On the classical and fractional control of a nonlinear inverted cart-pendulum system: a comparative analysis

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NUMERICO – **Nu**cleus of **M**odeling and **E**xperimentation with **Co**mputers

http://numerico.ime.uerj.br

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Section 1

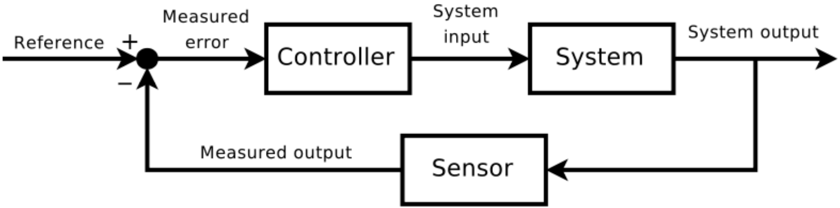
Introduction

The real world needs control

- Engineering
- Physics
- Chemistry
- Biology
- Medicine
- Economics
- Social Sciences

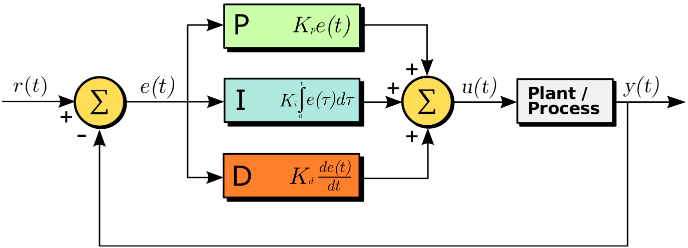
*Pictures obtainded from Google Images, several sources. If you are the owner of any one of these images, consider its use a compliment.

Feedback control system



*Picture from `https://en.wikipedia.org/wiki/Control_theory`

Classical PID-controller

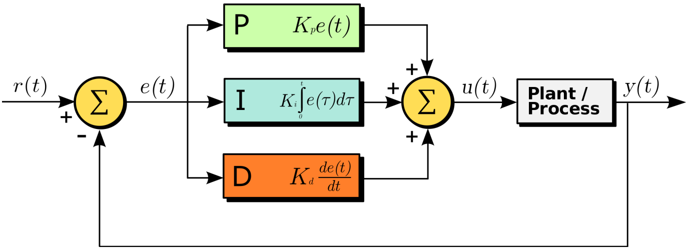


Control signal based on integer-order operators:

$$u(t) = K_P\, e(t) + K_I \int e(\tau)\mathrm{d}\tau + K_D \frac{\mathrm{d}}{\mathrm{d}t} e(t)$$

*Picture from https://en.wikipedia.org/wiki/Control_theory

Fractional-order PID-controller



Control signal based on fractional-order operators:

$$u(t) = K_P\, e(t) + K_I\, \mathcal{I}^{\alpha}\, e(t) + K_D\, \mathcal{D}^{\alpha}\, e(t)$$

*Picture from `https://en.wikipedia.org/wiki/Control_theory`

Classical vs Fractional PID-controller

Classical:

- ☺ Simple and easy to implement
- ☺ Robust to tuning mismatches
- ☹ Low robustness to uncertainties and disturbances
- ☹ Not suited for nonlinear systems

Fractional:

- ☹ Relatively complex and difficult to implement
- ☺ Takes into account input signal history (memory)
- ☺ Suited for nonlinear systems

*Picture from https://en.wikipedia.org/wiki/Control_theory

Research objectives

This research has the following objective:

Evaluate the performance of a fractional-order PID-controller in comparison with a integer-order PID-controller

Section 2

Fractional Calculus

What is a fractional operator?

$$\frac{d^{1/2} f}{dx^{1/2}} = ?$$

D.Ortigueira and J.A.Tenreiro Machado, What is a fractional derivative? **Journal of Computational Physics**, 293:4-13 2015.

*Picture from `https://medium.com/cantors-paradise/fractional-calculus-48192f4e9c9f`

What is a fractional operator?

- Classical calculus

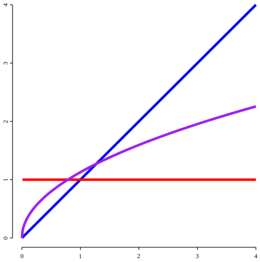
$$D^2(f) = D(D(f))$$

(powers $\iff$ composition)

- Fractional calculus

$$\sqrt{D} = D^{\frac{1}{2}}$$

(meaningful interpretation?)



Derivatives of $f(x) = x$

D.Ortigueira and J.A.Tenreiro Machado, What is a fractional derivative? **Journal of Computational Physics**, 293:4-13 2015.

The fractional integral

Several definitions are possible!

- Riemann-Liouville

$$\mathcal{I}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

- Hadamard

$$\mathcal{I}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}$$

D.Ortigueira and J.A.Tenreiro Machado, What is a fractional derivative? **Journal of Computational Physics**, 293:4-13 2015.

*Pictures from Wikipedia

The fractional derivative

Several definitions are also possible!


- Riemann-Liouville

$$\mathcal{D}^{\alpha} f(t) = \frac{d^n}{dt^n} \mathcal{I}^{n-\alpha} f(t)$$

- Caputo

$$\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{(n-\alpha-1)} f^{(n)}(u) du$$

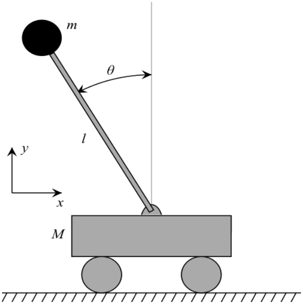
D.Ortigueira and J.A.Tenreiro Machado, What is a fractional derivative? **Journal of Computational Physics**, 293:4-13 2015.

*Pictures from Wikipedia and Research Gate

Section 3

Fractional Controller

Inverted cart-pendulum system

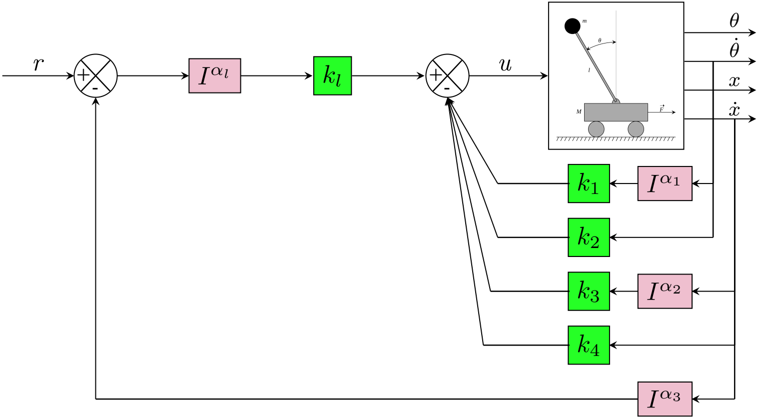


$$m\,l\,\cos\theta\,\ddot{x}(t) + (J + m\,l^2)\,\ddot{\theta}(t) - m\,g\,l\,\sin\theta(t) = 0$$

$$(m + M)\,\ddot{x}(t) + m\,l\,\cos\theta(t)\,\ddot{\theta}(t) - m\,l\,\sin\theta(t)\,\dot{\theta}^2(t) = 0$$

$m = 0.1\text{kg} \quad M = 2\text{kg} \quad l = 0.5\text{m} \quad J = 0.006\text{ kg } m^2$

Fractional-order PID-controller for cart-pendulum system



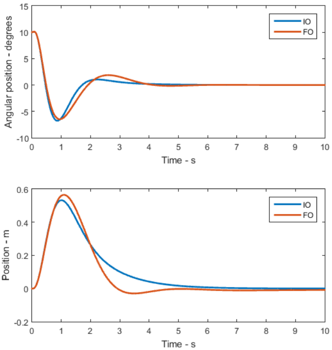
$\mathcal{I}^{\alpha}$ has an additional parameter to tune the controller

Section 4

Numerical Experiments

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 1.1 \qquad \alpha_3 = 1 \qquad \alpha_I = 1.1$

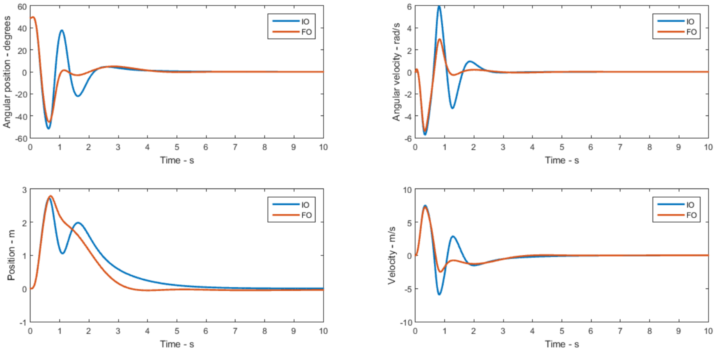


$\theta(0) = 10° \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 1.1 \qquad \alpha_3 = 1 \qquad \alpha_I = 1.1$

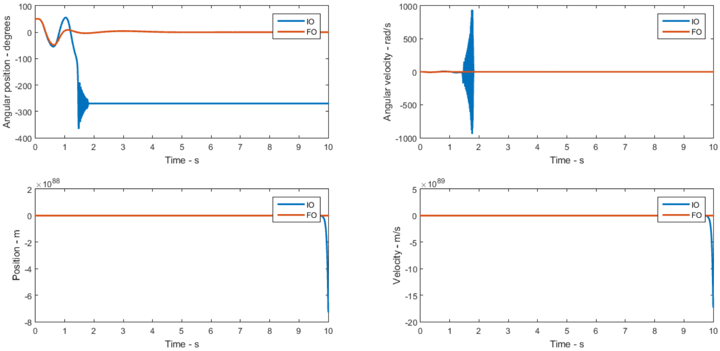


$\theta(0) = 49^\circ \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 1.1 \qquad \alpha_3 = 1 \qquad \alpha_I = 1.1$

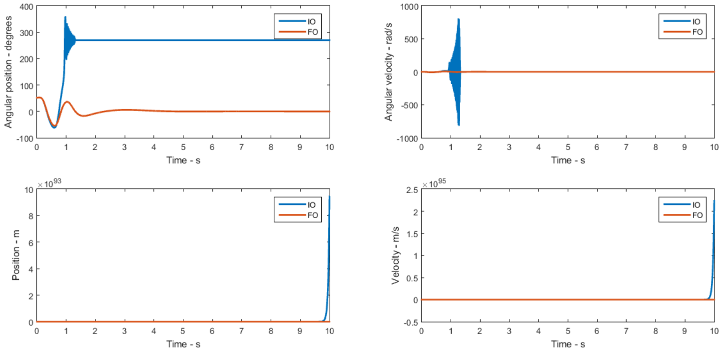


$\theta(0) = 50^\circ \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 1.1 \qquad \alpha_3 = 1 \qquad \alpha_I = 1.1$

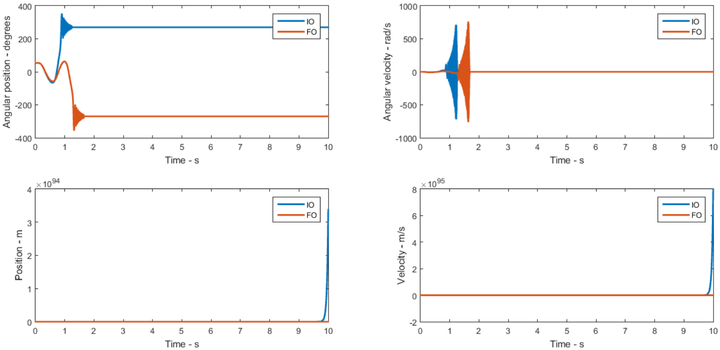


$\theta(0) = 52° \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 1.1 \qquad \alpha_3 = 1 \qquad \alpha_I = 1.1$

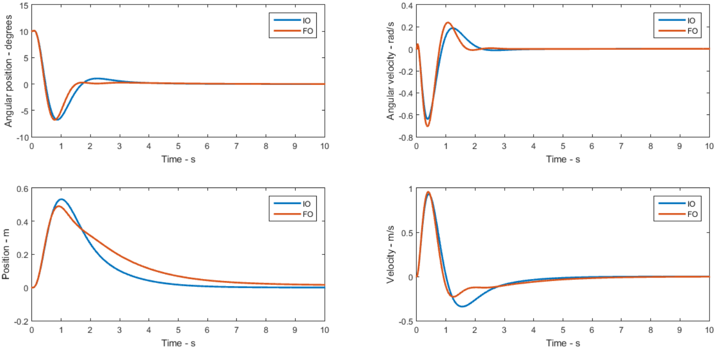


$\theta(0) = 53^\circ \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$$\alpha_1 = 1 \qquad \alpha_2 = 0.9 \qquad \alpha_3 = 1 \qquad \alpha_I = 0.9$$

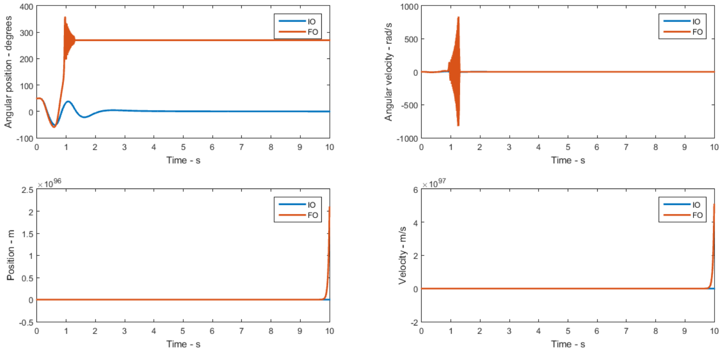


$$\theta(0) = 10° \qquad \dot{\theta}(0) = 0$$

$$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$$

Optimal gains for linear system

$\alpha_1 = 1 \quad \alpha_2 = 0.9 \quad \alpha_3 = 1 \quad \alpha_I = 0.9$

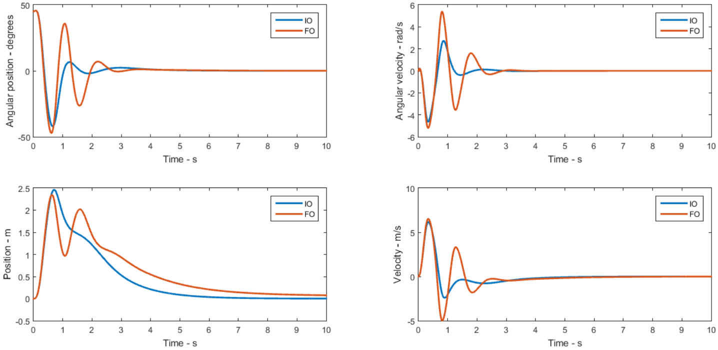


$\theta(0) = 49° \quad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 0.9 \qquad \alpha_3 = 1 \qquad \alpha_I = 0.9$

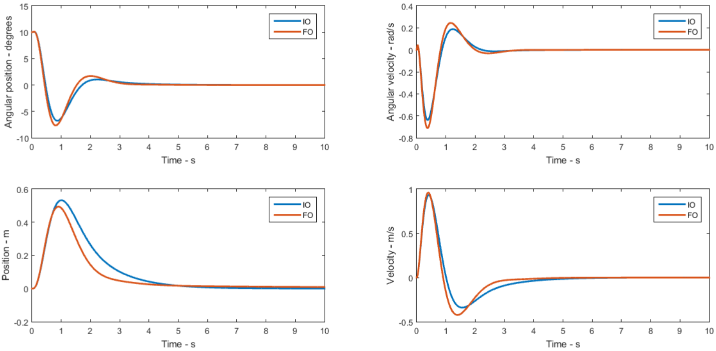


$\theta(0) = 45° \qquad \dot{\theta}(0) = 0$

$k_1 = -201 \;\; k_2 = -50 \;\; k_3 = -70 \;\; k_4 = -47 \;\; k_I = -35$

Non-optimal gains for linear system

$\alpha_1 = 1 \qquad \alpha_2 = 0.9 \qquad \alpha_3 = 1 \qquad \alpha_I = 0.9$

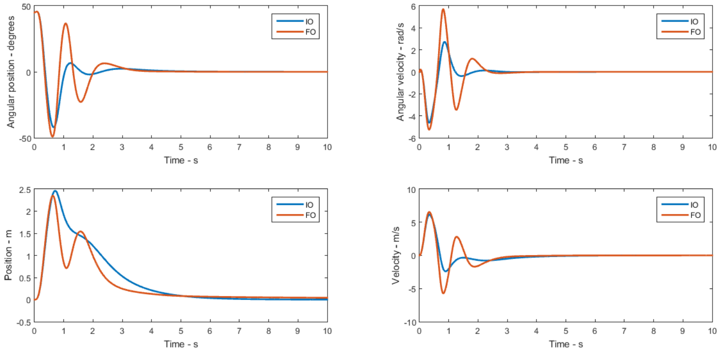


$\theta(0) = 10° \qquad \dot{\theta}(0) = 0$

$k_1 = -221 \;\; k_2 = -50 \;\; k_3 = -80 \;\; k_4 = -47 \;\; k_I = -65$

Non-optimal gains for linear system

$$\alpha_1 = 1 \qquad \alpha_2 = 0.9 \qquad \alpha_3 = 1 \qquad \alpha_I = 0.9$$



$$\theta(0) = 45^\circ \qquad \dot{\theta}(0) = 0$$

$$k_1 = -221 \;\; k_2 = -50 \;\; k_3 = -80 \;\; k_4 = -47 \;\; k_I = -65$$

Section 5

Final Remarks

Final remarks

Conclusion:

- Fractional-order PID-controller can enhance the control system performance
  - Improved transient
  - Larger attraction domains

Ongoing research:

- Application in a nonlinear energy harvesting system

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CAPES

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Thank you for your attention!

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