Prime Factorization

The Fundamental Theorem of Arithmetic

Every positive integer other than 1 can be expressed uniquely as a product of prime numbers where the prime factors are written in non-decreasing order

A prime number's prime factorization only consists of p.

To calculate greatest common divisor for x and y:

1.) Determine the prime factorization of x and y

2.) Express the prime factorization of both numbers using a common set of primes

ex: $x = 24 = 2^3 \cdot 3^1 \cdot 5^0$

$y = 50 = 2^1 \cdot 3^0 \cdot 5^2$

3.) Use the smallest exponent for each prime factor, then multiply the prime factors to their respective exponent to determine the gcd

ex: $\gcd(x,y) = \gcd(24,50)$

$= 2^{\min(3,1)} \cdot 3^{\min(1,0)} \cdot 5^{\min(0,2)} = 2^1 \cdot 3^0 \cdot 5^0$

$= 2$

To calculate the least common multiple for x and y:

3.) use the largest exponent for each prime factor

ex: $\text{lcm}(x,y) = \text{lcm}(24,50)$

$= 2^{\max(3,1)} \cdot 3^{\max(1,0)} \cdot 5^{\max(0,2)} = 2^3 \cdot 3^1 \cdot 5^2$

$= 600$

GCD and LCM from prime factorizations

Let x and y be two positive integers with prime factorizations expressed using a common set of primes as:

$$x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \, p_r^{\alpha_r}$$
$$y = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \ldots \, p_r^{\beta_r}$$

The p_i's are all distinct prime numbers. The exponents α_i's and β_i's are non-negative integers

Then:

- x divides y if and only if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq r$
- $\gcd(x,y) = p_1^{\min(\alpha_1,\beta_1)} \cdot p_2^{\min(\alpha_2,\beta_2)} \cdots p_r^{\min(\alpha_r,\beta_r)}$
- $\operatorname{lcm}(x,y) = p_1^{\max(\alpha_1,\beta_1)} \cdot p_2^{\max(\alpha_2,\beta_2)} \cdots p_r^{\max(\alpha_r,\beta_r)}$