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Dispersion relation for Landau Damping

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Use MATLABs symbolic toolbox, and define frequency ω, wave vector k

```
syms omega k x


% Set up grid of wave vectors the dispersion relation should be solved
 for
% kgrid=0.01:0.005:0.6;
kgrid=0.05:0.05:0.6;
```

Plasma dispersion function Z(x)

The plasma dispersion function can be found in many textbooks

$$Z(x) = \frac{1}{\sqrt{\pi}} \int_{\gamma} \frac{\mathrm{e}^{-z^2}}{z - x} \mathrm{d}z$$

```
Z=symfun( sqrt(sym(pi))*exp(-x^2)*(1j-erfi(x)),x);
```

Dispersion relation

The dispersion relation for the Vlasov-Poisson system with an initial state Maxwellian (set $\alpha = 1$)

$$f_0(t=0,x,v) = \frac{\alpha}{\sqrt{2\pi} v_{th}} \mathrm{e}^{-\frac{(v-v_0)^2}{2v_{th}^2}}$$

with thermal velocity v_{th}, mean velocity v_0 and plasma frequency ω_p is given as

$$D(\omega,k) = 1 + \alpha \left(\frac{\omega_p}{k v_{th}}\right)^2 \left[1 + \frac{1}{\sqrt{2} v_{th}} \left(\frac{\omega}{k} - v_0\right) Z\left(\frac{1}{\sqrt{2} v_{th}} \left(\frac{\omega}{k} - v_0\right)\right)\right].$$

Set parameters

```
vth=1;     % Thermal velocity of electrons
omegap=1; % Plasma frequency
alpha=1;  % Normalization
v0=0;     % Mean velocity

% Dispersion relation in symbolic variables
D=symfun(1+ alpha*(omegap/vth/k)^2*(1+ (omega/k-v0)/vth/sqrt(2)*...
    Z((omega/k-v0)/vth/sqrt(2))), ...
    [omega,k]);
```

Problem Parameters

Use MATLABS internal variable precision arithmetic to evaluate the Dispersion relation on the contour of a circle in the complex plane with radius *radius*. For all $ k \in$ *kgrid* we search for $\omega \in C$ with $|\omega| \leq$ *radius*.

```
radius=3; %
n=256;    % Number of unit roots, increase for precision
```

Roots of unity in symbolic expression zk=vpa(exp(1j*2*sym('pi')*(0:n-1)/n));

$$z_k = e^{2\pi i \frac{k}{n}}, k = 0, \ldots, n-1$$

```
zk=exp(sym('1j*2*pi')*(0:n-1)/n);

% Number of zeroes to retrieve K<<n
K=5;
K=min(K,n/2-1); %allowed maximum
% intK=(diff(D,omega)/D/(sym('2*pi*1j')));
% mean(double(intK(zk,kgrid(4))))


% Define double precision evaluations for vectorization
residual=matlabFunction(abs(D),'Vars',[omega,k]);

%Structure containing found roots in loose order
roots=struct('omega',[],'k',[],'residual',[]);
num_roots=0;

for idx=1:length(kgrid)

    % Print status
    fprintf('k= %08.4f,  %05.2f%%\n',...
        kgrid(idx), (idx-1)/length(kgrid)*100);

    % Alternative: use the ratdisk_K method
     %gk=double(vpa(1./D(zk*radius,kgrid(idx))));
     %[r,a,b,mu,nu,w] = ratdisk(gk, n-1-K, K);
```

```
k= 000.0500,  00.00%
```

k= 000.1000, 08.33%

k= 000.1500, 16.67%

k= 000.2000, 25.00%

k= 000.2500, 33.33%

k= 000.3000, 41.67%

k= 000.3500, 50.00%

k= 000.4000, 58.33%

k= 000.4500, 66.67%

k= 000.5000, 75.00%

k= 000.5500, 83.33%

k= 000.6000, 91.67%

Compute Zeroes and Poles by Cauchy Integrals

Algorithm [Cz2] taken from

NUMERICAL ALGORITHMS BASED ON ANALYTIC FUNCTION VALUES AT ROOTS OF UNITY, ANTHONY P. AUSTIN , PETER KRAVANJA , AND LLOYD N. TREFETHEN, SIAM J. NUMER. ANAL. Vol. 52, No. 4, pp. 1795–1821,

```
s = ifft(double(1./D( zk*radius, kgrid(idx)  )));
H = hankel(s(2:K+1), s(K+1:2*K));
H2 = hankel(s(3:K+2), s(K+2:2*K+1));
w = eig(H2,H);

w=w*radius;
```

Evaluate residual $|D(\omega,k)|$

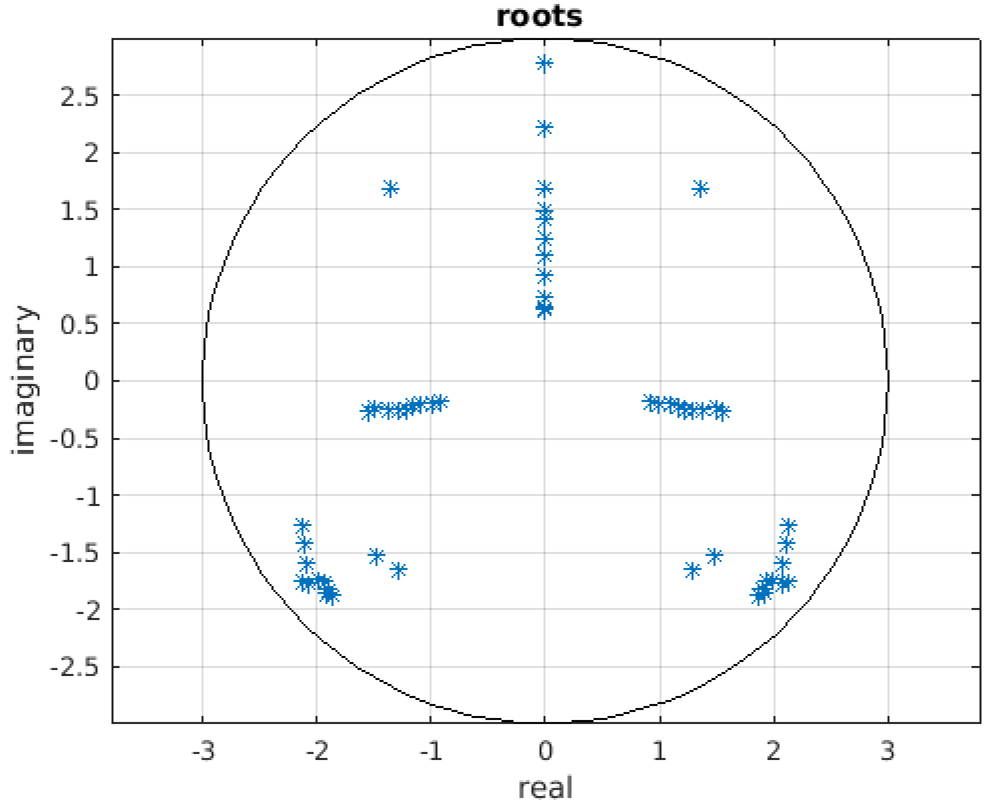
```
res=residual(w,kgrid(idx));


% Delete invalid zeroes
valid=(abs(w)<radius & ~isinf(res) & ~isnan(res));
res=res(valid);
w=w(valid);

% Sort by residual
[res,I]=sort(abs(res));
w=w(I);

% append new roots
roots=[roots, struct('omega',w.',...
    'k',ones(1,length(w))*kgrid(idx) ,...
    'residual',res.')];
```

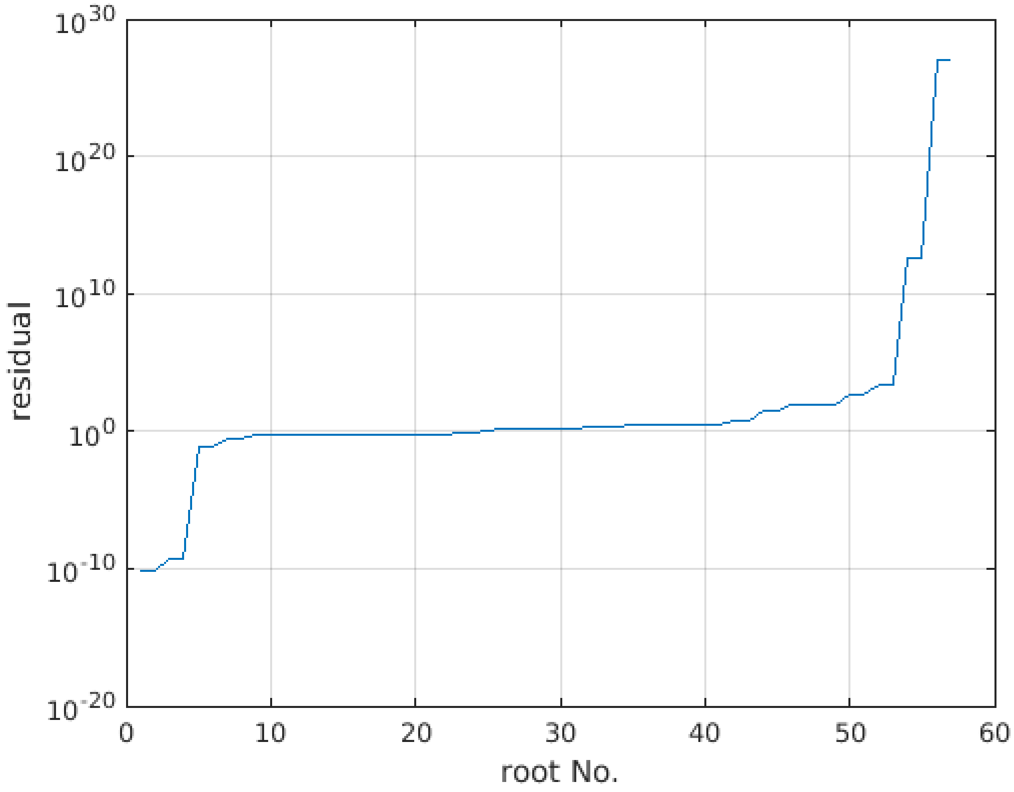
```
end

% Rearrange structure and sort by residual
roots=struct('omega',[roots.omega],'k',[roots.k],...
                    'residual',[roots.residual]);
[~,I]=sort(roots.residual);
roots.omega=roots.omega(I);
roots.residual=roots.residual(I);
roots.k=roots.k(I);
```

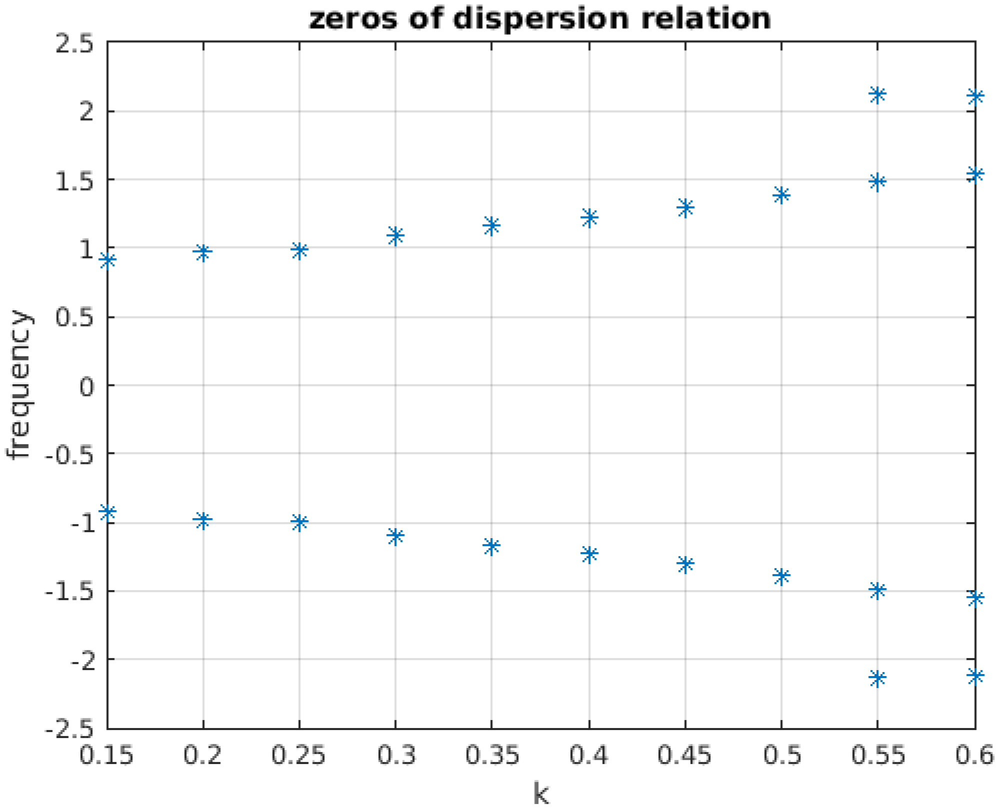
Discussion

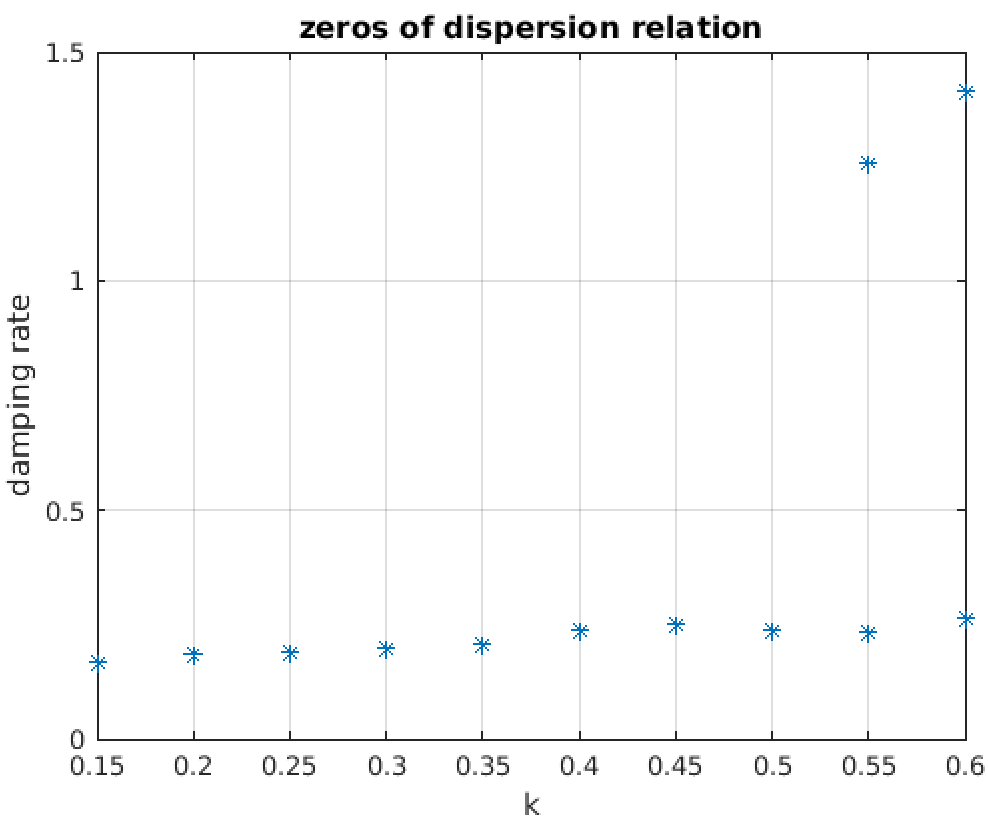
```
figure;
plot(real(roots.omega), imag(roots.omega),'*'); hold on;
plot(radius*cos(linspace(0,2*pi)),radius*sin(linspace(0,2*pi)),'k-');
title('roots ');
xlabel('real'); ylabel('imaginary');
axis equal; grid on;



figure;
semilogy(roots.residual);
ylabel('residual');
xlabel('root No.');
grid on;


tol=1;
valid=roots.residual<tol;



fprintf('Found %d roots \n', sum(valid));

figure;
 plot( roots.k(valid) ,real(roots.omega(valid)),'*'); grid on;
 xlabel('k'); ylabel('frequency');
 title('zeros of dispersion relation')

  figure;
 plot(roots.k(valid) ,-imag(roots.omega(valid)),'*'); grid on;
 xlabel('k'); ylabel('damping rate');
 title('zeros of dispersion relation')






 drawnow;
```

Found 24 roots


roots
imaginary
real
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3


residual
root No.
10^{30}
10^{20}
10^{10}
10^{0}
10^{-10}
10^{-20}
0
10
20
30
40
50
60


zeros of dispersion relation
frequency
k
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6


zeros of dispersion relation
damping rate
k
1.5
1
0.5
0
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6

Standard test case $k = 0.5$

Found the following zeros for the dispersion relation for the standard test case $k = 0.5$

```
disp(roots.omega(roots.k==0.5))
```

```
  Columns 1 through 4
   1.3784 - 0.2396i  -1.3784 - 0.2396i  -0.0000 + 1.4198i   2.0864 -
 1.6023i
  Column 5
  -2.0864 - 1.6023i
```

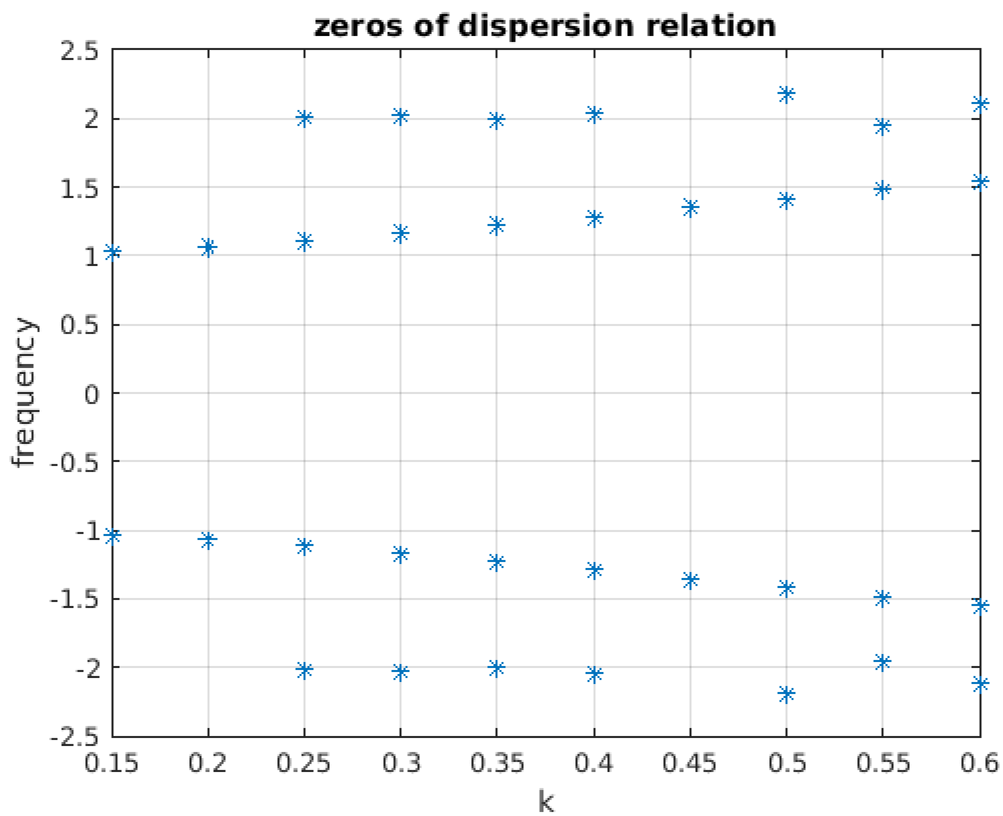
VPA Newton-Raphson refinement

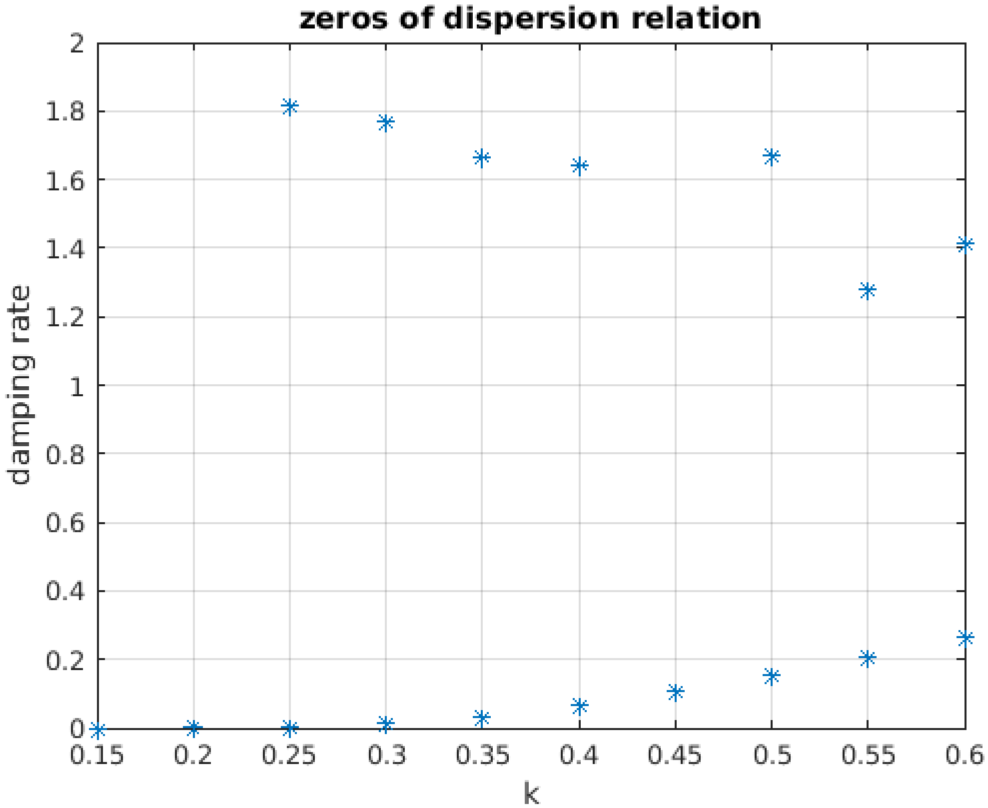
Set the number of significant decimal digits for variable-precision arithmetic

```
digits(64);

maxit=20;  % Maximum number of iterations

k=vpa(roots.k);
w=vpa(roots.omega);

newton_step=simplify(D/diff(D,omega));
for it=1:maxit
    tic;
    w=w-vpa(newton_step(w,k));
    toc;
end


roots_vpa=struct('k',k,'omega',w,'residual', vpa(abs(D(w,k))));

% Select a tolerance
valid=roots_vpa.residual<1e-7;


figure;
plot( roots_vpa.k(valid) , ...
    real(roots_vpa.omega(valid)),'*');
grid on;
xlabel('k'); ylabel('frequency');
title('zeros of dispersion relation')

figure;
plot(roots_vpa.k(valid) ,...
    -imag(roots_vpa.omega(valid)),'*');
grid on;
xlabel('k'); ylabel('damping rate');
title('zeros of dispersion relation')
drawnow;
```

```
Elapsed time is 0.186748 seconds.
```

```
Elapsed time is 0.242597 seconds.
Elapsed time is 0.296357 seconds.
Elapsed time is 0.223992 seconds.
Elapsed time is 0.217204 seconds.
Elapsed time is 0.220646 seconds.
Elapsed time is 0.309109 seconds.
Elapsed time is 0.288316 seconds.
Elapsed time is 0.146660 seconds.
Elapsed time is 0.220466 seconds.
Elapsed time is 0.288546 seconds.
Elapsed time is 0.219096 seconds.
Elapsed time is 0.294941 seconds.
Elapsed time is 0.150995 seconds.
Elapsed time is 0.289538 seconds.
Elapsed time is 0.212024 seconds.
Elapsed time is 0.141780 seconds.
Elapsed time is 0.288493 seconds.
Elapsed time is 0.274596 seconds.
Elapsed time is 0.147810 seconds.
```





Published with MATLAB® R2015a