WINTER CAMP LECTURE NOTES

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Contents

Chapter 1

NUMBER THEORY

1.1 DIVISIBILITY

Definition 1. *An integer A is divisible by B ($B \neq 0$) if there exists an integer q such that $A = Bq$.*

Properties: For integers a, b, c, d, m, n and primes p, q we have:

1. $n \mid n$.
2. If $a \mid b$ and $b \mid c$ then $a \mid c$.
3. If $d \mid n$ and $d \mid m$ then $d \mid an + bm$.
4. If $a \mid b$ and $b \mid a$ then $|a| = |b|$.
5. $1 \mid n$. (1 divides any number.)
6. $n \mid 0$. (0 is divisible by any number.)
7. If $p \mid ab$ then $p \mid a$ or $p \mid b$.
8. If $p \mid a$ and $q \mid a$ then $pq \mid a$.

Theorem 1. *(Fermat's Little Theorem) If p is a prime number such that $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.*

Theorem 2. *(Fermat's Theorem) For any integer a and prime p we have $a^p \equiv a \pmod{p}$.*

1.1.1 EXERCISES

1. Find all integers such that $n + 2 \mid n^2 + 4$.
2. Show that if $(a - c) \mid (ab + cd)$ then $(a - c) \mid (ad + bc)$.
3. Prove that for any integer n,
   (a) $6 \mid n^3 + 5n$
   (b) $30 \mid n^5 - n$
   (c) $2730 \mid n^{13} - n$
4. Prove that $A = 1 + 3 + 3^2 + 3^3 + ... + 3^{119}$ is divisible by 520.
5. Find all integers n such that $(n - 1)^3 + n^3 + (n + 1)^3$ is divisible by 18.
6. Prove that if p is a prime number greater than 3, then $42p \mid 3^p - 2^p - 1$.
7. Prove that $101 \cdot 102 \cdot 103 \cdot ... \cdot 200$ is divisible by 2^{100} but not divisible by 2^{101}.
8. Find a if for every $k \neq 27$ we have $(27 - k) \mid (a - k^{2007})$.
9. Prove that if $k \mid (ae + b)$ and $k \mid (ce + d)$ then $k \mid (ad - bc)$.
10. Find all values of a, for which the equation $x^2 + y^2 = axy$ has solution in Z.
11. Let m and n be natural numbers. Prove that if $\frac{m^2+n^2}{mn}$ is a natural number then $m = n$.
12. Prove that $\frac{12n+1}{30n+2}$ is irreducible (can't be simplified).
13. Prove that for integer x the fraction $\frac{x^3+2x}{x^4+3x^2+1}$ can't be simplified.
14. Given that m and n are relatively prime numbers. Find all numbers by which the fraction $\frac{2n-m}{3n+2m}$ can be simplified.
15. Given that n is an integer greater than 6. Prove that if $n - 1$ and $n + 1$ are prime numbers then $n^2(n^2 + 16)$ is divisible by 720.
16. Prove that if k is a natural number then $2^{2k-1} + 2^k - 1$ is not divisible by 7.
17. Solve the equation $xy = 2(x + y)$ in the set of integers.
18. Solve the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ in the set of integers.
19. Solve the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$ in the set of natural numbers, where p is prime.

1.1.2 SOLUTIONS

1. Find all integers such that $n+2 \mid n^2+4$.

   Solution: We can write $n^2+4 = n^2-4+8 = (n-2)(n+2)+8$.
   So $n+2 \mid n^2+4 \Longrightarrow (n+2) \mid 8$. That is, $n+2 \in \mp 1, \mp 2, \mp 4, \mp 8$.
   Therefore, $n \in \{-10, -6, -4, -3, -1, 0, 2, 6\}$.

2. Show that if $(a-c) \mid (ab+cd)$ then $(a-c) \mid (ad+bc)$.

   Solution: Observe that, $ab+cd-(ad+bc) = (a-c)(b-d)$ which is divisible by $a-c$. Since $(a-c) \mid (ab+cd)$ therefore, $(a-c) \mid (ad+bc)$.

3. Prove that for any integer n,

   (a) $6 \mid n^3+5n$

   (b) $30 \mid n^5-n$

   (c) $2730 \mid n^{13}-n$

   Solution: (a) and (b) exercise.
   (c) Knowing that $2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ we must show that $n^{13}-n$ is divisible by 2,3,5,7 and 13.
   i) Since n^{13} and n have the same parity, $n^{13}-n$ is even. So $2 \mid n^{13}-n$.
   ii) $n^{13}-n = n(n^{12}-1)$ if $3 \mid n$ then $3 \mid n^{13}-n$. If not, by Fermat's little thm. $n^2 \equiv 1 \pmod 3$. So, $n^{12}-1 \equiv 0 \pmod 3$ which means that $3 \mid n^{13}-n$.
   iii) Similarly, if $5 \mid n$ then $5 \mid n^{13}-n$. If not, by Fermat's little thm. $n^4 \equiv 1 \pmod 5$. So, $n^{12}-1 \equiv 0 \pmod 5$ hence, $5 \mid n^{13}-n$.
   By doing the same for 7 and 13 we prove that $2730 \mid n^{13}-n$.

4. Prove that $A = 1+3+3^2+3^3+...+3^{119}$ is divisible by 520.

   Solution: Since $520 = 8 \cdot 5 \cdot 13$ it is sufficient to show that A is divisible by 13 and 40. There are 120 terms in A. If we group them 3 by 3(we will have 40 groups),

   $$A = (1+3+3^2) + 3^3(1+3+3^2) + ... + 3^{117}(1+3+3^2)$$

   we see that each of those 40 groups is divisible by 13. So, $13 \mid A$. And if we group them 4 by 4(there will be 30 groups),

   $$A = (1+3+3^2+3^3) + 3^4(1+3+3^2+3^3) + ... + 3^{116}(1+3+3^2+3^3)$$

   we see that, each group is divisible by 40. That is $40 \mid A$.
   Therefore, $520 \mid A$.

5. Find all integers n such that $(n-1)^3+n^3+(n+1)^3$ is divisible by 18.

   Solution: We will show that $18 \mid (n-1)^3+n^3+(n+1)^3$ for every even n. Firstly, we will show that for any $n \in Z$, $9 \mid (n-1)^3+n^3+(n+1)^3$. Since $(n-1)^3+n^3+(n+1)^3 = 3n(n^2+2)$,

if $3 \mid n$ then $9 \mid (n-1)^3+n^3+(n+1)^3$. If $3 \nmid n$ then by Fermat's little thm. we have $n^2 \equiv 1 \pmod 3$. Which means that $3 \mid n^2+2$ so $9 \mid 3n(n^2+2)$.
$i)$ If n is even, then $3n(n^2+2)$ is divisible by 2.
So, $18 \mid (n-1)^3+n^3+(n+1)^3$.
$ii)$ If n is odd, then n^2+2 is odd as well. In that case $18 \nmid 3n(n^2+2)$.
Hence, $(n-1)^3+n^3+(n+1)^3$ is divisible by 18 for only even integers n.

6. Prove that if p is a prime number greater than 3, then $42p \mid 3^p - 2^p - 1$.

   Solution: To show that $42p \mid 3^p - 2^p - 1$, we must show that $3^p - 2^p - 1$ is divisible by 2,3,7 and p.
   $i)$ It is obvious that $3^p - 2^p - 1$ is even.
   $ii)$ Since p is odd, $2^p \equiv -1 \pmod 3$ so $3^p - 2^p - 1 \equiv 0 \pmod 3$. Which means that $3 \mid 3^p - 2^p - 1$.
   $iii)$ Since p is prime, by Fermat's thm. we have $3^p \equiv 3 \pmod p$ and $2^p \equiv 2 \pmod p$. So $3^p - 2^p - 1 \equiv 0 \pmod p$. That is, $p \mid 3^p - 2^p - 1$.
   $iv)$ Now, it remains to show that $3^p - 2^p - 1$ is divisible by 7. Since $p > 3$ we can write $p = 6k \mp 1$ for some $k \in Z$. If $p = 6k+1$, then

   $$3^p-2^p-1 = 3^{6k+1}-2^{6k+1}-1 \equiv (3^6)^k \cdot 3-(2^6)^k \cdot 2-1 \equiv 3-2-1 = 0 \pmod 7.$$

   And if $p = 6k+5$, then

   $$3^p-2^p-1 = 3^{6k+5}-2^{6k+5}-1 \equiv (3^6)^k \cdot 3^5-(2^6)^k \cdot 2^5-1 \equiv 5-4-1 = 0 \pmod 7.$$

   (Note that $3^6 \equiv 2^6 \equiv 1 \pmod 7$ by Fermat's little thm.)
   Therefore, $42p \mid 3^p - 2^p - 1$.

7. Prove that $101 \cdot 102 \cdot 103 \cdot ... \cdot 200$ is divisible by 2^{100} but not divisible by 2^{101}.

   Solution:

   $$\begin{aligned} 101 \cdot 102 \cdot 103 \cdot ... \cdot 200 = \frac{200!}{100!} &= \frac{(1 \cdot 3 \cdot 5 \cdot ... \cdot 199)(2 \cdot 4 \cdot 6 \cdot ... \cdot 200)}{100!} \\ &= \frac{(1 \cdot 3 \cdot 5 \cdot ... \cdot 199)(2^{100} \cdot 100!)}{100!} \\ &= 2^{100} \cdot 1 \cdot 3 \cdot 5 \cdot ... \cdot 199 \end{aligned}$$

   Since all the numbers in the product $1 \cdot 3 \cdot 5 \cdot ... \cdot 199$ are odd, $101 \cdot 102 \cdot 103 \cdot ... \cdot 200$ is divisible by 2^{100} but not divisible by 2^{101}.

8. Find a if for every $k \neq 27$ we have $(27-k) \mid (a-k^{2007})$.

   Solution: We know that $(27-k) \mid (27^{2007} - k^{2007})$. So

   $$(27-k) \mid [(a-k^{2007}) - (27^{2007} - k^{2007})] = (a - 27^{2007})$$

   for every $k \neq 27$. Which means that $a - 27^{2007}$ has infinetely many divisors. This is possible if and only if, $a - 27^{2007} = 0$ so $a = 27^{2007}$.

9. Prove that if $k \mid (ae+b)$ and $k \mid (ce+d)$ then $k \mid (ad-bc)$.

   Solution: $k \mid (ae+b)$ and $k \mid (ce+d) \Longrightarrow k \mid [a(ce+d)-c(ae+b)] = (ad-bc)$.

10. Find all values of a, for which the equation $x^2+y^2 = axy$ has solution in Z.

    Solution: Let $gcd(x,y) = d$ then we can write $x = x_1 \cdot d$ and $y = y_1 \cdot d$ where $gcd(x_1, y_1) = 1$. If we substitute $x = x_1 \cdot d$ and $y = y_1 \cdot d$ in the given equation, we will get $x_1^2 + y_1^2 = ax_1y_1$. Since $x_1 \mid y_1^2$ and $y_1 \mid x_1^2$ we have $|x_1| = |y_1|$. Knowing that $gcd(x_1, y_1) = 1$, we get $x = y = \pm d$. So $a = \pm 2$.

11. Let m and n be natural numbers. Prove that if $\frac{m^2+n^2}{mn}$ is a natural number then $m = n$.

    Solution: Let $\frac{m^2+n^2}{mn} = k$ so $m^2+n^2 = kmn$. By the previous exercise, we get that $|m| = |n|$ and since m, n are natural numbers, we have $m = n$.

12. Prove that $\frac{12n+1}{30n+2}$ is irreducible (can't be simplified).

    Solution: Let $gcd(12n+1, 30n+2) = d$, then $d \mid 5(12n+1) - 2(30n+2) = 1$. So $d = 1$. Namely, $\frac{12n+1}{30n+2}$ is irreducible.

13. Prove that for integer x the fraction $\frac{x^3+2x}{x^4+3x^2+1}$ can't be simplified.

    First Solution: Assume that $\frac{x^3+2x}{x^4+3x^2+1}$ can be simplified by a number d and let p be a prime divisor of d. Then $p \mid x^4+3x^2+1$ and $p \mid x^3+2x = x(x^2+2)$ which implies that $p \mid x$ or $p \mid x^2+2$.
    $i)$ If $p \mid x$ then $x^4+3x^2+1 \equiv 1 \pmod{p}$, hence contradiction.
    $ii)$ If $p \mid x^2+2$ then $x^2 \equiv -2 \pmod{p}$ and
    $x^4+3x^2+1 \equiv 4-6+1 \equiv -1 \pmod{p}$, contradiction again.
    So, $\frac{x^3+2x}{x^4+3x^2+1}$ is irreducible.
    Second Solution:

$$\begin{aligned} gcd(x^3+2x, x^4+3x^2+1) &= gcd(x^3+2x, x^4+3x^2+1-x(x^3+2x)) \\ &= gcd(x^3+2x, x^2+1) \\ &= gcd(x^3+2x-x(x^2+1), x^2+1) \\ &= gcd(x, x^2+1) = 1. \end{aligned}$$

    Therefore, $\frac{x^3+2x}{x^4+3x^2+1}$ is irreducible.

14. Given that m and n are relatively prime numbers. Find all numbers by which the fraction $\frac{2n-m}{3n+2m}$ can be simplified.

    Solution: Let $2n-m = ad$ and $3n+2m = bd$, where $gcd(a,b) = 1$.

    Solving the system for m and n, we get $n = \frac{d(2a+3b)}{7}$ and $m = \frac{d(2b-3a)}{7}$. Since m and n are relatively prime, if $7 \nmid (2a+3b)$, $7 \nmid (2b-3a)$ then $\frac{d}{7} = 1 \Rightarrow d = 7$. Otherwise, $d = 1$ namely the fraction can't be simplified. So, the given fraction can only be simplified by 7.

15. Given that n is an integer greater than 6. Prove that if $n-1$ and $n+1$ are prime numbers then $n^2(n^2+16)$ is divisible by 720.

    Solution: We must show that $n^2(n^2+16)$ is divisible by 2^4,5 and 9.
    i) Since $n > 6$ and $n-1$, $n+1$ are prime numbers, n must be even.
    So, $n^2(n^2+16)$ is divisible by 16.
    ii) Knowing that, one of $n-1, n, n+1$ is divisible by 3, we get that n is divisible by 3, because $n-1$ and $n+1$ are primes. Therefore, $9 \mid n^2(n^2+16)$.
    iii) If $5 \mid n$, then $5 \mid n^2(n^2+16)$.
    If $5 \nmid n$, then one of $n-2, n+2$ is divisible by 5 ($n-1$, $n+1$ are prime numbers). So $5 \mid n^2-4 \Longrightarrow 5 \mid n^2+16$. So, $5 \mid n^2(n^2+16)$.
    Therefore, $720 \mid n^2(n^2+16)$.

16. Prove that if k is a natural number then $2^{2k-1}+2^k-1$ is not divisible by 7.

    Solution: Since $2^3 \equiv 1 \pmod 7$ we get that,

    $$\begin{aligned} 2^{3n} &\equiv 1 \pmod 7 \\ 2^{3n+1} &\equiv 2 \pmod 7 \\ 2^{3n+2} &\equiv 4 \pmod 7 \end{aligned}$$

    i) So if $k = 3n$, then

    $$2^{2k-1}+2^k-1 = 2^{6n-1}+2^{3n}-1 \equiv 4+1-1 \equiv 4 \pmod 7$$

    ii) if $k = 3n+1$, then

    $$2^{2k-1}+2^k-1 = 2^{6n+1}+2^{3n+1}-1 \equiv 2+2-1 \equiv 3 \pmod 7$$

    iii) if $k = 3n+2$, then

    $$2^{2k-1}+2^k-1 = 2^{6n+3}+2^{3n+2}-1 \equiv 1+4-1 \equiv 4 \pmod 7.$$

    So in any case $7 \nmid (2^{2k-1}+2^k-1)$.

17. Solve the equation $xy = 2(x+y)$ in the set of integers.

    Solution: The given equation is equivalent to,

    $$xy-2x-2y+4 = 4 \Longrightarrow (x-2)(y-2) = 4.$$

    By considering all divisors of 4, we get the following systems:

    $$\begin{cases} x-2 = \pm 1 \\ y-2 = \pm 4 \end{cases}$$
    $$\begin{cases} x-2 = \pm 2 \\ y-2 = \pm 2 \end{cases}$$
    $$\begin{cases} x-2 = \pm 4 \\ y-2 = \pm 1 \end{cases}$$

Solving the systems and checking the equation, we get:

$$(x, y) \in \{(3, 6), (4, 4), (6, 3), (1, -2), (0, 0), (-2, 1)\}.$$

18. Solve the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ in $\mathbb{Z}$.
Solution: The original equation is equivalent to, $xy = 3(x + y)$, where $x, y \neq 0$. Like the previous exercise, we can rewrite this equation as:

$$(x - 3)(y - 3) = 9.$$

So, all possible systems are:

$$\begin{cases} x - 3 = \pm 1 \\ y - 3 = \pm 9 \end{cases}, \begin{cases} x - 3 = \pm 3 \\ y - 3 = \pm 3 \end{cases}, and \begin{cases} x - 3 = \pm 9 \\ y - 3 = \pm 1 \end{cases}$$

Therefore, $(x, y) \in \{(-4, -12), (4, 12), (6, 6), (12, 4), (-12, -4)\}$.

19. Solve the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$ in the set of natural numbers, where p is prime.

Solution: Like the previous exercises, we transform the given equation into $(x - p)(y - p) = p^2$.
And we get the systems:

$$\begin{cases} x - p = 1 \\ y - p = p^2 \end{cases}, \begin{cases} x - p = p \\ y - p = p \end{cases}, and \begin{cases} x - p = p^2 \\ y - p = 1 \end{cases}$$

Hence, $(x, y) \in \left\{(p + 1, p^2 + p), (2p, 2p), (p^2 + p, p + 1)\right\}$.

1.2 PRIME AND COMPOSITE NUMBERS

Definition 2. *A natural number which has exactly two divisors (1 and itself) is called a prime number.*

Definition 3. *An integer that has more than two divsors is called a composite number.*

Fact: Any prime number greater than 3, can be written as $6k \mp 1$ for some integer k.

Theorem 3. *Let* $N = p_1^{e_1} \cdot {}_2^{e_2} \cdot \ldots \cdot {}_n^{e_n}$ *then the number of divisors of* N *is:* $(e_1+1)(e_2+1)...(e_n+1)$.

1.2.1 EXERCISES

1. Find all prime numbers p such that $p+2$ and $p+4$ are all primes.
2. Find all prime numbers p such that $2p^2+1$ is also prime.
3. Find all prime numbers p for which $2^p + p^2$ is also a prime.
4. Let n be an integer greater than 1. Prove that if the number $2^n + n^2$ is prime, then $n-3$ is divisible by 6.
5. Find all prime numbers p, q, r such that $p^q + q^p = r$.
6. Find all primes p, q, r satisfying $p^3 - q^3 = r$.
7. Find all prime numbers which can be represented both as sums and as differences of two prime numbers.
8. Find all primes p, q and natural numbers n such that
$$p \mid (q-1) \text{ and } q^n \mid (p^2-1).$$
9. Find the prime number p such that $2p^2+1 = k^5$, where $k \in Z$.
10. Let p be a prime greater than 5. Prove that the equation $x^4 + 4^x = p$ has no solution in Z.
11. Prove that if p is a prime greater than 3, then $24 \mid p^2 - 1$.
12. Find all primes p such that $p^2 + 239$ has less than 16 divisors.
13. Find all primes p such that $2p^4 - 7p^2 + 1$ is a perfect square.
14. Let p be an odd prime number. Show that, there exist infinitely many natural number n such that $2p^n + 3$ is composite.
15. Find all primes p and q such that $24 \nmid (q+1)$ and p^2q+1 is a perfect square.

1.2.2 SOLUTIONS

1. Find all prime numbers p such that $p+2$ and $p+4$ are all primes.

   Solution: If $p=3$, then $p+2=5$ and $p+4=7$ are also primes. If $p>3$ then $p=6k\mp 1$, for some $k\in Z$.
   $i)$ If $p=6k+1$, then $p+2=6k+3=3(2k+1)$ can't be a prime.
   $ii)$ If $p=6k-1$, then $p+4=6k+3=3(2k+1)$ can't be a prime.
   Therefore, the only such prime is $p=3$.

2. Find all prime numbers p such that $2p^2+1$ is also prime.

   Solution: If $p=2$, then $2p^2+1=9$ which is not a prime.
   If $p=3$, then $2p^2+1=19$ is a prime.
   If $p>3$, then $p=6k\mp 1$, for some $k\in Z$.
   So, $2p^2+1=2(36k^2\mp 12k+1)+1=3(24k^2\mp 8k+1)$ can't be prime.
   Therefore, the only solution is $p=3$.

3. Find all prime numbers p for which 2^p+p^2 is also a prime.

   Solution: If $p=2$ then $2^p+p^2=8$ is not a prime.
   If $p=3$ then $2^p+p^2=17$ is a prime.
   If $p>3$ then $p=6k\mp 1$ for some $k\in Z$.
   $i)$ For $p=6k+1$ we have $2^p+p^2=2^{6k+1}+(6k+1)^2\equiv 2+1\equiv 0 \pmod 3$.
   So 2^p+p^2 can't be prime.
   $ii)$ For $p=6k-1$ we have $2^p+p^2=2^{6k+5}+(6k+5)^2\equiv 2+1\equiv 0 \pmod 3$ can't be prime.
   Therefore, the only solution is $p=3$.

4. Let n be an integer greater than 1. Prove that if the number 2^n+n^2 is prime then, $n-3$ is divisible by 6.

   Solution: We will prove the statement by its contrapositive. That is, if $n\neq 6k+3$ then 2^n+n^2 can't be prime.
   If $n=6k,6k+2,6k+4$ then 2^n+n^2 is even and greater than 2, so can't be prime.
   If $n=6k+1$ then $2^n+n^2=2^{6k+1}+(6k+1)^2\equiv 2+1\equiv 0 \pmod 3$. Knowing that $n>1\Longrightarrow 2^n+n^2>3$ and $3\mid 2^n+n^2$ we have that 2^n+n^2 can't be prime.
   If $n=6k+5$ then $2^n+n^2=2^{6k+5}+(6k+5)^2\equiv 2+1\equiv 0 \pmod 3$ by the same reason 2^n+n^2 can't be prime.
   Therefore, if 2^n+n^2 is prime then $n=6k+3$ which means that $n-3$ is divisible by 6.

5. Find all prime numbers p,q,r such that $p^q+q^p=r$.

   Solution: If both p and q are odd, then r is even. So can't be prime. Therefore, one of p, q must be even. Since p, q are primes, we have that either $p=2$ or $q=2$. Now the problem is the same as the third problem. So $q=2,p=3,r=17$ or $p=2,q=3,r=17$.

6. Find all primes p, q, r satisfying $p^3 - q^3 = r$.

   Solution: We know that $r = p^3 - q^3 = (p - q)(p^2 + pq + q^2)$. Since $p^2 + pq + q^2 > p - q$ and r is prime, we have that $p - q = 1$. So, $p = 3$ and $q = 2$ implying that $r = 17$ which is also a prime, as desired.

7. Find all prime numbers which can be represented both as sums and as differences of two prime numbers.

   Solution: Let p be such a prime. Since p is the sum of two primes, it is greater than 2. So p is odd and one of the primes in the sum and in the difference is 2. So let $p = q + 2 = r - 2$ where, q and r are primes. Therefore, we have $p - 2, p, p + 2$ are all primes. This is possible only if $p = 5$ because one of three consecutive odd numbers is divisible by 3. Hence, the only such prime is 5.

8. Find all primes p, q and natural numbers n such that

$$p \mid (q - 1) \text{ and } q^n \mid (p^2 - 1).$$

   Solution: Since p, q are prime numbers $q - 1, p^2 - 1$ are positive. So

$$p \mid q - 1 \Longrightarrow p \leq q - 1 \tag{1.1}$$
$$q^n \mid p^2 - 1 \Longrightarrow q^n \leq p^2 - 1 \tag{1.2}$$

   By (1.1) and (1.2) we get, $p < q \Longrightarrow p^n < q^n < p^2 - 1$. So $n = 1$. Now we have, $q \mid (p^2 - 1)$. Which implies that $q \mid p - 1$ or $q \mid p + 1$.
   $i)$ If $q \mid p - 1$ then $q \leq p - 1 \leq q - 2$ by (1.1). Hence, contradiction.
   $ii)$ If $q \mid p + 1$ then $q \leq p + 1 \leq q$ by (1.1) again. So $p + 1 = q$. Since p, q are primes we have, $p = 2$ and $q = 3$.

9. Find the prime number p such that $2p^2 + 1 = k^5$, where $k \in Z$.

   Solution: Obviously k is odd. Substituting $k = 2n + 1$ in the equation, we get $2p^2 = (2n + 1)^5 - 1$. Which implies that $p^2 = n(k^4 + k^3 + k^2 + 1)$. Knowing that p is prime and $n < k^4 + k^3 + k^2 + 1$ we get that $n = 1$ and $k^4 + k^3 + k^2 + 1 = p^2$. Implying that $k = 3$ and $p = 11$.

10. Let p be a prime greater than 5. Prove that the equation $x^4 + 4^x = p$ has no solution in Z.

    Solution: Observe that if $x < 0$, then $x^4 + 4^x$ is not an integer. If $x = 0$ we have $p = 1$ which is not prime. If $x = 1$ then $p = 5$. Now let $x > 1$. It is obvious that x is odd. So let $x = 2n + 1$. We get that

$$\begin{aligned} p = x^4 + 4^x = x^4 + 4^{2n+1} &= x^4 + 4^{2n+1} + 2 \cdot x^2 \cdot 2^{2n+1} - 2 \cdot x^2 \cdot 2^{2n+1} \\ &= (x^2 + 2^{2n+1})^2 - (2^{n+1}x)^2 \\ &= (x^2 + 2^{2n+1} - 2^{n+1}x)(x^2 + 2^{2n+1} + 2^{n+1}x) \end{aligned}$$

    Since p is a product of two numbers greater than 1, it can't be prime.

11. Prove that if p is a prime greater than 3, then $24 \mid p^2 - 1$.

    Solution: Since p is odd, $p^2-1 = (2k+1)^2-1 = 4k^2+4k+1-1 = 4k(k+1)$ is divisible by 8. Also, by Fermat's Little Thm. $p^2 \equiv 1 \pmod 3$ so $p^2 - 1$ is divisible by 3. Therefore, if $p > 3$ then $24 \mid p^2 - 1$.

12. Find all primes p such that $p^2 + 239$ has less than 16 divisors.

    Solution: If $p = 2$ then $p^2 + 239 = 243 = 3^5$. Which has 6 divisors.
    If $p = 3$ then $p^2 + 239 = 2^3 \cdot 31$ has $(3+1)(1+1) = 8$ divisors.
    For $p > 3$ we have $24 \mid p^2 - 1$ by the previous problem. So let $p^2 - 1 = 24k$ then $p^2 + 239 = p^2 - 1 + 240 = 24k + 240 = 24(k+10) = 2^3 \cdot 3(k+10)$.
    If $k + 10$ contains another prime, the number of divisors of $p^2 + 239$, $N \geq (3+1)(1+1)(1+1) = 16$.
    If $k + 10 = 2^a 3^b$, then we will have one of these cases:

    $$\begin{cases} a \geq 2 \\ b \geq 1 \end{cases}, \quad \begin{cases} a \geq 1 \\ b \geq 2 \end{cases}, \quad \begin{cases} a \geq 4 \\ b \geq 0 \end{cases} \quad or \quad \begin{cases} a \geq 0 \\ b \geq 3 \end{cases}$$

    since $k + 10 > 10$. In any case, we have $N \geq 16$.
    So, $p = 2$ and $p = 3$ are the only solutions.

13. Find all primes p such that $2p^4 - 7p^2 + 1$ is a perfect square.

    Solution: If $p = 3$ then $2p^4 - 7p^2 + 1 = 100 = 10^2$.
    If $p \neq 3$, then by Fermat's little thm. $p^2 \equiv 1 \pmod 3$.
    So, $2p^4 - 7p^2 + 1 \equiv 2 - 7 + 1 \equiv 2 \pmod 3$. But $n^2 \equiv 0$ or $1 \pmod 3$.
    So, $p = 3$ is the only solution.

14. Let p be an odd prime number. Show that, there exist infinitely many natural number n such that $2p^n + 3$ is composite.

    Solution: *i*) If p is a prime ending with the digits 1,3,7,9 then

    $$p^{4k} \equiv 1 \pmod{10} \Longrightarrow 2p^{4k} + 3 \equiv 5 \pmod{10}.$$

    So, $2p^n + 3$ is divisible by 5 and obviously greater than 5 for every $n = 4k$.
    Hence, $2p^n + 3$ is composite.
    ii) If $p = 2$ then for $n = 4k$ we have, $2p^n + 3 = 2^{4k+1} + 3 \equiv 5 \pmod{10}$.
    Because of the same reason for every $n = 4k$, $2p^n + 3$ is composite.
    iii) If $p = 5$ then for $n = 6k + 4$ we have

    $$2p^n + 3 = 2 \cdot 5^n + 3 \equiv 2 \cdot 2 + 3 \equiv 0 \pmod 7.$$

    So, if $p = 5$ for every $n = 6k + 4$, we have $2p^n + 3$ is composite.
    Therefore, $2p^n + 3$ is composite for infinitely many natural number n.

15. Find all primes p and q such that $24 \nmid (q+1)$ and p^2q+1 is a perfect square.

Solution: $i)$ If one of p or q is even:

Since $x^2 = p^2q+1$, x is odd, ie. $x^2 \equiv 1 \pmod 8$

$$\begin{aligned} p^2q+1 &\equiv 1 \pmod 8 \\ p^2q &\equiv 0 \pmod 8 \end{aligned}$$

Because both p and q are primes, and 2 is the only even prime, $p = q = 2$.
$ii)$ If both $p, q > 2$ in that case $x > 2$

$$p^2q+1 = x^2 \implies p^2q = (x-1)(x+1)$$

Since x is even, $x-1$ and $x+1$ will be both odd, and they are relatively prime because $\gcd(x-1, x+1) = \gcd(x-1, 2) = 1$. So,

$$\begin{cases} x-1 = q \\ x+1 = p^2 \end{cases} \tag{1.3}$$

or

$$\begin{cases} x-1 = p^2 \\ x+1 = q \end{cases} \tag{1.4}$$

Since $x > 2$ and the $gcd(x-1, x+1) = 1$ the other cases are not possible.
(1.3) $\implies q+1 = p^2 - 1$
$i)$ If $p = 3$, we have solution $q = 7$.
$ii)$ If $p \neq 3$, then $p^2 \equiv 1 \pmod 3$ and $p^2 \equiv 1 \pmod 8$ which contradicts with the condition that $24 \nmid q+1$.
(1.4) $\implies p^2 + 2 = q$ and for $p = 3$, we have solution $q = 11$.
If $p \neq 3$, then $q = p^2 + 2 \equiv 0 \pmod 3$ which cannot be prime.
So, all the solutions are: $(p, q) \in \{(2, 2), (3, 7), (3, 11)\}$.

1.3 SQUARES AND CUBES

Theorem 4. *There is no integer m such that $a^n < m^n < (a+1)^n$, where a is integer.*

Theorem 5. *If n is an odd integer, then $n^2 \equiv 1 \pmod 8$.*

Proof: Let $n = 2k+1$ then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 4k(k+1)+1$. Since one of k or $k+1$ is even, we have $n^2 = 8l+1$. Therefore, $n^2 \equiv 1 \pmod 8$.

1.3.1 EXERCISES

1. Prove that $4^n + 2^n$ can't be a perfect square for any natural number n.
2. Show that $m^2 + m + 1$ can't be a perfect square for any natural number m.
3. Prove that $4x(x+1) = y(y+1)$ has no solutions in Z.
4. Find all natural numbers n such that $n^2 + 2n + 2007$ is a perfect square.
5. Find all natural numbers n such that $n^2 + 2n + 2009$ is a perfect square.
6. Prove that the number
$$\underbrace{100...05}_{2009} \cdot \underbrace{111...1}_{2008} + 1$$
is a perfect square.
7. Prove that the product of four consecutive natural numbers can't be a perfect square.
8. Is the number $1\underbrace{00...0}_{49}5\underbrace{00...0}_{99}1$ a perfect cube?
9. Solve the equation $x^6 + 3x^3 + 1 = y^4$ in the set of integers.
10. Let $a_1, a_2, ..., a_{2008}$ be natural numbers such that
$$a_1^2 + a_2^2 + ... + a_{2007}^2 = a_{2008}^2.$$
Prove that at least two of $a_1, a_2, ..., a_{2008}$ are even numbers.
11. Prove that $x^2 - 2y^2 + 8z = 3$ has no solutions in integers.
12. Prove that $p_1 \cdot p_2 \cdot ... \cdot p_n + 1$ can't be a perfect square for any n, where p_i is the ith prime.
13. Solve the equation $2^n + 7 = m^2$ in the set of integers.
14. Let n be an integer greater than 9. Find all $n \in N$ such that $n^2 - 15n + 55$ is a perfect square.

1.3.2 SOLUTIONS

1. Prove that $4^n + 2^n$ can't be a perfect square for any natural number n.

   Solution: Since $(2^n)^2 < 4^n + 2^n < 4^n + 2 \cdot 2^n + 1 = (2^n + 1)^2$, so $4^n + 2^n$ can't be a perfect square.

2. Show that $m^2 + m + 1$ can't be a perfect square for any natural number m.

   Solution: Similarly, $m^2 < m^2 + m + 1 < (m + 1)^2$, therefore $m^2 + m + 1$ is not a perfect square for any m.

3. Prove that $4x(x + 1) = y(y + 1)$ has no solutions in Z.

   Solution: $4x(x+1) = y(y+1) \Longrightarrow y(y+1)+1 = 4x(x+1)+1 = (2x+1)^2$ but by the previous exercise we know that $y^2 + y + 1$ can't be a perfect square. So, $4x(x + 1) = y(y + 1)$ has no solutions in Z.

4. Find all natural numbers n such that $n^2 + 2n + 2007$ is a perfect square.

   Solution: Let $n^2 + 2n + 2007 = m^2$ then we have $n^2 + 2n + 1 + 2006 = m^2$ implying that $m^2 - (n + 1)^2 = 2006$.
   By factoring we get, $(m - n - 1)(m + n + 1) = 2006$. Since, $m - n - 1$ and $m + n + 1$ have the same parity, we must find two divisors of 2006 with the same parity. But 2006 is divisible by 2, but not divisible by 4. So, it is impossible. Therefore, there is no such n.

5. Find all natural numbers n such that $n^2 + 2n + 2009$ is a perfect square.

   Solution: Similarly, let $n^2 + 2n + 2009 = m^2$ then

   $$\begin{aligned} n^2 + 2n + 1 + 2008 = m^2 &\Longleftrightarrow m^2 - (n + 1)^2 = 2008 \\ &\Longleftrightarrow (m - n - 1)(m + n + 1) = 2008. \end{aligned}$$

   Considering positive divisors of 2008 with the same parity, we have the following systems:

   $$\begin{cases} m - n - 1 = 2 \\ m + n + 1 = 1004 \end{cases} \text{or} \begin{cases} m - n - 1 = 4 \\ m + n + 1 = 502 \end{cases}$$

   Giving the solutions: $(m, n) \in \{(503, 500), (253, 248)\}$.

6. Prove that the number $\underbrace{100...05}_{2009} \cdot \underbrace{111...1}_{2008} + 1$ is a perfect square.

   Solution: See that $\underbrace{100...05}_{2009} = 10^{2008} + 5$ and $\underbrace{111...1}_{2008} = \frac{10^{2008}-1}{9}$.
   Let $10^{2008} = t$. So

   $$\underbrace{100...05}_{2009} \cdot \underbrace{111...1}_{2008} + 1 = (t + 5)\left(\frac{t - 1}{9}\right) + 1 = \frac{t^2 + 4t + 4}{9} = \left(\frac{t + 2}{3}\right)^2.$$

   And it is obvious that $\frac{10^{2008}+2}{3}$ is an integer.

7. Prove that the product of four consecutive natural numbers can't be a perfect square.

   Solution: Let A be the product of four consecutive natural numbers, then

$$\begin{aligned} A = (n-1)n(n+1)(n+2) &= (n-1)(n+2)n(n+1) \\ &= (n^2+n-2)(n^2+n) \\ &= (n^2+n-1)^2 - 1. \end{aligned}$$

   So, $(n^2+n-2)^2 < A < (n^2+n-1)^2$ can't be a perfect square.

8. Is the number $A = 1\underbrace{00...0}_{49}5\underbrace{00...0}_{99}1$ a perfect cube?

   Solution: See that $A = 1\underbrace{00...0}_{49}5\underbrace{00...0}_{99}1 = 100^{150} + 5 \cdot 10^{100} + 1$.

   Because $(10^{50}+1)^3 = 10^{150} + 3 \cdot 10^{100} + 3 \cdot 10^{50} + 1 < A < (10^{50}+2)^3$, A can't be perfect cube.

9. Solve the equation $x^6 + 3x^3 + 1 = y^4$ in the set of integers.

   Solution: $i)$ If $x > 0$, then

$$(x^3+1)^2 = x^6 + 2x^3 + 1 < x^6 + 3x^3 + 1 = y^4 < x^6 + 4x^3 + 4 = (x^3+2)^2$$

   which is impossible. So, there is no solution for $x > 0$.
   $ii)$ If $x < -2$ then

$$(x^3+2)^2 = x^6 + 4x^3 + 4 < x^6 + 3x^3 + 1 = y^4 < x^6 + 2x^3 + 1 = (x^3+1)^2.$$

   So, no solution for $x < -2$.
   Now it remains to check the cases $x = 0$ and $x = -1$. When $x = 0$ we have a solution $y = \mp 1$. When $x = -1$ there is no solution.
   Therefore, $(x, y) \in \{(0, -1)(0, 1)\}$.

10. Let $a_1, a_2, ..., a_{2008}$ be natural numbers such that

$$a_1^2 + a_2^2 + ... + a_{2007}^2 = a_{2008}^2.$$

   Prove that at least two of $a_1, a_2, ..., a_{2008}$ are even numbers.

   Solution: Assume contrary, namely suppose that exactly one of $a_1, a_2, ..., a_{2008}$ is even or none of $a_1, a_2, ..., a_{2008}$ are even.
   $i)$ If exactly one of $a_1, a_2, ..., a_{2008}$ is even, then we have $Even = Odd$, contradiction.
   $ii)$ If none of $a_1, a_2, ..., a_{2008}$ are even, that is, all are odd, then reducing modulo 8 (by thm.5), we have $7 \equiv 1 \pmod 8$. Contradiction again.
   Therefore, at least two of $a_1, a_2, ..., a_{2008}$ are even.

11. Prove that $x^2 - 2y^2 + 8z = 3$ has no solutions in integers.

    Solution: It is obvious that x must be odd. So by Theorem.5, we have $x^2 \equiv 1 \pmod 8$. If we reduce the equation modulo 8, we will have:
    $i)$ $1 \equiv 3 \pmod 8$ if y is even.
    $ii)$ $-1 \equiv 3 \pmod 8$ if y is odd. In either case we have contradiction.
    Therefore, $x^2 - 2y^2 + 8z = 3$ has no solutions in integers.

12. Prove that $p_1 \cdot p_2 \cdot ... \cdot p_n + 1$ can't be a perfect square for any n, where p_i is the ith prime.

    Solution: Assume that $p_1 \cdot p_2 \cdot ... \cdot p_n + 1 = m^2$ for some $m \in Z$, then we have $p_1 \cdot p_2 \cdot ... \cdot p_n = m^2 - 1$. Since m is odd by Theorem.5, we have $m^2 \equiv 1 \pmod 8$. So $m^2 - 1$ is divisible by 8. But $p_1 \cdot p_2 \cdot ... \cdot p_n$ is divisible by 2, but not by 4. So $8 \mid RHS$ but $8 \nmid LHS$, hence contradiction.
    Therefore, $p_1 \cdot p_2 \cdot ... \cdot p_n + 1$ can't be a perfect square.

13. Solve the equation $2^n + 7 = m^2$ in the set of integers.

    Solution: If $n < 0$, then $2^n + 7$ is not an integer. And we can easily check the cases for $n = 0, 1, 2$. For $n = 1$ there is a solution $m = \mp 3$.
    Now if $n > 2$, 2^n is divisible by 8 and since m is odd, $m \equiv 1 \pmod 8$ we get that $7 \equiv 1 \pmod 8$, contradiction.
    So, the only solution is $n = 1, m = \mp 3$.

14. Let n be an integer greater than 9. Find all $n \in N$ such that $n^2 - 15n + 55$ is a perfect square.

    Solution: Observe that $n^2 - 15n + 55 = (n-7)(n-8) - 1$ and $(n-8)^2 < (n-7)(n-8) - 1 < (n-7)^2$ for $n > 9$. Therefore, for any integer n, $n^2 - 15n + 55$ can't be a perfect square.

1.4 REPUNITS

Definition 4. *A number that contains only 1's as digits is called repunit, namely the numbers of the form* 11...11.

Fact: $\underbrace{111...11}_{n} = \frac{10^n-1}{9}$.

1.4.1 EXERCISES

1. Prove that $\underbrace{11...1}_{n}\underbrace{22...2}_{n}$ can be written as the product of two consecutive natural numbers.

2. Prove that $\underbrace{11...1}_{2n} - \underbrace{22...2}_{n}$ is a perfect square.

3. Let $A = \underbrace{44...4}_{2n}$ and $B = \underbrace{88...8}_{n}$. Show that $A + 2B + 4$ is a perfect square.

4. Let $N = \underbrace{11...1}_{n}\underbrace{22...2}_{n+1}5$. Show that N is a perfect square.

5. Find all $n \in N$ such that $\underbrace{11...1}_{n}$ is a perfect square.

6. Prove that the number of digits of a prime repunit is prime. Is the converse true?

7. Find the smallest $n \in N$ such that $\underbrace{11...1}_{n}$ is divisible by 13.

1.4.2 SOLUTIONS

1. Prove that $\underbrace{11...1}_{n}\underbrace{22...2}_{n}$ can be written as the product of two consecutive natural numbers.

   Solution: See that $\underbrace{11...1}_{n}\underbrace{22...2}_{n} = \underbrace{111...11}_{2n} + \underbrace{111...11}_{n}$.

   So,

$$\underbrace{11...1}_{n}\underbrace{22...2}_{n} = \frac{10^{2n}-1}{9} + \frac{10^{n}-1}{9} = (\frac{10^{n}-1}{9})(10^{n}+1+1)$$
$$= (\frac{10^{n}-1}{3})(\frac{10^{n}+2}{3}).$$
$$= \underbrace{333...3}_{n} \cdot \underbrace{333...34}_{n}.$$

2. Prove that $\underbrace{11...1}_{2n} - \underbrace{22...2}_{n}$ is a perfect square.

   Solution: Similarly,

$$\underbrace{11...1}_{2n} - \underbrace{22...2}_{n} = \frac{10^{2n}-1}{9} - 2\cdot\frac{10^{n}-1}{9} = \frac{10^{2n}-2\cdot 10^{n}+1}{9}$$
$$= (\frac{10^{n}-1}{3})^{2}.$$

   And obviously $\frac{10^{n}-1}{3}$ is an integer.

3. Let $A = \underbrace{44...4}_{2n}$ and $B = \underbrace{88...8}_{n}$. Show that $A + 2B + 4$ is a perfect square.

   Solution:

$$A + 2B + 4 = 4\cdot(\frac{10^{2n}-1}{9}) + 2\cdot 8\cdot(\frac{10^{n}-1}{9}) + 4$$
$$= \frac{4}{9}(10^{2n} - 1 + 4\cdot 10^{n} - 4 + 9)$$
$$= (\frac{2(10^{n}-1)}{3})^{2}.$$

   And $\frac{10^{n}-1}{3}$ is an integer.

4. Let $N = \underbrace{11...1}_{n}\underbrace{22...2}_{n+1}5$. Show that N is a perfect square.

   Solution: We can write $N = \underbrace{111...11}_{2n+2} + \underbrace{111...11}_{n+2} + 3$. So

$$N = \frac{10^{2n+2}-1}{9} + \frac{10^{n}-1}{9} + 3 = \frac{10^{2n+2} + \text{‘}0^{n+2} + 25}{9} = (\frac{10^{n+1}+5}{3})^{2}.$$

   And $\frac{10^{n+1}+5}{3}$ is an integer.

5. Find all $n \in N$ such that $\underbrace{11...1}_{n}$ is a perfect square.

   Solution: Let $\underbrace{11...1}_{n} = m^2$. If $n = 1$ there is a solution. For $n = 2$ there is no solution, because 11 is not a perfect square.
   If $n \geq 3$ we have $\underbrace{11...1}_{n} \equiv 7 \pmod 8$ but since m is odd by Thm.5, $m^2 \equiv 1 \pmod 8$. So for $n \geq 3$ there is no solution. The only solution is $n = 1$.

6. Prove that the number of digits of a prime repunit is prime. Is the converse true?

   Solution: We will show that if the number of digits of a repunit is not prime, then the repunit can't be prime.
   Let $N = \underbrace{11...1}_{n}$ and n be composite. Say $n = ab$. We have that

$$\begin{aligned} N = \frac{10^n - 1}{9} &= \frac{10^{ab} - 1}{9} \\ &= (\frac{10^a - 1}{9}) \cdot (10^{a(b-1)} + 10^{a(b-2)} + ... + 10^a + 1) \\ &= \underbrace{11...1}_{a} \cdot (10^{a(b-1)} + 10^{a(b-2)} + ... + 10^a + 1) \end{aligned}$$

   So, can't be prime. The converse is not true. 111 is repunit with 3 digits which is a prime but $111 = 3 \cdot 37$ which is not prime.

7. Find the smallest $n \in N$ such that $\underbrace{11...1}_{n}$ is divisible by 13.

   Solution: Since 13 and 9 are relatively prime, $13 \mid \frac{10^n - 1}{9} \Longrightarrow 13 \mid (10^n - 1)$.
   By Fermat's little theorem, we know that $10^{12} \equiv 1 \pmod{13}$.
   So if there is a smaller n for which $10^n \equiv 1 \pmod{13}$, then n must be a divisor of 12.
   By checking $n = 2, 3, 4, 6$ we see that only for $n = 6$ we have $10^6 \equiv 1 \pmod{13}$. So, $n = 6$ is the smallest for which $\underbrace{11...1}_{n}$ is divisible by 13.

Chapter 2

ALGEBRAIC EXPRESSIONS

2.1 SOME USEFUL IDENTITIES

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
2. $(a + b)^2 = (a - b)^2 + 4ab$
3. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
4. $a^2 - b^2 = (a - b)(a + b)$
5. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
6. $a^3 \pm b^3 = (a \pm b)^3 \mp 3ab(a \pm b)$
7. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1})$
8. $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + ... + ab^{n-2} - b^{n-1})$ if n is odd.
9. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
10. $(a + b + c)(ab + bc + ca) - abc = (a + b)(b + c)(c + a)$
11. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
12. $(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)$

2.2 EXERCISES

1. Let $a+b=3$ and $ab=1$. Then compute:
   (a) a^2+b^2
   (b) $\frac{1}{a}+\frac{1}{b}$
   (c) $\frac{1}{a^2}+\frac{1}{b^2}$
   (d) $\frac{a}{b}+\frac{b}{a}$
   (e) $\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}$
2. Let $x, y>0$. Compute $x+y$ if $x-y=\sqrt{5}$ and $xy=1$.
3. Compute x^2+y^2 if $\frac{x}{y}+\frac{y}{x}=4$ and $x-y=2$.
4. Given that $y-x=1$ and $\frac{2}{x}-\frac{2}{y}=1$. Compute $(x+y)^2$.
5. Given that $x+\frac{1}{x}=3$. Find
   (a) $x^2+\frac{1}{x^2}$
   (b) $(x-\frac{1}{x})^2$
   (c) $x^3+\frac{1}{x^3}$
   (d) $x^4+\frac{1}{x^4}$
   (e) $x^5+\frac{1}{x^5}$
6. Compute $a^4+b^4+c^4$ if $a+b+c=0$ and $a^2+b^2+c^2=1$.
7. Let a, b, c be real numbers such that $abc \neq 0$ and $a+b+c \neq 0$. Prove that if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ then $(a+b)(b+c)(c+a)=0$.
8. Let a, b, c be real numbers such that $abc \neq 0$ and $a+b+c \neq 0$. Prove that if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ then $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}$.
9. Prove each statement if $x+y+z=0$.
   (a) $x^3+y^3+z^3=3xyz$
   (b) $x^3+y^3+z^3+3(x+y)(y+z)(z+x)=0$
   (c) $x^4+y^4+z^4=2(x^2y^2+y^2z^2+z^2x^2)$
   (d) $2(x^4+y^4+z^4)=(x^2+y^2+z^2)^2$
   (e) $\frac{x^7+y^7+z^7}{7}=\frac{x^3+y^3+z^3}{3}\cdot\frac{x^4+y^4+z^4}{2}$
   (f) $\frac{x^7+y^7+z^7}{7}\cdot\frac{x^3+y^3+z^3}{3}=\left(\frac{x^5+y^5+z^5}{5}\right)^2$.
10. Compute $x^{2009}+\frac{1}{x^{2009}}$ if $x^2+x+1=0$.
11. Let $x, y, z \neq 0$. Prove that if $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$ then $x=y=z$ or $(xyz)^2=1$

12. Prove that if $a+b+c=0$ then $2(a^5+b^5+c^5)=5abc(a^2+b^2+c^2)$.

13. Prove that if $a+b+c+d=0$ then $a^3+b^3+c^3+d^3=3(ab-cd)(c+d)$.

14. Let a,b,c be positive reals satisfying $2b^2=a^2+c^2$. Prove that
$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{2}{c+a}.$$

15. Prove that if $x+y+z=1$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ then $x^2+y^2+z^2=1$.

16. Calculate $xy+yz+zx$ if $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $a+b+c=1$ and $a^2+b^2+c^2=1$.

17. Given that $a^2+b^2=c^2+d^2=1$ and $ac+bd=0$. Find $ab+cd$.

18. Let $b,c\neq 1$. Prove that if $ac-a-c=b^2-2b$ and $bd-b-d=c^2-2c$ then $ad+b+c=bc+a+d$.

19. Prove that
$$b=\frac{2ac}{a+c}\iff(\frac{b}{2})^2=(a-\frac{b}{2})(c-\frac{b}{2}).$$

20. Prove that $(x+1)(y+1)=2$, where
$$x=\frac{b^2+c^2-a^2}{2bc}\text{ and }y=\frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)}.$$

21. Let x,y,z,a be real numbers satisfying $x+y+z=a$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}$. Prove that at least one of x,y,z equals to a.

22. Let a,b,c,x,y,z be real numbers such that $xyz\neq 0$, $abc\neq 0$. Prove that if $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$ then $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

23. Let x,y,z be real numbers such that
$$\begin{aligned}x+y+z&=1,\\x^2+y^2+z^2&=1,\\x^3+y^3+z^3&=1.\end{aligned}$$
Prove that $xyz=0$.

24. Find the minimum value of the expression $x^2+2xy+3y^2+2x+6y+4$, where $x,y\in\mathbb{R}$.

25. Positive real numbers x,y satisfy $x^2+y^2=6xy$. Compute $\frac{x+y}{x-y}$.

26. Let x,y,z be positive real numbers with $xyz=1$. Compute
$$\frac{x+1}{xy+x+1}+\frac{y+1}{yz+y+1}+\frac{z+1}{xz+z+1}.$$

27. Determine $ax^4 + by^4$ if the real numbers a, b, x, y satisfy the system of equations

$$\begin{aligned} a + b &= 6, \\ ax + by &= 10, \\ ax^2 + by^2 &= 24, \\ ax^3 + by^3 &= 62. \end{aligned}$$

28. Let a, b, c be real numbers such that $abc = 1$. Compute

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca}.$$

29. Let x, y be real numbers. Prove that if

$$(x + \sqrt{y^2+1})(y + \sqrt{x^2+1}) = 1,$$

then

$$(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}) = 1.$$

30. Let a, b, c be rational numbers satisfying $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Prove that $\sqrt{a^2 + b^2 + c^2}$ is also a rational number.

2.3 SOLUTIONS

1. Let $a+b=3$ and $ab=1$. Then compute:

   (a) a^2+b^2

   (b) $\frac{1}{a}+\frac{1}{b}$

   (c) $\frac{1}{a^2}+\frac{1}{b^2}$

   (d) $\frac{a}{b}+\frac{b}{a}$

   (e) $\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}$

   Solution: $a^2+b^2=(a+b)^2-2ab=3^2-2\cdot 1=7$.

   $$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=\frac{3}{1}=3, \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{a^2+b^2}{a^2b^2}=\frac{7}{1^2}=7, \quad \frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ab}=\frac{7}{1}=7.$$

   $$\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}=\frac{\sqrt{a}^3+\sqrt{b}^3}{\sqrt{ab}}=\frac{(\sqrt{a}+\sqrt{b})(a-\sqrt{ab}+b)}{1}=2(\sqrt{a}+\sqrt{b})=2\sqrt{a+2\sqrt{ab}+b}=2\sqrt{5}.$$

2. Let $x,y>0$. Compute $x+y$ if $x-y=\sqrt{5}$ and $xy=1$.
   Solution: $(x+y)^2=(x-y)^2+4xy=\sqrt{5}^2+4\cdot 1=9$. Since $x,y>0$ we have $x+y=3$.

3. Compute x^2+y^2 if $\frac{x}{y}+\frac{y}{x}=4$ and $x-y=2$.
   Solution: $\frac{x}{y}+\frac{y}{x}=4 \Longrightarrow x^2+y^2=4xy$. Squarring both sides of $x-y=2$, we get $2^2=(x-y)^2=x^2+y^2-2xy=4xy-2xy=2xy$. Implying that $xy=2$ and $x^2+y^2=8$.

4. Given that $y-x=1$ and $\frac{2}{x}-\frac{2}{y}=1$. Compute $(x+y)^2$.
   Solution: $\frac{2}{x}-\frac{2}{y}=1 \Longrightarrow \frac{2(y-x)}{xy}=\frac{2\cdot 1}{xy}=1 \Longrightarrow xy=2$.
   So, $(x+y)^2=(y-x)^2+4xy=1^2+4\cdot 2=9$.

5. Given that $x+\frac{1}{x}=3$. Find

   (a) $x^2+\frac{1}{x^2}$

   (b) $(x-\frac{1}{x})^2$

   (c) $x^3+\frac{1}{x^3}$

   (d) $x^4+\frac{1}{x^4}$

   (e) $x^5+\frac{1}{x^5}$

   Solution:

   (a) $x^2+\frac{1}{x^2}=(x+\frac{1}{x})^2-2\cdot x\cdot\frac{1}{x}=3^2-2=7$.

   (b) $(x-\frac{1}{x})^2=x^2-2\cdot x\cdot\frac{1}{x}+\frac{1}{x^2}=x^2+\frac{1}{x^2}-2=7-2=5$.

   (c) $x^3+\frac{1}{x^3}=(x+\frac{1}{x})(x^2-x\cdot\frac{1}{x}+\frac{1}{x^2})=3(7-1)=18$.

(d) $x^4 + \frac{1}{x^4} = (x^2 + \frac{1}{x^2})^2 - 2 \cdot x^2 \cdot \frac{1}{x^2} = 7^2 - 2 = 47.$

(e)

$$\begin{aligned} x^5 + \frac{1}{x^5} =&(x + \frac{1}{x})(x^4 - x^3 \cdot \frac{1}{x} + x^2 \cdot \frac{1}{x^2} - x \cdot \frac{1}{x^3} + \frac{1}{x^4}) \\ =&3(x^4 + \frac{1}{x^4} - (x^2 + \frac{1}{x^2}) + 1) = 3(47 - 7 + 1) = 123. \end{aligned}$$

6. Compute $a^4 + b^4 + c^4$ if $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$.
Solution:

$$\begin{aligned} 0^2 = (a + b + c)^2 =&a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ =&1 + 2(ab + bc + ca) \Longrightarrow ab + bc + ca = -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} (-\frac{1}{2})^2 = (ab + bc + ca)^2 =&a^2b^2 + b^2c^2 + c^2a^2 + 2(ab^2c + bc^2a + a^2bc) \\ =&a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \\ =&a^2b^2 + b^2c^2 + c^2a^2 \text{ since } a + b + c = 0. \end{aligned}$$

So, $a^4 + b^4 + c^4 = (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) = 1^2 - 2 \cdot \frac{1}{4} = \frac{1}{2}$.

7. Let a, b, c be real numbers such that $abc \neq 0$ and $a + b + c \neq 0$.
Prove that if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$ then $(a + b)(b + c)(c + a) = 0$.
Solution:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{a + b + c} \Longrightarrow \frac{ab + bc + ca}{abc} = \frac{1}{a + b + c} \Longrightarrow (a+b+c)(ab+bc+ca) = abc.$$

By the identity, $(a + b + c)(ab + bc + ca) - abc = (a + b)(b + c)(c + a)$ we have $(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc = 0$.

8. Let a, b, c be real numbers such that $abc \neq 0$ and $a + b + c \neq 0$.
Prove that if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$ then $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{1}{a^3+b^3+c^3}$.
Solution: By the previous exercise, we have $(a + b)(b + c)(c + a) = 0$. And

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{1}{a^3 + b^3 + c^3} \Longleftrightarrow (a^3 + b^3)(b^3 + c^3)(c^3 + a^3) = 0.$$

So it is sufficient to show that $(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) = 0$. That is true because

$$(a^3+b^3)(b^3+c^3)(c^3+a^3) = (a+b)(a^2-ab+b^2)(b+c)(b^2-bc+c^2)(c+a)(c^2-ca+a^2) = 0.$$

9. Prove each statement if $x + y + z = 0$.

(a) $x^3 + y^3 + z^3 = 3xyz$

(b) $x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) = 0$
(c) $x^4 + y^4 + z^4 = 2(x^2y^2 + y^2z^2 + z^2x^2)$
(d) $2(x^4 + y^4 + z^4) = (x^2 + y^2 + z^2)^2$
(e) $\frac{x^7+y^7+z^7}{7} = \frac{x^3+y^3+z^3}{3} \cdot \frac{x^4+y^4+z^4}{2}$
(f) $\frac{x^7+y^7+z^7}{7} \cdot \frac{x^3+y^3+z^3}{3} = \left(\frac{x^5+y^5+z^5}{5}\right)^2$.

Solution:

(a) $x^3+y^3+z^3 = (x+y)^3-3xy(x+y)+z^3 = -z^3-3xy(-z)+z^3 = 3xyz$
(b) By the previous exercise

$$x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) = 3xyz + 3(x + y)(y + z)(z + x)$$

since $x + y = -z, y + z = -x, z + x = -y$ we have

$$3xyz + 3(x + y)(y + z)(z + x) = 3xyz - 3xyz = 0.$$

(c) $x+y+z = 0 \Longrightarrow (x+y+z)^2 = 0 \Longrightarrow x^2+y^2+z^2 = -2(xy+yz+zx)$
$(x^2 + y^2 + z^2)^2 = (-2(xy + yz + zx))^2$
$\Longrightarrow x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) = 4(x^2y^2 + y^2z^2 + z^2x^2) + 8xyz(x + y + z)$
$\Longrightarrow x^4 + y^4 + z^4 = 2(x^2y^2 + y^2z^2 + z^2x^2)$ since $x + y + z = 0$.
(d) By using the previous exercise

$$\begin{aligned}(x^2 + y^2 + z^2)^2 &= x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2)\\ &= x^4 + y^4 + z^4 + (x^4 + y^4 + z^4) = 2(x^4 + y^4 + z^4).\end{aligned}$$

(e) $\frac{x^7+y^7+z^7}{7} = \frac{x^3+y^3+z^3}{3} \cdot \frac{x^4+y^4+z^4}{2}$
(f) $\frac{x^7+y^7+z^7}{7} \cdot \frac{x^3+y^3+z^3}{3} = \left(\frac{x^5+y^5+z^5}{5}\right)^2$.

10. Compute $x^{2009} + \frac{1}{x^{2009}}$ if $x^2 + x + 1 = 0$.
Solution: See that $x^2 + x + 1 = 0 \Longrightarrow x^3 - 1 = 0$. Replacing $x^3 = 1$ we get that $x^{2009} + \frac{1}{x^{2009}} = x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2x \cdot \frac{1}{x} = (-1)^2 - 2 = -1$ since $x^2 + x + 1 = 0 \Longrightarrow x + \frac{1}{x} = -1$. Here, note that x is a complex number that is why $x^3 = 1$ does not imply that $x = 1$ and we got $x^2 + \frac{1}{x^2} = -1$ which is impossible in $\mathbb{R}$.

11. Let $x, y, z \neq 0$. Prove that if $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$ then $x = y = z$ or $(xyz)^2 = 1$.
Solution:

$$x + \frac{1}{y} = y + \frac{1}{z} \Longrightarrow x - y = \frac{1}{z} - \frac{1}{y} = \frac{y - z}{yz} \quad (1)$$

$$x + \frac{1}{y} = z + \frac{1}{x} \Longrightarrow x - z = \frac{1}{x} - \frac{1}{y} = \frac{y - x}{xy} \quad (2)$$

$$y + \frac{1}{z} = z + \frac{1}{x} \Longrightarrow y - z = \frac{1}{x} - \frac{1}{z} = \frac{z - x}{zx} \quad (3)$$

Replacing, (3) in (1) and later (2), we get

$$x-y=\frac{y-z}{yz}=\frac{\frac{z-x}{zx}}{yz}=\frac{z-x}{xyz^2}=\frac{\frac{x-y}{xy}}{xyz^2}=\frac{x-y}{x^2y^2z^2}$$

$i)$ If $x-y\neq 0$ then $x^2y^2z^2=1$
$ii)$ If not then $x=y$ and therefore the above equality is now
$x-y=\frac{y-z}{yz}=0\Longrightarrow y=z$, so $x=y=z$.

12. Prove that if $a+b+c=0$ then $2(a^5+b^5+c^5)=5abc(a^2+b^2+c^2)$.
Solution:

$$\begin{aligned}
a+b+c=0 &\Longrightarrow a+b=-c\Longrightarrow (a+b)^5=-c^5\\
&\Longrightarrow a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5+c^5=0\\
&\Longrightarrow 2(a^5+b^5+c^5)=-2(5a^4b+10a^3b^2+10a^2b^3+5ab^4)\\
&=-5ab(2a^3+4a^2b+4ab^2+2b^3)\\
&=-5ab(2(a^3+b^3)+4ab(a+b))\\
&=-5ab(2(a+b)(a^2-ab+b^2)+4ab(a+b))\\
&=-5ab(a+b)(2a^2-2ab+2b^2+4ab)\\
&=5abc(a^2+2ab+b^2+a^2+b^2)=5abc(a^2+b^2+c^2)
\end{aligned}$$

13. Prove that if $a+b+c+d=0$ then $a^3+b^3+c^3+d^3=3(ab-cd)(c+d)$.
Solution: Remember the identity $(a+b)^3=a^3+b^3+3ab(a+b)$.

$$\begin{aligned}
a^3+b^3+c^3+d^3 &=(a+b)^3-3ab(a+b)+c^3+d^3\\
&=-(c+d)^3-3ab(a+b)+c^3+d^3\\
&=-[c^3+d^3+3cd(c+d)]+3ab(c+d)+c^3+d^3\\
&=-3cd(c+d)+3ab(c+d)=3(ab-cd)(c+d)
\end{aligned}$$

14. Let a,b,c be positive reals satisfying $2b^2=a^2+c^2$. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{2}{c+a}.$$

Solution:

$$\begin{aligned}
\frac{1}{a+b}+\frac{1}{b+c} &=\frac{a+2b+c}{b^2+ab+bc+ca}\\
&=\frac{2(a+2b+c)}{2b^2+2ab+2bc+2ac}\\
&=\frac{2(a+2b+c)}{a^2+c^2+2ab+2bc+2ac}\\
&=\frac{2(a+2b+c)}{(a+c)^2+2b(a+c)}\\
&=\frac{2(a+2b+c)}{(a+c)(a+c+2b)}=\frac{2}{c+a}.
\end{aligned}$$

15. Prove that if $x+y+z=1$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ then $x^2+y^2+z^2=1$.
Solution:
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0 \Longrightarrow \frac{xy+yx+zx}{xyz}=0 \Longrightarrow xy+yx+zx=0.$$
Therefore, $x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=1^2-2\cdot 0=1$.

16. Calculate $xy+yz+zx$ if $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $a+b+c=1$ and $a^2+b^2+c^2=1$.
Solution:
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{1}{k} \Longrightarrow a=xk, b=yk, c=zk.$$
$$a+b+c=1 \text{ and } a^2+b^2+c^2=1 \Longrightarrow ab+bc+ca=0$$
$$\Longrightarrow k^2(xy+yz+zx)=0 \Longrightarrow xy+yz+zx=0.$$

17. Given that $a^2+b^2=c^2+d^2=1$ and $ac+bd=0$. Find $ab+cd$.
Solution:
$$a^2+b^2=1, c^2+d^2=1 \Longrightarrow a^2+b^2-c^2-d^2=0$$
$$ac+bd=0 \Longrightarrow a^2c^2+2acbd+b^2d^2=0 \Longrightarrow 2abcd=-a^2c^2-b^2d^2$$
$$\begin{aligned}(ab+cd)^2=a^2b^2+2abcd+c^2d^2 &= a^2b^2+c^2d^2-a^2c^2-b^2d^2\\ &= a^2(b^2-c^2)-d^2(b^2-c^2)\\ &= (a^2-d^2)(b^2-c^2)=-(a^2-d^2)^2\end{aligned}$$
So, $(ab+cd)^2+(a^2-d^2)^2=0 \Longrightarrow (ab+cd)=0$.

18. Let $b,c\neq 1$. Prove that if $ac-a-c=b^2-2b$ and $bd-b-d=c^2-2c$ then $ad+b+c=bc+a+d$.
Solution: $ac-a-c=b^2-2b \Longrightarrow ac-a-c+1=b^2-2b+1=(b-1)^2$ and observe that $ac-a-c+1=(a-1)(c-1)$. So, $(a-1)(c-1)=(b-1)^2$. And similarly, we have $(b-1)(d-1)=(c-1)^2$. Multiplying these two, we get
$$(a-1)(b-1)(c-1)(d-1)=(b-1)^2(c-1)^2.$$
(We can simplify, since $b,c\neq 1$.)
Implying that, $(a-1)(d-1)=(b-1)(c-1) \Longrightarrow ad+b+c=bc+a+d$.

19. Prove that
$$b=\frac{2ac}{a+c} \Longleftrightarrow (\frac{b}{2})^2=(a-\frac{b}{2})(c-\frac{b}{2}).$$
Solution:
$$\begin{aligned}(\frac{b}{2})^2=(a-\frac{b}{2})(c-\frac{b}{2}) &\Longleftrightarrow ac-\frac{b}{2}(a+c)=0\\ &\Longleftrightarrow 2ac-b(a+c)=0\\ &\Longleftrightarrow 2ac=b(a+c)\\ &\Longleftrightarrow \frac{2ac}{a+c}=b.\end{aligned}$$

20. Prove that $(x+1)(y+1)=2$, where

$$x=\frac{b^2+c^2-a^2}{2bc} \text{ and } y=\frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)}.$$

Solution: Observe that

$$x+1=\frac{(b+c)^2-a^2}{2bc} \text{ and } y+1=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}+1=\frac{(b+c)^2-(b-c)^2}{(b+c)^2-a^2}$$

So

$$(x+1)(y+1)=\frac{(b+c)^2-a^2}{2bc}\cdot\frac{(b+c)^2-(b-c)^2}{(b+c)^2-a^2}=\frac{4bc}{2bc}=2.$$

21. Let x, y, z, a be real numbers satisfying $x+y+z=a$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}$. Prove that at least one of x, y, z equals to a.
Solution:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}=\frac{1}{x+y+z}\Longrightarrow(xy+yz+zx)(x+y+z)=xyz$$

And by the identity 10, we have
$0=(xy+yz+zx)(x+y+z)-xyz=(x+y)(y+z)(z+x)$. Therefore, one of $x+y, y+z, z+x$ is 0, and one of x, y, z is equal to a.

22. Let a, b, c, x, y, z be real numbers such that $xyz\neq 0$, $abc\neq 0$. Prove that if $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$ then $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
Solution: Let $\frac{x}{a}=X, \frac{y}{b}=Y, \frac{z}{c}=Z$. Now we have, $X+Y+Z=1$ and $\frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}=0$. We are to prove that $X^2+Y^2+Z^2=1$, which is the same as exercise 15.

23. Let x, y, z be real numbers such that

$$\begin{aligned} x+y+z&=1,\\ x^2+y^2+z^2&=1,\\ x^3+y^3+z^3&=1. \end{aligned}$$

Prove that $xyz=0$.
Solution: $x+y+z=1, x^2+y^2+z^2=1\Longrightarrow xy+yz+zx=0$.

$$\begin{aligned} (x+y+z)(x^2+y^2+z^2)=1\cdot 1=x^3+y^3+z^3\\ \Longrightarrow x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)=0\\ \Longrightarrow xy(x+y)+yz(y+z)+zx(z+x)=0\\ \Longrightarrow xy(1-z)+yz(1-x)+zx(1-y)=0\\ \Longrightarrow xy+yz+zx-3xyz=0\\ \Longrightarrow 3xyz=xy+yz+zx=0\\ \Longrightarrow xyz=0. \end{aligned}$$

24. Find the minimum value of the expression $x^2 + 2xy + 3y^2 + 2x + 6y + 4$, where $x, y \in \mathbb{R}$.
Solution:
$$\begin{aligned} x^2 + 2xy + 3y^2 + 2x + 6y + 4 &= x^2 + 2x(y+1) + (y+1)^2 + 2y^2 + 4y + 3 \\ &= (x+y+1)^2 + 2(y+1)^2 + 1 \geq 1. \end{aligned}$$
And for $x = y + 1 = 0$ it is equal to 1.

25. Positive real numbers x, y satisfy $x^2 + y^2 = 6xy$. Compute $\frac{x+y}{x-y}$.
Solution: $(x+y)^2 = x^2 + y^2 + 2xy = 6xy + 2xy = 8xy$ and $(x-y)^2 = 4xy$.
So, $\frac{x+y}{x-y} = \sqrt{\frac{(x+y)^2}{(x-y)^2}} = \sqrt{\frac{8xy}{4xy}} = \sqrt{2}$.

26. Let x, y, z be positive real numbers with $xyz = 1$. Compute
$$\frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{xz+z+1}.$$
Solution:
$$\frac{x+1}{xy+x+1} = \frac{x+1}{xy+x+1} \cdot \frac{z}{z} = \frac{xz+z}{1+xz+z}.$$
$$\frac{y+1}{yz+y+1} = \frac{y+xyz}{yz+y+xyz} = \frac{1+xz}{z+1+xz}.$$
So,
$$\frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{xz+z+1} = \frac{xz+z}{1+xz+z} + \frac{1+xz}{z+1+xz} + \frac{z+1}{xz+z+1} = 2.$$

27. Determine $ax^4 + by^4$ if the real numbers a, b, x, y satisfy the system of equations
$$\begin{aligned} a + b &= 6, \\ ax + by &= 10, \\ ax^2 + by^2 &= 24, \\ ax^3 + by^3 &= 62. \end{aligned}$$

28. Let a, b, c be real numbers such that $abc = 1$. Compute
$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca}.$$
Solution: The answer is 1. See 26.

29. Let x, y be real numbers. Prove that if
$$(x + \sqrt{y^2+1})(y + \sqrt{x^2+1}) = 1,$$
then
$$(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}) = 1.$$
Solution:

30. Let a, b, c be rational numbers satisfying $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Prove that $\sqrt{a^2 + b^2 + c^2}$ is also a rational number.
Solution: $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} \Longrightarrow c = \frac{ab}{a+b}$. So,

$$
\begin{aligned}
a^2 + b^2 + c^2 &= a^2 + b^2 + (\frac{ab}{a+b})^2 \\
&= \frac{(a^2 + b^2)(a+b)^2 + a^2b^2}{(a+b)^2} \\
&= \frac{(a^2 + b^2)^2 + 2ab(a^2 + b^2) + a^2b^2}{(a+b)^2} \\
&= (\frac{a^2 + b^2 + ab}{a+b})^2
\end{aligned}
$$

Therefore, $\sqrt{a^2 + b^2 + c^2} = \left| \frac{a^2+b^2+ab}{a+b} \right|$ is a rational number.

Chapter 3

QUADRATICS

3.1 QUADRATIC EQUATIONS

Definition 5. *Let a, b, c be real numbers with $a \neq 0$, an equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.*

Definition 6. *The quantity $\Delta = b^2 - 4ac$ is called the discriminant of the equation $ax^2 + bx + c = 0$.*

Theorem 6. *$i)$ If $\Delta > 0$ then $ax^2 + bx + c = 0$ has two distinct real solutions:*

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

$ii)$ If $\Delta = 0$ then $ax^2 + bx + c = 0$ has double root, $x_1 = x_2 = \frac{-b}{2a}$.
$iii)$ If $\Delta < 0$ then $ax^2 + bx + c = 0$ has no real roots.

Proof: Multiplying both sides of the equation $ax^2 + bx + c = 0$ by $4a$, we get:

$$\begin{aligned} 4a^2x^2 + 4abx + 4ac = 0 &\Longleftrightarrow (2ax)^2 + 2 \cdot 2ax \cdot b + b^2 = b^2 - 4ac \\ &\Longleftrightarrow (2ax + b)^2 = b^2 - 4ac \end{aligned}$$

So,
$i)$ if $b^2 - 4ac > 0$, then there are two solutions $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
$ii)$ If $b^2 - 4ac = 0$, then $2ax + b = 0 \Longrightarrow x = \frac{-b}{2a}$.
$iii)$ If $b^2 - 4ac < 0$, since no square is negative, there is no real solution.

Theorem 7. *(Vieta's Theorem) If x_1, x_2 are the roots of the equation $ax^2 + bx + c = 0$, then $x_1 + x_2 = \frac{-b}{a}$ and $x_1 \cdot x_2 = \frac{c}{a}$.*

3.1.1 EXERCISES

1. Solve the equation
$$\frac{3}{x}+\frac{1}{x-1}+\frac{4}{x-2}+\frac{4}{x-3}+\frac{1}{x-4}+\frac{3}{x-5}=0.$$

2. Find the real numbers x, y satisfying the equation
$$4(x^2-x+1)(y^2+2y+2)=3.$$

3. Solve the equation $a^2+2ab+2b^2=13$ in the set of integers.

4. Let x, y be real numbers. Prove that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1$ then $x+y=0$.

5. Solve the equation $(x^2-3)^3-(4x+6)^3+6^3=18(4x+6)(3-x^2)$.

6. Prove that if $a, b, c \in \mathbb{R}$, then
$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$
has real solution.

7. Solve the equations
   (a) $(x+2)^4+(x+3)^4=(2x+5)^4$
   (b) $3(x^2+\frac{1}{x^2})-7(x+\frac{1}{x})=0$
   (c) $\frac{1}{x^2}-\frac{1}{(x+1)^2}=1$
   (d) $(2x^2-3x+1)(2x^2+5x+1)=9x^2$
   (e) $(x+2)(x+3)(x+8)(x+12)=4x^2$
   (f) $x^4-2x^3+x-\frac{3}{4}=0$
   (g) $(x+2)^4+(x+4)^4=16$
   (h) $x^4+(x-1)^4=\frac{1}{8}$
   (i) $(1+x)^8+(1+x^2)^4=2x^4$

8. Let x_1, x_2 be the roots of $x^2+px-\frac{1}{2p^2}=0$, where $p \in \mathbb{R}\setminus\{0\}$. Prove that $x_1^4+x_2^4 \geq 2+\sqrt{2}$.

9. Let x_1, x_2 be the roots of $x^2+px-\frac{1}{2p^2}=0$, where $p \in \mathbb{R}\setminus\{0\}$. For what value of p, $x_1^4+x_2^4=2+\sqrt{2}$.

10. Given that a, b, c are positive real numbers such that $a+b+c=1$ and one of the roots of $ax^2+(b-1)x+c=0$ is in $(0,1)$. Prove that $2a+b>1$.

11. Given that the roots of the equation $x^2+px+q=0$ are integers. Find p, q and the roots of the equation, if $p+q=198$.

12. Find all pairs (a, b) of integers, for which $a+b$ is a solution to the equation $x^2 + ax + b = 0$.

13. Solve the equation $\sqrt{x - \sqrt{3}} + x^2y^2 + 2xy(\sqrt{6} - \sqrt{3}) = 6\sqrt{2} - 9$ in $\mathbb{R}$.

14. Find the real number a such that $x^2 + ax + 8 = 0$ and $x^2 + x + a = 0$ have a common root.

15. Find all values of a, for which the equation $x^2 + 4x - 2|x - a| + 2 - a = 0$ has exactly two different real roots.

16. Find all values of a, for which the equation $\left|x^2 + 2x + a\right| = 2$ has four different real solutions.

17. Find all values of a, for which the equation $\left|x^2 - 2x - 3\right| = a$ has exactly three different real solutions.

18. The real roots of the equation $x^2 + a_1x + b_1 = 0$ are x_0 and x_1, the real roots of the equation $x^2 + a_2x + b_2 = 0$ are x_0 and x_2 and the real roots of the equation $x^2 + a_3x + b_3 = 0$ are x_0 and x_3. Find the roots of the equation $x^2 + \frac{a_1+a_2+a_3}{3}x + \frac{b_1+b_2+b_3}{3} = 0$.

19. The product of one of the roots of the equation $ax^2 + bx + b = 0$ and one of the roots of the equation $ax^2 + ax + b = 0$ is 1. Find the roots of each equation.

20. Solve the equation $|x + 3| - a|x - 1| = 4$.

21. Let $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 \neq 0$. Is it possible for the equation

$$(a^2 + b^2 + c^2)x^2 + 2(a + b + c)x + 3 = 0$$

to have two different real roots?

22. Let $a, b, c \in \mathbb{R}^+$ such that $a^2 + b^2 = c^2$. Find $\frac{a}{b}$ if $\frac{12}{a} + \frac{12}{b} = \frac{35}{c}$.

23. Solve the equation $x^2 + 2y^2 + 2xy + 2x - 4y + 10 = 0$.

24. Let $a > 0$. Prove that for all real solutions x to the equation $x^2+px+q = 0$, we have $x \geq \frac{4q-(p+a)^2}{4a}$.

25. Let $x, y \in \mathbb{R}$. Find $x + y$ if $x^3 + y^3 + (x + y)^3 + 30xy = 2000$.

26. Solve the equation in $\mathbb{R}$:

$$(\frac{x^3 + x}{5})^3 + \frac{x^3 + x}{5} = 5x$$

27. Solve the equation $x + a^3 = \sqrt[3]{a - x}$.

28. Let x_1, x_2 be the roots of $(a-1)x^2 - (a+1)x + 2a - 1 = 0$, $a \in \mathbb{R}$, $a \neq 0$. Find all values of b such that the value of $(x_1 - b)(x_2 - b)$ does not depend on a.

29. Prove that the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$, where $a, b, c \in \mathbb{R}$ and $a, b \neq 0$, have common root if and only if, $a^3 + b^3 + c^3 = 3abc$.

30. Given that the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, are real and in $(0, 1)$. Prove that $a(2c+b) < 0$.

31. For $q \neq 0$, the equations $x^2 + px + q = 0$ and $x^2 + px - q = 0$ have integer solutions. Prove that there exist $a, b \in \mathbb{N}$ such that $p^2 = a^2 + b^2$.

32. Solve the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ where a is a real number.

33. Solve the equation $x^2 + (\frac{x}{x-1})^2 = 8$ in the set of integers.

34. Find all values of a for which the roots of the equation $x^2 - 2ax - (a+3) = 0$ are integers.

35. Find all values of a for which the roots of the equation $(a-1)x^2 - (a^2+1)x + a^2 + a = 0$ are integers.

36. Find the value of m such that $ax^2 + bx + c + m(x^2 + 1)$ is a perfect square.

37. Prove that $x^2 + (2n+1)x + (2n-1)$, where $n \in \mathbb{N}$, does not have integral solutions.(The roots are not integers.)

38. Prove that the roots of a quadratic equation all of whose coefficients are odd integers, can't be rational numbers.

39. Find all $k \in \mathbb{Z}$ such that the roots of $kx^2 + (2k-1)x + k - 2 = 0$ are rational numbers.

40. For what value of $p \in \mathbb{Z}$ the equations $3x^2 - 4x + p - 2 = 0$ and $x^2 - 2px + 5 = 0$ have a common root.

41. For what value of $k \in \mathbb{Z}$ the equations $2x^2 + (2k-1)x - 3 = 0$ and $2x^2 + (2k-3)x - 1 = 0$ have a common root.

42. For what value of $k \in \mathbb{Z}$ the equations $2x^2 - (3k+2)x + 12 = 0$ and $4x^2 - (9k-2)x + 36 = 0$ have a common root.

43. Prove that $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$,with $a \neq c$, have common solution if and only if $(a+c)^2 = b^2$.

44. Let b and c be two different real numbers. Prove that $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ have at least one common root if and only if $b + c = -1$.

45. Let a, b be two real numbers not both equal to 0. Prove that the equation $\frac{a^2}{x} + \frac{b^2}{x-1} = 1$ has real solution.

46. The roots of $x^2 + ax + b + 1 = 0$ are natural numbers. Prove that $a^2 + b^2$ is a composite number.

3.1.2 SOLUTIONS

1. Solve the equation

$$\frac{3}{x}+\frac{1}{x-1}+\frac{4}{x-2}+\frac{4}{x-3}+\frac{1}{x-4}+\frac{3}{x-5}=0.$$

Solution:

$$(\frac{3}{x}+\frac{3}{x-5})+(\frac{1}{x-1}+\frac{1}{x-4})+(\frac{4}{x-2}+\frac{4}{x-3})=0$$

$$\Longrightarrow \frac{3(2x-5)}{x^2-5x}+\frac{2x-5}{x^2-5x+4}+\frac{4(2x-5)}{x^2-5x+6}=0$$

$$\Longrightarrow (2x-5)\cdot(\frac{3(t+4)(t+6)+t(t+6)+4t(t+4)}{t(t+4)(t+6)})=0 \text{ where } t=x^2-5x.$$

$$\Longrightarrow (2x-5)\cdot(\frac{8t^2+52t+72}{t(t+4)(t+6)})=0$$

$$\Longrightarrow x=\frac{5}{2} \text{ or } t=-2 \text{ or } t=-\frac{9}{2}$$

$$\Longrightarrow x_1=\frac{5}{2}, x_{2,3}=\frac{5\pm\sqrt{17}}{2}, x_{4,5}=\frac{5\pm\sqrt{7}}{2}.$$

2. Find the real numbers x, y satisfying the equation

$$4(x^2-x+1)(y^2+2y+2)=3.$$

Solution: Observe that, $x^2-x+1=(x-\frac{1}{2})^2+\frac{3}{4}\geq\frac{3}{4}$ and similarly, $y^2+2y+2=(y+1)^2+1\geq 1$. So

$$4(x^2-x+1)(y^2+2y+2)\geq 4\cdot\frac{3}{4}\cdot 1=3.$$

Since $4(x^2-x+1)(y^2+2y+2)=3$ we must have $x^2-x+1=\frac{3}{4}$ and $y^2+2y+2=1$. That is $x=\frac{1}{2}$ and $y=-1$.

3. Solve the equation $a^2+2ab+2b^2=13$ in the set of integers.
Solution: $a^2+2ab+2b^2=(a+b)^2+b^2=13$.
If $|b|\geq 4$ then $a^2+2ab+2b^2=(a+b)^2+b^2\geq 16$ can't be equal to 13. So, $|b|\leq 3$.
$i)$ If $|b|=3$ then $(a\pm 3)^2=4 \Longrightarrow a\in\{-5,-1,1,5\}$.
$ii)$ If $|b|=2$ then $(a\pm 2)^2=9 \Longrightarrow a\in\{-5,-1,1,5\}$.
So all the solutions are:
$(a,b)\in\{(1,-3),(5,-3),(5,-2),(-1,-2),(1,2),(-5,2),(-1,3),(-5,3)\}$.

4. Let x, y be real numbers. Prove that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1$ then $x+y=0$.
Solution: Multiplying bith sides of $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1$ by $x-\sqrt{x^2+1}$ we get
$$-(y+\sqrt{y^2+1})=x-\sqrt{x^2+1} \tag{1}$$
and similarly multiplying both sides by $y-\sqrt{y^2+1}$ we get
$$-(x+\sqrt{x^2+1})=y-\sqrt{y^2+1}. \tag{2}$$
Adding (1) and (2) we get that $x+y=0$.

5. Solve the equation $(x^2-3)^3-(4x+6)^3+6^3=18(4x+6)(3-x^2)$.
Solution: Let $a=x^2-3$, $b=-4x-6$ and $c=6$. Now we have, $a^3+b^3+c^3=3abc$. By the identity
$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
we get that $a+b+c=0$ or $a^2+b^2+c^2-ab-bc-ca=0$.
$$a+b+c=0 \Longrightarrow x_{1,2}=2\pm\sqrt{7}$$
$a^2+b^2+c^2-ab-bc-ca=0 \Longrightarrow (a-b)^2+(b-c)^2+(c-a)^2=0 \Longrightarrow a=b=c$
and $a=b=c \Longrightarrow x=-3$.

6. Prove that if $a, b, c \in \mathbb{R}$, then
$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$
has real solution.
Solution:
$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0 \Longleftrightarrow 3x^2-2(a+b+c)x+ab+bc+ca=0$$
and the discriminant of $3x^2-2(a+b+c)x+ab+bc+ca=0$ is
$$\begin{aligned}\Delta=4(a+b+c)^2-4\cdot3\cdot(ab+bc+ca)&=4(a^2+b^2+c^2-ab-bc-ca)\\&=2[(a-b)^2+(b-c)^2+(c-a)^2]\geq 0.\end{aligned}$$
So the original equation has a real solution.

7. Solve the equations
   (a) $(x+2)^4+(x+3)^4=(2x+5)^4$
   Solution: $(x+2)^4+(x+3)^4=(2x+5)^4=[(x+2)+(x+3)]^4$
   $$=(x+2)^4+4(x+2)^3(x+3)+6(x+2)^2(x+3)^2+4(x+2)(x+3)^3+(x+3)^4$$
   $$\Longrightarrow 2(x+2)(x+3)[2(x+2)^2+3(x+2)(x+3)+2(x+3)^2]=0 \Longrightarrow x_{1,2}=-2,-3.$$

(b) $3(x^2+\frac{1}{x^2})-7(x+\frac{1}{x})=0$
Solution: Let $t=x+\frac{1}{x}$ then we have $3(t^2-2)-7t=3t^2-7t-6=0$ giving the solutions $t_1=3$ and $t_2=-\frac{2}{3}$. So,

$$x+\frac{1}{x}=3 \text{ or } x+\frac{1}{x}=-\frac{2}{3} \Longrightarrow x_{1,2}=\frac{3\pm\sqrt{5}}{2}.$$

(c) $\frac{1}{x^2}-\frac{1}{(x+1)^2}=1$
Solution: Multiplying both sides by x^2, we get

$$x^2+(\frac{x}{x+1})^2-1=0$$

which is equivalent to

$$(x-\frac{x}{x+1})^2+2\frac{x^2}{x+1}-1=0$$

$$\Longleftrightarrow (\frac{x^2}{x+1})^2+2\frac{x^2}{x+1}-1=0$$

Let $t=\frac{x^2}{x+1}$ then we have $t^2+2t-1=0$ giving the solutions $t=-1\pm\sqrt{2}$. So,

$$\frac{x^2}{x+1}=-1\pm\sqrt{2}\Longrightarrow x_{1,2}=\frac{\sqrt{2}-1\pm\sqrt{2\sqrt{2}-1}}{2}$$

(d) $(2x^2-3x+1)(2x^2+5x+1)=9x^2$
Solution: Since $x=0$ doesn't satisfy the equation we have that $x\neq 0$. So,

$$(2x^2-3x+1)(2x^2+5x+1)=9x^2 \Longleftrightarrow (2x-3+\frac{1}{x})(2x+5+\frac{1}{x})=9.$$

Let $t=2x+\frac{1}{x}$, then we have $(t-3)(t+5)=9$ giving $t=4$ or $t=-6$. So,

$$x_{1,2}=\frac{2\pm\sqrt{2}}{2}, x_{3,4}=\frac{-3\pm\sqrt{7}}{2}.$$

(e) $(x+2)(x+3)(x+8)(x+12)=4x^2$
Solution: $x=0$ is not a solution. So,

$$(x+2)(x+3)(x+8)(x+12)=4x^2 \Longleftrightarrow (\frac{x^2+11x+24}{x})(\frac{x^2+14x+24}{x})=4.$$

Now, let $t=x+\frac{24}{x}$. We get $(t+11)(t+14)=4\Longrightarrow t=-10$ or $t=-15$. Solving for x, we get the solutions:

$$x\in\{-6,-4,\frac{-15\pm\sqrt{129}}{2}\}.$$

(f) $x^4 - 2x^3 + x - \frac{3}{4} = 0$
Solution:

$$\begin{aligned} x^4 - 2x^3 + x - \frac{3}{4} &= (x^4 - 2x^3 + x^2) - x^2 + x - \frac{3}{4} \\ &= x^2(x^2 - 2x + 1) - (x^2 - x) - \frac{3}{4} \\ &= (x^2 - x)^2 - (x^2 - x) - \frac{3}{4} = 0 \end{aligned}$$

Letting $t = x^2 - x$, we get $t^2 - t - \frac{3}{4} = 0 \Longrightarrow t_1 = \frac{3}{2}, t_2 = -\frac{1}{2}$. Solving for x, we get $x = \frac{1 \mp \sqrt{7}}{2}$.

(g) $(x+2)^4 + (x+4)^4 = 16$
Solution: Let $t = x + 3$. Now, we have

$$(t-1)^4 + (t+1)^4 = 2t^4 + 12t^2 + 2 = 16$$

$$\Longleftrightarrow t^4 + 6t^2 - 7 = 0$$

$$\Longleftrightarrow (t^2+7)(t^2-1) = 0 \Longrightarrow t = \pm 1$$

So, $x = -4, -2$.

(h) $x^4 + (x-1)^4 = \frac{1}{8}$
Solution: Hint: Let $t = x - \frac{1}{2}$ and see the previous exercise.

(i) $(1+x)^8 + (1+x^2)^4 = 2x^4$
Solution:

$$(1+x)^8 + (1+x^2)^4 = 2x^4 \Longleftrightarrow (\frac{1+2x+x^2}{x})^4 + (\frac{1+x^2}{x})^4 = 2$$

$$(x + \frac{1}{x} + 2)^4 + (x + \frac{1}{x})^4 = 2$$

Now, let $t = x + \frac{1}{x} + 1$ (see the previous exercises apply the same trick) you will get a biquadratic equation in t, solve it for t and then solve for x.

8. Let x_1, x_2 be the roots of $x^2 + px - \frac{1}{2p^2} = 0$, where $p \in \mathbb{R} \setminus \{0\}$. Prove that $x_1^4 + x_2^4 \geq 2 + \sqrt{2}$.
Solution: By Vieta's Thm. we have $x_1 + x_2 = -p$ and $x_1 x_2 = -\frac{1}{2p^2}$. By using the identity $a^2 + b^2 = (a+b)^2 - 2ab$ twice, we get that

$$x_1^4 + x_2^4 = p^4 + \frac{1}{2p^4} + 2 \geq 2 \cdot \sqrt{p^4 \cdot \frac{1}{2p^4}} + 2 = 2 + \sqrt{2}.$$

By AM-GM inequality.

9. Let x_1, x_2 be the roots of $x^2 + px - \frac{1}{2p^2} = 0$, where $p \in \mathbb{R} \setminus \{0\}$. For what value of p, $x_1^4 + x_2^4 = 2 + \sqrt{2}$.
Solution: By the previous exercise we know that $x_1^4 + x_2^4 \geq 2 + \sqrt{2}$. In this case equality holds. In AM-GM inequality, equality holds iff the terms are equal. So we must have that $p^4 = \frac{1}{2p^4}$. That is, $p = \sqrt[8]{\frac{1}{2}}$.

10. Given that a, b, c are positive real numbers such that $a + b + c = 1$ and one of the roots of $ax^2 + (b - 1)x + c = 0$ is in $(0, 1)$. Prove that $2a + b > 1$.
Solution: See that $x_1 = 1$ is a solution. So $\frac{c}{a} = x_2 x_1 = x_2$ is in $(0, 1)$. Therefore,

$$\frac{c}{a} < 1 \Longrightarrow c < a \Longrightarrow 1 = a + b + c < a + b + a = 2a + b.$$

11. Given that the roots of the equation $x^2 + px + q = 0$ are integers. Find p, q and the roots of the equation, if $p + q = 198$.
Solution: By Vieta's thm. $x_1 + x_2 = -p$ and $x_1 x_2 = q$. Now the condition

$$p + q = x_1 x_2 - x_1 - x_2 = 198 \Longrightarrow (x_1 - 1)(x_2 - 1) = 199.$$

Since $x_1, x_2 \in \mathbb{Z}$ and 199 is a prime we have the following systems:

$$\left\{ \begin{array}{c} x_1 - 1 = \mp 1 \\ x_2 - 1 = \mp 199 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{c} x_1 - 1 = \mp 199 \\ x_2 - 1 = \mp 1 \end{array} \right.$$

Therefore, $(x_1, x_2) \in \{(2, 200), (0, -198), (200, 2), (-198, 0)\}$ and the corresponding values of $(p, q) \in \{(-202, 400), (198, 0), (-202, 400), (198, 0)\}$.

12. Find all pairs (a, b) of integers, for which $a + b$ is a solution to the equation $x^2 + ax + b = 0$.
Solution: Substituting $a + b$ in the original equation we get $2a^2 + 3ab + b^2 + b = 0$. This is a quadratic equation in b.(Also in a but that is a bit longer to work with.)Which is equivalent to $b^2 + (3a + 1)b + 2a^2 = 0$. Since a and b are integers, the discriminant must be a perfect square.

$$\Delta = a^2 + 6a + 1 = m^2$$

$$\Longleftrightarrow (a + 3)^2 - m^2 = 8$$

Considering that $a + 3 + m$ and $a + 3 - m$ are of the same parity we have the following systems:

$$\left\{ \begin{array}{l} a + 3 + m = \mp 4 \\ a + 3 - m = \mp 2 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} a + 3 + m = \mp 2 \\ a + 3 - m = \mp 4 \end{array} \right.$$

Giving $a = 0$ or $a = -6$. $a = 0 \Longrightarrow b = 0, -1$ and $a = -6 \Longrightarrow b = 8, 9$.
So all solutions are: $(a, b) \in \{(0, 0), (0, -1), (-6, 8), (-6, 9)\}$.

13. Solve the equation $\sqrt{x-\sqrt{3}}+x^2y^2+2xy(\sqrt{6}-\sqrt{3})=6\sqrt{2}-9$ in $\mathbb{R}$.
Solution:

$$\sqrt{x-\sqrt{3}}+x^2y^2+2xy(\sqrt{6}-\sqrt{3})-6\sqrt{2}+9=0$$

$$\iff \sqrt{x-\sqrt{3}}+\left(xy-(\sqrt{6}-\sqrt{3})\right)^2=0$$

The sum of two nonnegative expressions is 0. So $\sqrt{x-\sqrt{3}}=0$ and $xy-(\sqrt{6}-\sqrt{3})=0$. Giving that $x=\sqrt{3}$ and $y=\sqrt{2}-1$.

14. Find the real number a such that $x^2+ax+8=0$ and $x^2+x+a=0$ have a common root.
Solution: Let x_0 be the common root. Then we have $x_0^2+ax_0+8=0$ and $x_0^2+x_0+a=0$. Subtracting these two equations we get:

$$(a-1)x_0+8-a=0 \Longrightarrow x_0=\frac{a-8}{a-1}$$

Note that a can't be 1. Substituting this in the original equation we get $a^3-24a+72=0$. Implying that

$$a^3+6^3-24(a+6)=(a+6)(a^2-6a+12)=0 \Longrightarrow a=-6.$$

15. Find all values of a, for which the equation $x^2+4x-2\left|x-a\right|+2-a=0$ has exactly two different real roots.
Solution:
A) If $x\geq a$, then the original equation becomes $x^2+2x+a+2=0$.
i) If, $\Delta=-4a-4>0$ then $a<-1$. And the roots $x_{1,2}=-1\pm\sqrt{-a-1}$. So for a satisfying

$$-1+\sqrt{-a-1}>-1-\sqrt{-a-1}\geq a \Longrightarrow -1-a\geq\sqrt{-a-1} \Longrightarrow a\leq -2.$$

Together with the condition $a<-1$ we get that $a\in(-\infty,-2]$, there are two solutions.
ii) And for a satisfying

$$-1+\sqrt{-a-1}\geq a>-1-\sqrt{-a-1} \Longrightarrow \sqrt{-1-a}\geq a+1>-\sqrt{-a-1}$$

$$\Longrightarrow (a+1)^2<-a-1 \Longrightarrow a\in(-2,-1]$$

there is one solution.(Note that $a+1<0$. Also, for $a=-1$ we have double root $x=-1$ which satisfies $x\geq a$.)
iii) For $a>-1$, we have $\Delta<0$ so no solutions.
B) If $x<a$, then we have $x^2+6x+2-3a=0$.
i) If $\Delta=4(3a+7)>0$ then $a>\frac{-7}{3}$ there are two real roots which are $x_{1,2}=-3\pm\sqrt{3a+7}$. So for a satisfying

$$-3\pm\sqrt{3a+7}<a \Longrightarrow \sqrt{3a+7}<a+3 \Longrightarrow (a+1)(a+2)>0$$

which gives $a \in (-\infty, -2) \cup (-1, \infty)$ together with the condition $a > \frac{-7}{3}$ we get: $a \in (\frac{-7}{3}, -2) \cup (-1, \infty)$, there are two solutions.
ii) For a satisfying

$$-3 + \sqrt{3a+7} \geq a > -3 - \sqrt{3a+7} \Longrightarrow \sqrt{3a+7} \geq a+3 \Longrightarrow a \in [-2, -1]$$

(Note that $a + 3 > 0$) there is one solution.
iii) For $a < \frac{-7}{3}$, no solution. Since the original equation is A or B, we add the number of solutions.

a	-7/3	-2	-1	∞
Number of solns. Of A)	2	2	1	0
Number of solns. Of B)	0	2	1	2
Number of solns. Of original eqn.	2	4	2	2

So, for $a \in (-\infty, \frac{-7}{3}) \cup (-2, \infty)$ there are two solutions.

16. Find all values of a, for which the equation $\left|x^2 + 2x + a\right| = 2$ has four different real solutions.
Solution:

$$\left|x^2 + 2x + a\right| = 2 \Longrightarrow x^2 + 2x + a = 2 \text{ or } x^2 + 2x + a = -2$$

Since there are four solutions, both discriminants must be positive.
$4 - 4(a-2) > 0$ and $4 - 4(a+2) > 0$. So $a < -1$.

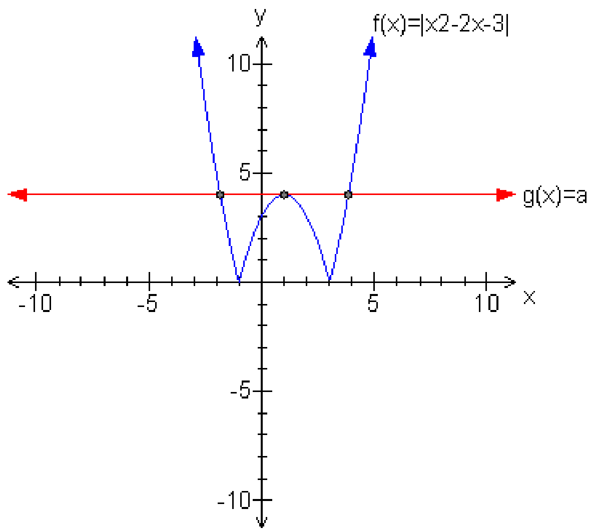
17. Find all values of a, for which the equation $\left|x^2 - 2x - 3\right| = a$ has exactly three different real solutions.
First Solution:

$$\left|x^2 - 2x - 3\right| = a \Longrightarrow x^2 - 2x - 3 = a \text{ or } x^2 - 2x - 3 = -a$$

Since there are three solutions, from one equation we must have one solution(double root) and from the other two. Also notice that, a must be positive, because it is absolute value of something. If the first equation has double root, then $4 - 4(-3-a) = 0 \Longrightarrow a = -4$ which is impossible. So the second equation has double root meaning that $-3 + a = 1 \Longrightarrow a = 4$.
Second Solution: If we draw the graphs of the functions

$$f(x) = \left|x^2 - 2x - 3\right| \text{ and } g(x) = a$$

the intersection points of the two functions will be the solution of the equation $\left|x^2 - 2x - 3\right| = a$. To draw the graph of $f(x) = \left|x^2 - 2x - 3\right|$, we first draw the graph of $y = x^2 - 2x - 3$ and reflect the negative part about the $x - axis$.



For $a > 4$ there are two intersection points.
For $0 < a < 4$ there are four intersection points.
Only for $a = 4$ there are three intersection points. So $a = 4$.

18. The real roots of the equation $x^2 + a_1x + b_1 = 0$ are x_0 and x_1, the real roots of the equation $x^2 + a_2x + b_2 = 0$ are x_0 and x_2 and the real roots of the equation $x^2 + a_3x + b_3 = 0$ are x_0 and x_3. Find the roots of the equation $x^2 + \frac{a_1+a_2+a_3}{3}x + \frac{b_1+b_2+b_3}{3} = 0$.
Solution: Since x_0 is a common root for the first three equations, by adding them, we get $3x_0^2 + (a_1 + a_2 + a_3)x_0 + b_1 + b_2 + b_3 = 0$. Implying that
$$x_0^2 + \frac{a_1 + a_2 + a_3}{3}x_0 + \frac{b_1 + b_2 + b_3}{3} = 0$$
So x_0 is a solution to the last equation. The other solution is, by Vieta's thm., $x = -\frac{a_1+a_2+a_3}{3} - x_0$. Again by Vieta's thm., we have that $x_1 = -a_1 - x_0$, $x_2 = -a_2 - x_0$ and $x_3 = -a_3 - x_0$. So the other root $x = \frac{x_1+x_2+x_3}{3}$.

19. The product of one of the roots of the equation $ax^2 + bx + b = 0$ and one of the roots of the equation $ax^2 + ax + b = 0$ is 1. Find the roots of each equation.
Solution: Let y be that root of the equation $ax^2 + bx + b = 0$, then $\frac{1}{y}$ is a root of $ax^2 + ax + b = 0$. So we have
$$ay^2 + by + b = 0 \text{ and } \frac{a}{y^2} + \frac{a}{y} + b = 0$$
Multiplying the second equation by y^2 and adding the new two equations we get:
$$(a + b)y^2 + (a + b)y + a + b = 0 \Longrightarrow (a + b)(y^2 + y + 1) = 0 \Longrightarrow a = -b$$

since $y^2 + y + 1 = 0$ has no real roots. Now, the equations becomes, $ax^2 - ax - a = 0$ and $ax^2 + ax - a = 0$. Roots of the first equation are

$$x_{1,2} = \frac{1 \pm \sqrt{5}}{2} \text{ and roots of second equation are } x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}.$$

20. Solve the equation $|x+3| - a\,|x-1| = 4$.
Solution: In such problems, seperating cases is quite helpful.
i) If $x < -3$ then both $x+3$ and $x-1$ are negative. So we have;

$$-x - 3 + a(x-1) = 4 \Longrightarrow (a-1)x = a+7 \Longrightarrow x = \frac{a+7}{a-1}$$

Note that a can't be equal to 1.
ii) If $-3 < x < 1$ then we have

$$x + 3 + a(x-1) = 4 \Longrightarrow (a+1)x = a+1 \Longrightarrow x = 1 \text{ or } a = -1$$

If $a = -1$ then we have

$$|x+3| + |x-1| = 4 \Longrightarrow x = -3 \text{ or } x = 1$$

iii) If $x > 1$ then we have

$$x + 3 - a(x-1) = 4 \Longrightarrow (1-a)x = 1-a \Longrightarrow x = 1 \text{ or } a = 1$$

If $a = 1$ then the oqiginal equation becomes $|x+3| - |x-1| = 4$ which gives the solution $x \in [1, \infty)$.

21. Let $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 \neq 0$. Is it possible for the equation

$$(a^2 + b^2 + c^2)x^2 + 2(a+b+c)x + 3 = 0$$

to have two different real roots?
Solution: The discriminant of the equation is

$$\begin{aligned}\Delta &= 4(a+b+c)^2 - 4 \cdot (a^2+b^2+c^2) \cdot 3 \\ &= 8(ab + bc + ca - a^2 - b^2 - c^2) \\ &= -4[(a-b)^2 + (b-c)^2 + (c-a)^2] \leq 0\end{aligned}$$

So the equation can't have two different solutions.

22. Let $a, b, c \in \mathbb{R}^+$ such that $a^2 + b^2 = c^2$. Find $\frac{a}{b}$ if $\frac{12}{a} + \frac{12}{b} = \frac{35}{c}$.
Solution: Squarring both sides we get

$$c^2\left(\frac{a^2 + b^2 + 2ab}{a^2b^2}\right) = \frac{35^2}{12^2}$$

By using $c^2 = a^2 + b^2$ and adding 1 to both sides we get:

$$\begin{aligned}\frac{(a^2+b^2)^2+2ab(a^2+b^2)+a^2b^2}{a^2b^2} &= \frac{35^2}{12^2}+1\\ \Longrightarrow (\frac{a^2+b^2+ab}{ab})^2 &= (\frac{37}{12})^2\\ \Longrightarrow \frac{a}{b}+\frac{b}{a} &= \frac{25}{12}\\ \Longrightarrow 12(\frac{a}{b})^2-25(\frac{a}{b})+12 &= 0\\ \Longrightarrow \frac{a}{b}=\frac{3}{4} \text{ or } \frac{a}{b} &= \frac{4}{3}\end{aligned}$$

23. Solve the equation $x^2+2y^2+2xy+2x-4y+10=0$.
Solution:

$$\begin{aligned}0 &= x^2+2y^2+2xy+2x-4y+10\\ &= x^2+2x(y+1)+2y^2-4y+10\\ &= x^2+2x(y+1)+(y^2+2y+1)+y^2-6y+9\\ &= (x+y+1)^2+(y-3)^2\end{aligned}$$

Since the sum of two nonnegative expressions is 0, both $y-3$ and $x+y+1$ must be 0. So $y=3$ and $x=-4$.

24. Let $a>0$. Prove that for all real solutions x to the equation $x^2+px+q=0$, we have $x \geq \frac{4q-(p+a)^2}{4a}$.
Solution: The roots of $x^2+px+q=0$ are $x_{1,2}=\frac{-p\pm\sqrt{p^2-4q}}{2}$. Since $\frac{-p+\sqrt{p^2-4q}}{2} > \frac{-p-\sqrt{p^2-4q}}{2}$ it is sufficient to show that

$$\frac{-p-\sqrt{p^2-4q}}{2} \geq \frac{4q-(p+a)^2}{4a}.$$

$$\begin{aligned}\frac{-p-\sqrt{p^2-4q}}{2} \geq \frac{4q-(p+a)^2}{4a} &\Longleftrightarrow -2ap-2a\sqrt{p^2-4q} \geq 4q-(a+p)^2\\ &\Longleftrightarrow a^2+p^2-2a\sqrt{p^2-4q}-4q \geq 0\\ &\Longleftrightarrow (a-\sqrt{p^2-4q})^2 \geq 0\end{aligned}$$

which is obviously true.

25. Let $x,y\in\mathbb{R}$. Find $x+y$ if $x^3+y^3+(x+y)^3+30xy=2000$.
Solution: By using the identity, $x^3+y^3=(x+y)^3-3xy(x+y)$ and letting $a=x+y$ and $b=xy$, we get:

$$2a^3-3ab+30b=2000 \Longrightarrow 2a^3-2000=3b(a-10)$$

$$\Longrightarrow (a-10)(2a^2+20a+200-3b)=0$$
$$a=10$$

Note that $2a^2+20a+200-3b$ can't be equal to 0, because it is equal to

$$(x+y+10)^2+\frac{1}{2}(x-y)^2+\frac{1}{2}(x^2+y^2)+10^2 \quad \ddot\smile$$

26. Solve the equation in $\mathbb{R}$:

$$(\frac{x^3+x}{5})^3+\frac{x^3+x}{5}=5x$$

Solution:Let $y=\frac{x^3+x}{5}$ then the original equation becomes $y^3+y=5y$. And by the assumption $x^3+x=5y$. By subtracting these two equations we get:

$$y^3-x^3+6y-6x=0 \Longrightarrow (y-x)(y^2+xy+x^2+6)=0$$

so $y-x=0$ or $y^2+xy+x^2+6=0$.
From the first equation we have $x=y$ giving the solutions $x=-2,0,2$ the second equation does not give any solution since

$$y^2+xy+x^2+6=0 \Longrightarrow (x+y)^2+x^2+y^2+12=0$$

which is impossible.

27. Solve the equation $x+a^3=\sqrt[3]{a-x}$.
Solution: Let $\sqrt[3]{a-x}=y$ then $a-x=y^3$. And the original equation becomes

$$a-y^3+a^3=y \Longrightarrow (a-y)(a^2+ay+y^2+1)=0$$

So $a=y$ or a^2+ay+y^2+1. Second equation has no real solution since

$$a^2+ay+y^2+1=0 \Longrightarrow (a+y)^2+a^2+y^2+2=0$$

which is impossible. Therefore, $a=y$ and $x=a-a^3$.

28. Let x_1,x_2 be the roots of $(a-1)x^2-(a+1)x+2a-1=0$, $a\in\mathbb{R}$, $a\neq 0$. Find all values of b such that the value of $(x_1-b)(x_2-b)$ does not depend on a.
Solution:

$$\begin{aligned}(x_1-b)(x_2-b)=x_1x_2-b(x_1+x_2)+b^2&=\frac{2a-1}{a-1}-b\cdot\frac{a+1}{a-1}+b^2\\ &=\frac{2a-1-ba-b+b^2a-b}{a-1}\\ &=\frac{(2-b+b^2)a-2b-1}{a-1}\end{aligned}$$

For the last fraction to be independent of a, it must be equal to a constant number say c.

$$\frac{(2-b+b^2)a-2b-1}{a-1}=c \Longrightarrow a(2-b+b^2)-2b-1=ac-c$$

since this is true for any a, we can consider this as a polynomial in a. So we must have that:

$$2-b+b^2=c \text{ and } -2b-1=-c$$

adding them side by side and solving the quadratic equation we get:

$$b=\frac{3\pm\sqrt{5}}{2}.$$

29. Prove that the quadratic equations $ax^2+bx+c=0$ and $bx^2+cx+a=0$, where $a,b,c\in\mathbb{R}$ and $a,b\neq 0$, have common root if and only if, $a^3+b^3+c^3=3abc$.
Solution: $\Longrightarrow$:
Let x_0 be the common root of the equations, then we have $ax_0^2+bx_0+c=0$ and $bx_0^2+cx_0+a=0$. Multiplying the first one by x_0 we get:

$$0=ax_0^3+bx_0^2+cx_0=ax_0^3-a=a(x_0^3-1)$$

since $a\neq 0$, we have $x_0^3-1=0$. Implying that $x_0=1$ or $x_0=\frac{-1\pm i\sqrt{3}}{2}$.
i) If $x_0=1$, then $a+b+c=0$. By the identity,

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0.$$

So $a^3+b^3+c^3=3abc$.
ii) If x_0 is equal to one of the complex roots of $x^2+x+1=0$ then $\overline{x_0}$ is a root also, since a,b,c are real numbers. In that case the two equations are equivalent, since they have the same roots. So $a:b:c=b:c:a$, implying that $a=b=c$. Which means that $a^3+b^3+c^3=3abc$.
$\Longleftarrow$:
Conversely, if $a^3+b^3+c^3=3abc$ then

$$a+b+c=0 \text{ or } a^2+b^2+c^2-ab-bc-ca=0.$$

If $a+b+c=0$, then $x_0=1$ is a common root for both equations.
If $a^2+b^2+c^2-ab-bc-ca=0$ then we have

$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0 \Longrightarrow a=b=c.$$

The two equations are the same, so they have the same roots.

30. Given that the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, are real and in $(0, 1)$. Prove that $a(2c+b) < 0$.
Solution:

$$a(2c + b) = a^2(2\frac{c}{a} + \frac{b}{a}) = a^2(2x_1x_2 - x_1 - x_2)$$

since $a \neq 0$, $a^2 > 0$ so it is sufficient to show that $2x_1x_2 - x_1 - x_2 < 0$. That is true due to AM-GM inequality

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1x_2} > x_1x_2$$

and the fact that $x_1x_2 < 1$. (Note that $\sqrt{t} > t$ if $t < 1$.)

31. For $q \neq 0$, the equations $x^2 + px + q = 0$ and $x^2 + px - q = 0$ have integer solutions. Prove that there exist $a, b \in \mathbb{N}$ such that $p^2 = a^2 + b^2$.
Solution: Since $x^2 + px + q = 0$ and $x^2 + px - q = 0$ have integer solutions, discrimants of both equations must be perfect squares. So, $p^2 - 4q = m^2$ and $p^2+4q = n^2$ for some $m, n \in \mathbb{N}$. By adding them, we get $2p^2 = m^2+n^2$.

$$2p^2 = m^2 + n^2 \Longrightarrow p^2 = \frac{m^2 + n^2}{2} = (\frac{m - n}{2})^2 + (\frac{m + n}{2})^2$$

Note that m and n are both even or both odd, and since $q \neq 0$ we have $m \neq n$, so $\frac{|m-n|}{2}, \frac{m+n}{2} \in \mathbb{N}$.

32. Solve the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ where a is a real number.
Solution:

$$x^4 - 2ax^2 + x + a^2 - a = 0 \Longleftrightarrow x^4 - (2a - 1)x^2 + a^2 - a = x^2 - x$$

$$\Longleftrightarrow x^4 - (2a - 1)x^2 + a^2 - a + \frac{1}{4} = x^2 - x + \frac{1}{4}$$

$$\Longleftrightarrow (x^2)^2 - 2(a - \frac{1}{2})x^2 + (a - \frac{1}{2})^2 = (x - \frac{1}{2})^2$$

$$\Longleftrightarrow [x^2 - (a - \frac{1}{2})]^2 = (x - \frac{1}{2})^2$$

$$\Longrightarrow x^2 - a + \frac{1}{2} = x - \frac{1}{2} \text{ or } x^2 - a + \frac{1}{2} = -x + \frac{1}{2}$$

$$\Longrightarrow x^2 - x - a + 1 = 0 \text{ and } x^2 + x - a = 0$$

Solving the parametric equations we get:

$$x_{1,2} = \frac{1 \pm \sqrt{4a - 3}}{2} \text{ or } x_{3,4} = \frac{-1 \pm \sqrt{4a + 1}}{2}.$$

33. Solve the equation $x^2 + (\frac{x}{x-1})^2 = 8$ in the set of integers.
Solution: We are to find $x \in \mathbb{Z}$. Since x and $x - 1$ are relatively prime $\frac{x}{x-1}$ is an integer only if $|x - 1| = 1$. So $x = 2$ or $x = 0$. $x = 0$ doesn't satisfy the equation. So $x = 2$ is the only solution.

34. Find all values of a for which the roots of the equation $x^2-2ax-(a+3)=0$ are integers.
Solution: Observe that $x_1+x_2=2a$, so $2a$ is an integer. For a quadratic equation to have integer roots, the discriminant must be a perfect square. So, $4a^2+4(a+3)=m^2$ for some $m\in\mathbb{Z}$.

$$m^2=4a^2+4a+12=4a^2+4a+1+11=(2a+1)^2+11$$

$$(m-2a-1)(m+2a+1)=11$$

since $2a$ is an integer, both $m-2a-1$ and $m+2a+1$ are integers. Considering all possible cases (there are 4 cases: $\pm1\cdot\pm11$ and $\pm11\cdot\pm1$) we get that $a=2$ or $a=-3$.

35. Find all values of a for which the roots of the equation $(a-1)x^2-(a^2+1)x+a^2+a=0$ are integers.
Solution: By Vieta's Thm. we have $x_1+x_2=\frac{a^2+1}{a-1}$ and $x_1x_2=\frac{a^2+a}{a-1}$. Subtracting we get:

$$x_1x_2-x_1-x_2=1\Longleftrightarrow x_1x_2-x_1-x_2+1=2\Longleftrightarrow(x_1-1)(x_2-1)=2$$

Since x_1 and x_2 are integers, considering all divisors of 2, we get that:

$$(x_1,x_2)\in\{(2,3),(3,2),(0,-1),(-1,0)\}$$

So, $x_1x_2=6$ or $x_1x_2=0$.
$i)$ If $6=x_1x_2=\frac{a^2+a}{a-1}$ then $a^2-5a+6=0\Longrightarrow a=2$ or $a=3$.
$ii)$ If $0=x_1x_2=\frac{a^2+a}{a-1}$ then $a^2+a=0\Longrightarrow a=0$ or $a=-1$.

36. Find the value of m such that $ax^2+bx+c+m(x^2+1)$ is a perfect square.
Solution: If a quadratic expression is a perfect square, then its discriminant is 0. $ax^2+bx+c+m(x^2+1)=(a+m)x^2+bx+c+m=0$. So,

$$\Delta=b^2-4(a+m)(c+m)=0\Longleftrightarrow 4m^2+4m(a+c)+4ac-b^2=0$$

Hence, $m=\frac{-a-c\pm\sqrt{b^2+(a-c)^2}}{2}$.

37. Prove that $x^2+(2n+1)x+(2n-1)$, where $n\in\mathbb{N}$, does not have integral solutions.(The roots are not integers.)
Solution: Assume contrary, then discriminant of $x^2+(2n+1)x+(2n-1)$ must be a perfect square.

$$\Delta=(2n+1)^2-4(2n-1)=4n^2-4n+5=4n(n-1)+5=m^2$$

Since m is odd, $m^2\equiv1 \pmod 8$, and one of $n,n-1$ is even we have $5\equiv1 \pmod 8$. Which is impossible.

38. Prove that the roots of a quadratic equation all of whose coefficients are odd integers, can't be rational numbers.
Solution: To have rational roots, Δ must be a perfect square. Because otherwise, $\frac{-b\pm\sqrt{\Delta}}{2a}$ would be irrational. Now, we have $\Delta = b^2 - 4ac = m^2$. Knowing that a, b, c are all odd, we get that m is odd. So
$$m^2 \equiv b^2 \equiv 1 \pmod 8 \quad \text{and } 4ac \equiv 4 \pmod 8.$$
($4ac$ is divisible by 4 but not by 8.) So,
$$b^2 - 4ac = m^2 \Longrightarrow -3 \equiv 1 \pmod 8.$$
Hence, contradiction.

39. Find all $k \in \mathbb{Z}$ such that the roots of $kx^2 + (2k-1)x + k - 2 = 0$ are rational numbers.
Solution: $\Delta = (2k-1)^2 - 4k(k-2) = m^2$ for some $m \in \mathbb{Z}$.
$$m^2 = (2k-1)^2 - 4k(k-2) = 4k+1$$
note that m is odd, let $m = 2n+1$.
$$4k+1 = m^2 = (2n+1)^2 = 4n^2 + 4n + 1 \Longrightarrow k = n(n+1)$$
where $n \in \mathbb{Z}$.

40. For what value of $p \in \mathbb{Z}$ the equations $3x^2 - 4x + p - 2 = 0$ and $x^2 - 2px + 5 = 0$ have a common root.
Solution: $p = 3$. See 14.

41. For what value of $k \in \mathbb{Z}$ the equations $2x^2 + (2k-1)x - 3 = 0$ and $2x^2 + (2k-3)x - 1 = 0$ have a common root.
Solution: $k = 1$. See 14.

42. For what value of $k \in \mathbb{Z}$ the equations $2x^2 - (3k+2)x + 12 = 0$ and $4x^2 - (9k-2)x + 36 = 0$ have a common root.
Solution: $k = 3$. See 14.

43. Prove that $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$,with $a \neq c$, have common solution if and only if $(a+c)^2 = b^2$.
Solution: $\Longrightarrow$: Let x_0 be the common root of the equations.
By subtracting the two equations we get:
$$(a-c)x_0^2 + c - a = 0 \Longrightarrow (a-c)(x_0^2 - 1) = 0 \Longrightarrow x_0^2 = 1 \Longrightarrow x_0 = \pm 1$$
So we have, $a \pm b + c = 0 \Longrightarrow (a+c)^2 = b^2$.
$\Longleftarrow$: If $(a+c)^2 = b^2$ then $a + c = \pm b \Longrightarrow a + b + c = 0$ or $a - b + c = 0$. In the first case $x_0 = 1$ is a common root, in the second case $x_0 = -1$ is a common root.

44. Let b and c be two different real numbers. Prove that $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ have at least one common root if and only if $b + c = -1$.
Solution: Let x_0 be the common root of the equations. By subtracting the equations we get

$$(b - c)x_0 + c - b = 0 \Longrightarrow (b - c)(x_0 - 1) = 0 \Longrightarrow x_0 = 1$$

Substituting this in either equation we get that $b+c+1 = 0 \Longrightarrow b+c = -1$. Conversely, if $b + c - 1$ then $b + c + 1 = 0$ so $x_0 = 1$ is a common root.

45. Let a, b be two real numbers not both equal to 0. Prove that the equation $\frac{a^2}{x} + \frac{b^2}{x-1} = 1$ has real solution.
Solution: Equating denominators we get: $x^2 - (a^2 + b^2 + 1)x + a^2 = 0$ whose discriminant is:

$$\begin{aligned}\Delta = (a^2 + b^2 + 1)^2 - 4a^2 &= a^4 + b^4 + 1 + 2a^2 + 2a^2b^2 + 2b^2 - 4a^2 \\ &= (a^4 - 2a^2 + 1) + b^4 + 2a^2b^2 + 2b^2 \\ &= (a^2 - 1)^2 + b^4 + 2a^2b^2 + 2b^2 \geq 0\end{aligned}$$

So the equation has real solution.

46. The roots of $x^2 + ax + b + 1 = 0$ are natural numbers. Prove that $a^2 + b^2$ is a composite number.
Solution: By Vieta's Thm., we have $-a = x_1 + x_2$ and $x_1x_2 = b + 1$. So,

$$\begin{aligned}a^2 + b^2 = (x_1 + x_2)^2 + (x_1x_2 - 1)^2 &= x_1^2 + 2x_1x_2 + x_2^2 + (x_1x_2)^2 - 2x_1x_2 + 1 \\ &= (x_1^2 + 1)(x_2^2 + 1)\end{aligned}$$

So it is composite.

3.2 SYSTEMS OF QUADRATIC EQUATIONS

3.2.1 EXERCISES

1. Solve each system.

(a) $\begin{cases} x + y = 5 \\ xy = 6 \end{cases}$

(b) $\begin{cases} x^2 + y^2 = 18 \\ xy = 9 \end{cases}$

(c) $\begin{cases} x^2 + y^2 = 13 \\ xy = 6 \end{cases}$

(d) $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$

(e) $\begin{cases} x^4 + y^4 = 82 \\ x + y = 4 \end{cases}$

(f) $\begin{cases} x^5 + y^5 = a^5 \\ x + y = a \end{cases}$

(g) $\begin{cases} x^2 - 3xy + 4y^2 = 2 \\ 3x^2 - xy - 5y^2 = 5 \end{cases}$

(h) $\begin{cases} x^2 + y^2 + x + y = 8 \\ x^2 + y^2 + xy = 7 \end{cases}$

2. Given the system $\begin{cases} x + y + \frac{x}{y} = 19 \\ \frac{x(x+y)}{y} = 60 \end{cases}$. Find $x + y$.

3. Find all $a \in \mathbb{R}$ such that the system $\begin{cases} x^2 - y^2 = 1 \\ y = ax + b \end{cases}$ has solution for any $b \in \mathbb{R}$.

4. Find all $a \in \mathbb{R}$ for which the system $\begin{cases} ax - y + 4 - 3a = 0 \\ x^2 + y^2 = 25 \end{cases}$ has two equal solutions.

5. Solve the system $\begin{cases} x^2 + 2yz = x \\ y^2 + 2zx = y \\ z^2 + 2xy = z \end{cases}$

6. Solve $\begin{cases} xy(x - y) = ab(a - b) \\ x^3 - y^3 = a^3 - b^3 \end{cases}$ $(a, b \in \mathbb{R}.)$

7. Solve $\begin{cases} (x^3 + y^3)(x^2 + y^2) = 2a^5 \\ x + y = a \end{cases}$ $(a \in \mathbb{R}.)$

8. For what value of a the system $\begin{cases} x + y = a^3 - a \\ xy = a^2 \end{cases}$ has solution.

3.2.2 SOLUTIONS

1. Solve each system.

   (a) $\begin{cases} x + y = 5 \\ xy = 6 \end{cases}$
   Solution: $x + y = 5 \Longrightarrow y = 5 - x$ and $6 = xy = x(5 - x) \Longrightarrow x^2 - 5x + 6 = 0 \Longrightarrow x = 2$ or $x = 3$. So $(x, y) \in \{(2, 3), (3, 2)\}$.

   (b) $\begin{cases} x^2 + y^2 = 18 \\ xy = 9 \end{cases}$
   Solution: $0 = 18 - 18 = x^2 + y^2 - 2xy = (x - y)^2 \Longrightarrow x = y$. So $(x, y) \in \{(-3, -3), (3, 3)\}$.

   (c) $\begin{cases} x^2 + y^2 = 13 \\ xy = 6 \end{cases}$
   Solution: Replacing $y = \frac{6}{x}$ in the first equation and rearranging, we get: $x^4 - 13x^2 + 36 = 0$. Solving the biquadratic equation we get $x = \pm 2$ or $x = \pm 3$. So $(x, y) \in \{(-3, -2), (-2, -3), (2, 3), (3, 2)\}$.

(d) $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$
Solution: $35 = x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 5(x^2 - xy + y^2) \Longrightarrow x^2 - xy + y^2 = 7$ and $x^2 + 2xy + y^2 = 25$. By subtracting these two we get $xy = 6$. Now by the first exercise, $(x, y) \in \{(2, 3), (3, 2)\}$.

(e) $\begin{cases} x^4 + y^4 = 82 \\ x + y = 4 \end{cases}$
Solution: $y = 4 - x \Longrightarrow 82 = x^4 + y^4 = x^4 + (4-x)^4 = x^4 + (x-4)^4$
Let $x - 2 = a$ then $x = a + 2$ and $x - 4 = a - 2$.

$$x^4+(x-4)^4 = 82 \Longrightarrow a^4+24a^2-25 = 0 \Longrightarrow a = \pm 1 \Longrightarrow x = 3 \text{ or } x = 1$$

So $(x, y) \in \{(3, 1), (1, 3)\}$.

(f) $\begin{cases} x^5 + y^5 = a^5 \\ x + y = a \end{cases}$
Solution: Substituting $y = a - x$ in the first equation

$$a^5 = x^5 + (a-x)^5 \Longrightarrow -5ax(a-x)(a^2 - ax + x^2) = 0$$

$x = 0$ or $x = a$. So $(x, y) \in \{(0, a), (a, 0)\}$.

(g) $\begin{cases} x^2 - 3xy + 4y^2 = 2 \\ 3x^2 - xy - 5y^2 = 5 \end{cases}$
Solution: Multiplying the first equation by -5 and the second one by 2, we get $x^2 + 13xy - 30y^2 = 0$. Solving this equation we get $x = -15y$ or $x = 2y$. Replacing this in either equations we get that: $(x, y) \in \{(\pm\frac{15}{\sqrt{137}}, \pm\sqrt{\frac{1}{137}}), (2, 1), (-2, -1)\}$.

(h) $\begin{cases} x^2 + y^2 + x + y = 8 \\ x^2 + y^2 + xy = 7 \end{cases}$
Solution: Let $x + y = a$ and $xy = b$ then $x^2 + y^2 = a^2 - 2b$. Now we have $a^2 - 2b + 1 = 8$ and $a^2 - 2b + b = 7$. Multiplying the second equation by -2 we get $a^2 - a - 6 = 0 \Longrightarrow a = 3$ or $a = -2$. Implying that $b = 2$ or $b = -3$. So $(x, y) \in \{(1, 2), (2, 1), (1, -3), (-3, 1)\}$.

2. Given the system $\begin{cases} x + y + \frac{x}{y} = 19 \\ \frac{x(x+y)}{y} = 60 \end{cases}$. Find $x + y$.
Solution: Let $x + y = a$ and $\frac{x}{y} = b$ then we have $a + b = 19$ and $ab = 60$. Giving $a = 15, b = 4$ or $a = 4, b = 15$. So $x + y = 4$ or 15.

3. Find all $a \in \mathbb{R}$ such that the system $\begin{cases} x^2 - y^2 = 1 \\ y = ax + b \end{cases}$ has solution for any $b \in \mathbb{R}$.
Solution: Replacing $y = ax + b$ in the first equation we get: $(a^2 - 1)x^2 + 2abx + b^2 + 1 = 0$. Since this equation has a solution its discriminant must be nonnegative.

$$\Delta = (2ab)^2 - 4(a^2 - 1)(b^2 + 1) = 4(b^2 - a^2 + 1) \geq 0 \Longrightarrow b^2 + 1 \geq a^2$$

for this inequality to hold for any $b \in \mathbb{R}$, $a^2 \leq 1$. So $a \in [-1, 1]$.

4. Find all $a \in \mathbb{R}$ for which the system $\begin{cases} ax - y + 4 - 3a = 0 \\ x^2 + y^2 = 25 \end{cases}$ has two equal solutions.
Solution: Replacing $y = ax + 4 - 3a$ in the second equation we get
$$(a^2+1)x^2 + (8a - 6a^2)x + 9a^2 - 24a - 9 = 0$$
for this equation to have two equal roots its discriminant must be 0. So
$$\Delta = (8a-6a^2)^2 - 4(a^2+1)(9a^2-24a-9) = 64a^2+96a+36 = 0 \Longrightarrow a = -\frac{3}{4}.$$

5. Solve the system $\begin{cases} x^2 + 2yz = x \\ y^2 + 2zx = y \\ z^2 + 2xy = z \end{cases}$
Solution: If one of x, y, z is 0, then the other two are also 0. So $x = y = z = 0$ is a solution. Now let $x, y, z \neq 0$. By adding all three equations we get that
$$(x+y+z)^2 = x+y+z \Longrightarrow x+y+z = 0 \text{ or } x+y+z = 1$$
By subtracting the second equation from the first one we get
$$(x-y)(x+y-2z-1) = 0$$
Now we have 4 cases: 1) $\begin{cases} x+y+z = 0 \\ x = y \end{cases}$ 2) $\begin{cases} x+y+z = 0 \\ x+y-2z = 1 \end{cases}$
3) $\begin{cases} x+y+z = 1 \\ x = y \end{cases}$ 4) $\begin{cases} x+y+z = 1 \\ x+y-2z = 1 \end{cases}$
So, $(x,y,z) \in \{(-\frac{2}{5}, -\frac{2}{5}, \frac{4}{5}), (\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}), (-\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})\}$.

6. Solve $\begin{cases} xy(x-y) = ab(a-b) \\ x^3 - y^3 = a^3 - b^3 \end{cases}$ $(a, b \in \mathbb{R}.)$
Solution: By the identity $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$ we get that $(x-y)^3 = (a-b)^3 \Longrightarrow x-y = a-b \Longrightarrow xy = ab$. Now we have $x-y = a-b$ and $xy = ab$. Giving the quadratic equation $y^2 + (a-b)y - ab = 0$. Solving this equation we get $x = a$ and $y = b$ or $x = -b$ and $y = -a$.

7. Solve $\begin{cases} (x^3+y^3)(x^2+y^2) = 2a^5 \\ x + y = a \end{cases}$ $(a \in \mathbb{R}.)$
Solution: Replacing $y = a - x$ we get
$$(x^3 - (x-a)^3)(x^2 + (x-a)^2) = 2a^5$$
letting $x - \frac{a}{2} = t$ we now have
$$(3t^2 + \frac{a^2}{4})(t^2 + \frac{a^2}{4}) = a^4 \Longrightarrow t^2 = \frac{5a^2}{12}$$
Solving for x we get $(x,y) \in \{(\frac{3+\sqrt{15}}{6}a, \frac{3-\sqrt{15}}{6}a), (\frac{3-\sqrt{15}}{6}a, \frac{3+\sqrt{15}}{6}a)\}$.

8. For what value of a the system $\begin{cases} x+y=a^3-a \\ xy=a^2 \end{cases}$ has solution.
   Solution: By Vieta's theorem, x and y are the roots of the equation $x^2-(a^3-a)x+a^2=0$. So for the system to have solution the quadratic equation must have solution. Hence

$$\Delta=(a^3-a)^2-4a^2=a^2(a^4-2a^2-3)=a^2(a^2+1)(a^2-3)\geq 0$$

   Therefore, $a\in(-\infty,-\sqrt{3}]\cup[\sqrt{3},\infty)\cup\{0\}$.

3.3 QUADRATIC FUNCTIONS

Definition 7. *A function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is called a quadratic function.*

Theorem 8. *A quadratic function $f(x) = ax^2 + bx + c$ has its minimum or maximum at $x_0 = \frac{-b}{2a}$ if $a > 0$ or $a < 0$, respectively.*

3.3.1 EXERCISES

1. If the real coefficients a, b, c where $a \neq 0$, of the equation $ax^2 + bx + c = 0$ satisfy $\frac{b+c}{a} \leq -1$ then the quadratic equation has a real solution.

2. Find the minimum and maximum value of the function $f(x) = \frac{x^2+1}{x^2-x+1}$.

3. For what values of m and n the function $f(m, n) = \frac{1}{4m^2+12mn+9n^2+1}$ has maximum value?

4. Find all values of a for which the equation $\left|x^2 - 2x - 3\right| = a$ has exactly three different real solutions.

5. Let $f(x) = ax^2 + bx + c$ be given such that the equation $f(x) = x$ has no real solutions. Prove that the equation $f(f(x)) = x$ has no real solutions, either.

6. The function $f(x) = ax^2 + bx + c$ takes integer values for integer x. Prove that $2a, a + b, c \in \mathbb{Z}$.

7. Find the minimum value of the function
$$f(x) = (x-1)(x-2)(x-3)(x-4) + 10$$
for $x \in \mathbb{R}$.

8. Find all values of m such that $f(x) = \left|x^2 - 6x\right| - m$ has exactly 3 zeros.

9. Find the minimum of the function $f(x) = x(x+1)(x+2)(x+3)$.

10. Let $f(x) = x^2 + px + q$ where $p, q \in \mathbb{R}$. Prove that if $|f(0)| > 1$ and $f(-1)f(1) > 0$ then $f(x)$ has no zero in $[-1, 1]$.

11. Find all $x \in \mathbb{Z}$ such that $2x^2 - x - 36$ is a square of a prime number.

12. Find all values of the function $f(x) = x^2 + px + q$ for $x \in [-1, 1]$.

13. Let $f(x) = ax^2 + bx + c$ where $a, b, c, x \in \mathbb{R}$. Prove that if $|f(x)| \leq 1$ whenever $|x| \leq 1$ then $|f(x)| \leq 7$ whenever $|x| \leq 2$.

14. Let $f(x) = (k+1)x^2 - 2(k-1)x + k - 5$ where $k \in \mathbb{R}$.

    (a) Prove that graphs of functions $y = f(x)$ for every k, pass through a common point.

(b) Prove that the common point is not an extreme point of any of those functions.

15. Let $f(x) = kx^2 - (k+2)x - 2k + 6$ where $k \in \mathbb{R}$.

    (a) Prove that graphs of functions $y = f(x)$ pass through two constant points, find them.

    (b) Find k such that $y = f(x)$ has an extreme value for those common points.

16. Let $f(x) = x^2 + (k-3)x - 1 - 2k$, where $k \in \mathbb{R}$. Prove that graphs of $y = f(x)$ cross the x-axis for any k.

17. Find the common points of parabolas $y = -x^2 + 2(k+1)x + 3k - 1$, $k \in \mathbb{R}$. Determine the relation between the coordinates of vertices of such parabolas.

18. Let $f(x) = (k+1)x^2 - 2kx + k - 1$, $k \in \mathbb{R}$.

    (a) Find the relation between the coordinates of vertices of $y = f(x)$.

    (b) Is there a common point of these parabolas?

19. Find the minimum and maximum value of $y = \frac{3x}{4x^2-x+1}$.

20. Find the minimum and maximum value of $y = \frac{x}{x^2+x+1}$.

3.3.2 SOLUTIONS

1. If the real coefficients a, b, c where $a \neq 0$, of the equation $ax^2 + bx + c = 0$ satisfy $\frac{b+c}{a} \leq -1$ then the quadratic equation has a real solution.
   Solution: $i)$ If $a > 0$ then the parabola opens up and

   $$\frac{b+c}{a} \leq -1 \Longrightarrow b + c \leq -a \Longrightarrow a + b + c \leq 0 \text{ or } f(1) \leq 0.$$

   Since $f(1) \leq 0$ and the parabola opens up there must be a point x_0 such that $f(x_0) = 0$.
   $ii)$ If $a < 0$ then the parabola opens down and

   $$\frac{b+c}{a} \leq -1 \Longrightarrow b + c \geq -a \Longrightarrow a + b + c \geq 0 \text{ or } f(1) \geq 0.$$

   And in that case, $f(1) \geq 0$ and parabola opens down again there is a point x_0 such that $f(x_0) = 0$.

2. Find the minimum and maximum value of the function $f(x) = \frac{x^2+1}{x^2-x+1}$.
   Solution:

3. For what values of m and n the function $f(m, n) = \frac{1}{4m^2+12mn+9n^2+1}$ has maximum value?

4. Find all values of a for which the equation $\left|x^2 - 2x - 3\right| = a$ has exactly three different real solutions.

5. Let $f(x) = ax^2 + bx + c$ be given such that the equation $f(x) = x$ has no real solutions. Prove that the equation $f(f(x)) = x$ has no real solutions, either.

6. The function $f(x) = ax^2 + bx + c$ takes integer values for integer x. Prove that $2a, a + b, c \in \mathbb{Z}$.

7. Find the minimum value of the function

   $$f(x) = (x - 1)(x - 2)(x - 3)(x - 4) + 10$$

   for $x \in \mathbb{R}$.

8. Find all values of m such that $f(x) = \left|x^2 - 6x\right| - m$ has exactly 3 zeros.

9. Find the minimum of the function $f(x) = x(x + 1)(x + 2)(x + 3)$.

10. Let $f(x) = x^2 + px + q$ where $p, q \in \mathbb{R}$. Prove that if $|f(0)| > 1$ and $f(-1)f(1) > 0$ then $f(x)$ has no zero in $[-1, 1]$.

11. Find all $x \in \mathbb{Z}$ such that $2x^2 - x - 36$ is a square of a prime number.

12. Find all values of the function $f(x) = x^2 + px + q$ for $x \in [-1, 1]$.

13. Let $f(x) = ax^2 + bx + c$ where $a, b, c, x \in \mathbb{R}$. Prove that if $|f(x)| \leq 1$ whenever $|x| \leq 1$ then $|f(x)| \leq 7$ whenever $|x| \leq 2$.

14. Let $f(x) = (k+1)x^2 - 2(k-1)x + k - 5$ where $k \in \mathbb{R}$.

    (a) Prove that graphs of functions $y = f(x)$ for every k, pass through a common point.

    (b) Prove that the common point is not an extreme point of any of those functions.

15. Let $f(x) = kx^2 - (k+2)x - 2k + 6$ where $k \in \mathbb{R}$.

    (a) Prove that graphs of functions $y = f(x)$ pass through two constant points, find them.

    (b) Find k such that $y = f(x)$ has an extreme value for those common points.

16. Let $f(x) = x^2 + (k-3)x - 1 - 2k$, where $k \in \mathbb{R}$. Prove that graphs of $y = f(x)$ cross the x-axis for any k.

17. Find the common points of parabolas $y = -x^2 + 2(k+1)x + 3k - 1$, $k \in \mathbb{R}$. Determine the relation between the coordinates of vertices of such parabolas.

18. Let $f(x) = (k+1)x^2 - 2kx + k - 1$, $k \in \mathbb{R}$.

    (a) Find the relation between the coordinates of vertices of $y = f(x)$.

    (b) Is there a common point of these parabolas?

19. Find the minimum and maximum value of $y = \frac{3x}{4x^2 - x + 1}$.

20. Find the minimum and maximum value of $y = \frac{x}{x^2 + x + 1}$.

3.4 QUADRATIC INEQUALITIES

Definition 8. *An inequality of the form $ax^2+bx+c \geq 0$ or $ax^2+bx+c > 0$, where $a \neq 0$, is called a quadratic inequality.*

Theorem 9. *i) If $a > 0$ and $\Delta \geq 0$, then $ax^2+bx+c < 0$ for $x \in (x_1, x_2)$ and $ax^2+bx+c > 0$ for $x \in (-\infty, x_1) \cup (x_2, \infty)$.*
ii) If $a < 0$ and $\Delta \geq 0$, then $ax^2+bx+c > 0$ for $x \in (x_1, x_2)$ and $ax^2+bx+c < 0$ for $x \in (-\infty, x_1) \cup (x_2, \infty)$.
iii) If $a > 0$ and $\Delta < 0$, then $ax^2+bx+c > 0$ for any real number x.
iv) If $a < 0$ and $\Delta < 0$, then $ax^2+bx+c < 0$ for any real number x.

3.4.1 EXERCISES

1. Solve each inequality.
   (a) $x^2 \geq 4$
   (b) $x^2 - 2x - 15 \leq 0$
   (c) $\frac{x^2-x-2}{x^2+x+1} > 0$
   (d) $\frac{x-2}{x^2+3x-4} > \frac{1}{3}$
   (e) $-1 \leq \frac{x^2+5x+12}{x^2+7x+12} \leq 1$
   (f) $\frac{x^3+15}{x^3+8} < 2$
   (g) $x^2 - 2|x| + 3 > 0$
   (h) $\frac{x^2-|x|-12}{x-3} \geq 2x$
   (i) $x^2 + 2|x+3| - 10 \leq 0$

2. Find all values of m for which the inequality
$$(4m^2-5m+1)x^2 + 2(m+1)x + 1 > 0$$
holds for all real number x.

3. Find all values of k for which the inequality
$$(k^2-1)x^2 + 2(k-1)x + 1 > 0$$
holds for all real number x.

4. Find all values of m for which the inequality
$$mx^2 - 4mx + m^2 + 2m - 3 > 0$$
holds for all real number x.

5. Find all values of a for which $x^2 < 4 - |x+a|$ has at least one negative solution.

6. Find all values of a, for which the system $\begin{cases} x^2 - y^2 > 1 \\ y = ax^2 + 1 \end{cases}$ has solution.

7. Find all values of b for which the system $\begin{cases} x^2 + (2-3b)x + 2b^2 - 2b < 0 \\ bx = 1 \end{cases}$ has solution.

8. Find all values of b for which the system $\begin{cases} 5x^2 - 4xy + 2y^2 \geq 3 \\ 7x^2 + 4xy + 2y^2 \leq \frac{2b-1}{2b+5} \end{cases}$ has solution.

9. Find all values of a for which, the solution of the inequality $x^2 + ax - 1 < 0$ is an interval of length 5.

10. Let a, b, c be positive real numbers. Prove that the inequality

$$(b-c)(x-a) + (\sqrt{bc} - \sqrt{ax})^2 > 0$$

holds for any $x > 0$ iff c is between a and b.

11. Prove that if $ax^2 + 2bx + c \geq 0$ and $px^2 + 2qx + r \geq 0$ hold for any real number x, where a, b, c, p, q, r are real numbers, then $apx^2 + 2bqx + cr \geq 0$ holds for any real number x.

12. Let a, b, c be real numbers such that $2ax^2 + bx + c = 0$ and $-2ax^2 + bx + c = 0$ have real solutions. Let α be a solution to the first equation and β be a solution to the second equation. Prove that $[\alpha, \beta]$ contains a solution of the equation $ax^2 + bx + c = 0$.

13. Prove that if a, b, c distinct real numbers, then the roots of

$$x^2 - 2(a+b+c)x + 3(ab+bc+ca) = 0$$

satisfy $x_1 < a + b + c < x_2$.

14. Find all values of k such that

    (a) $x^2 - 1 < 0 \Longrightarrow x^2 + 2k^2x - 3k^4 < 0$
    (b) $x^2 - k(1+k^2)x + k^4 < 0 \Longrightarrow x^2 + 4x + 3 > 0$
    (c) $2x^2 - (k+1)x + 2 < 0 \Longrightarrow 0 < x < 1$
    (d) $kx^2 - 2x - k + 1 > 0 \Longrightarrow -1 < x < 2$

15. Find $k \in \mathbb{R}$ such that the roots x_1, x_2 of $3x^2 - 4x - 4k = 0$ satisfy $\frac{1}{3} < x_1 < x_2 < 1$.

16. Solve the inequality $kx^2 - |x-1| \leq 0$, where $k \in \mathbb{R}$.

17. Solve the inequality $\frac{x}{x-a} - \frac{2a}{x+a} > \frac{8a^2}{x^2-a^2}$, where $a \in \mathbb{R}$.

18. Find all real values of a such that for any x satisfying the inequality

$$ax^2 + (1 - a^2)x - a > 0$$

we have $|x| \leq 2$.

19. Find all real values of a such that for any $x \in [-1, 1]$ the inequality $ax^2 + 2(a + 1)x + a - 4 \leq 0$ holds.

3.5 IRRATIONAL EQUATIONS-INEQUALITIES

1. Solve each equation.
   (a) $\sqrt{x-1}-\sqrt{x+1}=\sqrt{2x-2}$.
   (b) $\sqrt{2x+14}-\sqrt{x-7}=\sqrt{x+5}$
   (c) $\sqrt{x-4+\sqrt{x-2}}-\sqrt{x-3-\sqrt{x-2}}=1$
   (d) $(x-1)\sqrt[3]{\frac{x-1}{3-x}}+(3-x)\sqrt[3]{\frac{3-x}{x-1}}=0$
   (e) $\sqrt[3]{2-x}=1-\sqrt{x-1}$
   (f) $\sqrt{x+3-4\sqrt{x-1}}=a$, where $a\in\mathbb{R}^+$
   (g) $\sqrt{x^2+x+1}+2\sqrt{x+3}=\sqrt{6x^2-2x-18}$
   (h) $\sqrt[3]{\frac{x+3}{5x+2}}+\sqrt[3]{\frac{5x+2}{x+3}}=\frac{13}{6}$
   (i) $6\sqrt[3]{x-3}+\sqrt[3]{x-2}=5\sqrt[6]{(x-2)(x-3)}$
   (j) $x^3+x+\sqrt[3]{x^3+x-2}=12$
   (k) $\sqrt[3]{a+x}+\sqrt[3]{a-x}=\sqrt[3]{2a}$
   (l) $\sqrt[4]{x+8}-\sqrt[4]{x-8}=2$
   (m) $\sqrt[4]{x+62}-\sqrt[4]{6-x}=3\sqrt{2}$
   (n) $\sqrt[3]{x+2}=\sqrt{x-2}$

2. Solve each equation.
   (a) $\sqrt{x^2+x}+\sqrt{1+\frac{1}{x^2}}=x+3$
   (b) $x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}$
   (c) $\sqrt{4x-y^2}-\sqrt{y+2}=\sqrt{4x^2+y}$
   (d) $\sqrt{12-\frac{12}{x^2}}-x^2+\sqrt{x^2-\frac{12}{x^2}}=0$ for $x\neq 0$.
   (e) $\frac{1}{1-\sqrt{1-x}}-\frac{1}{1+\sqrt{1-x}}=\frac{\sqrt{3}}{x}$

3. Solve the equation $\sqrt{2\sqrt{3}-3}=\sqrt{x\sqrt{3}}-\sqrt{y\sqrt{3}}$ in the set of rational numbers.

4. Solve $\sqrt{x-\frac{1}{5}}+\sqrt{y-\frac{1}{5}}=\sqrt{5}$ in the set of integers.

5. Solve each equation interms of given parameter.
   (a) $\sqrt{x-4a+16}=2\sqrt{x-2a+4}-\sqrt{x}$
   (b) $x+\sqrt{x^2-4}=a$
   (c) $\sqrt{ax+1}-\sqrt{x-1}=\sqrt{(a-1)x}$
   (d) $x\sqrt{x}+\sqrt{x^3-2}=b$

(e) $\sqrt{x+\frac{1}{x}}+\sqrt{x-\frac{1}{x}}=a\sqrt{x}$

(f) $\sqrt[3]{(x+a)^2}+\sqrt[3]{(x-a)^2}+\sqrt[3]{x^2-a^2}=\sqrt[3]{a^2}$

6. Solve each inequality.

(a) $\sqrt{3x^2+5x+7}-\sqrt{3x^2+5x+2}>1.$

(b) $\sqrt{x+1}+\sqrt{x-2}\geq\sqrt{2-x}$

(c) $\sqrt{5x+3}-\sqrt{x+1}<2x+1.$

(d) $\frac{1-\sqrt{8x-3}}{4x}\geq 1$

(e) $\sqrt[3]{2-x}+\sqrt{x-1}>1$

7. Solve each inequality in terms of given parameter.

(a) $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}<2$

(b) $2\sqrt{x+a}>x+1$

(c) $\sqrt{x-b^2}+\sqrt{x}>2b$

(d) $2x+\sqrt{a^2-x^2}>0$

Chapter 4

COMPLEX NUMBERS

Definition 9. *A number of the form $a + bi$, where a, b are real numbers and $i^2 = -1$, is called a complex number.*

Definition 10. *Let $z = a + bi$. $Re(z) = a$ and $Im(z) = b$.*

Definition 11. *The norm of a complex number $z = a + bi$, is $|z| = \sqrt{a^2 + b^2}$.*

Definition 12. *Complex conjugate of $z = a + bi$, is $\bar{z} = a - bi$.*

USEFUL FACTS

1. A complex number z is real, if and only if, $z = \bar{z}$
2. $|z|^2 = z \cdot \bar{z}$
3. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
4. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
5. $\overline{(\frac{z_1}{z_2})} = \frac{\overline{z_1}}{\overline{z_2}}$ where $z_2 \neq 0$.
6. $\overline{z^{-1}} = (\overline{z})^{-1}$
7. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
8. $|z^n| = |z|^n$
9. $|z| = |\overline{z}|$
10. $|z_1 + z_2| \leq |z_1| + |z_2|$

4.1 EXERCISES

1. Prove that for any complex numbers z_1 and z_2 the following equalities hold.

   (a) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

   (b) $|z_1\overline{z_2} + 1|^2 + |z_1 - z_2|^2 = (|z_1|^2 + 1)(|z_2|^2 + 1)$

   (c) $|z_1 + z_2|^2 = (|z_1| + |z_2|)^2 - 2(|z_1\overline{z_2}| - Re(z_1\overline{z_2}))$

2. Solve the equations.

   (a) $\frac{z}{4-2i} = 5 - 4i$

   (b) $z + 2\overline{z} = 3 + 2i$

   (c) $|z| - z = 1 + 2i$

   (d) $|z + 1| + z + i = 0$

3. Solve the system $\left\{ \begin{array}{c} |z - 2i| = |z| \\ |z - 1| = |z + i| . \end{array} \right.$

4. Find all $n \in \mathbb{N}$ such that $(\frac{1+i}{1-i})^n = 1$.

5. Find all natural number n for which $(\frac{3+i}{2-i})^n$ is a real number.

6. Prove that $\frac{z-1}{z+1}$ is a real number if and only if z is a real number and $z \neq -1$.

7. Let $z_1, z_2 \in \mathbb{C}$ such that $Imz_1 \neq 0$ and $Imz_2 \neq 0$. Prove that $z_1 + z_2$ and z_1z_2 are both reals if and only if $z_1 = \overline{z_2}$.

8. Prove that $\frac{z-1}{z+1}$ is imaginary($z = yi,\ y \in \mathbb{R}$) if and only if $|z| = 1$ and $z \neq -1$.

9. Let $z_1, z_2 \in \mathbb{C}$ and $|z_2| = 1$. Prove that $\left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| = 1$.

10. Let u be a given complex number. Find all $z \in \mathbb{C}$ for which $\frac{u - \overline{u}z}{1-z}$ is a real number.

11. Prove that if $z \in \mathbb{Z}$, $z \neq -3 + 2i$ and $\left|\frac{z-2+3i}{z+3-2i}\right| = 1$ then $Im(z) = Re(z)$.

12. Find all complex numbers z such that $|z| = \left|\frac{1}{z}\right| = |z - 1|$.

13. Prove that the three complex solutions of the equation $(az + b)^3 = 1$ form an equilateral triangle.

14. Prove that if z_1, z_2 are complex numbers such that $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$, then $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

15. Let $z \in \mathbb{C}$, $z \neq 0$. Compute z^{2001} if $z^8 = \bar{z}$.

16. Let $z \in \mathbb{C}$, $z \neq 1$ and $n \in \mathbb{N}$, $n > 2$. Prove that if $z^n = 1$, then
$$1 + 2z + 3z^2 + ... + nz^{n-1} = \frac{n}{z-1}.$$

17. Solve $(x-3)^4 + (x-4)^4 = (2x-7)^4$ in $\mathbb{C}$.

18. Prove that if $z \in \mathbb{C}$, $|z| = 1$, then $\left|\frac{a_2 z^2 + a_1 z + a_0}{\bar{a_0} z^2 + \bar{a_1} z + \bar{a_2}}\right| = 1$ where $a_0, a_1, a_2 \in \mathbb{C}$.

19. Let a, b, c be complex numbers such that $|a| = |b| = |c| = 1$. Prove that
$$\left|\frac{ab + bc + ca}{a + b + c}\right| = 1.$$

20. Let $f(n) = (\frac{1+i}{\sqrt{2}})^n + (\frac{1-i}{\sqrt{2}})^n$. Find $f(2006) + f(2010)$.

21. Prove the equality
$$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 + z_2 + z_3|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2).$$

22. Prove that if $|z_1| = |z_2| = \cdots = |z_n| = 1$ and $\sum_{k=1}^{n} z_k = 0$, then
$$\sum_{k=1}^{n} |z - z_k|^2 = n(|z|^2 + 1).$$

Chapter 5

TRIGONOMETRY

5.1 EXERCISES

1. Prove that

   $$(1-\cos x\cdot\cos y)^2-2(1-\cos x\cdot\cos y)-\sin^2 x\cdot\sin^2 y+\sin^2 x+\sin^2 y=0.$$

2. For what value of t the expression $\sin^6 x+\cos^6 x+t(\sin^4 x+\cos^4 x)$ is independent of x.

3. Prove that if α is acute angle($\alpha<90^\circ$) then

   $$\sqrt{\sin\alpha}\cdot\sqrt[4]{\tan\alpha}+\sqrt{\cos\alpha}\cdot\sqrt[4]{\cot\alpha}\geq\sqrt[4]{8}.$$

4. Compute $\tan\alpha+\cot\alpha$ if $m=\frac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}$, where α is an acute angle and m is a real number.

5. Let α be an acute angle. Prove that

   $$\left(1+\frac{1}{\sin\alpha}\right)\left(1+\frac{1}{\cos\alpha}\right)\geq 3+2\sqrt{2}.$$

6. Prove the inequality

   $$\frac{1}{\sin^4 x}+\frac{1}{\cos^4 x}\geq 8$$

   where $\sin x\cos x\neq 0$.

7. Compute $\cos^2 1^\circ+\cos^2 2^\circ+\cdots+\cos^2 89^\circ$.

8. Prove that if α,β,γ are interior angles of a triangle, then

   $$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}.$$

9. Find the relation between α, β, and γ if
$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

10. Compute $x + 2y$, if $\tan x = \frac{1}{7}$ and $\sin y = \frac{1}{\sqrt{10}}$ where x, y are acute angles.

11. Prove that $\tan^2 20^\circ, \tan^2 40^\circ, \tan^2 80^\circ$ are the roots of $x^3 - 33x^2 + 27x - 3 = 0$.

12. Compute $\frac{1}{\cos^2 20^\circ} + \frac{1}{\cos^2 40^\circ} + \frac{1}{\cos^2 60^\circ} + \frac{1}{\cos^2 80^\circ}$.

13. Prove that if α and β are acute angles and $\sin^2\alpha + \sin^2\beta = \sin(\alpha + \beta)$ then $\alpha + \beta = 90^\circ$.

14. Let α, β, γ be interior angles of a triangle. Prove that if
$$\sin\alpha + \sin\beta + \sin\gamma = \sqrt{3}(\cos\alpha + \cos\beta + \cos\gamma)$$
then at least one of α, β, γ is equal to 60°.

15. Let α, β, γ be interior angles of a triangle. Prove that if
$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 0$$
then at least one of α, β, γ is equal to 60°.

16. Let α, β, γ be interior angles of a triangle. Prove that if
$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 1$$
then at least one of α, β, γ is equal to 120°.

17. α, β, γ are interior angles of a triangle and $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$. Prove that the triangle is a right triangle.

18. α, β, γ are interior angles of a triangle and
$$\frac{\cos\gamma + 2\cos\alpha}{\cos\gamma + 2\cos\beta} = \frac{\sin\beta}{\sin\alpha}.$$
Prove that the triangle is a right triangle or an equilateral triangle.

19. Prove that if α, β, γ are interior angles of a triangle and
$$\frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 2,$$
then the triangle is a right triangle.

20. Find the relation between α, β, and γ if
$$\cot\alpha + \frac{cos\beta}{\sin\alpha\cos\gamma} = \cot\beta + \frac{cos\alpha}{\sin\beta\cos\gamma}.$$

21. Find the minimum and the maximum of the function
$$f(x, y) = \cos x + \cos y - \cos(x + y).$$

22. Prove that there is no triangle whose angles α, β, and γ satisfy
$$\tan\alpha + \tan\beta + \tan\gamma = \cot\alpha + \cot\beta + \cot\gamma.$$

23. Prove that if $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 1$ then $\sin x = \sin y = \sin z = 0$ and $\cos x = \cos y = \cos z = 1$.

24. Prove each statement:
    (a) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$
    (b) $\cos 55^\circ \cos 65^\circ + \cos 65^\circ \cos 175^\circ + \cos 175^\circ \cos 55^\circ = -\frac{3}{4}$
    (c) $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}$
    (d) $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$
    (e) $\tan 55^\circ \tan 65^\circ \tan 75^\circ = \tan 85^\circ$
    (f) $\tan^2 10^\circ + \tan^2 50^\circ + \tan^2 70^\circ = 9$.

25. Solve the equation $\sin 2x + \tan x + 2 = 0$.

26. Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

27. Solve $\sin x + \cos x + 2\sin x \cos x = 1$.

28. Solve $\sin x + \cos x + \sin x \cos x = 1$.

29. Solve the equation $\tan^2 x = \frac{1-\cos x}{1-|\sin x|}$.

30. Solve $\sin^4 x - \cos^7 x = 1$.

31. Let a, b be the sides of a triangle and α, β be the corresponding interior angles in that triangle. Prove that if $(a^2+b^2)\sin(\alpha-\beta) = (a^2-b^2)\sin(\alpha+\beta)$, then the triangle is a right traiangle or equilateral.

32. Prove that if the sides a, b, c and the corresponding interior angles α, β, γ of a triangle satisfy $a + b = \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta)$, then the triangle is an equilateral triangle.

33. Let a, b, c be the sides and α, β, γ be the corresponding interior angles of a triangle. Prove that
$$a^2\cos^2\alpha = b^2\cos^2\beta + c^2\cos^2\gamma + 2bc\cos\beta\cos\gamma\cos 2\alpha.$$

34. Let a, b, c be the sides and α, β, γ be the corresponding interior angles of a triangle. Prove that
$$\frac{\tan\alpha}{\tan\beta} = \frac{a^2+c^2-b^2}{b^2+c^2-a^2}.$$

35. Solve the equation $\sqrt{1+3\sin^3 x} = 3 - \sqrt{1-\cos^4 x}$.

36. Solve the equation $\cos x \cos 2x \cos 4x \cos 8x = \frac{1}{16}$.

37. Find the maximum value of the function $f(x) = 3 - 2x - x^2$ where x is a solution of the equation $2\cos^2 x + \cos 4x = 0$.

38. Let a, b, c be the sides and α, β, γ be the corresponding interior angles of a triangle. Prove that

$$\frac{a^2 \sin(\beta-\alpha)}{\sin\alpha} + \frac{b^2 \sin(\gamma-\alpha)}{\sin\beta} + \frac{c^2 \sin(\alpha-\beta)}{\sin\gamma} = 0.$$

39. Solve the system

$$\begin{cases} xy = 1 \\ x + y + \cos^2 z = 2. \end{cases}$$

40. Let a, b, c be the sides and α, β, γ be the corresponding interior angles of a triangle. Prove that

$$\frac{1}{a}\cos^2\frac{\alpha}{2} + \frac{1}{b}\cos^2\frac{\beta}{2} + \frac{1}{c}\cos^2\frac{\gamma}{2} = \frac{(a+b+c)^2}{4abc}.$$

41. Solve the system

$$\begin{cases} \sin x \sin y = a \\ \tan x \tan y = b. \end{cases}$$

42. Find all pairs (a, b) of real numbers such that the equation $a(\cos x - 1) + b^2 = \cos(ax + b^2) - 1$ holds for all $x \in \mathbb{R}$.

43. Find all possible values of α for which the equation

$$x^2 - \frac{2x}{\sqrt{\sin\alpha}} - \frac{1}{\cos\alpha} - 2\sqrt{2} = 0$$

has double root.

44. Solve the system

$$\begin{cases} \sin^2 x + \cos^2 y = y^2 \\ \sin^2 y + \cos^2 x = x^2. \end{cases}$$

Chapter 6

LOGARITHM

6.1 EXERCISES

1. Solve the system

$$\begin{cases} (xy)^y x^{6x} = y^x \\ x^2 y = 1. \end{cases}$$

2. Solve the system

$$\begin{cases} (x)^x = (6x^y)^2 \\ 4x + \log_6 x = 9 + 8y. \end{cases}$$

3. Solve the equation $2^{\sin^2 x} = \sin x$.

4. Solve the equation $2^{\sin^2 x} = \cos x$.

5. Solve the equation $x^{\sqrt{x}} = (\sqrt{x})^x$.

6. Solve the system

$$\begin{cases} (1+y)^x = 100 \\ (y^4 - 2y^2 + 1)^{x-1} = \frac{(y-1)^{2x}}{(y+1)^2} \end{cases}$$

7. Solve the system

$$\begin{cases} x \log_2 3 + \log_2 y = y + \log_2 x \\ x \log_3 12 + \log_3 x = y + \log_3 y \end{cases}$$

8. Solve the inequality $\frac{2+\log_3 x}{x-1} < \frac{6}{2x-1}$.

9. Solve the equation $\log_2 (2^x - 7) = 3 - x$.

10. Prove that $(\log_2 3)^{-1} + (\log_4 3)^{-1} < 2$.

11. Solve the equation $4^{\log x} - 32 + x^{\log 4} = 0$.

12. Solve the system

$$\begin{cases} x^{\log y} + \sqrt{y^{\log x}} = 110 \\ xy = 1000 \end{cases}$$

13. Solve the equation

$$\log_{3x+4}(2x+1)^2 + \log_{2x+1}(6x^2 + 11x + 4) = 4.$$

14. Find all pairs (a, b) of real numbers for which $\ln(ax + b) = a\ln x + b$ for all $x > 0$.

15. Let a, b, c, x be real numbers such that $\log_a x = p$, $\log_b x = q$, and $\log_{abc} x = r$. Find $\log_c x$.

16. Prove that

$$\log_a x \log_b x + \log_b x \log_c x + \log_c x \log_a x = \frac{\log_a x \log_b x \log_c x}{\log_{abc} x}.$$

17. Solve the equation

$$\log_2(4^x + 4) = x + \frac{1}{\log_{2^{x+1}-3} 2}.$$