Prime Number Distribution

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Abstract

The purpose of this paper was to provide a differential equation based on Eratosthenes' sieve that describes the density of primes. It is of the form

$$f'(x) = \frac{-f(x)f(\sqrt{x})}{2x}$$

where $f(x)$ represents the local density of primes $\leq x$ and is considered a point density instead of average density. We derived a function for the relative error between a perturbed solution $f(x)$ and the ideal solution $1/lnx$. This allowed us to take its limit and verify that it goes to zero as x goes to infinity. Moreover, we proved that it also predicts the infinitely many jumps from Littlewood's theorem. Lastly, we look over the limitations of our differential equation as its absolute error predictions come in conflict with the Riemann hypothesis, a widely supported conjecture.

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1 Introduction

1.1 Background and Motivation

Number theory is a branch of mathematics that is useful to reveal the relationships between sequences of integer numbers from 1 to infinity. It particularly deals with natural numbers like prime numbers and distinguishes them from composite numbers. However, prime numbers have an element of randomness which makes them behave in unpredictable ways. We will use that as motivation for our paper, to try to understand the patterns held in prime number distribution with the help of differential equations.

1.2 Research Objective

One nonstandard way to understand the distribution of primes is to develop a differential equation, one which models the density of primes smaller than an arbitrary x. This equation reads

$$f'(x) = \frac{-f(x)f(\sqrt{x})}{2x} \tag{1}$$

where f(x) represents the density of primes at x for some x. The equation above provides an estimate for the actual prime number density at x tabulated by Gauss and denoted as $f(x) = 1/\ln x$. To prove that our model agrees with the previous findings and data, this paper can be broken down into four parts.

1. **Deriving our model in (1):**
   We derive it using the properties of Eratosthenes's sieve and basic differentiation techniques.
2. **Finding a differential-delay equation for the relative error and computing its limit.**
   We use a change of variables and common differentiation techniques.
3. **Finding solutions to our model.**
   We use numerical and analytical methods.
4. **Showing that our model predicts the infinitely many jumps from Littlewood's Theorem.**
   By applying the same change of variables to resolve the integral of the difference between a solution to our differential equation and the ideal solution.

Note that most of the information, data, and derivations presented in this paper are based on the paper from [1].

1.3 Previous Discoveries of our DE

It was first conjectured in 1792 by Carl Friedrich Gauss that

$$\pi(n) \approx \frac{n}{ln(n)}$$

where $\pi(n)$ is known as the 'prime-counting function' since it denotes the number of primes less than or equal to n. π is the asymptotic notation of the asymptotic law of distribution of primes. It signifies that the limit of the quotient of the two functions approaches 1. Thus,

$$lim_{x\to\infty} \frac{\pi(x)}{(\frac{x}{lnx})} = 1$$

This statement is equivalent to saying that

$$p_n \sim nln(n)$$

where p_n designates the $n'th$ prime number. Eratosthenes sieve is applied to convert this statement into a differential equation serving as a model for the density. The model we will study is of the form

$$f'(x) = \frac{-f(x)f(\sqrt{x})}{2x} \tag{2}$$

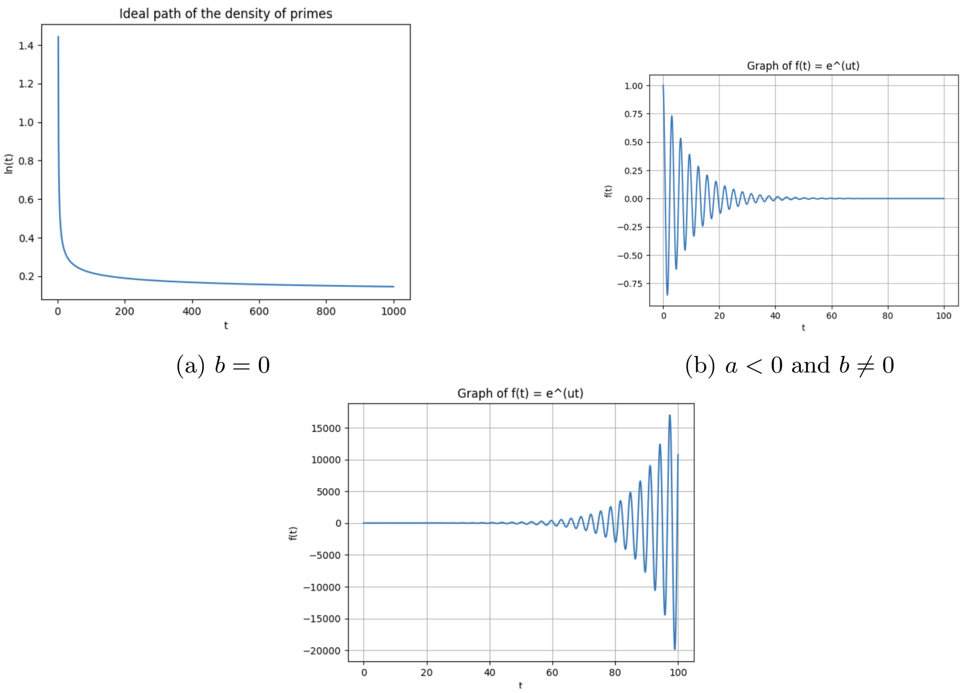
where $f(x)$ is the density of primes at x. That equation was discovered twice previously, first by the British physicist Cherwell in 1942 and in 1961 by the British electrical engineer G. Hoffman de Visme.

1.4 Possible Solutions of our DE

The solutions of most elementary first-order linear differential equations are of the standard form

$$f(t) = e^{\lambda t} = e^{at}(\cos bt + isin(bt))$$

Depending on whether λ is real or not with $\lambda = a + bi$, the density of primes can either follow the ideal path (a), oscillate and converge (b) or diverge(c) for $a > 0$ and $b \neq 0$ as shown in the graphs below.



(a) $b = 0$

(b) $a < 0$ and $b \neq 0$

(c) $a > 0$ and $b \neq 0$

Figure 1: All possible behaviors of the average density of primes

We will compare the behavior of the density and the accumulation of primes with the different asymptotic features presented in the graphs above.

2 Prime Number Theory

2.1 Gauss's conjectures comparison

The prime number theorem provides an asymptotic form of the density of primes defined by the function $\pi(x)$. It was first suggested by Legendre in 1808 that, for large x,

$$\pi(x) \sim \frac{x}{\ln x + B}$$

for $B = -1.08366$, otherwise known as Legendre's constant. However, the constant would only be applied in the leading term, such that:

$$\frac{x}{lnx + B} sin(\frac{x}{lnx}) - B\frac{x}{(lnx)^2} + B^2\frac{x}{(lnx)^3} + ...$$

[6]

Even before that, in 1792, 15-year-old Gauss hand-calculated the number of primes in intervals of length 1000, called *chiliads*. By plotting a graph of the densities, he noticed that they were asymptotic to the inverse of the natural logarithm function. He resumed his observations in the following conjecture

$$\pi(x) \sim \frac{x}{lnx}$$

which turned out to be a better estimate than Legendre's. Gauss later refined it to be the following,

$$\pi(x) \sim Li(x)$$

where the logarithmic integral $Li(x)$ considers the whole range of primes from 2 to x as follows:

$$Li(x) = \int_2^x \frac{1}{lnt}\,dt$$

To demonstrate Gauss's conjecture, we can plot the density of primes in successive *chiliads* and observe its similarity to the inverse natural logarithmic function.

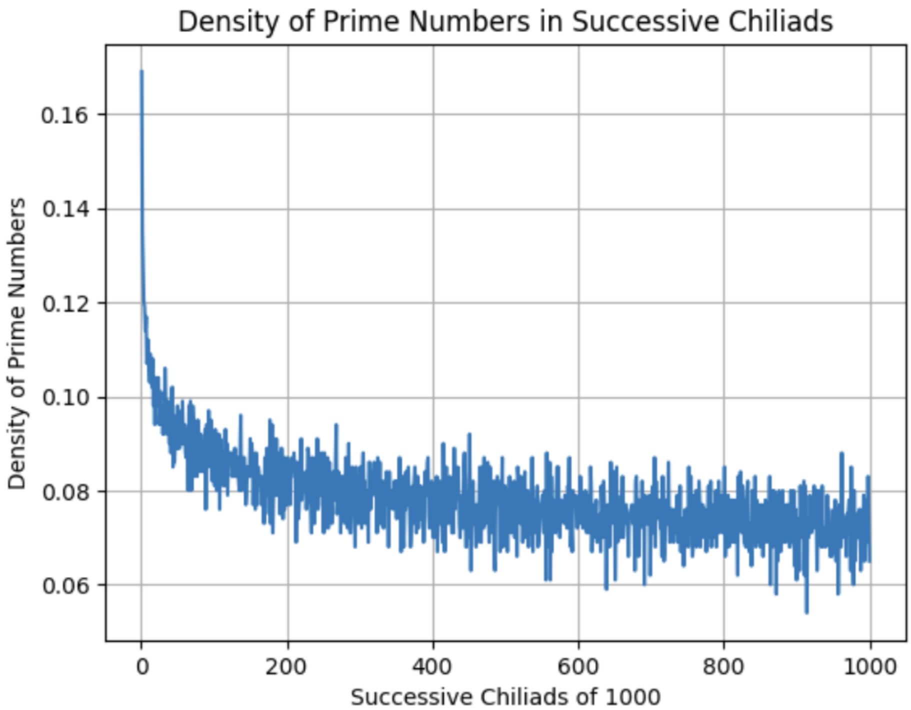


Figure 2: Average density of primes

Since each *chiliad* corresponds to intervals of 1000 values of x, Figure 2 shows us the density of prime numbers up to one million. Because the average density jumps above and below the idealized path line $\frac{1}{\ln x}$, there is a certain amount of error that can be calculated by comparison. Figure 3 below further portrays the accuracy of Gauss's estimates for x values up to 10^{20}.

Table 1: The total number of primes compared to Gauss's estimates			
x	л(x)	x/lnx	Li(x)
10	4	4	6
100	25	22	30
1E+03	168	145	178
1E+04	1 229	1 086	1 246
1E+05	9 592	8 686	9 630
1E+06	78 498	72 382	78 628
1E+07	664 579	620 421	664 918
1E+08	5 761 455	5 428 681	5 762 209
1E+09	50 847 534	48 254 942	50 849 235
1E+10	455 052 511	434 294 482	455 055 614
1E+11	4 118 054 813	3 948 131 654	4 118 066 401
1E+12	37 607 912 018	36 191 206 825	37 607 950 281
1E+13	346 065 536 839	334 072 678 387	346 065 645 810
1E+14	3 204 941 750 802	3 102 103 442 166	3 204 942 065 692
1E+15	29 844 570 422 669	28 952 965 460 217	29 844 571 475 288
1E+16	279 238 341 033 925	271 434 051 189 532	279 238 344 248 557
1E+17	2 623 557 157 654 230	2 554 673 422 960 300	2 623 557 165 610 820
1E+18	24 739 954 287 740 800	24 127 471 216 847 300	24 739 954 309 690 400
1E+19	234 057 667 276 344 000	228 576 043 106 975 000	234 057 667 376 222 000
1E+20	2 220 819 602 560 910 000	2 171 472 409 516 260 000	2 220 819 602 783 660 000

Figure 3: Gauss' estimates comparison with $\pi(x)$

Additional data obtained mainly from [] and [how any primes are there]

The logarithmic integral $Li(x)$ is an overestimate of $\pi(x)$ but seems to be getting closer and closer to it for large x. Meanwhile, the approximations by $\frac{x}{lnx}$ are most accurate for small x. However, both are good approximations since their relative errors tend to be 0 for $x \to \infty$:

$$\frac{\pi(x) - x/lnx}{\pi(x)} \qquad and \qquad \frac{\pi(x) - Li(x)}{\pi(x)}$$

That relationship can also be explained by the fact that those functions are asymptotic to one another, such that $\pi(x) \sim \frac{x}{lnx} \sim Li(x)$. The absolute error, $\pi(x) - Li(x)$, shows a completely different behavior of prime numbers for large x. Although the logarithmic integral always seems to exceed the actual number of primes, Littlewood proved that their difference changes sign infinitely often and that there exists a positive constant K such that some large x satisfy

$$\pi - Li(x) > \frac{k\sqrt{x}lnlnlnx}{lnx} \tag{3}$$

and other large x satisfy

$$\pi - Li(x) < \frac{-k\sqrt{x}lnlnlnx}{lnx} \tag{4}$$

Function (3) represents the state when $Li(x)$ is below the ideal path line, while function (3) represents the state when $Li(x)$ is above the ideal path line. Looking at the right-hand side of function (4), it shows a slowly increasing absolute error for sufficiently large positive x. This means that the values of the two functions show large oscillations and most importantly that the sign changes infinitely often. This theorem will be revisited once we learn more about the relative error.

2.2 Sieving Process

2.2.1 Eratosthenes' Sieve

As mentioned previously, our model is based on the sieve of Eratosthenes. Sieves are algorithms that calculate the number of primes in a set boundary by successfully crossing out numbers from an initial list of what are called "candidate primes". It begins with a list of candidate primes from 2 to n, meaning they are treated as primes until it is proven that they aren't. This works by marking the smallest number of the list as prime, then its multiples are eliminated before moving on to the next iteration.

The most crucial observation resulting from this sieve is that the presence of a prime number p seems to decrease the density of primes by a factor of $1 - \frac{1}{p}$. This factor will be further discussed further down the paper. For example, sieving by p=2 multiplies the whole range of numbers from 2 to n by 1/2 and decreases the density in that interval by $(1 - 1/2) = 1/2$. Another key remark is that primes only begin to affect the density after their square. For instance, p=3 first eliminates the composite number 9 from the list.

2.2.2 Pritchard's Sieve

This is a slightly more sophisticated method than the sieve of Eratosthenes, which examines the whole range of primes. The sieve of Pritchard instead examines the subset of the range of numbers from 2 to N which are the ones left after each iteration from Eratosthenes's sieving process. Those numbers are placed in growing successive wheels, such that the $i'th$wheelWi has circumference $Pi = pi1 * p2 * ... * pi$. The prime numbers between i and n (with n > 0), can be obtained by "rolling" the wheel along a numbered line. Beginning with the trivial wheel W0, it keeps generating wheels at each iteration until the square of the first prime number after 1 is at least N. It can be demonstrated as such:

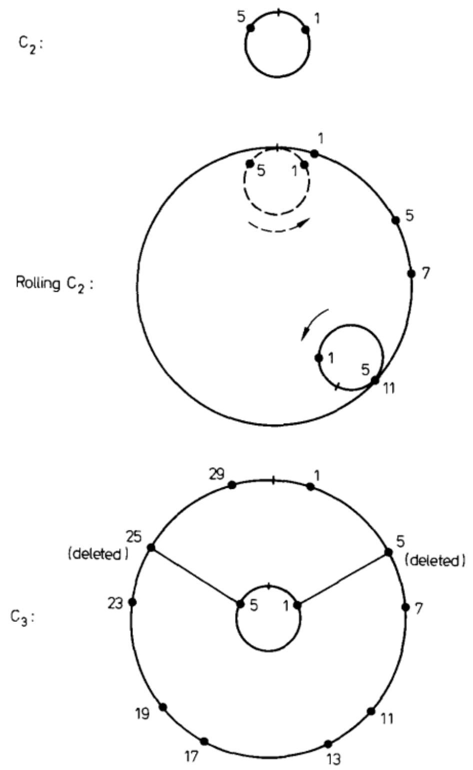


Figure 4: Average density of primes

Picture taken from [14]

The efficiency of Pritchard's sieve in finding prime numbers close to the ideal amount $\frac{1}{lnx}$ is related to its better asymptotic complexity. This means that it requires fewer steps to reach its goal and thus is easier to compute. A major improvement of Erastothenes's sieve is that Pritchard's runs on sublinear time within the set boundaries, which is the best we have so far. This means that the algorithm executes the density of primes slower than the growth of the size of the relative error obtained from this approximation. This is proved below, by computing the limit of the ratio of the relative error of the model and the model itself as x goes to infinity.

2.2.3 Observable Patterns

This sieving process underlines the spiraling nature of the distribution of primes. We will attempt to visualize it by graphing the primes in a two-dimensional Cartesian plane. We start by transforming each prime p into its polar coordinates

$$(r, \theta) = (p, p)$$

At first, the primes appear to be spiraling outward and form what is known as an "Archimedean spiral". However, when graphing the larger prime numbers, they start forming outward rays with gaps as such

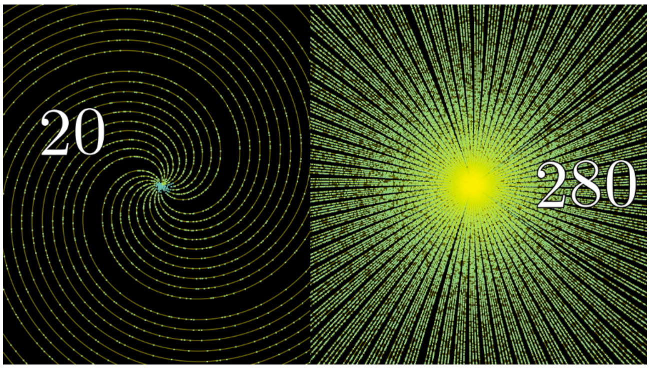


Figure 5: Spiraling Pattern of Primes

Figure 5, taken from [15], shows the graphing of primes creating 20 and 280 spirals. We can observe dark lines that signify the missing primes and how they too follow a pattern. However, no significant scientific interpretation has been extracted from this pattern.

3 Modeling the Density of Primes

3.1 Deriving the DE in (1)

To generate a differential equation that approximates the density of primes at x, the fundamental observations from the sieve of Eratosthenes will be accounted for. Since primes begin affecting the density only after their square, two intervals of integration are needed. Interval A accounts for the list of candidate primes from the whole range of 2 to N. Meanwhile, Interval B accounts for the primes that affect the density, which is what our focus will be.

- Interval A has boundaries x and $x + dx$
- Interval B has boundaries x^2 and $(x + dx)^2$

Approach 1 The function $f(x)$ should be considered a point density as it evaluates the primes immediately surrounding x. That means that it does not evaluate all the primes between 2 and x. Note that the density on Interval A affects the density on Interval B.
To begin with, we want to estimate the change in density on Interval B and we obtain $f((x + dx)^2) - f(x^2)$.

Next, we want to express the fact that each prime in Interval A factors by $1 - \frac{1}{x}$ the density in Interval B. Applying that to the initial boundary of Interval B: $f(x^2) * (1 - \frac{1}{x}) = f(x^2) -$

$\frac{f(x^2)}{x}$/. We are only focusing on the effect of primes from Interval A on Interval B. Now, taking the $f(x^2)$ to the other side of the equation, and considering $f(x)dx$ as the number of primes on Interval A, we obtain the first approximation of the change in density on Interval B:

$$f((x+dx)^2) - f(x^2) = -\frac{f(x^2)f(x)dx}{x} \tag{5}$$

Approach 2 Another method is to use the derivative $f'(x^2)$ to measure the change in density from the point where the primes begin to affect the density. That needs to be multiplied by the length of Interval B, which is the following:

$$(x+dx)^2 - x^2 = x^2 + 2xdx + d(x)^2 - x^2 = 2xdx + (dx)^2$$

We can ignore the quantity $(dx)^2$ without any repercussions because it represents an infinitesimal quantity compared to the prime numbers we're dealing with. The second approximation for the change in density on Interval B is:

$$f((x+dx)^2) - f(x^2) = f'(x^2) \cdot 2xdx \tag{6}$$

By equating the right-hand sides of the first and second approximations, we find

$$f'(x^2) \cdot 2xdx = -\frac{f(x^2)f(x)dx}{x}$$

Isolating the second-order term,

$$f'(x^2) = -\frac{f(x^2)f(x)}{2x^2}$$

and now changing the argument from x^2 to x,

$$f'(x) = -\frac{f(x)f(\sqrt{x})}{2x}$$

Here we have found the model of the system of primes, where $f(x)$ is the density of primes at x. It shows that the change in density at x depends on the density at x, the $\sqrt{x}$, and x itself.

A similar expression was derived by G. Hoffman de Visme in 1961 with essentially the same steps. However, his method is said to be probabilistic because it considers the density correction factor $1 - \frac{1}{p}$ to represent the probability that a number is not a product of the prime p rather than a deterministic factor.

Reducing factor of density The average density of primes is regulated by the factor $1 - \frac{1}{p}$. It can be shown by elevating the factor to a power corresponding to the density of primes $f(x)dx$ on the Interval A:

$$(1 - \frac{1}{x})^{f(x)dx} \approx 1 - \frac{1}{x} f(x)dx$$

Now, analyzing the density of Interval B under this new perspective:

$$f((x+dx)^2) = f(x^2)(1 - \frac{1}{x})^{f(x)dx}$$

$$= f(x^2)(1 - \frac{1}{x} f(x)dx)$$

$$= f(x^2) - \frac{1}{x} f(x^2)f(x)dx$$

By moving the $f(x^2)$ to the left side of the equation, we obtain the first approximation of the change in density on Interval B (4).

By accessing the link in GitHub in the Bibliography section, you can access a step-by-step illustration of the Eratosthenes sieve as well as a numerical check of the $\frac{1}{p}$ claim produced by Professor C.Stahn.

3.2 Computing the Relative Error Function

The prime number theorem states that the average density of primes smaller than x, denoted by $\frac{\pi(x)}{x}$, is asymptotic to the idealized path of the true density, described by the function $\frac{1}{lnx}$. As the logarithmic integral $Li(x)$ is similar to $\pi(x)$, we expect the density at x to be asymptotic to the ideal path as well, and $f(x) = \frac{1}{lnx}$ to be one solution, which it is indeed. To show that $f(x)$ is stable under perturbations, we consider an arbitrary function $f(x)$ that satisfies the differential equation until some $x = x_o$. After that, it jumps to some other value for which $f(x_o) \neq \frac{1}{ln(x_o)}$ and the function is supposed to bring back the values to the ideal slope.

We begin by computing the relative error of $f(x)$ compared to the ideal path to show that the limit is equal to zero:

$$\lim_{x\to\infty} \frac{f(x) - 1/lnx}{1/lnx} \tag{7}$$

To calculate the limit, a change of variables is made by using the *Anstaz* $x = e^{2^u}$ and letting $g(u) = f(e^{2^u})$. This change of variables provides a relationship between $g'(u)$ and the value of g at u and $u-1$, describing its delayed nature. It is advantageous to $f'(x)$, which depends on the value $f(x)$, $f(\sqrt{x})$, and x itself.

Change of variables We want a differential function for $g(u)$, thus we derive $f(e^{2^u})$ using the chain rule twice:

$$g'(u) = (ln2)2^u e^{2^u} f'(e^{2^u}) \tag{8}$$

Now, we want to substitute for the derivative $f'(e^{2^u})$ in equation (1).

$$g'(u) = \frac{-g(u)g(\sqrt{u})}{2e^{2^u}}$$

Plugging it back into equation (7),

$$g'(u) = \frac{(ln2)2^u e^{2^u} f'(e^{2^u}) \cdot -g(u)g(\sqrt{u})}{2e^{2^u}}$$

Simplifying the above equation, we obtain a differential equation for $g(u)$

$$g'(u) = -(ln2)2^{u-1}g(u-1)g(u) \tag{9}$$

Coming back to the limit equation, the ideal solution $\frac{1}{lnx}$ becomes 2^{-u}

$$\pi(x) = \frac{1}{ln(e^{2^u})} = \frac{1}{2^u lne} = 2^{-u}$$

The equation $g'(u)$ has a perturbed solution $g(u)$ and the limit function is equated to the relative error function.

$$\lim_{x\to\infty} \frac{g(u) - 2^{-u}}{2^{-u}} = 2^u g(u) - 1 \tag{10}$$

Relative error function The limit function in (6) is now expressed as a relative error function in (9). Substituting the left-hand-side of (9) by $h(u)$,

$$h(u) = 2^u g(u) - 1 \tag{11}$$

The $h(u)$ limit computation as $u \to \infty$ is done primarily with a differential-delay equation for $h(u)$. Finding it requires first differentiating $h(u)$ using the product rule,

$$h'(u) = 2^u g'(u) + (ln2)2^u g(u). \tag{12}$$

Substituting for g'(u) from (1),

$$f'(u) = \frac{-f(e^{2^u})f(e^{2^{u^{\frac{1}{2}}}})}{2e^{2^u}}$$

$$= -\frac{f(e^{2^u})f(e^{2^{\frac{u}{2}}})}{2e^{2^u}}$$

$$= -\frac{f(e^{2^u})f(e^{2^{u-1}})}{2e^{2^u}}$$

Converting back to g notation,

$$h'(u) = -2^u(ln2)2^{u-1}g(u-1)g(u) + (ln2)2^u g(u)$$

$$h'(u) = -(ln2)2^u g(u)(2^{u-1}g(u-1) - 1)$$

Substituting for $h(u)$,

$$h'(u) = -(ln2)2^u g(u)(2^u * 2^{-1}g(u-1) - 1)$$

$$= -(ln2)2^u g(u)(2^u * 2^{-1}g(u-1) - 1)$$

$$= -(ln2)2^u g(u) * 2^u * 2^{-1}g(u-1) + (ln2)2^u g(u))$$

$$= -(ln2)(2^u g(u) * 2^u * 2^{-1}g(u-1) - 2^u g(u))$$

$$= -(ln2)(2^{2u-1}g(u)g(u-1) - 2^u g(u))$$

$$= -(ln2)(2^u g(u))(2^{u-1}g(u-1) - 1)$$

$$= -(ln2)(2^u g(u) - 1 + 1)(2^{u-1}g(u-1) - 1)$$

and converting back to h notation, we obtain our differential-delay equation for the relative error:

$$h'(u) = -(ln2)(h(u) + 1)h(u-1) \tag{13}$$

Our DDE (differential-difference equation) $h'(u)$ depends on the value of $h(u)$ at u and $u-1$, which causes the time delay.

3.3 Taking the limit of the relative error function

To compute the limit, we assume that the relative error function $h(u) = 0$ from $u = 0$ to $u = u_o$ for some u_o. The solution is then perturbed by a single value, which produces a jump in the function to a nonzero value $h(u_o)$. From that point on, for $u > u_o$, the DDE in (12) dictates the behavior of $h(u)$. Due to Euler's observations, we expect $\frac{1}{lnx}$ to be stable under perturbations. To show that, we need to demonstrate that the limit of $u \to \infty$ equals zero.

If $h(u) \to l$ for some $l \in R$, the behavior of the function is defined by $h'(u)$. Using (12), we set the derivative equal to zero,

$$-(ln2)l(l+1) = 0$$

From the given information, we can say that l either equals -1 or 0. Wright demonstrated that the expression above has unique and continuous solutions when the initial conditions fall within the range of $-1 \leq l \leq 0$. He also proved that $h(0)$ determines the limiting value of h. Consequently, if, after a certain perturbation in u_o, $h(0) > -1$, then the limit of $h(u)$ is zero. That implies that the idealized line path, represented by the natural logarithm of x, remains stable under perturbations. By using the change of variables, specifically setting $x = e^{2^u}$ and $u = 0$, we find that $x = e$. Translating this back to h notation, we can conclude that $h(e) = 0$ when $h(0) = -1$, and when $h(0) > -1$, $h(u) \to 0$ as $u \to \infty$.

4 Solutions of the DDE

4.1 Numerical Solutions

Now that we know that the limit defined by $h(u)$ tends to zero as x goes to infinity, we want to know how exactly that happens. The equation for the relative error obtained in (12) is a non-linear differential-delay equation, which makes it difficult to solve analytically. Hence, numerical methods such as the tangent line approximation can be used to solve it.
First, we approximate at the point ($u = uo$,f($u = uo$)), which gives the point-slope equation:

$$h(u) \approx h(u_o) + h'(u_o)(u - u_o)$$

Now, we want to substitute for $h'(u_o)$ in $h'(u)$ in (12), resulting in

$$h(u) = h(u_o) - (ln2)(h(u_o) + 1)h(u_o - 1)(u - u_o) \tag{14}$$

This equation sets a relation between the derivative $h(u_o)$ and $h(u)$. If $h(u) = 0$ on an interval of length one, we can predict using (13) that the next interval will also be zero. This assumption is possible because of the linearity of this numerical approach. We do not expect this same outcome when the system is perturbed because nonzero initial conditions provide nonzero values for $h(u)$.

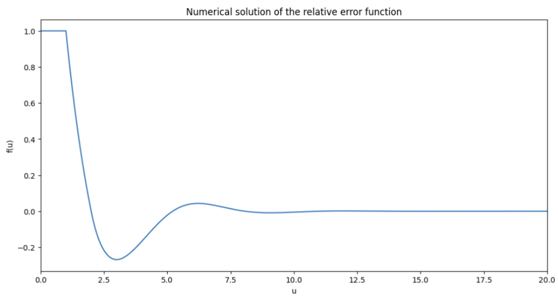
Plotting the numerical solution In order to graph how this perturbed system will behave, we first assume that $h(u) = 0$ for $u < -1$ and set $h(-1) = 1$. Equation (12) reacts the same way to a single perturbation no matter what value of u_o it occurs on. Only a shift occurs along the u-axis, so we can freely take $u_o = -1$ without losing generality. This value of u_o perturbs the ideal solution $1/lnx$ by a factor of two,

$$\pi(x) = \frac{1}{ln(e^{2^{-1}})}$$

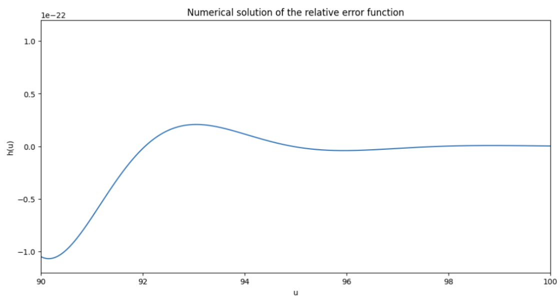
$$= \frac{1}{2^{-1}lne}$$

$$= \frac{2}{lne}$$

We know that $h(u) = 1$ on the whole interval $-1 \leq u \leq 0$ because (12) tells us that $h'(u) = 0$ on that interval. Using (13), we find the values of h for $u > -1$, and we plot the approximation on the interval $0 \leq u \leq 20$ (a) and $90 \leq u \leq 100$ (b).



(a)



(b)

Figure 6: Relative error of $Li(x)$

Figure 4 shows the precise path of the relative error $h'(u)$ before it goes to zero as $u \to \infty$. From the change in variables, we know that u stands for e^{2^u}, so this graph shows the perturbation of the density up to $e^{2^{20}} = e^{1048576}$.

4.2 Asymptotic Solutions

Now that we have found the numerical solutions to the nonlinear DDE in (12), we want to find analytical solutions to the associated *linear* DDE. We will see that both methods give very similar solutions as expected.
Since we know that $h(u) \to 0$ as $u \to \infty$, we can comfortably remove the quadratic term $h(u)+1$ of the equation (12) corresponding to $2^u f(e^{2^u})$. From this, we obtain

$$h'(u) = -(ln2)h(u-1) \tag{15}$$

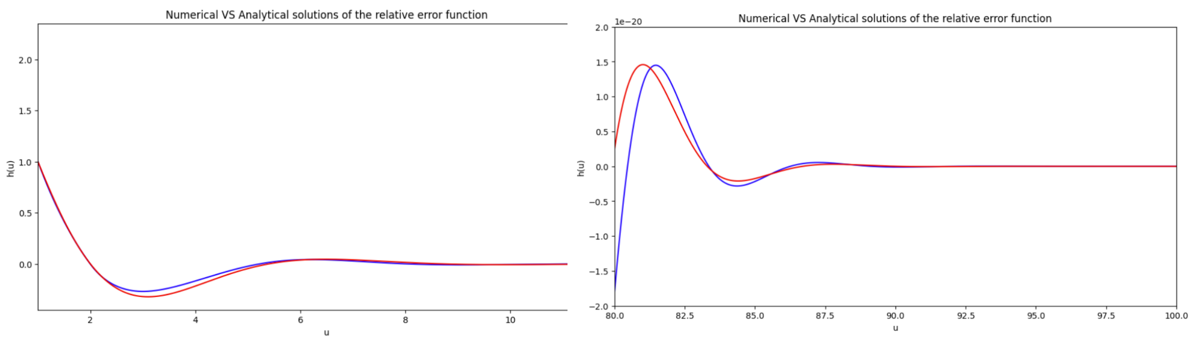
To solve this linear differential equation, we will use the *Ansatz* $h(u) = Ae^{wu}$, where $w = a + bi$. Substituting this *Ansatz* in (14), we get

$$\begin{aligned} Awe^{wu} &= -(ln2)Ae^{w(u-1)} \\ we^{wu} &= -(ln2)e^{wu}e^{-w} \\ we^{w} &= -ln2 \end{aligned} \tag{16}$$

The solutions arising from the expression (15) occur in conjugate pairs with negative real parts. Those lead to exponentially-damped oscillatory solutions, similar to our results from the numerical approach. Lambert-W found the solution with the smallest negative part using a multi-valued complex-valued function implemented in Maple [16]. That value is $w = -0.5716236091 \pm 1.086461157i$ and it is expected to dominate the behavior for large u. In fact, Wright proved that all solutions to (12) are asymptotic to

$$Ae^{-0.5716236091u}sin(1.086461157u + B) \tag{17}$$

. for A and B within the appropriate range provided that $h(0) > -1$. In Figure 5, we compare the numerical (in blue) and analytical (in red) results, where A and B satisfy $h(0) = 1$ such that the initial conditions match. Graphs (a) and (b) range from $0 \le u \le 10$ and $80 \le u \le 200$.



(a) (b)

Figure 7: Discrepancy between the numerical(blue) and analytical (red) solutions

Considering the small discrepancy between the analytical and numerical solutions for values up to $u = 200$ (resulting in $e^{1.6*10^{60}}$), we can conclude that both approaches are valid for approximating the relative error of our model $h'(u)$ and the ideal solution $\pi(x)$.

5 Littlewood's Theorem

Now that have more knowledge about the relative error $h(u)$, we can circle back to Littlewood's theorem. This theorem states that $\pi(x) - Li(x)$ changes sign infinitely often, and we want to show that our model agrees with that statement. To begin with, our differential equation (1) allows us to define $\pi(x)$ as the integral of f(t) from the interval 2 to x. With $f(t)$ being a solution of equation (1), we are integrating the density of primes at x. With $Li(x) = \frac{1}{lnt}$ on that interval, we get

$$\pi(x) - Li(x) \approx \int_2^x (f(t) - \frac{1}{lnt})\, dt \tag{18}$$

Note that $\pi(x)$ is the cumulative based on the solution of (1). is This integral can also be solved by the previously used change of variables, setting $t = e^{2^s}$.

$$\pi(x) - Li(x) \approx \int_2^x (f(e^{2^s}) - \frac{1}{ln(2^s)})\, ds$$

From (7), we know that

$$\int_2^x f(e^{2^s})\, ds = (ln2)2^s(g(s) - 2^{-s})ds$$

Using (10), we substitute for h and obtain this:

$$(f(t) - \frac{1}{lnt})dt = (ln2)e^{2^s}h(s)ds \tag{19}$$

from our model of the differential equation. For instance, h is asymptotic to the ideal path line, defined as an exponentially-damped trigonometric function. Therefore, the solution to $\pi(x) - Li(x)$ can be found by integrating an exponentially growing trigonometric function. The double exponential will eventually dominate and it will generate extreme variations from maximums to minimums. Integrating that function results in a series of cancelations from the positive area to the negative area and vice-versa. The highs and lows of this system create many transitions from positive to negative, thus alluding to infinitely many zeroes. This observation supports Littlewood's theorem, once again showing that there are infinitely many sign changes in the difference between the logarithmic integral and the cumulative number of primes $\leq x$.

6 Limitations of the model

6.1 Conflict with Riemann Hypothesis

We just showed why each zero of $h(u)$ creates a zero in the difference $\pi(x) - Li(x)$. We are currently interested in predicting the number of sign changes on the interval of integration, so from 2 to x. With knowledge of the specific frequency and period of the trigonometric function in (16), we can learn more about the frequency of the change. That frequency is the following:

$$freq = \frac{1.086461157u}{\pi}$$

Now, we want to convert u back to the initial variable $x = e^{2^u}$,

$$lnx = ln(e2^u)$$

$$lnx = 2^u lne$$

$$lnlnx = ln(2^u)l$$

$$lnlnx = uln2$$

$$u = \frac{lnlnx}{ln2}$$

Thus, the frequency becomes

$$freq = \frac{1.086461157lnlnx}{\pi ln2} \tag{20}$$

The frequency in (19) is expected to be a lower bound to the actual number of sign changes because there could be many sign changes accumulated within a single one predicted by this model. In fact, we assume it is a continuous model but it does have an effect of randomness. In 1985, Kaczorowski proposed that there are at least $clnx$ sign changes within the interval $(2, x)$

for some positive constant c. Nonetheless, the number where the first sign change occurs has not been found yet, partly due to the immensity of the numbers and the limited resources at our predisposition. The best clue that we have is that the first one happens for some $x < 1.39 * 10^{316}$, a number that no program on Earth can compute yet. Since this is as much information as we have on Kaczorowski's discoveries, we cannot confirm the reason for our model being a lower bound.

Littlewood also provided expressions that estimate exactly how much the absolute error $\pi(x) - Li(x)$ differs from zero. The precise discrepancies can be calculated using the inequalities in (2) and (3). We now want to show that our model predicts a similar outcome. To do so, we will use the solution to $h(u)$ in (16) to solve the integral in (17). The model provides two inequalities where $h\epsilon$ is a positive constant for small values of $\epsilon > 0$, such that large values of x satisfy

$$\pi(x) - Li(x) > k\epsilon x^{2^{-\epsilon}} (lnx)^{-0.825} \tag{21}$$

or

$$\pi(x) - Li(x) < -k\epsilon x^{2^{-\epsilon}} (lnx)^{-0.825} \tag{22}$$

Comparing the above expressions with (2) and (3), the right-hand sides of the functions indicate a slowly increasing relative error for large x. However, (20) and (21) predict even larger oscillations, which comes in conflict with the Riemann hypothesis.

The Riemann Zeta Function In fact, larger oscillations indicate a larger number of zeros on the interval $(2, x)$. The Riemann hypothesis, conjectured by the German mathematician Bernhard Riemann in 1859, treats the solutions to the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

[] where s is a positive and constant integer greater than 1. Calculating the first few values gives

$$zeta(s) = 1 + 2^{-s} + 3^{-s} + ... + n^{-s}$$

Since Euler was the first to discover the zeta function in the 18th century, it is sometimes called the Euler zeta function. It is known for its many applications in number theory. For example, the zeta function for even integers has a relation to the Bernoulli numbers, which appear as the coefficients of the Taylor series expansion of the function. More importantly, in 1737, Euler discovered a formula relating the zeta function to all prime numbers. It is of the form

$$zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi \frac{1}{1 - p^{-s}} \tag{23}$$

[9]

Knowing that (22) equals zero for all negative even integers such as -2,-4,-6,.. and those are called $trivialzeros$. Riemann's hypothesis is precisely that there is a critical line $x = 1/2$ where all the nontrivial zeros lie.

6.2 Comparing the bounds

There exists another version of the Riemann hypothesis that serves as bounds for the absolute error $\pi(x) - Li(x)$. For all $\epsilon > 0$, there exists a positive constant $C\epsilon$ such that

$$|\pi(x) - Li(x)| \leq C\epsilon x^{\frac{1}{2}+\epsilon} \tag{24}$$

To compare the bounds from Littlewood's theorem and Riemann hypothesis, we can graph the upper and lower bounds of the absolute error represented in (2) and (3) for Littlewood, and (23) for Riemann.

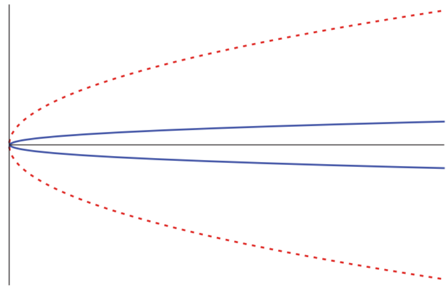


Figure 8: Comparison of bounds for $\pi(x) - Li(x)$; ; Littlewood (solid), Riemann (dotted)

Figure 6 shows that the Riemann hypothesis has larger bounds for the relative error than Littlewood's theorem. In turn, if we were to graph the bounds predicted by our model, they would exceed significantly the dotted line. Hence, this model could be flawed in that sense because of that discrepancy in the results, even though Riemann's hypothesis has not been proved yet this day but has a large body of evidence that supports it.

7 Conclusion

The model of the differential equation (1) we have studied is entirely based on the sieve of Eratosthenes. Our model agrees with the previous findings in many ways. We have seen that the density of primes predicted by that model is asymptotic to the natural logarithm, as shown in Figure 2. On top of that, the relative error between a perturbed solution of equation (1) and the ideal path line $\frac{1}{lnx}$ goes to zero as x approaches infinity, just like it was expected. As per the absolute error $\pi(x) - Li(x)$, our model agrees with Littlewood's theorem by predicting its infinitely many jumps.

As mentioned earlier, the sieve of Pritchard has better computational efficiency and runs in sublinear time. To produce a more accurate version of the model, we could instead create a differential equation for the density of primes, based on Pritchard's sieve. Doing so would be a significant contribution to number theory and provide new avenues for prime number theory and its applications.

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