Exercise 1: Christian Lehre & Emil Myhre

Problem 2 :

In this exercise we consider the following model problem.

$$-u_{xx} = f(x) \quad u(0) = u(1) = 0, f \in C(0,1) \tag{1}$$

For a given M, define the grid by

$$h = 1/M, \quad x_m = m * h, \quad m = 0, 1, \ldots, M$$

The problem is solved by the following central finite difference scheme for the inner gridpoints.

$$\frac{-U_{m+1} + 2U_m - U_{m-1}}{h^2} = f(x_m), \quad m = 1, 2, \ldots, M-1 \tag{2}$$

This results in the linear system of $M \times M$ equations, $A_h\mathbf{U} = \mathbf{F}$, where

$$A_{h1} = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & & & \\ & & \ddots & \ddots & \\ & 0 & & -1 & 2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_{M-1} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 + \frac{0}{h^2} \\ \vdots \\ f_{M-1} + \frac{0}{h^2} \end{bmatrix}$$

The green function is defined as

$$G(x,y) = \begin{cases} y(1-x) & 0 \leq y \leq x \\ x(1-y) & x \leq y \leq 1 \end{cases}$$

a)

The solution of the model problem (1) is given by

$$u(x) = \int_0^1 G(x,y)f(y)dy,$$

Proof.

$$\begin{aligned} u(x) &= \int_0^1 G(x,y)f(y)dy \\ &= (1-x)\int_0^x yf(y) + x\int_x^1 (1-y)ydy \end{aligned} \tag{3}$$

$$
\begin{aligned}
u_x(x) &= -\int_0^x yf(y)dy + \int_x^1 (1-y)f(y)dy \\
&\quad + (1-x)xf(x) - x(1-x)f(x) \\
&= -\int_0^x yf(y)dy + \int_x^1 (1-y)f(y)dy
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
u_{xx}(x) &= -xf(x) - (1-x)f(x) \\
&= -f(x)
\end{aligned} \tag{5}
$$

$\square$

In step (3) the definition of the green function is inserted into the given solution to (1). Further on, in step (4), $u(x)$ is differentiated with respect to x. The last two terms, which is a result of the fundamental theorem of calculus, cancels. In the last step (5), $u_x(x)$ is differentiated with respect to x to yield $u_{xx}(x)$. The calculation follows from the fundamental theorem of calculus.

Now the model problem (1) can be solved by the given relation. For $f = 1$ we have

$$u(x) = \int_0^1 G(x,y)dy,$$

Which is solved by integrating the function G(x,y) over two subintervals.

$$u(x) = \int_0^x y(1-x)dy + \int_x^1 x(1-y)dy$$

$$u(x) = (1-x)(\frac{x^2}{2}) + x(\frac{1}{2} - x + \frac{x^2}{2})$$

$$u(x) = \frac{1}{2}x(1-x) \tag{6}$$

Similarly, we find the solution to the problem for $f = sin(\pi x)$:

$$u(x) = \int_0^1 G(x,y)\sin(\pi y)dy$$

$$u(x) = (1-x)\int_0^x y\sin(\pi y)dy + x\int_x^1 (1-y)sin(\pi y)dy$$

$$u(x) = \frac{((1-x)+x)(\sin(\pi x) - \pi x\cos(\pi x))}{\pi^2}$$

$$u(x) = \frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2}$$

The calculations were done using integration by parts.

b)

If $f > 0$ then $u > 0$.

Proof. We are using the definition

$$u(x) = \int_0^1 G(x,y)f(y)dy,$$

and observing that $G > 0$ on $C(0,1)$. Thus, if $f > 0$, then clearly $u > 0$ as well. □

We also have the following relation

$$\|u(x)\|_\infty \leq \frac{1}{8}\|f\|_\infty$$

Proof.

$$\left|u(x)\right| = \left|\int_0^1 G(x,y)f(y)dy\right|$$

$$\left|u(x)\right| \leq \int_0^1 G(x,y)\left|f(y)\right| dy$$

$$\|u(x)\|_\infty \leq \int_0^1 G(x,y)\|f\|_\infty dy$$

$$\|u(x)\|_\infty \leq \|f\|_\infty \int_0^1 G(x,y)dy$$

where we solve $\int_0^1 G(x,y)dy$ as before

$$\int_0^1 G(x,y)dy = \int_0^x y(1-x)dy + \int_x^1 x(1-y)dy$$

$$\int_0^1 G(x,y)dy = \frac{1}{2}x(1-x)$$

which has its maximum value at $x = \frac{1}{2}$.

$$\int_0^1 G(x,y)dy \leq \frac{1}{8}$$

which yields

$$\|u(x)\|_\infty \leq \frac{1}{8}\|f\|_\infty$$

□

c)

Let $(G_h)_{m,n} = hG(x_m, x_n)$, then $A_h G_h = I_{M-1}$,
with $A_h = \frac{1}{h^2} tridiag(-1, 2-1)$ and I_{M-1} the $(M-1) \times (M-1)$ identity matrix.

Proof.

$$
\begin{aligned}
(A_h G_h)_{m,n} &= \sum_{k=1}^{M-1} a_{m,k} (G_h)_{k,n} \\
&= \frac{1}{h^2} h(-G(x_{m+1}, x_n) + 2G(x_m, x_n) - G(x_{m-1}, x_n)) \\
&\approx G''(x, x_n)
\end{aligned}
$$

- if n $\neq m$: $(\mathrm{A}_h G_h)_{m,n} = 0$,
  as the difference scheme solves $G''(x, x_n)$ exact, and $G(x, x_n)$ is piecewise linear
- if n = m:

  $$
  \begin{aligned}
  (A_h G_h)_{m,n} &= \frac{1}{h}(-G(x_{m+1}, x_m) + 2G(x_m, x_m) - G(x_{m-1}, x_m)) \\
  &= \frac{1}{h}(-(x_m(1-(x_m+h)) + 2(x_m(1-x_m)) - (x_m - h)(1-x_m))) \\
  &= \ldots = 1
  \end{aligned}
  $$

Hence, $A_h G_h = I_{M-1}$.

□

d)

In this subsection we are proving the stability of the difference scheme 2. That is, given that A_h is non-singular, show that A_h^{-1} is bounded by some positive constant, in this case $\left\| A_h^{-1} \right\|_\infty \leq \frac{1}{8}$.

Proof. We are using the following relation

$$A_h \mathbf{U} = \mathbf{F}$$

and considering the model problem with $f = 1$, we obtain

$$A_h^{-1} = u(x)$$

where $u(x) = \frac{1}{2} x(1-x)$ as derived in (6). Further, we obtain

$$\|A_h^{-1}\|_\infty \leq \max_x u(x)$$

with $x = \frac{1}{2}$ we get our desired result

$$\|A_h^{-1}\|_\infty \leq \frac{1}{8}$$

.

$\square$

e)

In this section, we are going to find an expression for the error term on the form

$$\|e_h\|_\infty \leq Ch^p \|u_{rx}\|_\infty$$

to do so, we make use of our stability proof in the previous section and express the error as

$$e_h = -A_h^{-1} \cdot \tau_h$$

by using the definition of the infinity norm we obtain

$$\|e_h\|_\infty \leq \|A_h^{-1}\|_\infty \|\tau_h\|_\infty$$

where $\tau_h = [\tau_1, ..., \tau_{M-1}]$ is the truncation error vector. The components are found by plugging the exact solution into our scheme (2):

$$\tau_m = \frac{-u(x_{m+1}) + 2u(x_m) - u(x_{m-h})}{h^2} + f(x_m)$$

$$\dots$$

$$\tau_m = \frac{1}{12} h^2 u_{4x}(x_m)$$

by combining $\|A_h^{-1}\|_\infty$ from the previous section and $\|\tau_h\|_\infty$ we obtain

$$\|e_h\|_\infty \leq \frac{1}{96} h^2 \|u_{4x}\|_\infty$$

that is, $C = \frac{1}{96}$, $p = 2$ and $r = 4$.

From our expression for the truncation error we observe that $\|\tau_h\|_\infty \Rightarrow 0$ as $h \Rightarrow 0$. This implies that our method is consistent. We also proved stability in section d). So we can conclude that the method is convergent, because it is both consistent and stable.

f)

The numerical approximations to the solution of the model problem (1) for three different test functions are shown in Figure 1 below, along with their exact solutions. The approximations were done using $M = 100$ grid points, or equivalently a step-length $h = 0.01$ for $x \in [0, 1]$ using the center difference scheme (2).

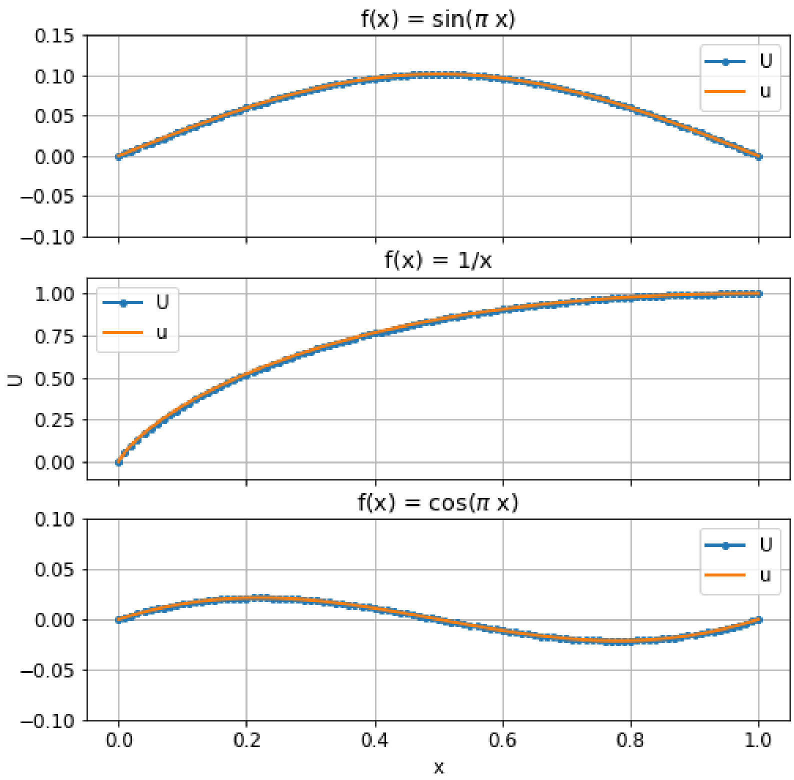

Figure 1: Numerical- (U) and exact (u) solutions for three different test functions

Approximations were calculated for a set of different step-sizes for each test-function, and the error was measured for each step length. Error-measurement was done in the infinity norm. Figure 2 below shows a convergence plot for each test function, where the measured error is plotted against step-size in a log-log fashion.

Assuming the error of the scheme depends on some parameter h, the step length, the method is said to be of order p if

$$\|e_h\| \approx Ch^p \Rightarrow log(\|e_h\|) = log(p) + log(C).$$

Hence, the rate of convergence is equivalent to the slope of the lines in the log-log plot of error against step length.
The slope of the curves in the convergence plot was approximated in a least squares fashion.
The test functions $f(x) = sin(\pi x)$ and $f(x) = cos(\pi x)$ both resulted in order $p = 2$, while $f(x) = \frac{1}{x}$ yielded a slope close to $p = 1$. As discussed above, the scheme should yield an order of $p = 2$. However, the lower order for $f(x) = 1/x$ results from the properties of the function itself, as the function-value limits towards a singularity at $x = 0$.

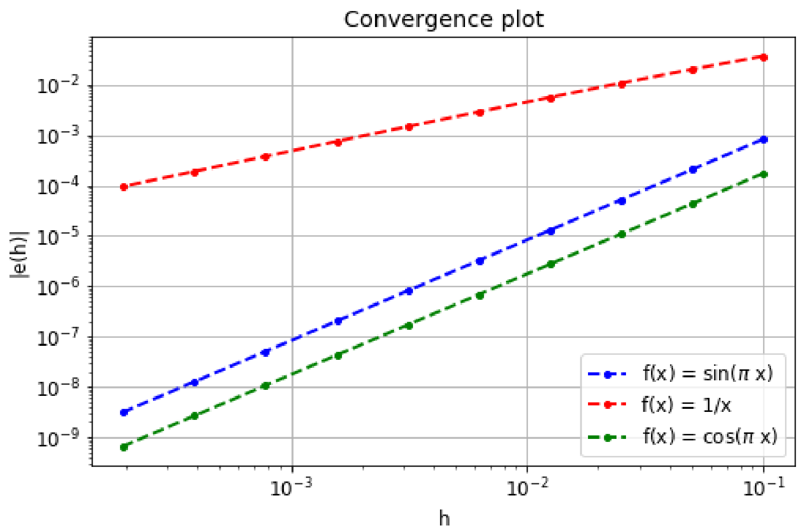

Figure 2: Convergence plot for the test functions, where error $e(h)$ is plotted against step-size h

Problem 3 :

Now we consider the model problem (1), but with mixed boundary conditions.

$$-u_{xx} = \sin(\pi x) \quad u(0) = 0, \quad u_x(1) + u(1) = 1 \tag{7}$$

We will consider three approximations of the derivative at the right boundary, namely

$$u_x(1) = \begin{cases} \frac{u_M - u_{M-1}}{h} + \mathcal{O}(h) \\ \frac{3u_M - 4u_{M-1} + u_{M-2}}{2h} + \mathcal{O}(h^2) \\ \frac{u_{M+1} - u_{M-1}}{2h} + \mathcal{O}(h^2) \end{cases}$$

In the first case, the right hand boundary condition is approximated as

$$\frac{U_M - U_{M-1}}{h} + U_M = 1 \Rightarrow \frac{U_M(1+h) - U_{M-1}}{h} = 1 \tag{8}$$

This approximation leads to the following $(M+1) \times (M+1)$ linear system of equations

$$A_h \mathbf{U} = \mathbf{F},$$

where

$$A_{h1} = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & -1 & 2 & -1 & \\ & & & -h & (h+1)h \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 \\ \vdots \\ U_M \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ \vdots \\ 1 \end{bmatrix}$$

In the second case, the boundary condition at $x = 1$ is approximated as

$$\frac{3U_M - 4U_{M-1} + U_{M-2}}{2h} + U_M = 1 \Rightarrow \frac{U_M(3+2h) - 4U_{M-1} + U_{M-2}}{2h} = 1 \tag{9}$$

Similar to the first case above, this approximation leads to a $(M+1) \times (M+1)$ system of linear equations

$$A_{h2} \mathbf{U} = \mathbf{F},$$

where

$$A_{h2} = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & -1 & 2 & -1 & \\ & & \frac{h}{2} & -2h & (3+2h)h \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 \\ \vdots \\ U_M \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ \vdots \\ 1 \end{bmatrix}$$

In the last case, the boundary condition is approximated as

$$\frac{U_{M+1} - U_{M-1}}{2h} + U_M = 1 \Rightarrow \frac{U_M(1+h) - U_{M-1}}{h} = 1 \tag{10}$$

This results in the same linear system of $(M+1) \times (M+1)$ equations as in case 1, i.e the first approximation.

The arrow in (10) is not obvious. The result is obtained by eliminating the node at index $(M+1)$ using the central difference scheme (2).

The numerical approximations along with exact solutions are given in figure 3 below. In the figure, the labels U1, U2, U3 corresponds to the three different cases in which the boundary condition at the right hand was approximated. The results was obtained by using a modified central difference scheme, as described above, with M = 100 grid points.

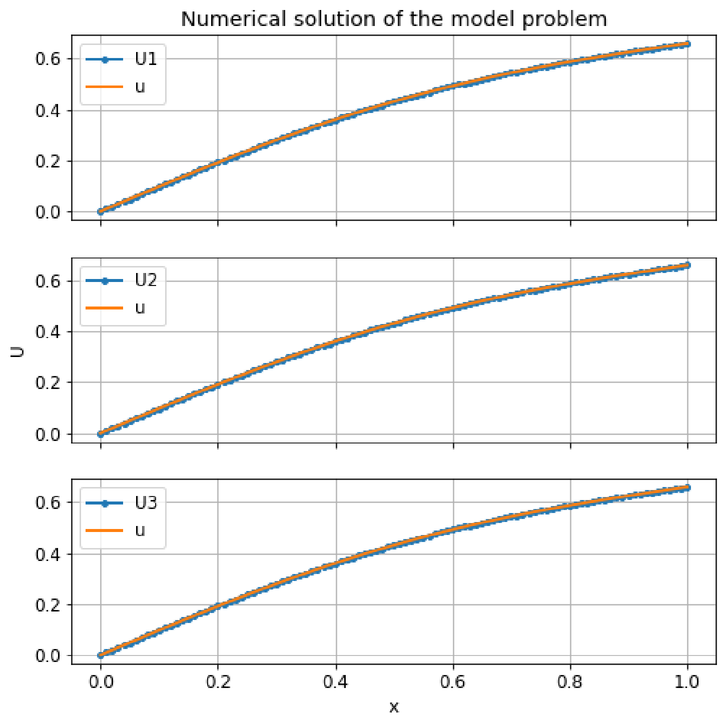

Figure 3: Numerical and exact solutions for the model problem with mixed boundary conditions

To measure the rate of convergence of each approximation, the error was measured over a set of different step-sizes (or equivalently, number of grid points). The error was measured in the infinity-norm, and plotted in a log-log plot against $n = 10$ different step sizes h. The results are depicted in the figure 4 below.

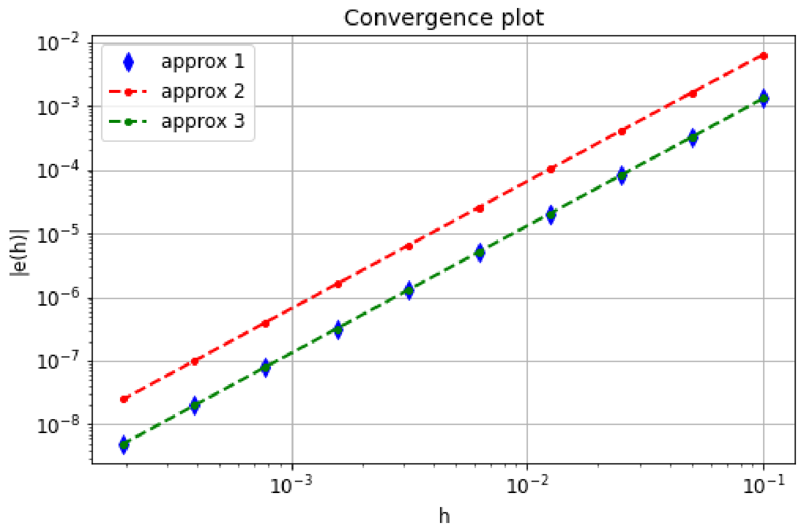

Figure 4: Convergence plot

The order of convergence was found using a least square approximations to the straight lines in the convergence plot above. All approximations results in an order of convergence of $p = 2$. Also note that the convergence curves for approximation 1 and 3 completely overlaps. This results from the fact that the linear system of equations are identical in these cases, as explained above.