Exercise 2: Christian Lehre & Emil Myhre

Problem 2 :

a)

In this exercise we are considering the heat equation.

$$u_t = u_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T = 0.5 \tag{1}$$

with boundary and initial values given by

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = f(x) \tag{2}$$

where we have chosen $f(x) = sin(\pi x)$.

We have used two different difference schemes to solve this problem. First, we used the Backward Euler method given by the linear system

$$\begin{bmatrix} 1+2r & -r & & & 0 \\ -r & \ddots & & & \\ & & \ddots & \ddots & \\ & & 0 & -r & 1+2r \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ \vdots \\ U_M^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n + rg_0^{n+1} \\ U_2^n \\ \vdots \\ U_{M-1}^n \\ U_M^n + rg_1^{n+1} \end{bmatrix}$$

Here, our boundary conditions imply that $g_0 = g_1 = 0$ which makes the computations even more simple.

Secondly, we used the Crank-Nicolson method gived by the following system of equations

$$\begin{bmatrix} 1+r & -\frac{r}{2} & & & 0 \\ -\frac{r}{2} & \ddots & & & \\ & & \ddots & \ddots & \\ & & 0 & -\frac{r}{2} & 1+r \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ \vdots \\ U_M^{n+1} \end{bmatrix} = \begin{bmatrix} d_1^{n+1} + \frac{r}{2}g_0^{n+1} \\ d_2^{n+1} \\ \vdots \\ d_{M-1}^{n+1} \\ d_M^{n+1} + \frac{r}{2}g_1^{n+1} \end{bmatrix}$$

Where d_m^{n+1} is defined as

$$d_m^{n+1} = \frac{r}{2}U_{m-1}^n + (1-r)U_m^n + \frac{r}{2}U_{m+1}^n$$

Again, $g_0 = g_1 = 0$.

By applying these schemes to our problem, we got several numerical solutions to compare with the exact solution given by

$$u(x,t) = e^{-\pi^2 t} sin(\pi x)$$

The error was considered in the Euclidean norm at $T = 0.1$, and convergence was analyzed in both space and time. For accurate results, we solved sparse systems, in order to allow us to use finer grids. The error was plotted as a function of stepsizes h and k, the resulting plots are presented in figure 1 and figure 2, respectively.

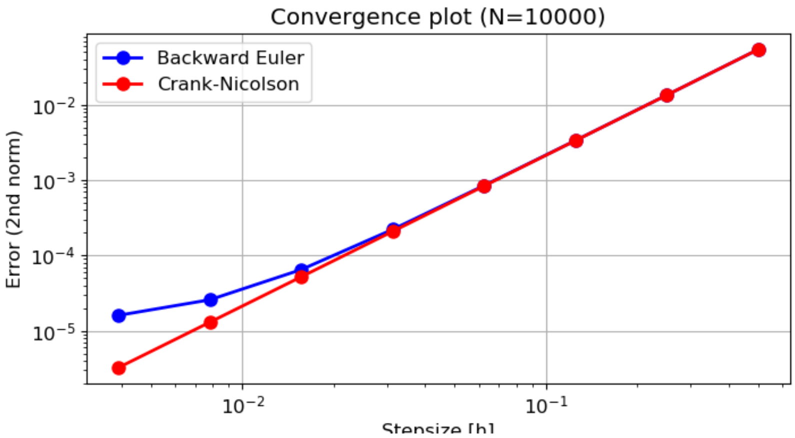

Figure 1: Convergence in space with fixed N = 10000

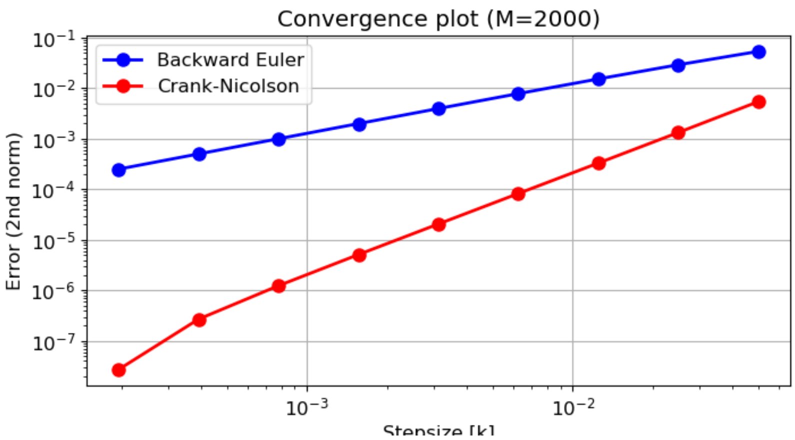

Figure 2: Convergence in time with fixed M = 2000

By studying the slope of these plots we can conclude that the order in space for both the Backward Euler and the Crank-Nicolson scheme is 2. In time, the order of the methods are 1 and 2, respectively. These results correspond to what we expected theoretically.

b)

In this problem we are also considering the heat equation, but with periodic boundary conditions, given by

$$\begin{aligned}
&u(x,0) = f(x), \quad -\infty < x < \infty \\
&f(x) = sin(\pi x), \quad 0 \leq x < 1 \\
&f(x+1) = f(x), \quad -\infty < x < \infty \\
&u(x+1,t) = u(x,t), \quad -\infty < x < \infty
\end{aligned}$$

We solved this problem on $x \in [0,1]$, in order to do so we made use of the last condition $u(x+1,t) = u(x,t)$ to design our scheme. We identify by central difference that the boundary points on this interval are dependent of each other. Hence we obtain the following Crank-Nicolson scheme for this problem

$$(\mathcal{I} - \frac{r}{2})AU^{n+1} = (\mathcal{I} + \frac{r}{2})AU^n$$

with A being given as

$$A = \begin{bmatrix} -2 & 1 & 0 & \dots & 1 \\ 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 1 & -2 \end{bmatrix}$$

The numerical solution was compared to the following exact solution calculated by Fourier analysis

$$u(x,t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi - 4\pi n^2} \cdot cos(2n\pi x) \cdot e^{-2^2 n^2 \pi^2 t}$$

and the error at $T = 0.1$ was considered again in the 2nd norm, and being plotted as a function of both h and k. Sparse matrices were again used for better accuracy. The convergence plots in space and time are presented in figure 3 and figure 4, respectively.

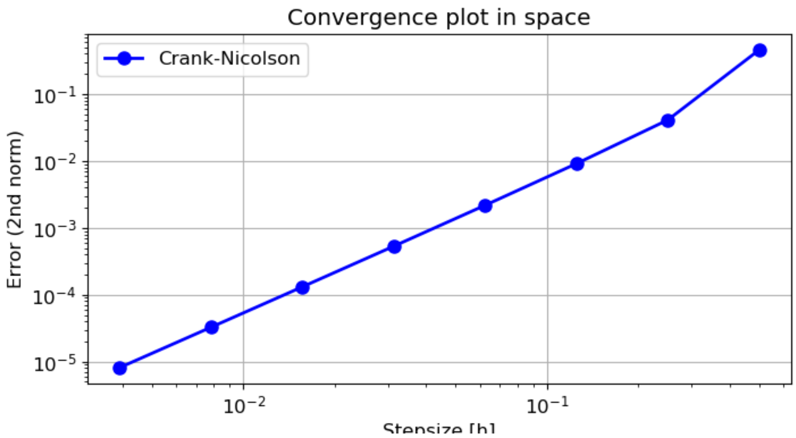

Figure 3: Convergence in space with fixed N = 10000

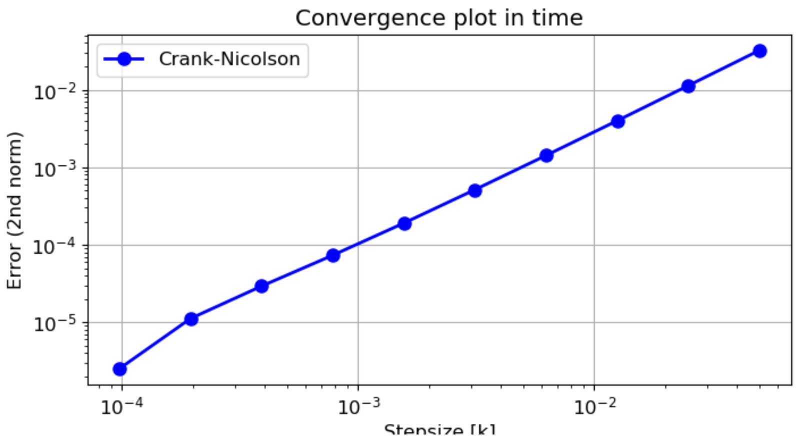

Figure 4: Convergence in time with fixed M = 3000

In space, our graph suggests the method is of order 2, as expected. In time however, we are achieving a slope of $\approx 1,7$, when we expect 2. This might be due to some numerical or implementing error.

Problem 3 :

In this problem we consider the following diffusion-reaction equation:

$$\begin{aligned} u_t &= du_{xx} + u(1-u), \quad x \in [0,1] \\ u_x(0,t) &= u_x(1,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad x \in [0,1] \end{aligned} \tag{3}$$

The problem is solved by so-called semi-discretization, where the above equation is discretized in the x-direction.

$$\begin{aligned} \partial_t u(x_m,t) &= d\partial_x^2 u(x_m,t) + u(x_m,t)(1-u(x_m,t)) \\ &= \frac{d}{h^2}\delta_x^2 u(x_m,t) + u(x_m,t)(1-u(x_m,t)), \end{aligned}$$

where $m = 0,\ldots,M$, using a three-point central difference scheme in x, denoted by δ_x^2.

We introduce a one-variable approximation $v_m(t)$ to $u(x_m,t)$, and rewrite the problem as

$$\dot{v}_m = \frac{d}{h^2}\delta_x^2 v_m(t) + v_m(t)(1-v_m(t))$$

.

The boundary conditions in (3) includes derivatives, and is approximated by second order central difference as follows, using the same one-variable approximation.

$$\frac{v_1 - v_{-1}}{2h} = 0, \quad \frac{v_{M+1} - v_{M-1}}{2h} = 0,$$

where we have included fictious gridlines at $m = -1$ and $m = M+1$. From the approximated boundary-conditions, we end up with the useful result

$$v_{-1} = v_1, \quad v_{M+1} = v_{M-1}$$

To get rid of the fictious gridlines, we plug this result into the scheme in the endpoints of our domain, i.e at $m = 0$ and $m = M$:

$$\begin{aligned} \dot{v_0}(t) &= \frac{d}{h^2}\delta_x^2 v_0(t) + v_0(t)(1-v_0(t)) \\ &= \frac{d}{h^2}(v_1(t) - 2v_0(t) + v_{-1}(t)) + v_0(t)(1-v_0(t)) \\ &= \frac{d}{h^2}(2v_1(t) - 2v_0(t)) + v_0(t)(1-v_0(t)) \end{aligned}$$

$$\begin{aligned} \dot{v_M}(t) &= \frac{d}{h^2}\delta_x^2 v_M(t) + v_M(t)(1-v_M(t)) \\ &= \frac{d}{h^2}(v_{M+1}(t) - 2v_M(t) + v_{M-1}(t)) + v_M(t)(1-v_M(t)) \\ &= \frac{d}{h^2}(2v_{M-1}(t) - 2v_M(t)) + v_M(t)(1-v_M(t)) \end{aligned}$$

Thus, the final semi-discretized scheme is given by

$$\begin{aligned}
\dot{v_m}(t) &= \frac{d}{h^2}\delta_x^2 v_m(t) + v_m(t)(1 - v_m(t)), \quad m \in [1, M-1] \\
\dot{v_0}(t) &= \frac{d}{h^2}(2v_1(t) - 2v_0(t)) + v_0(t)(1 - v_0(t)) \\
\dot{v_M}(t) &= \frac{d}{h^2}(2v_{M-1}(t) - 2v_M(t)) + v_M(t)(1 - v_M(t)) \\
v_m(0) &= f(x_m), \quad m \in [0, M]
\end{aligned}$$

For a more compact notation, we let $\mathbf{v(t)} = [v_0(t), \ldots, v_M(t)]^T$, and obtain a linear system of ordinary differential equations on the form

$$\dot{\mathbf{v(t)}} = \frac{d}{h^2}A\mathbf{v} + \mathbf{v}(I - \mathbf{v}), \tag{4}$$

where

$$A = \begin{bmatrix} -2 & 2 & & & 0 \\ 1 & -2 & 1 & \ddots & \\ & & \ddots & \ddots & \\ & 0 & & 2 & -2 \end{bmatrix} \in \mathbb{R}^{(M+1)\times(M+1)}$$

Equation (4) describes a set of ODES on the form $\dot{\mathbf{v}} = \mathbf{F(t,v)}$, with initial values $\mathbf{v(0)} = f(\mathbf{x}) = [f(x_0), \ldots, f(x_M)]^T$.
The system of ODEs was solved using a numerical solver, namely

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scipy.integrate.solve_ivp
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from the scipy module in python. The solution to the problem at different angles is depicted in figure 5 below. According to the problem description, we used $d = 0.01$ and $f(x) = \sin(\pi(x - 0.25))^{100}$. Integration was done over $t \in [0, 10]$

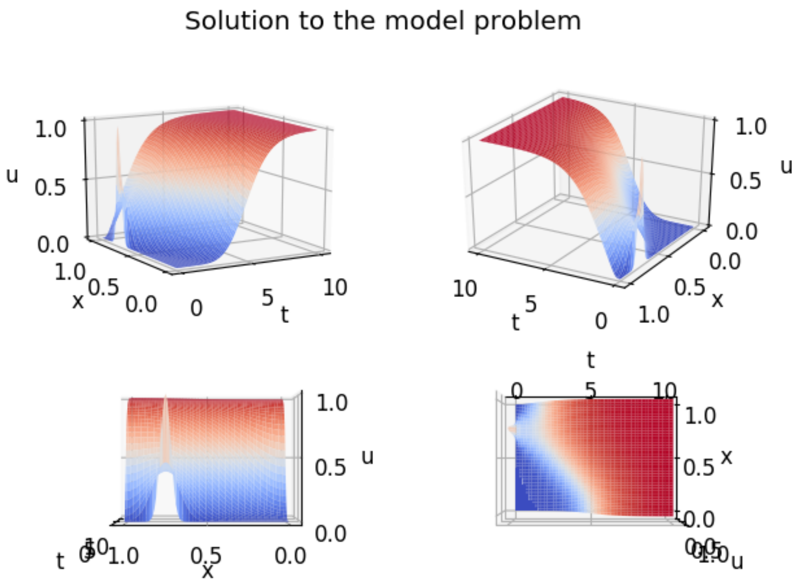

Figure 5: Solution to the model problem from different point of views

The problem we consider is a normalized version of Fisher's equation, where the goal is to model a population with diffusion taken into account. Note that the red areas in figure 5 above correspond to high denisty population, whereas the blue areas corresponds to less dense population.

Notice the spike in the initial solution, i.e at $t = 0$, at $x = 0.75$.
This correspond to the maximum value of the initial condition $f(x)$, namely when the argument of $\sin(\cdot)$ is zero. The boundary conditions, i.e no net flux of population at the boundaries, is also apparent from the plots.

In the right, lowermost plot one can see how the solution spreads out in space over time,i.e how the population varies in space and time. Initially, at $t = 0$, notice how the dense population at $x = 0.75$ spreads out to the less densely populated areas.

The population grows bigger with time, and one can observe that the population is dense over all space after $t \approx 5$.