Inequalities From Around the World 1995-2005

Solutions to 'Inequalities through problems' by Hojoo Lee

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Introduction

The aim of this work is to provide solutions to problems on inequalities proposed in various countries of the world in the years 1990-2005.

In the summer of 2006, after reading Hoojoo Lee's nice book, *Topics in Inequalities - Theorem and Techniques*, I developed the idea of demonstrating all the inequalities proposed in chapter 5, subsequently reprinted in the article *Inequalities Through Problems* by the same author. After a hard and tiresome work lasting over two months, thanks also to the help I mustered from specialised literature and from the http://www.mathlinks.ro website, I finally managed to bring this ambitious project to an end.

To many inequalities I have offered more than one solution and I have always provided the source and the name of the author. In the contents I have also marked with an asterisk all the solutions which have been devised by myself. Furthermore I corrected the text of the problems 5, 11, 32, 79, 125, 140, 159 which seems to contain some typos (I think !).

I would greatly appreciate hearing comments and corrections from my readers. You may email me at

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To Readers

This book is addressed to challenging high schools students who take part in mathematical competitions and to all those who are interested in inequalities and would like improve their skills in nonroutine problems. I heartily encourage readers to send me their own alternative solutions of the proposed inequalities: these will be published in the definitive version of this book. **Enjoy!**

Acknowledgement

I'm indebted to ***Hojoo Lee***, ***Vasile Cîrtoaje***, ***Massimo Gobbino***, ***Darij Grinberg*** and many other contributors of Mathlinks Forum for their nice solutions. Without their valuable help this work would not have been possible.

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1 Years 2001 $\sim$ 2005

1. (**BMO 2005, Proposed by Dušan Djukić, Serbia and Montenegro**)
$(a, b, c > 0)$

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{4(a-b)^2}{a+b+c}$$

First Solution. (*Andrei, Chang Woo-JIn - ML Forum*)
Rewrite the initial inequality to:

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a} \geq \frac{4(a-b)^2}{a+b+c}$$

This is equivalent to

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right) \geq 4(a-b)^2$$

Using Cauchy one can prove

$$(a+b+c)\left[\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right] \geq 4\left[\max(a,b,c)-\min(a,b,c)\right]^2$$

In fact

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right) \geq (|a-b|+|b-c|+|c-a|)^2$$

WLOG[1] assume $|c-a| = \max(|a-b|, |b-c|, |c-a|)$. Then, we get

$$|a-b|+|b-c| \geq |c-a|$$

[1] Without loss of generality.

So

$$|a-b|+|b-c|+|c-a| \geq 2|c-a|$$

Therefore,

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right) \geq 4\max\left(|a-b|,|b-c|,|c-a|\right)^2$$

Equality hold if and only if one of two cases occur : $a=b=c$ or $c=\omega b$, $a=\omega c$, where $\omega=\frac{\sqrt{5}-1}{2}$. $\square$

Second Solution. (*Ciprian - ML Forum*) With Lagrange theorem (for 3 numbers) we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{1}{a+b+c}\cdot\left[\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab}+\frac{\left(ab-c^2\right)^2}{ac}\right]$$

So we have to prove that

$$\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab}+\frac{\left(ab-c^2\right)^2}{ac} \geq 4\left(a-b\right)^2$$

But $\frac{\left(ab-c^2\right)^2}{ac} \geq 0$ and

$$\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab} \geq \frac{\left(ac-b^2-bc+a^2\right)^2}{b\left(a+c\right)}=\frac{\left(a-b\right)^2\left(a+b+c\right)^2}{b\left(a+c\right)}$$

By AM-GM we have

$$b\left(a+c\right) \leq \frac{\left(a+b+c\right)^2}{4} \quad \Longrightarrow \quad \frac{\left(a-b\right)^2\left(a+b+c\right)^2}{b\left(a+c\right)} \geq 4\left(a-b\right)^2$$

Then we get

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{4(a-b)^2}{a+b+c}$$

$\square$

Remark. The Binet-Cauchy identity

$$\left(\sum_{i=1}^{n} a_i c_i\right)\left(\sum_{i=1}^{n} b_i d_i\right)-\left(\sum_{i=1}^{n} a_i d_i\right)\left(\sum_{i=1}^{n} b_i c_i\right)=\sum_{1\leq i<j\leq n}\left(a_i b_j-a_j b_i\right)\left(c_i d_j-c_j d_i\right)$$

by letting $c_i = a_i$ and $d_i = b_i$ gives the Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2$$

It implies the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Equality holds if and only if $a_k b_j = a_j b_k$ for all $1 \le k, j \le n$.

2. (**Romania 2005, Cezar Lupu**) $(a, b, c > 0)$

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Solution. (*Ercole Suppa*) By using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{b+c}}{a}\frac{1}{\sqrt{b+c}}\right)^2 \le \\ &\le \left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right) \le \quad \text{(Cauchy-Schwarz)} \\ &\le \left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\end{aligned}$$

Therefore

$$\left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right) \ge \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

□

3. (**Romania 2005, Traian Tamaian**) $(a, b, c > 0)$

$$\frac{a}{b+2c+d} + \frac{b}{c+2d+a} + \frac{c}{d+2a+b} + \frac{d}{a+2b+c} \ge 1$$

First Solution. (*Ercole Suppa*) From the Cauchy-Schwartz inequality we have

$$(a+b+c+d)^2 \le \sum_{\text{cyc}} \frac{a}{b+2c+d} \sum_{\text{cyc}} a\left(b+2c+d\right)$$

Thus in order to prove the requested inequality is enough to show that

$$\frac{(a+b+c+d)^2}{\sum_{\text{cyc}} a\left(b+2c+d\right)} \ge 1$$

The last inequality is equivalent to

$$\begin{aligned}
&(a+b+c+d)^2 - \sum_{\text{cyc}} a\left(b+2c+d\right) \ge 0 && \iff \\
&a^2+b^2+c^2+d^2-2ac-2bd \ge 0 && \iff \\
&(a-c)^2+(b-d)^2 \ge 0
\end{aligned}$$

which is true. $\square$

Second Solution. (*Ramanujan - ML Forum*) We set $S = a+b+c+d$. It is

$$\begin{aligned}
&\frac{a}{b+2c+d} + \frac{b}{c+2d+a} + \frac{c}{d+2a+b} + \frac{d}{a+2b+c} = \\
=&\frac{a}{S-(a-c)} + \frac{b}{S-(b-d)} + \frac{c}{S+(a-c)} + \frac{d}{S+(b-d)}
\end{aligned}$$

But

$$\frac{a}{S-(a-c)} + \frac{c}{S+(a-c)} = \frac{(a+c)S+(a-c)^2}{S^2-(a-c)^2} \ge \frac{a+c}{S} \tag{1}$$

and

$$\frac{b}{S-(b-d)} + \frac{d}{S+(b-d)} \ge \frac{b+d}{S} \tag{2}$$

Now from (1) and (2) we get the result. $\square$

4. (**Romania 2005, Cezar Lupu**) $\left(a+b+c \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c},\ a,b,c>0\right)$

$$a+b+c \ge \frac{3}{abc}$$

Solution. (*Ercole Suppa*)
From the well-known inequality $(x+y+x)^2 \geq 3(xy+yz+zx)$ it follows that

$$\begin{aligned}(a+b+c)^2 &\geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq \\ &\geq 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \\ &= \frac{3(a+b+c)}{abc}\end{aligned}$$

Dividing by $a+b+c$ we have the desidered inequality. □

5. (**Romania 2005, Cezar Lupu**) $(1=(a+b)(b+c)(c+a),\ a,b,c>0)$

$$ab+bc+ca \leq \frac{3}{4}$$

Solution. (*Ercole Suppa*) From the identity

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$$

we have

$$ab+bc+ca = \frac{1+abc}{a+b+c} \tag{1}$$

From AM-GM inequality we have

$$1 = (a+b)(b+c)(c+a) \geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca} = 8abc \quad \Longrightarrow \quad abc \leq \frac{1}{8} \tag{2}$$

and

$$2\frac{a+b+c}{3} = \frac{(a+b)+(b+c)+(c+a)}{3} \geq \sqrt[3]{(a+b)(b+c)(c+a)} = 1 \quad \Longrightarrow$$

$$a+b+c \geq \frac{3}{2} \tag{3}$$

From (1),(2),(3) it follows that

$$ab+bc+ca = \frac{1+abc}{a+b+c} \leq \frac{1+\frac{1}{8}}{\frac{3}{2}} = \frac{3}{4}$$

□

6. (**Romania 2005, Robert Szasz - Romanian JBMO TST**) ($a+b+c = 3,\ a,b,c > 0$)

$$a^2b^2c^2 \geq (3-2a)(3-2b)(3-2c)$$

First Solution. (*Thazn1 - ML Forum*) The inequality is equivalent to

$$(a+b+c)^3(-a+b+c)(a-b+c)(a+b-c) \leq 27a^2b^2c^2$$

Let $x = -a+b+c$, $y = a-b+c$, $z = a+b-c$ and note that at most one of x, y, z can be negative (since the sum of any two is positive). Assume $x, y, z \geq 0$ if not the inequality will be obvious. Denote $x+y+z = a+b+c, x+y = 2c$, etc. so our inequality becomes

$$64xyz(x+y+z)^3 \leq 27(x+y)^2(y+z)^2(z+x)^2$$

Note that

$$9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx)$$

and

$$(xy+yz+zx)^2 \geq 3xyz(x+y+z)$$

Combining these completes our proof! $\square$

Second Solution. (*Fuzzylogic - Mathlink Forum*) As noted in the first solution, we may assume a, b, c are the sides of a triangle. Multiplying LHS by $a+b+c$ and RHS by 3, the inequality becomes

$$16\Delta^2 \leq 3a^2b^2c^2$$

where Δ is the area of the triangle. That is equivalent to $R^2 \geq \frac{1}{3}$ since $\Delta = \frac{abc}{4R}$, where R is the circumradius. But this is true since

$$R = \frac{a+b+c}{2(\sin A + \sin B + \sin C)}$$

and

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

by Jensen. $\square$

Third Solution. (*Harazi - Mathlink Forum*) Obviously, we may assume that a, b, c are sides of a triangle. Write Schur inequality in the form

$$\frac{9abc}{a+b+c} \geq a(b+c-a) + b(c+a-b) + c(a+b-c)$$

and apply AM-GM for the RHS. The conclusion follows. $\square$

7. (**Romania 2005**) ($abc \geq 1,\ a, b, c > 0$)

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1$$

First Solution. (*Virgil Nicula - ML Forum*)
The inequality is equivalent with the relation

$$\sum a^2(b+c) + 2abc \geq 2(a+b+c) + 2 \tag{1}$$

But

$$2abc \geq 2 \tag{2}$$

and

$$\begin{aligned} 3 + \sum a^2(b+c) &= \sum (a^2b + a^2c + 1) \geq \\ &\geq 3 \sum \sqrt[3]{a^2b \cdot a^2c \cdot 1} \geq \\ &\geq 3 \sum a \cdot \sqrt[3]{abc} \geq \\ &\geq 3 \sum a = \\ &= 2 \sum a + \sum a \geq \\ &\geq 2 \sum a + 3\sqrt[3]{abc} \geq \\ &\geq 2 \sum a + 3 \end{aligned}$$

Thus we have

$$\sum a^2(b+c) \geq 2 \sum a \tag{3}$$

From the sum of the relations (2) and (3) we obtain (1). $\square$

Second Solution. (*Gibbenergy - ML Forum*)
Clear the denominator, the inequality is equivalent to:

$$a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc \geq 2 + 2(a+b+c)$$

Since $abc \geq 1$ so $a+b+c \geq 3$ and $2abc \geq 2$. It remains to prove that

$$a^2(b+c)+b^2(a+c)+c^2(a+b) \geq 2(a+b+c)$$

It isn't hard since

$$\begin{aligned}
\sum(a^2b+a^2c+1)-3 &\geq \sum 3\sqrt[3]{a^4bc}-3 \geq \\
&\geq \sum 3\sqrt[3]{a^3}-3 = \\
&= 3\sum a-3 \geq \\
&\geq 2(a+b+c)+(a+b+c-3) \geq \\
&\geq 2(a+b+c)
\end{aligned}$$

□

Third Solution. (*Sung-yoon Kim - ML Forum*)
Let be $abc = k^3$ with $k \geq 1$. Now put $a = kx^3$, $b = ky^3$, $c = kz^3$, and we get $xyz = 1$. So

$$\begin{aligned}
\sum \frac{1}{1+a+b} &= \sum \frac{1}{1+k(x^3+y^3)} \leq \\
&\leq \sum \frac{1}{1+x^3+y^3} \leq \\
&\leq \sum \frac{1}{xyz+x^2y+xy^2} = \\
&= \sum \frac{1}{xy} \cdot \frac{1}{x+y+z} = \\
&= \left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\frac{1}{x+y+z} = \\
&= \frac{1}{xyz} = 1
\end{aligned}$$

and we are done. □

Remark 1. The problem was proposed in Romania at IMAR Test 2005, Juniors Problem 1. The same inequality with $abc = 1$, was proposed in Tournament of the Town 1997 and can be proved in the following way:

Solution. (*See* [66], *pag.* 161)
By AM-GM inequality

$$a+b+c \geq 3\sqrt[3]{abc} \geq 3 \quad \text{and} \quad ab+bc+ca \geq 3\sqrt[3]{ab \cdot bc \cdot ca} \geq 3$$

Hence

$$(a+b+c)(ab+bc+ca-2) \geq 3$$

which implies

$$2(a+b+c) \leq ab(a+b) + bc(b+c) + ca(c+a)$$

Therefore

$$\begin{aligned}
&\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} - 1 = \\
&= \frac{2+2a+2b+2c-(a+b)(b+c)(c+a)}{(1+a+b)(1+b+c)(1+c+a)} = \\
&= \frac{2a+2b+2c-ab(a+b)-bc(b+c)-ca(c+a)}{(1+a+b)(1+b+c)(1+c+a)} \leq 0
\end{aligned}$$

□

Remark 2. A similar problem was proposed in USAMO 1997 (problem 5) *Prove that, for all positive real numbers a, b, c we have*

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

The inequality can be proved with the same technique employed in the third solution (see problem 87).

8. (**Romania 2005, Unused**) $(abc = 1,\ a, b, c > 0)$

$$\frac{a}{b^2(c+1)} + \frac{b}{c^2(a+1)} + \frac{c}{a^2(b+1)} \geq \frac{3}{2}$$

First Solution. (*Arqady - ML Forum*) Let $a = \frac{x}{z}$, $b = \frac{y}{x}$ and $c = \frac{z}{y}$, where $x > 0$, $y > 0$ and $z > 0$. Hence, using the Cauchy-Schwarz inequality in the Engel form, we have

$$\begin{aligned}
\sum_{\text{cyc}} \frac{a}{b^2(c+1)} = \sum_{\text{cyc}} \frac{x^3}{yz(y+z)} &= \\
&= \sum_{\text{cyc}} \frac{x^4}{xyz(y+z)} \geq \\
&\geq \frac{(x^2+y^2+z^2)^2}{2xyz(x+y+z)}
\end{aligned}$$

Id est, remain to prove that

$$\frac{(x^2+y^2+z^2)^2}{2xyz(x+y+z)} \geq \frac{3}{2}$$

which follows easily from Muirhead theorem. In fact

$$\begin{aligned}
\frac{\left(x^2+y^2+z^2\right)^2}{2xyz\,(x+y+z)} \geq \frac{3}{2} & \qquad \Longleftrightarrow \\
2\sum_{\text{cyc}} x^4 + 4\sum_{\text{cyc}} x^2y^2 \geq 6\sum_{\text{cyc}} x^2yz & \qquad \Longleftrightarrow \\
\sum_{\text{sym}} x^4 + 2\sum_{\text{sym}} x^2y^2 \geq 3\sum_{\text{sym}} x^2yz &
\end{aligned}$$

and

$$\sum_{\text{sym}} x^4 \geq \sum_{\text{sym}} x^2yz \quad , \quad \sum_{\text{sym}} x^2y^2 \geq \sum_{\text{sym}} x^2yz$$

□

Second Solution. (*Travinhphuctk14 - ML Forum*) Let $a = \frac{x}{z}$, $b = \frac{y}{x}$ and $c = \frac{z}{y}$, where $x > 0$, $y > 0$ and $z > 0$. We need prove

$$\sum_{\text{cyc}} \frac{a}{b^2(c+1)} = \sum_{\text{cyc}} \frac{x^3}{yz(y+z)} \geq \frac{3}{2}$$

We have $x^3 + y^3 \geq xy(x+y)$, $\ldots$, etc. Thus the desidered inequality follows from Nesbit inequality:

$$\frac{x^3}{yz(y+z)} + \frac{y^3}{xz(x+z)} + \frac{z^3}{xy(x+y)} \geq \frac{x^3}{y^3+z^3} + \frac{y^3}{z^3+x^3} + \frac{z^3}{x^3+y^3} \geq \frac{3}{2}$$

□

9. (**Romania 2005, Unused**) ($a+b+c \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, $a,b,c > 0$)

$$\frac{a^3c}{b(c+a)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

Solution. (*Zhaobin - ML Forum*) First use Holder or the generalized Cauchy inequality. We have:

$$\left(\frac{a^3c}{b(a+c)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)}\right)(2a+2b+2c)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq (a+b+c)^3$$

so:

$$\frac{a^3c}{b(a+c)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)}\geq\frac{a+b+c}{2}$$

but we also have:

$$a+b+c\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

so the proof is over. $\square$

10. (**Romania 2005, Unused**) $(a+b+c=1,\ a,b,c>0)$

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}}$$

First Solution. (*Ercole Suppa*) By Cauchy-Schwarz inequality we have

$$\begin{aligned}1=(a+b+c)^2&=\left(\sum_{\text{cyc}}\frac{\sqrt{a}}{\sqrt[4]{b+c}}\sqrt{a}\sqrt[4]{b+c}\right)^2\leq\\ &\leq\left(\sum_{\text{cyc}}\frac{a}{\sqrt{b+c}}\right)\left(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{a+b}\right)\end{aligned}$$

Therefore

$$\left(\sum_{\text{cyc}}\frac{a}{\sqrt{b+c}}\right)\geq\frac{1}{a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{a+b}}\tag{1}$$

Since $a+b+c=1$ we have

$$\begin{aligned}a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{a+b}&=\left(\sum_{\text{cyc}}\sqrt{a}\sqrt{a\,(b+c)}\right)\leq\\ &\leq\sqrt{a+b+c}\sqrt{2ab+2bc+2ca}= \quad\text{(Cauchy-Schwarz)}\\ &=\sqrt{\frac{2}{3}}\sqrt{3ab+3bc+3ac}\leq\\ &\leq\sqrt{\frac{2}{3}}\sqrt{(a+b+c)^2}=\sqrt{\frac{2}{3}}\end{aligned}$$

and from (1) we get the result.

$\square$

Second Solution. (*Ercole Suppa*) Since $a+b+c=1$ we must prove that:

$$\frac{a}{\sqrt{1-a}}+\frac{b}{\sqrt{1-b}}+\frac{c}{\sqrt{1-c}}\geq\sqrt{\frac{3}{2}}$$

The function $f(x)=\frac{x}{\sqrt{1-x}}$ is convex on interval $[0,1]$ because

$$f''(x)=\frac{1}{4}(1-x)^{-\frac{5}{2}}(4-x)\geq 0\ ,\ \forall x\in[0,1]$$

Thus, from Jensen inequality, follows that

$$\sum_{\text{cyc}}\frac{a}{\sqrt{b+c}}=f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{1}{3}\right)=\sqrt{\frac{3}{2}}$$

□

11. (**Romania 2005, Unused**) $(ab+bc+ca+2abc=1,\ a,b,c>0)$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq\frac{3}{2}$$

Solution. (*See* [4], *pag.* 10, *problem* 19)
Set $x=\sqrt{ab}$, $y=\sqrt{bc}$, $z=\sqrt{ca}$, $s=x+y+z$. The given relation become

$$x^2+y^2+z^2+2xyz=1$$

and, by AM-GM inequality, we have

$$\begin{aligned}s^2-2s+1&=(x+y+z)^2-2(x+y+z)+1=\\&=1-2xyz+2(xy+xz+yz)-2(x+y+z)+1=\\&=2(xy+xz+yz-xyz-x-y-z+1)=\\&=2(1-x)(1-y)(1-z)\leq &&\text{(AM-GM)}\\&\leq 2\left(\frac{1-x+1-y+1-z}{3}\right)^3=2\left(\frac{3-s}{3}\right)^3\end{aligned}$$

Therefore

$$2s^3+9s^2-27\leq 0\quad\Leftrightarrow\quad(2s-3)(s+3)^2\leq 0\quad\Leftrightarrow\quad s\leq\frac{3}{2}$$

and we are done. □

12. (**Chzech and Slovak 2005**) ($abc = 1,\ a, b, c > 0$)

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}$$

Solution. (*Ercole Suppa*) The given inequality is equivalent to

$$4(ab + bc + ca) + 4(a + b + c) \geq 3(abc + ab + bc + ca + a + b + c + 1)$$

that is, since $abc = 1$, to

$$ab + bc + ca + a + b + c \geq 6$$

The latter inequality is obtained summing the inequalities

$$a + b + c \geq 3\sqrt[3]{abc} = 3$$

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3$$

which are true by AM-GM inequality. $\square$

13. (**Japan 2005**) ($a + b + c = 1,\ a, b, c > 0$)

$$a\,(1 + b - c)^{\frac{1}{3}} + b\,(1 + c - a)^{\frac{1}{3}} + c\,(1 + a - b)^{\frac{1}{3}} \leq 1$$

First Solution. (*Darij Grinberg - ML Forum*) The numbers $1+b-c$, $1+c-a$ and $1 + a - b$ are positive, since $a + b + c = 1$ yields $a < 1$, $b < 1$ and $c < 1$. Now use the weighted Jensen inequality for the function $f(x) = \sqrt[3]{x}$, which is concave on the positive halfaxis, and for the numbers $1 + b - c$, $1 + c - a$ and $1 + a - b$ with the respective weights a, b and c to get

$$\frac{a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b}}{a+b+c} \leq$$
$$\leq \sqrt[3]{\frac{a\,(1+b-c) + b\,(1+c-a) + c\,(1+a-b)}{a+b+c}}$$

Since $a + b + c = 1$, this simplifies to

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq$$
$$\leq \sqrt[3]{a\,(1+b-c) + b\,(1+c-a) + c\,(1+a-b)}$$

But

$$\sum_{\text{cyc}} a\,(1+b-c) = (a+ab-ca)+(b+bc-ab)+(c+ca-bc) = a+b+c = 1$$

and thus

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+c\sqrt[3]{1+a-b} \le \sqrt[3]{1} = 1$$

and the inequality is proven.

$\square$

Second Solution. (*Kunny - ML Forum*) Using A.M-G.M.

$$\sqrt[3]{1+b-c} = \sqrt[3]{1\cdot 1\cdot (1+b-c)} \leqq \frac{1+1+(1+b-c)}{3} = 1+\frac{b-c}{3}$$

Therefore by $a+b+c=1$ we have

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+c\sqrt[3]{1+a-b} \le$$

$$\leqq a\left(1+\frac{b-c}{3}\right)+b\left(1+\frac{c-a}{3}\right)+c\left(1+\frac{a-b}{3}\right) = 1$$

$\square$

Third Solution. (*Soarer - ML Forum*) By Holder with $p=\frac{3}{2}$ and $q=3$ we have

$$\begin{aligned}\sum_{\text{cyc}} a\,(1+b-c)^{\frac{1}{3}} &= \sum_{\text{cyc}} a^{\frac{2}{3}}\left[a\,(1+b-c)\right]^{\frac{1}{3}} \le \\ &\le \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}}\left[\sum_{\text{cyc}} a\,(1+b-c)\right]^{\frac{1}{3}} = \\ &= \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}}\left(\sum_{\text{cyc}} a\right)^{\frac{1}{3}} = \sum_{\text{cyc}} a = 1\end{aligned}$$

$\square$

14. (**Germany 2005**) ($a+b+c=1,\ a,b,c>0$)

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}$$

First Solution. (*Arqady - ML Forum*)

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c} \leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \quad \Leftrightarrow$$

$$\sum\left(\frac{2a}{b}-\frac{2a}{b+c}-1\right) \geq 0 \quad \Leftrightarrow$$

$$\sum_{\text{cyc}}(2a^4c^2-2a^2b^2c^2)+\sum_{\text{cyc}}(a^3b^3-a^3b^2c-a^3c^2b+a^3c^3) \geq 0$$

which is obviously true.

Second Solution. (*Darij Grinberg - ML Forum*) The inequality

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c} \leq 2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})$$

can be transformed to

$$\frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

or equivalently to

$$\frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)} \geq \frac{3}{2}$$

We will prove the last inequality by rearrangement. Since the number arrays

$$\left(\frac{ab}{c};\ \frac{bc}{a};\ \frac{ca}{b}\right)$$

and

$$\left(\frac{1}{a+b};\ \frac{1}{b+c};\ \frac{1}{c+a}\right)$$

are oppositely sorted (in fact, e. g., if $c \geq a \geq b$, we have $\frac{ab}{c} \leq \frac{bc}{a} \leq \frac{ca}{b}$ and $\frac{1}{a+b} \geq \frac{1}{b+c} \geq \frac{1}{c+a}$), we have

$$\frac{ab}{c}\cdot\frac{1}{b+c}+\frac{bc}{a}\cdot\frac{1}{c+a}+\frac{ca}{b}\cdot\frac{1}{a+b} \geq \frac{ab}{c}\cdot\frac{1}{a+b}+\frac{bc}{a}\cdot\frac{1}{b+c}+\frac{ca}{b}\cdot\frac{1}{c+a}$$

i.e.

$$\frac{ab}{c\,(b+c)}+\frac{bc}{a\,(c+a)}+\frac{ca}{b\,(a+b)}\geq\frac{ab}{c\,(a+b)}+\frac{bc}{a\,(b+c)}+\frac{ca}{b\,(c+a)}$$

Hence, in order to prove the inequality

$$\frac{ab}{c\,(b+c)}+\frac{bc}{a\,(c+a)}+\frac{ca}{b\,(a+b)}\geq\frac{3}{2}$$

it will be enough to show that

$$\frac{ab}{c\,(a+b)}+\frac{bc}{a\,(b+c)}+\frac{ca}{b\,(c+a)}\geq\frac{3}{2}$$

But this inequality can be rewritten as

$$\frac{ab}{ca+bc}+\frac{bc}{ab+ca}+\frac{ca}{bc+ab}\geq\frac{3}{2}$$

which follows from Nesbitt.

$\square$

Third Solution. (*Hardsoul and Darij Grinberg - ML Forum*) The inequality

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq 2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})$$

can be transformed to

$$\frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

or equivalently to

$$\frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)}\geq\frac{3}{2}$$

Now by Cauchy to

$$\left(\sqrt{\frac{ab}{c(b+c)}},\sqrt{\frac{bc}{a(c+a)}},\sqrt{\frac{ca}{b(a+b)}}\right)$$

and

$$\left(\sqrt{b+c},\sqrt{a+c},\sqrt{a+b}\right)$$

we have

$$LHS\cdot(2a+2b+2c)\geq\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)^2$$

To establish the inequality $LHS \geq \frac{3}{2}$ it will be enough to show that

$$\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)^2 \geq 3\left(a+b+c\right)$$

Defining $\sqrt{\frac{bc}{a}} = x$, $\sqrt{\frac{ca}{b}} = y$, $\sqrt{\frac{ab}{c}} = z$, we have

$$yz = \sqrt{\frac{ca}{b}}\sqrt{\frac{ab}{c}} = \sqrt{\frac{ca}{b}\cdot\frac{ab}{c}} = \sqrt{a^2} = a$$

and similarly $zx = b$ and $xy = c$, so that the inequality in question,

$$\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)^2 \geq 3\left(a+b+c\right)$$

takes the form

$$\left(x+y+z\right)^2 \geq 3\left(yz+zx+xy\right)$$

what is trivial because

$$\left(x+y+z\right)^2 - 3\left(yz+zx+xy\right) = \frac{1}{2}\cdot\left(\left(y-z\right)^2+\left(z-x\right)^2+\left(x-y\right)^2\right)$$

□

Fourth Solution. (*Behzad - ML Forum*) With computation we get that the inequality is equivalent to:

$$2(\sum a^3b^3+\sum a^4b^2) \geq 6a^2b^2c^2+\sum a^3b^2c+a^3bc^2$$

which is obvious with Muirhead and AM-GM.

□

15. (**Vietnam 2005**) $(a,b,c>0)$

$$\left(\frac{a}{a+b}\right)^3+\left(\frac{b}{b+c}\right)^3+\left(\frac{c}{c+a}\right)^3 \geq \frac{3}{8}$$

Solution. (*Ercole Suppa*) In order to prove the inequality we begin with the following Lemma

Lemma. Given three real numbers $x, y, z \geq 0$ such that $xyz = 1$ we have

$$\frac{1}{\left(1+x\right)^2}+\frac{1}{\left(1+y\right)^2}+\frac{1}{\left(1+z\right)^2} \geq \frac{3}{4}$$

Proof. WLOG we can assume that $xy \geq 1$, $z \leq 1$. The problems 17 yields

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy} = \frac{z}{z+1}$$

Thus it is easy to show that

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} \geq \frac{z}{z+1} + \frac{1}{(1+z)^2} \geq \frac{3}{4}$$

and the lemma is proved. $\square$

The power mean inequality implies

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} \geq \sqrt{\frac{a^2+b^2+c^2}{3}} \qquad \Leftrightarrow$$

$$a^3+b^3+c^3 \geq \frac{1}{\sqrt{3}}\left(a^2+b^2+c^2\right)^{3/2} \tag{1}$$

Thus setting $x = \frac{b}{a}$, $y = \frac{c}{b}$, $z = \frac{a}{c}$, using (1) and the Lemma we have:

$$\begin{aligned} LHS = \left(\frac{a}{a+b}\right)^3 + \left(\frac{b}{b+c}\right)^3 + \left(\frac{c}{c+a}\right)^3 \geq \\ \geq \frac{1}{\sqrt{3}}\left[\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2\right]^{3/2} \geq \\ \geq \frac{1}{\sqrt{3}}\left[\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2}\right]^{3/2} \geq \\ \geq \frac{1}{\sqrt{3}}\left(\frac{3}{4}\right)^{3/2} = \frac{3}{8} \end{aligned}$$

$\square$

Remark. The lemma can be proved also by means of problem 17 with $a = x$, $b = y$, $c = z$, $d = 1$.

16. (**China 2005**) $(a+b+c=1,\ a,b,c>0)$

$$10(a^3+b^3+c^3) - 9(a^5+b^5+c^5) \geq 1$$

Solution. (*Ercole Suppa*) We must show that for all $a, b, c > 0$ with $a+b+c = 1$ results:

$$\sum_{\text{cyc}} \left(10a^3 - 9a^5\right) \geq 1$$

The function $f(x) = 10x^3 - 9x^5$ is convex on $[0, 1]$ because

$$f''(x) = 30x\left(2 - 3x^2\right) \geq 0, \quad \forall x \in [0, 1]$$

Therefore, since $f\left(\frac{1}{3}\right) = \frac{1}{3}$, the Jensen inequality implies

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right) = 3 \cdot f\left(\frac{1}{3}\right) = 1$$

□

17. (**China 2005**) ($abcd = 1,\ a, b, c, d > 0$)

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1$$

First Solution. (*Lagrangia - ML Forum*) The source is Old and New Inequalities [4]. The one that made this inequality is Vasile Cartoaje.
I will post a solution from there:

The inequality obviously follows from:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}$$

and

$$\frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{1}{(1+cd}$$

Only the first inequality we are going to prove as the other one is done in the same manner: it's same as $1 + ab(a^2 + b^2) \geq a^2b^2 + 2ab$ which is true as

$$1 + ab(a^2 + b^2) - a^2b^2 - 2ab \geq 1 + 2a^2b^2 - a^2b^2 - 2ab = (ab - 1)^2$$

This is another explanation:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} - \frac{1}{1+ab} = \frac{ab(a-b)^2 + (ab-1)^2}{(1+a)^2(1+b)^2ab)} \geq 0$$

Then, the given expression is greater than

$$1/(1+ab) + 1/(1+cd) = 1$$

with equality if $a = b = c = d = 1$.

□

Second Solution. (*Iandrei - ML Forum*) I've found a solution based on an idea from the hardest inequality I've ever seen (it really is impossible, in my opinon!). First, I'll post the original inequality by Vasc, from which I have taken the idea.
Let $a, b, c, d > 0$ be real numbers for which $a^2 + b^2 + c^2 + d^2 = 1$. Prove that the following inequality holds:

$$(1-a)(1-b)(1-c)(1-d) \geq abcd$$

I'll leave its proof to the readers. A little historical note on this problem: it was proposed in some *Gazeta Matematica Contest* in the last years and while I was still in high-school and training for mathematical olympiads, I tried to solve it on a very large number of occasions, but failed. So I think I will always remember its difficult and smart solution, which I'll leave to the readers.

Now, let us get back to our original problem:

Let $a, b, c, d > 0$ with $abcd = 1$. Prove that:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1$$

Although this inequality also belongs to Vasc (he published it in the Gazeta Matematica), it surprisingly made the China 2005 TSTs, thus confirming (in my opinion) its beauty and difficulty. Now, on to the solution:
Let

$$x = \frac{1}{1+a},\ y = \frac{1}{1+b},\ z = \frac{1}{1+c},\ t = \frac{1}{1+d}$$

Then

$$abcd = 1 \Rightarrow \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} = 1 \Rightarrow (1-x)(1-y)(1-z)(1-t) = xyzt$$

We have to prove that $x^2+y^2+z^2+t^2 \geq 1$. We will prove this by contradiction. Assume that $x^2 + y^2 + z^2 + t^2 < 1$.
Keeping in mind that $x^2+y^2+z^2+t^2 < 1$, let us assume that $(1-x)(1-y) \leq zt$ and prove that it is not true (I'm talking about the last inequality here, which we assumed to be true). Upon multiplication with 2 and expanding, this gives:

$$1 - 2(x+y) + 1 + 2xy \leq 2zt$$

This implies that

$$2zt > x^2 + y^2 + z^2 + t^2 - 2(x+y) + 1 + 2xy$$

So, $2zt > (x+y)^2 - 2(x+y) + 1 + z^2 + t^2$, which implies $(z-t)^2 + (x+y-1)^2 < 0$, a contradiction. Therefore, our original assumption implies $(1-x)(1-y) > zt$. In a similar manner, it is easy to prove that $(1-z)(1-t) > xy$. Multiplying

the two, we get that $(1-x)(1-y)(1-z)(1-t) > xyzt$, which is a contradiction with the original condition $abcd = 1$ rewritten in terms of x, y, z, t.
Therefore, our original assumption was false and we indeed have

$$x^2 + y^2 + z^2 + t^2 \geq 1$$

□

18. (**China 2005**) $(ab + bc + ca = \frac{1}{3},\ a, b, c \geq 0)$

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \leq 3$$

First Solution. (*Cuong - ML Forum*) Our inequality is equivalent to:

$$\sum \frac{a}{3a(a+b+c)+2} \geq \frac{\frac{1}{3}}{a+b+c}$$

Since $ab + bc + ca = \frac{1}{3}$, by Cauchy we have:

$$\begin{aligned}
LHS &\geq \frac{(a+b+c)^2}{3\,(a+b+c)\,(a^2+b^2+c^2) + 2\,(a+b+c)} = \\
&= \frac{a+b+c}{3\,(a^2+b^2+c^2) + 2\,(a+b+c)} = \\
&= \frac{a+b+c}{3\,(a^2+b^2+c^2+2ab+2ac+2bc)} = \\
&= \frac{a+b+c}{3\,(a+b+c)^2} = \frac{\frac{1}{3}}{a+b+c}
\end{aligned}$$

□

Second Solution. (*Billzhao - ML Forum*) Homogenizing, the inequality is equivalent to

$$\sum_{\text{cyc}} \frac{1}{a(a+b+c)+2(ab+bc+ca)} \leq \frac{1}{ab+bc+ca}$$

Multiply both sides by $2(ab + bc + ca)$ we have

$$\sum_{\text{cyc}} \frac{2(ab+bc+ca)}{a(a+b+c)+2(ab+bc+ca)} \leq 2$$

Subtracting from 3, the above inequality is equivalent to

$$\sum_{\text{cyc}} \frac{a(a+b+c)}{a(a+b+c)+2(ab+bc+ca)} \geq 1$$

Now by Cauchy we have:

$$LHS = (a+b+c)\sum_{\text{cyc}} \frac{a^2}{a^2\left(a+b+c\right)+2a\left(ab+bc+ca\right)} \geq$$

$$\geq \frac{\left(a+b+c\right)^3}{\sum_{\text{cyc}}\left[a^2\left(a+b+c\right)+2a\left(ab+bc+ca\right)\right]} = 1$$

$\square$

19. (**Poland 2005**) $(0 \leq a, b, c \leq 1)$

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2$$

First Solution. (*See* [25] *pag.* 204 *problem* 95) WLOG we can assume that $0 \leq a \leq b \leq c \leq 1$. Since $0 \leq (1-a)(1-b)$ we have

$$\begin{aligned} &a+b \leq 1+ab \leq 1+2ab \quad \Rightarrow \\ &a+b+c \leq a+b+1 \leq 2\left(1+ab\right) \end{aligned}$$

Therefore

$$\frac{a}{1+bc} + \frac{b}{1+ac} + \frac{c}{1+ab} \leq \frac{a}{1+ab} + \frac{b}{1+ab} + \frac{c}{1+ab} \leq \frac{a+b+c}{1+ab} \leq 2$$

$\square$

Second Solution. (*Ercole Suppa*) We denote the LHS with $f\left(a,b,c\right)$. The function f is defined and continuous on the cube $C = [0,1]\times[0,1]\times[0,1]$ so, by Wierstrass theorem, f assumes its maximum and minimum on C. Since f is convex with respect to all variables we obtain that f take maximum value in one of vertices of the cube. Since f is symmetric in a, b, c it is enough compute the values $f(0,0,0), f(0,0,1), f(0,1,1), f(1,1,1)$. It's easy verify that f take maximum value in $(0,1,1)$ and $f\left(0,1,1\right) = 2$. The convexity of f with respect to variable a follows from the fact that

$$f\left(x,b,c\right) = \frac{x}{bc+1} + \frac{b}{cx+1} + \frac{c}{bx+1}$$

is the sum of three convex functions. Similarly we can prove the convexity with respect to b and c. $\square$

20. (**Poland 2005**) $(ab+bc+ca=3,\ a,b,c>0)$

$$a^3+b^3+c^3+6abc \geq 9$$

Solution. (*Ercole Suppa*) Since $ab+bc+ca=1$ by Mac Laurin inequality we have:

$$\frac{a+b+c}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} = 1 \tag{1}$$

By Schur inequality we have

$$\begin{gathered}\sum_{\text{cyc}} a\,(a-b)\,(a-c) \geq 0 \quad \Longrightarrow \\ a^3+b^3+c^3+3abc \geq \sum_{\text{sym}} a^2 b\end{gathered}$$

and, from (1), it follows that

$$\begin{aligned}a^3+b^3+c^3+6abc &\geq a^2b+a^2c+abc+b^2a+b^2c+abc+c^2a+c^2b+abc = \\ &= (a+b+c)\,(ab+bc+ca) = \\ &= 3\,(a+b+c) \geq 9\end{aligned}$$

□

21. (**Baltic Way 2005**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{a^2+2}+\frac{b}{b^2+2}+\frac{c}{c^2+2} \geq 1$$

Solution. (*Sailor - ML Forum*) We have

$$\sum \frac{a}{a^2+2} \leq \sum \frac{a}{2a+1}$$

We shall prove that $\sum \frac{a}{2a+1} \leq 1$ or

$$\sum \frac{2a}{2a+1} \leq 2 \quad \Longleftrightarrow \quad \sum \frac{1}{2a+1} \geq 1$$

Clearing the denominator we have to prove that:

$$2\sum a \geq 6$$

wich is true by AM-GM. □

22. (**Serbia and Montenegro 2005**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}(a+b+c)}$$

Solution. (*Ercole Suppa*) Putting

$$x = \frac{a}{a+b+c}, \qquad y = \frac{b}{a+b+c}, \qquad z = \frac{c}{a+b+c}$$

the proposed inequality is exctly that of problem 10. □

23. (**Serbia and Montenegro 2005**) $(a+b+c = 3,\ a, b, c > 0)$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

Solution. (*Suat Namly - ML Forum*) From AM-GM inequality we have

$$a^2 + \sqrt{a} + \sqrt{a} \geq 3a$$

By the same reasoning we obtain

$$b^2 + \sqrt{b} + \sqrt{b} \geq 3b$$

$$c^2 + \sqrt{c} + \sqrt{c} \geq 3c$$

Adding these three inequalities, we obtain

$$\begin{aligned} a^2 + b^2 + c^2 + 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) &\geq 3(a+b+c) = \\ &= (a+b+c)^2 = \\ &= a^2 + b^2 + c^2 + 2(ab+bc+ca) \end{aligned}$$

from which we get the required result. □

24. (**Bosnia and Hercegovina 2005**) $(a+b+c = 1, a, b, c > 0)$

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}$$

Solution. (*Ercole Suppa*) From the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left(a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\right)^2 &= \left(\sum_{\text{cyc}} \sqrt{a}\sqrt{ab}\right)^2 \le \\ &\le \left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}} ab\right) = \\ &= (ab+bc+ca) \le \\ &\le \frac{1}{3}(a+b+c)^2 = \frac{1}{3}\end{aligned}$$

Extracting the square root yields the required inequality. □

25. (**Iran 2005**) $(a, b, c > 0)$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Solution. (*Ercole Suppa*) After setting

$$x = \frac{a}{b}, \qquad y = \frac{b}{c}, \qquad z = \frac{c}{a}$$

the inequality become

$$\begin{aligned}&(x+y+z)^2 \ge +x+y+z+xy+xz+yz \qquad \Longleftrightarrow \\ &x^2+y^2+z^2+xy+xz+yz \ge 3+x+y+z\end{aligned}$$

where $xyz = 1$. From AM-GM inequality we have

$$xy+xz+yz \ge 3\sqrt[3]{x^2y^2z^2} = 3 \tag{1}$$

On the other hand we have

$$\begin{aligned}&\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2 \ge 0 \qquad \Longrightarrow \\ &x^2+y^2+z^2 \ge x+y+z+\frac{3}{4} \ge x+y+z \qquad (2)\end{aligned}$$

Adding (1) and (2) yields the required result. □

26. (**Austria 2005**) $(a, b, c, d > 0)$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3} \ge \frac{a+b+c+d}{abcd}$$

Solution. (*Ercole Suppa*) WLOG we can assume that $a \geq b \geq c \geq d$ so

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \geq \frac{1}{d}$$

Since the RHS can be written as

$$\frac{a+b+c+d}{abcd} = \frac{1}{bcd} + \frac{1}{acd} + \frac{1}{abd} + \frac{1}{abc}$$

from the rearrangement (applied two times) we obtain

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{1}{bcd} + \frac{1}{acd} + \frac{1}{abd} + \frac{1}{abc} = \frac{a+b+c+d}{abcd}$$

□

27. (**Moldova 2005**) $(a^4 + b^4 + c^4 = 3,\ a, b, c > 0)$

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ca} \leq 1$$

First Solution. (*Anto - ML Forum*) It is easy to prove that :

$$\sum \frac{1}{4-ab} \leq \sum \frac{1}{4-a^2} \leq \sum \frac{a^4+5}{18}$$

The first inequality follows from :

$$\frac{2}{4-ab} \leq \frac{1}{4-a^2} + \frac{1}{4-b^2}$$

The second :

$$\frac{1}{4-a^2} \leq \frac{a^4+5}{18}$$

is equivalent to :

$$0 \leq 4a^4 + 2 - a^6 - 5a^2 \Longleftrightarrow 0 \leq (a^2-1)^2(2-a^2)$$

which is true since $a^4 \leq 3$ and as a result $a^2 \leq 2$.

Thus

$$\sum_{\text{cyc}} \frac{1}{4-ab} \leq \sum_{\text{cyc}} \frac{a^4+5}{18} = \frac{a^4+b^4+c^4+15}{18} = \frac{3+15}{18} = 1$$

□

Second Solution. (*Treegoner - ML Forum*) By applying AM-GM inequality, we obtain

$$LHS \le \sum \frac{1}{4-\sqrt{\frac{a^4+b^4}{2}}} = \sum \frac{1}{4-\sqrt{\frac{3-c^4}{2}}}$$

Denote

$$u = \frac{3-c^4}{2} \quad , \quad v = \frac{3-c^4}{2} \quad , \quad w = \frac{3-c^4}{2}$$

Then $0 < u, v, w < \frac{3}{2}$ and $u + v + w = 3$. Let $f(u) = \frac{1}{4-\sqrt{u}}$. Then

$$f'(u) = \frac{1}{2\sqrt{u}(4-\sqrt{u})^2}$$

$$f''(u) = \frac{-u^{\frac{-3}{4}}(1-\frac{3}{4}u^{\frac{1}{2}})}{(4u^{\frac{1}{4}}-u^{\frac{3}{4}})^3}$$

Hence $f''(u) < 0$ for every $0 < u < \frac{3}{2}$. By apply Karamata 's inequality for the function that is concave down, we obtain the result. $\square$

28. (**APMO 2005**) ($abc = 8,\ a, b, c > 0$)

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \ge \frac{4}{3}$$

First Solution. (*Valiowk, Billzhao - ML Forum*) Note that

$$\frac{a^2+2}{2} = \frac{(a^2-a+1)+(a+1)}{2} \ge \sqrt{(a^2-a+1)(a+1)} = \sqrt{a^3+1}$$

with equality when $a = 2$. Hence it suffices to prove

$$\frac{a^2}{(a^2+2)(b^2+2)} + \frac{b^2}{(b^2+2)(c^2+2)} + \frac{c^2}{(c^2+2)(a^2+2)} \ge \frac{1}{3}$$

and this is easily verified. In fact clearing the denominator, we have

$$3\sum_{\text{cyc}} a^2(c^2+2) \ge (a^2+2)(b^2+2)(c^2+2)$$

Expanding, we have

$$6a^2 + 6b^3 + 6c^3 + 3a^2b^2 + 3b^2c^2 + 3c^2a^2 \ge$$
$$\ge a^2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 4a^2 + 4b^2 + 4c^2 + 8$$

Recalling that $abc = 8$, the above is equivalent to

$$2a^2 + 2b^2 + 2c^2 + a^2b^2 + b^2c^2 + c^2a^2 \geq 72$$

But $2a^2 + 2b^2 + 2c^2 \geq 24$ and $a^2b^2 + b^2c^2 + c^2a^2 \geq 48$ through AM-GM. Adding gives the result.

$\square$

Second Solution. (*Official solution*) Observe that

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}$$

In fact, this is equivalent to $(2+x^2)^2 \geq 4(1+x^3)$, or $x^2(x-2)^2 \geq 0$. Notice that equality holds if and only if if $x = 2$. Then

$$\begin{aligned}
&\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \\
&\geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+c^2)} \geq \\
&\geq \frac{2 \cdot S(a,b,c)}{36 + S(a,b,c)} = \\
&= \frac{2}{1 + \frac{36}{S(a,b,c)}}
\end{aligned}$$

where

$$S(a,b,c) = 2\left(a^2 + b^2 + c^2\right) + (ab)^2 + (bc)^2 + (ca)^2$$

By AM-GM inequality, we have

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{(abc)^2} = 12$$

$$(ab)^2 + (bc)^2 + (ca)^2 \geq 3\sqrt[3]{(abc)^4} = 48$$

The above inequalities yield

$$S(a,b,c) = 2\left(a^2 + b^2 + c^2\right) + (ab)^2 + (bc)^2 + (ca)^2 \geq 72$$

Therefore

$$\frac{2}{1 + \frac{36}{S(a,b,c)}} \geq \frac{2}{1 + \frac{36}{72}} = \frac{4}{3}$$

which is the required inequality. Note that the equalitiy hold if and only if $a = b = c = 2$. $\square$

29. (**IMO 2005**) ($xyz \geq 1, x, y, z > 0$)

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

First Solution. (*See* [32], *pag.* 26) It's equivalent to the following inequality

$$\left(\frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1\right) + \left(\frac{y^2 - y^5}{y^5 + z^2 + x^2} + 1\right) + \left(\frac{z^2 - z^5}{z^5 + x^2 + y^2} + 1\right) \leq 3$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 3.$$

With the Cauchy-Schwarz inequality and the fact that $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2 \quad \text{or} \quad \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum and $x^2 + y^2 + z^2 \geq xy + yz + zx$ give us

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{xy + yz + zx}{x^2 + y^2 + z^2} \leq 3.$$

$\square$

Second Solution. (*by an IMO 2005 contestant Iurie Boreico from Moldova, see* [32] *pag.* 28). We establish that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)}.$$

It follows immediately from the identity

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} - \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)} = \frac{(x^3 - 1)^2 x^2 (y^2 + z^2)}{x^3(x^2 + y^2 + z^2)(x^5 + y^2 + z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyc}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{1}{x^5 + y^2 + z^2} \sum_{\text{cyc}} \left(x^2 - \frac{1}{x}\right) \geq \frac{1}{x^5 + y^2 + z^2} \sum_{\text{cyc}} \left(x^2 - yz\right) \geq 0.$$

$\square$

30. (**Poland 2004**) ($a + b + c = 0, a, b, c \in \mathbb{R}$)

$$b^2c^2 + c^2a^2 + a^2b^2 + 3 \geq 6abc$$

First Solution. (*Ercole Suppa*) Since $a+b+c=0$ from the identity

$$(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$$

follows that

$$a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2$$

Then, from the AM-GM inequality we have

$$a^2b^2+b^2c^2+c^2a^2 \geq 3\cdot\left(\sqrt[3]{a^2b^2c^2}\right)^2$$

By putting $abc = P$ we have

$$\begin{aligned}
&a^2b^2+b^2c^2+c^2a^2+3-6abc \geq \\
&\geq 9\,(abc)^{\frac{4}{3}}+3-6abc = \\
&= 9P^4-6P^3+3 = \\
&= 3\left[\left(P^2-1\right)^2+2P^2\left(P-\frac{1}{2}\right)^2+\frac{3}{2}P^2\right] \geq 0
\end{aligned}$$

□

Second Solution. (*Darij Grinberg - ML Forum*)
For any three real numbers a, b, c, we have

$$\begin{aligned}
&\left(b^2c^2+c^2a^2+a^2b^2+3\right)-6abc = \\
&= (b+1)^2(c+1)^2+(c+1)^2(a+1)^2+(a+1)^2(b+1)^2 \\
&\quad -2\,(a+b+c)\,(a+b+c+bc+ca+ab+2)
\end{aligned}$$

so that, in the particular case when $a+b+c=0$, we have

$$\begin{aligned}
&\left(b^2c^2+c^2a^2+a^2b^2+3\right)-6abc = \\
&= (b+1)^2(c+1)^2+(c+1)^2(a+1)^2+(a+1)^2(b+1)^2
\end{aligned}$$

and thus

$$b^2c^2+c^2a^2+a^2b^2+3 \geq 6abc$$

□

Third Solution. (*Nguyenquockhanh, Ercole Suppa - ML Forum*)
WLOG we can assume that $a > 0$ e $b > 0$. Thus, since $c = -a - b$, we have

$$\begin{aligned}
&a^2b^2 + b^2c^2 + c^2a^2 + 3 - 6abc = \\
&= a^2b^2 + \left(a^2 + b^2\right)(a+b)^2 + 3 + 6ab(a+b) \geq \\
&\geq \left(a^2 + b^2\right)\left(a^2 + ab + b^2\right) + ab\left(a^2 + b^2\right) + a^2b^2 + 3 + 6ab(a+b) \geq \\
&\geq 3ab\left(a^2 + b^2\right) + 3\left(a^2b^2 + 1\right) + 6ab(a+b) = \\
&\geq 3ab\left(a^2 + b^2\right) + 6ab + 6ab(a+b) = \\
&= 3ab\left(a^2 + b^2 + 2 + 2a + 2b\right) = \\
&= 3ab\left[(a+1)^2 + (b+1)^2\right] \geq 0
\end{aligned}$$

□

31. (**Baltic Way 2004**) $(abc = 1, a, b, c > 0, n \in \mathbb{N})$

$$\frac{1}{a^n + b^n + 1} + \frac{1}{b^n + c^n + 1} + \frac{1}{c^n + a^n + 1} \leq 1$$

Solution. (*Ercole Suppa*) By setting $a^n = x$, $b^n = y$ e $c^n = z$, we must prove that

$$\frac{1}{1 + x + y} + \frac{1}{1 + y + z} + \frac{1}{1 + z + x} \leq 1$$

where $xyz = 1$. The above inequality is proven in the problem 7. □

32. (**Junior Balkan 2004**) $((x, y) \in \mathbb{R}^2 - \{(0, 0)\})$

$$\frac{2\sqrt{2}}{\sqrt{x^2 + y^2}} \geq \frac{x + y}{x^2 - xy + y^2}$$

First Solution. (*Ercole Suppa*) By using the two inequalities

$$x + y \leq \sqrt{2(x^2 + y^2)} \quad , \quad x^2 + y^2 \leq 2(x^2 - xy + y^2)$$

we have:

$$\begin{aligned}
\frac{(x+y)\sqrt{x^2 + y^2}}{x^2 - xy + y^2} &\leq \frac{\sqrt{2(x^2 + y^2)}\sqrt{x^2 + y^2}}{x^2 - xy + y^2} \leq \\
&\leq \frac{\sqrt{2}(x^2 + y^2)}{x^2 - xy + y^2} \leq \\
&\leq \frac{2(x^2 - xy + y^2)\sqrt{2}}{(x^2 - xy + y^2)} = 2\sqrt{2}
\end{aligned}$$

□

Second Solution. (*Darij Grinberg - ML Forum*) You can also prove the inequality by squaring it (in fact, the right hand side of the inequality is obviously ≥ 0; if the left hand side is ≤ 0, then the inequality is trivial, so it is enough to consider the case when it is ≥ 0 as well, and then we can square the inequality); this leads to

$$\frac{(x+y)^2}{(x^2-xy+y^2)^2} \leq \frac{8}{x^2+y^2}$$

This is obviously equivalent to

$$(x+y)^2(x^2+y^2) \leq 8(x^2-xy+y^2)^2$$

But actually, an easy calculation shows that

$$8(x^2-xy+y^2)^2-(x+y)^2(x^2+y^2) = (x-y)^2\left[2(x-y)^2+5x^2+5y^2\right] \geq 0$$

so everything is proven.

$\square$

33. (**IMO Short List 2004**) $(ab+bc+ca=1,\ a,b,c>0)$

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}$$

First Solution. (*Ercole Suppa*) The function $f(x)=\sqrt[3]{x}$ is concave on $(0,+\infty)$. Thus from Jensen inequality we have:

$$\sum_{\text{cyc}} f\left(\frac{1}{a}+6b\right) \leq 3\cdot f\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6a+6b+6c}{3}\right) \tag{1}$$

From the well-know inequality $3(xy+yz+zx) \leq (x+y+z)^2$ we have

$$3abc(a+b+c) = 3(ab\cdot ac+ab\cdot bc+ac\cdot bc) \leq (ab+bc+ca)^2 = 1 \quad \Longrightarrow$$

$$6(a+b+c) \leq \frac{2}{abc} \tag{2}$$

The AM-GM inequality and (2) yields

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6a+6b+6c\right) \leq \frac{ab+bc+ca}{abc}+\frac{2}{abc} = \frac{3}{abc} \tag{3}$$

Since $f(x)$ is increasing from (1) and (3) we get

$$f\left(\frac{1}{a}+6b\right)+f\left(\frac{1}{b}+6c\right)+f\left(\frac{1}{c}+6a\right) \leq 3\cdot f\left(\frac{1}{abc}\right) = \frac{3}{\sqrt[3]{abc}} \leq \frac{1}{abc}$$

where in the last step we used the AM-GM inequality

$$\frac{3}{\sqrt[3]{abc}} = \frac{3\sqrt[3]{(abc)^2}}{abc} = \frac{3\sqrt[3]{ab \cdot bc \cdot ca}}{abc} \le \frac{3 \cdot \frac{ab+bc+ca}{3}}{abc} = \frac{1}{abc}$$

□

Second Solution. (*Official solution*) By the power mean inequality

$$\frac{1}{3}(u+v+w) \le \sqrt[3]{\frac{1}{3}(u^3+v^3+w^3)}$$

the left-hand side does not exceed

$$\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{a}+6b+\frac{1}{b}+6c+\frac{1}{c}+6a} = \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{ab+bc+ca}{abc}+6(a+b+c)} \qquad (\star)$$

The condition $ab+bc+ca=1$ enables us to write

$$a+b = \frac{1-ab}{c} = \frac{ab-(ab)^2}{abc}, \quad b+c = \frac{bc-(bc)^2}{abc}, \quad c+a = \frac{ca-(ca)^2}{abc}$$

Hence

$$\begin{aligned}\frac{ab+bc+ca}{abc}+6(a+b+c) &= \frac{1}{abc}+3[(a+b)+(b+c)+(c+a)] = \\ &= \frac{4-3\left[(ab)^2+(bc)^2+(ca)^2\right]}{abc}\end{aligned}$$

Now, we have

$$3\big((ab)^2+(bc)^2+(ca)^2\big) \ge (ab+bc+ca)^2 = 1$$

by the well-known inequality $3(u^2+v^2+w^2) \ge (u+v+w)^2$. Hence an upper bound for the right-hand side of $(\star)$ is $3/\sqrt[3]{abc}$. So it suffices to check $3/\sqrt[3]{abc} \le 1/(abc)$, which is equivalent to $(abc)^2 \le 1/27$. This follows from the AM-GM inequality, in view of $ab+bc+ca=1$ again:

$$(abc)^2 = (ab)(bc)(ca) \le \left(\frac{ab+bc+ca}{3}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

Clearly, equality occurs if and only if $a=b=c=1/\sqrt{3}$. □

Third Solution. (*Official solution*)
Given the conditions $a,b,c>0$ and $ab+bc+ca=1$, the following more general result holds true for all $t_1,t_2,t_3>0$:

$$3abc(t_1+t_2+t_3) \le \frac{2}{3}+at_1^3+bt_2^3+ct_3^3. \qquad (1)$$

The original inequality follows from (2) by setting

$$t_1 = \frac{1}{3}\sqrt[3]{\frac{1}{a}+6b}, \quad t_2 = \frac{1}{3}\sqrt[3]{\frac{1}{b}+6c}, \quad t_3 = \frac{1}{3}\sqrt[3]{\frac{1}{c}+6a}.$$

In turn, (1) is obtained by adding up the three inequalities

$$3abct_1 \le \frac{1}{9}+\frac{1}{3}bc+at_1^3, \quad 3abct_2 \le \frac{1}{9}+\frac{1}{3}ca+bt_2^3, \quad 3abct_3 \le \frac{1}{9}+\frac{1}{3}ab+ct_3^3.$$

By symmetry, it suffices to prove the first one of them. Since $1-bc = a(b+c)$, the AM-GM inequality gives

$$(1-bc)+\frac{at_1^3}{bc} = a\left(b+c+\frac{t_1^3}{bc}\right) \ge 3a\sqrt[3]{bc\cdot\frac{t_1^3}{bc}} = 3at_1.$$

Hence $3abct_1 \le bc(1-bc)+at_1^3$, and one more application of the AM-GM inequality completes the proof:

$$\begin{aligned} 3abct_1 &\le bc(1-bc)+at_1^3 = bc\left(\frac{2}{3}-bc\right)+\frac{1}{3}bc+at_1^3 \\ &\le \left(\frac{bc+(\frac{2}{3}-bc)}{2}\right)^2+\frac{1}{3}bc+at_1^3 = \frac{1}{9}+\frac{1}{3}bc+at_1^3. \end{aligned}$$

□

34. (**APMO 2004**) $(a,b,c>0)$

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

First Solution. (*See* [32], *pag.* 14) Choose $A,B,C \in \left(0,\frac{\pi}{2}\right)$ with $a=\sqrt{2}\tan A$, $b=\sqrt{2}\tan B$, and $c=\sqrt{2}\tan C$. Using the well-known trigonometric identity $1+\tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \ge \cos A\cos B\cos C\left(\cos A\sin B\sin C+\sin A\cos B\sin C+\sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\begin{gathered}\cos(A+B+C) = \\ = \cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C.\end{gathered}$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \ge \cos A\cos B\cos C\left(\cos A\cos B\cos C-\cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying the AM-GM inequality and Jensen's inequality, we have

$$\cos A \cos B \cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3} \right)^3 \leq \cos^3 \theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3 \theta (\cos^3 \theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3 \theta - 3\cos\theta \ \text{ or } \ \cos 3\theta - \cos 3\theta = 3\cos\theta - 3\cos^3\theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4 \theta \left(1 - \cos^2 \theta\right),$$

which follows from the AM-GM inequality

$$\left(\frac{\cos^2 \theta}{2} \cdot \frac{\cos^2 \theta}{2} \cdot \left(1 - \cos^2 \theta\right) \right)^{\frac{1}{3}} \leq \frac{1}{3} \left(\frac{\cos^2 \theta}{2} + \frac{\cos^2 \theta}{2} + \left(1 - \cos^2 \theta\right) \right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. □

Second Solution. (*See* [32], *pag.* 34) After expanding, it becomes

$$8 + (abc)^2 + 2\sum_{\text{cyc}} a^2b^2 + 4\sum_{\text{cyc}} a^2 \geq 9\sum_{\text{cyc}} ab.$$

From the inequality $(ab-1)^2 + (bc-1)^2 + (ca-1)^2 \geq 0$, we obtain

$$6 + 2\sum_{\text{cyc}} a^2b^2 \geq 4\sum_{\text{cyc}} ab.$$

Hence, it will be enough to show that

$$2 + (abc)^2 + 4\sum_{\text{cyc}} a^2 \geq 5\sum_{\text{cyc}} ab.$$

Since $3(a^2 + b^2 + c^2) \geq 3(ab + bc + ca)$, it will be enough to show that

$$2 + (abc)^2 + \sum_{\text{cyc}} a^2 \geq 2\sum_{\text{cyc}} ab,$$

which is proved in [32], pag.33.

□

Third Solution. (*Darij Grinberg - ML Forum*)
First we prove the auxiliary inequality

$$1 + 2abc + a^2 + b^2 + c^2 \geq 2bc + 2ca + 2ab$$

According to the pigeonhole principle, among the three numbers $a-1$, $b-1$, $c-1$ at least two have the same sign; WLOG, say that the numbers $b-1$ and $c-1$ have the same sign so that $(b-1)(c-1) \geq 0$. Then according to the inequality $x^2 + y^2 \geq 2xy$ for any two reals x and y, we have

$$(b-1)^2 + (c-1)^2 \geq 2(b-1)(c-1) \geq -2(a-1)(b-1)(c-1)$$

Thus

$$\begin{aligned}
&(1 + 2abc + a^2 + b^2 + c^2) - (2bc + 2ca + 2ab) = \\
&= (a-1)^2 + (b-1)^2 + (c-1)^2 + 2(a-1)(b-1)(c-1) \geq \\
&\geq (a-1)^2 \geq 0
\end{aligned}$$

and the lemma is proved. Now, the given inequality can be proved in the following way:

$$\begin{aligned}
&(a^2+2)(b^2+2)(c^2+2) - 9(ab+bc+ca) = \\
&= \frac{3}{2}\left((b-c)^2 + (c-a)^2 + (a-b)^2\right) + 2\left((bc-1)^2 + (ca-1)^2 + (ab-1)^2\right) + \\
&+ (abc-1)^2 + \left((1 + 2abc + a^2 + b^2 + c^2) - (2bc + 2ca + 2ab)\right) \geq 0
\end{aligned}$$

□

Fourth Solution. (*Official solution.*)
Let $x = a+b+c$, $y = ab+bc+ca$, $z = abc$. Then

$$\begin{aligned}
a^2 + b^2 + c^2 &= x^2 - 2y \\
a^2b^2 + b^2c^2 + c^2a^2 &= y^2 - 2xz \\
a^2b^2c^2 &= z^2
\end{aligned}$$

so the inequality to be proved becomes

$$z^2 + 2\left(y^2 - 2xz\right) + 4\left(x^2 - 2y\right) + 8 \geq 9y$$

or

$$z^2 + 2y^2 - 4xz + 4x^2 - 17y + 8 \geq 0$$

Now from $a^2 + b^2 + c^2 \geq ab + bc + ca = y$, we get

$$x^2 = a^2 + b^2 + c^2 + 2y \geq 3y$$

Also

$$\begin{aligned}a^2b^2+b^2c^2+a^2c^2 &= (ab)^2+(bc)^2+(ca)^2 \geq \\ &\geq ab\cdot ac+bc\cdot ab+ac\cdot bc = \\ &= (a+b+c)abc = xz\end{aligned}$$

and thus

$$y^2a^2b^2+b^2c^2+a^2c^2+2xz \geq 3xz$$

Hence

$$\begin{aligned}&z^2+2y^2-4xz+4x^2-17y+8= \\ =&\left(z-\frac{x}{3}\right)^2+\frac{8}{9}(y-3)^2+\frac{10}{9}\left(y^2-3xz\right)+\frac{35}{9}\left(x^2-3y\right)\geq 0\end{aligned}$$

as required.

□

35. (**USA 2004**) $(a,b,c>0)$

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$$

Solution. (*See* [11] *pag.* 19) For any positive number x, the quantities x^2-1 and x^3-1 have the same sign. Thus, we have

$$0 \leq (x^3-1)(x^2-1) = x^5-x^3-x^2+1 \quad\Longrightarrow\quad x^5-x^2+3 \geq x^3+2$$

It follows that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a^3+2)(b^3+2)(c^3+2)$$

It suffices to show that

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3 \tag{$\star$}$$

Expanding both sides of inequality $(\star)$ and cancelling like terms gives

$$\begin{aligned}&a^3b^3c^3+3(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)+8\geq \\ \geq\ &3(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c)+6abc\end{aligned}$$

By AM-GM inequality, we have $a^3+a^3b^3+1 \geq 3a^2b$. Combining similar results, the desidered inequality reduces to

$$a^3b^3c^3+a^3+b^3+c^3+1+1 \geq 6abc$$

which is evident by AM-GM inequality. □

36. (**Junior BMO 2003**) $(x, y, z > -1)$

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$$

Solution. (*Arne - ML Forum*) As $x \leq \frac{1+x^2}{2}$ we have

$$\sum \frac{1+x^2}{1+y+z^2} \geq \sum \frac{2(1+x^2)}{(1+y^2)+2(1+z^2)}.$$

Denoting $1+x^2=a$ and so on we have to prove that

$$\sum \frac{a}{b+2c}\geq 1$$

but Cauchy tells us

$$\sum \frac{a}{b+2c}\sum a(2b+c)\geq \left(\sum a\right)^2$$

and as

$$\left(\sum a\right)^2 \geq 3(ab+bc+ca)=\sum a(2b+c)$$

we have the result. □

37. (**USA 2003**) $(a, b, c > 0)$

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

First Solution. (*See* [10] *pag.* 21) By multipliyng a, b and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desidered inequality read

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2}\leq 8$$

Set

$$f(x)=\frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c)\le 8$. Note that

$$\begin{aligned}
f(x) &= \frac{x^2+6x+9}{3\left(x^2-2x+3\right)} = \\
&= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right) = \\
&= \frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right) \le \\
&\le \frac{1}{3}\left(1+\frac{8x+6}{2}\right) = \frac{1}{3}\left(4x+4\right)
\end{aligned}$$

Hence

$$f(a)+f(b)+f(c)\le \frac{1}{3}\left(4a+4+4b+4+4c+4\right)=8$$

as desidered, with equality if and only if $a=b=c$. $\square$

Second Solution. (*See* [40]) We can assume, WLOG, $a+b+c=1$. Then the first term of LHS is equal to

$$f(a)=\frac{(a+1)^2}{2a^2+(1-a)^2}=\frac{a^2+2a+1}{3a^2-2a+1}$$

(When $a=b=c=\frac{1}{3}$, there is equality. A simple sketch of $f(x)$ on $[0,1]$ shows the curve is below the tangent line at $x=\frac{1}{3}$, which has the equation $y=\frac{12x+4}{3}$). So we claim that

$$\frac{a^2+2a+1}{3a^2-2a+1}\le\frac{12a+4}{3}$$

for $a<0<1$. This inequality is equivalent to

$$36a^3-15a^2-2a+1=(3a-1)^2(4a+1)\ge 0 \quad , \quad 0<a<1$$

hence is true. Adding the similar inequalities for b and c we get the desidered inequality. $\square$

38. (**Russia 2002**) $(x+y+z=3,\ x,y,z>0)$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\ge xy+yz+zx$$

Solution. (*Ercole Suppa*) See Problem 23. $\square$

39. (**Latvia 2002**) $\left(\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1,\ a,b,c,d>0\right)$

$$abcd \geq 3$$

First Solution. (*Ercole Suppa*) We first prove a lemma:

Lemma. For any real positive numbers x, y with $xy \geq 1$ we have

$$\frac{1}{x^2+1}+\frac{1}{y^2+1} \geq \frac{2}{xy+1}$$

Proof. The required inequality follows from the identity

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}-\frac{2}{xy+1}=\frac{(x-y)^2(xy-1)}{(x^2+1)(y^2+1)(xy+1)}$$

the proof of which is immediate. □

In order to prove the required inequality we observe at first that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4} \leq 1 \quad \Longrightarrow \quad a^4b^4 \geq 1 \quad \Longrightarrow \quad a^2b^2 \geq 1$$

Thus by previous lemma we have

$$\frac{1}{1+a^4}+\frac{1}{1+b^4} \geq \frac{2}{a^2b^2+1} \tag{1}$$

and similarly

$$\frac{1}{1+c^4}+\frac{1}{1+d^4} \geq \frac{2}{c^2d^2+1} \tag{2}$$

Since $ab \geq 1$ e $cd \geq 1$ we can add (1) and (2) and we can apply again the lemma:

$$\begin{aligned}1 &= \frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4} \geq \\ &\geq 2\left(\frac{1}{a^2b^2+1}+\frac{1}{c^2d^2+1}\right) \geq \\ &\geq \frac{4}{abdc+1}\end{aligned}$$

Thus $abcd+1 \geq 4$ so $abcd \geq 3$. □

Second Solution. (*See* [32], *pag.* 14) We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying the AM-GM inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3\left(\cos B \cos C \cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3\left(\cos C \cos D \cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3\left(\cos D \cos A \cos B\right)^{\frac{2}{3}}$$

and

$$\sin^2 D \geq 3\left(\cos A \cos B \cos C\right)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! □

40. (**Albania 2002**) $(a, b, c > 0)$

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq a+b+c+\sqrt{a^2+b^2+c^2}$$

Solution. (*Ercole Suppa*) From AM-GM inequality we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} \geq \frac{3\sqrt[3]{a^2b^2c^2}}{abc} = \frac{3}{\sqrt[3]{abc}} \tag{1}$$

From AM-QM inequality we have

$$a+b+c \leq 3\sqrt{\frac{a^2+b^2+c^2}{3}} \tag{2}$$

From (1) and (2) we get

$$\begin{aligned}
\frac{a+b+c+\sqrt{a^2+b^2+c^2}}{(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} &\leq \frac{\frac{3}{\sqrt{3}}\sqrt{a^2+b^2+c^2}+\sqrt{a^2+b^2+c^2}}{(a^2+b^2+c^2)\frac{3}{\sqrt[3]{abc}}} \leq \\
&\leq \frac{3+\sqrt{3}}{\sqrt{3}}\frac{\sqrt{a^2+b^2+c^2}\sqrt[3]{abc}}{3(a^2+b^2+c^2)} \leq \\
&\leq \frac{3+\sqrt{3}}{3\sqrt{3}}\frac{\sqrt{a^2+b^2+c^2}\sqrt{\frac{a^2+b^2+c^2}{3}}}{a^2+b^2+c^2} = \\
&= \frac{\sqrt{3}+1}{3\sqrt{3}}
\end{aligned}$$

Therefore

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq a+b+c+\sqrt{a^2+b^2+c^2}$$

□

41. (**Belarus 2002**) $(a, b, c, d > 0)$

$$\sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}} \geq \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \sqrt{(a+c)^2+(b+d)^2}$$

Solution. (*Sung-Yoon Kim, BoesFX*) Let $A(0,0)$, $B(a,b)$, $C(-c,-d)$ and let D be the foot of perpendicular from A to BC. Since

$$[ABC] = \frac{1}{2}\left|\det\begin{pmatrix} 0 & 0 & 1 \\ a & b & 1 \\ -c & -d & 1 \end{pmatrix}\right| = \frac{1}{2}\,|ad-bc|$$

we have that

$$AH = \frac{2\,[ABC]}{BC} = \frac{|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$$

So the inequality becomes:

$$BC + 2\cdot AH \geq AB + AC \geq BC$$

$\angle A$ is obtuse, since $A(0,0)$, B is in quadran I, and C is in the third quadrant. Since $\angle A$ is obtuse, $BD + DC$ must be BC. By triangle inequality,

$$AB + AC \geq BC, \qquad BD + AD \geq AB, \qquad DC + AD \geq AC$$

So, $AB + AC \leq BD + DC + 2AD = BC + 2AD$ and the inequality is proven.

□

42. (**Canada 2002**) $(a, b, c > 0)$

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c$$

First Solution. (*Massimo Gobbino - Winter Campus 2006*) We can assume WLOG that $a \geq b \geq c$. Then from the rearrangement inequality we have

$$a^3 \geq b^3 \geq c^3 \ , \ \frac{1}{bc} \geq \frac{1}{ac} \geq \frac{1}{ab} \ \Rightarrow \ \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}$$

and

$$a^2 \geq b^2 \geq c^2 \ , \ \frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a} \ \Rightarrow \ a + b + c \leq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Therefore

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}$$

□

Second Solution. (*Shobber - ML Forum*) By AM-GM, we have

$$\frac{a^3}{bc} + b + c \geq 3a$$

Sum up and done. □

Third Solution. (*Pvthuan - ML Forum*) The inequality is simple applications of $x^2 + y^2 + z^2 \geq xy + yz + zx$ for a^2, b^2, c^2 and $ab, bc, ca,$. We have

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

□

Fourth Solution. (*Davron - ML Forum*) The inequality

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

can be proved by Muirheads Theorem. □

43. (**Vietnam 2002, Dung Tran Nam**) ($a^2 + b^2 + c^2 = 9, \ a, b, c \in \mathbb{R}$)

$$2(a + b + c) - abc \leq 10$$

First Solution. (*Nttu - ML Forum*) We can suppose, WLOG, that

$$|a| \leq |b| \leq |c| \quad \Rightarrow \quad c^2 \geq 3 \quad \Rightarrow \quad 2ab \leq a^2 + b^2 \leq 6$$

We have

$$\begin{aligned}
[2(a+b+c) - abc]^2 &= [2(a+b) + c(2-ab)]^2 \leq && \text{(Cauchy-Schwarz)} \\
&\leq \left[(a+b)^2 + c^2\right]\left[2^2 + (2-ab)^2\right] = \\
&= 100 + (ab+2)^2(2ab-7) \leq 100
\end{aligned}$$

Thus

$$2(a+b+c) - abc \leq 10$$

□

Second Solution. (*See* [4], *pag.* 88, *problem* 93) □

44. (**Bosnia and Hercegovina 2002**) ($a^2 + b^2 + c^2 = 1, a, b, c \in \mathbb{R}$)

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \leq \frac{3}{5}$$

Solution. (*Arne - ML Forum*) From Cauchy-Schwartz inequality we have

$$1 = \left(a^2+b^2+c^2\right)^2 \leq \left(\sum_{\text{cyc}} \frac{a^2}{1+2bc}\right)\left(\sum_{\text{cyc}} a^2(1+2bc)\right) \tag{1}$$

From GM-AM-QM inequality we have:

$$\begin{aligned}
\left(\sum_{\text{cyc}} a^2(1+2bc)\right) &= a^2+b^2+c^2+2abc(a+b+c) \leq \\
&\leq 1 + 2\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)^3} \cdot 3\sqrt{\frac{a^2+b^2+c^2}{3}} = \qquad (2) \\
&= 1 + \frac{2}{3}\left(a^2+b^2+c^2\right)^2 = \frac{5}{3}
\end{aligned}$$

The required inequality follows from (1) and (2). □

45. (**Junior BMO 2002**) ($a, b, c > 0$)

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

Solution. (*Silouan, Michael Lipnowski - ML Forum*) From AM-GM inequality we have

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\geq\frac{3}{XY}$$

where $X=\sqrt[3]{abc}$ and $Y=\sqrt[3]{(a+b)(b+c)(c+a)}$. By AM-GM again we have that

$$X\leq\frac{a+b+c}{3}$$

and

$$Y\leq\frac{2a+2b+2c}{3}$$

So

$$\frac{3}{XY}\geq\frac{27}{2(a+b+c)^2}$$

and the result follows. □

46. (**Greece 2002**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\geq\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) From Cauchy-Schwarz inequality, $a^2+b^2+c^2=1$ and the well-knon inequality

$$a^2b^2+b^2c^2+c^2a^2\leq\frac{1}{3}\left(a^2+b^2+c^2\right)^2$$

we have

$$\begin{aligned}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2&\leq\left(\sum_{\text{cyc}}\frac{\sqrt{a}}{\sqrt{b^2+1}}\right)^2\leq\\&\leq\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left(\sum_{\text{cyc}}a^2b^2+a^2\right)=\\&=\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left(1+a^2b^2+b^2c^2+c^2a^2\right)\leq\\&\leq\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left[1+\frac{1}{3}\left(a^2+b^2+c^2\right)\right]=\\&=\frac{4}{3}\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\end{aligned}$$

Hence

$$\left(\sum_{\text{cyc}} \frac{a}{b^2+1}\right) \geq \frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

$\square$

47. (**Greece 2002**) ($bc \neq 0, \frac{1-c^2}{bc} \geq 0,\ a,b,c \in \mathbb{R}$)

$$10(a^2+b^2+c^2-bc^3) \geq 2ab+5ac$$

Solution. (*Ercole Suppa*) At first we observe that $\frac{1-c^2}{bc} \geq 0$ if and only if $bc\left(1-c^2\right) \geq 0$. Thus:

$$\begin{aligned} &10\left(a^2+b^2+c^2-bc^3\right)-2ab-5ac = \\ &= 5(b-c)^2+\frac{5}{2}(a-c)^2+(a-b)^2+10bc\left(1-c^2\right)+4b^2+\frac{5}{2}c^2+\frac{13}{2}a^2 \geq 0 \end{aligned}$$

$\square$

48. (**Taiwan 2002**) $\left(a,b,c,d \in \left(0,\frac{1}{2}\right]\right)$

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)} \leq \frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

Solution. (*Liu Janxin - ML Forum*) We first prove two auxiliary inequalities:

Lemma 1. If $a,b \in \left[0,\frac{1}{2}\right[$ we have

$$\frac{a^2+b^2}{ab} \geq \frac{(1-a)^2+(1-b)^2}{(1-a)(1-b)}$$

Proof. Since $1-a-b \geq 0$ (bacause $0 \leq a,b \leq \frac{1}{2}$) we get

$$\frac{a^2+b^2}{ab}-\frac{(1-a)^2+(1-b)^2}{(1-a)(1-b)} = \frac{(1-a-b)(a-b)^2}{ab(1-a)(1-b)} \geq 0$$

Lemma 2. If $a,b,c,d \in \left[0,\frac{1}{2}\right[$ we have

$$\frac{\left(a^2-b^2\right)^2}{abcd} \geq \frac{\left((1-a)^2-(1-b)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

Proof. Since $0 \leq c, d \leq \frac{1}{2}$ we get

$$\frac{(1-c)(1-d)}{cd} \geq 1 \tag{1}$$

Since $0 \leq a, b \leq \frac{1}{2}$ we get

$$\frac{\left(a^2-b^2\right)^2}{ab} - \frac{\left((1-a)^2-(1-b)^2\right)^2}{(1-a)(1-b)} = \frac{(a-b)^4(1-a-b)}{ab(1-a)(1-b)} \geq 0$$

Therefore

$$\frac{\left(a^2-b^2\right)^2}{ab} \geq \frac{\left((1-a)^2-(1-b)^2\right)^2}{(1-a)(1-b)} \tag{2}$$

Multiplying (1) and (2) we have

$$\frac{\left(a^2-b^2\right)^2}{abcd} \geq \frac{\left((1-a)^2-(1-b)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

and the Lemma 2 is proven. $\square$

Now we can prove the required inequality. By Lemma 2, we have

$$\begin{aligned}
&\frac{a^4+b^4+c^4+d^4}{abcd} - \frac{\left(a^2+b^2\right)\left(b^2+c^2\right)}{abcd} = \\
&= \frac{\left(a^2-c^2\right)^2+\left(a^2-d^2\right)^2+\left(b^2-c^2\right)^2+\left(b^2-d^2\right)^2}{2abcd} \geq \\
&\geq \frac{\left((1-a)^2-(1-c)^2\right)^2+\left((1-a)^2-(1-d)^2\right)^2+\left((1-b)^2-(1-c)^2\right)^2+\left((1-b)^2-(1-d)^2\right)^2}{2(1-a)(1-b)(1-c)(1-d)} = \\
&= \frac{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}{(1-a)(1-b)(1-c)(1-d)} - \frac{\left((1-a)^2+(1-b)^2\right)\left((1-c)^2+(1-d)^2\right)}{(1-a)(1-b)(1-c)(1-d)}
\end{aligned}$$

By Lemma 1, we have

$$\frac{\left(a^2+b^2\right)\left(b^2+c^2\right)}{abcd} \geq \frac{\left((1-a)^2+(1-b)^2\right)\left((1-c)^2+(1-d)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

Thus, addingthe last two inequalities, we get

$$\frac{a^4+b^4+c^4+d^4}{abcd} \geq \frac{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}{(1-a)(1-b)(1-c)(1-d)}$$

and the desidered inequality follows:

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)} \leq \frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

$\square$

49. (**APMO 2002**) ($\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,\ x,y,z>0$)

$$\sqrt{x+yz}+\sqrt{y+zx}+\sqrt{z+xy}\geq\sqrt{xyz}+\sqrt{x}+\sqrt{y}+\sqrt{z}$$

Solution. (*Suat Namly*) Multiplying by $\sqrt{xyz}$, we have

$$\sqrt{xyz}=\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}$$

So it is enough to prove that

$$\sqrt{z+xy}\geq\sqrt{z}+\sqrt{\frac{xy}{z}}$$

By squaring, this is equivalent to

$$\begin{aligned}
&z+xy\geq z+\frac{xy}{z}+2\sqrt{xy} && \Longleftrightarrow\\
&z+xy\geq z+xy\left(1-\frac{1}{x}-\frac{1}{y}\right)+2\sqrt{xy} && \Longleftrightarrow\\
&x+y\geq 2\sqrt{xy} && \Longleftrightarrow\\
&\left(\sqrt{x}-\sqrt{y}\right)^2\geq 0
\end{aligned}$$

$\square$

50. (**Ireland 2001**) ($x+y=2,\ x,y\geq 0$)

$$x^2y^2(x^2+y^2)\leq 2.$$

First Solution. (*Soarer - ML Forum*)

$$x^2y^2(x^2+y^2)=x^2y^2(4-2xy)=2x^2y^2(2-xy)\leq 2(1)\left(\frac{xy+2-xy}{2}\right)^2=2$$

$\square$

Second Solution. (*Pierre Bornzstein - ML Forum*) WLOG, we may assume that $x\leq y$ so that $x\in[0,1]$. Now

$$x^2y^2(x^2+y^2)=x^2(2-x)^2(x^2+(2-x)^2)=f(x)$$

Straighforward computations leads to

$$f'(x)=4x(1-x)(2-x)(2x^2-6x+4)\geq 0$$

Thus f is increasing on $[0;1]$. Since $f(1)=2$, the result follows. Note that equality occurs if and only if $x=y=1$.

$\square$

Third Solution. (*Kunny - ML Forum*) We can set

$$x = 2\cos^2\theta, y = 2\sin^2\theta$$

so we have

$$x^2y^2(x^2+y^2) = 2 - 2\cos^4 2\theta \leqq 2$$

$\square$

51. (**BMO 2001**) $(a+b+c \geq abc,\ a,b,c \geq 0)$

$$a^2+b^2+c^2 \geq \sqrt{3}abc$$

First Solution. (*Fuzzylogic - ML Forum*) From the well-know inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx)$$

by putting $x = bc, y = ca, z = ab$ we get

$$ab+bc+ca \geq \sqrt{3abc(a+b+c)}$$

Then

$$a^2+b^2+c^2 \geq ab+bc+ca \geq \sqrt{3abc(a+b+c)} \geq abc\sqrt{3}$$

$\square$

Second Solution. (*Cezar Lupu - ML Forum*) Let's assume by contradiction that

$$a^2+b^2+c^2 < abc\sqrt{3}$$

By applying Cauchy-Schwarz inequality, $3(a^2+b^2+c^2) \geq (a+b+c)^2$ and the hipothesys $a+b+c \geq abc$ we get

$$abc < 3\sqrt{3}$$

On the other hand , by AM-GM we have

$$abc\sqrt{3} > a^2+b^2+c^2 \geq 3\sqrt[3]{a^2b^2c^2}$$

We get from here $abc > 3\sqrt{3}$, a contradiction.

$\square$

Third Solution. (*Cezar Lupu - ML Forum*) We have

$$a+b+c \geq abc \Leftrightarrow \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \geq 1$$

We shall prove a stronger inequality

$$ab+bc+ca \geq abc\sqrt{3} \Leftrightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sqrt{3}$$

Now, let us denote $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$ and the problems becomes:
If x,y,z are three nonnegative real numbers such that $xy+yz+zx \geq 1$, then the following holds:

$$x+y+z \geq \sqrt{3}$$

But, this last problem follows immediately from this inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx)$$

□

52. (**USA 2001**) $(a^2+b^2+c^2+abc=4,\ a,b,c \geq 0)$

$$0 \leq ab+bc+ca-abc \leq 2$$

First Solution. (*Richard Stong, see* [9] *pag.* 22) From the given condition, at least one of a, b, c does not exceed 1, say $a \leq 1$. Then

$$ab+bc+ca-abc = a(b+c)+bc(1-a) \geq 0$$

It is easy to prove that the equality holds if and only if (a,b,c) is one of the triples $(2,0,0)$, $(0,2,0)$ or $(0,0,2)$.
To prove the upper bound we first note that some two of three numbers a, b, c are both greater than or equal to 1 or less than or equal to 1. WLOG assume that the numbers with this property are b and c. Then we have

$$(1-b)(1-c) \geq 0 \qquad (1)$$

The given equality $a^2+b^2+c^2+abc=4$ and the inequality $b^2+c^2 \geq 2bc$ imply

$$a^2+2bc+abc \leq 4 \quad \Longleftrightarrow \quad bc(2+a) \leq 4-a^2$$

Dividing both sides of the last inequality by $2+a$ yelds

$$bc \leq 2-a \qquad (2)$$

Combining (1) and (2) gives

$$\begin{aligned} ab + bc + ac - abc \le ab + 2 - a + ac(1-b) = \\ = 2 - a(1 + bc - b - c) = \\ = 2 - a(1-b)(1-c) \le 2 \end{aligned}$$

as desidered. The last equality holds if and only if $b = c$ and $a(1-b)(1-c) = 0$. Hence, equality for upper bound holds if and only if (a, b, c) is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$ and $(\sqrt{2}, \sqrt{2}, 0)$. □

Second Solution. (*See* [62]) Assume WLOG $a \ge b \ge c$. If $c > 1$, then $a^2 + b^2 + c^2 + abc > 1 + 1 + 1 + 1 = 4$, contradiction. So $c \le 1$. Hence $ab + bc + ca \ge ab \ge abc$.
Put $a = u + v$, $b = u - v$, so that $u, v = 0$. Then the equation given becomes

$$(2 + c)u^2 + (2 - c)v^2 + c^2 = 4$$

So if we keep c fixed and reduce v to nil, then we must increase u. But $ab+bc+ca-abc = (u^2-v^2)(1-c)+2cu$, so decreasing v and increasing u has the effect of increasing $ab+bc+ca-abc$. Hence $ab+bc+ca-abc$ takes its maximum value when $a = b$. But if $a = b$, then the equation gives $a = b = \sqrt{2-c}$. So to establish that $ab+bc+ca-abc \le 2$ it is sufficient to show that $2-c+2c\sqrt{2-c} = 2+c(2-c)$. Evidently we have equality if $c = 0$. If c is non-zero, then the relation is equivalent to $2\sqrt{2-c} \le 3-c$ or $(c-1)^2 \ge 0$. Hence the relation is true and we have equality only for $c = 0$ or $c = 1$. □

53. (**Columbia 2001**) $(x, y \in \mathbb{R})$

$$3(x + y + 1)^2 + 1 \ge 3xy$$

Solution. (*Ercole Suppa*) After setting $x = y$ we have

$$3(2x + 1)^2 + 1 - 3x^2 \ge 0 \quad \Longleftrightarrow \quad (3x + 2)^2 \ge 0 \tag{1}$$

where the equality holds if $x = -\frac{2}{3}$. This suggest the following change of variable

$$3x + 2 = a \qquad , \qquad 3y + 2 = b$$

Now for all $x, y \in \mathbb{R}$ we have:

$$\begin{aligned}
3(x+y+1)^2+1-3xy &= 3\left(\frac{a+b-4}{3}+1\right)^2+1-3\frac{(a-2)(b-2)}{9} = \\
&= \frac{(a+b-1)^2}{3}+1-\frac{ab-2a-2b+4}{3} = \\
&= \frac{a^2+b^2+ab}{3} = \\
&= \frac{a^2+b^2+(a+b)^2}{6} = \\
&= \frac{(3x+2)^2+(3y+2)^2+[3(x+y)+4]^2}{6} \geq 0
\end{aligned}$$

□

54. (**KMO Winter Program Test 2001**) $(a, b, c > 0)$

$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$

First Solution. (*See* [32], *pag.* 38) Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)} \geq abc+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, we obtain the constraint $xyz = 1$. It takes the form

$$\sqrt{(x+y+z)(xy+yz+zx)} \geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

From the constraint $xyz = 1$, we find two identities

$$\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right) = \left(\frac{x+z}{z}\right)\left(\frac{y+x}{x}\right)\left(\frac{z+y}{y}\right) = (z+x)(x+y)(y+z),$$

$$(x+y+z)(xy+yz+zx) = (x+y)(y+z)(z+x)+xyz = (x+y)(y+z)(z+x)+1.$$

Letting $p = \sqrt[3]{(x+y)(y+z)(z+x)}$, the inequality now becomes $\sqrt{p^3+1} \geq 1+p$. Applying the AM-GM inequality, we have $p \geq \sqrt[3]{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}} = 2$. It follows that $(p^3+1)-(1+p)^2 = p(p+1)(p-2) \geq 0$. □

Second Solution. (*Based on work by an winter program participant, see* [32] *pag.* 43).

$\square$

55. (**IMO 2001**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) Let T is the left hand side of the inequality. We have

$$\begin{aligned}
(a+b+c)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{a}}{\sqrt[4]{a^2+8bc}}\sqrt{a}\sqrt[4]{a^2+8bc}\right)^2 \leq & \text{(Cauchy-Schwarz)}\\
&\leq T \cdot \left(\sum_{\text{cyc}} a\sqrt{a^2+8bc}\right) = \\
&\leq T \cdot \left(\sum_{\text{cyc}} \sqrt{a}\sqrt{a}\sqrt{a^2+8bc}\right) \leq & \text{(Cauchy-Schwarz)}\\
&\leq T \cdot (a+b+c)^{\frac{1}{2}} \left(\sum_{\text{cyc}} a^3+8abc\right)^{\frac{1}{2}} = \\
&= T \cdot (a+b+c)^{\frac{1}{2}} \left(a^3+b^3+c^3+24abc\right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$T \geq \frac{(a+b+c)^{\frac{3}{2}}}{(a^3+b^3+c^3+24abc)^{\frac{1}{2}}} \geq 1$$

where in the last step we used the inequality

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc$$

which is true by BUNCHING, since

$$
\begin{aligned}
(a+b+c)^3 \geq a^3+b^3+c^3+24abc & \iff \\
3\left(\sum_{\text{sym}} a^2 b\right) + 6abc \geq 24abc & \iff \\
\sum_{\text{sym}} a^2 b \geq 6abc & \iff \\
\sum_{\text{sym}} a^2 b \geq \sum_{\text{sym}} abc &
\end{aligned}
$$

$\square$

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56. (**IMO 2000, Titu Andreescu**) $(abc = 1,\ a, b, c > 0)$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

Solution. (*See* [32], *pag.* 3) Since $abc = 1$, we make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for x, y, $z > 0$. We rewrite the given inequality in the terms of x, y, z :

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1 \quad \Leftrightarrow$$
$$xyz \geq (y + z - x)(z + x - y)(x + y - z)$$

This is true by Schur inequality. □

Remark. Alternative solutions are in [32], pag. 18, 19.

57. (**Czech and Slovakia 2000**) $(a, b > 0)$

$$\sqrt[3]{2(a + b)\left(\frac{1}{a} + \frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$$

First Solution. (*Massimo Gobbino - Winter Campus 2006*) After setting $a = x^3$ a $b = y^3$ the required inequality become

$$\frac{x}{y} + \frac{y}{x} \leq \sqrt[3]{2\left(x^3 + y^3\right)\left(\frac{1}{x^3} + \frac{1}{y^3}\right)}$$
$$\frac{x^2 + y^2}{xy} \leq \frac{1}{xy}\sqrt[3]{2\left(x^3 + y^3\right)^2}$$
$$\left(x^2 + y^2\right)^3 \leq 2\left(x^3 + y^3\right)^2$$
$$\left(x^2 + y^2\right)^{\frac{1}{2}} \leq 2^{\frac{1}{6}}\left(x^3 + y^3\right)^{\frac{1}{3}}$$
$$\sqrt{\frac{x^2 + y^2}{2}} \leq \sqrt[3]{\frac{x^3 + y^3}{2}}$$

which is true by Power Mean inequality. The equality holds if $x = y$, i.e. if $a = b$. □

Second Solution. (*Official solution.*) Elevating to the third power both members of the given inequality we get the equivalent inequality

$$\frac{a}{b} + 3\sqrt[3]{\frac{a}{b}} + 3\sqrt[3]{\frac{b}{a}} + \frac{b}{a} \geq 4 + 2\frac{a}{b} + 2\frac{b}{a}$$

that is

$$\frac{a}{b} + \frac{b}{a} + 4 \geq 3\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}\right)$$

The AM-GM inequality applied to the numbers $\frac{a}{b}, 1, 1$ implies

$$\frac{a}{b} + 1 + 1 \geq 3\sqrt[3]{\frac{a}{b}}$$

Similarly we have

$$\frac{b}{a} + 1 + 1 \geq 3\sqrt[3]{\frac{b}{a}}$$

Adding the two last inequalities we get the required result. $\square$

58. (**Hong Kong 2000**) ($abc = 1,\ a, b, c > 0$)

$$\frac{1 + ab^2}{c^3} + \frac{1 + bc^2}{a^3} + \frac{1 + ca^2}{b^3} \geq \frac{18}{a^3 + b^3 + c^3}$$

First Solution. (*Official solution*) Apply Cauchy-Scwarz Inequality, we have

$$\left(\frac{1 + ab^2}{c^3} + \frac{1 + bc^2}{a^3} + \frac{1 + ca^2}{b^3}\right)\left(c^3 + a^3 + b^3\right) \geq \left(\sum_{\text{cyc}} \sqrt{1 + ab^2}\right)^2$$

It remain to prove

$$\sum_{\text{cyc}} \sqrt{1 + ab^2} \geq \sqrt{18}$$

The proof goes as follows

$$\begin{aligned}
&\sqrt{1 + ab^2} + \sqrt{1 + bc^2} + \sqrt{1 + ca^2} \geq \\
&\geq \sqrt{(1 + 1 + 1)^2 + \left(\sqrt{ab^2} + \sqrt{bc^2} + \sqrt{ca^2}\right)^2} \geq && \text{(Minkowski Ineq)} \\
&\geq \sqrt{9 + \left(3\sqrt{abc}\right)^2} = && \text{(AM-GM Ineq)} \\
&= \sqrt{18}
\end{aligned}$$

$\square$

Second Solution. (*Ercole Suppa*) From AM-HM inequality we have

$$\frac{1}{c^3}+\frac{1}{a^3}+\frac{1}{b^3}\geq\frac{9}{a^3+b^3+c^3}\tag{1}$$

and

$$\frac{ab^2}{c^3}+\frac{bc^2}{a^3}+\frac{ca^2}{b^3}\geq 3\sqrt[3]{\frac{a^3b^3c^3}{a^3b^3c^3}}=\frac{9}{3\sqrt[3]{a^3b^3c^3}}\geq\frac{9}{a^3+b^3+c^3}\tag{2}$$

Adding (1) and (2) we get the required inequality. $\square$

59. (**Czech Republic 2000**) $(m,n\in N,\ x\in[0,1])$

$$(1-x^n)^m+(1-(1-x)^m)^n\geq 1$$

Solution. (*See* [61] *pag.* 83) The given inequality follow from the following most general result:

Let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ be nonnegative real numbers such that $x_i+y_i=1$ for each $i=1,2,\ldots,n$. Prove that

$$(1-x_1x_2\cdots x_n)^m+(1-y_1^m)(1-y_2^m)\cdots(1-y_n^m)\geq 1$$

We use the following probabilistic model suggested by the circumstance that $x_i+y_i=1$. Let n unfair coins. Let x_i be the probability that a toss of the $i-$th coin is a head $(i=1,2,\ldots,n)$. Then the probability that a toss of this coin is a tail equals $1-x_i=y_i$.

The probability of n heads in tossing all the coins once is $x_1x_2\cdots x_n$, because the events are independent. Hence $1-x_1x_2\cdots x_n$ is the probability of at least one tail. Consequently, the probability of *at least one tail in each of m consecutive tosses af all the coins* equals

$$(1-x_1x_2\cdots x_n)^m$$

With probability y_i^m, each of m consecutive tosses of the i-th coin is a tail; with probability $1-y_i^m$, we have at least one head. Therefore the probability that *after m tosses of all coins each coin has been a head at least once* equals

$$(1-y_1^m)(1-y_2^m)\cdots(1-y_n^m)$$

Denote the events given above in italics by A and B, respectively. It is easy to observe that *at leat one of them must occur* as a result of m tosses. Indeed, suppose A has not occurred. This means that the outcome of some toss has been n heads, which implies that B has occurred. Now we need a line more to finish the proof. Since one of the events A and B occurs as a result of m tosses, the sum of their probabilities is greater than or equal to 1, that is

$$(1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m) \cdots (1 - y_n^m) \geq 1$$

□

Remark. Murray Klamkin - Problem 68-1 (SIAM Review 11(1969)402-406).

60. (**Macedonia 2000**) $(x, y, z > 0)$

$$x^2 + y^2 + z^2 \geq \sqrt{2}\,(xy + yz)$$

Solution. (*Ercole Suppa*) By AM-GM inequality we have

$$\begin{aligned} x^2 + y^2 + z^2 &= x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 + z^2 \geq \\ &\geq 2x\frac{y}{\sqrt{2}} + 2\frac{y}{\sqrt{2}}z = \\ &= \sqrt{2}xy + \sqrt{2}yz = \\ &= \sqrt{2}\,(xy + yz) \end{aligned}$$

□

61. (**Russia 1999**) $(a, b, c > 0)$

$$\frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3$$

First Solution. (*Anh Cuong - ML Forum*)
First let $f(a, b, c) = \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ac}{a^2+c^2} + \frac{c^2+2ab}{a^2+b^2}$. We will prove that:

$$f(a, b, c) \geq \frac{2bc}{b^2 + c^2} + \frac{b}{c} + \frac{c}{b}$$

Suppose that: $b \geq c \geq a$. Since

$$\frac{a^2 + 2bc}{b^2 + c^2} \geq \frac{2bc}{b^2 + c^2}$$

we just need to prove that:

$$\frac{b^2 + 2ac}{a^2 + c^2} + \frac{c^2 + 2ab}{a^2 + b^2} \geq \frac{b}{c} + \frac{c}{b}$$

We have:

$$\begin{aligned}
&\frac{b^2+2ac}{a^2+c^2}+\frac{c^2+2ab}{a^2+b^2}-\frac{b}{c}-\frac{c}{b}= \\
&=\frac{b^3+2abc-c^3-ca^2}{b(c^2+a^2)}+\frac{c^3+2abc-b^3-ba^2}{c(b^2+a^2)}\geq \\
&\geq \frac{b^3-c^3}{b(a^2+c^2)}+\frac{c^3-b^3}{c(a^2+b^2)}= \\
&=\frac{\left(bc-a^2\right)(b-c)^2\left(b^2+bc+c^2\right)}{bc\left(a^2+b^2\right)\left(a^2+c^2\right)}\geq 0
\end{aligned}$$

Hence:

$$f(a,b,c)\geq \frac{2bc}{b^2+c^2}+\frac{b}{c}+\frac{c}{b}$$

But

$$\frac{2bc}{b^2+c^2}+\frac{b}{c}+\frac{c}{b}\geq 3 \Leftrightarrow (b-c)^2(b^2+c^2-bc)\geq 0.$$

So we have done now. $\square$

Second Solution. (*Charlie- ML Forum*)
Brute force proof: Denote $T(x,y,z)=\sum_{\text{sym}} a^x b^y c^z$. Expanding and simplifying yields

$$\frac{1}{2}\cdot T(6,0,0)+T(4,1,1)+2\cdot T(3,2,1)+T(3,3,0)\geq 2\cdot T(4,2,0)+\frac{1}{2}\cdot T(2,2,2)$$

which is true since

$$\frac{1}{2}\cdot T(6,0,0)+\frac{1}{2}\cdot T(4,1,1)\geq T(5,1,0)$$

by Schur's inequality, and

$$T(5,1,0)+T(3,3,0)\geq 2\cdot T(4,2,0)$$

by AM-GM ($a^5b+a^3b^3\geq 2a^4b^2$), and

$$2\cdot T(3,2,1)\geq 2\cdot T(2,2,2)\geq \frac{1}{2}\cdot T(2,2,2)$$

by bunching. $\square$

Third Solution. (*Darij Grinberg - ML Forum*)
Using the $\sum_{\text{cyc}}$ notation for cyclic sums, the inequality in question rewrites as

$$\sum_{\text{cyc}}\frac{a^2+2bc}{b^2+c^2}>3$$

But

$$\sum_{\text{cyc}} \frac{a^2+2bc}{b^2+c^2} - 3 = \sum_{\text{cyc}} \left(\frac{a^2+2bc}{b^2+c^2} - 1 \right) =$$

$$= \sum_{\text{cyc}} \frac{a^2}{b^2+c^2} - \sum_{\text{cyc}} \frac{(b-c)^2}{b^2+c^2}$$

Thus, we have to show that

$$\sum_{\text{cyc}} \frac{a^2}{b^2+c^2} > \sum_{\text{cyc}} \frac{(b-c)^2}{b^2+c^2}$$

Now, by the Cauchy-Schwarz inequality in the Engel form, we have

$$\sum_{\text{cyc}} \frac{a^2}{b^2+c^2} = \sum_{\text{cyc}} \frac{\left(a^2\right)^2}{a^2b^2+c^2a^2} \geq$$

$$\geq \frac{\left(a^2+b^2+c^2\right)^2}{\left(a^2b^2+c^2a^2\right)+\left(b^2c^2+a^2b^2\right)+\left(c^2a^2+b^2c^2\right)} =$$

$$= \frac{\left(a^2+b^2+c^2\right)^2}{2\left(b^2c^2+c^2a^2+a^2b^2\right)}$$

Hence, it remains to prove that

$$\frac{\left(a^2+b^2+c^2\right)^2}{2\left(b^2c^2+c^2a^2+a^2b^2\right)} > \sum_{\text{cyc}} \frac{(b-c)^2}{b^2+c^2}$$

i. e. that

$$\left(a^2+b^2+c^2\right)^2 > 2\left(b^2c^2+c^2a^2+a^2b^2\right) \sum_{\text{cyc}} \frac{(b-c)^2}{b^2+c^2}$$

Now,

$$2\left(b^2c^2+c^2a^2+a^2b^2\right) \sum_{\text{cyc}} \frac{(b-c)^2}{b^2+c^2} = \sum_{\text{cyc}} \frac{2\left(b^2c^2+c^2a^2+a^2b^2\right)}{b^2+c^2} (b-c)^2 =$$

$$= \sum_{\text{cyc}} \left(\frac{2b^2c^2}{b^2+c^2} + 2a^2 \right) (b-c)^2$$

The HM-GM inequality, applied to the numbers b^2 and c^2, yields

$$\frac{2b^2c^2}{b^2+c^2} \leq \sqrt{b^2c^2} = bc$$

thus,

$$2\left(b^2c^2+c^2a^2+a^2b^2\right)\sum_{\text{cyc}}\frac{(b-c)^2}{b^2+c^2}=\sum_{\text{cyc}}\left(\frac{2b^2c^2}{b^2+c^2}+2a^2\right)(b-c)^2\leq$$
$$\leq\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$$

Hence, instead of proving

$$\left(a^2+b^2+c^2\right)^2>2\left(b^2c^2+c^2a^2+a^2b^2\right)\sum_{\text{cyc}}\frac{(b-c)^2}{b^2+c^2}$$

it will be enough to show the stronger inequality

$$\left(a^2+b^2+c^2\right)^2>\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$$

With a bit of calculation, this is straightforward; here is a longer way to show it without great algebra:

$$\begin{aligned}
&\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2=\\
&=\sum_{\text{cyc}}\left(a\left(a+b+c\right)-\left(c-a\right)\left(a-b\right)\right)(b-c)^2=\\
&=\sum_{\text{cyc}}a\left(a+b+c\right)(b-c)^2-\sum_{\text{cyc}}\left(c-a\right)\left(a-b\right)(b-c)^2=\\
&=\left(a+b+c\right)\sum_{\text{cyc}}a\left(b-c\right)^2-\left(b-c\right)\left(c-a\right)\left(a-b\right)\underbrace{\sum_{\text{cyc}}\left(b-c\right)}_{=0}=\\
&=\left(a+b+c\right)\sum_{\text{cyc}}a\left(b-c\right)^2=\\
&=\left(a+b+c\right)\sum_{\text{cyc}}a\left(\left(b-c\right)\left(b-a\right)+\left(c-a\right)\left(c-b\right)\right)=\\
&=\left(a+b+c\right)\left(\sum_{\text{cyc}}a\left(b-c\right)\left(b-a\right)+\sum_{\text{cyc}}a\left(c-a\right)\left(c-b\right)\right)=\\
&=\left(a+b+c\right)\left(\sum_{\text{cyc}}c\left(a-b\right)\left(a-c\right)+\sum_{\text{cyc}}b\left(a-b\right)\left(a-c\right)\right)=\\
&=\left(a+b+c\right)\sum_{\text{cyc}}\left(b+c\right)\left(a-b\right)\left(a-c\right)
\end{aligned}$$

Thus, in order to prove that $\left(a^2+b^2+c^2\right)^2>\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$, we will show the equivalent inequality

$$\left(a^2+b^2+c^2\right)^2>\left(a+b+c\right)\sum_{\text{cyc}}\left(b+c\right)\left(a-b\right)\left(a-c\right)$$

In fact, we will even show the stronger inequality

$$\left(a^2+b^2+c^2\right)^2 > \sum_{\text{cyc}} a^2(a-b)(a-c) + (a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c)$$

which is indeed stronger since $\sum_{\text{cyc}} a^2(a-b)(a-c) \geq 0$ by the Schur inequality.
Now, this stronger inequality can be established as follows:

$$\sum_{\text{cyc}} a^2(a-b)(a-c) + (a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c) =$$

$$= \sum_{\text{cyc}}\left(a^2+(a+b+c)(b+c)\right)(a-b)(a-c) =$$

$$= \sum_{\text{cyc}}\left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)+bc\right)(a-b)(a-c) =$$

$$= \left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)\right)\underbrace{\sum_{\text{cyc}}(a-b)(a-c)}_{=(a^2+b^2+c^2)-(bc+ca+ab)} + \sum_{\text{cyc}} bc \underbrace{(a-b)(a-c)}_{=a^2+bc-ca-ab<2a^2+bc} <$$

$$< \left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)\right)\left(\left(a^2+b^2+c^2\right)-(bc+ca+ab)\right) + \sum_{\text{cyc}} bc\left(2a^2+bc\right) =$$

$$= \left(\left(a^2+b^2+c^2\right)^2-(bc+ca+ab)^2\right)+(bc+ca+ab)^2 = \left(a^2+b^2+c^2\right)^2$$

and the inequality is proven.

$\square$

62. (**Belarus 1999**) $(a^2+b^2+c^2=3,\ a,b,c>0)$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca} \geq \frac{3}{2}$$

Solution. (*Ercole Suppa*) From Cauchy-Schwartz inequality we have

$$9 = \left(a^2+b^2+c^2\right)^2 \leq \left(\sum_{\text{cyc}}\frac{1}{1+bc}\right)\left(\sum_{\text{cyc}} a^2(1+bc)\right) \tag{1}$$

From GM-AM-QM inequality we have:

$$\begin{aligned}\left(\sum_{\text{cyc}} a^2(1+bc)\right) &= a^2+b^2+c^2+abc(a+b+c) \leq \\ &\leq 3+\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)^3}\cdot 3\sqrt{\frac{a^2+b^2+c^2}{3}} = \qquad (2)\\ &= 3+\frac{1}{3}\left(a^2+b^2+c^2\right)^2 = 3+3=6\end{aligned}$$

The required inequality follows from (1) and (2). $\square$

63. (**Czech-Slovak Match 1999**) $(a, b, c > 0)$

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

Solution. (*Ercole Suppa*) Using Cauchy-Schwartz inequality and the well-know

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

we have

$$\begin{aligned} (a+b+c)^2 &\leq \sum_{\text{cyc}} \frac{a}{b+2c} \cdot \sum_{\text{cyc}} a(b+2c) = && \text{(Cauchy-Schwarz)} \\ &= \sum_{\text{cyc}} \frac{a}{b+2c} \cdot 3(ab+bc+ca) \leq \\ &\leq \sum_{\text{cyc}} \frac{a}{b+2c} \cdot (a+b+c)^2 \end{aligned}$$

Dividing for $(a+b+c)^2$ we get the result. □

64. (**Moldova 1999**) $(a, b, c > 0)$

$$\frac{ab}{c(c+a)} + \frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} \geq \frac{a}{c+a} + \frac{b}{b+a} + \frac{c}{c+b}$$

First Solution. (*Ghang Hwan, Bodom - ML Forum*)
After the substitution $x = c/a$, $y = a/b$, $z = b/c$ we get $xyz = 1$ and the inequality becomes

$$\frac{z}{x+1} + \frac{x}{y+1} + \frac{y}{z+1} \geq \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$$

Taking into account that $xyz = 1$, this inequality can be rewritten as

$$\frac{z-1}{x+1} + \frac{x-1}{y+1} + \frac{y-1}{z+1} \geq 0 \quad \Longleftrightarrow$$
$$yz^2 + zx^2 + xy^2 + x^2 + y^2 + z^2 \geq x + y + z + 3 \qquad (*)$$

The inequality $(*)$ is obtained summing the well-know inequality

$$x^2 + y^2 + z^2 \geq x + y + z$$

and

$$yz^2 + zx^2 + xy^2 \geq 3\sqrt[3]{x^3y^3z^3} = 3xyz = 3$$

which follows from the AM-GM inequality. □

Second Solution. (*Gibbenergy - ML Forum*) We have

$$L - R = \frac{abc\left[\left(\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ac}{b^2} - 3\right) + \left(\frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{a^2}{b^2}\right) - \left(\frac{b}{c} + \frac{c}{a} + \frac{a}{b}\right)\right]}{(a+b)(b+c)(c+a)} \geq 0$$

because

$$\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ac}{b^2} - 3 \geq 0$$

by AM-GM inequality and

$$\left(\frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{a^2}{b^2}\right) - \left(\frac{b}{c} + \frac{c}{a} + \frac{a}{b}\right) \geq 0$$

by the well-know inequality $x^2 + y^2 + z^2 \geq x + y + z$. □

65. (**United Kingdom 1999**) $(p + q + r = 1,\ p, q, r > 0)$

$$7(pq + qr + rp) \leq 2 + 9pqr$$

First Solution. (*Ercole Suppa*) From Schur inequality we have

$$(p + q + r)^3 + 9pqr \geq 4(p + q + r)(pq + qr + rp)$$

Therefore, since $p + q + r = 1$, we obtain

$$1 + 9pqr \geq 4(pq + qr + rp)$$

Hence

$$\begin{aligned} 2 + 9pqr - 7(pq + qr + rp) &\geq 2 + 4(pq + qr + rp) - 1 - 7(pq + qr + rp) = \\ &= 1 - 3(pq + qr + rp) = \\ &= (p + q + r)^2 - 3(pq + qr + rp) = \\ &= \frac{1}{2}\left[(p - q)^2 + (q - r)^2 + (r - p)^2\right] \geq 0 \end{aligned}$$

and the inequality is proven. □

Second Solution. (*See* [8] *pag.* 189)
Because $p + q + r = 1$ the inequality is equivalent to

$$\begin{aligned} &7(pq + qr + rp)(p + q + r) \leq 2(p + q + r)^3 + 9pqr && \Longleftrightarrow \\ &7\sum_{\text{cyc}}\left(p^2q + pq^2 + pqr\right) \leq 9pqr + \sum_{\text{cyc}}\left(2p^3 + 6p^2q + 6pq^2 + 4pqr\right) && \Longleftrightarrow \\ &\sum_{\text{cyc}} p^2q + \sum_{\text{cyc}} pq^2 \leq \sum_{\text{cyc}} 2p^3 = \sum_{\text{cyc}} \frac{2p^3 + q^3}{3} + \sum_{\text{cyc}} \frac{p^3 + 2q^3}{3} \end{aligned}$$

This last inequality is true by weighted AM-GM inequality. □

66. (**Canada 1999**) $(x+y+z=1,\ x,y,z \geq 0)$

$$x^2y+y^2z+z^2x \leq \frac{4}{27}$$

First Solution. (*See* [8] *pag.* 42)
Assume WLOG that $x = max(x,y,z)$. If $x \geq y \geq z$, then

$$x^2y+y^2z+z^2x \leq x^2y+y^2z+z^2x+z\left[xy+(x-y)(y-z)\right] =$$
$$= (x+y)^2 y = 4\left(\frac{1}{2}-\frac{1}{2}y\right)\left(\frac{1}{2}-\frac{1}{2}y\right)y \leq \frac{4}{27}$$

where the last inequality follows from AM-GM inequality. Equality occurs if and only if $z=0$ (from the first inequality) and $y=\frac{1}{3}$, in which case $(x,y,z) = \left(\frac{2}{3},\frac{1}{3},0\right)$.
If If $x \geq y \geq z$, then

$$x^2y+y^2z+z^2x \leq x^2z+z^2y+y^2x-(x-z)(z-y)(x-y) \leq$$
$$\leq x^2z+z^2y+y^2x \leq \frac{4}{27}$$

where the second inequality is true from the result we proved for $x \geq y \geq z$ (except with y and z reversed. Equality holds in the first inequality only when two of x, y, z are equal, and in the second inequality only when $(x,z,y) = \left(\frac{2}{3},\frac{1}{3},0\right)$. Because these conditions can't both be true, the inequality is actually strict in this case.
Therefore the inequality is indeed true, and the equality olds when (x,y,z) equals $\left(\frac{2}{3},\frac{1}{3},0\right)$, $\left(\frac{1}{3},0,\frac{2}{3}\right)$ or $\left(0,\frac{2}{3},\frac{1}{3}\right)$. $\square$

Second Solution. (*CMO Committee - Crux Mathematicorum 1999, pag. 400*)
Let $f(x,y,z) = x^2y+y^2z+z^2x$. We wish to determine where f is maximal. Since f is cyclic WLOG we may assume that $x = max(x,y,z)$. Since

$$f(x,y,z)-f(x,z,y) = x^2y+y^2z+z^2x-x^2z-z^2y-y^2x =$$
$$= (y-z)(x-y)(x-z)$$

we may also assume $y \geq z$. Then

$$f(x+z,y,0)-f(x,y,z) = (x+z)^2y-x^2y-y^2z-z^2x =$$
$$= z^2y+yz(x-y)+xz(y-z) \geq 0$$

so we may now assume $z=0$. The rest follows from AM-GM inequality

$$f(x,y,0) = \frac{2x^2y}{2} \leq \frac{1}{2}\left(\frac{x+x+2y}{3}\right)^3 = \frac{4}{27}$$

Equality occurs when $x=2y$, hence when (x,y,z) equals $\left(\frac{2}{3},\frac{1}{3},0\right)$, $\left(\frac{1}{3},0,\frac{2}{3}\right)$ or $\left(0,\frac{2}{3},\frac{1}{3}\right)$. $\square$

Third Solution. (*CMO Committee - Crux Mathematicorum 1999, pag. 400*)
With f as above, and $x = max(x, y, z)$ we have

$$f\left(x+\frac{z}{2}, y+\frac{z}{2}, 0\right) - f(x,y,z) = yz(x-y) + \frac{xz}{2}(x-z) + \frac{z^2y}{4} + \frac{z^3}{8}$$

so we may assume that $z = 0$. The rest follows as for second solution. $\square$

Fourth Solution. (*See* [4] *pag.* 46, *problem* 32)
Assume WLOG that $x = max(x, y, z)$. We have

$$x^2y + y^2z + z^2x \le \left(x+\frac{z}{2}\right)^2\left(y+\frac{z}{2}\right) \qquad (1)$$

because $xyz \ge y^2z$ and $\frac{x^2z}{2} \ge \frac{xz^2}{2}$.
Then by AM-GM inequality and (1) we have

$$\begin{aligned} 1 &= \frac{x+\frac{z}{2}}{2} + \frac{x+\frac{z}{2}}{2} + \left(y+\frac{z}{2}\right) \ge \\ &\ge 3\sqrt[3]{\frac{\left(y+\frac{z}{2}\right)\left(x+\frac{z}{2}\right)^2}{4}} \ge \\ &\ge 3\sqrt[3]{\frac{x^2y+y^2z+z^2x}{4}} \end{aligned}$$

from which follows the desidered inequality $x^2y + y^2z + z^2x \le \frac{4}{27}$. $\square$

67. (**Proposed for 1999 USAMO**, [AB, pp.25]) $(x, y, z > 1)$

$$x^{x^2+2yz}y^{y^2+2zx}z^{z^2+2xy} \ge (xyz)^{xy+yz+zx}$$

First Solution. (*See* [15] *pag.* 67)
The required inequality is equivalent to

$$\begin{aligned} &\left(x^2+2yz\right)\log x + \left(y^2+2xz\right)\log y + \left(z^2+2xy\right)\log z \ge \\ &\qquad\qquad \ge (xy+yz+x+zx)(\log x + \log y + \log z) \end{aligned}$$

that is

$$(x-y)(x-z)\log x + (y-z)(y-x)\log y + (z-x)(z-y)\log z \ge 0$$

We observe that $\log x, \log y, \log z > 0$ because $x, y, z > 1$. Furthermore, since the last inequality is symmetric, we can assume WLOG that $x \ge y \ge z$. Thus

$$(z-x)(z-y)\log z \ge 0 \qquad (1)$$

and, since the function $\log x$ is increasing on $x > 0$, we get

$$(x-y)(x-z)\log x \geq (y-z)(x-y)\log y \tag{2}$$

because each factor of LHS is greater or equal of a different factor of RHS. The required inequality follows from (1) and (2). $\square$

Second Solution. (*Soarer - ML Forum*)
The required equality is equivalent to

$$x^{x^2+yz-xy-xz}y^{y^2+xz-xy-yz}z^{z^2+xy-xz-yz} \geq 1$$
$$x^{(x-y)(x-z)}y^{(y-x)(y-z)}z^{(z-x)(z-y)} \geq 1$$
$$\left(\frac{x}{y}\right)^{x-y} \cdot \left(\frac{y}{z}\right)^{y-z} \cdot \left(\frac{x}{z}\right)^{x-z} \geq 1 \tag{$\star$}$$

By symmetry we can assume WLOG tath $x \geq y \geq z$. Therefore ($\star$) is verifyied. $\square$

68. (**Turkey, 1999**) ($c \geq b \geq a \geq 0$)

$$(a+3b)(b+4c)(c+2a) \geq 60abc$$

First Solution. (*ML Forum*) By AM-GM inequality we have

$$\begin{aligned}(a+3b)(b+4c)(c+2a) &\geq 4\sqrt[4]{ab^3} \cdot 5\sqrt[5]{bc^4} \cdot 3\sqrt[3]{ca^2} = \\ &= 60\left(a^{\frac{1}{4}}a^{\frac{2}{3}}\right)\left(b^{\frac{3}{4}}b^{\frac{1}{5}}\right)\left(c^{\frac{4}{5}}c^{\frac{1}{3}}\right) = \\ &= 60a^{\frac{11}{12}}b^{\frac{19}{20}}c^{\frac{17}{15}} = \\ &= 60abc \cdot a^{-\frac{1}{12}}b^{-\frac{1}{20}}c^{\frac{2}{15}} \geq \\ &\geq 60abc \cdot c^{-\frac{1}{12}}c^{-\frac{1}{20}}c^{\frac{2}{15}} = \\ &= 60abc\end{aligned}$$

where the last inequality is true because $c \geq b \geq a \geq 0$ and the function $f(x) = x^\alpha$ (with $\alpha < 0$) is decreasing. $\square$

Second Solution. (*See* [8] *pag.* 176) By the AM-GM inequality we have $a + b + b \geq 3\sqrt[3]{ab^2}$. Multiplying this inequality and the analogous inequalities for $b + 2c$ and $c + 2a$ yields $(a + 2b)(b + 2c)(c + 2a) \geq 27abc$. Then

$$\begin{aligned}&(a+2b)(b+2c)(c+2a) \geq \\ \geq &\left(a + \frac{1}{3}a + \frac{8}{3}b\right)\left(b + \frac{2}{3}b + \frac{10}{3}c\right)(c+2a) = \\ = &\frac{20}{9}(a+2b)(b+2c)(c+2a) \geq 60abc\end{aligned}$$

$\square$

69. (**Macedonia 1999**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$a+b+c+\frac{1}{abc}\geq 4\sqrt{3}$$

First Solution. (*Frengo, Leepakhin - ML Forum*) By AM-GM, we have

$$1=a^2+b^2+c^2\geq 3\sqrt[3]{(abc)^2}\Rightarrow (abc)^2\leq\frac{1}{27}$$

Thus, by AM-GM

$$\begin{aligned}a+b+c+\frac{1}{abc}&=a+b+c+\frac{1}{9abc}+\frac{1}{9abc}+\cdots+\frac{1}{9abc}\geq\\&\geq 12\sqrt[12]{\frac{1}{9^9(abc)^8}}\geq\\&\geq 12\sqrt[12]{\frac{1}{9^9(\frac{1}{27})^4}}=4\sqrt{3}\end{aligned}$$

Equality holds if and only if $a=b=c=\frac{1}{9abc}$ or $a=b=c=\frac{1}{\sqrt{3}}$. □

Second Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$abc\left(a+b+c-4\sqrt{3}\right)+1\geq 0$$

From Schur inequality we have

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca)$$

Since

$$ab+bc+ca=\frac{1}{2}\left[(a+b+c)^2-\left(a^2+b^2+c^2\right)\right]=\frac{1}{2}\left[(a+b+c)^2-1\right]$$

we get

$$abc\geq\frac{1}{9}\left[(a+b+c)^3-2(a+b+c)\right]$$

After setting $S=a+b+c$ from Cauchy-Schwarz inequality follows that

$$S=a+b+c\leq\sqrt{1+1+1}\sqrt{a^2+b^2+c^2}=\sqrt{3}$$

and, consequently

$$\begin{aligned}abc\left(a+b+c-4\sqrt{3}\right)+1&\geq\frac{1}{9}\left(S^3-2S\right)\left(S-4\sqrt{3}\right)+1=\\&=\frac{1}{9}\left[\left(S^3-2S\right)\left(S-4\sqrt{3}\right)+9\right]=\\&=\frac{1}{9}\left[\left(\sqrt{3}-S\right)^4+20S\left(\sqrt{3}-S\right)\right]\geq 0\end{aligned}$$

□

Third Solution. (*Ercole Suppa*)
After setting $S = a+b+c$, $Q = ab+bc+ca$, from the constraint $a^2+b^2+c^2 = 1$ we have $S^2 = 1+2Q \geq 1$. Then $S \geq 1$ and, by Cauchy-Schwarz inequality we get

$$S = a+b+c \leq \sqrt{1+1+1}\sqrt{a^2+b^2+c^2} = \sqrt{3}$$

From the well-know inequality $(ab+bc+ca)^2 \geq 3abc(a+b+c)$ follows that $\frac{1}{abc} \geq \frac{3S}{Q^2}$. Thus, to establish the required inequality is enough to show that

$$S + \frac{3S}{Q^2} \geq 4\sqrt{3} \quad \Longleftrightarrow \quad 4\left(S - 4\sqrt{3}Q^2\right)Q^2 + 12S \geq 0$$

Sine $1 < S \leq \sqrt{3}$ we have

$$\begin{aligned}
4\left(S-4\sqrt{3}\right)Q^2+12S &= \left(S-4\sqrt{3}\right)\left(S^2-1\right)^2+12S = \\
&= \left(\sqrt{3}-S\right)\left(-S^4+3\sqrt{3}S^3+11S^2+3\sqrt{3}S-4\right) \geq \\
&\geq \left(\sqrt{3}-S\right)\left(-S^4+3S^4+11S^2+3S^2-4\right) \geq \\
&\geq \left(\sqrt{3}-S\right)\left(2S^4+14S^2-4\right) \geq 12\left(\sqrt{3}-S\right) \geq 0
\end{aligned}$$

□

Fourth Solution. (*Ercole Suppa*) Since $a^2+b^2+c^2 = 1$, the inequality

$$abc\left(a+b+c-4\sqrt{3}\right)+1 \geq 0$$

can be tranformed into a homogeneous one in the following way

$$abc\,(a+b+c) - 4\sqrt{3}abc\sqrt{a^2+b^2+c^2} + \left(a^2+b^2+c^2\right)^2 \geq 0$$

Squaring and expanding the expression we get

$$\begin{aligned}
&\frac{1}{2}\sum_{\text{sym}} a^8 + 4\sum_{\text{sym}} a^6b^2 + \sum_{\text{sym}} a^6bc + 2\sum_{\text{sym}} a^5b^2c + 3\sum_{\text{sym}} a^4b^4 + \\
&+ \sum_{\text{sym}} a^4b^3c + \frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \geq 24\sum_{\text{sym}} a^4b^2c^2
\end{aligned}$$

The last inequality can be obtained addind the following inequalities which are true by Muirhead theorem:

$$\frac{1}{2}\sum_{\text{sym}} a^8 \ge \frac{1}{2}\sum_{\text{sym}} a^4b^2c^2 \tag{1}$$

$$4\sum_{\text{sym}} a^6b^2 \ge 4\sum_{\text{sym}} a^4b^2c^2 \tag{2}$$

$$\sum_{\text{sym}} a^6bc \ge \sum_{\text{sym}} a^4b^2c^2 \tag{3}$$

$$2\sum_{\text{sym}} a^5b^2c \ge 2\sum_{\text{sym}} a^4b^2c^2 \tag{4}$$

$$3\sum_{\text{sym}} a^4b^4 \ge 3\sum_{\text{sym}} a^4b^2c^2 \tag{5}$$

$$\sum_{\text{sym}} a^4b^3c \ge 4\sum_{\text{sym}} a^4b^2c^2 \tag{6}$$

$$\frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \ge \frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \tag{7}$$

□

Fifth Solution. (*Tiks - ML Forum*)

$$a+b+c+\frac{1}{abc} \ge 4\sqrt{3}$$
$$\Longleftrightarrow a+b+c+\frac{(a^2+b^2+c^2)^2}{abc} \ge 4\sqrt{3(a^2+b^2+c^2)}$$
$$\Longleftrightarrow \frac{(a^2+b^2+c^2)^2-3abc(a+b+c)}{abc} \ge 4(\sqrt{3(a^2+b^2+c^2)}-(a+b+c))$$
$$\Longleftrightarrow \frac{(a^2+b^2+c^2)^2-3abc(a+b+c)}{abc} \ge 4\frac{3(a^2+b^2+c^2)-(a+b+c)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$
$$\Longleftrightarrow \sum (a-b)^2\frac{(a+b)^2+3c^2}{2abc} \ge \sum (a-b)^2\frac{4}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

but we have that $\sqrt{3(a^2+b^2+c^2)}+a+b+c \ge 2(a+b+c)$ so we have to prove that

$$\sum(a-b)^2\frac{(a+b)^2+3c^2}{2abc} \ge \sum(a-b)^2\frac{2}{a+b+c}$$
$$\Longleftrightarrow \sum(a-b)^2\left[\frac{(a+b)^2+3c^2}{2abc}-\frac{2}{a+b+c}\right] \ge 0$$

We have that

$$(a+b+c)\left[(a+b)^2+3c^2\right] \ge c(a+b)^2 \ge 4abc$$

hence

$$\frac{(a+b)^2+3c^2}{2abc}-\frac{2}{a+b+c}\geq 0$$

So the inequality is done. $\square$

70. (**Poland 1999**) $(a+b+c=1,\ a,b,c>0)$

$$a^2+b^2+c^2+2\sqrt{3abc}\leq 1$$

Solution. (*Ercole Suppa*) From the well-know inequality

$$(x+y+z)^2\geq 3(xy+yz+xz)$$

by putting $x=ab$, $y=bc$ e $z=ca$ we have

$$(ab+bc+ca)^2\geq 3abc(a+b+c)\quad\Longrightarrow\quad ab+bc+ca\geq\sqrt{3abc}\qquad(1)$$

From the constraint $a+b+c=1$ follows that

$$1-a^2-b^2-c^2=(a+b+c)^2-a^2-b^2-c^2=2ab+2ac+2ca\qquad(2)$$

(1) and (2) implies

$$1-a^2-b^2-c^2-2\sqrt{3abc}=2ab+2bc+2ca-2\sqrt{3abc}\geq 0$$

$\square$

71. (**Canada 1999**) $(x+y+z=1,x,y,z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

Solution. (*Ercole Suppa*) See: problem n.66. $\square$

72. (**Iran 1998**) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2,\ x,y,z>1\right)$

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*)

$$\begin{aligned}\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1} &= \left(\sum_{\text{cyc}} \frac{\sqrt{x-1}}{\sqrt{x}}\sqrt{x}\right) \le \\ &\le \left(\sum_{\text{cyc}} \frac{x-1}{x}\right)^{\frac{1}{2}} (x+y+z)^{\frac{1}{2}} = \qquad \text{(Cauchy-Schwarz)} \\ &= \left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^{\frac{1}{2}} \sqrt{x+y+z} = \\ &= \sqrt{x+y+z}\end{aligned}$$

□

73. (**Belarus 1998**, I. Gorodnin) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{a+b}{b+c}+\frac{b+c}{a+b}+1$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$a^2b^3+ab^4+a^3c^2+b^3c^2+b^2c^3 \ge a^2b^2c+2ab^3c+2ab^2c^2 \qquad (\star)$$

From AM-GM inequality we have

$$a^2b^3+b^3c^2 \ge 2\sqrt{a^2b^6c^2} = 2ab^3c \qquad (1)$$

$$\frac{1}{2}ab^4+\frac{1}{2}a^3c^2 \ge 2\sqrt{\frac{a^4b^4c^2}{4}} = a^2b^2c \qquad (2)$$

$$\frac{1}{2}ab^4+\frac{1}{2}a^3c^2+b^2c^3 \ge a^2b^2c+b^2c^3 = 2\sqrt{a^2b^4c^4} = 2ab^2c^2 \qquad (3)$$

The $(\star)$ is obtained adding (1),(2) and (3). □

74. (**APMO 1998**) $(a, b, c > 0)$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Solution. (*See* [7] *pag.* 174) We have

$$\begin{aligned}
&\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)= \\
&\qquad = 2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}= \\
&\qquad = \left(\frac{a}{b}+\frac{a}{c}+\frac{a}{a}\right)+\left(\frac{b}{c}+\frac{b}{a}+\frac{b}{b}\right)+\left(\frac{c}{a}+\frac{c}{b}+\frac{c}{c}\right)-1\geq \\
&\qquad \geq 3\left(\frac{a}{\sqrt[3]{abc}}+\frac{b}{\sqrt[3]{abc}}+\frac{c}{\sqrt[3]{abc}}\right)-1= \\
&\qquad = 2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)+\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)-1\geq \\
&\qquad \geq 2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)+3-1= \\
&\qquad = 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)
\end{aligned}$$

by two applications of AM-GM inequality. $\square$

75. (**Poland 1998**) $\left(a+b+c+d+e+f=1,\ ace+bdf\geq\frac{1}{108}\ a,b,c,d,e,f>0\right)$

$$abc+bcd+cde+def+efa+fab\leq\frac{1}{36}$$

Solution. (*Manlio - ML Forum*) Put $A=ace+bdf$ and $B=abc+bcd+cde+def+efa+fab$.By AM-GM inequality we have

$$A+B=(a+d)(b+e)(c+f)\leq(((a+d)+(b+e)+(c+f))/3)^3=1/27$$

so

$$B\leq 1/27-A\leq 1/27-1/108=1/36$$

$\square$

Remark. (Arqady) This is a private case of Walther Janous's inequality: If $x_1+x_2+...+x_n=1$ where x_i are non-negative real numbers and $2\leq k<n, k\in\mathbb{N}$, then

$$x_1x_2...x_k+x_2x_3...x_{k+1}+...+x_nx_1...x_{k-1}\leq max\{\frac{1}{k^k},\frac{1}{n^{k-1}}\}$$

76. (**Korea 1998**) $(x+y+z=xyz,\ x,y,z>0)$

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\le\frac{3}{2}$$

First Solution. (*See* [32], *pag.* 14)
We can write $x=\tan A,\ y=\tan B,\ z=\tan C$, where $A,B,C\in\left(0,\frac{\pi}{2}\right)$. Using the fact that $1+\tan^2\theta=\left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of $A,\ B,\ C$:

$$\cos A+\cos B+\cos C\le\frac{3}{2}\qquad(\star)$$

It follows from $\tan(\pi-C)=-z=\frac{x+y}{1-xy}=\tan(A+B)$ and from $\pi-C,A+B\in(0,\pi)$ that $\pi-C=A+B$ or $A+B+C=\pi$.
Since $\cos x$ is concave on $\left(0,\frac{\pi}{2}\right)$, $(\star)$ a direct consequence of Jensen's inequality and we are done. $\square$

Second Solution. (*See* [32], *pag.* 17)
The starting point is letting $a=\frac{1}{x},\ b=\frac{1}{y},\ c=\frac{1}{z}$. We find that $a+b+c=abc$ is equivalent to $1=xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\le\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\le\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\le\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\begin{aligned}\frac{x}{\sqrt{(x+y)(x+z)}}&=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\le\\&\le\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\\&=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right)\end{aligned}$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\le\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)$$

and

$$\frac{z}{\sqrt{(z+x)(z+y)}} \le \frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right)$$

Adding these three yields the required result. $\square$

77. (**Hong Kong 1998**) $(a, b, c \ge 1)$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \le \sqrt{c(ab+1)}$$

First Solution. (*Ercole Suppa*) After setting $x = \sqrt{a-1}$, $y = \sqrt{b-1}$, $z = \sqrt{c-1}$, with $x, y, z \ge 0$, by easy calculations the required inequality in transformed in

$$\begin{aligned}
x+y+z \le \sqrt{\left(1+z^2\right)\left[\left(1+x^2\right)\left(1+y^2\right)+1\right]} \qquad & \Longleftrightarrow \\
(x+y+z)^2 \le \left(1+z^2\right)\left(x^2y^2+x^2+y^2+2\right) \qquad & \Longleftrightarrow \\
\left(x^2y^2+x^2+y^2+1\right)z^2-2(x+y)z+x^2y^2-2xy+2 \ge 0 \qquad & \qquad (\star)
\end{aligned}$$

The $(\star)$ is true for all $x, y, z \in \mathbb{R}$ because:

$$\begin{aligned}
\frac{\Delta}{4} &= (x+y)^2-\left(x^2y^2+x^2+y^2+1\right)\left(x^2y^2-2xy+2\right) = \\
&= (x+y)^2-\left[x^2y^2+(x+y)^2-2xy+1\right]\left[(xy-1)^2+1\right] = \\
&= (x+y)^2-\left[(xy-1)^2+(x+y)^2\right]\left[(xy-1)^2+1\right] = \\
&= -(xy-1)^4-(xy-1)^2-(x+y)^2(xy-1)^2 = \\
&= -(xy-1)^2\left(2+x^2+y^2+x^2y^2\right) \le 0
\end{aligned}$$

$\square$

Second Solution. (*Sung-Yoon Kim - ML Forum*) Use

$$\sqrt{x-1}+\sqrt{y-1} \le \sqrt{xy} \quad \Longleftrightarrow \quad 2\sqrt{(x-1)(y-1)} \le (x-1)(y-1)+1$$

Then

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \le \sqrt{ab}+\sqrt{c-1} \le \sqrt{c(ab+1)}$$

$\square$

Remark. The inequality used in the second solution can be generalized in the following way (see [25], pag. 183, n.51): given threee real positive numbers a, b, c con $a > c$, $b > c$ we have

$$\sqrt{c(a-c)}+\sqrt{c(b-c)} \le \sqrt{ab}$$

The inequality, squaring twice, is transformed in $(ab-ac-bc)^2 \ge 0$. The equality holds if $c = ab/(a+b)$.

78. (**IMO Short List 1998**) ($xyz = 1,\ x, y, z > 0$)

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

First Solution. (*IMO Short List Project Group - ML Forum*)
The inequality is equivalent to the following one:

$$x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{3}{4}(x+1)(y+1)(z+1).$$

In fact, a stronger inequality holds true, namely

$$x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{1}{4}[(x+1)^3 + (y+1)^3 + (z+1)^3].$$

(It is indeed stronger, since $u^3 + v^3 + w^3 \geq 3uvw$ for any positive numbers u, v and w.) To represent the difference between the left- and the right-hand sides, put

$$f(t) = t^4 + t^3 - \frac{1}{4}(t+1)^3, \qquad g(t) = (t+1)(4t^2 + 3t + 1).$$

We have $f(t) = \frac{1}{4}(t-1)g(t)$. Also, g is a strictly increasing function on $(0, \infty)$, taking on positive values for $t > 0$. Since

$$\begin{aligned}
&x^4 + x^3 + y^4 + y^3 + z^4 + z^3 - \frac{1}{4}[(x+1)^3 + (y+1)^3 + (z+1)^3] \\
=&f(x) + f(y) + f(z) \\
=&\frac{1}{4}(x-1)g(x) + \frac{1}{4}(y-1)g(y) + \frac{1}{4}(z-1)g(z),
\end{aligned}$$

it suffices to show that the last expression is nonnegative.
Assume that $x \geq y \geq z$; then $g(x) \geq g(y) \geq g(z) > 0$. Since $xyz = 1$, we have $x \geq 1$ and $z \leq 1$. Hence $(x-1)g(x) \geq (x-1)g(y)$ and $(z-1)g(y) \leq (z-1)g(z)$. So,

$$\begin{aligned}
&\frac{1}{4}(x-1)g(x) + \frac{1}{4}(y-1)g(y) + \frac{1}{4}(z-1)g(z) \\
\geq&\frac{1}{4}[(x-1) + (y-1) + (z-1)]g(y) \\
=&\frac{1}{4}(x+y+z-3)g(y) \\
\geq&\frac{1}{4}(3\sqrt[3]{xyz} - 3)g(y) = 0,
\end{aligned}$$

because $xyz = 1$. This completes the proof. Clearly, the equality occurs if and only if $x = y = z = 1$. $\square$

Second Solution. (*IMO Short List Project Group - ML Forum*)
Assume $x \le y \le z$ so that

$$\frac{1}{(1+y)(1+z)} \le \frac{1}{(1+z)(1+x)} \le \frac{1}{(1+x)(1+y)}.$$

Then Chebyshev's inequality gives that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)}$$
$$\ge \frac{1}{3}(x^3+y^3+z^3)\left[\frac{1}{(1+y)(1+z)} + \frac{1}{(1+z)(1+x)} + \frac{1}{(1+x)(1+y)}\right]$$
$$= \frac{1}{3}(x^3+y^3+z^3)\frac{3+(x+y+z)}{(1+x)(1+y)(1+z)}.$$

Now, setting $(x+y+z)/3 = a$ for convenience, we have by the AM-GM inequality

$$\frac{1}{3}(x^3+y^3+z^3) \ge a^3,$$
$$x+y+z \ge 3\sqrt[3]{xyz} = 3,$$
$$(1+x)(1+y)(1+z) \le \left[\frac{(1+x)+(1+y)+(1+z)}{3}\right]^3 = (1+a)^3.$$

It follows that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \ge a^3 \cdot \frac{3+3}{(1+a)^3}.$$

So, it suffices to show that

$$\frac{6a^3}{(1+a)^3} \ge \frac{3}{4};$$

or, $8a^3 \ge (1+a)^3$. This is true, because $a \ge 1$. Clearly, the equality occurs if and only if $x = y = z = 1$. The proof is complete. □

Third Solution. (*Grobber - ML Forum*)
Amplify the first, second and third fraction by x, y, z respectively. The LHS becomes

$$\sum \frac{x^4}{x(1+y)(1+z)} \ge \frac{\left(x^2+y^2+z^2\right)^2}{x+y+z+2(xy+yz+zx)+3} \ge \frac{\left(x^2+y^2+z^2\right)^2}{4\left(x^2+y^2+z^2\right)} \ge \frac{3}{4}$$

I used the inequalities

$$x^2+y^2+z^2 \ge xy+yz+zx$$
$$x^2+y^2+z^2 \ge 3$$
$$x^2+y^2+z^2 \ge \frac{(x+y+z)^2}{3} \ge x+y+z$$

□

Fourth Solution. (*MysticTerminator - ML Forum*) First, note that

$$\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)} \geq \frac{\left(x^2+y^2+z^2\right)^2}{x(1+y)(1+z)+(1+x)y(1+z)+(1+x)(1+y)z}$$

by Cauchy, so we need to prove:

$$4\left(x^2+y^2+z^2\right)^2 \geq 3\left(x(1+y)(1+z)+(1+x)y(1+z)+(1+x)(1+y)z\right)$$

Well, let $x=a^3$, $y=b^3$, $z=c^3$ (with $abc=1$), and homogenize it to find that we have to prove:

$$\sum_{\text{cyc}}\left(4a^{12}+8a^6b^6\right) \geq \sum_{\text{cyc}}\left(3a^6b^6c^3+6a^5b^5c^2+3a^4b^4c^4\right)$$

which is perfectly Muirhead. $\square$

Remark. None of the solutions 1 and 2 above actually uses the condition $xyz=1$. They both work, provided that $x+y+z \geq 3$. Moreover, the alternative solution also shows that the inequality still holds if the exponent 3 is replaced by any number greater than or equal to 3.

79. (**Belarus 1997**) $(a,x,y,z>0)$

$$\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z \geq x+y+z \geq \frac{a+z}{a+z}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z$$

First Solution. (*Soarer - ML Forum*) First one

$$\begin{aligned}\sum x\frac{a+z}{a+x} &= \sum a+z-a\left(\sum\frac{a+z}{a+x}\right)=\\ &= x+y+z-a\left(\sum\frac{a+z}{a+x}-3\right)\leq\\ &\leq x+y+z\end{aligned}$$

Second one

$$\begin{aligned} & \sum x\frac{a+y}{a+z} \geq x+y+z\\ \Leftrightarrow\quad & \sum\frac{y-z}{a+z}x \geq 0\\ \Leftrightarrow\quad & \sum\frac{xy}{a+z} \geq \sum\frac{xz}{a+z}\\ \Leftrightarrow\quad & \sum\frac{1}{z(a+z)} \geq \sum\frac{1}{y\,(a+z)}\end{aligned}$$

which is rearrangement. $\square$

Second Solution. (*Darij Grinberg- ML Forum*)
Let's start with the first inequality:

$$\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\leq x+y+z$$

It is clearly equivalent to

$$\left(\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\right)-(x+y+z)\leq 0$$

But

$$\begin{aligned}
&\left(\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\right)-(x+y+z)=\\
&=\left(\frac{a+z}{a+x}-1\right)x+\left(\frac{a+x}{a+y}-1\right)y+\left(\frac{a+y}{a+z}-1\right)z=\\
&=\frac{z-x}{a+x}x+\frac{x-y}{a+y}y+\frac{y-z}{a+z}z=(z-x)\frac{x}{a+x}+(x-y)\frac{y}{a+y}+(y-z)\frac{z}{a+z}=\\
&=\left(z\frac{x}{a+x}-x\frac{x}{a+x}\right)+\left(x\frac{y}{a+y}-y\frac{y}{a+y}\right)+\left(y\frac{z}{a+z}-z\frac{z}{a+z}\right)=\\
&=\left(z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\right)-\left(x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}\right)
\end{aligned}$$

thus, it is enough to prove the inequality

$$\left(z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\right)-\left(x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}\right)\leq 0$$

This inequality is clearly equivalent to

$$z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\leq x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}$$

And this follows from the rearrangement inequality, applied to the equally sorted number arrays $(x;\ y;\ z)$ and $\left(\frac{x}{a+x};\ \frac{y}{a+y};\ \frac{z}{a+z}\right)$ (proving that these arrays are equally sorted is very easy: if, for instance, $x\leq y$, then $\frac{a}{x}\geq\frac{a}{y}$, so that $\frac{a+x}{x}=\frac{a}{x}+1\geq\frac{a}{y}+1=\frac{a+y}{y}$, so that $\frac{x}{a+x}\leq\frac{y}{a+y}$).

Now we will show the second inequality:

$$x+y+z\leq\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z$$

It is equivalent to

$$0\leq\left(\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z\right)-(x+y+z)$$

Since

$$\left(\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z\right)-(x+y+z)=$$
$$=\left(\frac{a+y}{a+z}-1\right)x+\left(\frac{a+z}{a+x}-1\right)y+\left(\frac{a+x}{a+y}-1\right)z=$$
$$=\frac{y-z}{a+z}x+\frac{z-x}{a+x}y+\frac{x-y}{a+y}z=(xy-zx)\frac{1}{a+z}+(yz-xy)\frac{1}{a+x}+(zx-yz)\frac{1}{a+y}=$$
$$=\left(xy\frac{1}{a+z}-zx\frac{1}{a+z}\right)+\left(yz\frac{1}{a+x}-xy\frac{1}{a+x}\right)+\left(zx\frac{1}{a+y}-yz\frac{1}{a+y}\right)=$$
$$=\left(xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}\right)-\left(zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\right)$$

thus, it is enough to verify the inequality

$$0\leq\left(xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}\right)-\left(zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\right)$$

This inequality is equivalent to

$$zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\leq xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}$$

But this follows from the rearrangement inequality, applied to the equally sorted number arrays $(yz;\ zx;\ xy)$ and $\left(\frac{1}{a+x};\ \frac{1}{a+y};\ \frac{1}{a+z}\right)$ (proving that these arrays are equally sorted is almost trivial: if, for instance, $x\leq y$, then $y\geq x$ and $yz\geq zx$, while on the other hand $a+x\leq a+y$ and thus $\frac{1}{a+x}\geq\frac{1}{a+y}$).
This completes the proof of your two inequalities.

□

80. (**Ireland 1997**) $(a+b+c\geq abc,\ a,b,c\geq 0)$

$$a^2+b^2+c^2\geq abc$$

Solution. (*Ercole Suppa*) See problem 51. □

81. (**Iran 1997**) $(x_1x_2x_3x_4=1,x_1,x_2,x_3,x_4>0)$

$$x_1^3+x_2^3+x_3^3+x_4^3\geq max\left(x_1+x_2+x_3+x_4,\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

Solution. (*See* [15] *pag.* 69)

After setting $A = \sum\limits_{i=1}^{4} x_i^3$, $A_i = A - x_i^3$, from AM-GM inequality we have

$$\frac{1}{3}A_1 \geq \sqrt[3]{x_2^3 x_3^3 x_4^3} = x_2 x_3 x_4 = \frac{1}{x_1}$$

Similarly can be proved that $\frac{1}{3}A_i \geq \frac{1}{x_i}$ for all $i = 2, 3, 4$. Therefore

$$A = \frac{1}{3}\sum_{i=1}^{4} A_i \geq \sum_{i=1}^{4} \frac{1}{x_i}$$

On the other hand by Power Mean inequality we have

$$\begin{aligned} \frac{1}{4}A = \frac{1}{4}\sum_{i=1}^{4} x_i^3 \geq \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)^3 = \\ = \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)\left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)^2 \geq \\ \geq \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right) \end{aligned}$$

(in the last step we used AM-GM inequality: $\sum\limits_{i=1}^{4} x_i \geq \sqrt[4]{x_1 x_2 x_3 x_4} = 1$). Thus

$$A \geq \sum_{i=1}^{4} x_i$$

and the inequality is proven. $\square$

82. (**Hong Kong 1997**) $(x, y, z > 0)$

$$\frac{3+\sqrt{3}}{9} \geq \frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)}$$

Solution. (*Ercole Suppa*) From QM-AM-GM inequality we have

$$x+y+z \geq \sqrt{3}\sqrt{x^2+y^2+z^2} \quad (1)$$

$$xy+yz+zx \geq 3\sqrt[3]{(xyz)^2} \quad (2)$$

$$\sqrt{x^2+y^2+z^2} \geq \sqrt{3}\sqrt[3]{xyz} \quad (3)$$

Therefore

$$
\begin{aligned}
\frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)} &\le \frac{xyz\sqrt{x^2+y^2+z^2}\left(\sqrt{3}+1\right)}{(x^2+y^2+z^2)3\sqrt[3]{(xyz)^2}} \le \\
&\le \frac{xyz\left(\sqrt{3}+1\right)}{3\sqrt{x^2+y^2+z^2}\sqrt[3]{(xyz)^2}} \le \\
&\le \frac{xyz\left(\sqrt{3}+1\right)}{3\sqrt{3}\sqrt[3]{xyz}\sqrt[3]{(xyz)^2}} = \\
&= \frac{\sqrt{3}+1}{3\sqrt{3}} = \frac{3+\sqrt{3}}{9}
\end{aligned}
$$

□

Remark. See: Crux Mathematicorum 1988, pag. 203, problem 1067.

83. (**Belarus 1997**) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

First Solution. (*Ghang Hwan - ML Forum, Siutz - ML Contest 1st Ed. 1R*) The inequality is equivalent with

$$\frac{1+\frac{b}{a}}{1+\frac{c}{a}}+\frac{1+\frac{c}{b}}{1+\frac{a}{b}}+\frac{1+\frac{a}{c}}{1+\frac{b}{c}} \le \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Let $x = a/b$, $y = c/a$, $z = b/c$ and note that $xyz = 1$. After some boring calculation we see that the inequality become

$$\left(x^2+y^2+z^2-x-y-z\right)+\left(x^2z+y^2x+z^2y-3\right) \ge 0$$

This inequality is true. In fact the first and second term are not negative because

$$x^2+y^2+z^2 \ge (x+y+z)\frac{x+y+z}{3} \ge x+y+z \qquad \text{(by CS and AM-GM)}$$

and

$$x^2z+y^2x+z^2y \ge 3\sqrt[3]{x^3y^3z^3} = 3 \qquad \text{(by AM-GM)}$$

□

Second Solution. (*See* [4], *pag.* 43, *problem* 29) Let us take $x = a/b$, $y = c/a$, $z = b/c$ and note that $xyz = 1$. Observe that

$$\frac{a+c}{b+c} = \frac{1+xy}{1+y} = x + \frac{1-x}{1+y}$$

Using similar relations, the problem reduces to proving that if $xyz = 1$, then

$$\frac{x-1}{y+1}\frac{y-1}{z+1} + \frac{z-1}{x+1} \geq 0 \quad \Longleftrightarrow$$
$$\left(x^2-1\right)(z+1) + \left(y^2-1\right)(x+1) + \left(z^2-1\right)(y+1) \geq 0 \quad \Longleftrightarrow$$
$$\sum x^2 z + \sum x^2 \geq \sum x + 3$$

But this inequality is very easy. Indeed, using the AM-GM inequality we have $\sum x^2 z \geq 3$ and so it remains to prove that $\sum x^2 \geq \sum x$, which follows from the inequalities

$$\sum x^2 \geq \frac{\left(\sum x\right)^2}{3} \sum x$$

$\square$

Third Solution. (*Darij Grinberg, ML Forum*) We first prove a lemma:

Lemma. Let a, b, c be three reals; let x, y, z, u, v, w be six nonnegative reals. Assume that the number arrays $(a; b; c)$ and $(x; y; z)$ are equally sorted, and

$$u(a-b) + v(b-c) + w(c-a) \geq 0$$

Then,

$$xu(a-b) + yv(b-c) + zw(c-a) \geq 0$$

Proof Since the statement of Lemma is invariant under cyclic permutations (of course, when these are performed for the number arrays $(a; b; c)$, $(x; y; z)$ and $(u; v; w)$ simultaneously), we can WLOG assume that b is the "medium one" among the numbers a, b, c; in other words, we have either $a \geq b \geq c$, or $a \leq b \leq c$. Then, since the number arrays $(a; b; c)$ and $(x; y; z)$ are equally sorted, we get either $x \geq y \geq z$, or $x \leq y \leq z$, respectively. What is important is that $(x-z)(a-b) \geq 0$ (since the numbers $x - z$ and $a - b$ have the same sign: either both ≥ 0, or both ≤ 0), and that $(y-z)(b-c) \geq 0$ (since the numbers $y - z$ and $b - c$ have the same sign: either both ≥ 0, or both ≤ 0). Now,

$$\begin{aligned} & xu(a-b) + yv(b-c) + zw(c-a) = \\ = {} & u\underbrace{(x-z)(a-b)}_{\geq 0} + v\underbrace{(y-z)(b-c)}_{\geq 0} + z\underbrace{(u(a-b) + v(b-c) + w(c-a))}_{\geq 0} \geq 0 \end{aligned}$$

and the Lemma is proven. $\square$

Proof of inequality. The inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

can be written as

$$\sum \frac{c+a}{c+b} \leq \sum \frac{a}{b} \quad \Longleftrightarrow \quad \sum \frac{a}{b}-\sum \frac{c+a}{c+b} \geq 0$$

But

$$\sum \frac{a}{b}-\sum \frac{c+a}{c+b}=\sum\left(\frac{a}{b}-\frac{c+a}{c+b}\right)=\sum \frac{1}{b+c} \cdot \frac{c}{b} \cdot (a-b)$$

So it remains to prove that

$$\sum \frac{1}{b+c} \cdot \frac{c}{b} \cdot (a-b) \geq 0$$

In fact, denote $u=c/b$; $v=a/c$; $w=b/a$. Then,

$$\sum u\,(a-b)=\sum \frac{c}{b}\,(a-b)=\sum\left(\frac{ca}{b}-c\right)=\sum \frac{ca}{b}-\sum c=$$
$$=\frac{\sum c^2a^2-\sum c^2ab}{abc}=\frac{\frac{1}{2}\sum (ca-ab)^2}{abc} \geq 0$$

Now, denote

$$x=\frac{1}{b+c} \quad ; \quad y=\frac{1}{c+a} \quad ; \quad z=\frac{1}{a+b}$$

Then, the number arrays $(a;b;c)$ and $(x;y;z)$ are equally sorted (in fact, e. g., if $a \geq b$, then $c+a \geq b+c$, so that $\frac{1}{b+c} \geq \frac{1}{c+a}$, or, equivalently, $x \geq y$); thus, according to the Lemma, the inequality

$$\sum u\,(a-b) \geq 0$$

implies

$$\sum xu\,(a-b) \geq 0$$

In other words, $\sum \frac{1}{b+c} \cdot \frac{c}{b} \cdot (a-b) \geq 0$. And the problem is solved. $\square$

84. (**Bulgaria 1997**) $(abc=1,\ a,b,c>0)$

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$$

Solution. (*Official solution*) Let $x = a + b + c$ and $y = ab + bc + ca$. It follows from CS inequality that $x \geq 3$ and $y \geq 3$. Since both sides of the given inequality are symmetric functions of a, b and c, we transform the expression as a function of x, y. Taking into account that $abc = 1$, after simple calculations we get

$$\frac{3 + 4x + y + x^2}{2x + y + x^2 + xy} \leq \frac{12 + 4x + y}{9 + 4x + 2y}$$

which is equivalent to

$$3x^2y + xy^2 + 6xy - 5x^2 - y^2 - 24x - 3y - 27 \geq 0$$

Write the last inequality in the form

$$\left(\frac{5}{3}x^2y - 5x^2\right) + \left(\frac{xy^2}{3} - y^2\right) + \left(\frac{xy^2}{3} - 3y\right) + \left(\frac{4}{3}x^2y - 12x\right) +$$
$$+ \left(\frac{xy^3}{3} - 3x\right) + (3xy - 9x) + (3xy - 27) \geq 0$$

When $x \geq 3$, $y \geq 3$, all terms in the left hand side are nonnegative and the inequality is true. Equality holds when $x = 3, y = 3$, which implies $a = b = c = 1$. $\square$

Remark. 1 The inequality can be proved by the general result: if $\prod x_i = 1$ then

$$\sum \frac{1}{n - 1 + x_i} \leq 1$$

PROOF. $f(x_1, x_2, ..., x_n) = \sum \frac{1}{n-1+x_i}$.As $\prod x_i = 1$ we may assume $x_1 \geq 1, x_2 \leq 1$. We shall prove that $f(x_1, x_2, ..., x_n) \leq f(1, x_1x_2, ..., x_n)$. And this is true because after a little computation we obtain $(1-x_1)(x_2-1)(x_1x_2+(n-1)^2) \geq 0$ which is obviously true. So we have $f(x_1, x_2, ..., x_n) \leq f(1, x_1x_2, ..., x_n) \leq ... \leq f(1, ..., 1) = 1$. $\square$

Remark 2. (Darij Grinberg) I want to mention the appearance of the inequality with solution in two sources:

1. Titu Andreescu, Vasile Cîrtoaje, Gabriel Dospinescu, Mircea Lascu, Old and New Inequalities, Zalau: GIL 2004, problem 99.
2. American Mathematics Competitions: Mathematical Olympiads 1997-1998: Olympiad Problems from Around the World, Bulgaria 21, p. 23.

Both solutions are almost the same: Brute force. The inequality doesn't seem to have a better proof.

85. (**Romania 1997**) ($xyz = 1,\ x, y, z > 0$)

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}\geq 2$$

Solution. (*Ercole Suppa*) By setting $a = x^3$, $b = y^3$, $c = z^3$ we have $abc = 1$. From the know inequality $a^3 + b^3 \geq ab(a + b)$ follows that

$$\begin{aligned}\frac{a^3+b^3}{a^2+ab+b^2} &= \frac{a^3+b^3+2\left(a^3+b^3\right)}{3\left(a^2+ab+b^2\right)}\geq\\ &\geq \frac{a^3+b^3+2ab(a+b)}{3\left(a^2+ab+b^2\right)}=\\ &= \frac{(a+b)\left(a^2+ab+b^2\right)}{3\left(a^2+ab+b^2\right)}=\\ &= \frac{a+b}{3}\end{aligned}$$

Similarly can be proved the following inequalities:

$$\frac{b^3+c^3}{b^2+bc+c^2}\geq\frac{b+c}{3}\quad;\quad\frac{c^3+a^3}{c^2+ca+a^2}\geq\frac{c+a}{3}$$

Then, by AM-GM inequality we have:

$$\begin{aligned}&\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}=\\ &=\frac{a^3+b^3}{a^2+ab+b^2}+\frac{b^3+c^3}{b^2+bc+c^2}+\frac{c^3+a^3}{c^2+ca+a^2}=\\ &=\frac{a+b}{3}+\frac{b+c}{3}+\frac{c+a}{3}=\\ &=\frac{2(a+b+c)}{3}\geq\\ &\geq\frac{2\cdot 3\sqrt[3]{abc}}{3}=2\end{aligned}$$

□

86. (**Romania 1997**) ($a, b, c > 0$)

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\geq 1\geq\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}$$

Solution. (*Pipi - ML Forum*) Let

$$I=\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab} \quad , \quad J=\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}$$

We wish to show that $I \geq 1 \geq J$. Since $x^2+y^2 \geq 2xy$ we have

$$\frac{a^2}{a^2+2bc} \geq \frac{a^2}{a^2+b^2+c^2}$$

Similarly,

$$\frac{b^2}{b^2+2ca} \geq \frac{b^2}{a^2+b^2+c^2} \quad , \quad \frac{c^2}{c^2+2ab} \geq \frac{c^2}{a^2+b^2+c^2}$$

Then it is clear that $I \geq 1$. Next, note that $I+2J=3$ or $I=3-2J$. By $I \geq 1$, it is easy to see that $J \leq 1$. $\square$

87. (**USA 1997**) $(a, b, c > 0)$

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Solution. (*ML Forum*) By Muirhead (or by factoring) we have

$$a^3+b^3 \geq ab^2+a^2b$$

so we get that:

$$\sum_{\text{cyc}} \frac{abc}{a^3+b^3+abc} \leq \sum_{\text{cyc}} \frac{abc}{ab(a+b+c)} = \sum_{\text{cyc}} \frac{c}{a+b+c} = 1$$

$\square$

88. (**Japan 1997**) $(a, b, c > 0)$

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

Solution. (*See* [40])
WLOG we can assume that $a+b+c=1$. Then the first term on the left become

$$\frac{(1-2a)^2}{(1-a)^2+a^2} = 2-\frac{2}{1+(1-2a)^2}$$

Next, let $x_1 = 1 - 2a$, $x_2 = 1 - 2b$, $x_3 = 1 - 2c$, then $x_1 + x_2 + x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desidered inequality is

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \frac{1}{1+x_3^2} \leq \frac{27}{10}$$

We consider the equation of the tangent line to $f(x) = \frac{1}{1+x^2}$ at $x = 1/3$ which is $y = \frac{27}{50}(-x+2)$. We have $f(x) \leq \frac{27}{50}(-x+2)$ for $-1 < x < 1$ because

$$\frac{27}{50}(-x+2) - \frac{1}{1+x^2} = \frac{(3x-1)^2(4-3x)}{50(x^2+1)} \geq 0$$

Then

$$f(x_1) + f(x_2) + f(x_3) \leq \frac{27}{10}$$

and the desidered inequality follows. □

89. (**Estonia 1997**) $(x, y \in \mathbb{R})$

$$x^2 + y^2 + 1 > x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

Solution. (*Ercole Suppa*) We have:

$$\left(x - \sqrt{y^2+1}\right) + \left(y - \sqrt{x^2+1}\right) \geq 0$$

and, consequently,

$$x^2 + y^2 + 1 \geq x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

The equality holds if and only if $x = \sqrt{y^2+1}$ and $y = \sqrt{x^2+1}$, i.e.

$$x^2 + y^2 = x^2 + y^2 + 2$$

Since this last equality is impossible, the result is proven. □

90. (**APMC 1996**) $(x + y + z + t = 0, x^2 + y^2 + z^2 + t^2 = 1, x, y, z, t \in \mathbb{R})$

$$-1 \leq xy + yz + zt + tx \leq 0$$

Solution. (*Ercole Suppa*) After setting $A = xy + yz + zt + tx$ we have

$$0 = (x + y + z + t)^2 = 1 + 2A + 2(xz + yt) \quad \Longrightarrow \quad A = -\frac{1}{2} - xz - yt$$

The required inequality is equivalent to

$$-1 \leq -\frac{1}{2} - xz - yt \leq 0 \quad \Longleftrightarrow \quad -\frac{1}{2} \leq xz + yt \leq \frac{1}{2} \Longleftrightarrow \quad |xz + yt| \leq \frac{1}{2}$$

and can be proved by means of Cauchy-Schwarz and AM-GM inequalities

$$\begin{aligned} |xz + yt| &\leq \sqrt{x^2 + y^2} \cdot \sqrt{t^2 + z^2} = && \text{(CS)} \\ &= \sqrt{(x^2 + y^2)(t^2 + z^2)} \leq && \text{(AM-GM)} \\ &\leq \frac{x^2 + y^2 + t^2 + z^2}{2} = \frac{1}{2} \end{aligned}$$

$\square$

91. (**Spain 1996**) $(a, b, c > 0)$

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a - b)(b - c)$$

Solution. (*Ercole Suppa*) We have:

$$\begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca - 3(a - b)(b - c) &= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac = \\ &= (a - 2b + c)^2 \geq 0 \end{aligned}$$

$\square$

92. (**IMO Short List 1996**) $(abc = 1,\ a, b, c > 0)$

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

Solution. (*by IMO Shortlist Project Group - ML Forum*)
We have

$$\begin{aligned} a^5 + b^5 &= (a + b)\left(a^4 - a^3b + a^2b^2 - ab^3 + b^4\right) = \\ &= (a + b)\left[(a - b)^2\left(a^2 + ab + b^2\right) + a^2b^2\right] \geq \\ &\geq a^2b^2(a + b) \end{aligned}$$

with equality if and only if $a = b$. Hence

$$\begin{aligned} \frac{ab}{a^5 + b^5 + ab} \leq \frac{ab}{ab(a+b)+1} &= \\ = \frac{1}{ab(a+b+c)} &= \\ = \frac{c}{a+b+c} & \end{aligned}$$

Taking into account the other two analogous inequalities we have

$$\sum \frac{ab}{a^5 + b^5 + ab} \leq \frac{c}{a+b+c} + \frac{a}{a+b+c} + \frac{b}{a+b+c} = 1$$

and the required inequality is established. Equality holds if and only if $a = b = c = 1$. $\square$

93. (**Poland 1996**) $\left(a + b + c = 1,\ a, b, c \geq -\frac{3}{4}\right)$

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}$$

Solution. (*Ercole Suppa*) The equality holds if $a = b = c = 1/3$. The line tangent to the graph of $f(x) = \frac{x}{x^2+1}$ in the point with abscissa $x = 1/3$ has equation $y = \frac{18}{25}x + \frac{30}{50}$ and the graph of $f(x)$, per $x > -3/4$, if entirely below that line, i.e.

$$\frac{x}{x^2+1} \leq \frac{18}{25}x + \frac{30}{50} \quad , \quad \forall x > -\frac{3}{4}$$

because

$$\frac{18}{25}x + \frac{30}{50} - \frac{x}{x^2+1} = (3x-1)^2(4x+3) \geq 0 \quad , \quad \forall x > -\frac{3}{4}$$

Therefore

$$\sum_{\text{cyc}} \frac{a}{a^2+1} \leq f(a) + f(b) + f(c) = \frac{9}{10}$$

$\square$

Remark. It is possible to show that the inequality

$$\sum_{\text{cyc}} \frac{a}{a^2+1} \leq \frac{9}{10}$$

is true for all $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. See ML Forum.

94. (**Hungary 1996**) $(a+b=1,\ a,b>0)$

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}$$

Solution. (*See* [32], *pag.* 30) Using the condition $a+b=1$, we can reduce the given inequality to homogeneous one, i. e.,

$$\frac{1}{3}\leq\frac{a^2}{(a+b)(a+(a+b))}+\frac{b^2}{(a+b)(b+(a+b))}\ \text{ or }\ a^2b+ab^2\leq a^3+b^3,$$

which follows from $(a^3+b^3)-(a^2b+ab^2)=(a-b)^2(a+b)\geq 0$. The equality holds if and only if $a=b=\frac{1}{2}$.

$\square$

95. (**Vietnam 1996**) $(a,b,c\in\mathbb{R})$

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}\left(a^4+b^4+c^4\right)$$

First Solution. (*Namdung - ML Forum*) Let

$$f(a,b,c)=(a+b)^4+(b+c)^4+(c+a)^4-\frac{4}{7}\left(a^4+b^4+c^4\right).$$

We will show that $f(a,b,c)\geq 0$ for all a, b, c. Among a, b, c, there exist at least one number which has the same sign as $a+b+c$, say a. By long, but easy computation, we have

$$f(a,b,c)-f(a,\frac{b+c}{2},\frac{b+c}{2})=3a(a+b+c)(b-c)^2+\frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2\geq 0$$

So, it sufficient (and necessary) to show that $f(a,t,t)>=0$ for all a, t. Is equivalent to $f(0,t,t)\geq 0$ and $f(1,t,t)\geq 0$ (due homogeneousness). The first is trivial, the second because

$$\begin{aligned}f(1,t,t)&=59t^4+28t^3+42t^2+28t+5=\\&=\frac{6}{59}(20t+7)^2+\left(\sqrt{59}t^2+\frac{14t}{\sqrt{59}}-\frac{1}{\sqrt{59}}\right)^2>0\end{aligned}$$

$\square$

Remark. To find the identity

$$f(a,b,c)-f(a,\frac{b+c}{2},\frac{b+c}{2}) = 3a(a+b+c)(b-c)^2+\frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2 \geq 0$$

we can use the following well-known approach. Let

$$h(a,b,c) = f(a,b,c) - f(a,\frac{b+c}{2},\frac{b+c}{2}) \geq 0$$

The first thing we must have is $h(0,b,c) \geq 0$. $h(0,b,c)$ is symmetric homogenus polynomial of b, c and it's easily to find that

$$h(0,b,c) = \frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2$$

Now, take $h(a,b,c) - h(0,b,c)$ and factor, we will get

$$h(a,b,c) - h(0,b,c) = 3a(a+b+c)(b-c)^2$$

Second Solution. (*Iandrei - ML Forum*)
Let $f(a,b,c) = (a+b)^4 + (b+c)^4 + (c+a)^4 - \frac{4}{7}\left(a^4+b^4+c^4\right)$. It's clear that $f(0,0,0) = 0$. We prove that $f(a,b,c) \geq 0$. We have

$$f(a,b,c) = \frac{10}{7}\sum a^4 + 4\sum ab\left(a^2+b^2\right) + 6\sum a^2b^2 \geq 0 \quad \Longleftrightarrow$$
$$\frac{5}{7}\sum a^4 + 2\sum ab\left(a^2+b^2\right) + 3\sum a^2b^2 \geq 0 \quad \Longleftrightarrow$$
$$5\sum a^4 + 14\sum ab\left(a^2+b^2\right) + 21\sum a^2b^2 \geq 0$$

We prove that

$$\frac{5}{2}\left(a^4+b^4\right) + 14ab\left(a^2+b^2\right) + 21a^2b^2 \geq 0 \qquad (\star)$$

Let $x = ab$, $y = a^2+b^2$. Thus

$$\frac{5}{2}\left(y^2-2x^2\right) + 14xy + 21x^2 \geq 0 \quad \Longleftrightarrow \quad 16x^2 + 14xy + \frac{5}{2}y^2 \geq 0$$

If $x \neq 0$, we want prove that

$$32 + 28\frac{y}{x} + 5\left(\frac{y}{x}\right)^2 \geq 0.$$

If $y/x = t$ with $|t| > 2$, we must prove $32 + 28t + 5t^2 \geq 0$. The latter second degree function has roots

$$r_1 = \frac{-28+12}{10} = -1,6 \quad , \quad r_2 = \frac{-28-12}{10} = -4.$$

It's clear that $|t| > 2$ implies $32 + 28t + 5t^2 \geq 0$. If $x = 0$ then $a = 0$ or $b = 0$ and $(\star)$ is obviously verified. $\square$

Remark. A different solution is given in [4], pag. 92, problem 98.

96. (**Belarus 1996**) ($x + y + z = \sqrt{xyz},\ x, y, z > 0$)

$$xy + yz + zx \geq 9(x + y + z)$$

First Solution. (*Ercole Suppa*)
From the well-know inwquality $(xy + yz + zx)^3 \geq 3xyz(x + y + z)$ and AM-GM inequality we have

$$\begin{aligned}(xy + yz + zx)^3 &\geq 3xyz(x + y + z) = \\ &= 3(x + y + z)^3 \geq && \text{(AM-GM)} \\ &\geq 3\left(3\sqrt[3]{xyz}\right)^3 = \\ &= 81xyz = 81(x + y + z)^2\end{aligned}$$

The required inequality follows extracting the square root. □

Second Solution. (*Cezar Lupu - ML Forum*)
We know that $x + y + z = \sqrt{xyz}$ or $(x + y + z)^2 = xyz$. The inequality is equivalent with this one:

$$xyz(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9(x + y + z),$$

or

$$(x + y + z)^2(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9(x + y + z).$$

Finally, our inequality is equivalent with this well-known one:

$$(x + y + z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9.$$

□

97. (**Iran 1996**) ($a, b, c > 0$)

$$(ab + bc + ca)\left(\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2}\right) \geq \frac{9}{4}$$

First Solution. (*Iurie Boreico, see* [4], *pag.* 108, *problem* 114) With the substitution $y+z=a$, $z+x=b$, $x+y=c$ the inequality becomes after some easy computations

$$\sum\left(\frac{2}{ab}-\frac{1}{c^2}\right)(a-b)^2$$

Assume LOG that $a\geq b\geq c$. If $2c^2\geq ab$, each term in the above expression is positive and we are done. So, suppose $2c^2<ab$. First, we prove that $2b^2\geq ac$, $2a^2\geq bc$. Suppose that $2b^2<ac$. Then $(b+c)^2\leq 2(b^2+c^2)<a(b+c)$ and so $b+c<a$, false. Clearly, we can write the inequality like that

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)(a-c)^2+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2\geq\left(\frac{1}{c^2}-\frac{2}{ab}\right)(a-b)^2$$

We can immediately see that the inequality $(a-c)^2\geq(a-b)^2+(b-c)^2$ holds and thus it suffices to prove that

$$\left(\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\right)(b-c)^2\geq\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}\right)(a-b)^2$$

But is clear that

$$\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}\right)<\left(\frac{1}{b}-\frac{1}{c}\right)^2$$

and so the right hand side is at most

$$\frac{(a-b)^2(b-c)^2}{b^2c^2}$$

Also, it is easy to see that

$$\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\geq\frac{1}{ac}+\frac{1}{bc}>\frac{(a-b)^2}{b^2c^2}$$

which show that the left hand side is at least

$$\frac{(a-b)^2(b-c)^2}{b^2c^2}$$

and this ends the solution. □

Second Solution. (*Cezar Lupu - ML Forum*) We take $x=p-a, y=p-b, z=p-c$ so the inequality becomes:

$$(p-a)(p-b)+(p-b)(p-c)+(p-c)(p-a)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\frac{9}{4}\quad\Longleftrightarrow$$

$$(p^2-16Rr+5r^2)\left[(4R+r)(p^2-16Rr+5r^2)+4r\left(3R(5R-r)+r(R-2r)\right)\right]+$$
$$+4R^3(R-2r)^2\geq 0$$

But using Gerrestein's inequality $p^2\geq 16Rr-5r^2$ and Euler's inequality $R\geq 2r$ we are done. Hope I did not make any stupid mistakes in my calculations. □

Remark. Gerrestein's inequality.
In the triangle ABC we have $p^2 + 5r^2 \geq 16Rr$.

Put $a = x + y, b = y + z, c = z + x, x, y, z > 0$. The inital inequality becomes

$$(x + y + z)^3 \geq 4(x + y)(y + z)(z + x) - 5xyz$$

This one is homogenous so consider $x + y + z = 1$. So we only must prove that

$$1 \geq 4(1 - x)(1 - y)(1 - z) - 5xyz \Leftrightarrow 1 + 9xyz \geq 4(xy + yz + zx)$$

which is true by Shur . Anyway this one is weak , it also follows from

$$s^2 \geq 2R^2 + 8Rr + 3r^2$$

which is little bit stronger.

Third Solution. (*payman_pm - ML Forum*)

$$(xy + yz + zx)(\frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2} =$$
$$= (xy + yz + xz)(\frac{(x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2}{(x + y)^2(y + z)^2(z + x)^2}$$

but we have

$$(xy + yz + zx)((x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2) =$$
$$\sum(x^5y + 2x^4y^2 + \frac{5}{2}x^4yz + 13x^3y^2z + 4x^2y^2z^2)$$

and

$$(x + y)^2(y + z)^2(z + x)^2 = \sum(x^4y^2 + x^4yz + x^3y^3 + 6x^3y^2z + \frac{5}{3}x^2y^2z^2)$$

and by some algebra

$$\sum(4x^5y - x^4y^2 - 3x^3y^3 + x^4yz - 2x^3y^2z + x^2y^2z^2) \geq 0$$

and by using Sschur inequality we have $\sum(x^3 - 2x^2y + xyz) \geq 0$ and if multiply this inequality to xyz :

$$\sum(x^4yz - 2x^3y^2z + x^2y^2z^2) \geq 0 \qquad (1)$$

and by using$AM - GM$ inequality we have

$$\sum((x^5y - x^4y^2) + 3(x^5y - x^3y^3)) \geq 0 \qquad (2)$$

and by using (1), (2) the problem is solved. $\square$

Fourth Solution. (*Darij Grinberg - ML Forum*) I have just found another proof of the inequality which seems to be a bit less ugly than the familiar ones. We first prove a lemma:

Lemma 1. If a, b, c, x, y, z are six nonnegative reals such that $a \geq b \geq c$ and $x \leq y \leq z$, then

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+ \\ +z(a-b)^2\left(3ab+bc+ca-c^2\right) \geq 0.$$

Proof of Lemma 1. Since $a \geq b$, we have $ab \geq b^2$, and since $b \geq c$, we have $bc \geq c^2$. Thus, the terms $3ca+ab+bc-b^2$ and $3ab+bc+ca-c^2$ must be nonnegative. The important question is whether the term $3bc+ca+ab-a^2$ is nonnegative or not. If it is, then we have nothing to prove, since the whole sum

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+ \\ +z(a-b)^2\left(3ab+bc+ca-c^2\right)$$

is trivially nonnegative, as a sum of nonnegative expressions. So we will only consider the case when it is not; i. e., we will consider the case when $3bc+ca+ab-a^2<0$. Then, since $(b-c)^2 \geq 0$, we get $(b-c)^2\left(3bc+ca+ab-a^2\right) \leq 0$, and this, together with $x \leq y$, implies that

$$x(b-c)^2\left(3bc+ca+ab-a^2\right) \geq y(b-c)^2\left(3bc+ca+ab-a^2\right)$$

On the other hand, since $3ab+bc+ca-c^2 \geq 0$ and $(a-b)^2 \geq 0$, we have $(a-b)^2\left(3ab+bc+ca-c^2\right) \geq 0$, which combined with $y \leq z$, yields

$$z(a-b)^2\left(3ab+bc+ca-c^2\right) \geq y(a-b)^2\left(3ab+bc+ca-c^2\right)$$

Hence,

$$\begin{aligned}
&x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+z(a-b)^2\left(3ab+bc+ca-c^2\right) \\
&\geq y(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+y(a-b)^2\left(3ab+bc+ca-c^2\right) \\
&= y\left((b-c)^2\left(3bc+ca+ab-a^2\right)+(c-a)^2\left(3ca+ab+bc-b^2\right)+(a-b)^2\left(3ab+bc+ca-c^2\right)\right)
\end{aligned}$$

But

$$\begin{aligned}
&(b-c)^2\left(3bc+ca+ab-a^2\right)+(c-a)^2\left(3ca+ab+bc-b^2\right)+(a-b)^2\left(3ab+bc+ca-c^2\right) \\
&\geq (b-c)^2\left(-bc+ca+ab-a^2\right)+(c-a)^2\left(-ca+ab+bc-b^2\right)+(a-b)^2\left(-ab+bc+ca-c^2\right)
\end{aligned}$$

(since squares of real numbers are always nonnegative)

$$\begin{aligned}
&= (b-c)^2(c-a)(a-b)+(c-a)^2(a-b)(b-c)+(a-b)^2(b-c)(c-a) \\
&= (b-c)(c-a)(a-b)\left(\underbrace{(b-c)+(c-a)+(a-b)}_{=0}\right)=0
\end{aligned}$$

thus,

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+$$
$$+z(a-b)^2\left(3ab+bc+ca-c^2\right)\geq 0$$

and Lemma 1 is proven.

Now to the proof of the Iran 1996 inequality:

We first rewrite the inequality using the $\sum$ notation as follows:

$$\sum\frac{1}{(b+c)^2}\geq\frac{9}{4(bc+ca+ab)}$$

Upon multiplication with $4(bc+ca+ab)$, this becomes

$$\sum\frac{4(bc+ca+ab)}{(b+c)^2}\geq 9$$

Subtracting 9, we get

$$\sum\frac{4(bc+ca+ab)}{(b+c)^2}-9\geq 0$$

which is equivalent to

$$\sum\left(\frac{4(bc+ca+ab)}{(b+c)^2}-3\right)\geq 0$$

But

$$\frac{4(bc+ca+ab)}{(b+c)^2}-3=\frac{(3b+c)(a-b)}{(b+c)^2}-\frac{(3c+b)(c-a)}{(b+c)^2}$$

Hence, it remains to prove

$$\sum\left(\frac{(3b+c)(a-b)}{(b+c)^2}-\frac{(3c+b)(c-a)}{(b+c)^2}\right)\geq 0$$

But

$$\sum\left(\frac{(3b+c)(a-b)}{(b+c)^2}-\frac{(3c+b)(c-a)}{(b+c)^2}\right)=$$
$$=\sum\frac{(3b+c)(a-b)}{(b+c)^2}-\sum\frac{(3c+b)(c-a)}{(b+c)^2}=$$
$$=\sum\frac{(3b+c)(a-b)}{(b+c)^2}-\sum\frac{(3a+c)(a-b)}{(c+a)^2}=$$
$$=\sum\left(\frac{(3b+c)(a-b)}{(b+c)^2}-\frac{(3a+c)(a-b)}{(c+a)^2}\right)=$$
$$=\sum\frac{(a-b)^2\left(3ab+bc+ca-c^2\right)}{(b+c)^2(c+a)^2}$$

Thus, the inequality in question is equivalent to

$$\sum\frac{(a-b)^2\left(3ab+bc+ca-c^2\right)}{(b+c)^2(c+a)^2}\geq 0$$

Upon multiplication with $(b+c)^2(c+a)^2(a+b)^2$, this becomes

$$\sum(a+b)^2(a-b)^2\left(3ab+bc+ca-c^2\right)\geq 0$$

In other words, we have to prove the inequality

$$(b+c)^2(b-c)^2\left(3bc+ca+ab-a^2\right)+(c+a)^2(c-a)^2\left(3ca+ab+bc-b^2\right)+$$
$$+(a+b)^2(a-b)^2\left(3ab+bc+ca-c^2\right)\geq 0$$

But now it's clear how we prove this - we WLOG assume that $a\geq b\geq c$, and define $x=(b+c)^2$, $y=(c+a)^2$ and $z=(a+b)^2$; then, the required inequality follows from Lemma 1 after showing that $x\leq y\leq z$ (what is really easy: since $a\geq b\geq c$, we have $(a+b+c)-a\leq(a+b+c)-b\leq(a+b+c)-c$, what rewrites as $b+c\leq c+a\leq a+b$, and thus $(b+c)^2\leq(c+a)^2\leq(a+b)^2$, or, in other words, $x\leq y\leq z$).
This completes the proof of the Iran 1996 inequality. Feel free to comment or to look for mistakes (you know, chances are not too low that applying a new method one can make a number of mistakes). $\square$

Remark. For different solutions proof see: [19], pag.306; [65], pag.163; Crux Mathematicorum [1994:108; 1995:205; 1996:321; 1997:170,367].

98. (**Vietnam 1996**) $(2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab=16, a,b,c,d\geq 0)$

$$a+b+c+d\geq\frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

Solution. (*Mohammed Aassila - Crux Mathematicorum 2000, pag.332*)
We first prove a lemma:

Lemma If x, y, z are real positive numbers such that $x+y+z+xyz=4$, then

$$x+y+z \geq xy+yz+zx$$

Proof. Suppose that $x+y+z < xy+yz+zx$. From Schur inequality we have

$$\begin{aligned} 9xyz &\geq 4(x+y+z)(xy+yz+zx)-(x+y+z)^3 \geq \\ &\geq 4(x+y+z)^2-(x+y+z)^3 = \\ &= (x+y+z)^2\left[4-(x+y+z)\right] = \\ &= xyz(x+y+z)^2 \end{aligned}$$

Thus

$$(x+y+z)^2 < 9 \quad \Longrightarrow \quad x+y+z < 3$$

and AM-GM implies

$$xyz < \left(\frac{x+y+z}{3}\right)^3 = 1$$

Hence $x+y+z+xyz < 4$ and this is impossible. Therefore we have $x+y+z \geq xy+yz+zx$ and the lemma is proved. □

Now, the given inequality can be proved in the following way. Put

$$S_1 = \sum a \quad , \quad S_2 = \sum ab \quad , \quad S_3 = \sum abc.$$

Let

$$P(x) = (x-a)(x-b)(x-c)(x-d) = x^4 - S_1x^3 + S_2x^2 - S_3x + abcd.$$

Rolle's theorem says that $P'(x)$ has 3 positive roots u, v, w. Thus

$$P'(x) = 4(x-u)(x-v)(x-w) = 4x^3-4(u+v+w)x^2+4(uv+vw+wu)x-4uvw$$

Since $P'(x) = 4x^3 - 3S_1x^2 + 2S_2x - S_3$ we have:

$$S_1 = \frac{4}{3}(u+v+w) \quad , \quad S_2 = 2(uv+vw+wu) \quad , \quad S_3 = 4uvw \tag{1}$$

From (1) we have

$$\begin{gathered} 2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab = 16 \quad \Longleftrightarrow \\ 2S_2+S_3 = 16 \quad \Longleftrightarrow \quad 4(uv+vw+wu)+4uvw = 16 \quad \Longleftrightarrow \\ uv+vw+wu+uvw = 4 \end{gathered} \tag{2}$$

From the Lemma and (1) follows that

$$u+v+w \geq uv+vw+wu \quad \Longleftrightarrow \quad \frac{3}{4}S_1 \geq \frac{1}{2}S_2 \quad \Longleftrightarrow \quad S_1 \geq \frac{2}{3}S_2$$

i.e. $a+b+c+d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$ and we are done. □

Remark. A different solution is given in [6] pag. 98.

3 Years 1990 $\sim$ 1995

99. (**Baltic Way 1995**) $(a, b, c, d > 0)$

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geq 4$$

Solution. (*Ercole Suppa*) From HM-AM inequality we have

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y} \quad ; \quad \forall x, y \in \mathbb{R}_0^+.$$

Therefore:

$$\begin{aligned} LHS &= (a+c)\left(\frac{1}{a+b} + \frac{1}{c+d}\right) + (b+d)\left(\frac{1}{a+b} + \frac{1}{c+d}\right) = \\ &= (a+b+c+d)(a+c)\left(\frac{1}{a+b} + \frac{1}{c+d}\right) \geq && \text{(HM-AM)} \\ &\geq (a+b+c+d)\frac{4}{a+b+c+d} = 4 \end{aligned}$$

$\square$

100. (**Canada 1995**) $(a, b, c > 0)$

$$a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

First Solution. (*Ercole Suppa*) From Weighted AM-GM inequality applied to the numbers $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ with weights $p_1 = \frac{a}{a+b+c}$, $p_2 = \frac{b}{a+b+c}$, $p_3 = \frac{c}{a+b+c}$ we have

$$\begin{gathered} p_1 \cdot \frac{1}{a} + p_2 \cdot \frac{1}{b} + p_3 \cdot \frac{1}{c} \geq \left(\frac{1}{a}\right)^{p_1} \cdot \left(\frac{1}{b}\right)^{p_2} \cdot \left(\frac{1}{c}\right)^{p_3} \qquad \Longleftrightarrow \\ \frac{3}{a+b+c} \geq \frac{1}{\sqrt[a+b+c]{a^a b^b c^c}} \end{gathered}$$

Thus, the AM-GM inequality yields:

$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \Longrightarrow \quad a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

$\square$

Second Solution. (See [56]*pag.* 15)
We can assume WLOG that $a \leq b \leq c$. Then

$$\log a \leq \log b \leq \log c$$

and, from Chebyshev inequality we get

$$\frac{a+b+c}{3} \cdot \frac{\log a + \log b + \log c}{3} \leq \frac{a \log a + b \log b + c \log c}{3}$$

Therefore

$$a \log a + b \log b + c \log c \geq \frac{a+b+c}{3} (\log a + \log b + \log c) \qquad \Longrightarrow$$

$$a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

$\square$

Third Solution. (*Official solution - Crux Mathematicorum 1995, pag. 224*)
We prove equivalently that

$$a^{3a} b^{3b} c^{3c} \geq (abc)^{a+b+c}$$

Due to complete symmetry in a, b and c, we may assume WLOG that $a \geq b \geq c > 0$. Then $a - b \geq 0$, $b - c \geq 0$, $a - c \geq 0$ and $a/b \geq 1$, $b/c \geq 1$, $a/c \geq 1$. Therefore

$$\frac{a^{3a} b^{3b} c^{3c}}{(abc)^{a+b+c}} = \left(\frac{a}{b}\right)^{a-b} \left(\frac{b}{c}\right)^{b-c} \left(\frac{a}{c}\right)^{a-c} \geq 1$$

$\square$

101. (**IMO 1995, Nazar Agakhanov**) ($abc = 1,\ a, b, c > 0$)

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

First Solution. (*See* [32], *pag.* 17)
After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

It follows from the Cauchy-Schwarz inequality that

$$[(y+z) + (z+x) + (x+y)] \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \right) \geq (x+y+z)^2$$

so that, by the AM-GM inequality,

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$

$\square$

Second Solution. (*See* [32], *pag.* 36)
It's equivalent to

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2(abc)^{4/3}}.$$

Set $a=x^3, b=y^3, c=z^3$ with $x,y,z>0$. Then, it becomes

$$\sum_{\text{cyc}}\frac{1}{x^9(y^3+z^3)}\geq\frac{3}{2x^4y^4z^4}.$$

Clearing denominators, this becomes

$$\sum_{\text{sym}}x^{12}y^{12}+2\sum_{\text{sym}}x^{12}y^9z^3+\sum_{\text{sym}}x^9y^9z^6\geq 3\sum_{\text{sym}}x^{11}y^8z^5+6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}}x^{12}y^{12}-\sum_{\text{sym}}x^{11}y^8z^5\right)+2\left(\sum_{\text{sym}}x^{12}y^9z^3-\sum_{\text{sym}}x^{11}y^8z^5\right)+$$
$$+\left(\sum_{\text{sym}}x^9y^9z^6-\sum_{\text{sym}}x^8y^8z^8\right)\geq 0$$

and every term on the left hand side is nonnegative by Muirhead's theorem. $\square$

102. (**Russia 1995**) $(x,y>0)$

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}$$

Solution. (*Ercole Suppa*) From AM-GM inequality we have:

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{x}{2\sqrt{x^4y^2}}+\frac{y}{2\sqrt{x^2y^4}}=$$
$$=\frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

$\square$

103. (**Macedonia 1995**) $(a,b,c>0)$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2$$

Solution. (*Manlio Marangelli - ML Forum*)
After setting $A = 1$, $B = \frac{a}{b+c}$ the HM-AM yields

$$\sqrt{AB} \geq \frac{2}{\frac{1}{A} + \frac{1}{B}}$$

so

$$\sqrt{\frac{a}{b+c}} \geq \frac{2}{1 + \frac{b+c}{a}} = \frac{2a}{a+b+c}$$

Similar inequalities are true also for the other two radicals. Therefore

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} = 2$$

□

104. (**APMC 1995**) ($m, n \in \mathbb{N},\ x, y > 0$)

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n) \geq nm(x^{n+m-1}y+xy^{n+m-1})$$

Solution. (*See* [5] *pag.* 147)
We rewrite the given inequality in the form

$$mn(x-y)\left(x^{n+m-1} + y^{n+m-1}\right) \geq (n+m-1)\left(x^n - y^n\right)\left(x^m - y^m\right)$$

and divide both sides by $(x-y)^2$ to get the equivalent form

$$\begin{aligned} &nm\left(x^{n+m-2} + x^{n+m-3}y + \cdots + y^{n+m-2}\right) \geq \\ \geq &(n+m-1)\left(x^{n-1} + \cdots + y^{n-1}\right)\left(x^{m-1} + \cdots + y^{m-2}\right) \end{aligned}$$

We now will prove a more general result. Suppose

$$P(x,y) = a_d x^d + \cdots + a_{-d} y^d$$

is a homogeneous polynomial of degree d with the following properties

(a) For $i = 1, \ldots, d$, $a_i = a_{-i}$ (equivalently $P(x,y) = P(y,x)$)

(b) $\sum\limits_{i=-d}^{d} a_d = 0$, (equivalently $P(x,x) = 0$)

(c) For $i = 0, \ldots, d-1$, $a_d + \cdots + a_{d-i} \geq 0$.

Then $P(x,y) \geq 0$ for all $x, y > 0$. (The properties are easily verified for $P(x,y)$ equal to the difference of the two sides in our desidered inequality. The third property follows from the fact that in this case, $a_d \geq a_{d-1} \geq \cdots \geq a_0$). We prove the general result by induction on d, as it is obvious for $d = 0$. Suppose P has the desidered properties, and let

$$Q(x,y) = (a_d + a_{d-1})\, x^{d-1} + a_{d-2}x^{d-2}y + \cdots \\ + a_{-d+2}xy^{d-2} + (a_{-d} + a_{-d+1})\, y^{d-1}.$$

Then Q has smaller degree and satisfies the required properties, so by the induction hypotesis $Q(x,y) \geq 0$. Moreover,

$$P(x,y) - Q(x,y) = a_d \left(x^d - x^{d-1}y - xy^{d-1} + y^d\right) = \\ = a_d(x-y)\left(x^d - y^d\right) \geq 0$$

since $a_d \geq 0$ and the sign of $x - y$ is the same as the sign of $x^d - y^d$. Adding these two inequalities give $P(x,y) \geq 0$, as desidered. □

105. (**Hong Kong 1994**) $(xy + yz + zx = 1,\ x, y, z > 0)$

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}$$

First Solution. (*Grobber - ML Forum*)
What we must prove is

$$x + y + z + xyz(xy + yz + zx) \leq \frac{4\sqrt{3}}{9} + xy(x+y) + yz(y+z) + zx(z+x).$$

By adding $3xyz$ to both sides we get (we can eliminate $xy + yz + zx$ since it's equal to 1)

$$x + y + z + 4xyz \leq \frac{4\sqrt{3}}{9} + x + y + z.$$

By subtracting $x + y + z$ from both sides and dividing by 4 we are left with $xyz \leq \frac{\sqrt{3}}{9}$, which is true by AM-GM applied to xy, yz, zx. □

Second Solution. (*Murray Klamkin - Crux Mathematicorum 1998, pag.394*)
We first convert the inequality to the following equivalent homogeneous one:

$$x\left(T_2 - y^2\right)\left(T_2 - z^2\right) + yx\left(T_2 - z^2\right)\left(T_2 - x^2\right) + z\left(T_2 - x^2\right)\left(T_2 - y^2\right) \leq \\ \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}}$$

where $T_2 = xy + yz + zx$, and for subsequent use $T_1 = x + y + z$, $T_3 = xyz$. Expanding out, we get

$$T_1 T_2^2 - T_2 \sum x\left(y^2 + z^2\right) + T_2 T_3 \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}}$$

or

$$T_1 T_2^2 - T_2\left(T_1 T_2 - 3T_3\right) + T_2 T_3 = 4T_2 T_3 \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}}$$

Squaring, we get one of the know Maclaurin inequalities for symmetric functions:

$$\sqrt[3]{T_3} \leq \sqrt[3]{\frac{T_2}{3}}$$

There is equality if and only if $x = y = z$. $\square$

106. (**IMO Short List 1993**) $(a, b, c, d > 0)$

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \geq \frac{2}{3}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) From Cauchy-Schwarz inequality we have:

$$\begin{aligned}
(a+b+c+d)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{a}}{\sqrt{b+2c+3d}}\sqrt{a}\sqrt{b+2c+3d}\right)^2 \leq \\
&\leq \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\left(\sum_{\text{cyc}} ab + 2ac + 3ad\right) = \\
&= \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\left(\sum_{\text{sym}} ab\right) \leq \\
&\leq \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\frac{3}{2}(a+b+c+d)^2 \qquad (1)
\end{aligned}$$

The inequality of the last step can be proved by BUNCHING principle (Muirhead Theorem) in the following way:

$$\sum_{\text{sym}} ab \le \frac{3}{2}(a+b+c+d)^2$$
$$\sum_{\text{sym}} ab \le \frac{3}{2}\sum_{\text{cyc}} a^2 + \frac{3}{4}\sum_{\text{sym}} ab$$
$$\frac{1}{4}\sum_{\text{sym}} ab \le \frac{3}{2}\sum_{\text{cyc}} a^2$$
$$\sum_{\text{sym}} ab \le 6\sum_{\text{cyc}} a^2$$
$$\sum_{\text{sym}} ab \le \sum_{\text{sym}} a^2$$

From (1) follows that

$$\sum_{\text{cyc}} \frac{a}{b+2c+3d} \ge \frac{2}{3}$$

□

107. (**APMC 1993**) $(a, b \ge 0)$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \le \frac{a+\sqrt{ab}+b}{3} \le \sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

Solution. (*Tsaossoglou - Crux Mathematicorum 1997, pag. 73*)
Let $A = \sqrt[6]{a}$, $B = \sqrt[6]{b}$. The first inequality

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4}$$

is equivalent to

$$\begin{aligned}
&\left(\sqrt{a}+\sqrt{b}\right)^2 \le \left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\\
\iff\quad &\left(A^3+B^3\right)^2 \le \left(A^4+B^4\right)\left(A^2+B^2\right)
\end{aligned}$$

which holds by the Cauchy inequality.
The second inequality

$$\frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \le \frac{a+\sqrt{ab}+b}{3}$$

is equivalent to

$$\begin{aligned}
& 3(a+b)+3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \leq 4\left(a+\sqrt{ab}+b\right) \\
\iff\quad & a+3\sqrt[3]{a^2b}+3\sqrt[3]{ab^2}+b \leq 2\left(a+\sqrt{ab}+b\right) \\
\iff\quad & \left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3 \leq 2\left(\sqrt{a}+\sqrt{b}\right)^2 \\
\iff\quad & \left(\frac{A^2+B^2}{2}\right)^3 \leq \left(\frac{A^3+B^3}{2}\right)^2
\end{aligned}$$

which holds by the power mean inequality.
The third inequality

$$\frac{a+\sqrt{ab}+b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

is equivalent to

$$\left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \leq \left(\frac{A^4+B^4}{2}\right)^3$$

For this it is enough to prove that

$$\left(\frac{A^4+B^4}{2}\right)^3 - \left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \geq 0$$

or

$$\begin{aligned}
& 9\left(A^4+B^4\right)^3-8\left(A^6+A^3B^3+B^6\right)^2 = \\
=& (A-B)^4(A^8+4A^7B+10A^6B^2+4A^5B^3-2A^4B^4+ \\
& +4A^3B^5+10A^2B^6+4AB^7+B^8) \geq \\
\geq& (A-B)^4\left(A^8-2A^4B^4+B^8\right) = \\
=& (A-B)^4\left(A^4-B^4\right)^2 \geq 0
\end{aligned}$$

□

108. (**Poland 1993**) $(x,y,u,v>0)$

$$\frac{xy+xv+uy+uv}{x+y+u+v} \geq \frac{xy}{x+y}+\frac{uv}{u+v}$$

Solution. (*Ercole Suppa*)
Is enough to note that

$$\begin{aligned}&\frac{xy+xv+uy+uv}{x+y+u+v}-\frac{xy}{x+y}+\frac{uv}{u+v}=\\=&\frac{(x+u)(y+v)}{x+y+u+v}-\frac{xy}{x+y}+\frac{uv}{u+v}=\\=&\frac{(vx-uy)^2}{(x+y)(u+v)(x+y+u+v)}\geq 0\end{aligned}$$

$\square$

109. (**IMO Short List 1993**) $(a+b+c+d=1,\ a,b,c,d>0)$

$$abc+bcd+cda+dab\leq\frac{1}{27}+\frac{176}{27}abcd$$

Solution. (*See* [23] *pag.* 580)
Put $f(a,b,c,d)=abc+bcd+cda+dab-\frac{176}{27}abcd$ and note that f is symmetric with respect to the four variables a, b, c, d. We can write

$$f(a,b,c,d)=ab(c+d)+cd(a+b-\frac{176}{27}ab)$$

If $a+b-\frac{176}{27}ab\leq 0$, then using AM-GM for a, b, $c+d$, we have

$$f(a,b,c,d)\leq ab(c+d)=\frac{1}{27}$$

If $a+b-\frac{176}{27}ab>0$ by AM-GM inequality applied to c, d we get

$$f(a,b,c,d)\leq ab(c+d)+\frac{1}{4}(c+d)^2\left(a+b-\frac{176}{27}ab\right)=f\left(a,b,\frac{c+d}{2},\frac{c+d}{2}\right)$$

Consider now the following fourtplets:

$$P_0(a,b,c,d)\quad,\quad P_1\left(a,b,\frac{c+d}{2},\frac{c+d}{2}\right)\quad,\quad P_2\left(\frac{a+b}{2},\frac{a+b}{2},\frac{c+d}{2},\frac{c+d}{2}\right)$$
$$P_3\left(\frac{1}{4},\frac{a+b}{2},\frac{c+d}{2},\frac{1}{4}\right)\quad,\quad P_4\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$$

From the above considerations we deduce that for $i=0,1,2,3$ either $f(P_i)=1/27$ or $f(P_i)\leq f(P_{i+1})$. Since $f(P_4)=1/27$, in every case we are led to

$$f(a,b,c,d)=f(P_0)=\frac{1}{27}$$

Equality occurs only in the cases $\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (with permutations) and $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$.
$\square$

110. (**Italy 1993**) $(0 \le a, b, c \le 1)$

$$a^2 + b^2 + c^2 \le a^2 b + b^2 c + c^2 a + 1$$

First Solution. (*Ercole Suppa*) The given inequality is equivalent to

$$a^2(1-b) + b^2(1-c) + c^2(1-a) \le 1$$

The function

$$f(a, b, c) = a^2(1-b) + b^2(1-c) + c^2(1-a)$$

after setting b, c is convex with respect to the variable a so take its maximum value in $a = 0$ or in $a = 1$. A similar reasoning is true if we fix a, c or a, b. Thus is enough to compute $f(a, b, c)$ when $a, b, c \in \{0, 1\}$. Since f is symmetric (with respect to a, b, c) and:

$$f(0,0,0) = 0, \quad f(0,0,1) = 1, \quad f(0,1,1) = 1, \quad f(1,1,1) = 0$$

the result is proven. $\square$

Second Solution. (*Ercole Suppa*) We have

$$\begin{aligned} a^2(1-b) + b^2(1-c) + c^2(1-a) &\le a(1-b) + b(1-c) + c(1-a) = \\ &= a + b + c - (ab + bc + ca) = \\ &= 1 - (1-a)(1-b)(1-c) - abc \le 1 \end{aligned}$$

$\square$

111. (**Poland 1992**) $(a, b, c \in \mathbb{R})$

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \ge (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

Solution. (*Harazi - ML Forum*) It can be proved observing that

$$(a+b-c)^2(c+a-b)^2 \ge (a^2+b^2-c^2)(c^2+a^2-b^2)$$

which is true because:

$$(a+c-b)^2(b+a-c)^2 = (a^2-(b-c)^2)^2 = a^4 - 2a^2(b-c)^2 + (b-c)^4$$

But $(a^2+c^2-b^2)(b^2+a^2-c^2) = a^4 - (b^2-c^2)^2$. So, it is enough to prove that

$$(b^2-c^2)^2 + (b-c)^4 \ge 2a^2(b-c)^2 \iff (b+c)^2 + (b-c)^2 \ge 2a^2 \iff b^2+c^2-a^2 \ge 0$$

We can assume that $b^2+c^2-a^2 \ge 0$, $c^2+a^2-b^2 \ge 0$, $a^2+b^2-c^2 \ge 0$ (only one of them could be negative and then it's trivial), so these inequalities hold. Multiply them and the required inequality is proved. $\square$

112. (**Vietnam 1991**) ($x \geq y \geq z > 0$)

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

First Solution. (*Gabriel - ML Forum*) Since $x \geq y \geq z \geq 0$ we have that,

$$\begin{aligned}\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} = \frac{x^3y^2 + y^3z^2 + z^3x^2}{xyz} &\geq \\ \geq \frac{(x^3 + y^3 + z^3)(x^2 + y^2 + z^2)}{3(xyz)} &\geq \\ \geq \frac{3xyz(x^2 + y^2 + z^2)}{3(xyz)} &= \\ = x^2 + y^2 + z^2 &\end{aligned}$$

by Chebyshev's inequality □

Second Solution. (*Murray Klamkin - Crux Mathematicorum 1996, pag.111*) Let $z = a$, $y = a + b$, $x = a + b + c$ where $a > 0$ and $b, c \geq 0$. Substituting back in the inequality, multiplying by the least common denominator, we get

$$\begin{aligned}&\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} - x^2 - y^2 - z^2 = \\ =&\frac{1}{a(a+b)(a+b+c)}(a^3b^2 + 3a^2b^3 + 3ab^4 + b^5 + a^3bc + 6a^2b^2c + 8ab^3c + \\ &+ 3b^4c + a^3c^2 + 3a^2bc^2 + 6ab^2c^2 + 3b^3c^2 + abc^3 + b^2c^3 \geq 0\end{aligned}$$

and the inequality is proved. □

Third Solution. (*ductrung - ML Forum*) First, note that

$$\sum \frac{ab(a-b)}{c} = \frac{(a-b)(a-c)(b-c)(ab+bc+ca)}{abc} \geq 0$$

Hence

$$\sum \frac{a^2b}{c} \geq \sum \frac{ab^2}{c}$$

and so

$$2\sum \frac{a^2b}{c} \geq \sum \frac{ab(a+b)}{c}$$

It remains to show that

$$\sum \frac{ab(a+b)}{c} \geq 2(a^2 + b^2 + c^2)$$

or equivalently

$$\sum a^2b^2(a+b) \geq 2abc(a^2+b^2+c^2)$$

But

$$\sum a^2b^2(a+b) - 2abc(a^2+b^2+c^2) = \sum c^3(a-b)^2 \geq 0$$

Remark. Different solutions are given in *Crux Mathematicorum* 1994, pag.43.

113. (**Poland 1991**) ($x^2+y^2+z^2=2,\ x,y,z \in \mathbb{R}$)

$$x+y+z \leq 2+xyz$$

First Solution. (*See* [4], *pag.* 57, *problem* 50)
Using the Cauchy-Schwarz inequality, we find that

$$x+y+z-xyz = x(1-yz)+(y+z) \leq \sqrt{[x^2+(y+z)^2]\,[1+(1-yz)^2]}$$

So, it is enough to prove that this last quantity is at most 2, which is equivalent to

$$(2+2yz)\left[2-2yz+(yz)^2\right] \leq 4 \quad \Longleftrightarrow \quad (2yz)^3 \leq (2yz)^2$$

which is clearly true because $2yz \leq y^2+z^2 \leq 2$. □

Second Solution. (*Crux Mathematicorum 1989, pag. 106*)
Put $S = x+y+z$, $P = xyz$. It is enough to show that

$$E = 4-(S-P)^2 \geq 0$$

Now using $x^2+y^2+z^2=2$ we have

$$\begin{aligned}
4E &= 2^3 - 2^2\left(S^2-2\right) + 2(4SP) - 4P^2 = \\
&= 2^3 - 2^2(2xy+2yz+2zx) + 2\left(4x^2yz+4xy^2z+4xyz^2\right) - 8x^2y^2z^2 + 4P^2 = \\
&= (2-2xy)(2-2yz)(2-2zx) + 4P^2
\end{aligned}$$

Since

$$\begin{aligned}
2-2xy &= z^2+(x-y)^2 \\
2-2yz &= x^2+(x-z)^2 \\
2-2zx &= y^2+(z-x)^2
\end{aligned}$$

the above quantities are nonnegative. Thus, so also is E, completing the proof.
□

Third Solution. (*Crux Mathematicorum 1989, pag. 106*)
Lagrange multipliers provide a straighforward solution. Here the Lagrangian is

$$\mathcal{L} = x + y + z - xyz - \lambda\left(x^2 + y^2 + z^2 - 2\right)$$

Now setting the partial derivatives equal to zero we obtain

$$1 - yz = 2\lambda x$$
$$1 - xz = 2\lambda y$$
$$1 - xy = 2\lambda z$$

On subtraction, we get

$$(x - y)(z - 2\lambda) = 0 = (y - z)(x - 2\lambda)$$

Thus the critical points are $x = y = z$ and $x = y$, $z = \left(1 - x^2\right)/x$ and any cyclic permutation. The maximum value corresponds to the critical point $x = y$, $z = \left(1 - x^2\right)/x$. Since $x^2 + y^2 + z^2 = 2$ this leads to

$$\left(3x^2 - 1\right)\left(x^2 - 1\right) = 0$$

Finally, the critical point $(1, 1, 0)$ and permutations of it give the maximum value of $x + y + z - xyz$ to be 2. □

Fourth Solution. (*See* [1] *pag.* 155)
If one of x, y, z is nagative, for example $z < 0$ then

$$2 + xyz - x - y - z = (2 - x - y) - z(1 - xy) \geq 0$$

since $x + y \leq \sqrt{2\left(x^2 + y^2\right)} \leq 2$ and $2xy \leq x^2 + y^2 \leq 2$. Thus, WLOG, we can suppose $0 < x \leq y \leq z$. If $z \leq 1$ then

$$2 + xyz - x - y - z = (1 - x)(1 - y) + (1 - z)(1 - xy) \geq 0$$

If, on the contrary $z > 1$ then by Cauchy-Schwartz inequality we have

$$x + y + z \leq \sqrt{2\left[(x + y)^2 + z^2\right]} = 2\sqrt{xy + 1} \leq xy + 2 \leq xyz + 2$$

□

Remark 1. This inequality was proposted in IMO shortlist 1987 by United Kingdom.

Remark 2. The inequality admit the following generalization: Given real numbers x, y, z such that $x^2 + y^2 + z^2 = k$, $k > 0$, prove the inequality

$$\frac{2}{k}xyz - \sqrt{2k} \leq x + y + z \leq \frac{2}{k}xyz + \sqrt{2k}$$

When $k = 2$, see problem 113.

114. (**Mongolia 1991**) $(a^2+b^2+c^2=2,\ a,b,c\in\mathbb{R})$

$$|a^3+b^3+c^3-abc|\le 2\sqrt{2}$$

Solution. (*ThAzN1 - ML Forum*) It suffices to prove

$$(a^3+b^3+c^3-abc)^2\le 8=(a^2+b^2+c^2)^3.$$

This is equivalent to

$$(a^2+b^2+c^2)^3-(a^3+b^3+c^3)^2+2abc(a^3+b^3+c^3)-a^2b^2c^2\ge 0$$
$$\sum(3a^4b^2+3a^2b^4-2a^3b^3)+2abc(a^3+b^3+c^3)+5a^2b^2c^2\ge 0$$
$$\sum(a^4b^2+a^2b^4+a^2b^2(a-b)^2+a^4(b+c)^2)+5a^2b^2c^2\ge 0$$

□

115. (**IMO Short List 1990**) $(ab+bc+cd+da=1,\ a,b,c,d>0)$

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\ge\frac{1}{3}$$

First Solution. (*See* [23] *pag.* 540)
Let A, B, C, D denote $b+c+d$, $a+c+d$, $a+b+d$, $a+b+c$ respectively. Since $ab+bc+cd+da=1$ the numbers A, B, C, D are all positive. By Cauchy-Schwarz inequality we have

$$a^2+b^2+c^2+d^2\ge ab+bc+cd+da=1$$

We'll prove the required inequality under a weaker condition that A, B, C, D are all positive and $a^2+b^2+c^2+d^2\ge 1$. We may assume, WLOG, that $a\ge b\ge c\ge d\ge 0$. Hence $a^3\ge b^3\ge c^3\ge d^3\ge 0$ and $\frac{1}{A}\ge\frac{1}{B}\ge\frac{1}{C}\ge\frac{1}{D}\ge 0$. Using Chebyshev inequality and Cauchy inequality we obtain

$$\frac{a^3}{A}+\frac{b^3}{B}+\frac{c^3}{C}+\frac{d^3}{D}\ge\frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)\ge$$
$$\ge\frac{1}{16}\left(a^2+b^2+c^2+d^2\right)(a+b+c+d)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)=$$
$$=\frac{1}{48}\left(a^2+b^2+c^2+d^2\right)(A+B+C+D)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)\ge\frac{1}{3}$$

This complete the proof. □

Second Solution. (*Demetres Christofides - J. Sholes WEB site*)
Put

$$
\begin{aligned}
&S = a + b + c + d \\
&A = \frac{a^3}{S-a} + \frac{b^3}{S-b} + \frac{c^3}{S-c} + \frac{d^3}{S-d} \\
&B = a^2 + b^2 + c^2 + d^2 \\
&C = a(S-a) + b(S-b) + c(S-c) + d(S-d) = 2 + 2ac + 2bd
\end{aligned}
$$

By Cauchy-Scwarz we have

$$AC \geq B^2 \tag{1}$$

We also have

$$
\begin{aligned}
&(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 \geq 0 \quad \Longrightarrow \\
&B \geq ab + bc + cd + da = 1 \qquad (2)
\end{aligned}
$$

and

$$(a-c)^2 + (b-d)^2 \geq 0 \quad \Longrightarrow \quad B \geq 2ac + 2bd \tag{3}$$

If $2ac + 2bd \leq 1$ then $C \leq 3$, so by (1) and (2) we have

$$A \geq \frac{B^2}{C} \geq \frac{1}{C} \geq \frac{1}{3}$$

If $2ac + 2bd > 1$ then $C > 3$, so by (1), (2), and (3) we have

$$A \geq \frac{B^2}{C} \geq \frac{B}{C} \geq \frac{2ac+2bd}{C} \geq \frac{C-2}{C} = 1 - \frac{2}{C} > \frac{1}{3}$$

This complete the proof. $\square$

Third Solution. (*Campos - ML Forum*) By Holder we have that

$$
\begin{aligned}
&\left(\sum \frac{a^3}{b+c+d}\right)\left(\sum a(b+c+d)\right)\left(\sum 1\right)^2 \geq \left(\sum a\right)^4 \quad \Longrightarrow \\
&\sum \frac{a^3}{b+c+d} \geq \frac{(a+b+c+d)^4}{16(\sum a(b+c+d))}
\end{aligned}
$$

but it's easy to verify from the condition that

$$(a+b+c+d)^2 \geq 4(a+c)(b+d) = 4$$

and

$$3(a+b+c+d)^2 \geq 4\sum a(b+c+d)$$

because

$$\begin{aligned}
&3(a+b+c+d)^2-4\sum a(b+c+d)= \\
=&3(a+b+c+d)^2-8(ab+ac+ad+bc+bd+cd)= \\
=&3\left(a^2+b^2+c^2+d^2\right)-2(ab+ac+ad+bc+bd+cd)= \\
=&4\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2\geq 0 \qquad \text{(by Cauchy-Schwarz)}
\end{aligned}$$

This complete the proof. $\square$

4 Supplementary Problems

116. (**Lithuania 1987**) $(x, y, z > 0)$

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\geq\frac{x+y+z}{3}$$

Solution. (*Gibbenergy - ML Forum*) Since $3xy \leq x^2+xy+y^2$ we have

$$\frac{x^3}{x^2+xy+y^2}=x-\frac{xy(x+y)}{x^2+xy+y^2}\geq x-\frac{x+y}{3}$$

Then doing this for all other fractions and summing we obtain the inequality we want to prove. □

Remark. This inequality was proposed in Tournament of the Towns 1998.

117. (**Yugoslavia 1987**) $(a, b > 0)$

$$\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)\geq a\sqrt{b}+b\sqrt{a}$$

First Solution. (*Ercole Suppa*)
From AM-GM inequality follows that $\frac{a+b}{2}\geq\sqrt{ab}$. Therefore

$$\begin{aligned}
&\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-a\sqrt{b}-b\sqrt{a}=\\
&=\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)\geq\\
&\geq\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\frac{a+b}{2}\left(\sqrt{a}+\sqrt{b}\right)=\\
&=\frac{a+b}{2}\left[a+b+\frac{1}{2}-\sqrt{a}-\sqrt{b}\right]=\\
&=\frac{a+b}{2}\left[\left(\sqrt{a}-\frac{1}{2}\right)^2+\left(\sqrt{b}-\frac{1}{2}\right)^2\right]\geq 0
\end{aligned}$$

□

Second Solution. (*Arne- ML Forum*)
The left-hand side equals

$$\frac{a^2}{2}+\frac{b^2}{2}+ab+\frac{a}{4}+\frac{b}{4}$$

Now note that, by AM-GM inequality

$$\frac{a^2}{2}+\frac{ab}{2}+\frac{a}{8}+\frac{b}{8}\geq 4\sqrt[4]{\frac{a^2}{2}\cdot\frac{ab}{2}\cdot\frac{a}{8}\cdot\frac{b}{8}}=a\sqrt{b}$$

Similarly

$$\frac{b^2}{2}+\frac{ab}{2}+\frac{a}{8}+\frac{b}{8}\geq 4\sqrt[4]{\frac{b^2}{2}\cdot\frac{ab}{2}\cdot\frac{a}{8}\cdot\frac{b}{8}}=b\sqrt{a}$$

Adding these inequalities gives the result. □

118. (**Yugoslavia 1984**) $(a,b,c,d>0)$

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2$$

Solution. (See [65] *pag.* 127)
From Cauchy-Schwarz inequality we have

$$(a+b+c+d)^2\leq\sum_{\text{cyc}}a(b+c)\sum_{\text{cyc}}\frac{a}{b+c}=$$
$$=(ab+2ac+bc+2bd+cd+ad)\cdot\sum_{\text{cyc}}\frac{a}{b+c}$$

Then, to establish the required inequality it will be enough to show that

$$(ab+2ac+bc+2bd+cd+ad)\leq\frac{1}{2}(a+b+c+d)^2$$

This inequality it is true because

$$\frac{1}{2}(a+b+c+d)^2-(ab+2ac+bc+2bd+cd+ad)=$$
$$=\frac{1}{2}(a-c)^2+\frac{1}{2}(b-d)^2\geq 0$$

The equality holds if and only if $a=c$ e $b=d$. □

119. (**IMO 1984**) $(x+y+z=1,\ x,y,z\geq 0)$

$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}$$

First Solution. (*See* [32], *pag.* 23)
Let $f(x,y,z) = xy + yz + zx - 2xyz$. We may assume that $0 \le x \le y \le z \le 1$. Since $x+y+z=1$, we find that $x \le \frac{1}{3}$. It follows that $f(x,y,z) = (1-3x)yz + xyz + zx + xy \ge 0$. Applying the AM-GM inequality, we obtain $yz \le \left(\frac{y+z}{2}\right)^2 = \left(\frac{1-x}{2}\right)^2$. Since $1-2x \ge 0$, this implies that

$$f(x,y,z) = x(y+z)+yz(1-2x) \le x(1-x)+\left(\frac{1-x}{2}\right)^2(1-2x) = \frac{-2x^3+x^2+1}{4}.$$

Our job is now to maximize $F(x) = \frac{1}{4}(-2x^3+x^2+1)$, where $x \in \left[0, \frac{1}{3}\right]$. Since $F'(x) = \frac{3}{2}x\left(\frac{1}{3}-x\right) \ge 0$ on $\left[0,\frac{1}{3}\right]$, we conclude that $F(x) \le F(\frac{1}{3}) = \frac{7}{27}$ for all $x \in \left[0, \frac{1}{3}\right]$. $\square$

Second Solution. (*See* [32], *pag.* 31)
Using the condition $x+y+z=1$, we reduce the given inequality to homogeneous one, i. e.,

$$0 \le (xy+yz+zx)(x+y+z) - 2xyz \le \frac{7}{27}(x+y+z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \le xyz + \sum_{\text{sym}} x^2y.$$

The right hand side inequality simplifies to

$$7\sum_{\text{cyc}} x^3 + 15xyz - 6\sum_{\text{sym}} x^2y \ge 0.$$

In the view of

$$7\sum_{\text{cyc}} x^3 + 15xyz - 6\sum_{\text{sym}} x^2y = \left(2\sum_{\text{cyc}} x^3 - \sum_{\text{sym}} x^2y\right) + 5\left(3xyz + \sum_{\text{cyc}} x^3 - \sum_{\text{sym}} x^2y\right),$$

it's enough to show that

$$2\sum_{\text{cyc}} x^3 \ge \sum_{\text{sym}} x^2y \quad\text{and}\quad 3xyz + \sum_{\text{cyc}} x^3 \ge \sum_{\text{sym}} x^2y.$$

We note that

$$2\sum_{\text{cyc}} x^3 - \sum_{\text{sym}} x^2y = \sum_{\text{cyc}}(x^3+y^3) - \sum_{\text{cyc}}(x^2y+xy^2) = \sum_{\text{cyc}}(x^3+y^3-x^2y-xy^2) \ge 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyc}} x(x-y)(x-z) \ge 0,$$

which is a particular case of Schur's theorem. $\square$

120. (**USA 1980**) ($a, b, c \in [0, 1]$)

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c) \leq 1.$$

Solution. (See [43] *pag.* 82)
The function

$$f(a, b, c)=\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c)$$

is convex in each of the three variables a, b, c, so f takes its maximum value in one of eight vertices of the cube $0 \leq a \leq 1$, $0 \leq b \leq 1$, $0 \leq c \leq 1$. Since $f(a, b, c)$ takes value 1 in each of these points, the required inequality is proven. $\square$

121. (**USA 1979**) ($x+y+z=1,\ x, y, z>0$)

$$x^{3}+y^{3}+z^{3}+6 x y z \geq \frac{1}{4}.$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$\begin{gathered}
4\left(x^{3}+y^{3}+z^{3}\right)+24 x y z \geq(x+y+z)^{3} \quad \Longleftrightarrow \\
3\left(x^{3}+y^{3}+z^{3}\right)+18 x y z \geq 3 \sum_{\text{sym}} x^{2} y \quad \Longleftrightarrow \\
\sum x^{3}+3 \sum x y z \geq \sum_{\text{sym}} x^{2} y
\end{gathered}$$

which is true for all $x, y, z>0$ by Schur inequality. $\square$

122. (**IMO 1974**) ($a, b, c, d>0$)

$$1<\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}<2$$

Solution. (*Ercole Suppa*) We have

$$\begin{aligned}
&\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}< \\
&<\frac{a}{a+b}+\frac{b}{b+a}+\frac{c}{c+d}+\frac{d}{c+d}=2
\end{aligned}$$

and

$$\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}>$$
$$>\frac{a}{a+b+c+d}+\frac{b}{a+b+c+d}+\frac{c}{a+b+c+d}+\frac{d}{a+b+c+d}=1$$

$\square$

Remark. In the problem 5 of IMO 1974 is requested to find all possible values of

$$S=\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}$$

for arbitrary positive reals a, b, c, d. A detailed solution is given in [59], pag. 203.

123. (**IMO 1968**) $(x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in \mathbb{R}, x_1y_1 > {z_1}^2, x_2y_2 > {z_2}^2)$

$$\frac{1}{x_1y_1-{z_1}^2}+\frac{1}{x_2y_2-{z_2}^2}\geq\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2}$$

Solution. ([23]*pag.* 369)
Define $u_1=\sqrt{x_1y_1}+z_1$, $u_2=\sqrt{x_2y_2}+z_2$, $v_1=\sqrt{x_1y_1}-z_1$ and $v_2=\sqrt{x_2y_2}-z_2$. By expanding both sides we can easily verify

$$(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2=(u_1+u_2)(v_1+v_2)+\left(\sqrt{x_1y_2}-\sqrt{x_2y_1}\right)^2$$

Thus

$$(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2\geq(u_1+u_2)(v_1+v_2)$$

Since $x_iy_i-z_i^2=u_iv_i$ for $i=1,2$, it suffices to prove

$$\frac{8}{(u_1+u_2)(v_1+v_2)}\leq\frac{1}{u_1v_1}+\frac{1}{u_2v_2}$$
$$\iff\quad 8u_1u_2v_1v_2\leq(u_1+u_2)(v_1+v_2)(u_1v_1+u_2v_2)$$

which trivially follows from the AM-GM inequalities

$$2\sqrt{u_1u_2}\leq u_1+u_2\quad,\quad 2\sqrt{v_1v_2}\leq v_1+v_2\quad,\quad 2\sqrt{u_1v_1u_2v_2}\leq u_1v_1+u_2v_2$$

Equality holds if and only if $x_1y_2=x_2y_1$, $u_1=u_2$ and $v_1=v_2$, i.e. if and only if $x_1=x_2$, $y_1=y_2$ and $z_1=z_2$. $\square$

124. (**Nesbitt's inequality**) $(a, b, c > 0)$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution. (*See* [32], *pag.* 18) After the substitution $x = b + c,\ y = c + a,\ z = a + b$, it becomes

$$\sum_{\text{cyc}} \frac{y+z-x}{2x} \geq \frac{3}{2} \quad \textit{or} \quad \sum_{\text{cyc}} \frac{y+z}{x} \geq 6,$$

which follows from the AM-GM inequality as following:

$$\sum_{\text{cyc}} \frac{y+z}{x} = \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} \geq 6\left(\frac{y}{x} \cdot \frac{z}{x} \cdot \frac{z}{y} \cdot \frac{x}{y} \cdot \frac{x}{z} \cdot \frac{y}{z}\right)^{\frac{1}{6}} = 6.$$

Remark. In [32], are given many other proofs of this famous inequality.

125. (**Polya's inequality**) $(a \neq b, a, b > 0)$

$$\frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right) > \frac{a-b}{\ln a - \ln b}$$

Solution. (*Kee-Wai Lau - Crux Mathematicorum 1999, pag.253*)
We can assume WLOG that $a > b$. The required inequality is equivalent to

$$\frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right) \geq \frac{a-b}{\ln \frac{a}{b}}$$

or, dividing both members by b :

$$\frac{1}{3}\left(2\sqrt{\frac{a}{b}} + \frac{\frac{a}{b}+1}{2}\right) \geq \frac{\frac{a}{b}-1}{\ln \frac{a}{b}}$$

After setting $x = \sqrt{\frac{a}{b}}$ we must show that

$$\ln x - \frac{3\left(x^2-1\right)}{x^2+4x+1} \geq 0 \quad , \quad \forall x \geq 1$$

By putting

$$f(x) = \ln x - \frac{3\left(x^2-1\right)}{x^2+4x+1}$$

a direct calculation show that

$$f'(x) = \frac{(x-1)^4}{x\left(x^2+4x+1\right)^2}$$

Thus $f'(x) > 0$ for all $x > 1$ (i.e. $f(x)$ is increasing for all $x > 1$). Since $f(1) = 0$, we have $f(x) > 0$ for all $x > 1$ and the the result is proven. $\square$

126. (**Klamkin's inequality**) $(-1 < x, y, z < 1)$

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2$$

Solution. (*Ercole Suppa*)
From AM-GM inequality we have

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq \frac{2}{\sqrt{(1-x^2)(1-y^2)(1-z^2)}} \geq 2$$

$\square$

127. (**Carlson's inequality**) $(a, b, c > 0)$

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}$$

First Solution. (*P. E. Tsaoussoglou - Crux Mathematicorum 1995, pag. 336*)
It is enough to prove that for all positive real numbers a, b, c the following inequality holds

$$64(ab+bc+ca)^3 \leq 27(a+b)^2(b+c)^2(c+a)^2$$

or

$$64 \cdot 3(ab+bc+ca)(ab+bc+ca)^2 \leq 81\left[(a+b)(b+c)(c+a)\right]^2$$

It is know that $3(ab+bc+ca) \leq (a+b+c)^2$. Thus, it is enough to prove one of the following equivalent inequalities

$$\begin{gathered} 8(a+b+c)(ab+bc+ca) \leq 9(a+b)(b+c)(c+a) \\ 8(a+b)(b+c)(c+a) + 8abc \leq 9(a+b)(b+c)(c+a) \\ 8abc \leq (a+b)(b+c)(c+a) \end{gathered}$$

The last inequality is well-know and this complete the proof. $\square$

Second Solution. (See [49] *pag.* 141)
It is enough to prove that for all positive real numbers a, b, c the following inequality holds

$$64(ab+bc+ca)^3 \leq 27(a+b)^2(b+c)^2(c+a)^2$$

Write $s=a+b+c$, $u=ab+bc+ca$, $v=abc$. Since $a^2+b^2+c^2>ab+bc+ca=u$ we have

$$s=\sqrt{a^2+b^2+c^2+2u}\geq\sqrt{3u}$$

By AM-GM inequality

$$s\geq 3\sqrt[3]{abc}\quad,\quad u\geq 3\sqrt[3]{(ab)(bc)(ca)}=3\sqrt[3]{v^2}$$

and hence $su\geq 9v$. Consequently,

$$\begin{aligned}(a+b)(b+c)(c+a)&=(s-a)(s-b)(s-c)=s^3+su-v\geq\\&\geq su-\frac{1}{9}su=\frac{8}{9}su\geq\frac{8}{9}u\sqrt{3u}=\\&=\frac{8\sqrt{3}}{9}\sqrt{(ab+bc+ca)^3}\end{aligned}$$

and raising both sides to the second power we obtain the asserted inequality. Equality holds if and only if $a=b=c$. □

Remark. The problem was proposed in Austrian-Polish Competition 1992, problem 6.

128. (*See* [4], *Vasile Cirtoaje*) $(a,b,c>0)$

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)+\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)+\left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right)\geq 3$$

Solution. (*See* [4], *pag.* 89, *problem* 94) Assume WLOG that $x=\max\{x,y,z\}$. Then

$$x\geq\frac{1}{3}(x+y+z)=\frac{1}{3}\left(a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}-3\right)\geq\frac{1}{3}(2+2+2-3)=1$$

On the other hand,

$$\begin{aligned}(x+1)(y+1)(z+1)&=abc+\frac{1}{abc}+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\\&\geq 2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=5+x+y+z\end{aligned}$$

and hence

$$xyz+xy+yz+zx\geq 4$$

Since

$$y+z=\frac{1}{a}+b+\frac{(c-1)^2}{c}>0$$

two cases are possible.

(a) Case $yz \le 0$. We have $xyz \le 0$, and from $xy + yz + zx \ge 4$ it follows that $xy + yz + zx \ge 4 > 3$.

(b) Case $y, z > 0$. Let $d = \sqrt{\frac{xy+yz+zx}{3}}$. We have to show that $d \ge 1$. By AM-GM we have $xyz \le d^3$. Thus $xyz + xy + yz + zx \ge 4$ implies $d^3 + 3d^2 \ge 4$, $(d-1)(d+2)^2 \ge 0$, $d \ge 1$. Equality occurs for $a = b = c = 1$.

□

129. (**[ONI], Vasile Cirtoaje**) $(a, b, c, d > 0)$

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \ge 0$$

Solution. (*See* [4], *pag.* 60, *n.* 54)
By AM-HM inequality we have

$$\begin{aligned}
&\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} = \\
=&\frac{a+c}{b+c} + \frac{b+d}{c+d} + \frac{c+a}{d+a} + \frac{d+b}{a+b} - 4 = \\
=&(a+c)\left(\frac{1}{b+c} + \frac{1}{d+a}\right) + (b+d)\left(\frac{1}{c+d} + \frac{1}{a+b}\right) - 4 \ge \\
\ge&\frac{4(a+c)}{(b+c)+(d+a)} + \frac{4(b+d)}{(c+d)+(a+b)} - 4 = 0
\end{aligned}$$

□

130. (**Elemente der Mathematik, Problem 1207, S̃efket Arslanagić**) $(x, y, z > 0)$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \ge \frac{x+y+z}{\sqrt[3]{xyz}}$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$x^2z + y^2x + z^2y \ge (x+y+z)\sqrt[3]{(xyz)^2}$$

The above inequality is obtained by adding the following

$$\begin{aligned}
\frac{1}{3}x^2z + \frac{1}{3}x^2z + \frac{1}{3}xy^2 &\ge x\sqrt[3]{(xyz)^2} \\
\frac{1}{3}xy^2 + \frac{1}{3}xy^2 + \frac{1}{3}yz^2 &\ge y\sqrt[3]{(xyz)^2} \\
\frac{1}{3}yz^2 + \frac{1}{3}yz^2 + \frac{1}{3}x^2z &\ge z\sqrt[3]{(xyz)^2}
\end{aligned}$$

which follows from AM-GM inequality. □

131. ($\sqrt{WURZEL}$**, Walther Janous**) $(x+y+z=1,\ x,y,z>0)$

$$(1+x)(1+y)(1+z) \geq (1-x^2)^2+(1-y^2)^2+(1-z^2)^2$$

First Solution. (*Ercole Suppa*) By setting $A = xy+yz+zx$, $B = xyz$, since $x+y+z=1$, we get

$$x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = 1-2A$$

and

$$\begin{aligned} x^4+y^4+z^4 &= \left(x^2+y^2+z^2\right)^2 - 2(x^2y^2+y^2z^2+z^2x^2) = \\ &= (1-2A)^2 - 2\left[(xy+yz+zx)^2 - 2\left(x^2yz+xy^2z+xyz^2\right)\right] = \\ &= (1-2A)^2 - 2\left[A^2 - 2B(x+y+z)\right] = \\ &= (1-2A)^2 - 2A^2 + 4B = \\ &= 2A^2 - 4A + 4B + 1 \end{aligned}$$

The required inequality is equivalent to

$$\begin{aligned} 1+x+y+z+xy+yz+zx+xyz \geq x^4+y^4+z^4-2\left(x^2+y^2+z^2\right)+3 \quad &\iff \\ 2+A+B \geq 2A^2-4A+4B+1-2(1-2A)+3 \quad &\iff \\ A \geq 2A^2+3B \quad &\iff \\ xy+yz+zx \geq 2(xy+yz+zx)^2+3xyz \quad &\iff \\ (x+y+z)^2(xy+yz+zx) \geq 2(xy+yz+zx)^2+3xyz(x+y+z) \quad &\iff \\ \left(x^2+y^2+z^2\right)(xy+yz+zx) \geq 3xyz(x+y+z) \quad & (\star) \end{aligned}$$

The inequality $(\star)$ follows from Muirhead theorem since

$$\left(x^2+y^2+z^2\right)(xy+yz+zx) \geq 3xyz(x+y+z) \quad \iff$$

$$x^3y+x^3z+xy^3+y^3z+xz^3+yz^3 \geq 2x^2yz+2xy^2z+2xyz^2 \quad \iff$$

$$\sum_{\text{sym}} x^3y \geq \sum_{\text{sym}} x^2yz$$

Alternatively is enough to observe that for all $x,y,z \geq 0$ we get

$$\begin{aligned} &\left(x^2+y^2+z^2\right)(xy+yz+zx) - 3xyz(x+y+z) \geq \\ \geq&(xy+yz+zx)^2 - 3xyz(x+y+z) = \\ =&\frac{1}{2}\left[x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2\right] \geq 0 \end{aligned}$$

□

Second Solution. (*Yimin Ge - ML Forum*) Homogenizing gives

$$(x+y+z)(2x+y+z)(x+2y+z)(x+y+2z) \geq \sum((y+z)(2x+y+z))^2$$

By using the Ravi-substitution, we obtain

$$(a+b+c)(a+b)(b+c)(c+a) \geq 2\sum(a(b+c))^2$$

which is equivalent to

$$\sum_{\text{sym}} a^3b \geq \sum_{\text{sym}} a^2b^2$$

which is true. $\square$

Remark. This inequality was proposed in Austrian-Polish Competition 2000, problem 6.

132. ($\sqrt{WURZEL}$**, Heinz-Jürgen Seiffert**) ($xy > 0, x, y \in \mathbb{R}$)

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2}$$

Solution. (*Campos - ML Forum*)
We have

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2} \quad \Leftrightarrow$$

$$\sqrt{\frac{x^2+y^2}{2}} - \sqrt{xy} \geq \frac{x+y}{2} - \frac{2xy}{x+y} \quad \Leftrightarrow$$

$$\frac{(x-y)^2}{2} \geq \frac{(x-y)^2}{2(x+y)} \cdot \left(\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}\right) \quad \Leftrightarrow$$

$$x+y \geq \sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy} \quad \Leftrightarrow$$

$$\frac{(x+y)^2}{2} \geq 2\sqrt{\frac{x^2+y^2}{2}}\sqrt{xy}$$

and this is AM-GM. $\square$

133. ($\sqrt{WURZEL}$**, Šefket Arslanagić**) $(a, b, c > 0)$

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3\,(x+y+z)}$$

Solution. (*Ercole Suppa*) First we prove the following lemma:

Lemma. If $a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n$ are real positive numbers, the following inequality holds

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^3 \leq \left(\sum_{i=1}^{n} {a_i}^3\right)\left(\sum_{i=1}^{n} {b_i}^3\right)\left(\sum_{i=1}^{n} {c_i}^3\right)$$

Proof. By Holder and Cauchy-Schwarz inequalities we have

$$\begin{aligned}\sum a_i b_i c_i &\leq \left(\sum a_i^3\right)^{\frac{1}{3}} \left(\sum (b_i c_i)^{\frac{3}{2}}\right)^{\frac{2}{3}} \leq \\ &\leq \left(\sum a_i^3\right)^{\frac{1}{3}} \left[\left(\sum b_i^3\right)^{\frac{1}{2}} \left(\sum c_i^3\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} = \\ &= \left(\sum a_i^3\right)^{\frac{1}{3}} \cdot \left(\sum b_i^3\right)^{\frac{1}{3}} \cdot \left(\sum c_i^3\right)^{\frac{1}{3}}\end{aligned}$$

□

In order to show the required inequality we put

$$(a_1, a_2, a_3) = \left(\frac{a}{\sqrt[3]{x}}, \frac{b}{\sqrt[3]{y}}, \frac{c}{\sqrt[3]{z}}\right)$$

$$(b_1, b_2, b_3) = \left(\sqrt[3]{x}, \sqrt[3]{y}, \sqrt[3]{z}\right)$$

$$(c_1, c_2, c_3) = (1, 1, 1)$$

and we use the Lemma:

$$\begin{aligned}(a+b+c)^3 &= \left(\sum a_i b_i c_i\right)^3 \leq \\ &\leq \sum a_i^3 \sum b_i^3 \sum c_i^3 = \\ &= \left(\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}\right)(x+y+z)(1+1+1)\end{aligned}$$

Finally, dividing by $(x+y+z)$ we have

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3\,(x+y+z)}$$

□

134. ($\sqrt{WURZEL}$, **Šefket Arslanagić**) ($abc = 1, a, b, c > 0$)

$$\frac{1}{a^2(b+c)} + \frac{1}{b^2(c+a)} + \frac{1}{c^2(a+b)} \geq \frac{3}{2}.$$

Solution. (*Ercole Suppa*) After setting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$ we have $xyz = 1$ and the required inequality is equivalent to

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$$

which is the well-know Nesbitt inequality (see Problem 124). □

135. ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) ($abc = 1, a, b, c > 0$)

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{2}(a+b)(c+a)(b+c) - 1.$$

Solution. (*Ercole Suppa*) After setting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$ we have $xyz = 1$ and the required inequality is equivalent to

$$x^3 + y^3 + z^3 \geq \frac{1}{2} \cdot \frac{x+y}{xy} \cdot \frac{y+z}{yz} \cdot \frac{z+x}{zx} - 1 \quad \Longleftrightarrow$$

$$2\left(x^3 + y^3 + z^3\right) \geq x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 \quad \Longleftrightarrow$$

$$\sum_{\text{sym}} x^3 \geq \sum_{\text{sym}} x^2y$$

The above inequality follows from Muirhead theorem or can be obtained adding the three inequalities

$$x^3 + y^3 \geq x^2y + xy^2, \qquad y^3 + z^3 \geq y^2z + yz^2, \qquad z^3 + x^3 \geq z^2x + zx^2$$

□

136. ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) ($x+y+z = 3, x^2+y^2+z^2 = 7, x, y, z > 0$)

$$1 + \frac{6}{xyz} \geq \frac{1}{3}\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)$$

Solution. (*Ercole Suppa*)
From the constraints $x+y+z=3$, $x^2+y^2+z^2=7$ follows that

$$9=(x+y+z)^2=7+2(xy+yz+zx) \quad\Longrightarrow\quad xy+yz+zx=1$$

The required inequality is equivalent to

$$\begin{aligned}
&3xyz+18\geq x^2y+y^2z+z^2x && \Longleftrightarrow\\
&3xyz+6(x+y+x)(xy+yz+zx)\geq x^2y+y^2z+z^2x && \Longleftrightarrow\\
&21xyz+5\left(x^2y+y^2z+z^2x\right)+6\left(x^2z+xy^2+yz^2\right)\geq 0
\end{aligned}$$

which is true for all $x,y,z>0$. □

137. ($\sqrt{WURZEL}$**, Šefket Arslanagić**) $(a,b,c>0)$

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\geq\frac{3\,(a+b+c)}{a+b+c+3}.$$

Solution. (*Ercole Suppa*)
We can assume WLOG that $a+b+c=1$. The required inequality is equivalent to

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\geq\frac{3}{4} \qquad (\star)$$

From Cauchy-Scwarz inequality we have

$$1=(a+b+c)^2\leq\left(\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\right)[a(b+1)+b(c+1)+c(a+1)]$$

Thus, by using the well-know inequality $(a+b+c)^2\geq 3(ab+bc+ca)$, we get

$$\begin{aligned}
\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}&\geq\frac{1}{ab+bc+ca+a+b+c}=\\
&=\frac{1}{ab+bc+ca+1}\geq\\
&\geq\frac{1}{\frac{1}{3}+1}=\frac{3}{4}
\end{aligned}$$

and $(\star)$ is proven. □

138. (**[ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva**) $(a,b,c>0)$

$$a^2+b^2+c^2+2abc+3\geq(1+a)(1+b)(1+c)$$

Solution. (*See* [4], *pag.* 75, *problem* 74)
Let $f(a,b,c) = a^2+b^2+c^2+2abc+3-(1+a)(1+b)(1+c)$. We have to prove that all values of f are nonnegative. If $a,b,c > 3$, then we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1 \quad \Longrightarrow \quad ab+bc+ca < abc$$

hence

$$\begin{aligned} f(a,b,c) &= a^2+b^2+c^2+abc+2-a-b-c-ab-bc-ca > \\ &> a^2+b^2+c^2+2-a-b-c > 0 \end{aligned}$$

So, we may assume that $a \le 3$ and let $m = \frac{b+c}{2}$. Easy computations show that

$$f(a,b,c) - f(a,m,m) = \frac{(3-a)(b-c)^2}{4} \ge 0$$

and so it remains to prove that

$$f(a,m,m) \ge 0 \quad \Longleftrightarrow \quad (a+1)m^2 - 2(a+1)m + a^2 - a + 2 \ge 0$$

This is cleary true, because the discriminant of the quadratic equation is

$$\Delta = -4(a+1)(a-1)^2 \le 0$$

□

139. (**Gazeta Matematicã**) $(a,b,c > 0)$

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \ge a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}$$

Solution. (*See* [32], *pag.* 43) We obtain the chain of equalities and inequalities

$$\begin{aligned} \sum_{\text{cyc}} \sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyc}} \sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)} \ge \\ &\ge \frac{1}{\sqrt{2}} \sum_{\text{cyc}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) = && \text{(Cauchy-Schwarz)} \\ &= \frac{1}{\sqrt{2}} \sum_{\text{cyc}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right) \ge \\ &\ge \sqrt{2} \sum_{\text{cyc}} \sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} \ge && \text{(AM-GM)} \\ &\ge \sqrt{2} \sum_{\text{cyc}} \sqrt{a^4+\frac{a^2bc}{2}} = && \text{(Cauchy-Schwarz)} \\ &= \sum_{\text{cyc}} \sqrt{2a^4+a^2bc} \end{aligned}$$

□

140. (**C**[2]**2362, Mohammed Aassila**) $(a, b, c > 0)$

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

Solution. (*Crux Mathematicorum 1999, pag. 375, n.2362*) We use the well-know inequality $t + 1/t \geq 2$ for $t > 0$. Equality occurs if and only if $t = 1$. Note that

$$\frac{1+abc}{a(1+b)} = \frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} - 1$$
$$\frac{1+abc}{b(1+c)} = \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} - 1$$
$$\frac{1+abc}{c(1+a)} = \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a} - 1$$

Then

$$\frac{1+abc}{a(1+b)} + \frac{1+abc}{b(1+c)} + \frac{1+abc}{c(1+a)} \geq 2 + 2 + 2 - 3 = 3$$

by the above inequality. Equality holds when

$$\frac{1+a}{a(1+b)} = \frac{1+b}{b(1+c)} = \frac{1+c}{c(1+a)} = 1$$

that is, when $a = b = c = 1$. $\square$

141. (**C2580**) $(a, b, c > 0)$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab}$$

Solution. (*Crux Mathematicorum 2001, pag. 541, n.2580*)
Let $D = abc\left(a^2+bc\right)\left(b^2+ac\right)\left(c^2+ab\right)$. Clearly $D > 0$ and

$$\begin{aligned}
&\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{b+c}{a^2+bc} - \frac{c+a}{b^2+ac} + \frac{a+b}{c^2+ab} = \\
=&\frac{a^4b^4 + b^4c^4 + c^4a^4 - a^4b^2c^2 - b^4c^2a^2 - c^4a^2b^2}{D} = \\
=&\frac{\left(a^2b^2 - b^2c^2\right)^2 + \left(b^2c^2 - c^2a^2\right)^2 + \left(c^2a^2 - a^2b^2\right)^2}{2D} \geq 0
\end{aligned}$$

which shows that the given inequality is true. Equality holds if and only if $a = b = c$. $\square$

[2]CRUX with MAYHEM

142. (**C2581**) $(a, b, c > 0)$

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c$$

Solution. (*Crux Mathematicorum 2001, pag. 541, n.2581*)
Let $D = (a+b)(b+c)(c+a)$. Clearly $D > 0$. We show that the difference between the left-hand side and the right-hand side of the inequality is nonnegative.

$$\begin{aligned}
&\frac{a^2+bc}{b+c}-a+\frac{b^2+ca}{c+a}-b+\frac{c^2+ab}{a+b}-c= \\
=&\frac{a^2+bc-ab-ac}{b+c}+\frac{b^2+ac+ab-bc}{a+c}+\frac{c^2+ab-ac-bc}{a+b}= \\
=&\frac{(a-b)(a-c)}{b+c}+\frac{(b-a)(b-c)}{a+c}+\frac{(c-a)(c-b)}{a+b}= \\
=&\frac{\left(a^2-b^2\right)\left(a^2-c^2\right)+\left(b^2-a^2\right)\left(b^2-c^2\right)+\left(c^2-a^2\right)\left(c^2-b^2\right)}{D}= \\
=&\frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{D}= \\
=&\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{2D} \geq 0
\end{aligned}$$

Equality holds if and only if $a = b = c$. □

143. (**C2532**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 3+\frac{2(a^3+b^3+c^3)}{abc}$$

Solution. (*Crux Mathematicorum 2001, pag. 221, n.2532*)
We have

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-3-\frac{2(a^3+b^3+c^3)}{abc}= \\
=&\frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2}-3-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)= \\
=&a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)= \\
=&a^2\left(\frac{1}{b}-\frac{1}{c}\right)^2+b^2\left(\frac{1}{c}-\frac{1}{a}\right)^2+c^2\left(\frac{1}{a}-\frac{1}{b}\right)^2 \geq 0
\end{aligned}$$

Equality holds if and only if $a = b = c$. □

144. (**C3032, Vasile Cirtoaje**) ($a^2+b^2+c^2=1,\ a,b,c>0$)

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\le\frac{9}{2}$$

Solution. (*Crux Mathematicorum 2006, pag. 190, problem 3032*)
Note first that the given inequality is equivalent to

$$3-5(ab+bc+ca)+7abc(a+b+c)-9a^2b^2c^2\ge 0$$
$$3-5(ab+bc+ca)+6abc(a+b+c)+abc(a+b+c-9abc)\ge 0 \qquad (1)$$

By the AM-GM inequality we have

$$a+b+c-9abc=(a+b+c)\left(a^2+b^2+c^2\right)-9abc\ge$$
$$\ge 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}-9abc=0 \qquad (2)$$

On the other hand,

$$\begin{aligned}
&3-5(ab+bc+ca)+6abc(a+b+c)=\\
=&3\left(a^2+b^2+c^2\right)^2-5(ab+bc+ca)\left(a^2+b^2+c^2\right)+6abc(a+b+c)=\\
=&3\left(a^4+b^4+c^4\right)+6\left(a^2b^2+b^2c^2+c^2a^2\right)+abc(a+b+c)\\
&-5\left[ab\left(a^2+b^2\right)+bc\left(b^2+c^2\right)+ca\left(c^2+a^2\right)\right]=\\
=&\left[2\sum a^4+6\sum a^2b^2-4\sum ab\left(a^2+b^2\right)\right]+\\
&+\left[a^4+b^4+c^4+abc(a+b+c)-ab\left(a^2+b^2\right)-bc\left(b^2+c^2\right)-ca\left(c^2+a^2\right)\right]=\\
=&\left[(a-b)^4+(b-c)^4+(c-a)^4\right]+\\
&+a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\ge 0 \qquad (3)
\end{aligned}$$

since

$$(a-b)^4+(b-c)^4+(c-a)^4\ge 0$$

and

$$a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\ge 0$$

is the well-knom Schur's inequality. Now (1) follows from (2) and (3). The equality holds if and only if $a=b=c=\sqrt{3}/3$. □

145. (**C2645**) ($a,b,c>0$)

$$\frac{2(a^3+b^3+c^3)}{abc}+\frac{9(a+b+c)^2}{(a^2+b^2+c^2)}\ge 33$$

First Solution. (*Darij Grinberg - ML Forum*)
Equivalently transform our inequality:

$$\frac{2\left(a^3+b^3+c^3\right)}{abc}+\frac{9\left(a+b+c\right)^2}{a^2+b^2+c^2}\geq 33 \Longleftrightarrow$$
$$\left(\frac{2\left(a^3+b^3+c^3\right)}{abc}-6\right)+\left(\frac{9\left(a+b+c\right)^2}{a^2+b^2+c^2}-27\right)\geq 0 \Longleftrightarrow$$
$$2\frac{a^3+b^3+c^3-3abc}{abc}+9\frac{\left(a+b+c\right)^2-3\left(a^2+b^2+c^2\right)}{a^2+b^2+c^2}\geq 0$$

Now, it is well-known that

$$a^3+b^3+c^3-3abc=\left(a+b+c\right)\left(a^2+b^2+c^2-bc-ca-ab\right)$$

and

$$\left(a+b+c\right)^2-3\left(a^2+b^2+c^2\right)=-2\left(a^2+b^2+c^2-bc-ca-ab\right),$$

so the inequality above becomes

$$2\frac{\left(a+b+c\right)\left(a^2+b^2+c^2-bc-ca-ab\right)}{abc}+9\frac{-2\left(a^2+b^2+c^2-bc-ca-ab\right)}{a^2+b^2+c^2}\geq 0$$

Now, according to the well-known inequality $a^2+b^2+c^2\geq bc+ca+ab$, we have $a^2+b^2+c^2-bc-ca-ab\geq 0$, so that we can divide this inequality by $a^2+b^2+c^2-bc-ca-ab$ to obtain

$$\begin{aligned}
&2\frac{a+b+c}{abc}+9\frac{-2}{a^2+b^2+c^2}\geq 0\\
\Longleftrightarrow\quad &2\frac{a+b+c}{abc}-\frac{2\cdot 9}{a^2+b^2+c^2}\geq 0\\
\Longleftrightarrow\quad &\frac{a+b+c}{abc}\geq\frac{9}{a^2+b^2+c^2}\\
\Longleftrightarrow\quad &\left(a+b+c\right)\left(a^2+b^2+c^2\right)\geq 9abc
\end{aligned}$$

But this is evident, since AM-GM yields $a+b+c\geq 3\sqrt[3]{abc}$ and $a^2+b^2+c^2\geq 3\sqrt[3]{a^2b^2c^2}$, so that $\left(a+b+c\right)\left(a^2+b^2+c^2\right)\geq 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}=9abc$.
Proof complete. $\square$

Second Solution. (*Crux Mathematicorum 2002, pag. 279, n.2645*)
On multiplying by the common denominator and performing the necessary calculations, we have that the given inequality is equivalent to

$$2\left(a^3+b^3+c^3\right)\left(a^2+b^2+c^2\right)+9abc(a+b+c)^2-33abc\left(a^2+b^2+c^2\right)\geq 0$$

The left side of this is the product of

$$a^2+b^2+c^2-ab-bc-ca \tag{1}$$

and

$$2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right) \tag{2}$$

The product of (1) and (2) is nonnegative because

$$a^2+b^2+c^2-ab-bc-ca=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\geq 0$$

and (by AM-GM)

$$\begin{aligned}&2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right)\geq\\ \geq&2\left(9\sqrt[9]{a^9b^9c^9}-9abc\right)=0\end{aligned}$$

Equality holds if and only if $a=b=c$. $\square$

Remark. In order to prove that (2) is positive we can use also the S.O.S method (=sum of squares):

$$\begin{aligned}&2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right)=\\ =&(a-b)^2(a+b+3c)+(b-c)^2(b+c+3a)+(c-a)^2(c+a+3b)\geq 0\end{aligned}$$

$\square$

146. $(x,y\in\mathbb{R})$

$$-\frac{1}{2}\leq\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\leq\frac{1}{2}$$

First Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$\begin{aligned}-\left(1+x^2\right)\left(1+y^2\right)\leq 2(x+y)(1-xy)\leq\left(1+x^2\right)\left(1+y^2\right)\quad\Longleftrightarrow\\ -(x+y)^2-(1-xy)^2\leq 2(x+y)(1-xy)\leq(x+y)^2+(1-xy)^2\end{aligned}$$

which is true by the well-know inequalitie $a^2+b^2\pm 2ab\geq 0$. $\square$

Second Solution. (*See* [25], pag. 185, n.79)
Let

$$\vec{a}=\left(\frac{2x}{1+x^2},\frac{1-x^2}{1+x^2}\right)\quad,\quad\vec{b}=\left(\frac{1-y^2}{1+y^2},\frac{2y}{1+y^2}\right)$$

Then it is easy to verify that $|\vec{a}|=|\vec{b}|=1$. The Cauchy-Schwarz $|\vec{a}\cdot\vec{b}|\leq|\vec{a}|\cdot|\vec{b}|$ inequality implies that

$$|\vec{a}\cdot\vec{b}|=\left|2\cdot\frac{x\left(1-y^2\right)+y\left(1-x^2\right)}{\left(1+x^2\right)\left(1+y^2\right)}\right|=\left|2\cdot\frac{(x+y)(1-xy)}{\left(1+x^2\right)\left(1+y^2\right)}\right|\leq 1$$

Dividing by 2, we get the result. $\square$

147. $(0 < x, y < 1)$
$$x^y + y^x > 1$$

Solution. (*See* [25], *pag.* 198, *n.* 66) First we prove the following lemma:

LEMMA. If u, x are real numbers such that $u > 0$, $0 < x < 1$, we have
$$(1+u)^x < 1 + ux$$

PROOF. Let $f(u) = 1 + xu - (1+u)^x$. We have $f(0) = 0$ and f is increasing in the interval $]0, 1[$ because
$$f'(u) = x - x(1+u)^{x-1} = x\left[1 - \frac{1}{(1+u)^{1-x}}\right] > x > 0$$
Thus $f(u) > 0$ for all $x \in \mathbb{R}$ and the lemma is proved. $\square$

Now, the given inequality can be proved in the following way:

Let $x = \frac{1}{1+u}$, $y = \frac{1}{1+v}$, $u > 0$, $v > 0$. Then, by the LEMMA, we have
$$x^y = \frac{1}{(1+u)^y} > \frac{1}{1+uy} = \frac{1+v}{1+u+v}$$
$$y^x > \frac{1}{(1+v)^x} > \frac{1}{1+vx} = \frac{1+u}{1+u+v}$$
Thus
$$x^y + y^x > \frac{1+v}{1+u+v} + \frac{1+u}{1+u+v} = 1 + \frac{1}{1+u+v} > 1$$
and the inequality is proven. $\square$

148. $(x, y, z > 0)$
$$\sqrt[3]{xyz} + \frac{|x-y| + |y-z| + |z-x|}{3} \geq \frac{x+y+z}{3}$$

Solution. (*Ercole Suppa*)
We can assume WLOG that $x \leq y \leq z$. Let a, b, c be three real numbers such that $x = a$, $y = a + b$, $z = a + b + c$ con $a > 0$, $b, c \geq 0$. The required inequality

is equivalent to:

$$\sqrt[3]{a(a+b)(a+b+c)} + \frac{b+c+b+c}{3} \geq \frac{3a+2b+c}{3} \quad \Longleftrightarrow$$

$$3\sqrt[3]{a(a+b)(a+b+c)} \geq 3a-c \quad \Longleftrightarrow$$

$$54a^2b + 54a^2c + 27ab^2 + 27abc - 9ac^2 + c^3 \geq 0 \quad \Longleftrightarrow$$

$$54a^2b + 27ab^2 + 27abc + \left(54a^2c + c^3 - 9ac^2\right) \geq 0$$

The above inequality is satisfied for all $a > 0$, $b, c \geq 0$ since AM-GM inequality yields

$$54a^2c + c^3 \geq 2\sqrt{54a^2c^4} = 6\sqrt{6}ac^2 \geq 9ac^2$$

$\square$

149. $(a, b, c, x, y, z > 0)$

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) By generalized Holder inequality we have

$$\sum_{i=1}^{n} a_i b_i c_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} c_i^r\right)^{\frac{1}{r}}$$

which is true for all $p, q, r \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. After setting $a_1 = a^{\frac{1}{3}}, b_1 = b^{\frac{1}{3}}, c_1 = c^{\frac{1}{3}}$ e $a_2 = x^{\frac{1}{3}}, b_2 = y^{\frac{1}{3}}, c_2 = z^{\frac{1}{3}}$ we get:

$$\begin{aligned} \sqrt[3]{abc} + \sqrt[3]{xyz} &= \sum a_i b_i c_i \leq \\ &\leq \left(\sum a_i^3\right)^{\frac{1}{3}} \left(\sum b_i^3\right)^{\frac{1}{3}} \left(\sum c_i^3\right)^{\frac{1}{3}} = \\ &= \sqrt[3]{(a+x)(b+y)(c+z)} \end{aligned}$$

$\square$

150. $(x, y, z > 0)$

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1$$

Solution. (*Walther Janous, see* [4], *pag.* 49, *problem* 37)
We have

$$(x+y)(x+z)=xy+\left(x^2+yz\right)+xz\geq xy+2x\sqrt{yz}+xz=\left(\sqrt{xy}+\sqrt{xz}\right)^2$$

Hence

$$\sum\frac{x}{x+\sqrt{(x+y)(x+z)}}\leq\sum\frac{x}{x+\sqrt{xy}+\sqrt{xz}}=$$
$$=\sum\frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}=1$$

and the inequality is proved. □

151. $(x+y+z=1,\ x, y, z > 0)$

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}}\geq\sqrt{\frac{3}{2}}$$

First Solution. (*Ercole Suppa*)
The function $f(t)=\frac{t}{\sqrt{1-t}}$ is convex on $]0,1[$ because

$$f''(t)=\frac{4-t}{4(1-t)^{\frac{5}{2}}}\geq 0$$

Then by Jensen inequality

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{1}{3}\right)\quad\Longleftrightarrow$$
$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}}\geq\sqrt{\frac{3}{2}}$$

□

Second Solution. (*Ercole Suppa*) After setting $a = \sqrt{1-x}$, $b = \sqrt{1-y}$, $c = \sqrt{1-z}$, we have $0 < a, b, c < 1$, $a^2 + b^2 + c^2 = 2$ and the required inequality is equivalent to:

$$\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-b^2}{c} \geq \sqrt{\frac{3}{2}} \quad \Longleftrightarrow$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{\frac{3}{2}} + a + b + c \tag{$\star$}$$

From Cauchy-Schwarz inequality we have

$$2 = a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \quad \Longrightarrow \quad a + b + c \leq 2\sqrt{\frac{3}{2}} \tag{1}$$

From AM-HM inequality we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c} \geq \frac{9}{2}\sqrt{\frac{2}{3}} = 3\sqrt{\frac{3}{2}} \tag{2}$$

By adding (1) and (2) we get ($\star$) and the result is proven. □

Third Solution. (*Campos - ML Forum*) Assume WLOG that $x \geq y \geq z$. Then

$$\frac{1}{\sqrt{1-x}} \geq \frac{1}{\sqrt{1-y}} \geq \frac{1}{\sqrt{1-z}}$$

and, from Chebyshev and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
\sum \frac{x}{\sqrt{1-x}} &\geq \frac{\sum x \cdot \sum \frac{1}{\sqrt{1-x}}}{3} = \\
&= \frac{1}{3} \cdot \sum \frac{1}{\sqrt{1-x}} = \\
&= \frac{1}{3} \cdot \frac{9}{\sum \sqrt{1-x}} \geq \\
&\geq \frac{3}{\sqrt{3 \cdot \sum (1-x)}} = \\
&= \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}
\end{aligned}$$

□

Remark.
The inequality can be generalized in the following way (India MO 1995):

If $x_1, x_2, ..., x_n$ are n real positive numbers such that $x_1 + x_2 + x_3 + ... + x_n = 1$ the following inequality holds

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + ... + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

152. $(a, b, c \in \mathbb{R})$

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

Solution. (*Ercole Suppa*) After setting $a + b + c = t$, from the Minkowski inequality we have:

$$\begin{aligned}
&\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \\
\geq &\sqrt{(a+b+c)^2 + (3-a-b-c)^2} = \\
= &\sqrt{t^2 + (3-t)^3} \geq \frac{3\sqrt{2}}{2}
\end{aligned}$$

The last step is true since

$$\sqrt{t^2 + (3-t)^3} \geq \frac{3\sqrt{2}}{2} \quad \Longleftrightarrow \quad t^2 + (3-t)^2 \geq \frac{9}{2} \quad \Longleftrightarrow \quad (2t-3)^2 \geq 0$$

$\square$

153. $(a, b, c > 0)$

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

First Solution. (*Ercole Suppa*)
We have:

$$\begin{gathered}
\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2} \quad \Longleftrightarrow \\
a^2 - ab + b^2 + b^2 - bc + c^2 + 2\sqrt{(a^2 - ab + b^2)(b^2 - bc + c^2)} \geq a^2 + ac + c^2 \quad \Longleftrightarrow \\
2\sqrt{(a^2 - ab + b^2)(b^2 - bc + c^2)} \geq ab + bc + ac - 2b^2 \quad \Longleftrightarrow \\
4(a^2 - ab + b^2)(b^2 - bc + c^2) \geq \left(ab + bc + ac - 2b^2\right)^2 \quad \Longleftrightarrow \\
3(ab - ac + bc)^2 \geq 0
\end{gathered}$$

and we are done. $\square$

Second Solution. (*Albanian Eagle - ML Forum*)
This inequality has a nice geometric interpretation:
let O, A, B, C be four points such that $\angle AOB = \angle BOC = 60^\circ$ and $OA = a$, $OB = b$, $OC = c$ then our inequality is just the triangle inequality for $\triangle ABC$.

Remark. The idea of second solution can be used to show the following inequality (given in a Singapore TST competition):

Let a, b, c be real positive numbers. Show that

$$c\sqrt{a^2 - ab + b^2} + a\sqrt{b^2 - bc + c^2} \geq b\sqrt{a^2 + ac + c^2}$$

Proof. By using the same notations of second solution, the required inequality is exactly the Tolomeo inequality applied to the quadrilateral $OABC$. $\square$

Third Solution. (*Lovasz - ML Forum*)
We have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} = \sqrt{\left(\frac{a}{2} - b\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2} + \sqrt{\left(b - \frac{c}{2}\right)^2 + \left(\frac{c\sqrt{3}}{2}\right)^2}$$

In Cartesian Coordinate, let the two vectors $\left(\frac{a}{2} - b, \frac{b\sqrt{3}}{2}\right)$ and $\left(b - \frac{c}{2}, \frac{c\sqrt{3}}{2}\right)$. Then

$$\vec{a} + \vec{b} = \left(\frac{a - c}{2}, \frac{(a + c)\sqrt{3}}{2}\right).$$

Now use $\|a\| + \|b\| \geq \|a + b\|$, we get:

$$\begin{aligned}\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} &\geq \sqrt{\frac{(a - c)^2}{4} + \frac{3(a + c)^2}{4}} = \\ &= \sqrt{a^2 + ac + c^2}\end{aligned}$$

$\square$

154. $(xy + yz + zx = 1,\ x, y, z > 0)$

$$\frac{x}{1 + x^2} + \frac{y}{1 + y^2} + \frac{z}{1 + z^2} \geq \frac{2x(1 - x^2)}{(1 + x^2)^2} + \frac{2y(1 - y^2)}{(1 + y^2)^2} + \frac{2z(1 - z^2)}{(1 + z^2)^2}$$

Solution. (*See* [25], pag.185, n.89)
After setting $x = \tan\alpha/2$, $y = \tan\beta/2$, $z = \tan\gamma/2$, by constraint $xy+yz+zx = 1$ follows that

$$\begin{aligned}\tan\frac{\gamma}{2} = \frac{1-xy}{x+y} = \frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{1}{\tan\frac{\alpha+\beta}{2}} = \\ = \cot\frac{\alpha+\beta}{2} = \tan\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)\end{aligned}$$

Thus $\alpha+\beta+\gamma = \pi$, so we can assume that α, β, γ are the angles of a triangle. The required inequality is equivalent to

$$\cos\alpha\,\mathrm{sen}\,\alpha + \cos\beta\,\mathrm{sen}\,\beta + \cos\gamma\,\mathrm{sen}\,\gamma \le \frac{\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma}{2}$$
$$\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma \le \mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma \tag{1}$$

By sine law, using the common notations, we have

$$\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma = \frac{a+b+c}{2R} = \frac{2s}{2R} = \frac{sr}{Rr} = \frac{S}{rR} \tag{2}$$

If x, y, z are the distances of circumcenter O fromi BC, CA, AB we have

$$\begin{aligned}\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma = 2\left(\mathrm{sen}\,\alpha\cos\alpha + \mathrm{sen}\,\beta\cos\beta + \mathrm{sen}\,\gamma\cos\gamma\right) = \\ = \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{R} = \\ = \frac{a\cdot\frac{x}{R} + b\cdot\frac{y}{R} + c\cdot\frac{z}{R}}{R} = \frac{2S}{R^2}\end{aligned} \tag{3}$$

From (2), (3) and Euler inequality $R \ge 2r$ we get

$$\frac{\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma}{\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma} = \frac{R}{2r} \ge 1$$

and (1) is proven. □

155. $(x, y, z \ge 0)$

$$xyz \ge (y+z-x)(z+x-y)(x+y-z)$$

Solution. (*See* [32], *pag.* 2)
The inequality follows from Schur's inequality because

$$\begin{aligned}xyz - (y+z-x)(z+x-y)(x+y-z) = \\ = x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \ge 0\end{aligned}$$

The equality hols if and only if $x = y = z$ or $x = y$ and $z = 0$ and cyclic permutations. □

156. $(a, b, c > 0)$

$$\sqrt{ab(a+b)} + \sqrt{bc(b+c)} + \sqrt{ca(c+a)} \geq \sqrt{4abc + (a+b)(b+c)(c+a)}$$

Solution. (*Ercole Suppa*) Squaring both members with easy computations we get that the required inequality is equivalent to:

$$a\sqrt{bc(a+b)(a+c)} + b\sqrt{ac(b+a)(b+c)} + c\sqrt{ab(c+a)(c+b)} \geq 3abc$$

which is true by AM-GM inequality:

$$\begin{aligned} & a\sqrt{bc(a+b)(a+c)} + b\sqrt{ac(b+a)(b+c)} + c\sqrt{ab(c+a)(c+b)} \geq \\ \geq & 3\sqrt[3]{(abc)^2(a+b)(b+c)(c+a)} \geq \\ \geq & 3\sqrt[3]{8(abc)^3} = 6abc \geq 3abc \end{aligned}$$

$\square$

157. (**Darij Grinberg**) $(x, y, z \geq 0)$

$$\left(\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)}\right) \cdot \sqrt{x+y+z} \geq 2\sqrt{(y+z)(z+x)(x+y)}$$

First Solution. (*Darij Grinberg - ML Forum*)
Consider the triangle with sides $a = y+z$, $b = z+x$, $c = x+y$ and semiperimeter $s = \frac{a+b+c}{2} = x+y+z$. Then, our inequality becomes

$$\left(\sqrt{(s-a)a} + \sqrt{(s-b)b} + \sqrt{(s-c)c}\right) \cdot \sqrt{s} \geq 2\sqrt{abc}$$

or

$$\sqrt{\frac{s(s-a)}{bc}} + \sqrt{\frac{s(s-b)}{ca}} + \sqrt{\frac{s(s-c)}{ab}} \geq 2$$

If we call A, B, C the angles of our triangle, then this simplifies to

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq 2$$

i. e.

$$\sin\left(90^\circ - \frac{A}{2}\right) + \sin\left(90^\circ - \frac{B}{2}\right) + \sin\left(90^\circ - \frac{C}{2}\right) \geq 2$$

But $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$ and $90^\circ - \frac{C}{2}$ are the angles of an acute triangle (as one can easily see); hence, we must show that if A, B, C are the angles of an acute triangle, then

$$\sin A + \sin B + \sin C \geq 2$$

(Actually, for any non-degenerate triangle, $sinA+sinB+sinC>2$, but I don't want to exclude degenerate cases.) Here is an elegant proof of this inequality by Arthur Engel: Since triangle ABC is acute, we have $A-B\leq C$, and $\cos\frac{A-B}{2}\geq\cos\frac{C}{2}$, so that

$$\begin{aligned}\sin A+\sin B&=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=\\&=2\cos\frac{C}{2}\cos\frac{A-B}{2}\geq\\&\geq 2\cos^2\frac{C}{2}=1+\cos C\end{aligned}$$

and

$$\sin A+\sin B+\sin C\geq 1+\cos C+\sin C\geq 2$$

Hereby, we have used the very simple inequality $\cos C+\sin C\geq 1$ for any acute angle C.
(I admit that I did not find the proof while trying to solve the problem, but I rather constructed the problem while searching for a reasonable application of the $\sin A+\sin B+\sin C\geq 2$ inequality, but this doesn't matter afterwards...)

□

Second Solution. (*Harazi - ML Forum*)
Take $x+y+z=1$. Square the inequality

$$\sum\sqrt{x(1-x)}\geq 2\cdot\sqrt{(1-x)(1-y)(1-z)}$$

and reduce it to

$$\sum xy-2xyz\leq\sum\sqrt{xy(y+z)(z+x)}$$

But

$$\sum xy-2xyz\leq\sum xy$$

and

$$\sum\sqrt{xy(x+z)(y+z)}\geq\sum xy+\sum z\cdot\sqrt{xy}$$

□

Third Solution. (*Zhaobin, Darij Grinberg - ML Forum*)
We have

$$\sqrt{\frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}}\geq\frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}$$

then we get:

$$\sum \sqrt{\frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}} \geq \sum \frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)} =$$

$$= 2\frac{\left(y+z\right)\left(z+x\right)\left(x+y\right)+xyz}{\left(y+z\right)\left(z+x\right)\left(x+y\right)} \geq 2$$

□

158. (**Darij Grinberg**) $(x, y, z > 0)$

$$\frac{\sqrt{y+z}}{x}+\frac{\sqrt{z+x}}{y}+\frac{\sqrt{x+y}}{z} \geq \frac{4\left(x+y+z\right)}{\sqrt{\left(y+z\right)\left(z+x\right)\left(x+y\right)}}.$$

Solution. (*See* [54], *pag.* 18)
By Cauchy, we have $\sqrt{(a+b)(a+c)} \geq a+\sqrt{bc}$. Now,

$$\sum \frac{\sqrt{b+c}}{a} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}} \quad \Longleftrightarrow$$

$$\sum \frac{b+c}{a}\sqrt{(a+b)(a+c)} \geq 4(a+b+c)$$

Substituting our result from Cauchy, it would suffice to show

$$\sum (b+c)\frac{\sqrt{bc}}{a} \geq 2(a+b+c)$$

Assume WLOG $a \geq b \geq c$, implying $b+c \leq c+a \leq a+b$ and $\frac{\sqrt{bc}}{a} \leq \frac{\sqrt{ca}}{b} \leq \frac{\sqrt{ab}}{c}$.
Hence, by Chebyshev and AM-GM,

$$\sum (b+c)\frac{\sqrt{bc}}{a} \geq \frac{(2(a+b+c))\left(\frac{\sqrt{bc}}{a}+\frac{\sqrt{ca}}{b}+\frac{\sqrt{ab}}{c}\right)}{} \geq 2(a+b+c)$$

as desidered. □

159. (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2\left(b+c\right)}{\left(b^2+c^2\right)\left(2a+b+c\right)}+\frac{b^2\left(c+a\right)}{\left(c^2+a^2\right)\left(2b+c+a\right)}+\frac{c^2\left(a+b\right)}{\left(a^2+b^2\right)\left(2c+a+b\right)} > \frac{2}{3}.$$

Solution. (*Zhaobin - ML Forum*)
Just notice

$$(b+c)(a^2+bc) = ba^2+ca^2+b^2c+bc^2 = b(a^2+c^2)+c(a^2+b^2)$$

then let $x = a\left(b^2+c^2\right)$, $y = \frac{b}{a^2+c^2}$, $z = \frac{c}{a^2+b^2}$. The given inequality is equivalent to the well-know Nesbitt inequality.

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y} \geq \frac{3}{2}$$

□

160. (**Darij Grinberg**) $(a,b,c>0)$

$$\frac{a^2}{2a^2+(b+c)^2}+\frac{b^2}{2b^2+(c+a)^2}+\frac{c^2}{2c^2+(a+b)^2}<\frac{2}{3}.$$

Solution. (*Darij Grinberg - ML Forum*) The inequality in question,

$$\sum \frac{a^2}{2a^2+(b+c)^2}<\frac{2}{3}$$

rewrites as

$$\frac{2}{3}-\sum \frac{a^2}{2a^2+(b+c)^2}>0$$

But

$$\begin{aligned}
&\frac{2}{3}-\sum \frac{a^2}{2a^2+(b+c)^2}= \\
=&\frac{2}{3}\cdot\sum \frac{a}{a+b+c}-\sum \frac{a^2}{2a^2+(b+c)^2}= \\
=&\sum\left(\frac{2}{3}\cdot\frac{a}{a+b+c}-\frac{a^2}{2a^2+(b+c)^2}\right)= \\
=&\sum \frac{a\left(b+c-a\right)^2+a\left(b+c\right)\left(b+c-a\right)}{3\left(a+b+c\right)\left(2a^2+(b+c)^2\right)}= \\
=&\sum \frac{a\left(b+c-a\right)^2}{3\left(a+b+c\right)\left(2a^2+(b+c)^2\right)}+\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{3\left(a+b+c\right)\left(2a^2+(b+c)^2\right)}
\end{aligned}$$

Now, it is obvious that

$$\sum \frac{a\left(b+c-a\right)^2}{3\left(a+b+c\right)\left(2a^2+(b+c)^2\right)}\geq 0$$

What remains to be proven is the inequality

$$\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{3\left(a+b+c\right)\left(2a^2+\left(b+c\right)^2\right)} > 0$$

which simplifies to

$$\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} > 0$$

Now,

$$\begin{aligned}
\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} &= \sum \frac{ab\left(b+c-a\right)+ca\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} = \\
&= \sum \frac{ab\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} + \sum \frac{ca\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} = \\
&= \sum \frac{bc\left(c+a-b\right)}{2b^2+\left(c+a\right)^2} + \sum \frac{bc\left(a+b-c\right)}{2c^2+\left(a+b\right)^2} = \\
&= \sum bc\left(\frac{c+a-b}{2b^2+\left(c+a\right)^2} + \frac{a+b-c}{2c^2+\left(a+b\right)^2}\right) = \\
&= \sum bc\frac{a\left(\left(a+b+c\right)^2+a^2+2bc\right)+\left(b+c\right)\left(b-c\right)^2}{\left(2b^2+\left(c+a\right)^2\right)\left(2c^2+\left(a+b\right)^2\right)} > 0
\end{aligned}$$

$\square$

161. (**Vasile Cirtoaje**) $(a, b, c \in \mathbb{R})$

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$

Solution. (*Darij Grinberg - ML Forum*)
Vasile Cartoaje established his inequality

$$\left(a^2+b^2+c^2\right)^2 \geq 3\left(a^3b+b^3c+c^3a\right)$$

using the identity

$$\begin{aligned}
&4\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)\left(\left(a^2+b^2+c^2\right)^2-3\left(a^3b+b^3c+c^3a\right)\right) = \\
&= \left(\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(b^2a+c^2b+a^2c\right)\right)^2 + \\
&\quad + 3\left(\left(a^3+b^3+c^3\right)-\left(a^2b+b^2c+c^2a\right)-2\left(b^2a+c^2b+a^2c\right)+6abc\right)^2
\end{aligned}$$

Actually, this may look a miracle, but there is a very natural way to find this identity. In fact, we consider the function

$$g\left(a,b,c\right)=\left(a^2+b^2+c^2\right)^2-3\left(a^3b+b^3c+c^3a\right)$$

over all triples $(a,b,c)\in\mathbb{R}^3$. We want to show that this function satisfies $g(a,b,c)\geq 0$ for any three reals a, b, c. Well, fix a triple (a,b,c) and translate it by some real number d; in other words, consider the triple $(a+d,b+d,c+d)$. For which $d\in\mathbb{R}$ will the value $g(a+d,b+d,c+d)$ be minimal? Well, minimizing $g(a+d,b+d,c+d)$ is equivalent to minimizing $g(a+d,b+d,c+d)-g(a,b,c)$ (since (a,b,c) is fixed), but

$$\begin{aligned}
&g\left(a+d,b+d,c+d\right)-g\left(a,b,c\right)=\\
=&d^2\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)+\\
&+d\left(\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(b^2a+c^2b+a^2c\right)\right)
\end{aligned}$$

so that we have to minimize a quadratic function, what is canonical, and it comes out that the minimum is achieved for

$$d=-\frac{\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(b^2a+c^2b+a^2c\right)}{2\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)}$$

So this is the value of d such that $g(a+d,b+d,c+d)$ is minimal. Hence, for this value of d, we have $g(a,b,c)\geq g(a+d,b+d,c+d)$. Thus, in order to prove that $g(a,b,c)\geq 0$, it will be enough to show that $g(a+d,b+d,c+d)\geq 0$. But, armed with the formula

$$d=-\frac{\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(b^2a+c^2b+a^2c\right)}{2\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)}$$

and with a computer algebra system or a sufficient patience, we find that

$$\begin{aligned}
&g\left(a+d,b+d,c+d\right)=\\
=&\frac{3\left(\left(a^3+b^3+c^3\right)-\left(a^2b+b^2c+c^2a\right)-2\left(b^2a+c^2b+a^2c\right)+6abc\right)^2}{4\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)}
\end{aligned}$$

what is incontestably ≥ 0. So we have proven the inequality. Now, writing

$$g\left(a,b,c\right)=g\left(a+d,b+d,c+d\right)-\left(g\left(a+d,b+d,c+d\right)-g\left(a,b,c\right)\right)$$

and performing the necessary calculations, we arrive at Vasc's mystic identity. $\square$

A Classical Inequalities

Theorem 1. (**AM-GM inequality**)
Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 2. (**Weighted AM-GM inequality**)
Let $\lambda_1, \cdots, \lambda_n$ real positive numbers with $\lambda_1 + \cdots + \lambda_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\lambda_1 \cdot x_1 + \cdots + \lambda_n \cdot x_n \geq {x_1}^{\lambda_1} \cdots {x_n}^{\lambda_n}.$$

Theorem 3. (**GM-HM inequality**)
Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have

$$\sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Theorem 4. (**QM-AM inequality**)
Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n}$$

Theorem 5. (**Power Mean inequality**)
Let $x_1, \cdots, x_n > 0$. The power mean of order p is defined by

$$M_0\left(x_1, x_2, \ldots, x_n\right) = \sqrt[n]{x_1 \cdots x_n},$$

$$M_p\left(x_1, x_2, \ldots, x_n\right) = \left(\frac{x_1^p + \cdots + {x_n}^p}{n}\right)^{\frac{1}{p}} \qquad (p \neq 0).$$

Then the function $M_p\left(x_1, x_2, \ldots, x_n\right) : \mathbb{R} \to \mathbb{R}$ is continuous and monotone increasing.

Theorem 6. (**Rearrangement inequality**)
Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

Theorem 7. (**The Cauchy[3]-Schwarz[4]-Bunyakovsky[5] inequality**)
Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Remark. This inequality apparently was firstly mentioned in a work of A.L. Cauchy in 1821. The integral form was obtained in 1859 by V.Y. Bunyakovsky. The corresponding version for inner-product spaces obtained by H.A. Schwartz in 1885 is mainly known as Schwarz's inequality. In light of the clear historical precedence of Bunyakovsky's work over that of Schwartz, the common practice of referring to this inequality as CS-*inequality* may seem unfair. Nevertheless in a lot of modern books the inequality is named CSB-*inequality* so that both Bunyakovsky and Schwartz appear in the name of this fundamental inequality.

By setting $a_i = \frac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$ the CSB inequality takes the following form

Theorem 8. (**Cauchy's inequality in Engel's form**)
Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Then,

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}$$

Theorem 9. (**Chebyshev's inequality[6]**)
Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 10. (**Hölder's inequality[7]**)
Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

[3]Louis Augustin Cauchy (1789-1857), french mathematician
[4]Hermann Amandus Schwarz (1843-1921), german mathematician
[5]Viktor Yakovlevich Bunyakovsky (1804-1889), russian mathematician
[6]Pafnuty Lvovich Chebyshev (1821-1894), russian mathematician.
[7]Otto Ludwig Hölder (1859-1937), german mathematician

Theorem 11. (**Minkowski's inequality**[8])
If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ and $p > 1$, then

$$\left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {y_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}}$$

Definition 1. (**Convex functions.**)
We say that a function $f(x)$ is convex on a segment $[a,b]$ if for all $x_1, x_2 \in [a,b]$

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

Theorem 12. (**Jensen's inequality**[9])
Let $n \geq 2$ and $\lambda_1, \ldots, \lambda_n$ be nonnegative real numbers such that $\lambda_1 + \cdots + \lambda_n = 1$. If $f(x)$ is convex on $[a,b]$ then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n x_n$$

for all $x_1, \ldots, x_n \in [a,b]$.

Definition 2. (**Majorization relation for finite sequences**)
Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two (finite) sequences of real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. We say that the sequence a majorizes the sequence b and we write

$$a \succ b \qquad \text{or} \qquad b \prec a$$

if the following two conditions are satisfyied

(i) $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k$, *for all k, $1 \leq k \leq n-1$;*

(ii) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$.

Theorem 13. (**Majorization inequality | Karamata's inequality**[10])
Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a,b]$. Then, we obtain

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

[8] Hermann Minkowski (1864-1909), german mathematician.
[9] Johan Ludwig William Valdemar Jensen (1859-1925), danish mathematician.
[10] Jovan Karamata (1902-2967), serbian mathematician.

Theorem 14. (**Muirhead's inequality**[11] | **Bunching Principle**)
If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are two nonincreasing sequences of nonnegative real numbers such that a majorizes b, then we have

$$\sum_{sym} x_1^{a_1} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} \cdots x_n^{b_n}$$

where the sums are taken over all $n!$ permutations of variables $x_1, x_2, \ldots, x_n$.

Theorem 15. (**Schur's inequality**[12])
Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyc}} x^r(x-y)(x-z) \geq 0.$$

Remark. The case $r = 1$ of Schur's inequality is

$$\sum_{\text{sym}} \left(x^3 - 2x^2y + xyz\right) \geq 0$$

By espanding both the sides and rearranging terms, each of following inequalities is equivalent to the $r = 1$ case of Schur's inequality

- $x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$
- $xyx \geq (x+y-z)(y+z-x)(z+x-y)$
- $(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx)$

Theorem 16. (**Bernoulli's inequality**[13])
For all $r \geq 1$ and $x \geq -1$, we have

$$(1+x)^r \geq 1 + rx.$$

Definition 3. (**Symmetric Means**)
For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t+x_1)\cdots(t+x_n)$ is called the i-th elementary symmetric function σ_i. This means that

$$(t+x_1)\cdots(t+x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1}t + \sigma_n.$$

[11] Robert Muirhead (1860-1941), english matematician.

[12] Issai Schur (1875-1941), was Jewish a mathematician who worked in Germany for most of his life. He considered himself German rather than Jewish, even though he had been born in the Russian Empire in what is now Belarus, and brought up partly in Latvia.

[13] Jacob Bernouilli (1654-1705), swiss mathematician founded this inequality in 1689. However the same result was exploited in 1670 by the english mathematician Isaac Barrow.

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 17. (**Newton's inequality**[14])
Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have

$$S_i^2 \geq S_{i-1} \cdot S_{i+1}$$

Theorem 18. (**Maclaurin's inequality**[15])
Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n}$$

[14]Sir Isaac Newton (1643-1727), was the greatest English mathematician of his generation. He laid the foundation for differential and integral calculus. His work on optics and gravitation make him one of the greatest scientists the world has known.

[15]Colin Maclaurin (1698-1746), Scottish mathematican.

B Bibliography and Web Resources

References

[1] M. Aassila, *300 défis mathématiques*, Ellipses (2001)

[2] R. Alekseyev, L. Kurlyandchik, *The sum of minima and the minima of the sums*, Quantum, January (2001)34-36

[3] T. Andreescu, E. Bogdan, *Mathematical Olympiad Treasures*, Birkhauser, Boston (2004)

[4] T. Andreescu, V. Cîrtoaje, G. Dospinescu, M. Lascu, *Old and New Inequalities*, GIL Publishing House, Zalau, Romania (2004)

[5] T. Andreescu, K. Kedlaya, *Mathematical Contests 1995-1996*, American Mathematics Competitions (1997)

[6] T. Andreescu, K. Kedlaya, *Mathematical Contests 1996-1997*, American Mathematics Competitions (1998)

[7] T. Andreescu, Z. Feng, *Mathematical Olympiads From Around the World 1998-1999*, Mathematical Association of America (2000)

[8] T. Andreescu, Z. Feng, *Mathematical Olympiads From Around the World 1999-2000*, Mathematical Association of America (2002)

[9] T. Andreescu, Z. Feng, P.S. Loh, *USA and International Mathematical Olympiads 2001*, Mathematical Association of America (2002)

[10] T. Andreescu, Z. Feng, P.S. Loh, *USA and International Mathematical Olympiads 2003*, Mathematical Association of America (2003)

[11] T. Andreescu, Z. Feng, P.S. Loh, *USA and International Mathematical Olympiads 2004*, Mathematical Association of America (2004)

[12] *Art of Problem Solving*, http://www.artofproblemsolving.com

[13] I. Boreico, *An original method of proving inequalities*, Mathematical Reflections N.3(2006),http://reflections.awesomemath.org/

[14] I. Boreico, I. Borsenco *A note on the breaking point of a simple inequality*, Mathematical Reflections N.5(2006), http://reflections.awesomemath.org/

[15] P. Bornsztein, *Inégalités Classiques*, http://shadowlord.free.fr/server/Maths/Olympiades/Cours/Inegalites/inegalites.pdf

[16] O. Bottema, R. Z̃. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen 1969

[17] V. Q. B. Can *On a class of three-variable inequalities*, Mathematical Reflections N.2(2007), http://reflections.awesomemath.org/

[18] E. Carneiro *As desigualdades do Rearranjo e Chebychev*, (2004) www.ma.utexas.edu/users/ecarneiro/RearranjoChebychev.pdf

[19] V. Cîrtoaje, *Algebraic Inequalities*, GIL Publishing House, Zalau, Romania (2006)

[20] M.A.A. Cohen, R.V. Milet *Contas com desigualdades*, Eureka! N.23 (2006) http://www.obm.org.br/eureka/eureka23.pdf

[21] B.N.T.Cong, N.V. Tuan, N.T. Kien *The SOS-Schur method*, Mathematical Reflections N.5(2007), http://reflections.awesomemath.org/

[22] D. A. Holton, *Inequalities*, University of Otago, Problem Solving Series, Booklet No.12 (1990)

[23] D. Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium*, Springer-Verlag, New York (2006)

[24] S. Dvoryaninov, E. Yasinovyi *Obtaining symmetric inequalities*, Quantum, november(1998)44-48

[25] A. Engel, *Problem-Solving Startegies*, Springer-Verlag, New York (1998)

[26] R.T. Fontales, *Trigonometria e desigualdades em problemas de olimpiadas*, Eureka! N.11 (2001), http://www.obm.org.br/eureka/eureka11.pdf

[27] G. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge (1999)

[28] L. K. Hin, *Muirhead Inequality*, Mathematical Excalibur, Vol.11, N.1 (2006) http://www.math.ust.hk/mathematical_excalibur/

[29] D. Hrimiuc, *The Rearrangement Inequality*, Π in the Sky, N. 2, december (2000), http://www.pims.math.ca/pi

[30] D. Hrimiuc, *Inequalities for Convex Functions (Parti I)*, Π in the Sky, N. 4, december (2001), http://www.pims.math.ca/pi

[31] D. Hrimiuc, *Inequalities for Convex Functions (Parti II)*, Π in the Sky, N. 5, september (2002), http://www.pims.math.ca/pi

[32] H. Lee, *Topics in Inequalities - Theorems and Techniques*, http://www.imomath.com

[33] H. Lee, *Inequalities Through Problems*, www.artofproblemsolving.com/Forum/viewtopic.php?mode=attach&id=8077

[34] P.K. Hung, *The entirely mixing variables method*, Mathematical Reflections N.5(2006), http://reflections.awesomemath.org/

[35] P. K. Hung *On the AM-GM inequality*, Mathematical Reflections N.4(2007), http://reflections.awesomemath.org/

[36] P. K. Hung *Secrets in Inequalities, vol.I*, GIL Publishing House, Zalau, Romania (2007)

[37] K. J. Li, *Rearrangement Inequality*, Mathematical Excalibur, Vol.4, N.3 (1999), http://www.math.ust.hk/mathematical_excalibur/

[38] K. J. Li, *Jensen's Inequality*, Mathematical Excalibur, Vol.5, N.4 (2000) http://www.math.ust.hk/mathematical_excalibur/

[39] K. J. Li, *Majorization Inequality*, Mathematical Excalibur, Vol.5, N.4 (2000), http://www.math.ust.hk/mathematical_excalibur/

[40] K. J. Li, *Using Tangent Lines to Prove Inequalities*, Mathematical Excalibur, Vol.10, N.5 (2005), http://www.math.ust.hk/mathematical_excalibur/

[41] Z.Kadelburg, D. Dukić, M. Lukić, I. Matić, *Inequalities of Karamata, Schur and Muirhead, and some applications*, http://elib.mi.sanu.ac.yu/pages/browse_journals.php?

[42] K. S. Kedlaya, $A < B$, http://www.unl.edu/amc/a-activities/a4-for-students/s-index.html

[43] M. S. Klamikin *USA Mathematical Olympiads 1972-1986*, Mathematical Association of America (1988)

[44] P.P. Korovkin, *Media aritmetica, media geometrica y otras medias* http://www.rinconmatematico.com.ar

[45] P.P. Korovkin, *Desigualdad de Bernoulli* http://www.rinconmatematico.com.ar

[46] P.P. Korovkin, *Medias potenciales* http://www.rinconmatematico.com.ar

[47] P.P. Korovkin, *Parte entera del numero* http://www.rinconmatematico.com.ar

[48] J. Herman, R. Kučera, J. Šimša, *Equations and Inequalities*, Springer-Verlag, New York (2000)

[49] M. E. Kuczma, *144 problems of the austrian-polish matematics competition 1978-1993*, The Academic Distribution Center, Freeland, Maryland (1994)

[50] C. Lupu, C. Pohoata *About a nice inequality*, Mathematical Reflections N.1(2007), http://reflections.awesomemath.org/

[51] S. Malikic, *Inequalities with product condition*, Mathematical Excalibur, Vol.12, N.4 (2007), http://www.math.ust.hk/mathematical_excalibur/

[52] I. Matic, *Classical inequalities*, The Imo Compendium, Olympiad Training Materials (2007), http://www.imocompendium.com/

[53] *MathLinks*, http://www.mathlinks.ro

[54] T. J. Mildorf, *Olympiad Inequalities*,http://web.mit.edu/tmildorf/www

[55] D. S. Mitrinović, P.M. Vasić *Analytic Inequalities*, Springer-Verlag, New York (1970)

[56] J. H. Nieto, *Desigualdades*, http://www.acm.org.ve/desigual.pdf

[57] A. C. M. Neto, *Desigualdades Elementares*, Eureka! N.5 (1999) http://www.obm.org.br/eureka/eureka5.pdf

[58] L. Pinter, I. Khegedysh *Ordered sets*, Quantum, July(1998)44-45

[59] I. Reiman, *International Mathematical Olympiad 1959-1999*, Anthem Press Publishing Company, Wimbledon (2001)

[60] N. Sato, *Tips on Inequalities*, Crux Matematicorum, (1998)161-167

[61] S. Savchev, T. Andreescu, *Matematical Miniatures*, Mathematical Association of America, (2003)

[62] *John Scholes WEB site*, http://www.kalva.demon.co.uk/

[63] A. Slinko, *Algebraic inequalities*, www.nzamt.org.nz/mapbooklet/Inequalities.pdf

[64] J. M. Steele, *The Cauchy Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Mathematical Association of America, (2004)

[65] T. B. Soulami, *Les olympiades de mathématiques*, Ellipses Editions, (1999)

[66] P. J. Taylor, *Tournament of towns, 1993-1997*, Australian Mathematics Trust Pubblication, (1998)

[67] P. V. Tuan, *Square it !*, Mathematical Excalibur, Vol.12, N.5 (2007) http://www.math.ust.hk/mathematical_excalibur/

[68] P.V. Tuan, T.V. Hung *Proving inequalities using linear functions*, Mathematical Reflections N.4(2006), http://reflections.awesomemath.org/

[69] V. Verdiyan, D.C. Salas, *Simple trigonometric substitutions with broad results*, Mathematical Reflections N.6(2007), http://reflections.awesomemath.org/

[70] G. West, *Inequalities for the Olympiad Enthusiast*, The South African Mathematical Society, (1996)