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OPTIMIZING TRUSS STRUCTURES WITH NATURAL FREQUENCY CONSTRAINTS USING THE CROSS-ENTROPY METHOD

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Abstract. *This work addresses the challenging problem of structural optimization, specifically focusing on minimizing the mass of a structure while satisfying constraints related to natural frequencies. Traditional optimization methods that rely on gradient information are not suitable for such complex problems. To overcome this limitation, metaheuristic methods have emerged as effective alternatives. In this study, we propose a novel framework that employs the Cross-Entropy (CE) method, a powerful Monte Carlo technique for handling non-convex optimization problems. By applying the CE method to optimize structural trusses as benchmark tests, we compare its performance with the genetic algorithm (GA). Through numerical experiments, we demonstrate that the CE method provides accurate and computationally efficient solutions, outperforming other metaheuristic optimization methods. This research highlights the efficacy of the CE method as a valuable tool for addressing non-convex structural optimization problems, enabling efficient solutions with natural frequency constraints.*

Keywords: *structural optimization; non-convex optimization; stochastic simulation*

1. INTRODUCTION

Structural optimization plays a crucial role in engineering design, aiming to find the optimal configuration of a structure that meets specific performance criteria (**???**). In many real-world applications, the objective is to minimize the mass of the structure while satisfying constraints related to natural frequencies. Achieving such optimization poses significant challenges due to the non-convex nature of the problem and the complex interplay between mass and natural frequency constraints.

Traditional optimization techniques, such as gradient-based methods, rely on the availability of explicit gradients of the objective function and constraints (**????**). However, in several structural optimization problems with non-convex objectives and complex constraints, obtaining these gradients can be impractical or even impossible. Moreover, even if gradients are available, the presence of multiple local optima makes it challenging to guarantee finding the global optimum. To address these challenges, metaheuristic methods have emerged as promising approaches for solving non-convex optimization problems (**??????**). Metaheuristics, unlike traditional methods, do not rely on gradient information and are capable of exploring complex search spaces efficiently. These methods have been successfully applied in various fields, including structural optimization, due to their ability to handle non-convexity and constraints without explicit gradient information.

In the context of structural optimization, the minimization of mass while respecting natural frequency constraints is of paramount importance. However, this problem presents additional difficulties, as it requires balancing conflicting objectives and dealing with complex nonlinear system. The optimization process must strike a delicate balance between achieving a lightweight structure while ensuring avoiding undesirable vibration modes.

In this paper, we address the challenges associated with structural optimization, particularly focusing on the problem of minimizing mass subject to natural frequency constraints. To tackle this problem, we propose the utilization of the Cross-Entropy (CE) method, a powerful metaheuristic algorithm capable of handling non-convex optimization problems (**????**). The CE method, originally developed for rare event simulation, provides an efficient approach to explore complex search spaces and find near-optimal solutions. The primary motivation for employing the CE method in this context is twofold. Firstly, the CE method does not require explicit gradient information, making it suitable for non-convex problems where gradients are unavailable or difficult to compute. Secondly, the CE method offers a balance between accuracy and computational efficiency, providing competitive solutions for optimization problems with constraints.

In this study, we compare the performance of the CE method with the genetic algorithm (GA) on benchmark structural truss problems (**??**). We aim to demonstrate that the CE method can effectively handle the non-convex structural optimization problem of minimizing mass while satisfying natural frequency constraints. Through numerical experiments, we evaluate the accuracy and computational efficiency of the CE method, highlighting its potential as a valuable tool in the field of structural optimization.

2. Structural optimization

The optimization problem of interest in this work seeks to find a configuration of areas for the cross sections of a truss structure with n bars that minimize its total mass, respecting certain restrictions imposed on the k first natural frequencies. This problem can be mathematically formulated as finding a vector of design variables $\boldsymbol{x} = \{A_1, \cdots, A_n\}$ that minimizes the objective function

$$\mathcal{J}(\boldsymbol{x}) = \int_{\mathfrak{Truss}} \rho(\boldsymbol{x})\, dV \,, \tag{1}$$

respecting the constraints defined by

$$\omega_1 \geq \omega_1^*, \;\cdots\; , \omega_k \geq \omega_k^* \,, \tag{2}$$

for natural frequencies obtained from

$$\mathbf{K}\, \boldsymbol{\phi}_i = \omega_i^2\, \mathbf{M}\, \boldsymbol{\phi}_i \quad (i = 1, \cdots, k) \,, \tag{3}$$

where ρ is the truss material density, ω_i is the i-th natural frequency, ω_i^* is the i-th natural frequency constraint, and A_e is the cross-sectional in e-th bar element. The natural frequencies of a structural model are obtained by the eigenvalue problem, being, $\mathbf{M}$ the mass matrix, ω^2 the eigenvalue, and $\boldsymbol{\phi}$ the eigenvector, where ω is the value of the natural frequency.

3. Cross-Entropy method

The Cross-Entropy (CE) method is a Monte Carlo technique used for estimation and optimization. In the estimation setting, the CE provides a form of searching for the sampling of optimal importance. After formulating an optimization problem as an estimation problem, CE becomes a powerful stochastic search method. The method is based on a simple iterative procedure and in each iteration it contains only two phases: generating the random data samples (trajectories, vectors, etc.) and updating the parameters of random mechanisms based on the data in order to produce a better sample in the next iteration (**?**).

The CE has its origin in the adaptation of the algorithm to estimate a rare event based on variance minimization. This procedure was soon modified to an algorithm adapted for the estimation of rare events and combinatorial optimization, where the minimum variation programs were replaced by the CE minimization program (**?**).

Let $\mathcal{J}$ be the objective function in $\mathbb{R}^n$. Suppose one wants to find the maximum of $\mathcal{J}$ in $\mathbb{R}^n$, i.e.,

$$\boldsymbol{x^*} = \arg\max_{\boldsymbol{x} \in \mathbb{R}^n} \mathcal{J}(\boldsymbol{x}) \,, \tag{4}$$

and the achieved maximum is denoted by $\gamma^* = \mathcal{J}(\boldsymbol{x^*})$.

Associating it with the probability estimation problem, $\mathcal{P}(\mathcal{J}(\boldsymbol{X}) \geq \gamma)$, where $\boldsymbol{X}$ is a randomized version of the design variables vector with $f(x; \boldsymbol{v})$ in $\mathbb{R}^n$, the maximum can be approximated by the estimation of this probability. In fact, if γ is a near-unknown choice γ^*, this is typically the probability of a rare-event, and the CE approach for estimation can be used to search for the distribution of importance sampling near the sampling density of theoretical greatest importance, which concentrates all its mass on the point $\boldsymbol{x}^*$ (**?**). Sampling from this distribution produces optimal or close to optimal results. A schematic of CE method is represented in Figure 1, and the algorithm is presented in the sequence.

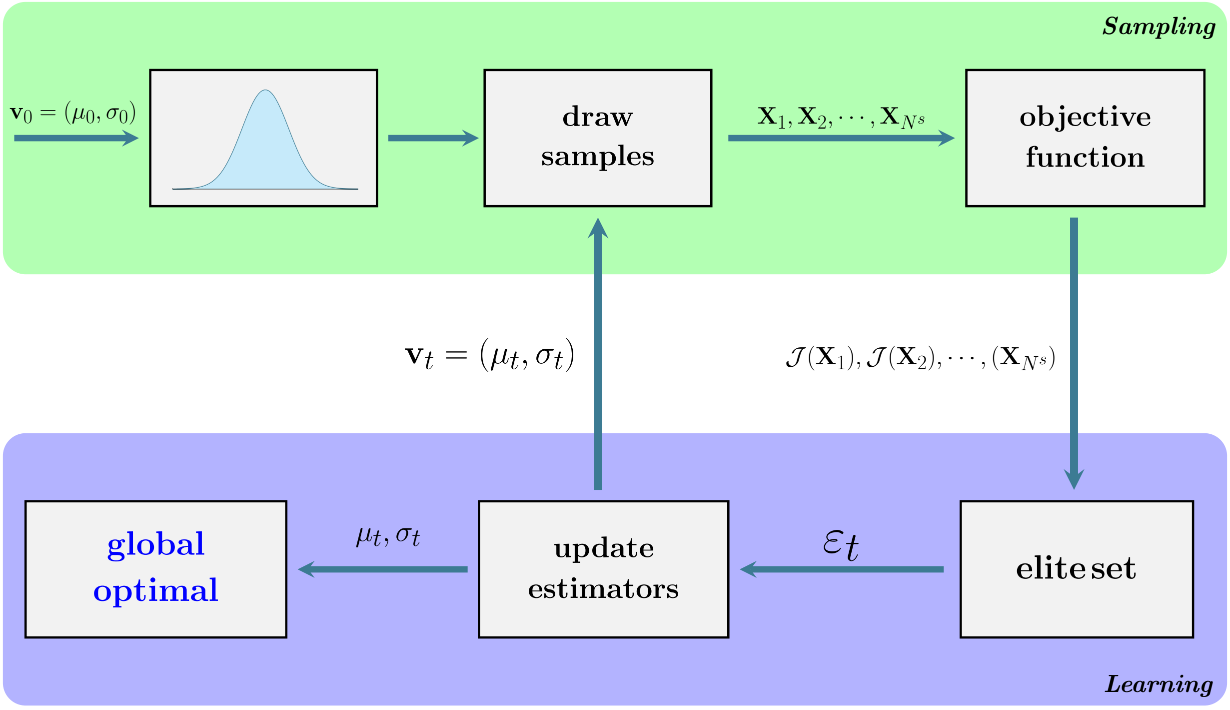

Figure 1: Schematic representation of the CE algorithm for optimization.

CE Algorithm for optimization

1. Choose an initial parameter vector $\boldsymbol{v}_0$. Let $N^e = [\varrho N^s]$.
Set t = 1. (iteration counter);

2. Generate $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{N^s} \sim$ iid. Calculate $\mathcal{J}_i = \mathcal{J}(\boldsymbol{X}_i)$ for all i,
and order these from smallest to largest: $\mathcal{J}_{(1)} \leq \ldots \leq \mathcal{J}_{(N)}$;
Let $\hat{\gamma}_t$ be the sample $(1-\varrho)$ - quantile of performances; that is, $\hat{\gamma}_t = \mathcal{J}_{(N^s - N^e + 1)}$.

3. Use the same sample $\boldsymbol{X}_{(1)}, \ldots, \boldsymbol{X}_{(N^e)}$ to solve the stochastic program
$\max\limits_{\boldsymbol{v}} \dfrac{1}{N} \sum\limits_{X_k \epsilon \mathcal{E}}^{N} \ln f(\boldsymbol{X}_k; \boldsymbol{v})$
Denote the solution by $\hat{\boldsymbol{v}}_t$;

4. If some stopping criterion is met, stop; otherwise, set $t = t + 1$,
and return to Step 2.

Algorithm reproduced integrally from (**?**).

Whenever the optimization problem has constraints $q_i(\boldsymbol{X}) = 0 \;\; i = q, \cdots, k$, a penalization strategy is used, where the objective function is modified to

$$\tilde{\mathcal{J}}(\boldsymbol{X}) = \mathcal{J}(\boldsymbol{X}) + \nu \sum_{i=1}^{k} \max\{q_i(\boldsymbol{X}), 0\} \;, \tag{5}$$

being $\nu < 0$ the i-th penalty parameter (**?**).

4. Numerical Results

This section presents the numerical results for the optimization of a 10-bar truss structure, as illustrated in Figure 2. The objective of the optimization is to find the optimal configuration of the truss that satisfies specific natural frequency requirements. Specifically, the target natural frequencies are set as $\omega_1^* = 7\ Hz$, $\omega_2^* = 15\ Hz$, and $\omega_3^* = 20\ Hz$. Additionally, the truss structure must accommodate an added mass m_{ad} of 454 kg distributed at nodes 1, 2, 3, and 4. The optimization problem also includes constraints on the cross-sectional areas, where each area A_e must fall within the range of 65.4 mm^2 to 5000 mm^2. The aim of the optimization is to find a truss configuration that minimizes the overall mass while satisfying these frequency and cross-sectional area constraints. To reduce the mass of the structural system while satisfying the defined natural frequency constraints, the cross-sectional areas of the bars are systematically reduced. The optimization methods aim to find the minimum mass configuration that achieves or exceeds the target natural frequency values.

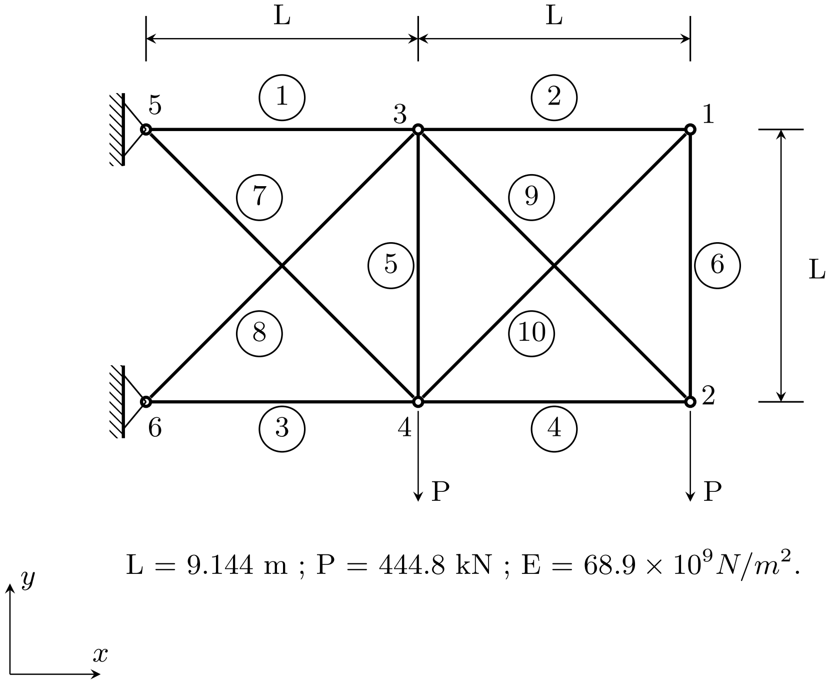

Figure 2: Illustration of the truss structure to be optimized.

Figure 3 presents the box plot with the statistics of the values ends of the truss mass in each of the two metaheuristics of the 100 obtained values where the red line represents the median, the lower and upper limits of the blue box represent the 25% quartile, and the 75% quartile, respectively, the dashed upper and lower limits represent the most extreme values, disregarding the outliers, which are the red crosses. CE obtained a median value better than GA.

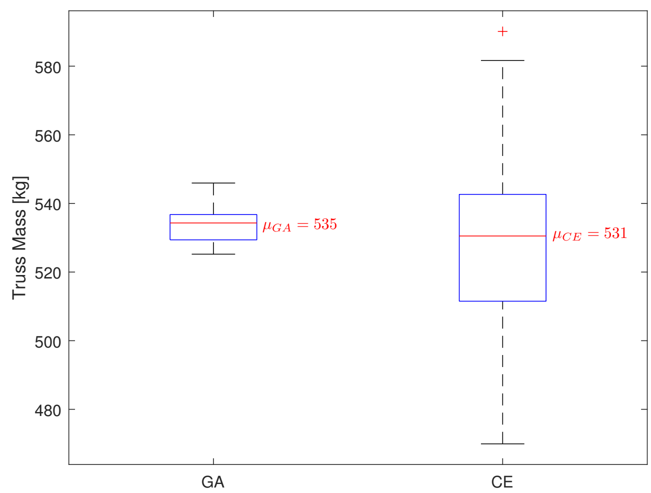

Figure 3: Boxplot with GA and CE.

Table 1 displays the cross-sectional areas obtained by the optimization methods with mean values, while Table 2 provides an illustration of the corresponding area distributions. The achieved natural frequencies resulting from the optimization methods are presented in Table 3. It can be observed that all the optimization methods have successfully respected the constraints, with the first three natural frequencies (ω_1, ω_2, and ω_3) matching or exceeding the defined constraints (ω_1^*, ω_2^*, and ω_3^*). This demonstrates the effectiveness of the optimization methods in achieving the desired structural performance while minimizing the mass.

Table 1: Values of cross-sectional area in the Truss found by optimization methods GA and CE in mm^2.

Method	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
GA	3476	1427	3339	1398	131	466	2521	2441	1236	1516
CE	3558	1490	3558	1490	65	462	2403	2402	1256	1257

Table 2: Areas obtained by the two optimization methods considering natural frequencies constraints.

Method	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
GA										
CE										

Table 3: Natural frequencies in the Truss found by optimization methods GA and CE in Hz.

Method	ω_1	ω_2	ω_3
GA	7.0	16.6	20.0
CE	7.0	16.2	20.0

In the past decade, two metaheuristic methods, Harmony Search (HS) and Firefly Algorithm (FA), have gained attention as effective optimization techniques (**?**). In this study, these two methods are applied to the optimization problem with constraints on natural frequencies, which involves nonlinear dynamic optimization. It is worth noting that this was the first time these methods were utilized for sizing and shape optimization with natural frequency constraints. The results of these methods, along with the CE method (median), are presented in Table 4.

Table 4: Comparison between the results obtained by CE, HS and FA optimizing the area of each bar in Truss.

Method	mass (kg)
CE	531
HS	535
FA	531

Table 6 provides an overview of the cross-sectional areas obtained by the CE, HS, and FA optimization methods. Additionally, Table 7 illustrates the corresponding area distributions obtained by these methods. The natural frequencies resulting from the optimization are shown in Table 5. It is important to note that the values for HS and FA are taken from (**?**). It is worth mentioning that CE, HS, and FA are relatively new optimization methods that hold potential for further improvement and future advancements in this example.

Table 5: Natural frequencies in the Truss found by optimization methods CE, HS and FA in Hz.

Method	ω_1	ω_2	ω_3
CE	7.0	16.2	20.0
HS	7.0	16.7	20.1
FA	7.0	16.1	20.0

Table 6: Values of cross-sectional area in the Truss found by optimization methods CE, HS and FA in mm^2.

Method	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
CE	3558	1490	3558	1490	65	462	2403	2402	1256	1257
HS	3428	1565	3764	1606	107	474	2250	2460	1287	1210
FA	3620	1403	3475	1490	65	467	2347	2551	1271	1235

Table 7: Illustration of the areas obtained by CE, HS and FA considering natural frequencies constraint in Truss.

Method	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
CE										
HS										
FA										

5. Concluding remarks

Numerical experiments evaluate the effectiveness and robustness of the CE in the context of structural optimization. The results show that the evaluated method is very competitive, proving to be an appealing tool for optimization problems.

The results indicate that CE performs favorably compared to GA, HS and FA. This highlights the efficiency and viability of CE as a technique for structural optimization. Despite this, the comparative analysis suggests that CE, HS and FA have potential for improvement and may yield better results in future studies. However, it is important to highlight that, for computational expensive problems like structural optimization, the use of metaheuristics is only justified if, despite being expensive, it is still feasible, and it is not viable to use gradient-based methods, either because the problem is non-differentiable, or there is a great need to obtain a global optimum and the non-convexity of the problem makes a solution obtained by gradient-based methods unattractive.

In summary, CE demonstrates strong performance and efficiency in the considered structural optimization problem. Furthermore, in future works CE will be applied in other structural optimization scenarios such as in 3D structural models.

6. ACKNOWLEDGEMENTS

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