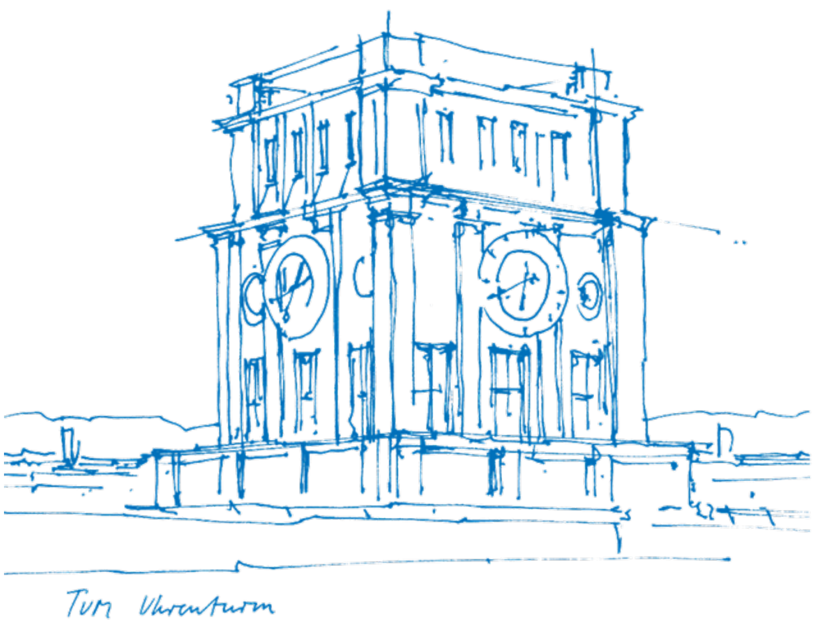
# Bifurcation theory and visualization

Dr. Felix Dietrich

2019-11-14



# Organizational issues

## Groups, Moodle, Reports

- Did every group upload their reports for the second exercise?
- All of you should have received their points for the first exercise. Let me know if you have questions about my feedback.
- You will receive the points for the second exercise in the next two weeks.

# Highlights

## Exercise 1

- Many different simulators, languages, GUIs, ...
- Some students got 100 points even without the 50 additional points
- Nobody implemented pedestrians walking the same speed in different directions
- The Dijkstra algorithm was a challenge for many groups (diagonal cells have a different distance!)

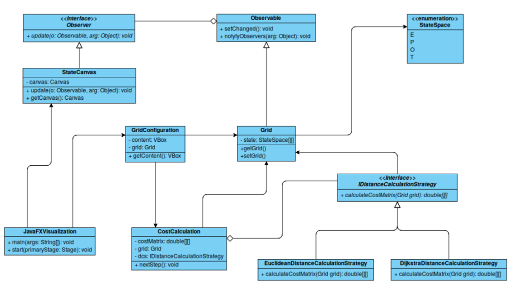
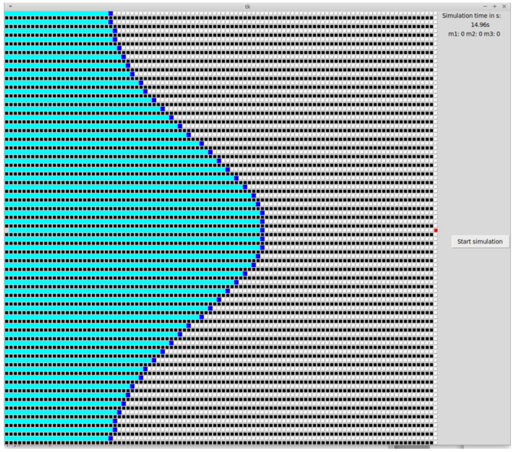


Figure 1: UML diagram of the project.



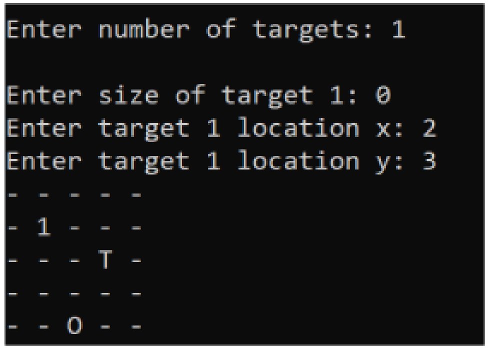



Figure 4: Grid with one target

# Highlights

## Exercise 1

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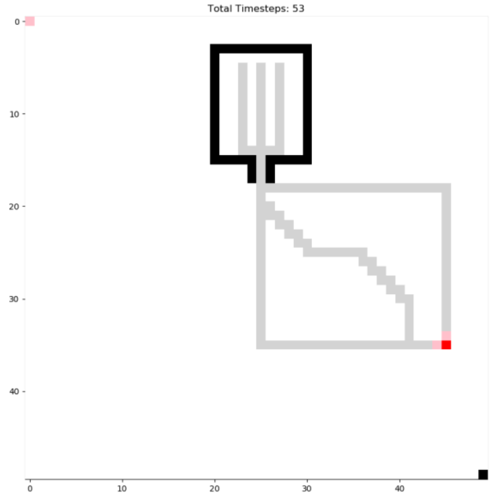


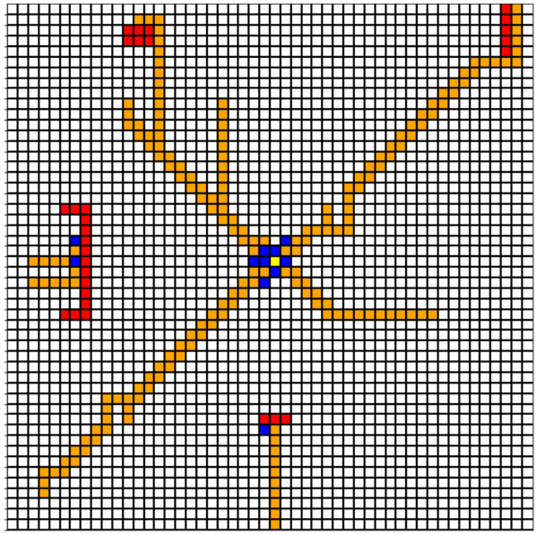

# Recap

## Lecture 1: Modeling crowd dynamics

- Modeling approaches, verification and validation
- In detail: cellular automata (first exercise)

## Lecture 2: Simulation software

- Introduction to the Vadere software (guest lecture from Benedikt Zönnchen)
- Using and analyzing simulation software (second exercise)

# Today

## Lecture 3: Bifurcation theory and visualization

We will study qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called *bifurcations*, “to divide into two, like a fork”.

## Parameters in crowd dynamics

- Desired speed for individuals
- Distance to obstacles
- Width of doors
- Number of pedestrians in a room
- Target distribution or sequence
- ...

All of them influence the crowd, and by changing them the behavior changes.
When is a change “relevant”?

# Today

## Outline

1. Dynamical systems: introduction
2. Phase portraits, orbits, topological equivalence
3. Bifurcation theory
4. Exercise 3

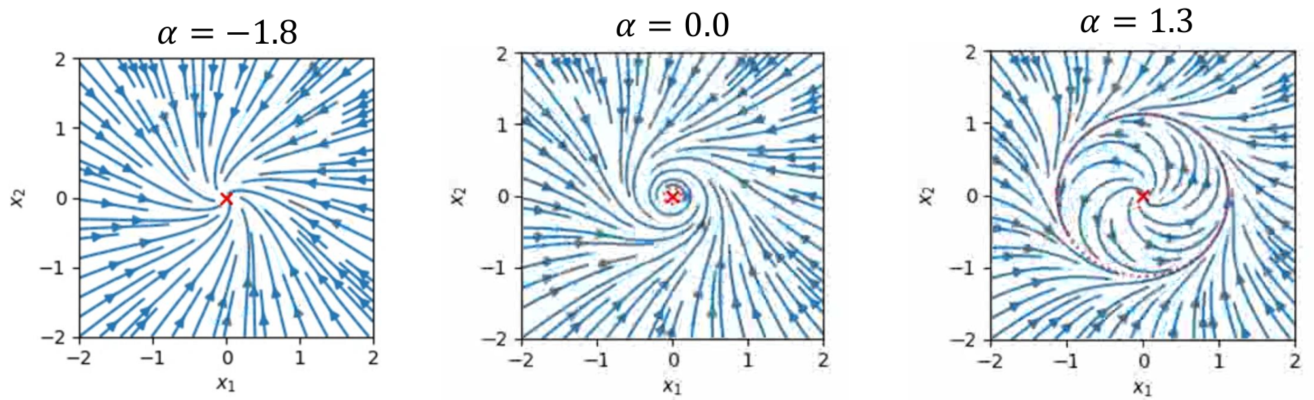


Figure: Bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of $a$, a limit cycle exists.

# Dynamical systems

## What is a dynamical system?

"The notion of a dynamical system is the mathematical formalization of the general scientific concept of a deterministic process." [Kuznetsov, 2004]

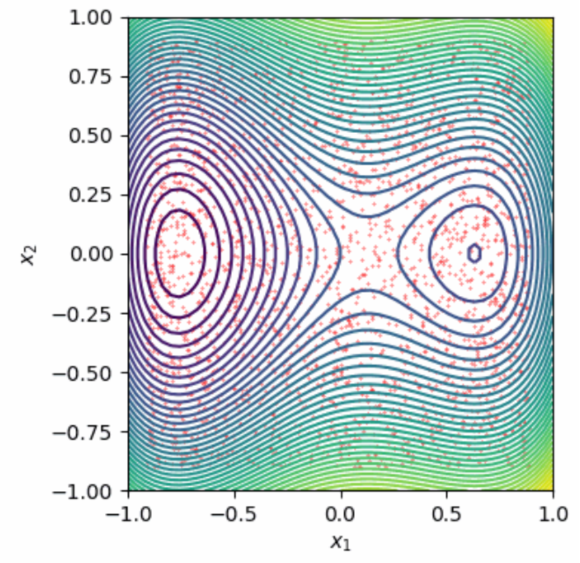


Figure: Each point represents a state of the system, and is moving towards one of the three steady states over time.

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A dynamical system is a triple $(T, X, \phi)$, with

- the state space $X$ (Euclidean space $\mathbb{R}^n$, manifold $\mathscr{M}$, metric space,...),
- the time $T$ (continuous $\mathbb{R}$, $\mathbb{R}_0^+$, discrete $\mathbb{Z}$, $\mathbb{N}$, ...), and
- the evolution operator $\phi : T \times X \to X$, with the following properties for all $x \in X$:

P1 $\phi(0, x) = \mathrm{Id}(x) = x$,

P2 $\phi(t+s, x) = \phi(t, \phi(s, x)) = (\phi_t \circ \phi_s)(x)$ for all $t, s \in T$.

A good book covering dynamical systems and numerical analysis is written by [Stuart and Humphries, 1996].

# Dynamical systems

## Notation

The evolution operator can be specified explicitly (as a map), or implicitly, as a

1. recurrence relation (here, $\phi(n, x_0)$ may be difficult to state explicitly):
$$x_{n+1} = \phi(1, x_n)$$
2. differential equation (here, $\phi(t, x)$ may be difficult to state explicitly):
$$\left.\frac{d\phi}{dt}\right|_{t=0}(x) = v(x),$$
where $v$ is called *vector field*.

The time derivative of the evolution operator at $t = 0$ has several notations,
$$\frac{d}{dt}\phi^t(x),\ \frac{d}{dt}x,\ \text{or } \dot{x}. \tag{1}$$

# Dynamical systems

## Vector fields

For more details on differential geometry, see [Lee, 2012]. On a manifold $\mathscr{M}$, the tangent bundle is the disjoint union of the tangent spaces at every point,

$$T\mathscr{M} := \cup_{x\in\mathscr{M}} T_x\mathscr{M}. \tag{2}$$

A vector field $v$ is a section of the tangent bundle: $v : \mathscr{M} \to T\mathscr{M}$, such that $x \mapsto v(x) \in T_x\mathscr{M}$. That means the vector field assigns every point on the manifold a vector in the local tangent space.

With this prerequisite, an evolution operator $\phi$ of a dynamical system on a manifold can be defined implicitly and without coordinates through

$$\left.\frac{d\phi}{dt}\right|_{t=0}(x) = v(x). \tag{3}$$

Note that $v(x) \in T_x\mathscr{M}$, the vectors are not elements of $\mathscr{M}$!

However: $T\mathbb{R}^n \simeq \mathbb{R}^n$.

# Dynamical systems

## Orbits

Given a system $(I, X, \phi)$ and a state $x \in X$, the orbit containing $x$ is the set

$$\mathscr{O}(x) := \{\phi(t, x) \in X | t \in I\}. \tag{4}$$

Orbits are also called trajectories (usually if the time information is kept).

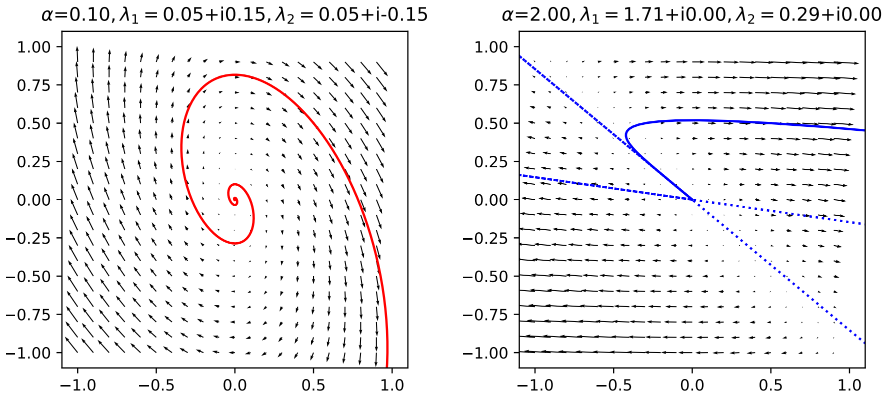


Figure: Vector fields and orbits of a parametrized dynamical system with state $x \in \mathbb{R}^2$. The vector field is $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25 x_1)$, with $\alpha = 0.1$ (left) and $\alpha = 2.0$ (right). Orbits are shown in color, the eigenvalues of the matrix $A_\alpha$ are shown in the title.

# Dynamical systems

## Phase portraits

A phase portrait visualizes qualitative features of a dynamical system by showing representative orbits and vectors.

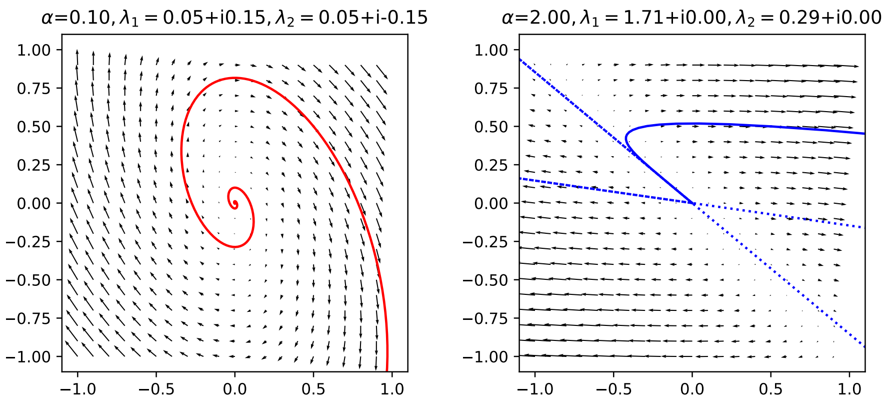


Figure: Phase portraits of a dynamical system with state $x \in \mathbb{R}^2$ and parametrized vector field $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25x_1)$, with $\alpha = 0.1$ (left) and $\alpha = 2.0$ (right). Orbits are shown in color, the eigenvalues of the matrix $A_\alpha$ are shown in the title.

# Dynamical systems

## Topological equivalence

The notion of qualitative difference is formalized through the notion of topological equivalence, i.e. a system is *qualitatively the same* as another system, if it is *topologically equivalent*:

A dynamical system $(I, X, \phi)$ is topologically equivalent to another dynamical system $(I, Y, \psi)$ if there is a homeomorphism $h : X \rightarrow Y$ (continuous, and with continuous inverse), such that $h$ is mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

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## Examples in crowd dynamics

1. If two models produce exactly the same trajectories, they are topologically equivalent systems.
2. If the trajectories are the same, but the speed along them is different, the systems are still topologically equivalent.
3. If one model moves one pedestrian in a circle, but the other one moves them in a straight line, the models are not topologically equivalent (one is recurrent, the other is not).

# Bifurcation theory

## What is a bifurcation?

1. Introduction
2. Normal forms
3. Examples: 1D, 2D, one and more parameters

# Bifurcation theory

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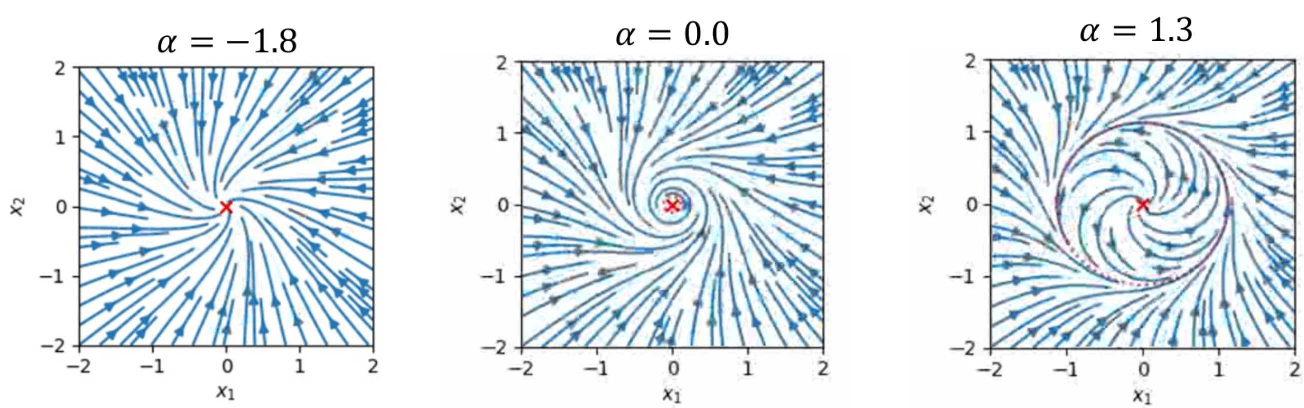


Figure: Bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of $a$, a limit cycle exists.

# Bifurcation theory

## Normal forms (informal definition)

The normal form of a dynamical system around a fixed point is a polynomial vector field, with

- the smallest degree,
- the smallest number of coordinates, and
- the smallest number of parameters,
- such that the two systems are topologically conjugate locally around the fixed point (in state space), and locally around the bifurcation point (in parameter space).

For formal definitions of normal forms and topological equivalence including the parameter space, see [Kuznetsov, 2004, p.63ff].

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Most important statement of the lecture today:

The normal form is locally topologically equivalent to **all dynamical systems with that normal form**. That means once we know that a particular system has a normal form, we do not need to study the specific system anymore - we understand it! At least around the given steady state...

# Bifurcation theory

## Examples - 1D space, 1 parameter

**The Pitchfork bifurcation**

$$\dot{x} = x\alpha - x^3 \tag{5}$$

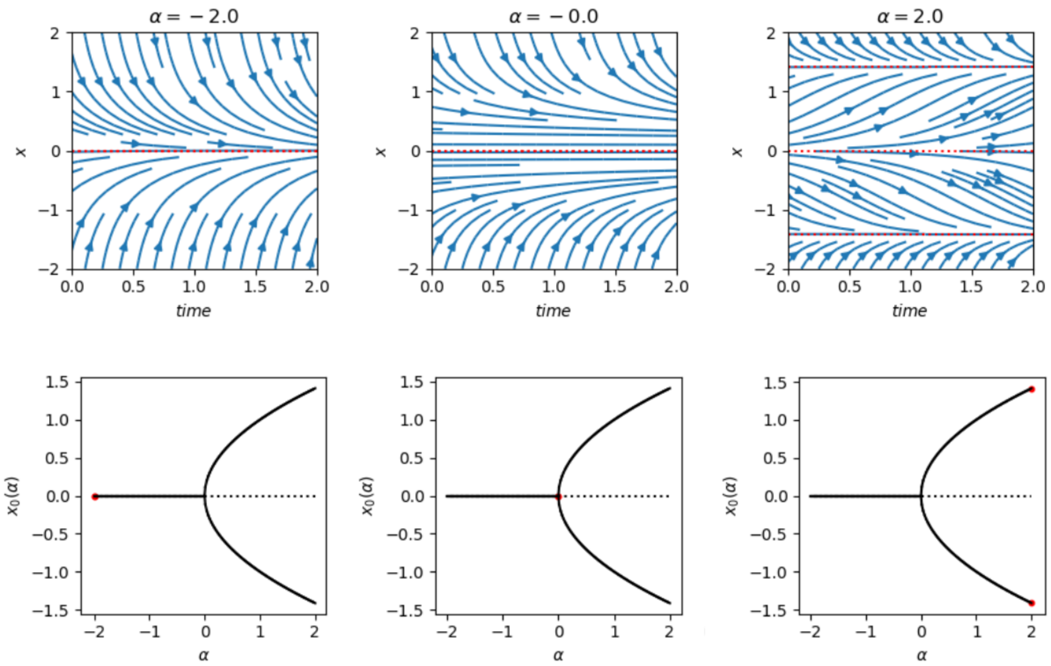


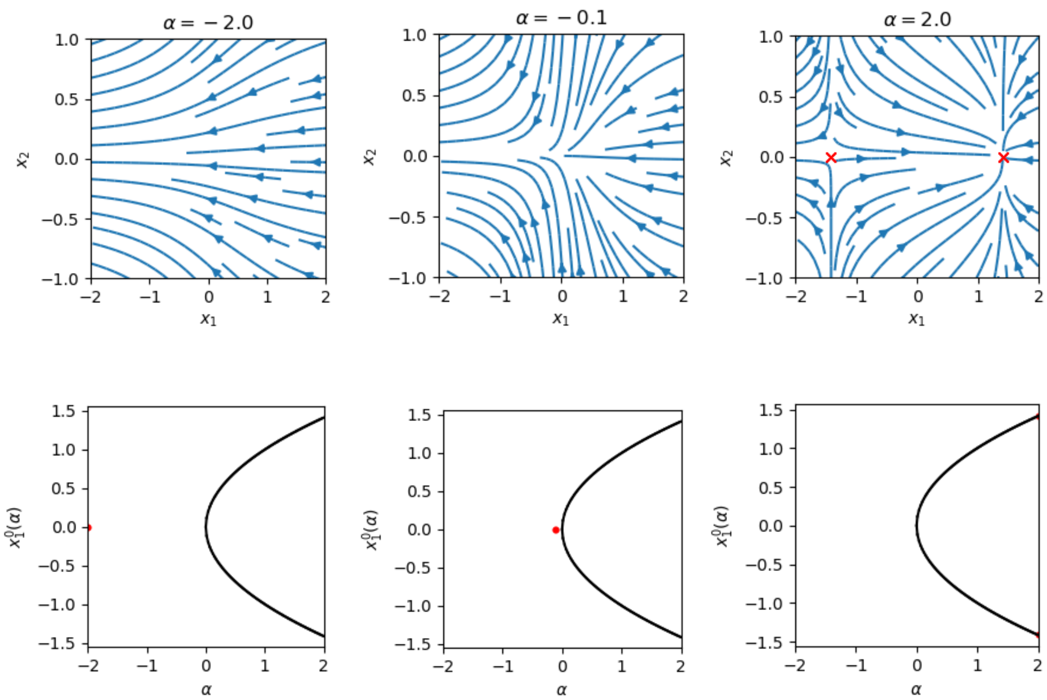

Figure: Phase portraits and bifurcation diagram for the pitchfork bifurcation.

# Bifurcation theory

## Examples - 1D space, 1 parameter

The **Saddle-node bifurcation**

$$\dot{x} = \alpha - x^2 \tag{6}$$




Phase portraits and bifurcation diagram for the saddle-node bifurcation. Here, the $x_2$ direction does not contribute to the bifurcation (its dynamic is $\dot{x}_2 = -x_2$).

# Bifurcation theory

## Examples - 2D space, 1 parameter

The **Andronov-Hopf bifurcation**

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2)^2, \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2)^2.\end{aligned} \tag{7}$$

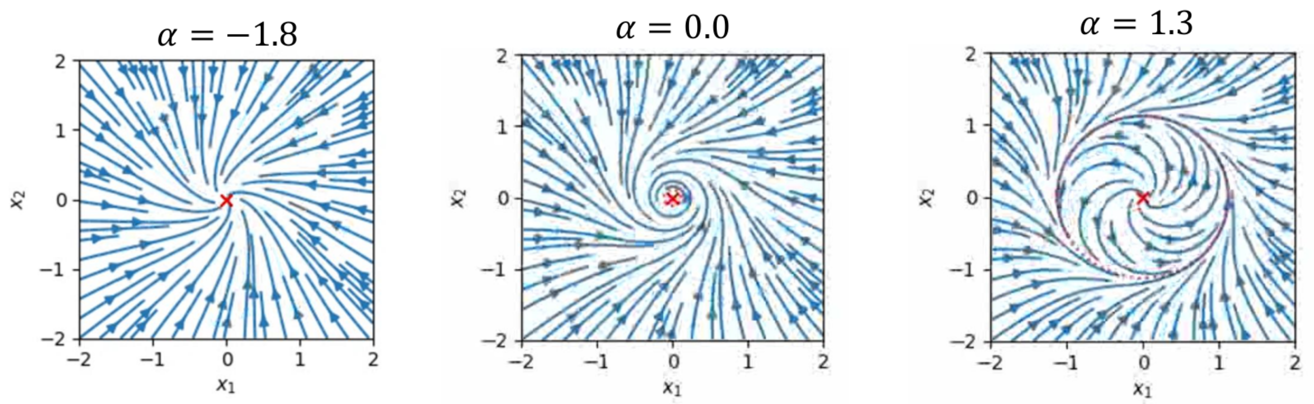



Figure: Hopf bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of $a$, a limit cycle exists.

# Bifurcation theory

## Examples - 3D space, 2 parameters

The **Blue sky catastrophe.** A stable limit cycle disappears, its length and period tend to infinity, while it remains bounded, and located at a finite distance from all equilibrium points. The normal form is

$$
\begin{aligned}
\dot{x} &= x[2+\alpha-10(x^2+y^2)]+z^2+y^2+2y, \\
\dot{y} &= -z^3-(y+1)(z^2+y^2+2y)-4x+\alpha y, \\
\dot{z} &= y^2(y+1)+x^2-\beta.
\end{aligned} \tag{8}
$$

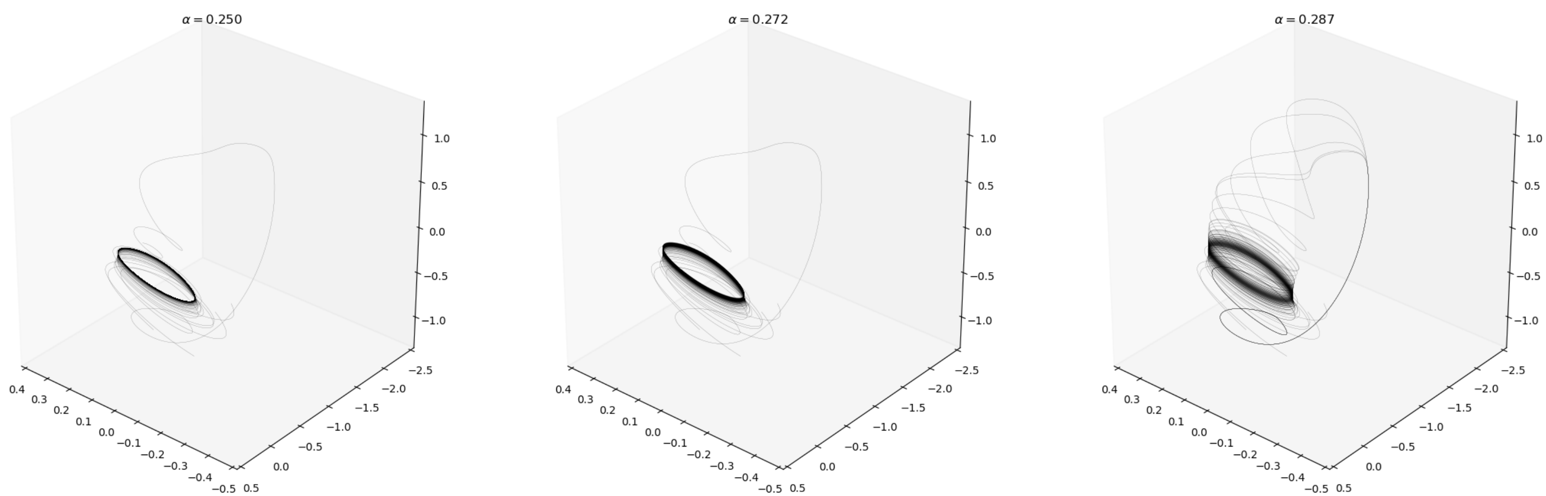



Figure: Stable limit cycle disappears into bounded orbit of infite length and period.

# Bifurcation theory and visualization

## Exercise 3

**Your tasks in the exercise:**

1. Familiarize yourself with the mathematical notation of bifurcation theory,
2. Understand topological equivalence between systems,
3. Know several basic bifurcations present in almost all dynamical systems in the world,
4. Visualize qualitative changes of a dynamical system in a bifurcation diagram, and
5. Apply these ideas to crowd dynamics.

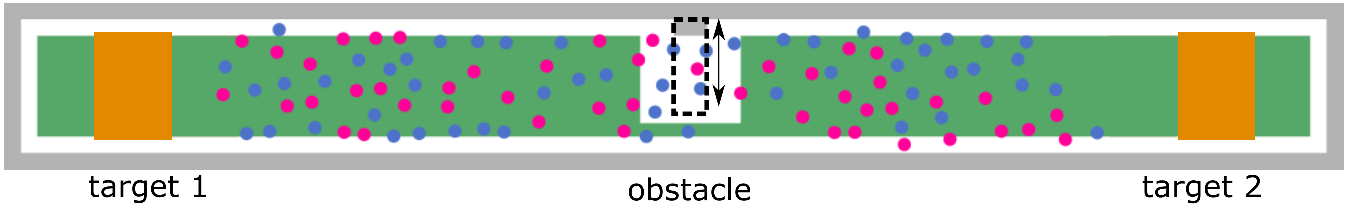
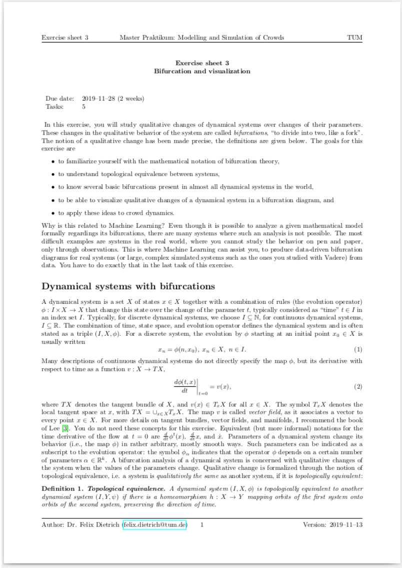



Figure: Setup of the bifurcation scenario in Vadere. 100 pedestrians move back and forth between two targets, while an obstacle in the center blocks movement. The red pedestrians are currently moving to the left, the blue ones move to the right, their color changes once they reach the next target and they switch to the other one.

# Bifurcation theory and visualization

## Exercise 3

You can find the exercise sheet on Moodle



Exercise sheet 3 Master Praktikum: Modelling and Simulation of Crowds TUM

**Exercise sheet 3**
**Bifurcation and visualization**

Due date: 2019–11–28 (2 weeks)
Tasks: 5

In this exercise, you will study qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called *bifurcations*, "to divide into two, like a fork". The notion of a qualitative change has been made precise, the definitions are given below. The goals for this exercise are

- to familiarize yourself with the mathematical notation of bifurcation theory,
- to understand topological equivalence between systems,
- to know several basic bifurcations present in almost all dynamical systems in the world,
- to be able to visualize qualitative changes of a dynamical system in a bifurcation diagram, and
- to apply these ideas to crowd dynamics.

Why is this related to Machine Learning? Even though it is possible to analyze a given mathematical model formally regardings its bifurcations, there are many systems where such an analysis is not possible. The most difficult examples are systems in the real world, where you cannot study the behavior on pen and paper, only through observations. This is where Machine Learning can assist you, to produce data-driven bifurcation diagrams for real systems (or large, complex simulated systems such as the ones you studied with Vadere) from data. You have to do exactly that in the last task of this exercise.

### Dynamical systems with bifurcations

A dynamical system is a set $X$ of states $x \in X$ together with a combination of rules (the evolution operator) $\phi : I \times X \to X$ that change this state over the change of the parameter $t$, typically considered as "time" $t \in I$ in an index set $I$. Typically, for discrete dynamical systems, we choose $I \subseteq \mathbb{N}$, for continuous dynamical systems, $I \subseteq \mathbb{R}$. The combination of time, state space, and evolution operator defines the dynamical system and is often stated as a triple $(I, X, \phi)$. For a discrete system, the evolution by $\phi$ starting at an initial point $x_0 \in X$ is usually written

$$x_n = \phi(n, x_0),\ x_n \in X,\ n \in I. \tag{1}$$

Many descriptions of continuous dynamical systems do not directly specify the map $\phi$, but its derivative with respect to time as a function $v : X \to TX$,

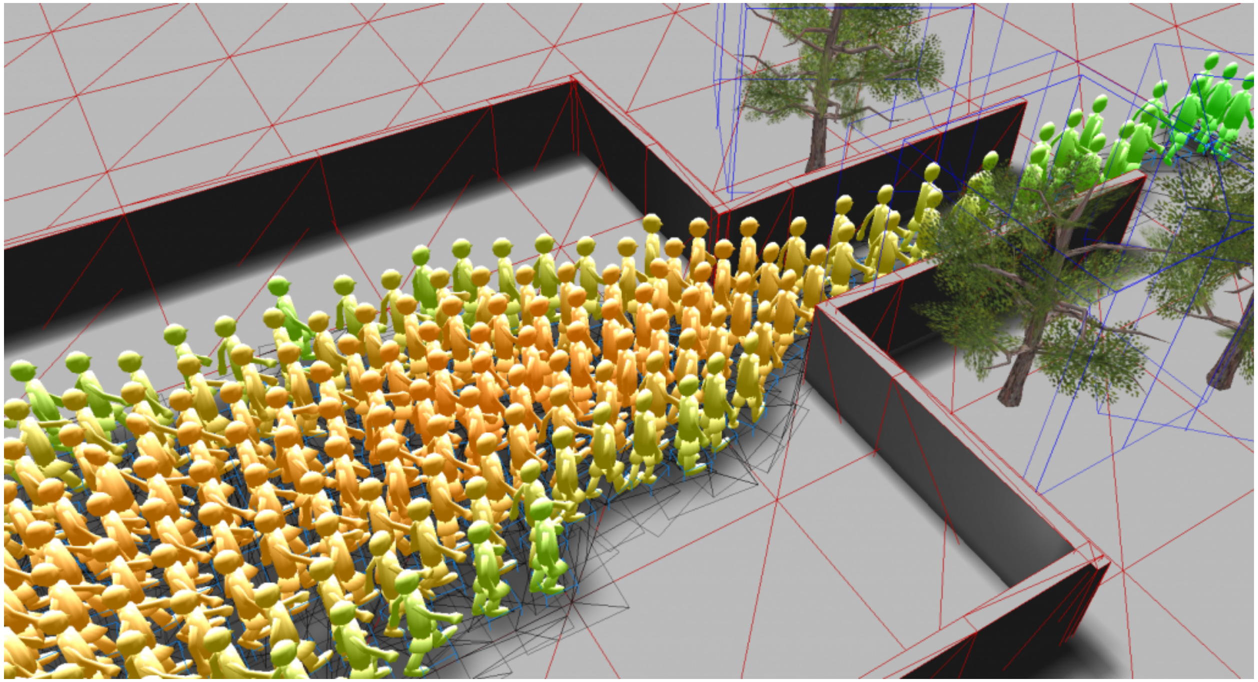
$$\left.\frac{d\phi(t,x)}{dt}\right|_{t=0} = v(x), \tag{2}$$

where $TX$ denotes the tangent bundle of $X$, and $v(x) \in T_xX$ for all $x \in X$. The symbol $T_xX$ denotes the local tangent space at $x$, with $TX = \cup_{x\in X} T_xX$. The map $v$ is called *vector field*, as it associates a vector to every point $x \in X$. For more details on tangent bundles, vector fields, and manifolds, I recommend the book of Lee [3]. You do not need these concepts for this exercise. Equivalent (but more informal) notations for the time derivative of the flow at $t = 0$ are $\frac{d}{dt}\phi^t(x)$, $\frac{d}{dt}x$, and $\dot{x}$. Parameters of a dynamical system change its behavior (i.e., the map $\phi$) in rather arbitrary, mostly smooth ways. Such parameters can be indicated as a subscript to the evolution operator: the symbol $\phi_\alpha$ indicates that the operator $\phi$ depends on a certain number of parameters $\alpha \in \mathbb{R}^k$. A bifurcation analysis of a dynamical system is concerned with qualitative changes of the system when the values of the parameters change. Qualitative change is formalized through the notion of topological equivalence, i.e. a system is *qualitatively the same* as another system, if it is *topologically equivalent*:

**Definition 1.** ***Topological equivalence.*** *A dynamical system $(I, X, \phi)$ is topologically equivalent to another dynamical system $(I, Y, \psi)$ if there is a homeomorphism $h : X \to Y$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.*

Author: Dr. Felix Dietrich (felix.dietrich@tum.de) 1 Version: 2019–11–13

# Questions?



Homework: finish third exercise & upload report until 2019-11-28.
For questions / appointments: please ask via email, `felix.dietrich@tum.de`.

# Literature I


Kuznetsov, Y. A. (2004).
*Elements of Applied Bifurcation Theory.*
Springer New York.

Lee, J. M. (2012).
*Introduction to Smooth Manifolds.*
Springer New York.

Stuart, A. M. and Humphries, A. R. (1996).
*Dynamical Systems and Numerical Analysis.*
Cambridge University Press.