'A brilliant gourmet feast of what maths is really about'
Ian Stewart, author of *Professor Stewart's Incredible Numbers*

CAKES, CUSTARD + CATEGORY THEORY

Easy recipes for understanding complex maths

EUGENIA CHENG

CAKES, CUSTARD + CATEGORY THEORY

Eugenia Cheng is Senior Lecturer in Pure Mathematics at the University of Sheffield. She was educated at the University of Cambridge and has done post-doctoral work at the Universities of Cambridge, Chicago and Nice. Since 2007 her YouTube lectures and videos have been viewed over a million times. A concert pianist, she also speaks French, English and Cantonese, and her mission in life is to rid the world of maths phobia.

CAKES, CUSTARD + CATEGORY THEORY

Easy recipes for understanding complex mathematics

EUGENIA CHENG

PROFILE BOOKS

CONTENTS

To
my parents
and Martin Hyland

In memory of
Christine Pembridge

They say mathematics is a glorious garden. I know I would certainly lose my way in it without your guidance. Thank you for walking us through the most beautiful entrance pathway.

From a student's letter to the author
University of Chicago, June 2014

PROLOGUE

Here is a recipe for clotted cream.

Ingredients

Cream

Method

1. Pour the cream into a rice cooker.
2. Leave it on ‘warm’ with the lid slightly open, for about 8 hours.
3. Cool it in the fridge for about 8 hours.
4. Scoop the top part off: that’s the clotted cream.

What on earth does this have to do with maths?

Maths myths

Maths is all about numbers.

You might think that rice cookers are for cooking rice. This is true, but this same piece of equipment can be used for other things as well: making clotted cream, cooking vegetables, steaming a chicken. Likewise, maths is about numbers, but it’s about many other things as well.

Maths is all about getting the right answer.

Cooking is about ways of putting ingredients together to make delicious food. Sometimes it’s more about the method than the ingredients, just as in the recipe for clotted cream, which only has one ingredient – the entire recipe is just a method. Maths is about ways of putting ideas together, to make exciting new ideas. And sometimes it's more about the method than the 'ingredients'.

Maths is all either right or wrong.

Cooking can go wrong – your custard can curdle, your soufflé can collapse, your chicken can be undercooked and give everyone food poisoning. But even if it doesn't poison you, some food tastes better than other food. And sometimes when cooking goes 'wrong' you have actually accidentally invented a delicious new recipe. Fallen chocolate soufflé is deliciously dark and squidgy. If you forget to melt the chocolate for your cookies, you get chocolate chip cookies. Maths is like this too. At school if you write 10 + 4 = 2 you will be told that is wrong, but actually that's correct in some circumstances, such as telling the time – four hours later than 10 o'clock is indeed 2 o'clock. The world of maths is more weird and wonderful than some people want to tell you…

You're a mathematician? You must be really clever.

Much as I like the idea that I am very clever, this popular myth shows that people think maths is hard. The little-understood truth is that the aim of maths is to make things easier. Herein lies the problem – if you need to make things easier it gives the impression that they were hard in the first place. Maths *is* hard, but it makes hard things easier. In fact, since maths is a hard thing, maths also makes maths easier.

Many people are either afraid of maths, or baffled by it, or both. Or they were completely turned off it by their lessons at school. I understand this – I was completely turned off sport by my lessons at school, and have never really recovered. I was so bad at sport at school, my teachers were incredulous that anybody so bad at sport could exist. And yet I'm quite fit now, and have even run the New York Marathon. At least I now appreciate physical exercise, but I still have a horror of any kind of team sport.

How can you do research in maths? You can't just discover a new number.

This book is my answer to that question. It's hard to answer it quickly at a cocktail party, without sounding trite, or taking up too much of someone’s time, or shocking the gathered company. Yes, one way to shock people at a polite party is to talk about maths.

It’s true, you can’t just discover a new number. So what can we discover that’s new in maths? In order to explain what this ‘new maths’ could possibly be about, I need to clear up some misunderstandings about what maths is in the first place. Indeed, not only is maths not just about numbers, but the branch of maths I’m going to describe is actually not about numbers at all. It’s called *category theory* and it can be thought of as the ‘mathematics of mathematics’. It’s about relationships, contexts, processes, principles, structures, cakes, custard.

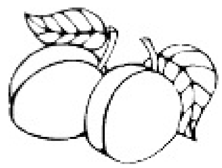
Yes, even custard. Because mathematics is about drawing analogies, and I’m going to be drawing analogies with all sorts of things to explain how maths works. Including custard, cake, pie, pastry, doughnuts, bagels, mayonnaise, yoghurt, lasagne, sushi.

Whatever you think maths is … let go of it now.
This is going to be different.

part one

MATHEMATICS

5 GENERALISATION



Olive oil plum cake

Ingredients

2–4 plums
1 egg
100 g ground almonds
75 g agave or maple syrup
75 ml olive oil

Method

1. Slice the plums quite thinly and arrange them cut face down in a pretty pattern on a lined cake tin.
2. Whisk the rest of the ingredients together and pour gently into the tin over the plums.
3. Bake at 180°C for 20 minutes or until golden brown and set.
4. Turn out upside down so that the plums are on top.

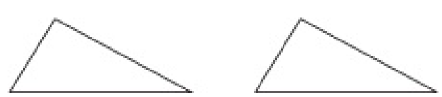
If you've ever invented a new recipe, you might well have started with one from a book, or online, and modified it to your own tastes, whims or allergies. That is, you start with a situation you know and love, and see what you can do that's a bit similar but different – and maybe even better.

When I was little I was allergic to food colouring, so my parents lovingly worked out how to make jelly from scratch instead of from the appealingly (or appallingly) brightly coloured jelly packets. Later, I was going out with someone allergic to wheat, so I invented a lot of wheat-free desserts. (It's a bit easier to make wheat-free main courses.) Later on I started avoiding sugar, and I had other friends who were avoiding dairy… A modern complaint about cooking for friends is that so many people are following strange restrictive diets that it's impossible to cook for all of them at once. If you're faced with such friends you have several choices. You can refuse to invite them for dinner, you can ignore their dietary preferences and cook whatever you like, you can ask them to bring their own food, or you can rise to the challenge.

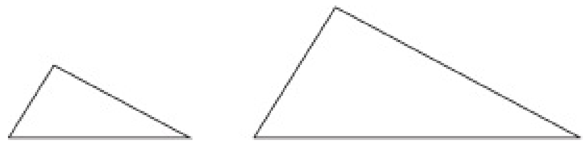
I invented the olive oil plum cake to be gluten-free, dairy-free, sugar-free and paleo-compatible. The only party guest who couldn't eat it was the one who was at that time only eating courgettes and ghee. Everyone said it was delicious, but when they asked me what it was I didn't know what to call it, because it's not really quite like a 'cake' – it's a *generalisation* of a cake. It has things in common with a cake, looks like a cake, is made like a cake, plays the role of a cake, but is still somehow not quite the same as a cake. It is useful in situations that an ordinary cake would not be able to handle.

This is the point of generalisation in mathematics as well – you start with a familiar situation, and you modify it a bit so that it can become useful in more situations. It's called a generalisation because it makes a concept more general, so that the notion of 'cake' can encompass some other things that aren't exactly cakes, but are close. It's not the same as a sweeping statement, which is a different use of the word, as we'll see later.

One example of generalisation is where we move from *congruent* triangles to *similar* triangles. Congruent triangles are ones that are exactly the same – they have the same angles and the same lengths of sides, that is, they're the same 'shape' and the same 'size'.



For *similar* triangles we only demand that they're the same shape, but not necessarily the same size – that is, they still have to have the same angles, but we drop the rule about having the same lengths of sides.



Because we've relaxed a rule, there are now more triangles that satisfy these conditions, but it still isn't total anarchy.

Flourless chocolate cake

Inventing things by omission

Imagine trying to 'prove' that you really need to boil water to make tea. You would probably just try to make tea without boiling the water. You discover that it tastes disgusting (or has no taste at all) and conclude that, yes, you do need to boil water to make tea.

Or you might try to 'prove' that you need petrol to make your car go. You try running it on an empty tank and discover it doesn't go anyway. So yes, you do need petrol to make your car go.

In maths this is called *proof by contradiction* – you do the opposite of what you're trying to prove, and show that something would go horribly wrong in that case, so you conclude that you were right all along.

Here's an example of a small proof by contradiction. Suppose n is a whole number and n^2 is odd. We're going to prove that n has to be odd as well.

We begin by assuming the opposite is true, so we suppose that n^2 is odd, but that n is even. However, an even number times an even number is always even, so this would make n^2 even. This contradicts the fact that n^2 was supposed to be odd, so we must have been wrong to assume the opposite. So the original statement that n is odd must be true.

Sometimes, proof by contradiction can be very unsatisfying, because it doesn't really explain why something *is* true, it just explains why something *can't be false*. We'll come back to this later when we talk about the difference between 'illuminating' and 'unilluminating' proofs, and the background assumption that if something isn't false then it must be true.

A famous, longer, proof by contradiction proves that $\sqrt{2}$ is *irrational*, which means that it can't be written as a fraction a/b, where a and b are integers (whole numbers). You might know that $\sqrt{2} = 1.4142135\ldots$ and that this decimal expansion 'goes on forever without repeating itself'. This is related to being irrational, but is not a proof. Here is a proof.

Proof. We start by assuming the *opposite* of what we are trying to prove, so we assume that there are actually two whole numbers a and b where $\sqrt{2} = \frac{a}{b}$. The trick is also to assume that this fraction is in its lowest terms, which means you can't divide the top and bottom by something to make a simpler fraction.

Now we square both sides to get

$$\begin{aligned} 2 &= \frac{a^2}{b^2} \\ \text{so} \quad 2b^2 &= a^2. \end{aligned}$$

So far so good. Now we know that a^2 is two times something, which means it is an *even* number. This means that a has to be an even number as well, because if a were odd then a^2 would also be odd.

What does it mean for a to be even? It means it is divisible by 2, which means that $\frac{a}{2}$ is still a whole number. Let's say

$$\begin{aligned} \frac{a}{2} &= c \\ \text{so} \quad a &= 2c \end{aligned}$$

and now substitute that into the equation above, so we get

$$\begin{aligned} 2b^2 &= (2c)^2 \\ &= 4c^2 \\ \text{so } b^2 &= 2c^2. \end{aligned}$$

Now we can do the same reasoning on b that we just did for a. We know b^2 is two times something, so it's even, which means b must be even.

Now we've discovered that a and b are both even. But right at the beginning we assumed that $\frac{a}{b}$ was a fraction *in its lowest terms*, which means that a and b *can't* both be even. This is a contradiction.

So it was wrong to assume $\sqrt{2} = \frac{a}{b}$ in the first place, which means that $\sqrt{2}$ cannot be written as a fraction, so is irrational.

□

Proof by contradiction can be very efficient, and mathematicians sometimes use it as a last resort when they can't work out how to prove that something is true directly – they instead try to prove that it can't be false. Sometimes that kind of proof doesn't turn out the way you're expecting it to. Maybe you try to prove that you really need flour to make a chocolate cake. So you make it without flour … and you discover that it's really not that bad. In fact, you've invented a whole new kind of cake: the flourless chocolate cake, now popular in many fancy restaurants.

Likewise yeast and bread. You might try to prove (no pun intended) that you definitely need yeast to make bread. So you try making it without yeast – and you've 'invented' unleavened bread.

This can happen in mathematics as well – you set out to try and prove that something can't be done, and you accidentally discover that it actually can, although maybe something slightly different results. This is one way that generalisation can turn up, almost by accident. One of the most important examples of this is from geometry, involving parallel lines.

Parallel lines

The genius of Euclid

Once upon a time Euclid set out to write down the rules of geometry. The idea was to axiomatise geometry, that is, write down a short list of rules from which all geometrical facts could be deduced. The idea is that your basic rules should be absolutely fundamental, things so basic that you can't imagine deriving them from anything else – they simply *are true*.

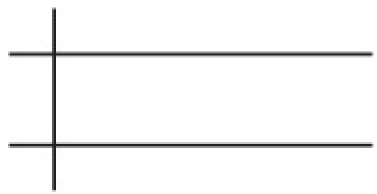
Anyway, Euclid came up with four very simple and obvious sounding rules, and one annoyingly complicated one. They went something like this:

1. There's exactly one way to draw a straight line between any two points.
2. There's exactly one way to extend a finite straight line to an infinitely long one.
3. There's exactly one way to draw a circle with a given centre and radius.
4. All right angles are equal.

   Those sound quite obvious, don't they? And then comes the fifth one:

5. If you draw three random straight lines they will make a triangle somewhere, if you draw them long enough, unless they meet each other at right angles.

The idea is that if your three straight lines meet each other at right angles, then two of them will be parallel, and no matter how long you draw them they will never meet up to form a triangle.



This is why the fifth law is called the ‘parallel postulate’, even though it doesn’t explicitly mention parallel lines. The fifth law is also what tells us that the angles of a triangle always add up to 180°.

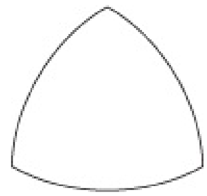
This last rule sounds so much more complicated than the others that people spent hundred of years trying to show it was redundant as a law, that is, that it could be deduced from the other four. Everyone knew they wanted it to be true; the only question was whether it needed to be enforced out loud, or whether it would automatically follow from the other laws, even if you didn’t say it out loud.

People went round and round in loops and often thought they had proved it from the first four laws, when really they had accidentally used some assumption about geometry that seemed very obvious to them but was subtly equivalent to the fifth law. So, implicitly, they were *using* the fifth law to *prove* the fifth law – not that earth-shattering.

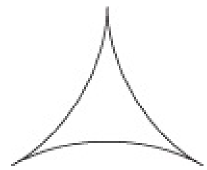
In the end people decided to try to prove it by contradiction, that is, they assumed that the first four rules held but that the parallel postulate did not, and then set about looking for things that would go horribly wrong elsewhere.

And the funny thing was, like with the flourless chocolate cake, nothing ever went wrong. It was just different – they had invented a new form of geometry.

We now know there are two types of geometry that don’t satisfy the parallel postulate. There’s the type where you imagine you’re on the surface of something round like a sphere or a rugby ball. Here, the angles of a triangle add up to *more* than 180°. This is called *elliptical geometry*.



The other type is where you imagine you're on a surface curved the other way. Here the angles of a triangle add up to *less* than 180°. This is called *hyperbolic geometry*.



The original case where the parallel postulate does hold is like being on a flat surface, and is called *Euclidean geometry*.

Taxi cab

Generalising the notion of distance

We talk about distance 'as the crow flies', but when you're actually travelling it's unlikely you'll ever travel as the crow flies – so the distance from A to B will change depending on how you're travelling. How much you care about this will probably change too.

If you take a train, you usually buy your ticket at the beginning and then you don't worry about exactly how far the train is going. But if you take a taxi, it really matters how far the taxi is going. However, instead of the distance-as-the-crow-flies, we're thinking about the distance 'as the taxi drives'. The trouble is this can be affected by questions such as: is the taxi driver going the long way round? So we'd better assume we have an honest taxi driver, just like we assume the crow is going to take the shortest route rather than take some scenic detour. The important difference is that distance now depends on things like one-way systems, and suddenly the principles that are followed by crow-distance might not hold for taxi-distance. (Perhaps one day we'll have flying taxis that will really take us as the crow flies, but not yet.)

Here's an example. For a crow, the distance from A to B is the same as the distance from B to A. But this is not true for a taxi. For example, if you hail the cab at one end of a one-way street and get it to take you to the other end, that will be a much shorter journey than when you try to go home again and have to go the long way round.

If I get directions on Google Maps between Sheffield train station and Sheffield town hall I get this:

Station to town hall by car	1.4 miles
Town hall to station by car	0.9 miles
As the crow flies	0.5 miles

In a place like London it's quite hard to work out the taxi-distance from A to B, because the one-way system is complex, because the streets are so bendy, and because you're so concerned about how expensive the whole thing is becoming that you can't really focus on distances. So let's talk about Chicago, where it's much easier to work out taxi-distance for several reasons.

1. Mostly, it's a grid system, so the roads are all long and straight and meet at right angles.
2. The addresses are numbered according to distance, so '5734 South' (the number of the Maths Department of the University of Chicago) tells you how far south of zero the building is, not that it's the 5734th building down. This blew my mind when it was first explained to me. Since 800 = 1 mile, you can calculate how far your taxi has to go relatively easily.
3. The one-way system is fairly sensible, so that it's mostly possible to get where you're going without doubling back on yourself too much, as long as you know the system and make your turns at well-timed moments.
4. Taxis are much cheaper than in London, so I don't get quite so worked up about how much it's going to cost.

Aside from getting worked up about the cost, this doesn't really depend on being in a taxi rather than any other kind of car. However, it is a genuine mathematical concept called the *taxicab metric*. It might be because it's the kind of thing mathematicians think about when sitting in a taxi, whereas if they're in a car one hopes they're concentrating on the traffic. We are gradually building up to the notion of 'metric', by investigating what sorts of properties distance-like notions should have.

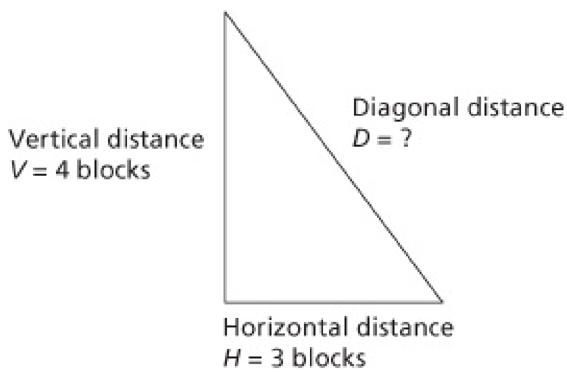
Of course, Chicago isn't *precisely* a grid system at all times, and there are big highways that cut across the grid system at diagonals. So we're throwing away the details about diagonals for the time being. This process of throwing away inconvenient details is a form of 'idealisation' that is a key part of mathematics. This can seem frustrating (there simply *are* diagonal highways in Chicago) but the point is to shed light on something rather than to model it precisely. Our aim now is to shed light on the notion of 'distance'. Now that we've turned Chicago into an 'ideal grid' that taxis drive across, making only right-angled turns, the taxi-distance from A to B is simply

$$\text{horizontal distance} + \text{vertical distance}$$

That is, no matter what clever route the taxi driver takes, it can't get any shorter than simply driving all the way across first, and then all the way down afterwards. Even if we make the turns in different places, say like this:

the distance is still the same, because we're not taking into account the time it takes to turn a corner. However, it would be longer if we did something really bizarre, like this:

If you remember anything about Pythagoras' theorem, you may remember that it tells us how to calculate the length of the diagonal edge of a right-angled triangle. In our case, that's the distance as the crow flies.



and in Pythagoras' case, that's called the 'hypotenuse'. Pythagoras' theorem says:

> *The square of the hypotenuse is equal to the sum of the squares of the other two sides.*

What this means on our diagram is:

$$D^2 = V^2 + H^2$$

and we can work out what the diagonal, crow-flying distance is:

$$
\begin{aligned}
D &= \sqrt{V^2 + H^2} \\
&= \sqrt{4^2 + 3^2} \\
&= \sqrt{16 + 9} \\
&= \sqrt{25} \\
&= 5.
\end{aligned}
$$

The crow only has to fly the distance of five blocks. However, the taxi has to go the vertical distance and the horizontal distance:

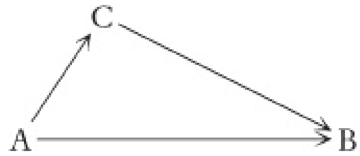
$$
\begin{aligned}
\text{Taxi distance} &= V + H \\
&= 4 + 3 \\
&= 7.
\end{aligned}
$$

The taxi has to drive the distance of seven blocks. The crow knows that taking some sort of diagonal route across the grid would definitely be shorter. But as a taxi, even if we tried to wiggle in a diagonal sort of fashion across the grid, it wouldn't help us – we'd still have to wiggle in only horizontal and vertical straight lines, and it would still add up to the same total horizontal distance and total vertical distance. And worse: we'd have to turn a lot of corners in the process.

Still, the taxi-distance is a perfectly good notion of 'distance', and is an example of generalisation. Again, we have taken a notion that we know and love, and we can now see what other notions are a bit like it but somehow different. What sorts of things should also count as 'distance'? This idealised taxi-distance obeys two crucial rules that crow-distance obeys:

1. The distance from A to A is zero, and that's the *only* way of getting a zero distance.
2. The distance from A to B is the same as the distance from B to A.

But there's also a third rule that is related to Pythagoras' triangle. It says that if you're trying to go from A to B, it can't be any better to go via some random other place C. Usually that will make it worse:

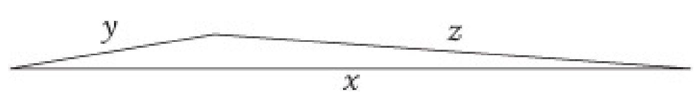


At best, C was on the way from A to B anyway, and going via C made no difference.

(You might have trouble trying to persuade a taxi driver of this though.) This rule about stopping off on the way is called the ‘triangle inequality’ because it’s about the edges of a triangle – not necessarily a right-angled one any more. It’s like a very puny version of Pythagoras’ theorem.

> **Pythagoras:** Yes! If we have a right-angled triangle then we can precisely work out the exact length of any side from the other two!
>
> **Triangle inequality:** Um, if we have a non-right-angled triangle then we know that the length of the third side will be *at worst* the sum of the other two.

Here ‘worst’ means ‘longest’ (because we’re thinking taxis), so what we’re saying is that if the sides of the triangle are x, y and z then the biggest x can be is $y + z$. You can imagine this as being an extremely long and thin triangle where the y and z edges have pretty much done the splits so that x has to be really long to accommodate them, like this:



Now if we think of the edges of that triangle as the distances between our three places A, B and C then we get the ‘intermediate stopping place’ rule from before.

There are two curious things about this triangle inequality rule, I think. The first is that the taxi-distance still obeys this rule. The second is that there’s a perfectly common ‘distance-like’ situation that does not, which is the cause of endless frustration to me: train tickets.

Train tickets

Generalising the notion of distance a bit more

If you've taken many trains around the UK you'll know exactly what I mean. It's the infuriating fact that sometimes, if you want to take a train from A to B, it's cheaper to buy two singles, via somewhere else. It's particularly stupid because you don't even have to take a different route – you just have to split the ticket in two. You don't always even have to get off the trains. Remember here we're not thinking about the actual distance covered in going from A to B, but the *cost* of going from A to B. In a sensible world, this would obey the triangle inequality – it would not cost less to go via some other place C. But in reality it does, or at least, it can.

For example, to go from Sheffield to Cardiff it can be cheaper to buy a single from Sheffield to Birmingham and a single from Birmingham to Cardiff.

To go from Sheffield to Gatwick it can be cheaper to buy a single from Sheffield to London and another from London to Gatwick.

To go from Sheffield to Bristol it can be cheaper to buy a single from Sheffield to Cheltenham and a single from Cheltenham to Bristol.

This is aside from the various other anomalies of UK train ticket prices, such as:

- Sometimes it's cheaper to go first class than standard.
- Sometimes it's cheaper to go further, for example London–Ely can be cheaper than London–Cambridge although the Ely train stops at Cambridge on the way.
- Sometimes it's cheaper to get a flexible ticket (where you can travel at any time of day) rather than one where you can only travel off-peak.

These last points are harder to explain in relation to the three rules of distance, because they're more to do with the interaction between cost and distance, or cost and time. So we'll leave those for now. Often in mathematics we focus on the easier things first, not because we're being wimps, but because the harder things are often built up from the easier things, and so we have to get the easier things right first.

In order to see why rules are imposed, it's often helpful to look at situations where they are *not* obeyed. Why is drinking alcohol not allowed on the Tube? Because it caused havoc. Why is smoking not allowed in Tube stations? Because there was a huge fire that killed people. This is similar to wanting to understand the principles behind things, rather than just memorising the rules or blindly following instructions in a recipe.

Now our three rules of distance are:

1. The distance from A to B is zero when A and B are the same place, and this is the *only* way the distance from A to B can be zero.
2. The distance from A to B is the same as the distance from B to A.
3. The distance from A to B can’t be made any shorter by going via C.

Now that we’ve come up with a proposed list of axioms for the notion of distance, we’ll do what is often the temptation when presented with a list of rules: we’ll try to break them. The point of trying to break rules in mathematics is not to be arbitarily rebellious, but to test the strength and the boundaries of the world that we have set up.

We’ve seen distance-like situations that break rule 3 (train tickets) and 2 (one-way systems) but what about 1? You might think there’s no real situation that violates rule 1 but here is one.

Online dating

Generalising the notion of distance yet further

GPS is marvellous technology. It means I get lost a lot less than I used to, especially on buses, where I can follow my position along the map on my phone, and then miraculously get off the bus in the right place.

GPS has also made online dating rather immediate. In the old, slow model, you could see if someone lived in the same city as you, or within say 100 miles, or 200 miles. With GPS, you can see how many *metres* away this person is *right now*. I’ve watched friends of mine do this in bars (just for a laugh, of course …) and the excitement of seeing how close someone is is palpable, especially when they’re getting closer. ‘Ooh, this one is only 200 metres away … 150 metres away … 50 metres away – wait, doesn’t that mean he’s in here?’

However, this can cause great disappointment because the distances are based only on GPS and don’t take into account how far off the ground you are. A friend of mine was lonely in a hotel room somewhere and was perplexed at the number of interested parties who were supposedly ‘zero metres away’. ‘And yet,’ he lamented, ‘here I am alone in my hotel room.’

This is an example of a distance-like notion that does not obey the first rule of distance – that you can only be zero distance away if you're actually in the same place. This is relevant to some slightly more useful situations than lamenting your online-dating problems as well. For example, if your 'distance-like notion' is not actually the distance from A to B, but the amount of energy you need to expend to transport something from A to B. Then if A is directly above B you can just drop it, so the energy used getting it from A to B is zero, even though A and B are not in the same place.

A 'distance-like notion' is called a *metric* in mathematics. There's one more rule it has to satisfy that we didn't bother mentioning: that the distance from A to B is never negative. There are even situations where it's useful to relax this rule, for example, if we're studying how much it will cost to transport something from A to B. Not only might it cost you nothing (so the 'distance' would be zero), but someone might even pay you to do it. Coffee growers in Costa Rica are *paid* to send their coffee to Europe to be decaffeinated, because the caffeine that is extracted is so valuable to the makers of energy drinks.

Relaxing one or more of the usual rules for metrics is one way to generalise the notion of distance in mathematics. Another way combines generalisation with abstraction, and gives us the notion of *topology*, which we'll look at later in this chapter.

Three-dimensional pen

Generalising by adding dimensions

The problem with using GPS for online dating, as we saw above, is that it assumes we're only in a two-dimensional world. This usually works fine for finding your way around in a car, but not for finding a potential date inside a skyscraper, where the third dimension is rather important.

Increasing the number of dimensions is an important form of mathematical generalisation. There's a joke that if you're at a maths research seminar you can ask an intelligent-sounding question even if you don't understand anything, which is 'Can this be generalised to higher dimensions?'

A sphere is a higher-dimensional generalisation of a circle, if you think about a circle in the right way. Let's think about drawing a circle with a pair of compasses (although these days we all just draw circles by selecting a circle function on a computer). With compasses, you first choose a size (radius) for your circle, so let's say you open the compasses to 5 cm. Next you fix the pointy tip on the page where the centre of your circle will be, and then with the drawing end you essentially mark every point on the page that's exactly 5 cm from the centre.

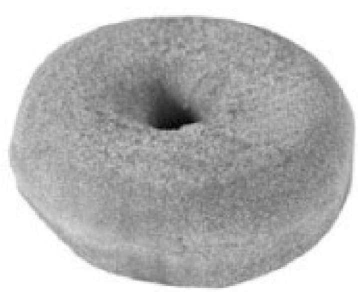
Now imagine you have a pen that can draw in mid-air, which is something I've always dreamt of. Then you could fix your compass point somewhere, and use your mid-air pen to mark every point in the air that was exactly 5 cm from your chosen centre, *in all directions*. This would be a sphere.

At this point mathematicians are perfectly happy to generalise this to four, five, or even more dimensions, although we don't exactly know what that means. A sphere of radius 5 cm in four-dimensional space is 'all the points in that space that are exactly 5 cm from a fixed centre'. Because it's an *idea* rather than a physical object, it doesn't matter that we don't know what it looks like. It only matters that the idea makes sense. But just because one generalisation makes sense doesn't mean there aren't others that make sense, too.

Doughnut

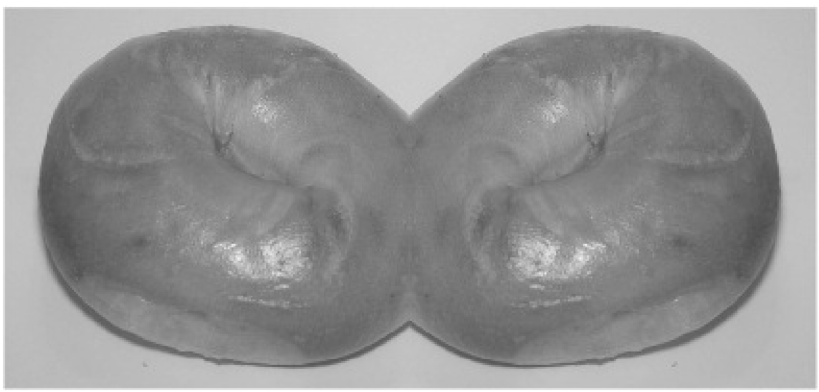
A different generalisation of a circle

Imagine a doughnut. A ring doughnut.



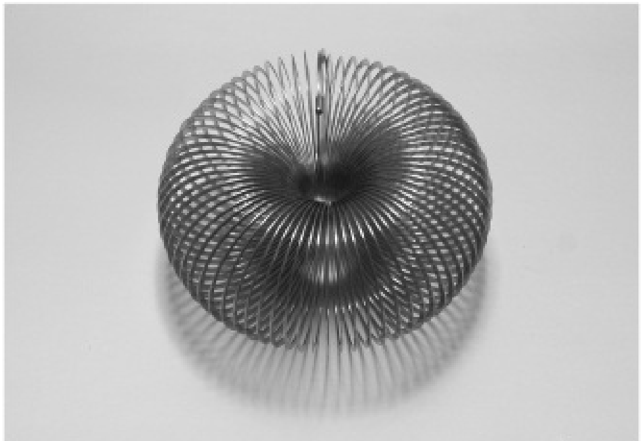
When mathematicians say 'doughnut' they always mean a ring doughnut, at least when they're talking about maths. Perhaps they should start saying 'bagel' instead.

How would you generalise a bagel? The most obvious way is to give it more holes. A two-holed bagel!



But there is another way to generalise it. For this we have to be a bit more careful about this bagel/doughnut of ours. When mathematicians think about doughnuts they're usually only thinking about the *surface* of the doughnut, not the solid doughnut. Just like when they say 'sphere' they only mean the surface of the ball, like the skin of an orange, not the whole orange. A sphere is like a balloon, with empty space on the inside.

Likewise for doughnuts. Perhaps you can imagine taking a toilet roll, magically turning it into stretchy rubber, and bending it round into a little hoop. Or perhaps imagine taking a Slinky and bending it round so that the ends meet up. It will look like a ring doughnut, but be hollow.



This is technically called a *torus*.

Now, let's think about how we made it from a toilet roll. You could also imagine trying to make it out of bubbles – the kind that come in a big bottle, with a big hoop that you can drag through the air to make bubbles, instead of blowing. Now imagine taking this hoop and dragging it through the air for a while – you make a sort of bubble tube as you go. Now imagine dragging it in a big circle so that it comes back to meet itself. It will be like a doughnut – a hollow doughnut. A hollow bubble doughnut.

We made this by dragging a hoop through the air in a circle, which shows that the torus is a generalisation of a circle – all we've done is draw in the air with a hoop instead of a mid-air pen. Now for the generalisation of the torus things are going to get a bit weird. Imagine dragging *an entire doughnut* through the air in a circle. It's pretty difficult to imagine what this looks like, because it doesn't really fit into three-dimensional space, but perhaps you can at least imagine that it's definitely not the same as a two-holed doughnut.

Sweeping statements

A different kind of generalisation

'It always rains in England.'

'The trains never run on time.'

'Opera is really expensive.'

‘You always say that.’

These are all *sweeping statements*, or generalisations. But this is a different kind of generalisation from the kind where you turn a bagel into a two-holed bagel. This kind is not about relaxing conditions to allow more people in, but is more like ignoring outlying cases temporarily, to focus on the central part of the bell curve.

Of course, these sweeping statements aren’t *entirely* true. Occasionally, trains do run on time. And sometimes it stops raining in England. And you can easily get opera tickets in London for under ten pounds. And you don’t really say ‘that’ (whatever that is) all the time, just under certain situations. The question is, do these exceptions matter? Do we study exceptions or do we study the main body of behaviour?

The answer, surely, is both. We can’t really study one without studying the other. There are interesting things to be learnt from the extremities of behaviour, even if those extremities are rare, and so not at all representative. But how can we know in what way something is unusual if we don’t also study what is usual? That involves temporarily ignoring the extremities.

Bagels, doughnuts and coffee cups

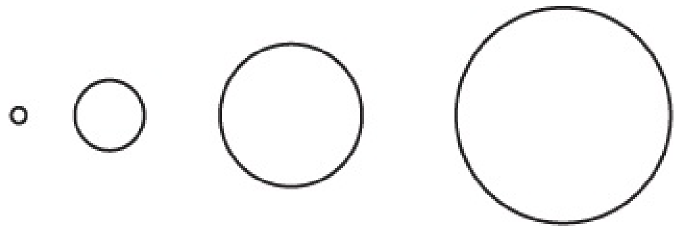
An introduction to topology

Combining our previous discussions about distances and bagels brings us to a branch of mathematics called ‘topology’, which studies the shapes of things. We’ve already seen ways of generalising the notion of ‘distance’ so that we have something a bit like distance, but not necessarily satisfying all of the usual rules that distance does.

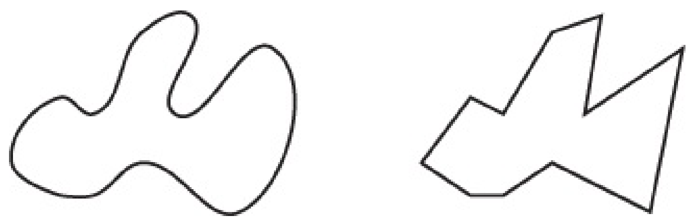
But now we can generalise this even more, because there are times when we don’t mind so much exactly how far apart two things are, but only whether we can get from one point to the other, and how. If you live in the south of England, the Isle of Wight is probably closer than Scotland, but the fact is that you can’t just drive there – so it’s a whole different kind of hassle.

Something similar can happen with neighbourhoods of a city. Some cities, like Chicago, can change rather abruptly from one block to the next, where one ‘neighbourhood’ ends and another begins. It doesn’t matter that you’ve only travelled one street over – the distance is very small, but you’ve gone into a completely different neighbourhood.

When we don't care about distance it means we also don't care about size, just like with the similar triangles, so all these are 'the same':



Another related thing we might not worry about is *curvature*, so these two shapes also count as 'the same':



In fact the only thing we're really worried about is the number of holes something has. So now we have a system under which not only are all triangles 'the same', but triangles are also 'the same' as squares and circles: they're all shapes with one hole. However, a figure 8 is 'different' because it has two holes.

One way to think about this is to imagine that everything is made of plasticine or playdough, and you want to know if you can bend one shape into another without making any new holes or sticking anything together.

Question: Which capital letters of the alphabet are 'the same' in this bendy sense?

- There are letters with no holes: C E F G H I J K L M N S T U V W X Y Z.
- There are letters with one hole: A D O P Q R.
- There is just one letter with two holes: B.

What this says is that *topologically* almost all letters are the same. This is one of the reasons that computer recognition of handwriting is so hard.

We can also try this in higher dimensions. Imagine trying to make a bagel (a solid one, not a hollow one) out of a lump of playdough. There are basically two ways of doing it: you could either make a sausage shape and stick the ends together, or you could poke a hole in the lump. Either way, you've done something that shows that a bagel is not topologically the same as a plain lump. However, once you have your bagel/doughnut shape, you can make a coffee cup *without* making any new holes or sticking anything together. The doughnut's hole can turn into the handle of the coffee cup, and then you just need to squash an indentation in the rest of it to make the cup part. What this says is:

Topologically, a bagel is the same as a coffee cup.

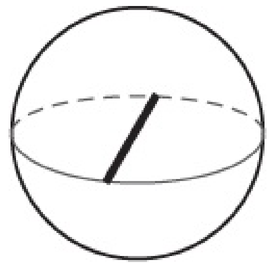
However, the 'two-holed bagel' pictured earlier, is definitely different. The study of which things are topologically the same and different has many applications. For example, we talked about the mathematics of knots earlier on, and these are studied using topology. The amazing idea here is like the kind of drawing where instead of drawing on a blank page, you colour in an entire page and then erase parts to make a picture in white. Now we'll imagine doing this in three dimensions.

Imagine your mid-air pen again, and imagine that you have 'coloured in' the whole inside space of a box. Now you take a 'mid-air eraser' and erase a knot from what you coloured in. What is left is something with a curious shape that's almost impossible to imagine, but very handy to study mathematically.

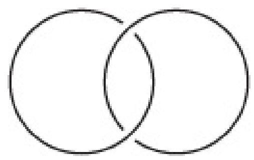
A challenge for your imagination

The process of erasing something in three dimensions that we just described is called taking the 'complement'. Once we've done it, we can imagine that we are allowed to squash what's left just as if it were playdough, again without making any new holes or sticking things together. Can you imagine the following complements?

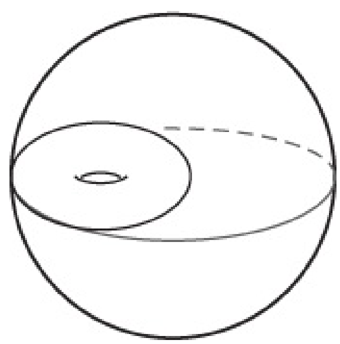
- The complement of a circle ○ is topologically the same as a sphere with a bar stuck across the middle of its empty insides:



- The complement of two interlocking circles

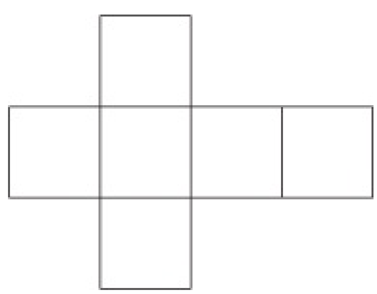


is topologically the same as a sphere with a torus stuck on the inside of the surface, in the empty space:

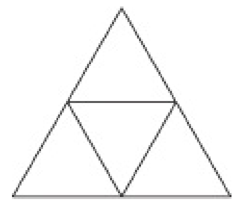


Those were only very simple shapes, and already it's very hard to imagine them in your head. The power of mathematics is that it enables us to study these things rigorously without having to imagine them at all.

Here's another example, involving cutting out shapes and sticking the sides together to make something three-dimensional. You may remember how to make a cube starting from a flat shape.

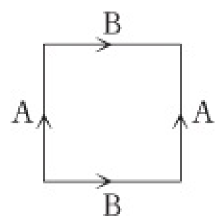


If you cut this out and fold it along the lines, you can stick the edges together to make a cube. If you try it with this one:

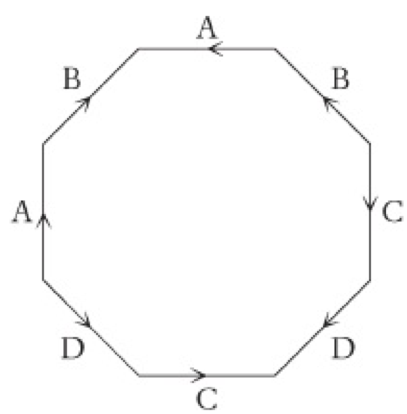


you will get a triangular pyramid which is technically called a *tetrahedron*.

Now imagine that you're actually making this out of bendy playdough paper. Now we can make a bagel/doughnut/torus out of a square like this – here we have to make sure we stick the edges labelled A to one another, with the arrows matching up, and likewise the edges labelled B:



Now here's a serious challenge. Can you imagine what shape you'll get if you cut out this octagon and stick it together according to the labels?



The answer is: a *two-holed bagel*.

Now imagine trying to generalise this for even more holes – it's pretty hopeless to try and do this in your head, but topology gives us a way of studying these things rigorously, for shapes much harder than those that our imaginations can ever visualise.

A generalisation game

What do the following shapes have in common?

square, trapezium, rhombus, quadrilateral, parallelogram

The answer is that they all have four sides. Now can you see how to arrange them in order of *increasing* generality? And what is the process of generalisation to go from each one to the next?

The answer is:

square, rhombus, parallelogram, trapezium, quadrilateral

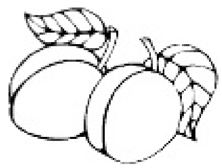
The processes of generalisation are like this:

- A square has all four sides the same length, and all four angles the same.
- A rhombus only has all four sides the same, so the step of generalisation is to allow the angles to be different. However, they will be forced to be in pairs – the angles opposite one another have to be the same as one another just because the sides are all the same length.
- A parallelogram is like a rhombus but now only the sides opposite one another have to be the same length. There's no generalisation regarding the angles, which will still be forced to be the same in opposite pairs. Note that opposite sides are forced to be parallel, because of the opposite angles being the same.
- A trapezium only has the condition that one pair of opposite sides has to be parallel. So there's no longer any condition on the lengths of the sides or the sizes of the angles. They can now all be different.
- A quadrilateral is any old shape with four sides, so in this step we have generalised by dropping the condition that one pair of sides has to be parallel.

In this example we can see that each step of generalisation occurred by dropping some conditions on the shape in question, so that more shapes were allowed into the picture. Relaxing conditions slightly is one of the common ways of performing a generalisation in maths.

You might have noticed that there's another possible step in this generalisation, via another type of four-sided shape that we didn't mention above: the rectangle. A rectangle is a different way of generalising a square – where a rhombus still has the same lengths of sides, but possibly different angles, a rectangle has the same angles, but possibly different lengths of sides. When we relax rules one by one, we get different routes to generalisation depending on the order in which we relax the rules. Generalisation is not an automatic process. There are always different possible generalisations depending not just on how far you go, but on what point of view you take. This is one of the reasons mathematics as a subject keeps growing at an ever-increasing rate, as each generalisation gives rise to a multitude of others.

6 INTERNAL VS EXTERNAL



Chocolate and prune bread-and-butter pudding

Ingredients

250 g stale bread, without crusts
350 g chopped prunes
100 g dark chocolate
2 eggs
75 g caster and dark muscovado sugar in total
50 g melted butter
300 ml milk

Method

1. Break the bread into small pieces and make into bread crumbs in a food processor.
2. Beat the eggs and the sugar.
3. Melt the chocolate gently with the milk, and mix it into the eggs.
4. Pour over the bread and prunes in a large bowl, and leave it to soak for a few hours.
5. Mix in the melted butter.
6. Bake in a lined 8-inch square cake tin, at 180°C for 45 minutes or until set and slightly crispy on top.
7. Serve warm with chocolate sauce or chocolate custard.

This recipe for chocolate bread-and-butter pudding is something I came up with after making Christmas pudding one year. I had leftover bread (which I don't usually eat, and which had gone stale because I'd cut the crusts off) and prunes (which quickly go rock hard once you've opened the packet). And, of course, I always have plenty of chocolate in the house.

There are many dishes invented by our more frugal ancestors for using up leftovers. Cottage pie and shepherd's pie for using up leftover roast meat from Sunday lunch. Bread-and-butter pudding and French toast, or as the French call it *pain perdu*, literally 'lost (or wasted) bread', make use of stale bread by softening it up in egg and milk. There's the Chinese version, egg fried rice, where leftover rice is similarly fried with egg to soften it up again. Black bananas can be made into delicious banana cake. And everyone has their favourite dish to make out of the mountains of leftover turkey that are somewhat inevitable at Christmas. Curry? Pie? My favourite was my mother's turkey spaghetti salad with peanut sauce.

In all these cases it's sort of the wrong way round if you go and and deliberately look for the ingredients to make a dish that was supposed to be there to use up leftovers. Something similar can happen even if you're deliberately making a dish from new ingredients, as we mentioned in Chapter 1: you could pick a recipe and go shopping for the ingredients you need, or you could buy some ingredients that look interesting and invent something with them.

All this illustrates the difference between what I call *internal* and *external* motivation. If you set out with a recipe in mind, this is an external motivation. If you make something up from the ingredients you have, it's an internal motivation. Sometimes you set out with something in mind, but make it up as you go along, to see what will happen. If it then matches up with whatever you had in mind to make, your internal and external motivations have gloriously come together. Sometimes things turn out completely differently from how you were expecting them but are still fantastic. Or maybe you had no idea what to expect at all (like when I first tried making raw chocolate energy bars) but it's fantastic anyway. This is what we might call a 'happy accident'. That is different from the internal and the external matching up.

Funnily enough, in the kitchen I'm much more externally motivated. In maths I'm very internally motivated.

Here's a small mathematical example. If I give you the following numbers: 25, 50, 75, 100, 3, 6 you could mess around and see what other numbers you can make, by adding subtracting, multiplying and dividing, like on *Countdown*. That would be like an *internal* motivation, where you start with some ingredients and see what you can build with them.

Or if you were actually on *Countdown*, you might try to use these numbers to make a given number, such as 952, like the mathematician James Martin did rather spectacularly some years ago, as follows: $\frac{(100+6)\times 3\times 75-50}{25}=952.$

That was like *external* motivation, where you try to build something specific in whatever way you can.

Tourism

Using a map vs following your nose

When you're visiting a new city, do you set out to look for particular attractions that you've heard about, or do you just plonk yourself in the middle of the city and follow your nose? People often say that their favourite thing about a holiday was when they were just wandering around and discovered some little hidden gem down a backstreet. Sometimes this happens when you're trying to get to the Eiffel Tower or the Empire State Building or some other much-trumpeted destination, and you stumble upon a fantastic little cafe on the way.

Maths is like this too. A lot of maths happens by trying to answer a particular question or solve a particular problem. That is, you have a particular destination in mind and you just want to get there. This is external motivation. Many of the great problems in the history of maths have been like this: a particular question that needs answering, and nobody really minds how it's answered as long as it gets answered.

One of the problems with learning maths at school is that almost everything – or maybe everything – is externally motivated. You're always just trying to solve a problem, and worse, it's a problem that somebody else set for you, which you probably have no need to solve apart from for your maths homework, or maths exam, or something.

Take solving quadratic equations. You might remember from your past, or from Chapter 2, that if you're given an equation like $ax^2 + bx + c = 0$

the solutions are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula was produced *just* for solving that equation. It's not exactly something that you'd just come up with for fun and think 'I wonder what I can do with this?'

In real research maths, it often happens the other way, where you just give yourself a starting point in the mathematical world, and see where it takes you. I call this 'internal motivation'. It's a bit less dramatic, and so tends to get less attention. Just as your little gem down a backstreet is much less dramatic than the Eiffel Tower, and probably won't get a mention in the guidebooks. But what is it that makes Paris what it is – the Eiffel Tower or all the little gems down backstreets? Surely both, and indeed, the way they are juxtaposed.

One of the most famous instances of this is that the study of prime numbers was not thought to have any useful applications for hundreds of years. And yet, mathematicians were fascinated by them just because they're intrinsically fascinating, and seem so fundamental. How could they have known that a theorem proposed by Fermat in 1640 and proved by Euler in 1736 would become the basis for internet cryptography several centuries later? Even computers were hundreds of years away. Incidentally this is the same Fermat of 'Fermat's last theorem' fame, but the theorem in question is known as 'Fermat's little theorem' to distinguish it from the 'big' one.

In fact, Fermat's last theorem itself is an example of the curious ways in which the internal and external motivation can interact. First, there are the discoveries you can make along the way to the question you're trying to answer. Along the way to proving Fermat's last theorem, Andrew Wiles made many important discoveries about elliptic curves – a particular type of curve that doesn't sound like it should have anything to do with Fermat's last theorem. Remember, this theorem says it is impossible to make the equation $a^n + b^n = c^n$

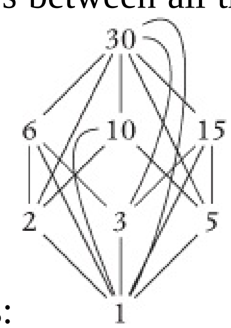
work for any whole numbers a, b, c if n is a whole number bigger than 2.

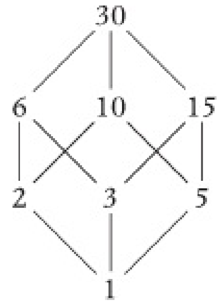
But there's also the interaction the other way, the way that I find the most satisfying and beautiful. This is where you put yourself in the middle of a city and have in mind that you'd like to see the Notre Dame, let's say, but instead of just going straight there following a map, you follow your nose down the interesting winding streets in the way that interests you. And then, lo and behold, you find yourself at the Notre Dame. In the case of Fermat's last theorem, mathematicians were also working on elliptic curves for their own sake, in a way that also happened to help with proving the theorem.

When maths is done purely by external motivation, it might be like taking such a determined route to the Notre Dame that you end up walking up a horrible main road for ages. You could say that this is maths that is overly utilitarian or pragmatic. When it's done purely by internal motivation, you might go on a very pretty journey but never arrive at anything notable. You could say this is maths that is overly idealistic or aesthetic. When the two coincide you get a journey that is interesting in its own right, with a destination that is also interesting in its own right – the best of both worlds, and the most beautiful of mathematics.

Different areas of maths have a different emphasis. Number theory has many famous unsolved problems that mathematicians are trying to solve in whatever way they can. Category theory is a bit different. One of its aims is to find the internal motivation behind everything, or to find the point of view that illuminates the internal motivation that was secretly already there. In Part II we'll see various ways in which category theory does this. Here's an example. We can think about all the possible factors of 30, that is, all the whole numbers that go into 30 without leaving a remainder. These are: 1, 2, 3, 5, 6, 10, 15, 30.

However, just listing them all in a row like this is not as illuminating as it might be, because in fact some of these factors are also factors of each other. If we draw lines between all the ones that are factors of each other we get a picture like this:



But this is a bit of a mess. We can clear it up if we decide only to draw lines where there isn't another factor *in between*. So we'll put a line between 6 and 30, but not a line directly from 2 to 30, because 6 is in between. In that case we get this more satisfying picture: 

We'll come back to this sort of picture later and see that this is exactly how category theory brings out structure, making concepts visible in geometrical diagrams.

Jungle

Invention vs discovery

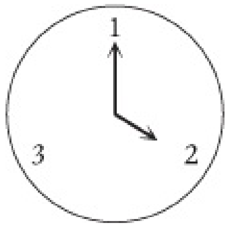
Sometimes I think about how different the world of 'research' was when there were still parts of the earth unmapped, still new large animals to be discovered – at least by Europeans. I suppose there are still new insects and bacteria and plants being discovered, but imagine being the first Europeans to see a platypus. And nobody believed them – when the specimen and drawing arrived in Great Britain, in 1798, it was suspected of being a hoax, perhaps created by a skilled taxidermist attaching a duck's beak to some other animal.

Here's some maths that some people think is a hoax. People often say to me 'Maths is always just right or wrong, I mean 2 + 2 just *is* 4.' And yet, I'm now going to explain that sometimes 2 + 2 = 1.

Do you think I'm just pulling your leg? I'm actually not. There is a world of numbers in which this is true. It's like being on a three-hour clock instead of a twelve-hour clock. We're quite used to the fact that if it's now 11 o'clock, then two hours later it will be 1 o'clock. In other words,

$$11 + 2 = 1.$$

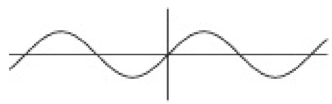
If we were on a three-hour clock



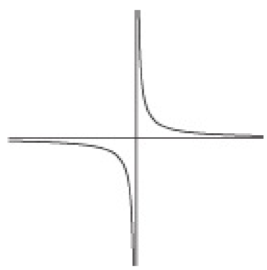
then two hours later than 2 o'clock it would be 1 o'clock. In other words 2 + 2 = 1.

This example might seem a bit contrived, like I'd invented it for the sole purpose of making a silly answer for 'two add two'. That is, I made it with *external* motivation. But later on we'll see that this 'three-hour clock' number system arises quite naturally from *internal* motivations, and is quite important.

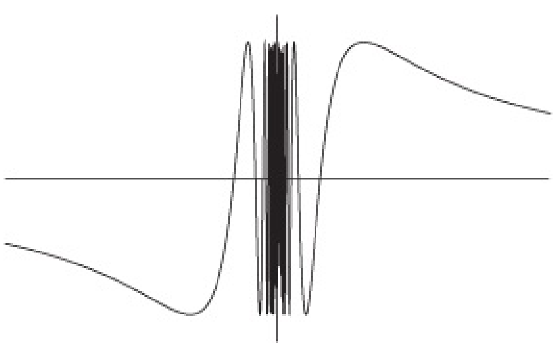
Here's an internally motivated example of a strange mathematical creature. You probably remember what the graph of $y = \sin x$ looks like:



and what the graph of $y = \frac{1}{x}$ looks like:



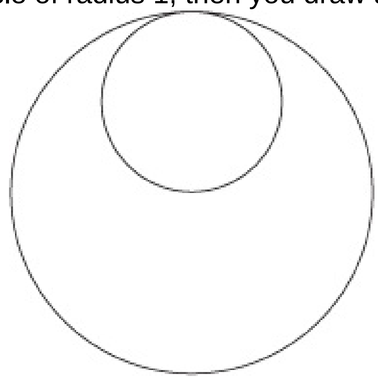
Now we might blithely try combining these, to look at the graph of $y = \sin(\frac{1}{x})$. This function is very wild.



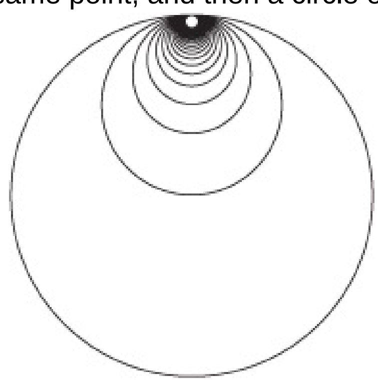
On the other hand, sometimes mathematicians set out deliberately looking for wild functions, like looking for the Loch Ness monster. What usually happens is that they want a particularly wild example of a function or a space or something, so they deliberately make one up.

Here's an example of a wild function that's been 'made up' with external motivation. We say $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. This function is basically impossible to draw because it leaps up and down between 0 and 1 all the time.

An example of a space that's been deliberately made up to confuse everyone is known as the 'Hawaiian earring'. You start with a circle of radius 1, then you draw a circle of radius $\frac{1}{2}$ stuck to it somewhere on the inside:



Then you add a circle of radius $\frac{1}{3}$ attached at the same point, and then a circle of radius $\frac{1}{4}$, and then $\frac{1}{5}$, and you keep going 'forever':



Remember, this is maths, so you don't actually have to sit there drawing forever: you just have to imagine that you did. Anyway the Hawaiian earring has very strange and wild properties which are quite exciting to topologists.

Jigsaw puzzle

Fitting pieces together vs looking at the picture

When you sit down to do a jigsaw puzzle, do you look at the picture on the box first, and match up all the pieces to the picture? Or do you put the picture away and just work out how the pieces fit together by comparing them to each other?

If you use the picture on the box, that's like an external motivation in maths. You have a clear aim, you know what the aim is, and you're trying to get there. If you don't look at the picture, that's like internal motivation. You're trying to see how the pieces fit together based on their own structure and their relationships with each other, not their relationships with something external.

I've found that a small child's first instinct is often the internal rather than the external, with jigsaw puzzles. They seem more likely to just keep trying to fit pieces with each other if they look vaguely similar, rather than comparing the pieces with the picture on the box. In fact I've found it quite hard to persuade small children there's any point at all in looking at the picture on the box; I suppose there is some stage of development where they make the connection with the internal and the external. There's also a more literal sense in which they seem more interested by the internal than the external: they tend to start with the middle of the puzzle, where the interesting part is. Most adults learn, at some point when they're growing up, that the sensible way to start a puzzle (at least, assuming it's rectangular) is to find the four corners, and then find all the edge pieces, and put the edge in place. Children, at least the children I know, don't seem to want to do that at all.

When I took physics A-level we were given a formula sheet that made the whole thing more like a jigsaw puzzle than a test of physics knowledge. So we had a list of helpful formulae that we weren't expected to remember, such as:

force between two point charges	$F = \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q_1 Q_2}{r^2}$
force on a charge	$F = EQ$
field strength for a uniform field	$E = \dfrac{V}{d}$
field strength for a radial field	$E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

Now, I'll be the first to admit that a lot of it didn't really mean anything to me. In fact I was quite proud of the fact that I found a way of doing extremely well at physics A-level without really having to understand any physics. I just read the question, wrote down all the letters corresponding to the quantities given in the question, and then scanned the formula sheet for a formula containing all the correct letters. This is like the efficient adult way of doing a jigsaw puzzle by ‘external’ processes rather than ‘internal’ ones. I felt I had worked out the most efficient way to get an A at physics A-level with the least possible work.

Later we’ll see that category theory often bridges the gap between internal and external processes. It makes the internal processes more geometrical, so that sometimes it really is like fitting a jigsaw puzzle together.

Here’s an example of a jigsaw puzzle in category theory. You can try fitting the pieces together even without knowing what they mean. We have these two pieces:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd} \qquad \begin{tikzcd} GA \arrow[r, "\beta_A"] \arrow[d, "Gf"'] & HA \arrow[d, "Hf"] \\ GB \arrow[r, "\beta_B"'] & HB \end{tikzcd}$$

and we want to make this picture:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[r, "\beta_A"] & HA \arrow[d, "Hf"] \\ FB \arrow[r, "\alpha_B"'] & GB \arrow[r, "\beta_B"'] & HB \end{tikzcd}$$

We can just fit the two pieces together sideways to make the picture, like this:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[r, "\beta_A"] \arrow[d, "Gf"] & HA \arrow[d, "Hf"] \\ FB \arrow[r, "\alpha_B"'] & GB \arrow[r, "\beta_B"'] & HB \end{tikzcd}$$

This is a very typical calculation in category theory. The pictures get bigger and bigger and there are more and more pieces we get to use. However, because the pieces are *abstract* pieces, we have an endless supply of them and we can use each one as many times as we want.

In case you’re wondering, this is part of the proof that *composing natural transformations* *component-wise yields another natural transformation*. More generally, this sort of jigsaw puzzle in category theory is called ‘making diagrams commute’, and is something I find fun and satisfying.

Marathon

Getting fit vs training for a race

If you work out or do something to keep physically fit, are you always training for a specific event? Some people always aim for a specific event, like a marathon, a triathlon or an expedition, to keep themselves motivated. Others do it for general fitness, enjoyment or stress release. Of course, it’s probably some kind of combination of those things – if you don’t enjoy it in the first place, then aiming for a marathon is hardly going to help.

When I ran the New York Marathon I had to change my workout dramatically. I had read various articles saying that you can run a half marathon without really specifically training for it, but not a marathon. Indeed I had already run the London Half Marathon without specifically training, other than my usual every-other-day gym routine. Also, I had a reasonably fit friend who tried to run the New York Marathon without specifically training, and he had damaged his knee.

So I did much longer workouts, building up my stamina, and following a pattern of fortnightly long runs that I found online somewhere, tapering off in the last few weeks so that the longest long run occurred something like a month before the actual marathon. It all worked fine, and I finished in exactly the time I planned (which was in fact extremely slow, but I had very realistic expectations of myself).

This is all to say that for about six months my workout became externally motivated – I had a specific aim in mind and everything was geared towards that aim. Before that, however, and ever since, my workout has been internally motivated, without a specific aim (‘general fitness and weight loss’ not counting as a specific aim here). The point was the workout itself, and how much I enjoy that process in its own right.

Maths is often sold for its external motivations – it is useful for getting a job, it is useful for real-life situations. But just as with the marathon, if you don’t enjoy it in the first place, then imposing some contrived ‘real-life’ situation on it won’t help. Take this example that a friend of mine gave me recently – she was trying to help her son with his homework, but needed help herself.

George drove 764 miles last week and his car used 15 gallons of petrol. If George averages 54 miles per gallon on motorways and 31 miles per gallon in town, how many miles did he drive in town?

The sad thing about this question is that it *tries* to give an external motivation, but the scenario is completely contrived. Why would you need to know how many miles George drove in town unless you're his wife and trying to see if he's having an affair? And otherwise wouldn't it be easier to remember how far you drove on motorways and just subtract that from 764?

However, the internal motivation behind this question is much more interesting to me. This problem has two unknown quantities: the number of miles driven in town and the number of miles driven on motorways. It also has two pieces of information relating them: the total miles driven and the total petrol used. This is a jigsaw puzzle that has the right number of pieces.

The first step is abstraction – turning the wordy problem into a piece of maths with some letters, numbers, equations, and so on. If we write M for the number of miles driven on motorways and T for the number of miles driven in town, we can then turn our two pieces of information into equations.

- The total miles driven is 764, which means

$$M + T = 764.$$

- On motorways he gets 54 miles to the gallon, so the number of gallons used on motorways is $\frac{M}{54}$.
- In town he gets only 31 miles to the gallon, so the number of gallons used in town is $\frac{T}{31}$.
- The total gallons used is 15, which means if we add up the gallons used on motorways and in town, we should get 15, that is:

$$\frac{M}{54} + \frac{T}{31} = 15.$$

So we have two unknowns and two equations governing them. Intuitively you probably realise that if we had 200 unknown quantities and only one equation governing them, we would have not nearly enough information to work out what all the unknown quantities are. But in general if we have the same number of equations as unknowns, then we're in good shape.[1]

Personally I think that actually finding the answer at this point is the least interesting part, but that's because I particularly enjoy the process of abstraction, and enjoy that more than the process of doing calculations. In fact, did you recognise this situation from earlier in the book? Now that we have turned George's situation into two linear equations, it's just another example of the pair of equations we looked at in Chapter 2, which came from a question about my father's age. We have abstracted far enough to get to a situation that we've already solved, so we definitely don't need to do any more work.

But here's the calculation in any case.

> Start with the second equation: we get rid of the fractions by multiplying by 54 and 31, to give
>
> $$\begin{aligned} 31M + 54T &= 15 \times 54 \times 31 \\ &= 25\,110. \end{aligned}$$
>
> Now subtracting T from both sides of the first equation we get $M = 764 - T$
>
> which we can now substitute in to get
>
> $$\begin{aligned} 31(764 - T) + 54T &= 25\,110 \\ \text{so } 23\,684 - 31T + 54T &= 25\,110 && \text{(multiply out the bracket)} \\ 23\,684 + 23T &= 25\,110 && \text{(gather the } T\text{'S)} \\ 23T &= 25\,110 - 23\,684 && \text{(}-23\,684 \text{ from both sides)} \\ &= 1426 \\ T &= 1426 \div 23 && \text{(divide both sides by 23)} \\ &= 62. \end{aligned}$$

So the answer is that George drove 62 miles in town. Bully for him. Perhaps he was having an affair?

Dreaming up some new mathematics

All through this chapter I've been discussing two different ways of coming up with a new piece of mathematics. There's the internal way, where you follow your nose, dig inside your imagination and dream up something that feels good or makes sense. And then there's the external way, where you have a specific problem that you want to solve, and so you build the tools to solve it.

We'll now compare these two approaches to come up with the notion of *imaginary numbers*.

The internal way

You might remember being told the important rule that you 'can't take the square root of a negative number'. The reason is that a positive number times a positive number is positive, but a negative number times a negative number is also positive. So if you times a number by itself it is always positive (or zero). That means whenever you *square* a number, the answer will never be negative. Taking a square root is the reverse of the process of taking a square. So to find the square root of a negative number, we have to find a number whose square is negative – and we've just decided there aren't any.

The key to the internal motivation at this point is to feel a bit dissatisfied, frustrated, irritated or even outraged that you can't take the square root of a negative number. Imagine seeing a sign saying you're not allowed to do something that you think is completely harmless – do you immediately want to do that thing? Similarly, you're now faced with a sign saying you're not allowed to take the square root of a negative number. But what harm would it do? In mathematics, 'harm' means 'causing a logical contradiction'. If something doesn't cause a logical contradiction, you might as well do it.

Now, the only way that taking the square root of a negative number would cause this kind of 'harm' would be if you tried to claim the answer was a positive or negative number, because we know that this cannot be true.

So how can there possibly be a square root of, say, –1? Well, what if there was a whole different type of number such that when you times it by itself the answer is a negative number. You might immediately say – but this doesn't exist. Just like the platypus?

The key in maths is that things exist as soon as you imagine them, as long as they don't cause a contradiction. Having a square root of –1 is not a contradiction, as long as it's a completely new number, and not any of the positive or negative numbers we already knew about. It's like having a completely new Lego piece. To make sure we don't get it mixed up with our old numbers, we call it something completely different: i. This letter i stands for 'imaginary', because it's some kind of new number that isn't 'real'. We'll come back to it later.

The external way

A more 'external' way to come up with imaginary numbers is by trying to solve quadratic equations. Remember a quadratic equation is one involving x and x^2, like $x^2 + x - 2 = 0$

or

$$2x^2 - 7x + 3 = 0.$$

You might be able to remember how to go about solving these, that is, finding all the values of x that make the left-hand side equal 0. Or if you don't remember, I can tell you the answers and you can just check that substituting $x = 1$ or $x = -2$ makes the first equation true, and for the second one $x = 3$ or $x = \frac{1}{2}$ will do. Moreover, if you try any other number, it won't work.

But what about this one?

$$x^2 + x + 1 = 0.$$

No matter what number you put in, positive or negative or 0, you are doomed – the left-hand side can never equal 0. At this point you might shrug and say you never really cared about solving quadratic equations anyway. But mathematicians don't like leaving problems unsolved. Coming up with 'imaginary numbers' is a way of fabricating solutions to the equations that previously had no solutions. In this case the internal and the external have got quite close to meeting up.

Do you think it's cheating to solve a problem by inventing a whole new concept and declaring it to be the answer? For me this is one of the most exciting aspects of maths. As long as your new idea doesn't cause a contradiction, you are free to invent it. The key is to balance out the external and internal motivations for it. If you invent a new concept that is obviously contrived only to solve one problem, then it's unlikely to be a good mathematical concept in the long run, even though it won't actually be *wrong*. The best new mathematical inventions are the ones that make internal sense and also solve some existing problems.

Footnote

[1] There are two potential problems though: the two equations could be contradictory, or they could be essentially the same. We won't go into that here.

part two

CATEGORY THEORY

9 WHAT IS CATEGORY THEORY?

Not much mathematics was needed before people started doing trade. Numbers themselves weren’t even necessary, let alone the more complicated things you can do with them. Negative numbers don’t make much sense if you haven’t thought about the possibility of going into debt.

Children don’t really need numbers in the early part of their lives. If we deliberately teach them numbers then they have the capacity to pick them up when they’re one or two, but if we don’t actively teach them the concept, I’m not sure when they’d pick it up. Plenty of children arrive at primary school at the age of five, being able to recite their ‘number poem’ without being able to use it to count anything. In everyday adult life it’s hard to avoid numbers even if it’s nothing other than prices at the supermarket, but small children get by just fine without numbers.

Likewise, mathematics got by just fine without category theory for thousands of years, but now, in everyday mathematical life it’s hard to avoid it – at least in *pure* maths.

The distinction between ‘pure maths’ and ‘applied maths’ is a bit spurious, or at least the grey area where they meet is pretty grey and quite large. But broadly speaking applied maths is a bit closer to normal life. Applied maths is more likely to be modelling real things in life like the sun, water flowing through pipes, traffic flow. It could be thought of as the theory behind things in real life.

Pure maths is one step more abstract: it is the theory behind applied maths. This is a simplification, but it will do for now.

Lego, yet again

The difference between pure and applied Lego

Are you more interested in using basic Lego bricks to build fantastic big sculptures? Or are you be interested in buying all the complicatedly designed little pieces to build machines, or working robots, or train sets, or spaceships? Even if you don't do Lego yourself, what do you find more fascinating – a Lego version of the Eiffel Tower built only from basic 2-by-4 pieces, or a fantastic articulated robot built from complex high-tech pieces? Using special pieces will be quicker, and you'll get a more realistic model. For example, you can have real wheels with tyres, instead of sort of bumpy angular ones. But there's something immensely satisfying and impressive about having whole buildings and towns built from basic pieces. The creativity and ingenuity required to do it are fascinating.

Pure maths is like using only the basic Lego bricks and building everything from scratch. Applied maths is like using special pieces. Applied maths more closely models real life, but pure maths is at the heart of it, just as you can't get away from the 'pure' Lego-building techniques just because you've acquired wheel pieces.

Topology is a part of pure mathematics that studies the shapes of things like surfaces. We've talked about how topology studies which shapes can be morphed into other shapes without breaking them or sticking them together, but it also studies what happens when you *do* cut them up and stick them together, and how you can build more complicated shapes from simpler ones. It is in fact quite a lot like Lego.

Topology gets used in quantum mechanics, to build models of subatomic particle behaviour. This is called 'topological quantum field theory' and is probably somewhere in the grey area between applied mathematics and theoretical physics. A more large-scale part of applied topology is in cosmology, where the shape of space-time is studied.

Even further along the applied scale is where topology is applied to the study of knots in DNA and configurations that robotic arms can get into. This takes us into biology and engineering.

Another example along the scale of pureness comes from calculus. At root, calculus is the study of infinitesimally small things, or things that are changing continuously rather than in jumps. This is an important area of pure mathematics. As a field of pure study, it is concerned with things like whether a quantity is changing smoothly, and what its rate of change is.

It leads to the question of solving equations involving quantities *and* their rate of change at the same time. For example, if something is moving, we might know about the force being applied to it and the speed it is going, as well as the position it is in. This sort of equation is called a *differential equation* and this takes us further towards applied mathematics than pure. It relates to things like gravitational pull, radioactive decay and fluid flow.

When these things get applied to specific real-world situations, we have gone out of the realm of applied mathematics and right into engineering or medicine or even finance. Differential equations are one of the most widely applied pieces of mathematics all over the place, as almost all measurements of things in real life are somehow fluctuating at some rate or other.

Lego Lego

When it is possible to build things out of themselves

Have you ever tried making a Lego brick … out of Lego? It would be a sort of meta-Lego brick. Instead of a Lego train or a Lego car or a Lego house, you'd have built 'Lego Lego'. I have seen pictures of cakes made out of Lego bricks – a Lego cake. And I've seen Lego bricks made out of cake: cake Lego. And, inevitably, there are cakes made out of Lego bricks that are themselves made out of cake: cake Lego cake.

Category theory is the mathematics of mathematics, a sort of 'meta-mathematics', like Lego Lego. Whatever mathematics does for the world, category theory does it for mathematics. This means that category theory is closely related to logic. Logic is the study of the reasoning that holds mathematics together. Category theory is the study of the structures that hold mathematics up.

At the end of the last chapter I suggested that mathematics is 'the process of working out exactly what is easy, and the process of making as many things easy as possible'. Category theory, then, is:

> *The process of working out exactly which parts of maths are easy,*
> *and the process of making as many parts of maths easy as possible.*

In order to understand this we need to know what 'easy' means inside the context of mathematics. That's really at the heart of the matter, and is what we'll be investigating in this second part of the book. In the first part we saw that mathematics works by abstraction, that it seeks to study the principles and processes behind things, and that it seeks to axiomatise and generalise those things.

We will now see that category theory does the same thing, but entirely inside the mathematical world. It works by abstraction of *mathematical* things, it seeks to study the principles and processes behind *mathematics*, and it seeks to axiomatise and generalise those things.

Mathematics is, if you like, an organising principle. Category theory is also an organising principle, just one that operates *inside* the world of mathematics. It serves to organise mathematics. Just as you don't need a classification system for your books until you have quite a vast collection, mathematics didn’t need this kind of organising until the middle of the twentieth century, which is when category theory grew up. Systematising things can be time-consuming and complicated, but the idea is that in the end it’s supposed to help you think more clearly.

Category theory is the study of the mathematical notion of ‘categories’. Although this is a word taken from normal life, it has a different and carefully formulated meaning in mathematics. These mathematical things called categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in the 1940s. They were studying algebraic topology, which turns shapes and surfaces into pieces of algebra in order to study them more rigorously. Originally this involved relating all those shapes to *groups*, the notion that we introduced and axiomatised in the previous part of the book. They realised that in order to keep a clear head while doing this, a more powerful and expressive type of algebra was needed, a bit like groups but with some further subtleties. Mathematics had become vast enough to need its own system of organisation. Mathematics needed to think more clearly. And so category theory was born.

Then something wonderful happened. Just as mathematics began as the study of numbers, but then people realised the same techniques could be used to study all sorts of other things, category theory began as a study of topology, but then mathematicians rapidly realised that the same techniques could be used across huge swathes of mathematics. Category theory grew up to have greater influence than its ‘parents’ ever imagined.

12 STRUCTURE

Baked Alaska

Ingredients

1 flat 8-inch round sponge cake
200 g raspberries
1 pint vanilla ice cream
4 egg whites
175 g caster sugar

Method

1. Whisk the egg whites and sugar until very stiff to make the meringue topping.
2. Put the cake on an ovenproof dish and pile the raspberries on top, leaving plenty of space around the edge. Then pack the ice cream on top of that in a dome shape, still leaving a good space at the edge of the cake.
3. Pile the stiff egg whites over the ice cream, making sure there are no gaps, and that the egg whites make a good seal around the cake and all the way down to the dish.
4. Bake in a hot oven (220°C) until the meringue has browned. Eat immediately.

Baked Alaska is not just food – it's science. The various parts of it are not just there for taste, they serve a *structural* purpose. The meringue and the sponge insulate the ice cream from the heat of the oven, so that we get the exciting sensation of eating hot meringue and cold ice cream at the same time.

There are plenty of other types of food that have important structural features. Sandwiches and sushi, devised to be conveniently edible on the go. Yorkshire puddings the Yorkshire way, where the pudding is essentially an edible plate containing your food. Vol-au-vent, another type of edible food container. Battered fish, where the batter protects the fish from being overcooked on the outside. Or that amazing way of baking a cake on a campfire, inside a hollowed-out orange skin. Not only does the skin hold the cake mix and protect the cake from the fire, but it also gives the cake a lovely subtle orange flavour.

All these are examples of food where the structure is integral to the food, and in some cases where the taste of the food is affected or even determined by the structure. This is different from a cake in the shape of a dinosaur, where the shape is more or less independent of the taste.

One important aspect of category theory is that it examines what part of a mathematical idea is *structural* – more like a baked alaska than a dinosaur cake. It looks very carefully at what role everything is playing in holding the structure together.

Multistorey car park

What the structural part of a building looks like

I was looking at a half-built building with some friends. Actually it was probably even less than half-built – it was just a shell of a structure. We were speculating about what sort of building it was going to be. Some of us were trying to work it out by remembering what we'd read recently about new buildings in the area. But being a mathematician (and a pure one at that) I was staring at it and trying to work it out from 'first principles', that is: what does the thing in front of me actually look like?

I suddenly realised two things. First, it looked like a multistorey car park. Secondly, that *every* building must look like a multistorey car park at that stage in the building process. Usually when I think about the basic structure of a building I think about stripping things away: first the furniture and decorations such as wallpaper and pictures, then windows and doors, then any walls that aren't bearing any load.

But there's the opposite way of thinking about the structure of a building: building it up, rather than stripping it down. Because the structure has to be put in place before any of the decorations go on.

A lot of maths is about structures, and category theory is particularly about structures. What is holding something up? Which parts could you remove, without making the whole thing fall down?

This is a bit like the tale of the parallel postulate, where mathematicians spent hundreds of years trying to work out whether that fifth axiom was actually necessary or not. Would geometry fall apart without it, or would geometry be just the same? In category theory we like to understand exactly what part of the axioms is making everything work in any given mathematical world. This is important as it helps us *generalise* the situation and take it to a slightly different world, if we know exactly what is holding it up.

Here's a thought experiment we can do to see what is holding the integers together. Imagine that the number 2 no longer exists. Which numbers are *now* prime? Remember a prime number is one that is only divisible by 1 and itself, and 1 doesn't count as prime.

Now 3 is still prime, as it's only divisible by 1 and itself. But what about 4? 4 used to be divisible by 2 as well, but 2 *no longer exists*. So 4 is now only divisible by 1 and 4, so it has become 'prime'.

And 5 is still prime, and you might be able to *generalise* this fact now, and realise that any number that used to be prime will still be prime, because it can't suddenly become divisible by new things – there aren't any new things here. (We removed the number 2, but we didn't invent any new things in its place.) The problem will be with *even* numbers because they are no longer going to be divisible by 2 – because 2 doesn't exist.

So 6 is now prime, because it's no longer divisible by 3. Here we have to be a bit more careful about what 'divisible' by 3 means: it means that $6 = 3 \times k$ where k is any whole number. But 6 is no longer 3 times anything, because 2 doesn't exist. So 6 is only divisible by 1 and itself. Likewise 8 and 10.

We now have a curious fact – numbers can now be expressed as a product of 'primes' in different ways. Can you think of an example? Here's one:

$$24 = 3 \times 8 = 4 \times 6.$$

In our new 2-less world, 3, 8, 4 and 6 are all 'prime'. So by throwing away the number 2 we have destroyed one of the fundamental principles of numbers, that every number can be expressed as a product of prime numbers in a *unique* way.

St Paul's Cathedral

Three versions of one structure

I quite often watch television with no sound at the gym, as I prefer listening to music to make me work out harder, but the screens are in front of my face so I can't help watching. One time I watched a really naff docu-drama about the building of St Paul's Cathedral, with stilted auto-subtitles in that crude type-face that brings to mind a robot voice.

I didn't know much about St Paul's at the time except that it was designed by Sir Christopher Wren, and I particularly didn't know how the dome was constructed or how long it took to build or how nearly it didn't get finished. I'm not even sure I appreciated its great and majestic beauty at the time; I just knew it was large and famous.

What I learnt from all this was that the dome is actually made of *three* domes: an inner dome and an outer dome, both visible, and a third, hidden, dome in between them that's actually supporting the structure. The outer dome is the one that's visible across London, still proudly dominating the skyline after all these years despite the arrival of the Shard, the Gherkin, and other, taller buildings. It is not the sheer height of the dome that makes it imposing – it was surpassed as the tallest building in London in 1962, and the Shard is almost three times as tall. The dome is imposing because of its overall size, presenting a severe engineering problem at the time: how to hold such a thing up without the base collapsing?

The inner dome serves the aesthetic desires of the inside of the cathedral, that is, the interior of the cathedral needs a certain balance in its proportions, without a ridiculously huge dome overpowering the main body of space. Until I saw this docu-drama, I didn't realise that the dome visible on the inside was not the same as the one seen from the outside.

But the genius of the construction is the third, hidden, structural dome that 'mediates' in between them. The other two domes are much too broad and flat to be able to support the heavy structure of the lantern at the top of the dome, so in between them is a much pointier brick construction, which would not be very beautiful to look at, but which is strong and secure enough to support the necessary load.

I was a PhD student at the time, and I had this epiphany that this was exactly like the thesis that I was in the process of writing. My thesis involved three expressions of the same structure, one with 'internal' motivation (the internal logic of the situation), one with 'external' motivation (the applications), and the other which was 'hidden' and whose only purpose was structural, to mediate between the two.

The personal part of this drama was that, apparently, Wren had no idea how he was going to achieve the effect he wanted. The building of the cathedral was actually under way, and Wren still had no idea how he was going to construct the dome; he just had a vision of what he was going to accomplish. The idea of three domes came later.

I now have a strong belief in the difference between internal and external motivation, structural mediation between the two, and the idea that if one has a genuinely good idea, the means of accomplishing or justifying it will come later. And that one can be on the point of spectacular failure just before spectacular success. And that I love St Paul's Cathedral.

Category theory often studies different aspects of the same structure. It can be fascinating to turn things inside out and see them from the other way up – understanding something from only one point of view is far too restrictive. The greatest leaps forward in the history of mathematics have often been when connections are made between apparently unrelated subjects, enabling communication and the transfer of both information and techniques. It's like the difference between building a bridge between two islands, and a bridge to nowhere.

Category theory grew up from the study of *algebraic topology*. We have already met various ideas from topology, including surfaces, knots, bagels, doughnuts and the idea of 'morphing' shapes into other shapes as if they were made of playdough. We've also met various ideas from algebra, including groups, relations, associativity, and so on.

Algebraic topology is like a road between those two 'cities', algebra and topology. The original aim was to use algebra to study topology, but then it turned out to be a two-way road, so topology can also be used to study algebra. Category theory helps translate between the two cities. It enables us to ask questions like:

- Are there features in one city that resemble features in another city?
- If we take our tools and techniques from one city to the other, will they still work?
- Are the relationships between things in one city at all like relationships between things in the other city?

Category theory doesn't necessarily answer all those questions, but it gives us a way of posing the questions, and helps us see which ideas are important and which are irrelevant to finding the answers.

CD

Which part makes it a CD?

I once decided to try and remove the label from a CD. I can't remember why – perhaps it was so ugly I couldn't bear to look at it any more? I had been making my own CDs for the first time and so had a pack of self-adhesive CD labels that I really liked using. I think my plan was to design a new label and stick it on the CD myself. I tried sticking a new label on top, but I could still see the old one underneath.

If you think this sounds like a made-up story, I sympathise; I feel a bit like I'm making this part up myself. The thing is, I now can't for the life of me remember why I was trying to remove the label from the CD, but I definitely remember what happened next. I took the label off, and all I was left with was: a transparent piece of plastic.

I felt very foolish. Was it obvious to everyone on earth except me that the crucial part of a CD, the shiny part, was actually structurally part of the label? That apart from this the CD was just a piece of plain plastic?

Similar things have happened to me with dresses, when I've thought 'That is a great dress apart from that ugly flower attachment on it.' But when I investigate the possibility of simply removing the flower, I discover that it's so deeply attached to the dress it's actually part of its structure. The dress stays in the shop.

In category theory one of the important aspects of looking at structure is to see what will go wrong if you discard parts of the structure. This is all part of finding out exactly how something works in case you find yourself in a (mathematical) world with less structure. It's a bit like learning how to whisk egg whites by hand as well as doing it using an electric whisk. It means you'll be able to do it even when you're in a kitchen with no electric whisk. Or no electricity. Perhaps you're in the forest and you really need stiff egg whites? Oh never mind.

One mathematical version of the 'electric whisk' is related to how we solve quadratic equations. We saw in Chapter 7 that we can try to solve the quadratic equation

$$x^2 - 3x + 2 = 0$$

by recognising that the left-hand side can be factorised:

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

Then we conclude that one of the two brackets must be 0, in order for the answer to be 0, so either $x - 1 = 0$, in which case $x = 1$, or $x - 2 = 0$, in which case $x = 2$. So these are the two solutions.

However, suppose we were doing this on a six-hour clock. You can try putting in some other values for x to see what the answer is. For example, if you put $x = 4$, you'll get

$$\begin{aligned} x^2 - 3x + 2 &= (4 \times 4) - (3 \times 4) + 2 \\ &= 16 - 12 + 12 \\ &= 6 \end{aligned}$$

but on the six-hour clock, 6 *is the same as* 0, so $x = 4$ actually gives 0 as the answer here. You can check that $x = 5$ gives 12, which is also the same as 0. This means that 1, 2, 4 and 5 are *all* solutions to this quadratic equation on the six-hour clock. What is going on? Where are these 'extra' solutions coming from? How can we look for them and how can we be sure we've found them all?

The key is to go back and carefully look at how this argument works. The crucial moment is where we declare that 'one of the brackets has to equal 0'. What we're saying there is that if we multiply two things together and get 0, one of them had to be 0 already. However, while this is true with normal numbers, it is *not* true on the six-hour clock. For example,

$$3 \times 2 = \ \ 6 = 0$$

$$4 \times 3 = 12 = 0.$$

This is why some new solutions have popped up even though when $x = 4$ neither of those brackets $(x - 1)$ and $(x - 2)$ is 0. The point is that when $x = 4$ the brackets work out to be 3 and 2, and when $x = 5$ they work out to be 4 and 3. So those two 'extra' ways of multiplying numbers to get 0 give two 'extra' solutions to the quadratic equation.

We have gone to a mathematical world without a piece of structure that we're rather used to: the fact that the only way to multiply numbers to get 0 is if one of the numbers we're multiplying was already 0. So we have to be careful how we proceed in this other world, and also in *any* other world that doesn’t have this structure. We have isolated an important piece of structure that it’s important to look for if we want to go round solving quadratic equations in different worlds. Although there might be more solutions floating around for us find, we have to work a lot harder to make sure we’ve found all the right ones if we don’t have this rather useful piece of equipment.

Money

Being careful how you spend it

If you have a lot of money – I mean, really a lot of money – you don’t ever have to find out how anything works. If it goes wrong, you can just throw money at it to get it fixed. You can either pay someone else to fix it, or you can just go right ahead and buy a new one. If you’re rich, you also don’t have to worry about exactly how much money you’re spending on things every day, although some rich people apparently still do.

But if you’re a normal person, you do have to worry about these things, at least if you want to avoid financial catastrophe. Even if you’re not extremely frugal all the time, it’s good to be aware of what you’re spending money on, so that you can rein it in if necessary.

Some mathematics is done in the ‘rich’ way – with no fear of ever running out of (mathematical) resources, so without really paying attention to which resources are being used. In contrast, category theory is like being frugal or at least aware of your mathematical spending. That is, the aim is to study mathematics always being aware of what structures you are using to get by at any given moment. You might not be using them explicitly, but sometimes the hidden usage is even more important, precisely because it’s hidden, so you’re likely to use it without noticing. A bit like when people accidentally get huge credit card bills because their children have bought extras in a game on their mobile phone. Or when you run up a huge roaming bill because your phone has connected to the internet when you’re abroad.

Category theory aims to keep track of resources, not because resources might suddenly run out in mathematics (that’s not how mathematical resources work, fortunately), but so that you can deliberately go to a planet with fewer resources. The aim is to make connections between different mathematical worlds, and develop techniques that can be used without extra effort in those different worlds.

This is just like the example with quadratic equations that we just saw. The resource in this case is the property

If $a \times b = 0$, *then* $a = 0$ *or* $b = 0$ (*or both*).

Now if you think you will *never* end up in a world without this resource, then you will not care about how many times you use it. But if you care about *modular arithmetic* (on a clock face) or even just the possibility of going to worlds without your resource, then you have to go back through all the techniques you love, and work out when you used this principle, and how to get round it.

A more profound mathematical example involves something called the 'axiom of choice'. This axiom says it is possible to make an infinite number of arbitrary choices. In normal life you might think it's perfectly possible to make an arbitrary choice – it's just like picking a raffle ticket out of a hat. The axiom of choice says it's possible to pick a raffle ticket out of each of an infinite number of hats, which might seem a bit odd to you. Mathematicians don't really agree on whether this is odd or not.

Processes involving some notion of 'infinity' always require great care if we're trying to make them rigorous, and this one about arbitrary choices turns out to be particularly difficult to pin down, which is why it is an axiom all by itself. People are a bit undecided about whether it should be assumed to be true or not, and so the best approach is to *be aware* every time you need to use it.

One branch of category theory deliberately goes to worlds where this axiom is *not* true, to see how much of mathematics can still be done.

Skeleton

Not a whole person, but the last part that remains when all is stripped away

A wonderful old professor sat next to me at dinner in Cambridge one day when he was about ninety. It was round about the time of the scandal at Alder Hey hospital, when it was discovered that, shockingly, organs from dead children had been removed and kept by the hospital without authorisation. The professor told us he was worried that this scandal would put people off organ donation, and that this had moved him to contact Addenbrooke's, the Cambridge teaching hospital, to ask if there was anything at all useful they could do with his old body after he died. He was too old for organ donation, but they told him that his skeleton would be useful for teaching medical students, so he should try not to die mangled in a road accident. (He told us this with typical glee and a twinkle in his eye. I wonder if I will be able to speak with such cheekiness about my own future death.) A few years later I heard that he had passed away at home; I hope that his skeleton is indeed now being used for teaching purposes.

A skeleton is not a whole person, but it's an important part to understand, in order to understand how a person functions. It gives a person their structure. It has little to do with thought, emotions, feelings, and so on, but it's the frame on which everything hangs. This is the point of studying structure in mathematics as well.

Logic is a branch of mathematics that studies the structure of mathematical arguments. Category theory, on the other hand, studies the structure of the mathematical objects themselves. They are similar in a way, in that they're both even more abstract than mathematics itself, as they study the way mathematics is done. However, logic is more obviously used in ordinary daily life – or rather, it is *usable* even if it's often used rather badly. Any time you construct an argument, justify your point of view or make a decision, some element of logic could (or should) come into it, when you start from some more basic thoughts and proceed to some more complex ones.

It is less obvious how the study of mathematical structure could arise in daily life. However, it is the mental exercise of stripping away layers to reveal important structure that is usable everywhere. It also goes the other way round, as we have the mental process of starting with simple structures and carefully building up more complex ones. Category theory formally only does this for mathematical structure, just like formal logic – it only really applies to *mathematical* arguments and not normal arguments in every day life. However, the mental exercise in the abstract mathematical environment prepares us for the concrete non-mathematical environment, just as working out in a gym can make us more generally fit for the world outside the gym.

Battenberg cake

An example of a ubiquitous piece of structure

Here's an example of a mathematical structure that pops up all over the place in different guises. Let's start by thinking about addition on a two-hour clock, or, to use the technical term, ‘addition modulo 2’. This means that there are only two numbers, 0 and 1. Now 2 counts as the same as 0, as do 4, 6, 8, 10, …; and 3 counts as the same as 1, as do all the odd numbers.

We can now draw an addition table for this. We only need the numbers 0 and 1 (because all other numbers are the same as one of these). And we need to remember that 1 + 1 = 2 but that 2 is the same as 0, so in fact 1 + 1 = 0. The addition table then looks like this:

+	0	1
0	0	1
1	1	0

In fact, this is the second smallest possible *group*. We have already seen that the smallest possible group has only one object, the identity. Now we have a group with two objects. This is related to the question we posed at the end of the chapter on principles, about filling in the squares with colours, where each colour can only appear once in each row and each column.

Now here’s another way that this pattern appears. We can think about just using the two numbers 1 and –1, and combining them using multiplication. What table does that give us?

×	1	−1
1	1	−1
−1	−1	1

If you compare this with the previous table, you’ll see that it has the same pattern, just with different labels in the boxes. We can also think about the rotational symmetry of a rectangle. A rectangle only has two forms of rotational symmetry: the rotation by 0° and the rotation by 180°. If we do the 0° rotation followed by the 180° one, then the result is rotation by a total of 180°. Likewise if we do it in the opposite order. However, if we do a rotation by 180° and then another, we have gone round 360° and we get back to exactly where we started – the same as doing a rotation by 0°, or nothing at all. We can now put these in a table as well:

rotation	0	180
0	0	180
180	180	0

You might not be surprised to see that it's the same table again. We have already seen this pattern in the chapter on context, when we thought about multiplying positive and negative numbers, or real and imaginary numbers, and we drew up the following tables of results:

×	positive	negative
positive	positive	negative
negative	negative	positive

×	real	imaginary
real	real	imaginary
imaginary	imaginary	real

In fact, all the inside part of these tables have the same pattern as a Battenberg cake:

which is designed for the same reasons – we don't want two squares of the same coloured cake to touch each other.

Battenberg challenge

Here's a challenge: can you draw a picture of a Battenberg cake, each of whose mini-cakes is itself a Battenberg cake? I call this the 'iterated Battenberg'. This means you have to start with two types of Battenberg cake, in different colours. So there are four colours altogether and they need to fit together in a 4 × 4 grid. In fact, we've seen one of these already at the end of Chapter 3. There we had four examples of 4 × 4 grids of colours, and the first one was an iterated Battenberg.

This pattern comes up if we look at the rotations *and* reflections of a rectangle, instead of just the rotations. Another place this comes up is if we draw a multiplication table for the odd numbers, modulo 8. We only need to consider the numbers 1, 3, 5, 7, as all other odd numbers will be the same as these on the eight-hour clock. You can try filling in this multiplication table, remembering that every time you get to 8 you go back to 0. So 3 × 3 is 9, which is the same as 1, and so on.

×	1	3	5	7
1				
3				
5				
7				

You should get the following table, with the iterated Battenberg pattern:

×	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Now that we have found Battenberg-type structures all over the place, we need to see what it means to say that all of these structures are 'really the same'. One of the easiest ways to see that they were all the same was to isolate the structure and put it in the tables as we did above. Category theory does something similar for more general forms of structure. We have already seen how we draw relationships between objects using arrows. We can now boil a piece of structure down to a little diagram of arrows.

For example, we might go round looking for diagrams like this:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

or like this:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

In the latter case, we might use rotation by 180° for *f* and *g*, which gives us this diagram:

$$\begin{tikzcd} & B \arrow[dr, "180^\circ"] & \\ A \arrow[ur, "180^\circ"] \arrow[rr, "0^\circ"'] & & A \end{tikzcd}$$

showing that if we do rotation by 180° twice, it's the same as doing nothing.

In the same way as putting all the above examples in 2 × 2 tables, these category theory diagrams help us to see the structure in different situations, and we can then more easily see if it's 'the same' as some other piece of structure in an otherwise completely different situation. But what does 'the same' mean? This is the subject of the next chapter.

15 WHAT CATEGORY THEORY IS

We said, in the first half of the book, that mathematics is there to make difficult things easy. We have now seen that category theory is the mathematics of mathematics. So, category theory is there to make difficult *mathematics* easy.

In the second half we have discussed various ways in which it goes about this, but I want to conclude by characterising category theory as a category theorist would: what is the glass slipper that fits category theory exactly? That is, instead of saying what category theory looks like, we're going to say what role it fills.

Truth

People often think that mathematics is all either right or wrong. That's not true – even if a piece of maths is right, it can still be good or bad, it can be illuminating or not, it can be helpful or not, and so on.

However, there's a grain of truth in this business of right and wrong. One of the remarkable qualities of mathematics is that, because it's all built from logic and nothing else, mathematicians can readily agree when something is right. This is very different from other fields, in which opposing theories can be argued forever. As philosopher Michael Dummett remarked in *The Philosophy of Mathematics*:

> Mathematics makes a steady advance, while philosophy continues to flounder in unending bafflement at the problems it confronted at the outset.

Mathematical fact has an elevated status over other kinds of fact. We've already discussed the fact that scientists revere the so-called scientific method, experimental method and evidence-based knowledge, where facts are deduced from hard evidence that can be experimentally repeated. Maths isn't like that at all – it doesn't use *evidence*, because evidence isn't logically water-tight. Evidence is the foundation of science, but it isn't enough to give us mathematical truth. This is why mathematics is sort of a part of science, but also isn't a part of science.

Mathematics uses the 'logical method', where facts are deduced only using cold, bright logic. Mathematical truth is revered because of proof: everything is rigorously proved, and once it has been proved, it cannot be refuted. You can find a mistake in a proof, but that means it was never really proved in the first place. Thanks to the notion of 'proof', we have an utterly unassailable way of knowing what is and isn't true in mathematics. How do we show that something is true? We prove it.

Or do we?

The wonderful thing about formal mathematical proof is that it eliminates the use of intuition in an argument. You don't have to guess what someone is trying to say, or interpret their words carefully, or listen to the inflection in their voice, or look at the expression on their face, or respond to their body language. You don't have to take into account the nature of your relationship with them, the stress they're under at the moment, the fact that they might be drunk, or the way their past experiences might be affecting them now. You don't have to be able to imagine what something looks like, you don't have to be able to imagine eight-dimensional space, or what a pile of two million apples would be like, or how it feels to be at the North Pole. All of these problematic subtleties are gone.

And the trouble with formal mathematical proof is that all of these subtleties are gone. The subtleties that can cause problems are also useful, but useful for something different. They are useful for getting a personal insight into something. You might think that mathematics shouldn't be about personal insight, but in the end *all* of understanding is about personal insight. It's the difference between understanding and knowledge. Formal mathematical proofs may be wonderfully water-tight and unambiguous, but they are difficult to understand.

Imagine being led, step by step, through a dark forest, but having no idea of the overall route. If you were abandoned at the start of that route again, you would not be able to find your way. And yet, when you're led there step by step, you do make it to the other side.

Mathematicians and maths students have all had the experience of reading a proof and thinking, 'Well, I see how each step follows from the previous one, but I don't have a clue what's going on.' We can read a correct proof, and be completely convinced of each logical step of the proof, but still not have any understanding of the whole. Here's a completely formal proof of a very trivial-sounding fact: *any statement implies itself*. Note that by 'implies' here we mean logical implication. In mathematical logic 'implies' doesn't mean quite the same as in normal life – it means something much stricter. 'A implies B' means that if A is true then B is *definitely* true without any room for doubt. In normal life we say things like 'Are you implying that I'm stupid?' and implication is more of a suggestion or an insinuation, not a hard and fast fact.

Back to our example of statements implying themselves. This is a bit like things equalling themselves – the most obvious equation is

$$x = x.$$

Surely this is true about logical implication as well?

For example:

- If I’m a girl, then I’m a girl.
- If it’s raining, then it’s raining.
- If 1 + 1 = 2, then 1 + 1 = 2.

And yet, look how absurd and convoluted the rigorous proof of this is. Here the little arrow sign means ‘implies’, and this is the completely rigorous proof that any statement *p* implies itself, using the axioms of formal logic.

Proof of $(p \Rightarrow p)$

$$(p \Rightarrow ((p \Rightarrow p) \Rightarrow p)) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$$
$$p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$$
$$(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$$
$$p \Rightarrow (p \Rightarrow p)$$
$$p \Rightarrow p$$

I admit that I find this proof extremely exciting and satisfying, but not even all mathematicians will agree with me. I only included it here so that you could marvel at how ludicrously complicated it seems to be to prove the most basic logical statement. Non-mathematicians think they’ll never understand what mathematicians do, but half the time mathematicians don’t understand each other either. Does this proof convince mathematicians that any statement really does imply itself? No, of course not.

So, if the proof by itself doesn’t convince them of the truth, then what does?

The trinity of truth

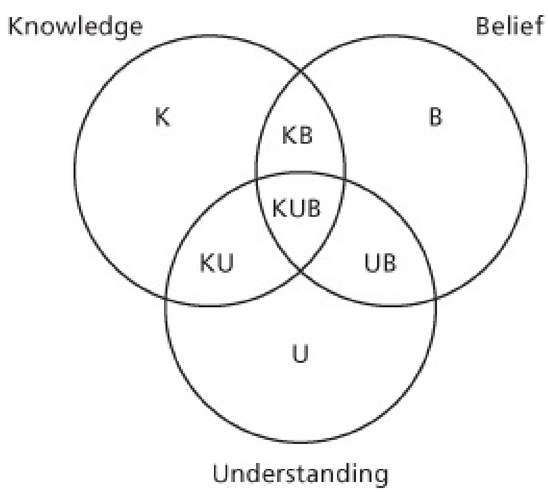
There is something else that plays the role of convincing mathematicians that something is true. I think of it as an *illumination*.

I'm going to talk about three aspects of truth:

1. belief
2. understanding
3. knowledge.

This is a bit like the three domes of St Paul's Cathedral. We have knowledge, which is what the outside world sees, belief, which is what we feel inside ourselves, and understanding, which holds them together.

The interplay between these three types of truth is complex. We can start by drawing a Venn diagram for them:



I've marked the different areas of overlap, so we have:

KUB: Things we know, believe and understand. The most secure of truths.

KB: Things we know and believe, but do not understand. This includes scientific facts that are certainly true, even if we don't understand them. For example, I don't really understand how gravity works, but I know and believe it works. I know and believe that the earth is round, but I don't understand *why*.

B: Things we believe, but do not understand or know. These are our axioms, where everything else begins – the things we can't justify using anything else. For example, for me, there are things like love and the preciousness of life. I believe that love is the most important thing of all. I can't explain why, and I can't say I know for sure it is true – because what does that even mean?

After this things get a bit trickier.

K: Things we know, but do not understand or believe. Is this at all possible? I think if you've ever known sudden grief or heartbreak you might know what this is like. Those numb days after the event when you know, rationally, that it really has happened, but you simply can't believe it, you can't feel it to be true in your stomach. And you certainly don't understand it. Perhaps extremes of good emotions feel like too. Perhaps if I won the lottery I would, for a while, know that it had happened without understanding or believing it. Winning the lottery of love feels like that too, at the height of its ecstasy.

KU: Things we know and understand, but do not believe. Perhaps this is where we get to the next stage of grieving, when we have come to understand that this terrible thing really has happened, but we still don't believe it. But if you're in this state you're probably in some state of denial, because usually knowing and understanding something would make you really believe it's true.

Finally we have the following sections which I suspect are empty.

U: Things we understand, but do not know or believe.

UB: Things we understand and believe, but do not know.

I don't think it's possible (or rather, reasonable) to understand something without knowing it. In this way, understanding is different from the other two forms of truth, which do seem to be able to exist by themselves. Truth flows through this diagram in one direction only – from understanding flows everything else.

Of course, it all depends somewhat on exactly how we define these things, but just try thinking for a second about some things you believe. Here are some things you might believe:

- $1 + 1 = 2$.
- The earth is round.
- The sun will rise tomorrow morning.
- It is very cold at the North Pole.
- My name is Eugenia.

Why do you believe these things? Perhaps you think you understand why $1 + 1 = 2$, except when it doesn't, as we've discussed earlier. If we are working in the natural numbers or integers, $1 + 1 = 2$, mostly because that's the *definition* of the number 2. But $1 + 1 = 0$ if we're working in two-hour clock arithmetic, that is, the integers modulo 2.

But why is the earth round? Why will the sun rise tomorrow morning? Why is it cold at the North Pole? These are things that most of us know, but without really understanding them. I think a lot of our personal scientific knowledge is just that – knowledge that we believe because somebody we trust has told it to us. We have taken it on trust, or on authority.

Why is my name Eugenia – if it is? That last one is fairly easy, assuming that is my name: it is so because my parents chose it. But are you going to believe that, just because it's on the cover of this book? Or would you have to go and look up the record of my birth before believing it? (I hope not.) This is more complex. You might believe it's true without really knowing if it's true or not.

Understanding is a mediator between knowledge and belief. In the end the aim is to get as many things as possible into the central part of the picture, where knowledge, understanding and belief all meet.

Here's a mathematical example of the difference between knowledge and understanding. Suppose you are trying to solve the equation

$$x + 3 = 5.$$

Perhaps you remember that you can 'take the 3 to the other side and switch the sign'. So the next step is

$$x = 5 - 3$$

and we see that x is 2.

However, knowing that this works is not the same as understanding it. Why does it work? It's because we have an equality between the left-hand side and the right-hand side, and so we can do the same thing to both sides and they'll still be equal. Now, we want to get the x isolated by itself on one side, which means we want to get rid of the 3 on the left. How do we do that? We subtract 3. But if we do that on the left we have to do that on the right as well. So what we're really doing is

$$x + 3 = 5$$
$$x + 3 - 3 = 5 - 3$$
$$x = 2.$$

Understanding this *principle* rather than merely knowing the rule makes the knowledge more transferable to other situations.

Pickpocket/putpocket

Remember the strange case of the 'putpocketing' from Chapter 4? You had a ten-pound note in your pocket. Someone pickpocketed you, but also someone else slipped a ten-pound note into your pocket afterwards. So you believe you have a ten-pound note in your pocket.

But do you actually know you do? Perhaps you then check to see if your ten-pound note is still there. At this point, you now also know you have a ten-pound note in your pocket.

But until someone enlightens you about the whole story, you will not actually understand *why* you have a ten-pound note in your pocket.

Why? Why? Why?

Why did the chicken cross the road?

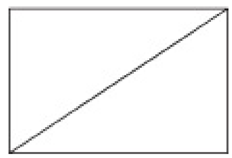
The key to understanding is the question, Why? Why is such-and-such true? 'Because we've proved it' is not a satisfactory answer, from a *human* point of view. Why is that glass broken? 'Because I dropped it' or 'Because the molecular bonds between the glass molecules are no longer in place'. We've all heard, 'We apologise for the late departure of this flight. This is due to the late arrival of the incoming flight …' And, of course, why did the chicken cross the road? Asking why is like asking what the moral of the story is.

Let us try asking some mathematical whys.

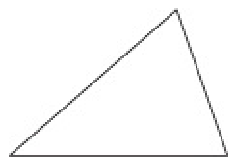
1. Why is the area of a triangle half the base times the height?
2. Why is minus minus one equal to one?
3. Why is zero times anything zero?

4. Why can’t you divide by zero?
5. Why is the ratio of the circumference of a circle to its diameter always the same (it’s π)?
6. Why does the decimal expansion of π go on forever?

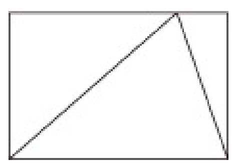
Let’s try answering these now. The area of a triangle is quite easy to think about if it’s a *right-angled* triangle, because then the triangle is obviously half of a rectangle:



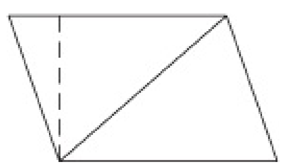
If it’s a more random-looking triangle, like this one:



then we have to fill it in to a rectangle a bit more cleverly, say like this:



and then work out why the extra parts can be pieced together to make the same triangle that we started with:



That’s pretty convincing, but it’s not quite a proof.

For the next one, we could do a proof using the axioms for numbers. Formally it looks like this.

The additive inverse of x is defined to be $-x$, that is,

$$-x + x = 0$$

and it is unique with this property. We need to show that 1 is the additive inverse of –1. That is,

$$1 + (-1) = 0.$$

But this is true since –1 is the additive inverse of 1.

This is mathematically correct, but not exactly *convincing*. Are you more convinced if I say something like 'putting a minus sign flips which way we're facing, and if we flip twice we get back to the direction we started'? Not mathematical at all, but possibly more convincing. Perhaps it would be more convincing to put it like this. Whenever we have $a + b = 0$, this tells us that a and b are additive inverses of each other, that is,

$$a = -b \text{ and } b = -a.$$

We know that –1 is the additive inverse of 1, so we can put $a = -1$ and $b = 1$ and we get $a + b = 0$. Now we can conclude that $b = -a$, which in this case means

$$1 = -(-1).$$

This is essentially the same proof as before, but written out a bit less elegantly. Did you find it more convincing?

As for multiplying by 0 giving 0, there is a similarly technical and even more unilluminating proof from the axioms that looks like this:

Let x be any real number. Then

$$0x + 0x = (0 + 0)x$$

distributive law

$= 0x$ definition of 0

Subtracting $0x$ from both sides, we get $0x = 0$.

We have already discussed the fact that 'you can't divide by 0' really means '0 has no multiplicative inverse according to the axioms'. But with all of these proofs from the axioms for the real numbers, we are not really trying to justify *why* these things are true. Rather, the proofs are only to check that the things we *feel* are true really are true according to the axioms we've chosen. It's not actually an explanation of anything.

The fact about circles can be proved using calculus, but you can also try to convince yourself like this: both the circumference and the diameter are *lengths*, and when you scale a shape up or down all its *lengths* stay in proportion.

As for the decimal expansion of π going on forever, you might remember it's because π is irrational. But why is π irrational? I don't know of a particularly convincing *explanation* of that, except that circles are curved, and diameters are straight, and it would seem a bit oddly neat and tidy if the ratio was something rational.

Actually some rational numbers have decimal expansions that go on forever too, such as $\frac{1}{9}$, which is 0.1111111… However, the decimal expansion of a rational number always ends up repeating in cycles, whereas the decimal expansion of an irrational number like π or $\sqrt{2}$ never repeats itself.

You can always keep on asking why, because there is always another level of 'why' that can be asked. Every child knows that the question why is actually an infinite sequence of questions with which to harass an adult.

The point of the above examples was to illustrate the fact that if you ask *why* a mathematical fact is true, the mathematical proof is often not something that will convince you why it is true. Instead, it might convince you *that* it is true. And there's the crucial difference.

Proof vs illumination

Proof has a sociological role; illumination has a personal role.

Proof is what convinces society; illumination is what convinces us.

In a way, mathematics is like an emotion, which can't ever be described precisely in words – it's something that happens inside an individual. What we write down is merely a language for communicating those ideas to others, in the hope that they will be able to reconstruct the feeling within their own mind.

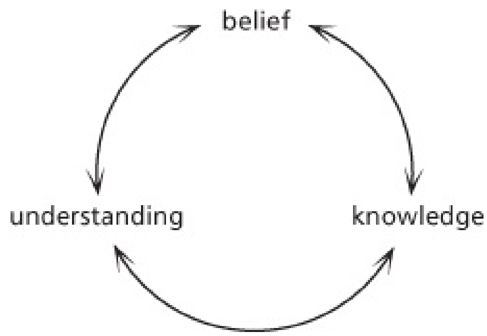
When I'm doing maths I often feel like I have to do it twice – once in my head, and then a second time to translate it into a form that can actually be communicated to anyone else. It's like having something you want to say to someone, which seems perfectly clear in your head, but then you find you can't quite put it into words. The translation is not a trivial process; why do we try to do it at all? Why do we not just stick to the things that are illuminating?

a. Illumination is very difficult to define.
b. Different people can have different notions of what is illuminating.

So illumination by itself doesn't make a very good organisational tool for mathematics. In the end, doing mathematics is not just about convincing onself that things are true; the point is to advance the knowledge of the world around us, not just the knowledge inside our own head.

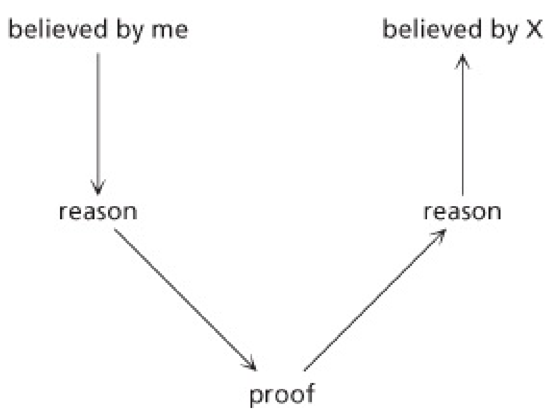
The circle of truth

I am going to describe mathematical activity in terms of moving around between these three kinds of truth:



In maths, knowledge comes from proof – we know something is true by proving it. Usually we think that the big aim of doing maths is to prove theorems, that is, move things into the 'proved' area. But I think the deeper aim is to get things into the 'believed' area – believed by as many mathematicians as possible. But how do we do that? If I have proved something is true, how do I really come to believe it? It's as if I have some sort of illuminating reason for believing it, rather than just following the proof through step by step. However, once I believe it, how do I convince someone else? I show them my proof.

We need the proof to enable us to move from the realm of *my* believed things to anyone else's:



So the procedure is:

- I start with a truth I believe that I wish to communicate to X.
- I find a reason for it to be true.
- I turn that reason into a rigorous proof.
- I send the proof to X.
- X reads the proof and turns it into a convincing reason.
- X then accepts the truth into his realm of believed truth.

In fact, it's not so much a circle of truth as a valley; attempting to fly directly from belief to belief is inadvisable. We've all seen people try to transmit beliefs directly, by yelling. So if transmitting beliefs directly is unfeasible, why don't I just send the reason directly to X, thus eliminating what are probably the two hardest parts of this process: turning a reason into a proof, and turning a proof into a reason?

The answer is that a 'reason' is harder to communicate than a proof.

I think that the key characteristic about proof is not its infallibility, but its sturdiness in transit. Proof is the best medium for communicating my argument to X in a way that will not be in danger of ambiguity, misunderstanding or distortion. Proof is the bridge for getting from one person to another, but some translation is needed on both sides.

When I read someone else's maths, I always hope that the author will have included a reason and not just a proof. When this does happen, the benefits are very great. Unfortunately a lot of maths is taught without any attempt at illumination. Even worse, it's sometimes taught without any explanation at all. But even if it is explained, not every explanation is illuminating. For example, we mentioned earlier that when you learnt how to solve something like

$$x + 2 = 5$$

you might have been told, as I was: 'You take the 2 over to the other side of the equals sign and the plus becomes a minus.' This gives

$$x = 5 - 2$$

so

$$x = 3.$$

This is correct, but unilluminating. Why does that trick with the equals sign work? Apparently one way of teaching this is that the plus sign moves through the equals sign and the vertical bar gets stuck, so + turns into –. This is a pretty absurd way of teaching it, because then what happens when you send a minus sign through? A very unilluminating explanation.

At least in the UK and US, many people grow up feeling great antipathy towards maths, probably because of how they were taught it at school, as a set of facts you're supposed to believe and a set of rules you have to follow.

You're not supposed to ask why, and when you're wrong you're wrong, end of story. The important stage in between the belief and the rules has been omitted: the illuminating reasons. An illuminated approach is much less baffling, much less autocratic, and much less frightening.

But is there always an illuminating explanation for every piece of maths? Probably not, just as there is not an illuminating explanation for everything that happens in life. Some things so incredible or tragic happen that no explanation is possible.

Category theory seeks to illuminate maths. In fact, category theory could be thought of as the universal way of illuminating maths – it seeks to illuminate, and that's all it does. That's its role. That's the glass slipper into which it perfectly fits. I'm not claiming category theory explains everything in mathematics, any more than mathematics explains everything in the world.

Mathematics can seem like an autocratic state with strict unbending rules that seem arbitrary to the citizens of this 'state': the pupils and students. Schoolchildren try to follow the rules, but are sometimes abruptly told that they have broken a rule. They didn't do it deliberately – most students who get some maths questions wrong didn't do it on purpose, they really thought they had the right answer. And yet they're told they've broken the law and will be punished – being marked wrong feels like a punishment to them. Perhaps it is never really explained to them what they did wrong, or perhaps it was not explained to them in an illuminating way that could actually make sense to them. As a result they don't know when they will next be found to have broken a rule, and they will creep around in fear. Eventually they'll simply want to escape to a more 'democratic' place, a subject in which many different views are valid.

***

‘Knowledge is power’, or so the adage goes. But understanding is more powerful power. We have moved on from the age when knowledge was a secret, passed around in mysterious books that could only be deciphered by a small number of people. We have moved on from the age when there were so few books that even those who did know how to read them were at the mercy of those who owned them, the age when students seeking knowledge had to gather around somebody who would read the book out loud to them, a ‘lecturer’ – remember that the word ‘lecture’ comes from the act of reading, not the act of pontificating to an audience. Anyway, we have moved on from that age.

We are now in the age where information is everywhere. Literacy rates still leave room for improvement, but most adults can read, and in some countries most of them have access to the internet. Many of us essentially have the internet in our pocket at all times. Knowledge is no longer a secret. But understanding is still kept a secret, at least in mathematics. Students of all levels are shown the rules but kept in the dark about the reasons. We encourage children to ask the question why, but only up to a point, because beyond that point we might not understand it ourselves. So we stifle their quest for illumination to match our own inability to provide it. Instead of being afraid of that darkness, we should bring everyone to the edge of it and say: ‘Look! Here’s an area that needs illumination.’ Bring fire, torches, candles – anything you can think of that will cast light. Then we can lay down our foundations and build our great buildings, cure diseases, invent fabulous new machines and do whatever else we think the human race should be doing. But first of all we need some light.

ACKNOWLEDGEMENTS

I am deeply grateful to so many people that I'm beginning to wonder if it wouldn't be better to thank nobody at all rather than omit people, but perhaps that's taking logic to extremes in a way that I don't advocate.

So first I'll thank my friends and collaborators in the category theory research community. My conversations with them, mathematical or otherwise, are a continued source of inspiration and excitement. Some of them will recognise where they are thanked implicitly in the text. I would also like to thank my non-mathematician friends who have been sufficiently curious about my work to have given me years of practice at describing it by way of analogy, anecdote and anything other than technicalities.

Credit is due to my friends and family whose curious incidents or insightful observations on life I mentioned in the text, with or without their names: my mother, father (who also took the bagel, slinky and knot photos), sister, little nephews Jack and Liam, Tyen-Nin Tay, Noo Saro-Wiwa, Brandon Fogel, James Martin, Mike Mitchell, Celia Cobb, Karla Rohde, Marina Cronin, the late Professor Philip Grierson, James Fraser, Jim Dolan, Amaia Gabantxo and first year student Frank Luan, who wrote me the letter from which I quoted at the start.

I would like to thank all my students who ever failed to understand something so that I had to explain it better, my agent Diane Banks, Nick Sheerin and Andrew Franklin at Profile, and TJ Kelleher and Lara Heimert at Basic Books. I also thank Sarah Gabriel, for always being a beacon when my brain was getting foggy, and I thank Jason Grunebaum, Oliver Camacho and James Allen Smith for their love.

Chapter 5 is dedicated to Gregory Peebles, who is so moved by the concept of a torus.

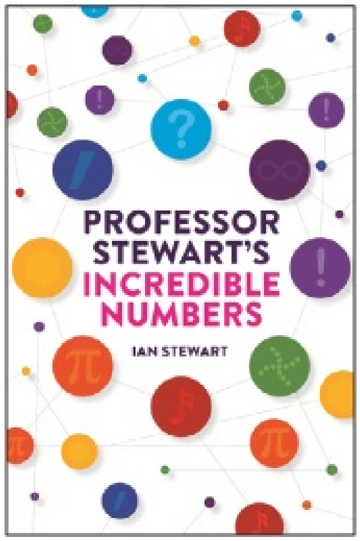
Finally I'd like to thank Kevin × 2, Natalie, Slava, Ryan and Tim at the Travelle in Chicago, for keeping me nourished while I did the final edits, all small children for brightening the world, and everyone else.

There, I think that covers it.

INDEX

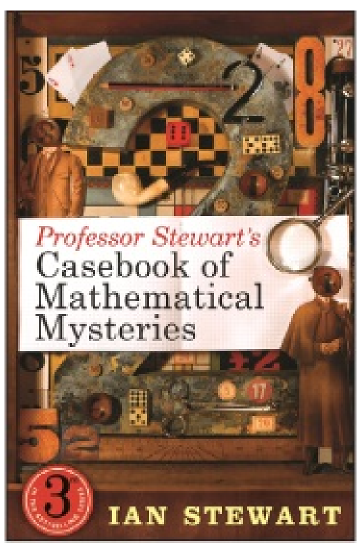
ALSO AVAILABLE FROM PROFILE

PROFILE BOOKS



Professor Stewart's Incredible Numbers Ian Stewart

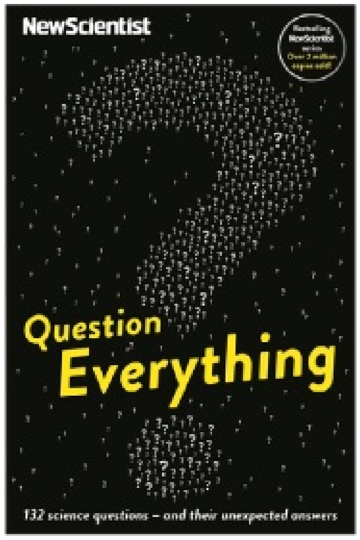
ISBN 978 1 78125 410 3
eISBN 978 1 78283 158 7



Professor Stewart's Casebook of Mathematical Mysteries Ian

Stewart

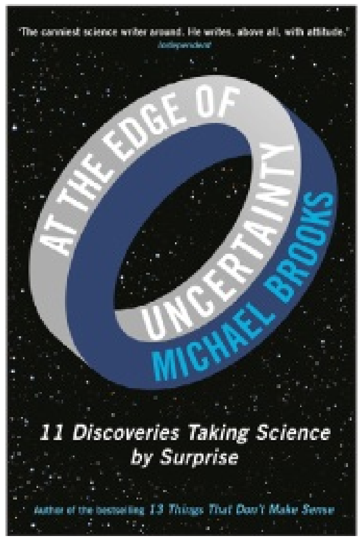
ISBN 978 1 84668 347 3
eISBN 978 1 84765 432 8



Question Everything

132 science questions – and their unexpected answers

New Scientist ISBN 978 1 78125 164 5
eISBN 978 1 84765 984 2



At the Edge of Uncertainty

11 Discoveries Taking Science by Surprise

Michael Brooks ISBN 978 1 78125 128 7
eISBN 978 1 84765 966 8