Exercise 3: Christian Lehre & Emil Myhre

Problem 1 :

In this problem we consider the Poisson problem with Dirichlet boundary conditions on a unit square Ω

$$\begin{aligned} &u_{xx} + u_{yy} = f \quad x, y \in [0,1] \\ &u(x,y) = g(x,y) \text{ on } \partial\Omega \end{aligned} \tag{1}$$

The PDE is discretized by the so-called 5-point formula, which results from 3-point central difference approximations for both second derivatives (note that we use equal step-sized in the x- and y-direction, i.e h = k).

$$\begin{aligned} &\frac{1}{h^2}\delta_x^2 U_p + \frac{1}{k^2}\delta_y^2 U_p = f_p \\ \\ &U_w + U_s + U_n + U_e - 4U_p = h^2 f_p, \quad \text{for } h = k \end{aligned} \tag{2}$$

In this problem we set up a test problem to verify the rate of convergence. The local truncation error for the 5-point formula above is given by

$$\tau_p = \frac{1}{h^2}\left(\partial_x^4 u_p + \partial_y^4 u_p\right) + \dots$$

From the above truncation error, it is clear that the formula gives the exact solution if $\partial_x^4 u = \partial_y^4 u = 0$. To be able to look into the convergence rate of the scheme, we choose a function $u(x,y)$ that is not solved exactly, i.e where $\partial_x^4 u,\ \partial_y^4 u \neq 0$.

Let $u(x,y) = x^5 + 3y^4$, then $u_{xx} + u_{yy} = f = 20x^3 + 36y^2$. Implementing this in the handed-out program poisson.py, we end up with the following solution and error

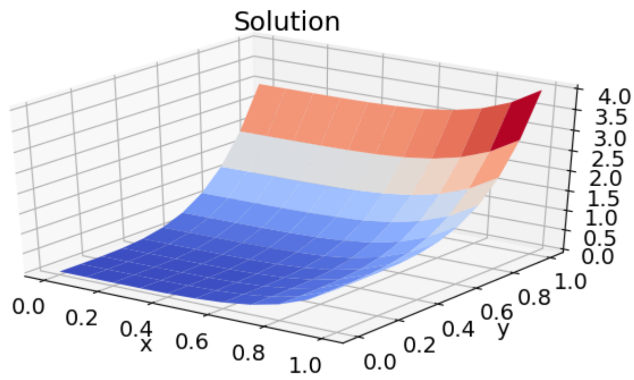

Figure 1: Solution to $u_{xx} + u_{yy} = 20x^3 + 36y^2$ on the unit square $x, y \in [0, 1]$ with Dirichlet conditions

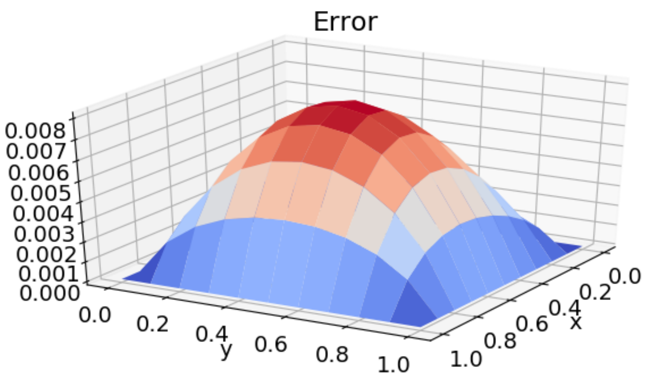

Figure 2: Error u-U over the unit square, observe that the maximum error occurs in the middle of the grid

As we consider an equal amount of gridlines in both directions, i.e $h = k$, the convergence rate in both directions are identical. The convergence rate is found by measuring the error along the gridline in which the error is greatest. This is done for a set of varying step-sizes, and the error is plotted against step-size in a loglog-fashion. The convergence rate can then be extracted as the slope of the resulting curve.

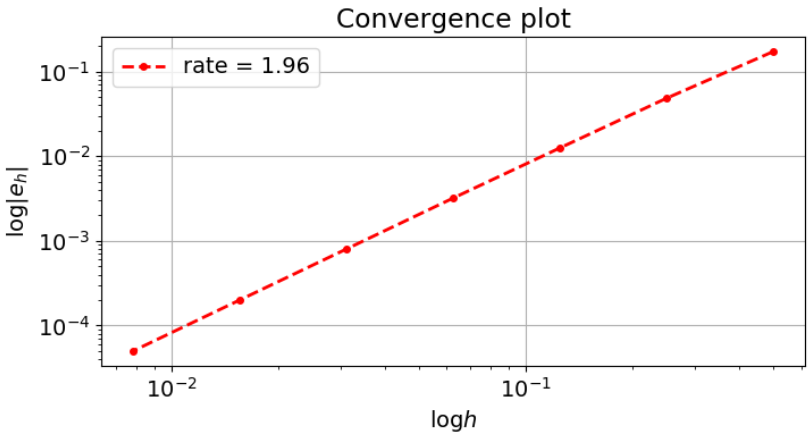

Figure 3: Error $|e_h|$ in the x-direction for varying step-sizes h

The slope was approximated in a least-squares fashion, and the result is a convergence rate of $1.96 \approx 2$.

Problem 2 :

In this problem we solve the Poisson equation yet again, but with a Neumann condition $\partial_y u(x, 1) = g = 12$ on the upper boundary at $y = 1$. This condition was chosen to get the same solution as in Problem 1 above.

To solve this problem, we increase the amount of inner gridpoints to include the Neumann condition at the upper boundary. The boundary condition is discretized using a two-point central difference:

$$\begin{aligned} \partial_y u_p &= \frac{u_{n'} - u_s}{2h} + \mathcal{O}(h^2) \\ \Rightarrow \frac{U_{n'} - U_s}{2h} &= g \end{aligned} \tag{3}$$

where we have included a fictious node $u_{n'}$ outside the grid. The fictious node is eliminated by inserting (3) into (2):

$$\begin{aligned} U_e + U_{n'} + U_w + U_s - 4U_p &= h^2 f \\ U_e + 2U_s + 2hg + U_w - 4U_p &= h^2 f \end{aligned} \tag{4}$$

The problem was solved using (4) on the extra gridpoints corresponding to the Neumann conditions on the upper boundary at $y = 1$, and the usual five-point formula (2) elsewhere. The resulting solution is depicted in figure

4 below. Note that we use the same exact solution $u(x, y) = x^5 + 3y^4$ as in problem 1 above, with neumann condition $\partial_y u(x, y) = 12y^3$ at $y = 1$.

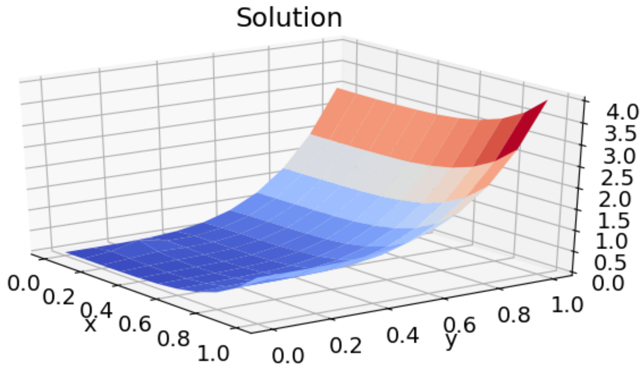

Figure 4: Solution to $u_{xx} + u_{yy} = 20x^3 + 36y^2$ on the unit square $x, y \in [0, 1]$ with Neumann-condition at $y = 1$

Problem 3 :

In this problem we are considering the Poisson equation on a less regular grid. The domain we are analyzing is given by $\Omega = \Omega_2 - \Omega_1$, where

$$\Omega_1 = (x, y) : |x| \leq 1, |y| \,| \leq 1$$

$$\Omega_2 = (x, y) : |x| < 3, |y| < 3$$

We let u denote the height of a membrane over the xy-plane, and the solution satisfies

$$u_{xx} + u_{yy} = 0, \qquad (x, y) \in \Omega$$
$$u = 0, \qquad (x, y) \in \partial\Omega_1$$
$$u = 1, \qquad (x, y) \in \partial\Omega_2$$

Again, we are using central difference to approximate the second derivatives, yielding the 5 points formula. However, here the issue is how to deal with the inner square Ω_1 in the calculations. We decided to design our algorithm in the following way. First, we design a grid over the whole square, that is $\Omega_1 + \Omega_2$. Using the 5 point formula, we designed the numerical scheme including all grid points. Since the points within Ω_1 were considered to be 0, we localized these points and removed the rows and columns corresponding to these points, from the A matrix and the solution vector b. This leaves us with a smaller system, only solving for grid points in Ω. After finding this solution, we add the solution $u = 0$ for all $(x, y) \in \Omega_1$, and add the boundary $u = 1$ for $(x, y) \in \partial\Omega_2$. The resulting solutions are presented in figure 5, 6, 7.

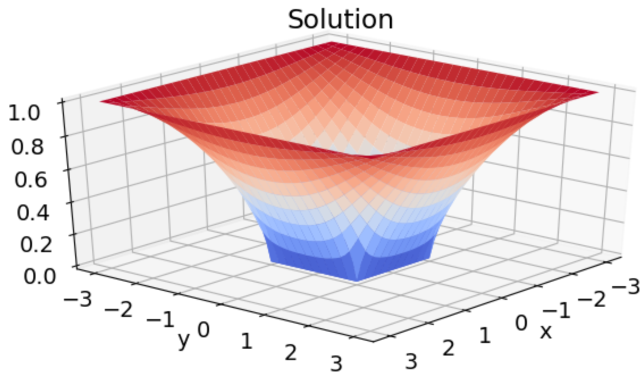

Figure 5: Visual representation of the height of the membrane

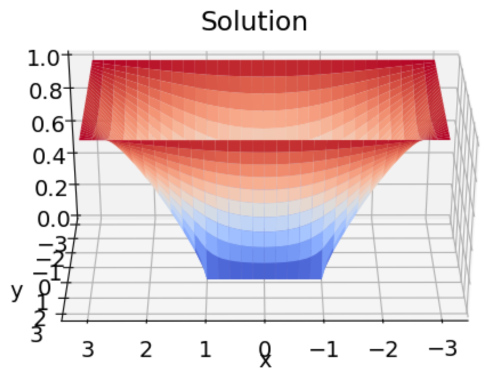

Figure 6: View of the solution perpendicular to the x-axis

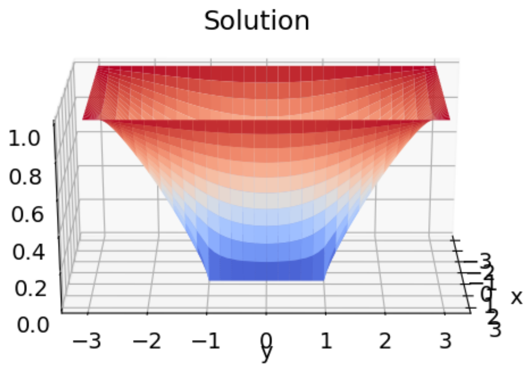

Figure 7: View of the solution perpendicular to the y-axis

As expected, the height is 0 in Ω_1 and increasing towards 1 when getting closer to the boundary $\partial\Omega_2$. By solving the solution first with a very fine grid, and comparing this solution with solutions with greater stepsizes, we made a convergence plot for the method, presented in figure 8.

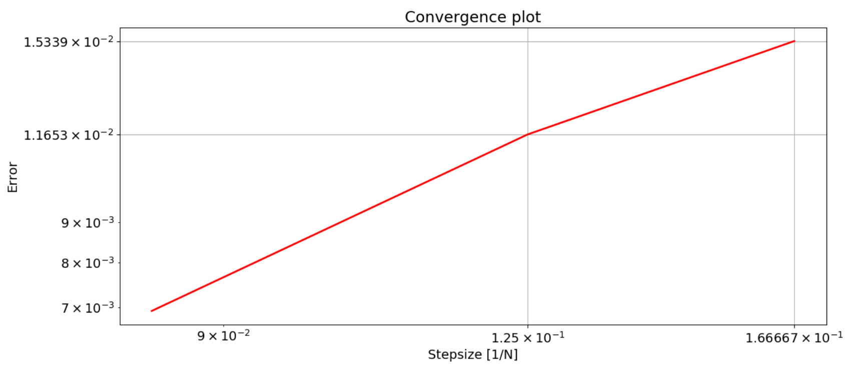

Figure 8: Convergence plot for the method

Due to irregularity in the corners, the error is large in these areas, which also leads to a reduction in order. We managed to get a slope of convergence equal to $\approx 1,2$, but we would obviously theoretically expect order 2.