Modeling Project: Small climate model

TMA4195 Mathematical Modeling
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Question 1

Earth's Energy Balance

In this part we derive a mathematical model for the temperature of both the earth and the atmosphere.

Approach

For deducing and constructing a set of equations for the temperatures, we consider the different sources of radiation. These energy radiations will hit and propagate within the atmosphere, consisting of 1) clouds, 2) molecules, 3) ozone. Then we consider the possibilities of the incoming radiations either being *reflected*, *absorbed* or *transmitted*. Lets denote the fractions of incoming propagation, P^{refl}, P^{abs}, P^{trans} as reflected, absorbed and transmitted energy respectively. These fractions we obtain by the following relations

$$\begin{aligned} P^{\text{abs}} &= a(1-r)P^{\text{in}} \\ P^{\text{refl}} &= rP^{\text{in}} \\ P^{\text{trans}} &= (1-a)(1-r)P^{\text{in}} \end{aligned} \tag{1}$$

where a and r are given absorption and reflection coefficients for the different mediums.

The sources of radiations that we consider, categorized as short waves (sw) or long waves (lw), are the following

- Solar flux (sw)
- Emission of the earth (lw)
- Emission of the atmosphere (lw)
- Heat and latent flux from the earth (lw)

The solar flux and earth emission are assumed to be either absorbed, reflected or transmitted in the atmosphere, while the latter two sources are assumed not to be affected by these phenomenons. Short waves can also be reflected by the earth. This leads to many different energy contributions in different directions, as we showcase in Figure 1. The complexity of the model can easily be adjusted, and some contributions might be close to negligible. However, in Figure 1, we illustrate all the radiation that we decided to add to the model. The notation and physical interpretation of our choice of contributions will be presented in Table 1.

Notation	
S	The incoming energy radiation from the solar flux
S_R	The fraction of solar radiation being reflected back from the atmosphere
S_T	The fraction of solar radiation being transmitted through the atmosphere
S_{TR}	The transmitted solar radiation being reflected back again by the earth
S_{TRR}	Solar radiation coming back from earth and again being reflected back by the atmosphere
S_{TRT}	Solar radiation that has been reflected by the earth, which then transmits again through the atmosphere
L	Latent heat from condensed water being brought to the atmosphere
H	Heat flux induced by the temperature difference between the earth and the atmosphere
E	The self emission from the earth
E_R	The fraction of earths emission being reflected back from the atmosphere
E_t	The fraction of earths emission being transmitted through the atmosphere
A_U	The self emission from the atmosphere in the direction of space
A_D	The self emission from the atmosphere towards earth

Table 1: Explanation of included energy contributions.

Assumptions

When describing the dynamics in the atmosphere, we have made a couple of simplifying assumptions, namely the following

- The clouds, ozone particles and other molecules are all included in the same layer, able to interfere with each other,
- The clouds and the molecules are weighted according to the cloud coverage in the atmosphere. That is, we assume that incoming radiation either hits clouds or molecules, and the probability of the two outcomes sum to 1,
- For the molecules, the ozone and the other molecules are also weighted, yielding a similar interpretation as above.

Derivation of the model

The final mathematical model is constructed by an assumption of energy balance. In fact, we consider energy balance for both the earth and the atmosphere. That is, the incoming radiation to the earths surface, must equal the outgoing radiation from the surface. Similarly for the atmosphere. This would equal looking at the energy balance for a control volume right outside of the atmosphere on both sides, which is illustrated by the dotted lines in Figure 1. The physical constants of interest are presented in figure 2. Given our assumptions, let us first derive total absorption and reflection coefficients for the atmosphere, a_s^{tot}, a_l^{tot}, r_s^{tot}, r_l^{tot}, being absorption and reflection for short and long waves respectively.

$$a_s^{\text{tot}} = (1 - Cc) \cdot (\frac{a_{SW} + a_{O3}}{2}) + Cc \cdot a_{SC}$$

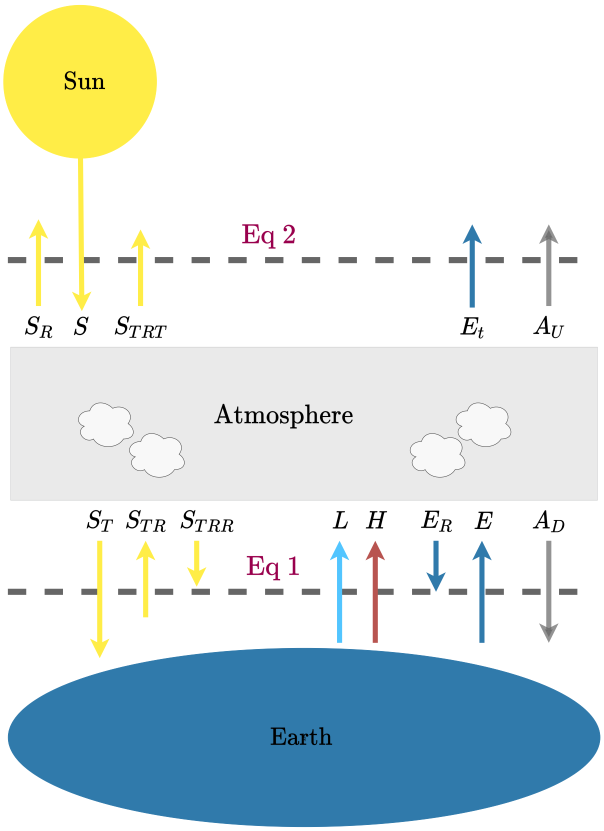

Figure 1: An illustration of the simplified model designed for estimating the temperatures T_A and T_E. The arrows illustrate the energy radiation between the sun, the atmosphere and the Earth. The notation is explained in Table 1.

$$a_l^{\text{tot}} = (1 - Cc) \cdot a_{LW} + cc \cdot a_{LC}$$
$$r_s^{\text{tot}} = (1 - Cc) \cdot r_{SM} + cc \cdot r_{SC}$$
$$r_l^{\text{tot}} = Cc \cdot r_{LC}$$

Using these coefficients, we can now derive the system of equations.

Equation 1

Conservation of energy flow over the first control volume, as indicated in Figure 1, leads to the following equilibrium

$$S_T + S_{TRR} + E_R + A_D = S_{TR} + L + H + E$$

Using the definition of these quantities, and exploiting the derived absorption and reflection coefficients and the relations (1), we obtain the following mathematical expressions

$$\left(1+r_{SE}\cdot r_s^{\text{tot}}\right)P_S^0(1-a_s^{\text{tot}})(1-r_s^{\text{tot}})+r_l^{\text{tot}}\epsilon_e\sigma T_E^4+\epsilon_a\sigma T_A^4 f = r_{SE}P_S^0(1-a_s^{\text{tot}})(1-r_s^{\text{tot}})-(\alpha+\beta)(T_A-T_E)+\epsilon_e\sigma T_E^4$$

parameter	symbol	value	unit
averaged solar flux	P_S^0	341.3	$\mathrm{W\,m^{-2}}$
Cloud cover	C_C	66.0	%
sw molecular scattering coefficient	r_{SM}	10.65	%
sw cloud scattering coefficient	r_{SC}	22.0	%
sw Earth reflectivity	r_{SE}	17.0	%
sw absorptivity: ozone	a_{O3}	8.0	%
sw cloud absorptivity	a_{SC}	12.39	%
sw absorptivity: H2O-CO2-CH4	a_{SW}	14.51	%
lw cloud scattering coefficient	r_{LC}	19.5	%
lw Earth reflectivity	r_{LE}	0.0	%
lw cloud absorptivity	a_{LC}	62.2	%
lw absorptivity: H2O-CO2-CH4-O3	a_{LW}	82.58	%
Earth emissivity	$\varepsilon_E = 1 - r_{LE}$	100.0	%
atmosph. emissivity	ε_A	87.5	%
asymmetry factor	f_A	61.8	%
sensible heat flux	α	3	$\mathrm{W\,m^{-2}\,K^{-1}}$
latent heat flux	β	4	$\mathrm{W\,m^{-2}\,K^{-1}}$

Figure 2: Physical constants used for the mathematical model

$$\big(1 + r_{SE} \cdot r_s^{\mathrm{tot}} - r_{SE}\big) P_S^0 (1 - a_s^{\mathrm{tot}})(1 - r_s^{\mathrm{tot}}) + \big(r_l^{\mathrm{tot}} - 1\big) \epsilon_e \sigma T_E^4 + \epsilon_a \sigma T_A^4 f + (\alpha + \beta)(T_A - T_E) = 0 \tag{2}$$

Equation 2

Using the same approach, and considering the second control volume, we obtain the following expression for the second equation

$$S_R + S_{TRT} + E_t + A_U = S$$

Again, expressing this mathematically in a similar manner as for equation 1, we obtain

$$P_S^0 \cdot r_s^{tot} + r_{SE} P_S^0 \Big((1 - a_s^{tot})(1 - r_s^{tot})\Big)^2 + \epsilon_e \sigma T_E^4 (1 - a_l^{tot})(1 - r_l^{tot}) + \epsilon_a \sigma T_A^4 = P_S^0$$

$$P_S^0 \Big(r_s^{tot} + r_{SE}\big((1 - a_s^{tot})(1 - r_s^{tot})\big)^2 - 1\Big) + \epsilon_e \sigma T_E^4 (1 - a_l^{tot})(1 - r_l^{tot}) + \epsilon_a \sigma T_A^4 = 0 \tag{3}$$

This system of equation can be solved numerically for T_A and T_E, for which we used a numerical solver in Python (see the attached codes) which yielded the following estimations of the temperatures

$$T_A \approx -8.53°\mathrm{C} \qquad \text{(Atmosphere temperature)}$$
$$T_E \approx 10.66°\mathrm{C} \qquad \text{(Earth temperature).}$$

Question 2

Sensitivity with respect to modeling parameters

We wish to obtain the sensitivity with respect to certain modeling parameters. Some are deemed unchangeable and thus omitted. Those included are the absorption and reflection coefficients for all molecules, as well as the earth reflectivity.

The sensitivity analysis could be done analytically, but this is a tedious approach, and the results are cumbersome to interpret and visualize. Thus, the sensitivity analysis was done numerically, producing results that are easier to understand.

One by one, the coefficients were changed by a small size Δc, while the other coefficients were held constant. This changes the values for T_A and T_E, the respective temperatures of the atmosphere and Earth. The relative change was measured, $\Delta T = T_{\text{new}}/T_{\text{old}}$, with T_{new} being the temperature when a coefficient c is incremented by Δc, and T_{old} being the original temperature.
The results for $\Delta c = 0.01$, or 1 percent, are visualized in Figure 3. We see that out of all the coefficients we have chosen to inspect, our model is most sensitive to changes in a_{LW} and r_{SE}.

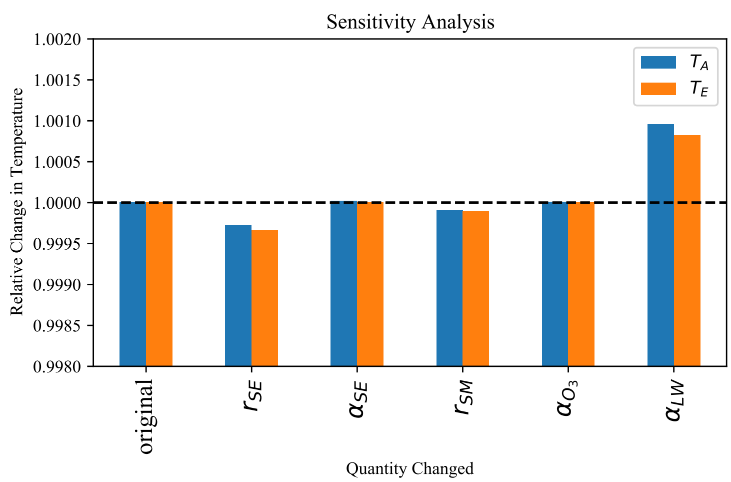

Figure 3: The sensitivity of the two temperatures T_A and T_E, to a 1% increase for the desired parameters.

a_{LW} is the percentage of long wave energy radiation absorbed by $H_2O - CO_2 - CH_4 - O_3$. An increase of $H_2O - CO_2 - CH_4 - O_3$ in the atmosphere would increase the temperature of both Earth and the atmosphere.

r_{SE} is the percentage of short wave energy radiation reflected by Earth. A reduction of Earth's reflectivity would increase both T_A and T_E. As the ice caps are melting, the reflectivity is decreasing.

This analysis shows the relative change in temperature for a small perturbation of some of the quantities, but the nature of the relationship between the temperature and the coefficients is unknown. This can be explored by plotting how the temperatures change while letting a coefficient c go from 0 to 1. This is done in Figures 4 and 5.

The relationship between T_A, T_E and a_{LW} seem to be almost linear, at least in the region around the true value of a_{LW}. The same applies for r_{SE}, but when r_{SE} gets close to 1, the relationships seem to be closer to quadratic.

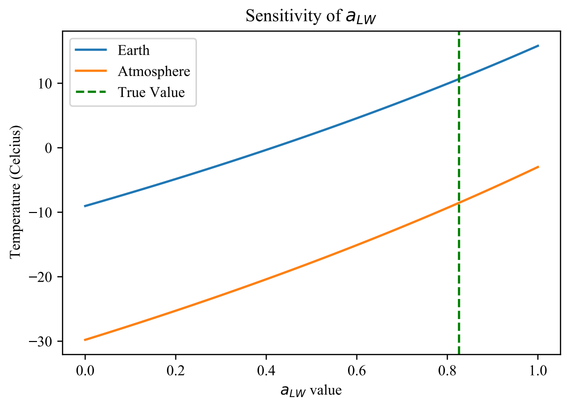

Figure 4: The temperature of both the earth and the atmosphere, for all possible values of a_{LW}, while the other constants are being held constant to the actual values.

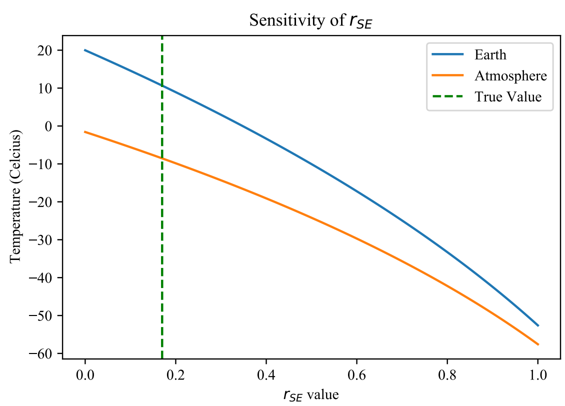

Figure 5: The temperature of both the earth and the atmosphere, for all possible values of r_{SE}, while the other constants are being held constant to the actual values.