

$$\underset{\sim}{\beta}_j \mid \sigma_j^2 \sim N(0, g\sigma_j^2 D_j) \quad D_j \text{ is a } p_j \times p_j \text{ matrix}$$

$$\sigma_j^2 \sim IG(a, b)$$

Then

$$P(\underset{\sim}{\beta}_j \mid \sigma_j^2) = \frac{1}{(2\pi g\sigma_j^2)^{p_j/2}\, |D_j|^{1/2}} \exp -\frac{1}{2}\frac{\beta_j' D_j^{-1} \underset{\sim}{\beta}_j}{g\sigma_j^2}$$

$$p_j = |A_j| + 1$$

$$\& \; P(\sigma_j^2) = \frac{b^a \,\sigma_j^{2-(1+a)}\, e^{-b/\sigma_j^2}}{\Gamma(a)}$$

We will first integrate over $\underset{\sim}{\beta}_j$ to get

$$P(\underset{\sim}{x}_j \mid X_{A_j}, \sigma_j^2) = \int_{\mathbb{R}^{|A_j|+1}} P(\underset{\sim}{x}_j \mid X_{A_j}, \sigma_j^2)\, P(\beta_j \mid \sigma_j^2)\, d\underset{\sim}{\beta}_j$$

$$= \int_{\mathbb{R}^{p_j}} \frac{1}{(2\pi\sigma_j^2)^{n/2}} \exp -\frac{1}{2}\frac{(\underset{\sim}{x}_j - X_{A_j}\beta_j)'(\underset{\sim}{x}_j - X_{A_j}\beta_j)}{\sigma_j^2} \times \frac{|D|^{-1/2}}{(2\pi\sigma_j^2 g)^{p_j/2}} \exp -\frac{1}{2}\frac{\beta_j' D_j^{-1}\beta_j}{g\sigma_j^2}$$

$$= \frac{|D|^{-1/2}\, g^{-(p_j/2)}}{(2\pi\sigma_j^2)^{n/2}} \exp -\frac{1}{2\sigma_j^2}\underset{\sim}{x}_j'\underset{\sim}{x}_j \;\times$$

$$\int \frac{1}{(2\pi\sigma_j^2)^{p_j/2}} \exp -\frac{1}{2}\left[\frac{\beta_j' X_{A_j}' X_{A_j}\beta_j}{\sigma_j^2} + \frac{\beta_j' D_j^{-1}\beta_j}{g\sigma_j^2} - \frac{2\beta_j' X_{A_j}' x_j}{\sigma_j^2}\right]$$

$$= \frac{|D|^{-1/2}\, g^{-p_j/2}}{(2\pi\sigma_j^2)^{n/2}} \exp -\frac{1}{2\sigma_j^2} x_j' x_j \;\times$$

$$\int \frac{1}{(2\pi\sigma_j^2)^{p_j/2}} \exp -\frac{1}{2\sigma_j^2}\left\{\underset{\sim}{\beta}_j'\left[\frac{X_{A_j}'X_{A_j}g + D_j^{-1}}{g}\right]\underset{\sim}{\beta}_j - 2\underset{\sim}{\beta}_j' X_{A_j}\underset{\sim}{x}_j\right\}$$

Consider the integrand of the equation above & let

$$V_{\beta_j} = \left[\frac{X_{A_j}'X_{A_j}g + D^{-1}}{g}\right]^{-1} \quad \& \text{ let}$$

$$\underset{\sim}{m}_{\beta_j} = V_{\beta_j} X_{A_j}' \underset{\sim}{x}_j \quad \text{then}$$

$$\exp -\frac{1}{2\sigma_j^2}\left\{\underset{\sim}{\beta}_j'\left[\frac{X_{A_j}'X_{A_j}g + D}{g}\right]\underset{\sim}{\beta}_j' - 2\underset{\sim}{\beta}_j' X_{A_j}' \underset{\sim}{x}_j\right\}$$

$$= \exp -\frac{1}{2\sigma_j^2}\left[\underset{\sim}{\beta}_j' V_{\beta_j}^{-1}\underset{\sim}{\beta}_j - 2\underset{\sim}{\beta}_j' V_{\beta_j}^{-1}\underset{\sim}{m}_{\beta_j}\right]$$

which is Gaussian in $\underset{\sim}{\beta}_j$ with variance $\sigma_j^2 V_{\beta_j}$ & mean $V_{\beta_j} X_{A_j}' \underset{\sim}{x}_j$

What is missing is a term $\dfrac{\underset{\sim}{m}_{\beta_j}' V_{\beta_j}^{-1} \underset{\sim}{m}_{\beta_j}}{\sigma_j^2}$

which we can add & subtract so that

$$\exp -\frac{1}{2\sigma_j^2}\left\{\underset{\sim}{\beta}_j'\left[\frac{X_{A_j}'X_{A_j}g + D_j^{-1}}{g}\right]\beta_j - 2\beta_j' X_{A_j}' \underset{\sim}{x}_j\right\} =$$

$$\exp -\frac{1}{2}\left\{\frac{\beta_j' V_\beta^{-1}\beta_j}{\sigma_j^2} - \frac{2\beta_j' V_\beta^{-1} m_{\beta_j}}{\sigma_j^2} + \frac{m_\beta' V_\beta^{-1} m_{\beta_j}}{\sigma_j^2} - \frac{m_{\beta_j}' V_{\beta_j}^{-1} m_{\beta_j}}{\sigma_j^2}\right\}$$

& hence the integral

$$\int_{\mathbb{R}^{p_j}} \frac{1}{(2\pi\sigma_j^2)^{p_j/2}} \exp -\frac{1}{2\sigma_j^2}\left\{\beta_j'\left(\frac{X_{A_j}'X_{A_j}g + D^{-1}}{g}\right)\beta_j - 2\beta_j' X_{A_j}'\underset{\sim}{x}_j\right\}$$

$$= |V_{\beta_j}|^{1/2} \exp -\frac{1}{2\sigma_j^2} - \underset{\sim}{m}_{\beta_j}' V_{\beta_j}^{-1} \underset{\sim}{m}_{\beta_j} \;\times$$

$$\int_{\mathbb{R}^{p_j}} \frac{1}{(2\pi\sigma_j^2)^{p/2}\, |V_{\beta_j}|^{1/2}} \exp -\frac{1}{2\sigma_j^2}(\beta_j - \underset{\sim}{m}_{\beta_j})' V_{\beta_j}^{-1}(\beta_j - m_{\beta_j})$$

$$= |V_\beta|^{1/2} \exp -\frac{1}{2\sigma_j^2}(m_\beta V_{\beta_j}^{-1} m_\beta)$$

$$\& \therefore P(\underset{\sim}{x}_j \mid X_{A_j}, \sigma_j^2) = \frac{|D|^{-1/2}}{(2\pi\sigma_j^2)^{n/2}}\, g^{-p_j/2} \;\times$$

$$\exp -\frac{1}{2\sigma_j^2}(\underset{\sim}{x}_j' x_j - m_{\beta_j}' V_{\beta_j}^{-1} m_\beta)$$

$$\& \; \underset{\sim}{m}_{\beta_j}' V_{\beta_j}^{-1} \underset{\sim}{m}_{\beta_j} = \underset{\sim}{x}_j' X_{A_j} V_{\beta_j} X_{A_j}' \underset{\sim}{x}_j$$

$$\therefore P(\underset{\sim}{x}_j \mid X_{A_j}, \sigma_j^2) = \frac{|D|^{-1/2}}{(2\pi\sigma_j^2)^{n/2}}\, g^{-\frac{p_j}{2}} \;\times$$

$$\exp -\frac{1}{2\sigma_j^2}(\underset{\sim}{x}_j' H_j \underset{\sim}{x}_j)$$

$$H_j = I - X_{A_j} V_{\beta_j} X_{A_j}'$$

with

$$V_{\beta_j} = \left(\frac{X_{A_j}'X_{A_j}g + D_j^{-1}}{g}\right)^{-1}$$

$$= g(g X_{A_j}' X_{A_j} + D_j^{-1})^{-1}.$$

Integrating over σ_j^2

$$P(\underset{\sim}{x}_j \mid X_{A_j}) = \int_0^\infty P(\underset{\sim}{x}_j \mid X_{A_j}, \sigma_j^2)\, P(\sigma_j^2)\, d\sigma_j^2$$

$$= \frac{|D_j|^{-1/2}\, g^{-p_j/2}}{(2\pi)^{n/2}} \int_0^\infty \sigma^{2-\frac{p_j}{2}} \exp -\frac{1}{2\sigma_j^2} \underset{\sim}{x}_j' H_j \underset{\sim}{x}_j \times \frac{\sigma^{2(-1+a)}\, b^a}{\Gamma(a)} e^{-b/\sigma_j^2}\, d\sigma_j^2$$

$$= \frac{|D|^{-1/2}\, g^{-p_j/2}}{(2\pi)^{n/2}\,\Gamma(a)}\, b^a \int_0^\infty \sigma^{2-(1+a+\frac{p_j}{2})} \exp -\left[\frac{x_j' H_j x_j}{2} + b\right]\Big/\sigma_j^2 \; d\sigma_j^2$$

$$= \frac{|D|^{-1/2}\, g^{-\frac{p_j}{2}}\, b^a\, \Gamma(a^*)}{(2\pi)^{n/2}\,\Gamma(a)\, b^{*a^*}} \int_0^\infty \frac{\sigma^{2-(1+a^*)}}{\Gamma(a^*)}\, b^{*a^*} \exp -\frac{b^*}{\sigma_j^2}\, d\sigma_j^2$$

$$= \frac{|D_j|^{-1/2}\, g^{-p_j/2}\, b^a\, \Gamma(a_j^*)}{(2\pi)^{n/2}\,\Gamma(a)\, b_j^{*a_j^*}}$$

where $a^* = a + \dfrac{p_j}{2}$ & $b^* = \dfrac{x_j' H_j x_j}{2} + b$

So that for each $j = 1, \dots, d$

$$P(\underset{\sim}{x}_j \mid X_{A_j}) \propto |D_j|^{-1/2}\, g^{-p_j/2}\, \Gamma(a_j^*)\, b_j^{*\,-a_j^*}$$

where

$$b_j^* = \frac{\underset{\sim}{x}_j' H_j \underset{\sim}{x}_j}{2} + b \quad \&$$

$$H_j = I - X_{A_j} V_{\beta_j} X_{A_j}'$$

$$V_{\beta_j} = g(g X_{A_j}' X_{A_j} + D^{-1})^{-1}$$

$$a_j^* = \frac{p_j}{2} + a \qquad p_j = |A_j| + 1$$

$|D_j|^{-1/2}$ is the determinant of a $p_j \times p_j$ covariance matrix