Sum and Product Rules

Product Rule:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

- the Product Rule provides a way to count sequences.

example: A meal consists of a drink, a main, and a side.

$D = \{$coffee, orange juice$\}$
$M = \{$pancakes, eggs$\}$
$S = \{$bacon, sausage, hash browns$\}$

$D \times M \times S$ = cartesian product $= \{(d, m, s) : d \in D, m \in M, s \in S\}$

$|D \times M \times S| = |D| \cdot |M| \cdot |S| = 2 \cdot 2 \cdot 3 = 12$

*** the set order matters ***

Counting Strings

Σ is an alphabet, n is the string length, Σ
Σ^n is the set all n-length strings in the alphabet

$$|\Sigma^n| = |\underbrace{\Sigma \times \Sigma \ldots \times \Sigma}_{n \text{ times}}| = \underbrace{|\Sigma| \cdot |\Sigma| \ldots |\Sigma|}_{n \text{ times}} = |\Sigma|^n$$

ex: binary alphabet where $\Sigma = \{0,1\}$ and string length $= 6$
$|\Sigma^6| = |\{0,1\}^6| = |\{0,1\}|^6 = 2^6 = 64$

When one or more characters are restricted in a string:

Let S be a set of binary strings of length 5 that start and end with 0.

$S = 0{*}{*}{*}0$ where $*$ is a wild card $* = 0$ xor $* = 1$

$S = \{0\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$

$|S| = |\{0\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0\}|$

$|S| = |\{0\}| \times |\{0,1\}| \times |\{0,1\}| \times |\{0,1\}| \times |\{0\}| = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 8$

$|S| = |\{0\}| \times |\{0,1\}|^3 \times |\{0\}| = 1 \cdot 2^3 \cdot 1 = 8$

The sum rule

consider n sets $A_1, A_2, \dots A_n$
if the sets are mutually disjoint ($A_i \cap A_j \neq \emptyset$ for $i \neq j$)

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \quad + |A_n|$$

- Sum rule is applied when there are multiple choices but only one selection is made

Ex: Hot drinks = H = {coffee, tea, cocoa}
Cold drinks = C = {milk, orange juice}
customer buys one drink
since $H \cap C$ is $\emptyset$ (an empty set), then
$|H \cup C| = |H| + |C| = 3 + 2 = 5$

Example: Product and sum rule combined: counting passwords

- The password string length 6-8 (inclusive)
- each character can be a lowercase letter or digit

L = all lower case characters $|L| = 26$
D = all digits 0-9 $|D| = 10$

Since $L \cap D = \emptyset$ (an empty set), then the set of all possible characters, $C = L \cup D$

$$|C| = |L \cup D| = |L| + |D| = 26 + 10 = 36$$

- a string of length n with $\Sigma = C$
  a set of all possible combinations of strings with length n in alphabet C is represented by Σ^n

  • the cardinality of $\Sigma^n = |\Sigma^n| = |\Sigma|^n = |C|^n$

- since Σ^6 and Σ^7 and Σ^8 are all mutually disjointed with each other,
  the set of all possible passwords $= A = \Sigma^6 \cup \Sigma^7 \cup \Sigma^8$

  the cardinality of $A = |A| = |\Sigma^6 \cup \Sigma^7 \Sigma^8|$
  $= |\Sigma^6| + |\Sigma^7| + |\Sigma^8| = |\Sigma|^6 + |\Sigma|^7 + |\Sigma|^8$
  since $|\Sigma| = |C| = 36$, then

  $$= |\Sigma|^6 + |\Sigma|^7 + |\Sigma|^8 = 36^6 + 36^7 + 38^8$$