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Abstract. A stochastic version of the restoring force surface method is proposed and used to identify the parameters of a clamped-free beam with nonlinear effects induced by the presence of a magnet near to the free extremity. This system recalls a Duffing oscillator, which is used as a single-degree-of-freedom mathematical model to represent the mechanical system. Experimental and theoretical responses are compared taking into account a probabilistic band of confidence. The results show that the stochastic model identified can predict the beam's vibration responses, which ensure the robustness of the stochastic identification method.

Keywords: Nonlinear dynamics, stochastic model, restoring force surface, Duffing oscillator

1 Introduction

It is known that many engineering structures can present nonlinear behavior caused by geometric effects, operating conditions, materials with complex structure and others. So, to perform a reliable analysis of a structure, the nonlinear effects have to be taken into account [1]. In this sense, Masri and Caughey [2] presented the method of restoring force surface (RFS), that showed to be effective [3]. Many other approaches can be used to describe nonlinear systems, such as Hilbert transform, Narmax Models, High-Order Frequency Response Functions [4, 5], Volterra series [6], Harmonic balance or Artificial neural network [7]. However, once the approaches described above are deterministic, they are not robust to variations in the system parameters, neither offer a confidence interval to the identified model. Since any real system is uncertain with regard to the nominal project values (due to material imperfections, noise, etc. [8]), a reliable system identification technique must take into account the model parameters uncertainties, also known as *data uncertainties*.

Techniques of stochastic system identification are available in the literature, for instance using convex analysis [9], Bayesian statistics [10–15] or a nonparametric probabilistic approach [16–18]. These methods are very sophisticated and powerful tools, generally used to identify a mechanical system with a large number of degree of freedoms (DoFs). Although these techniques can be used to identify systems with one or a few DoFs, the low dimension of these systems allows one to develop a more simple framework for stochastic identification. It is proposed to use a stochastic version of the RSF method to identify a single degree-of-freedom (DoF) system, developed in a probabilistic framework, which models the system parameters as random variables assuming underlying uncertainties. In this way, the main contribution is to propose a stochastic version of the RFS method, where the probability density functions (PDFs) of model parameters are identified, instead of the parameters deterministic values, as made by conventional RFS methods. The conclusions show that the method is simple and reaches useful results, so that it is suitable for application in simple systems, with low order, where the use of more sophisticated techniques may be complicated.

2 Experimental apparatus

The experimental setup is composed by a clamped-free beam ($300 \times 18 \times 3$ [mm^3]) with a steel mass glued in the free extremity, which is connected to cause a magnetic interaction between the beam and a magnet (Fig. 1). A shaker is used to excite the structure considering different levels of voltage amplitude. A vibrometer laser is utilized to measure the beam free extremity velocity. It is important to note that, the input signal considered in this work is the voltage applied to the shaker. By using this strategy, the input signal is kept constant over a range of frequencies. The magnetic interaction of the system generates a hardening nonlinear behavior showed in Fig. 2(a), which presents the jump phenomenon, that is represented by a sudden drop in the amplitude of the response with a low increment in the excitation frequency. Additionally, the spectrogram of the system response can be seen in Fig. 2(b) where it is observed the presence of the second and third order harmonics in the response.

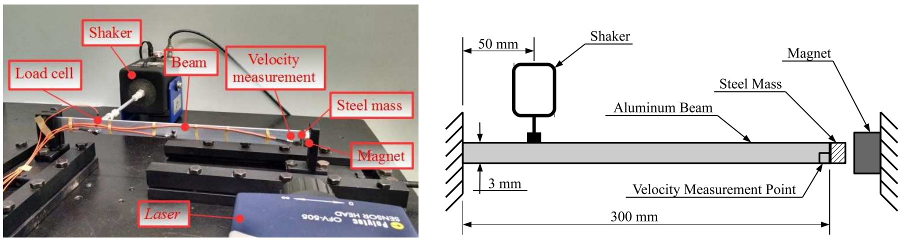
3 Mechanical-mathematical modeling

The experimental setup presents nonlinear behavior only for large displacements, so a Duffing oscillator can well approximate its dynamic behavior [19]

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) + k_2\,x(t)^2 + k_3\,x(t)^3 = U(t), \tag{1}$$

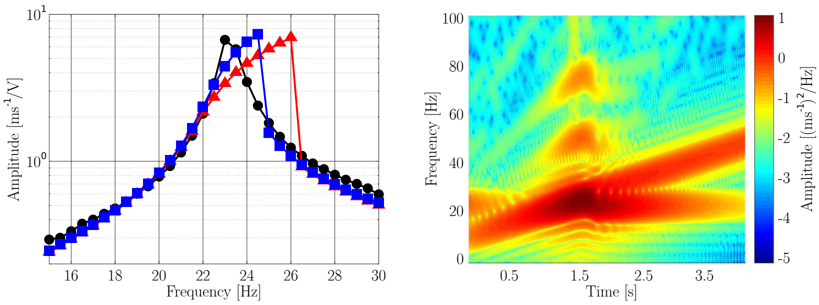
where m is the system equivalent mass, c is the damping coefficient, k is the linear stiffness, k_2 is the quadratic stiffness, k_3 is the cubic stiffness, and $U(t)$ is the external force. The displacement, velocity and acceleration in the free extremity of the beam are represented, respectively, by $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$. Once in the application of the RFS method no form for the restoring force is assumed initially, Eq. (1) is rewritten in terms of the restoring force $\mathcal{F}(x,t)$ as

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + \mathcal{F}(x,t) = U(t). \tag{2}$$



(a) Experimental apparatus. (b) Schematic representation.

Fig. 1: Illustration of the experimental apparatus used.



(a) Stepped sine test. (b) System response spectrogram.

Fig. 2: Illustration of the system nonlinear behavior: (a) Stepped sine test for different levels of voltage applied in the shaker. • - 0.01 V, ■ - 0.10 V and ▲ - 0.15 V; (b) Spectrogram of the system response.

The parameters uncertainties are induced by measurements noise, variation in the boundary conditions, the position of the shaker, sensor and magnet, uncertainties related to the methods of parameters estimation [8]. Thus, the model parameters are random variables or random processes, defined on the probability space $(\Theta, \Sigma, \mathbb{P})$, where Θ is sample space, Σ is a σ-algebra over Θ, and $\mathbb{P}$ is a probability measure. Thus, the stochastic equivalent of Eq. (2) is given by

$$\mathbb{m}(\theta)\,\ddot{\mathbb{x}}(\theta,t) + \mathbb{c}(\theta)\,\dot{\mathbb{x}}(\theta,t) + \mathbb{F}(\mathbb{x}(\theta,t),t) = U(t), \tag{3}$$

where the random processes $(\theta,t) \in \Theta \times \mathbb{R} \mapsto \mathbb{x}(\theta,t)$, $(\theta,t) \in \Theta \times \mathbb{R} \mapsto \dot{\mathbb{x}}(\theta,t)$, and $(\theta,t) \in \Theta \times \mathbb{R} \mapsto \ddot{\mathbb{x}}(\theta,t)$, respectively represent the displacement, velocity and acceleration in the beam free extremity. The stochastic model of Eq. (3) is used to describe the nonlinear random dynamics of the mechanical system emulated by the experimental apparatus.

4 Stochastic system parameters identification

Two types of experimental tests were performed. The first one excites the mechanical system with a chirp signal with a low level (0.01 V) of the constant voltage amplitude, while in the second test, the level is high (0.15 V). The two tests were executed in sequence, so that chirp signal range of frequencies varied with a rate of 10 Hz/s, from 10 to 50 Hz. Each test was repeated 200 times on different days.

The identification of system parameters m and c uses the underlying linear dynamics of the beam assuming the low level of input amplitude. The system equivalent mass and damping coefficient are estimated using the Impulse Response Function. After, identifying several realizations of these parameters (200 in fact), their PDFs are nonparametrically estimated through histograms and kernel smoothed curves [20].

Then, using the nonlinear dynamics of the beam, obtained when the input signal has a high level of amplitude (0.15 V), the restoring force $\mathbb{F}(\mathbb{x}(\theta,t),t)$ is estimated to each realization θ. In this case, the RFS method defines the restoring force from the equation

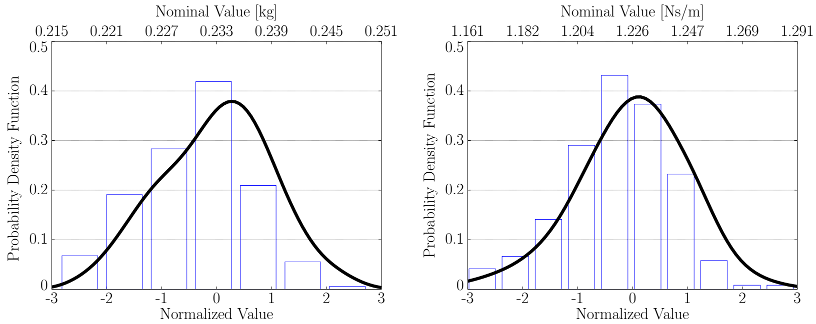
$$\mathbb{F}(\mathbb{x}(\theta,t),t) = U(t) - \left[\mathbb{m}\,\ddot{\mathbb{x}}(\theta,t) + \mathbb{c}\,\dot{\mathbb{x}}(\theta,t)\right], \tag{4}$$

where all objects of the equation right-hand side are known. Note that the nonlinear function $\mathbb{F}$ is a stochastic process, once it is defined as the difference between the excitation U and the stochastic process $\mathbb{m}\ddot{\mathbb{x}} + \mathbb{c}\dot{\mathbb{x}}$. In practice, realizations of $\mathbb{F}$ are constructed utilizing realizations of the system parameters as well as from velocity and acceleration time series. Additionally, the reader can observe that, for a fixed time t, each experimental realization of $\mathbb{F}$ defines a three-dimensional surface, i.e., $\mathcal{F} = g(x,\dot{x})$ for some scalar map $g : \mathbb{R}^2 \to \mathbb{R}$. Thus, the polynomial coefficients (k, k_2 and k_3), to each realization, can be estimated through the polynomial regression based on the minimization of the squared error (least squares method). As performed with the mass and damping, the PDFs of $\mathbb{k}$, $\mathbb{k}_2$ and $\mathbb{k}_3$ are nonparametrically estimated.

5 Results and discussion

The nonparametric estimations for mass and damping coefficient PDFs can be seen in Fig. 3. The figures show PDFs of normalized random variables, i.e., random variables with zero mean and unit standard deviation, in addition to the nominal values. The PDFs show that both parameters have unimodal behavior. It is possible to observe that $\mathbb{m}$ has mean value of $\mu_{\mathbb{m}} = 0.233$ [kg] with low dispersion around the nominal value. The coefficient of variation, standard deviation divided by the mean, is $\delta_{\mathbb{m}} = 2.44$ %. The damping coefficient $\mathbb{c}$ has concentration across the mean value $\mu_{\mathbb{c}} = 1.226$ [Ns/m] and $\delta_{\mathbb{c}} = 1.77$ %.

With the surfaces $\mathcal{F} = g(x,\dot{x})$ nonparametric estimated, a parametric identification of this force to fit a function whose shape resembles the curve raised by RFS method. A polynomial form was chosen to describe the nonlinear force,



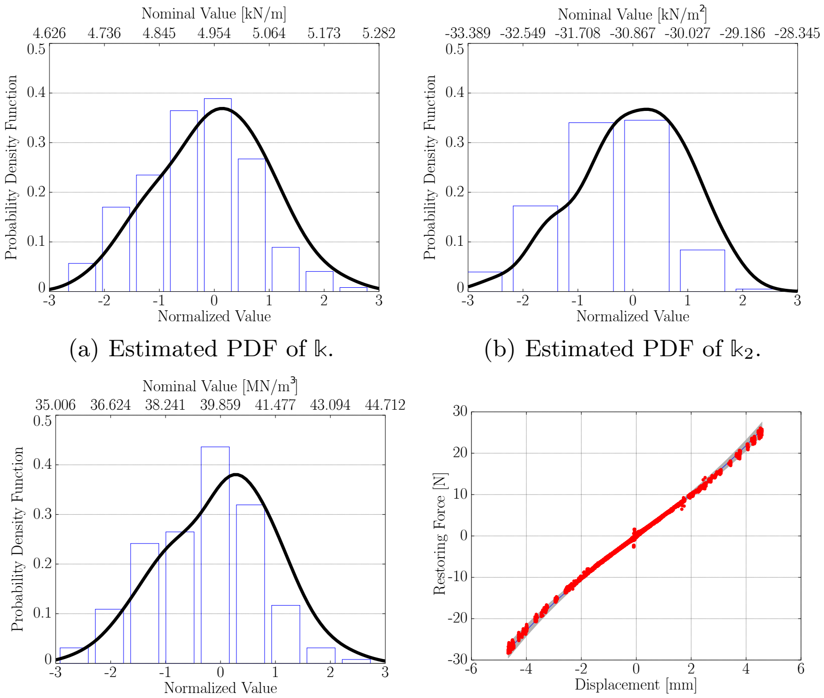
(a) Estimated PDF for $\mathbb{m}$. (b) Estimated PDF for $\mathbb{c}$.

Fig. 3: PDFs for mass and damping parameters of the mechanical system. The PDF is represented by the solid line and the histogram by the bars.

as described in the Duffing equation of Eq. (1). Since the nonlinear restoring force is random, it should be assumed that the stiffnesses are also aleatory, being modeled by random variables. The nonparametric estimations for the PDFs of $\mathbb{k}$, $\mathbb{k}_2$ and $\mathbb{k}_3$ are present in Fig. 4. It can be seen in Fig. 4(a) the PDF of the linear stiffness. The behavior is unimodal with the values concentrated around the mean value $\mu_{\mathbb{k}} = 4.954 \times 10^3$ N/m and $\delta_{\mathbb{k}} = 2.21\%$. Fig. 4(b) shows the PDF of the quadratic stiffness. The mean value is equal to $\mu_{\mathbb{k}_2} = -30.867$ N/m^2 and $\delta_{\mathbb{k}_2} = 2.72\%$. The PDF of the cubic stiffness, presented in Fig. 4(c), has also unimodal distribution with $\mu_{\mathbb{k}_3} = 39.859 \times 10^7$ [N/m^3] and $\delta_{\mathbb{k}_3} = 4.06\%$. The large variation of these parameters is related with the uncertainties present in the RFS method applied considering underlying variabilities (e.g. noise, the magnet, shaker and sensor position, etc). Finally, Fig. 4(d) shows the experimental $\mathbb{F}$ and the polynomial modeling identified with 99% of confidence bands. The results are satisfactory considering that the model can predict the behavior of the restoring force, mainly when it has high amplitude.

Once the stochastic model of Eq. (3) is identified, it can be used to make predictions about the beam nonlinear dynamics behavior, offering probabilistic limits of confidence in the response. The calculation of the model response is done using Monte Carlo (MC) method [21]. First of all, the experimental nonparametric PDFs estimated are used to generate samples of the system parameters. In the procedure, the Metropolis-Hastings Markov Chain Monte Carlo (MCMC) algorithm is applied [21]. Additionally, the sampling is made considering the correlation between the random variables, through the Cholesky decomposition of the correlation matrix. Comparisons between experimental and simulated beam velocity, in the time domain, can be seen in Fig. 5, considering the same chirp signal used in the model identification process. One can observe that the experimental response is inside the limits with 99% of confidence, that indicates the adequate performance of the stochastic model.

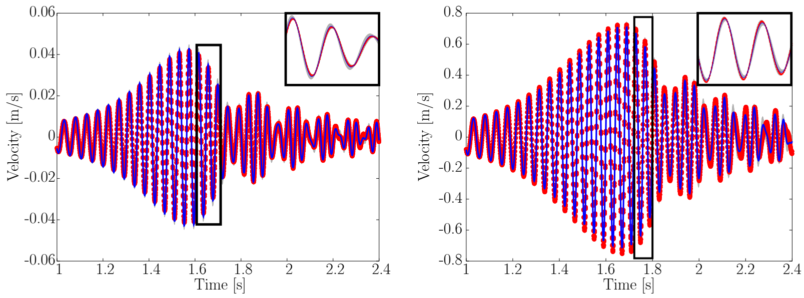
The model validation was performed considering the stepped sine test, and the results are shown in Fig. 6. It is possible to see that the stochastic model describes well the system behavior, both in linear as nonlinear regime of motion. The difference between the curves saw in the linear case is related to the difficulty



(a) Estimated PDF of $\Bbbk$. (b) Estimated PDF of $\Bbbk_2$.

(c) Estimated PDF of $\Bbbk_3$. (d) Fitted restoring force.

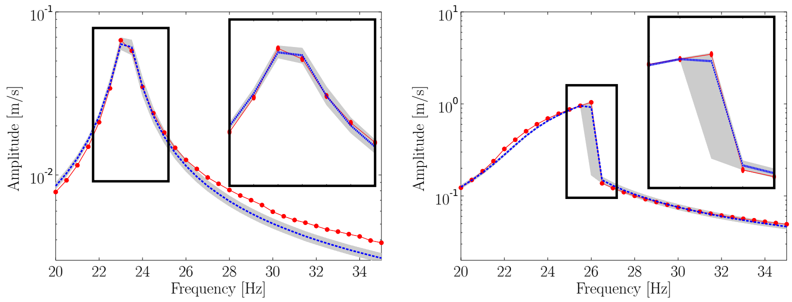
Fig. 4: PDFs for mechanical system stiffnesses and the fitted restoring force. (a, b and c) The PDF is represented by the solid line and the histogram by the bars; (d) The model mean is presented as –, the confidence band as grey shown, and the experimental realization as •.



(a) Low level of input (linear). (b) High level of input (nonlinear).

Fig. 5: System response comparison. The model mean is presented as – –, the confidence band as gray fill, and the experimental realization as – ○ –.

of conducting the stepped sine test with very low excitation amplitude and the possible influence of the second vibration mode shape, this can be confirmed observing Fig. 2(a). In the nonlinear case, the stochastic model is also able to describe the experimental behavior. The large dispersion in the nonlinear regime of motion in consequence of the nonlinear restoring force variation makes with the nonlinear stiffness varies, as seen in Figs. 4(b) and 4(c).



(a) Low level of input (linear). (b) High level of input (nonlinear).

Fig. 6: Stepped sine curves comparison. The model mean is presented as −, the confidence band as gray fill, and the experimental realization as − ○ −.

6 Final remarks

A stochastic version of the restoring force surface method was proposed to identify the parameters of a Duffing oscillator. The formulation of this method was done in terms of a stochastic process and able to take into account the intrinsic variability of the system parameters. In the analysis of non-complex nonlinear systems, the proposed method can be applied without the use of more sophisticated mathematical tools. The effectiveness of this methodology was tested and verified in the parameters estimation of a clamped-free beam, presenting nonlinear behavior. The results showed that the identified stochastic model is robust, once it describes well the structure behavior and specifies a reliability envelope.

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