The Inverse of a function

If a function $f: X \to Y$ is a bijection, then the inverse of f is obtained by exchanging the first and second entries in each pair in f.

Notation: f^{-1}

$$f^{-1} = \{ (y, x) : (x, y) \in f \}$$

i.e. if $f(x) = y$, then $f^{-1}(y) = x$

- Because reversing each pair in a function doesn't always result in a well-defined function, some functions do NOT have an inverse.

- A function $f: X \to Y$ has an inverse if and only if reversing each pair in f results in a well-defined function from Y to X.

  i.e. f^{-1} is a well-defined function if every $y \in Y$ maps to exactly one element $x \in X$

$\Rightarrow$ A function f has an inverse if and only if f is a bijection.

Solving for inverse of function analytically when a function is defined on an infinite domain:

if $f(x) = y$, then $f^{-1}(y) = x$ $\Rightarrow$ solve for x to get f^{-1}

ex:

$f: \mathbb{R} \to \mathbb{R}$, where $f(x) = 3x - 2$

① check $f(x)$ is 1-to-1: $(x \neq x') \to (3x - 2 \neq 3x' - 2)$

contrapositive proof $(3x - 2 = 3x' - 2)$, then $(x = x')$

② check $f(x)$ is onto: for every y in $\mathbb{R}$, there is an x

$f(x) = 3x - 2 = y$

③ solve for x in terms of y:

$3x - 2 = y$

$3x = y + 3$ $\qquad\qquad f^{-1}: \mathbb{R} \to \mathbb{R}$

$x = \frac{y+3}{2}$ $\Rightarrow$ $f^{-1} = \frac{y+3}{2}$ $\Rightarrow$ $f^{-1}(y) = \frac{y+3}{2} = f^{-1}(x) = \frac{x+3}{2}$