# Exercise 4: Christian Lehre & Emil Myhre

## Problem 1 :

In this problem we consider the transport problem

$$u_t + au_x = 0, \quad x \in [0,3], \quad u(x,0) = f(x), u(0,t) = g(t) \tag{1}$$

with a > 0.

We solve the problem by implementing the following schemes.

$$\begin{aligned}
&\text{FTBS}: \quad U_m^{n+1} = U_m^n - r\big(U_m^n - U_{m-1}^n\big) \\
&\text{Lax-Wendroff}: \quad U_m^{n+1} - \frac{r}{2}\big(U_{m+1}^n - U_{m-1}^n\big) + \frac{r^2}{2}\big(U_{m+1}^n - 2U_m^n + U_{m-1}^n\big) \\
&\text{Wendroff}: \quad U_m^{n+1} = U_{m-1}^n - \frac{1-r}{1+r}\big(U_{m-1}^{n+1} - U_m^n\big)
\end{aligned}$$

We are solving the problem given by (1), with initial and boundary values given by

$$u(x,0) = 0, \quad u(0,t) = 1$$

This problem has the following exact solution

$$u(x,t) = \begin{cases} 1, & x < at \\ 0, & x > at \end{cases}$$

We solved the problem for all three methods, and parameters $a = 1$, $t \leq 2$, $h = \frac{1}{160}$ and $k = rh$, where h and r are stepsizes in x and t direction respectively. The numerical solution was found for both $r = 1$ and $r = 0.5$, presented in figure 1 and 2 respectively. The solution is plotted at different time intervals

$$t = 0 + 0,5 \cdot i, \quad i \in \{0,1,2,3,4\},$$

illustrated by the different colors. Since the velocity $a = 1$, the solution travels one unit along the x-axis per second.

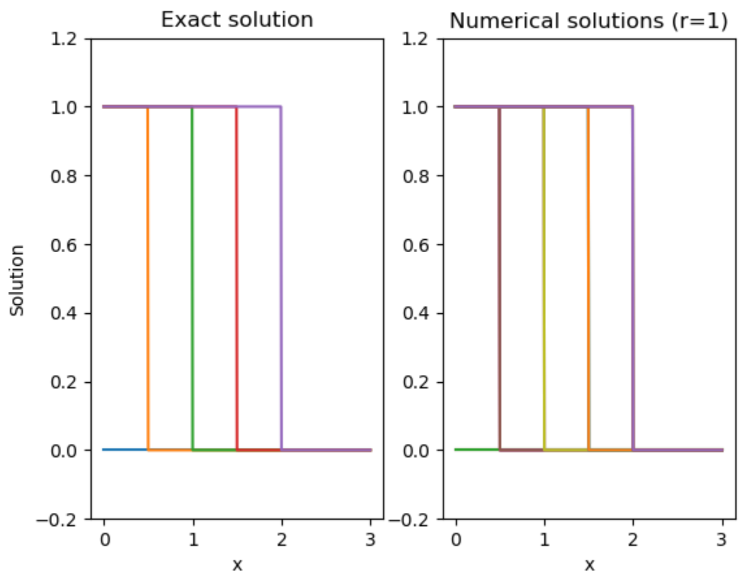


Figure 1: Exact solution (left panel) and numerical solutions for all schemes with r=1 (right panel). The numerical solutions were all identical and also equal to the exact solution.

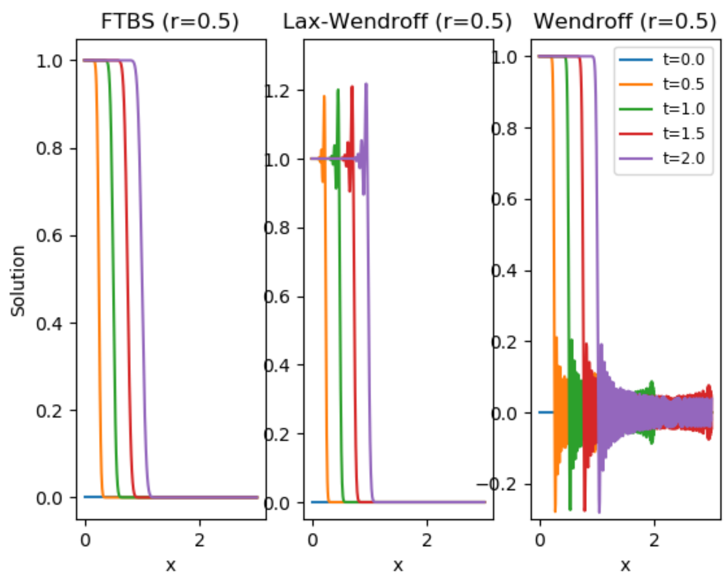


Figure 2: Numerical solution with FTBS scheme (left panel), Lax-Wendroff scheme (central panel) and the Wendroff Scheme (right panel) with Courant number $r = 1.0$

We observe that in the case of $r = 1$, all numerical solutions equals the exact solution. However, for $r = 0.5$, we see that all numerical solutions deviates in some manner from the exact solution from figure 1. The properties of these schemes for different Courant numbers will be further discussed in the next problem.

## Problem 2 :

### a)

In this problem we look into dispersion and dissipation of the methods, i.e how the phase and amplitude of the solution varies with time, respectively.

A method is dissipative of order $2s$ if there is an upper bound for the time step $k_0 > 0$ and a constant $\sigma > 0$ such that

$$|\xi| \leq 1 - \sigma(\beta h)^{2s} \quad \text{for} |\beta h| < \pi$$

Moreover, a method is dispersive if $\alpha$ depends on $\beta$.

Lets look at the Wendroff-scheme first. If we let $U_m^n = \xi^n e^{i\beta x_m}$ and combine this numerical approximation with the scheme for the method, we end up with

$$\xi = e^{-i\beta h} - \frac{1-r}{1+r}\left(\xi e^{-i\beta h} - 1\right)$$

$$= \frac{1 + \frac{1-r}{1+r} e^{i\beta h}}{e^{i\beta h} + \frac{1-r}{1+r}}$$

$$= e^{i\beta h}\left(\frac{e^{-i\beta h} + \gamma}{e^{i\beta h} + \gamma}\right), \quad \gamma = \frac{1-r}{1+r}.$$

From the equation above, it is clear that $|\xi| = 1$, as $\left|e^{ix}\right| = 1$ for all x, and that the fraction can be written as $\bar{z}/z$ where $z$ is an arbitrary complex number, and $\bar{z}$ is its complex conjugate. Hence, using the definition of dissipation above, we expect the method to be dissipative of order 0, i.e not dissipative. Further on, if we write $\xi$ in its polar form,

$$\xi = |\xi|\, e^{-i\varphi},$$

and require $\varphi = \beta\alpha k = \beta\alpha r h$, we can find an explicit expression for $\alpha$.

$$\varphi = \tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right) = \tan^{-1}\left(\frac{(1-r^2)\cos\beta h + (1-r)^2}{(1-r^2)\cos\beta h + (1+r)^2}\right)$$

$$\alpha = \frac{a}{r\beta h}\tan^{-1}\left(\frac{(1-r^2)\cos\beta h + (1-r)^2}{(1-r^2)\cos\beta h + (1+r)^2}\right)$$

In the above expression for $\alpha$, we have used the following complex form of $\xi$

$$\xi = \frac{(1-r)\cos\beta h + (1+r)}{(1+r)\cos\beta h + (1-r)} + i\left(\frac{1-r}{1+r}\right).$$

Note that $\alpha$ is constant and equal to 0 in the stability limit $r = 1$, i.e independent of $\beta$. Thus, the method is expected to be non-dispersive in this case. For $r \neq 1$, it is clear that $\alpha$ depends on $\beta$, and the method is expected to be dispersive.

As for the Lax-Wendroff method, the following expression for $|\xi|$ and $\alpha$ are derived in the notes by Brynjulf Owren.

$$|\xi| \leq 1 - \frac{4r^4(1-r^2)}{2\pi^4}(\beta h)^4$$

$$\alpha = \frac{a}{r\beta h}\tan^{-1}\left(\frac{r\sin\beta h}{1 - 2r^2\sin^2\frac{\beta h}{2}}\right)$$

These results show that the method is dissipative of order 2 ($2s = 4$). Also, we get that in the stability limit $r = 1$, we get $\alpha = 1 = a$ independent of $\beta$. Thus, the method is non-dispersive in this case.

Below are plots showing how $|\xi|$ and $\alpha$ varies with $\beta h$ for two values of the Courant number $r$, namely $r = \{0.5, 1\}$.

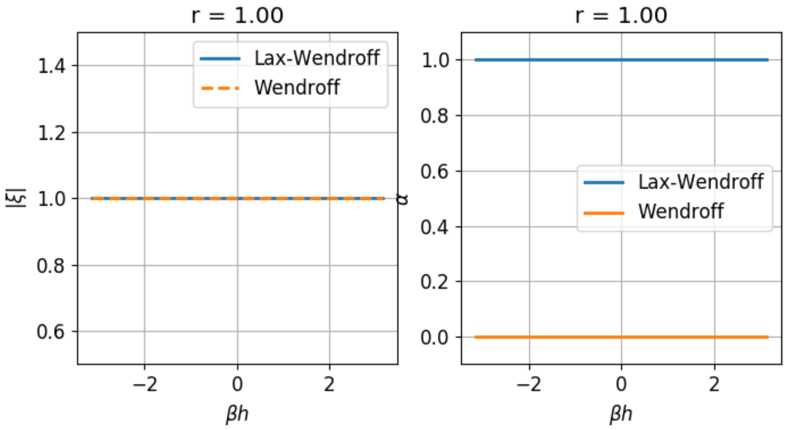


Figure 3: $|\xi|$ (left panel) and $\alpha$ (right panel) against $\beta h$ for both methods with Courant number $r = 1.0$

Observe in figure 3 above that $|\xi|$ is constant and equal to 1 for all $\beta h \in [-\pi, \pi]$ for both methods. That is, both methods are non-dissipative for Courant number $r = 1.0$. It is also clear that $\alpha$ is independent of $\beta h$ for both methods, i.e both methods are non-dispersive in the stability limit $r = 1$.

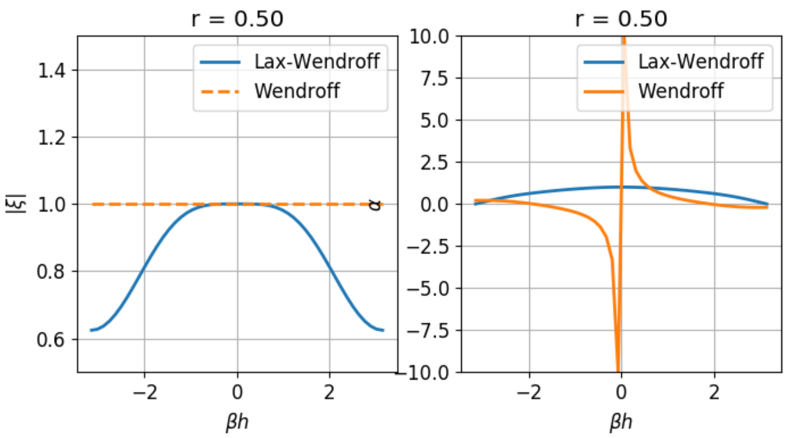


Figure 4: $|\xi|$ (left panel) and $\alpha$ (right panel) against $\beta h$ for both methods with Courant number $r = 0.5$

For the Wendroff method, looking at figure 4 above, we see that $|\xi| = 1$ for all $\beta h$ and that $\alpha$ varies with $\beta h$. Hence, the method is non-dissipative and dispersive for Courant number $r = 0.5$.

As for the Lax-Wendroff method, we see that $|\xi|$ as well as $\alpha$ varies with $\beta h$.

Hence, the method is both dissipative and dispersive.

## b)

In this problem we solve the transport equation with inital value

$$f(x) = \exp -64(x - 0.5)^2 \sin 32\pi x,$$

using Lax-Wendroff's and Wendroff's methods with $h = 1/200$, $k = 1/400$ and $a = 1$, i.e for Courant number $r = 0.5$.
In the solution plots below we restricted ourselves to two time-steps to avoid messy plots.

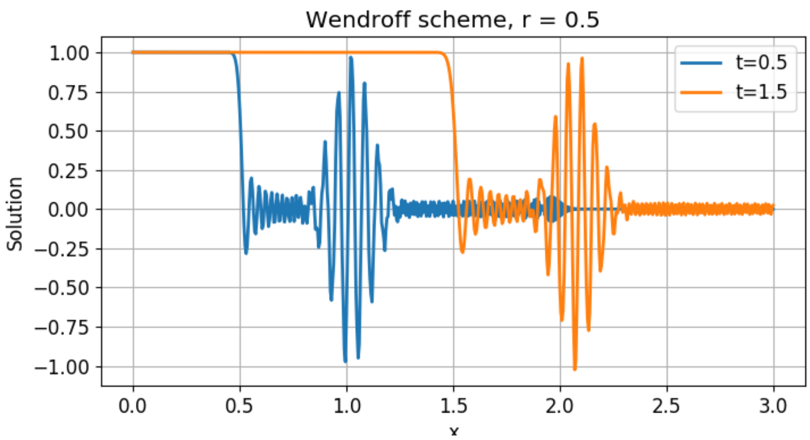


Figure 5: Solution using Wendroffs method at two timesteps

In figure 5 above we have plotted the solution of the transport equation using the Wendroff scheme at two different time-steps, namely $t = 0.5$ and $t = 1.5$. Note how the amplitude stays constant, while the phase of the solution shifts. These observations supports our claim that the Wendroff method is non-dissipative but dispersive at $r = 0.5$.

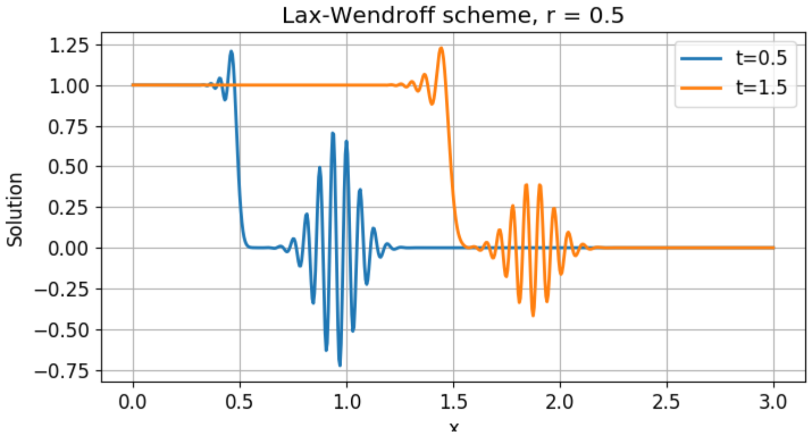


Figure 6: Solution using Lax-Wendroff's method at two timesteps

Looking at the solution computed with Lax-Wendroffs method in figure 6 above, we see clearly that amplitude of the solution varies with time. It is also apparent that the phase is shifted with time (note that he initial "wave packet" moves closer to the discontinuity imposed by the boundary condition). These observations shows that the method is both dissipative and dispersive, and supports our claims from figure 4.

Below are animations showing how the solutions computed from Wendroff's and Lax-Wendroff's methods evolve with time. Note that you, the reader, have to open the pdf in a multimedia compatible pdf-reader, e.g adobe acrobat, in order to view the content.

The animation above holds for Wendroffs method with Courant number

$r = 0.5$. Observe how the phase changes with time, while the amplitude stays constant. This supports our claim that Wandroffs method is dispersice and non-dissipative at $r = 0.5$

The above animaion corresponds to the Lax-Wendroff method with Courant number $r = 0.5$. Observe how both the phase and amplitude changes with time. This supports our above claim.