Elements of Probability Theory

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Probability in Dimension 1

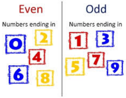
Random experiment

An experiment which repeated under same fixed conditions produce different results is called random experiment.

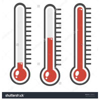
Examples:

1. Rolling a cube-shaped fare die

2. Choosing an integer even number randomly



3. Measuring temperature



Probability space

The mathematical framework in which a random experiment is described consists of a triplet $(\Omega, \Sigma, \mathcal{P})$, called probability space.

The elements of a probability space are:

- Ω: sample space
  (set with all possible events)

- Σ: σ-algebra on Ω
  (set with relevant events only)

- $\mathcal{P}$: probability measure
  (measure of expectation of an event occurrence)

Sample space

A non-empty set which contains all possible events for a certain random experiment is called sample space, being represented by Ω.

Examples:

1. Rolling a cube-shaped fare die (finite Ω)

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

2. Choosing an integer even number randomly (denumerable Ω)

$$\Omega = \{\cdots, -8, -6, -4, -2,\ 0,\ 2,\ 4,\ 6,\ 8, \cdots\}$$

3. Measuring temperature in Kelvin (non-denumerable Ω)

$$\Omega = [a, b] \subset [0, +\infty)$$

σ-algebra of events

In general, not all of the events in Ω are of interest.

Intuitively, a σ-algebra on Ω is the set of relevant outcomes for a random experiment. Formally, Σ is a σ-algebra on Ω if

- $\phi \in \Sigma$
  (contains the empty set)
- $\mathcal{A}^c \in \Sigma$ for any $\mathcal{A} \in \Sigma$
  (closed under complementation)
- $\displaystyle\bigcup_{i=1}^{\infty} \mathcal{A}_i \in \Sigma$ for any $\mathcal{A}_i \in \Sigma$
  (closed under denumerable unions)

Probability measure

A probability measure is a function $\mathcal{P} : \Sigma \to [0,1] \subset \mathbb{R}$ such that

- $\mathcal{P}\{\mathcal{A}\} \geq 0$ for any $\mathcal{A} \in \Sigma$
  (probability is nonnegative)
- $\mathcal{P}\{\Omega\} = 1$
  (entire space has probability one)
- $\mathcal{P}\left\{\bigcup_{i=1}^{\infty} \mathcal{A}_i\right\} = \sum_{i=1}^{\infty} \mathcal{P}\{\mathcal{A}_i\}$ for any $\mathcal{A}_i$ mutually disjoint
  (σ-additivity)

Remark:
$\mathcal{P}\{\phi\} = 0$ (empty set has probability zero)

An example in discrete probability

A fair coin is thrown twice.

The number of faces is of interest.

A. C. Morgado, J. B. Pitombeira, P. C. P. Carvalho, P. J. Fernandez, **Análise Combinatória e Probabilidade**, SBM, 2016

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Probability space 1:

$$\Omega_1 = \{(H,H), (H,T), (T,H), (T,T)\}$$

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Probability space 1:

$$\Omega_1 = \{(H,H),(H,T),(T,H),(T,T)\}$$

$$\mathcal{P}_1\{(H,H)\} = 1/4, \quad \mathcal{P}_1\{(H,T)\} = 1/4,$$
$$\mathcal{P}_1\{(T,H)\} = 1/4, \quad \mathcal{P}_1\{(T,T)\} = 1/4$$

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Probability space 2:

$$\Omega_2 = \left\{0,1,2\right\}$$

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Probability space 2:

$$\Omega_2 = \big\{0, 1, 2\big\}$$

$$\mathcal{P}_2 \{0\} = 1/4, \quad \mathcal{P}_2 \{1\} = 1/2, \quad \mathcal{P}_2 \{2\} = 1/4$$

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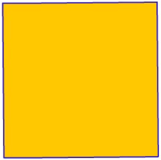
An example in continuous probability



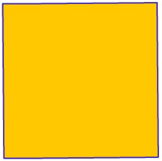
A point is randomly choosen in a square (side b).

Event A: point lie above main diagonal

Event B: point lie in main diagonal

Event C: point lie outside main diagonal

An example in continuous probability



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$$\mathcal{P}\left\{A\right\} = \frac{\text{area of upper triangle}}{\text{area of square}} = \frac{0.5\, b^2}{b^2} = \frac{1}{2}$$

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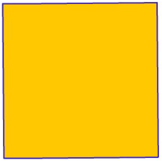

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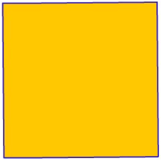

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$$\mathcal{P}\left\{B\right\} = \frac{\text{area of main diagonal}}{\text{area of square}} = \frac{0}{b^2} = 0$$

$$\mathcal{P}\left\{C\right\} = \frac{\text{area outside main diagonal}}{\text{area of square}} = \frac{b^2 - 0}{b^2} = 1$$

Remarks on probability

Probability zero

- An impossible event has probability zero
  (e.g. roll a six faces dice, numbered from 1 to 6, and get 7)
- Not every event with probability zero is impossible
  (e.g. randomly pick a point on the main diagonal of a square)

Probability one

- An event which occurrence is certain has probability one
  (e.g. throw a coin and obtain head or tail)
- Not every event with probability one occurs
  (e.g. randomly pick a point outside square's main diagonal)

Conditional probability

Consider a pair of random events A and B such that $\mathcal{P}\left\{B\right\} > 0$.

The conditional probability of A, given the occurrence of B, denoted as $\mathcal{P}\left\{A|B\right\}$, is defined as

$$\mathcal{P}\left\{A \mid B\right\} = \frac{\mathcal{P}\left\{A \cap B\right\}}{\mathcal{P}\left\{B\right\}}.$$

It follows that

$$\mathcal{P}\left\{A \cap B\right\} = \mathcal{P}\left\{A \mid B\right\} \times \mathcal{P}\left\{B\right\}.$$

An example in conditional probability

Somebody rolls a pair of six-sided dice.

$A =$ value rolled on die 1

$B =$ value rolled on die 2

What is the probability that $A = 2$ given that $A + B \leq 5$?

Conditional probability — Wikipedia, The Free Encyclopedia, 2017.

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A=2

+		B					
		1	2	3	4	5	6
A	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
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$$\mathcal{P}\left\{A\right\} = 6/36$$

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$A{+}B{\leq}5$

+		B					
		1	**2**	**3**	**4**	**5**	**6**
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$\mathcal{P}\left\{A{+}B{\leq}5\right\} = 10/36$

$A \,|\, A{+}B{\leq}5$

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$$\mathcal{P}\left\{A\right\} = 6/36$$

$A{+}B{\leq}5$

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$$\mathcal{P}\left\{A{+}B{\leq}5\right\} = 10/36$$

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$$\mathcal{P}\left\{A \,|\, A{+}B{\leq}5\right\} = 3/10$$

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Independence of events

If the occurrence of an event B does not affect the occurrence of an event A one has

$$\mathcal{P}\left\{A \mid B\right\} = \mathcal{P}\left\{A\right\}.$$

In this way, once $\mathcal{P}\left\{A \cap B\right\} = \mathcal{P}\left\{A \mid B\right\} \times \mathcal{P}\left\{B\right\}$ it is true that

$$\mathcal{P}\left\{A \cap B\right\} = \mathcal{P}\left\{A\right\} \times \mathcal{P}\left\{B\right\}.$$

Events A and B in which the latter holds are said to be independent.

Remark:
This notion generalizes itself naturally to n events.

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

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Probability space 1 (fair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit \end{array} \right\}$$

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$$\mathcal{P}_1\{Q\} = 4/52$$

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$\mathcal{P}_1 \{Q\} = 4/52$

$\mathcal{P}_1 \{\spadesuit\} = 13/52$

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$$\mathcal{P}_1 \{\spadesuit\} = 13/52$$
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$$\mathcal{P}_1\{Q\} = 4/52$$

$$\mathcal{P}_1\{\spadesuit\} = 13/52$$

$$\mathcal{P}_1\{Q\spadesuit\} = 1/52 = \underbrace{4/52}_{\mathcal{P}_1\{Q\}} \times \underbrace{13/52}_{\mathcal{P}_1\{\spadesuit\}}$$

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$$\mathcal{P}_1\{Q\} = 4/52$$

$$\mathcal{P}_1\{\spadesuit\} = 13/52$$

$$\mathcal{P}_1\{Q\spadesuit\} = 1/52 = \underbrace{4/52}_{\mathcal{P}_1\{Q\}} \times \underbrace{13/52}_{\mathcal{P}_1\{\spadesuit\}}$$

Events are independent!

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

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Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit,3\diamondsuit,4\diamondsuit,5\diamondsuit,6\diamondsuit,7\diamondsuit,8\diamondsuit,9\diamondsuit,10\diamondsuit,J\diamondsuit,Q\diamondsuit,K\diamondsuit,A\diamondsuit, \\ 2\heartsuit,3\heartsuit,4\heartsuit,5\heartsuit,6\heartsuit,7\heartsuit,8\heartsuit,9\heartsuit,10\heartsuit,J\heartsuit,Q\heartsuit,K\heartsuit,A\heartsuit, \\ Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit, \\ Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit,Q\spadesuit, \end{array} \right\}$$

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A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: $\spadesuit$ "spade"

Are these events independent?

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$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$\mathcal{P}_2 \{ Q \} = 28/52$

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$$\mathcal{P}_2 \{Q\spadesuit\} = 1/2 \neq \underbrace{28/52}_{\mathcal{P}_2\{Q\}} \times \underbrace{1/2}_{\mathcal{P}_2\{\spadesuit\}}$$

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$$\mathcal{P}_2\{Q\spadesuit\} = 1/2 \neq \underbrace{28/52}_{\mathcal{P}_2\{Q\}} \times \underbrace{1/2}_{\mathcal{P}_2\{\spadesuit\}}$$

Events are not independent!

Further remarks on probability

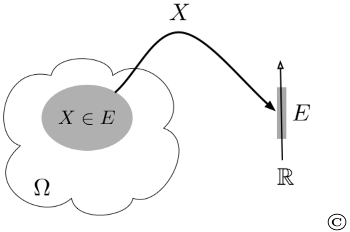
The last example shows that:

- Different probability spaces, for the same random experiment, can produce different predictions
- A probability space that does not accurately describe a random event can produce completely erroneous predictions
- The notion of independence strongly depends on the probability measure employed

Random variable

A mapping $X : \Omega \to \mathbb{R}$ is called a random variable (RV) if the preimage of every real number under X is a relevant event, i.e.,

$$X^{-1}(x) = \left\{\omega \in \Omega : \; X(\omega) \leq x\right\} \in \Sigma, \quad \text{for every } x \in \mathbb{R}.$$



A collection of events in Ω is mapped to an interval E on the real line under such mapping.

RV are numerical characteristics of interesting events.

Remark:
A random variable is a function from Ω to $\mathbb{R}$, not a real number.

Examples of random variables

1. Rolling die experiment

$\Omega = \{1, 2, 3, 4, 5, 6\}$

- $X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is even} \\ 0 & \text{if } \omega \text{ is odd} \end{cases}$ $\qquad \Sigma = \{\phi, \{1,3,5\}, \{2,4,6\}, \Omega\}$
  (random variable)

2. Temperature (in Kelvin) measurement experiment

$\Omega = [a, b] \subset [0, +\infty)$ $\qquad \Sigma = \mathcal{B}_{[a,b]}$ (Borel σ-algebra)

- $X(\omega) = -459.67 + 1.8\,\omega$
  (random variable)

G. Grimmett and D. Welsh, **Probability: An Introduction**. Oxford University Press, 2 edition, 2014.

Probability distribution

The probability distribution of X, denoted by F_X, is defined as the probability of the elementary event $\{X \leq x\}$, i.e.,

$$F_X(x) = \mathcal{P}\left\{X \leq x\right\}.$$

F_X is also known as cumulative distribution function (CDF) and has the following properties:

- $0 \leq F_X(x) \leq 1$
- F_X is a monotonic, non decreasing, right continuous function
- $\mathcal{P}\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} dF_X(x)$
- $\int_{\mathbb{R}} dF_X(x) = 1$
- $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$

Probability density function

If the function F_X is differentiable, its derivative $p_X(x) = dF_X(x)/dx$ is called probability density function (PDF) of X, and one has

$$F_X(x) = \int_{-\infty}^{x} p_X(\xi)\, d\xi.$$

Note also that:

- $p_X(x) \geq 0$ for every $x \in \mathbb{R}$
- $\displaystyle\int_{\mathbb{R}} p_X(x)\, dx = 1$

Remark:
Intuitively, $p_X(x)\, dx$ can be thought of as the probability of X falling within the infinitesimal interval $[x, x + dx]$.

Types of random variables

1. Discrete random variable
   Distribution is discrete.
   Assumes a denumerable number of values.
   Typically associated with counting processes.
2. Continuous random variable
   Distribution is continuous.
   Assumes a non-denumarable number of values.
   Typically associated with measuring processes.
3. Mixed random variable
   Distribution has points of discontinuity.
   Assumes a non-denumarable number of values.
   A "mixture"of the two previous types.
4. Singular random variable
   Distribution is not differentiable at any point.
   It has theoretical interest only.

Mathematical expectation operator

The mathematical expectation of a random variable X is defined as

$$\mathbb{E}\{X\} = \int_{\mathbb{R}} x \, dF_X(x).$$

The mathematical expectation is a linear operator since

$$\mathbb{E}\{\alpha_1 X_1 + \alpha_2 X_2\} = \alpha_1 \mathbb{E}\{X_1\} + \alpha_2 \mathbb{E}\{X_2\},$$

for any pairs of number α_1, α_2 and random variables X_1, X_2.

Theorem (law of the unconscious statistician):

Given a measurable mapping $h : \mathbb{R} \to \mathbb{R}$ and a random variable X the expected value of $h(X)$ is given by

$$\mathbb{E}\{h(X)\} = \int_{\mathbb{R}} h(x) \, dF_X(x).$$

Mean value

The mean value of the random variable X is defined as

$$\begin{aligned} \mu_X &= \mathbb{E}\left\{X\right\} \\ &= \int_{\mathbb{R}} x \, dF_X(x) \\ &= \int_{\mathbb{R}} x \, p_X(x) \, dx. \end{aligned}$$

(measure of the central tendency)

Remark:
The mean value μ_X is the constant which best approximate the random variable X. The error of this approximation is the standard deviation σ_X.

Variance

The variance of the random variable X is defined as

$$\begin{aligned}\sigma_X^2 &= \mathbb{E}\left\{(X-\mu_X)^2\right\} \\ &= \int_{\mathbb{R}} (x-\mu_X)^2 \, dF_X(x) \\ &= \int_{\mathbb{R}} (x-\mu_X)^2 \, p_X(x)\, dx.\end{aligned}$$

(measure of dispersion about the mean)

Note that variance can also be written as

$$\sigma_X^2 = \mathbb{E}\left\{X^2\right\} - \left(\mathbb{E}\left\{X\right\}\right)^2.$$

Remark:
σ_X^2 has the same unit as X^2.

Standard deviation and variation coefficient

Other second-order statistics of X are the standard deviation

$$\sigma_X = \sqrt{\sigma_X^2},$$

and the variation coefficient

$$\delta_X = \sigma_X / \mu_X, \quad \mu_X \neq 0.$$

(both are measures of dispersion about the mean)

Remark:
σ_X has the same unit as X and δ_X is dimensionless.

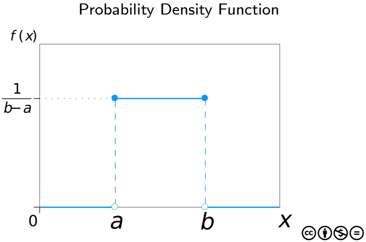
Probability Distributions

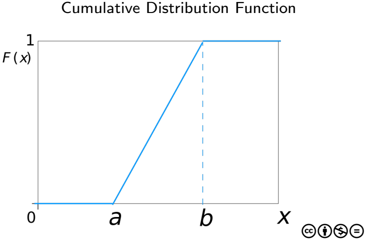
Uniform

- Notation: $\mathcal{U}(a, b)$
- Support: $[a, b]$
- Parameters:
  - $-\infty < a < b < +\infty$ — boundaries
- PDF:

$$p_X(x) = \frac{1}{b-a} \, \mathbb{1}_{[a,b]}(x)$$

- Statistics:
  - $\mu = \frac{1}{2}(a + b)$
  - $\sigma^2 = \frac{1}{12}(b - a)^2$



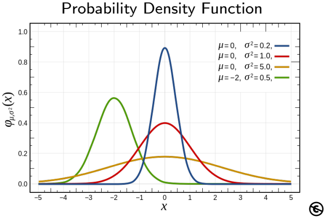


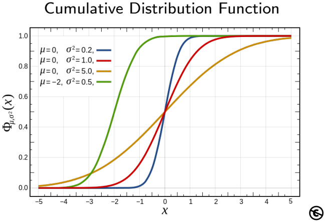
Uniform distribution — Wikipedia, The Free Encyclopedia, 2021.

Gaussian

- Notation: $\mathcal{N}(\mu, \sigma^2)$
- Support: $(-\infty, +\infty)$
- Parameters:
  - μ — mean
  - σ^2 — variance
- PDF:

$$p_X(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\,\sigma^2} \right\}$$





Normal distribution — Wikipedia, The Free Encyclopedia, 2021.

Characterization of a probability distribution

Given the mean μ and standard deviation σ of a random variable.
Is the distribution well-defined?

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$$p_X(x) = \frac{1}{\sqrt{6\,\pi}} \exp\left\{ -\frac{(x-1)^2}{6} \right\} \quad \text{and} \quad p_X(x) = \frac{1}{6}\,\mathbb{1}_{[-2,4]}(x)$$

have $\mu = 1$ and $\sigma = \sqrt{3}$.

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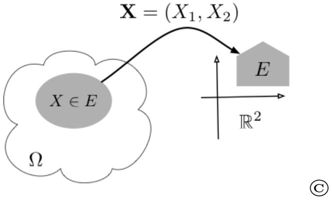
What type of information determines a distribution?

- cumulative distribution function
- probability density function (if exists)
- quantile function
- characteristic function
- moment-generating function (if exists)

Probability in Dimension n

Random vector

Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. A random vector $\mathbf{X} = (X_1, \cdots, X_n)$ is a collection of n random variables $X_i : \Omega \to \mathbb{R}$ that together may be considered a (measurable) mapping $\mathbf{X} : \Omega \to \mathbb{R}^n$.



A collection of event in Ω is mapped into a region on the Euclidean space under such mapping.

Joint probability distribution

The joint probability distribution of random vector $\mathbf{X} = (X_1, \cdots, X_n)$, denoted by $F_{\mathbf{X}}$, is defined as

$$F_{\mathbf{X}}(x_1, \cdots, x_n) = \mathcal{P}\left\{\{X_1 \leq x_1\} \cap \cdots \cap \{X_n \leq x_n\}\right\}.$$

Thus,

$$\mathcal{P}\left\{\mathbf{a} < \mathbf{X} \leq \mathbf{b}\right\} = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} dF_{\mathbf{X}}(x_1, \cdots, x_n),$$

in which $\{\mathbf{a} < \mathbf{X} \leq \mathbf{b}\} = \{a_1 < X_1 \leq b_1\} \cap \cdots \cap \{a_n < X_n \leq b_n\}$.

$F_{\mathbf{X}}$ is also known as joint cumulative distribution function.

Joint probability density function

If $p_{\mathbf{X}}(x_1, \cdots, x_n) = \partial^n F_{\mathbf{X}} / \partial x_1 \cdots \partial x_n$ exists, for any $x_1, \cdots, x_n$, then it is called joint probability density function of $\mathbf{X}$, and one has

$$F_{\mathbf{X}}(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p_{\mathbf{X}}(\xi_1, \cdots, \xi_n)\, d\xi_1 \cdots d\xi_n.$$

Note also that:

- $p_{\mathbf{X}}(x_1, \cdots, x_n) \geq 0$ for every $(x_1, \cdots, x_n) \in \mathbb{R}^n$
- $\displaystyle\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{X}}(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = 1$

Marginal probability density function

The marginal probability density function of X_i is defined as

$$p_{X_i}(x_i) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n-1 \; times} p_{\mathbf{X}}(x_1, \cdots, x_n) \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n,$$

for $i = 1, \cdots, n$.

Conditional distribution

Consider a pair of jointly distributed random variables X and Y. The conditional distribution of X, given the occurrence of the value y of Y, is defined as

$$F_{X|Y}(x \mid y) = \frac{F_{X,Y}(x,y)}{F_Y(y)}.$$

Thus

$$F_{X,Y}(x,y) = F_{X|Y}(x|y) \times F_Y(y),$$

and

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) \times p_Y(y).$$

Remark:
This definition extends naturally to the n-dimensional case.

Independence of distributions

The random variables X and Y are said to be independent if the realization of X does not affect the probability distribution of Y, i.e.,

$$F_{X|Y}(x|y) = F_X(x).$$

Therefore, for independent random variable one has

$$F_{X\,Y}(x, y) = F_X(x) \times F_Y(y),$$

and

$$p_{X\,Y}(x, y) = p_X(x) \times p_Y(y).$$

Remark:
This definition extends naturally to the n-dimensional case.

Statistics of random vectors

- second-order random vector

$$\mathbb{E}\left\{\parallel \mathbf{X} \parallel^2\right\} = \int_{\mathbb{R}^n} \parallel \mathbf{x} \parallel^2 \, dF_{\mathbf{X}}(\mathbf{x}) < +\infty$$

- mean vector

$$\mathbf{m}_{\mathbf{X}} = \mathbb{E}\left\{\mathbf{X}\right\} = \int_{\mathbb{R}^n} \mathbf{x} \, dF_{\mathbf{X}}(\mathbf{x}) \in \mathbb{R}^n$$

- correlation matrix

$$[R_{\mathbf{XY}}] = \mathbb{E}\left\{\mathbf{X}\mathbf{Y}^T\right\} = \int_{\mathbb{R}^n} \mathbf{x}\mathbf{x}^T \, dF_{\mathbf{XY}}(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n \times n}$$

- covariance matrix

$$\begin{aligned} [K_{\mathbf{XY}}] &= \mathbb{E}\left\{\left(\mathbf{X} - \mathbf{m}_{\mathbf{X}}\right)\left(\mathbf{Y} - \mathbf{m}_{\mathbf{Y}}\right)^T\right\} \in \mathbb{R}^{n \times n} \\ &= [R_{\mathbf{XY}}] - \mathbf{m}_{\mathbf{X}}\mathbf{m}_{\mathbf{Y}}^T \end{aligned}$$

Remark:

Matrices $[R_{\mathbf{XY}}]$ and $[K_{\mathbf{XY}}]$ are symetric positive semi-definite when $\mathbf{X} = \mathbf{Y}$.

Correlation of random variables

The random vectors $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ are said to be uncorrelated if covariance matrix $[K_{\mathbf{XY}}]$ is null, i.e.,

$$[R_{\mathbf{XY}}] = \mathbf{m}_{\mathbf{X}} \mathbf{m}_{\mathbf{Y}}^T.$$

If two random vectors are independent, then they are uncorrelated.

$$\text{independence} \Longrightarrow \text{uncorrelation}$$

But uncorrelated random vectors are not independent in general.

$$\text{uncorrelation} \not\Longrightarrow \text{independence}$$

Remark:
Uncorrelated random vectors which the joint distribution is Gaussian are independent.

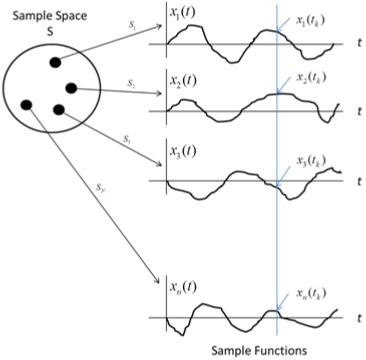
Notions of Random Processes

Random process

A real-valued random process (also called stochastic process) defined on probability space $(\Omega, \Sigma, \mathcal{P})$, indexed by $t \in \mathcal{T}$, is a mapping

$$(t, \omega) \in \mathcal{T} \times \Omega \to X(t, \omega) \in \mathbb{R},$$

such that, for fixed t, the output is a random variable $X(t, \cdot)$, while for fixed ω, $X(\cdot, \omega)$ is a function of t (sample function).



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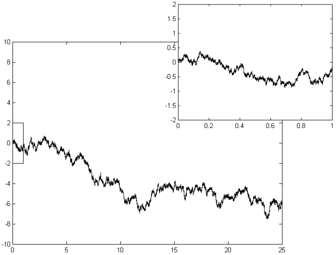
Interpretation and classification

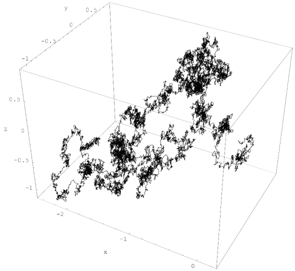
Note that, by definition, a real-valued random process $X(t, \omega)$ is a collection of real-valued random variables indexed by a parameter, and can be thought of as a time-dependent random variable.

According to the nature of the set of indices $\mathcal{T}$, a stochastic process is classified as:

- discrete – if $\mathcal{T}$ is countable
- continuous – if $\mathcal{T}$ is uncountable

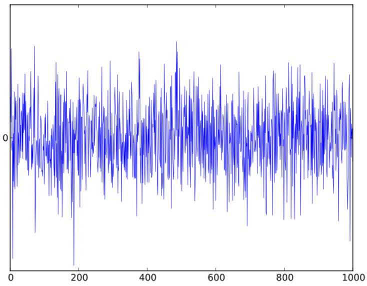
Wiener process (Brownian motion process)

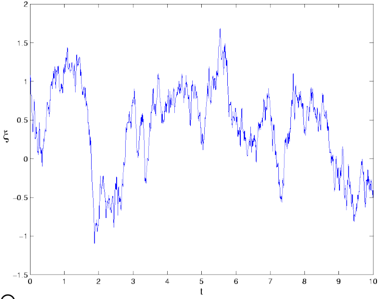




Wiener process — Wikipedia, The Free Encyclopedia, 2021.

White noise and colored noise





White noise

Colored noise

White noise — Wikipedia, The Free Encyclopedia, 2021.

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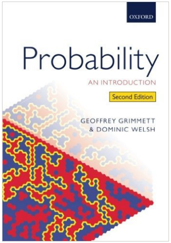

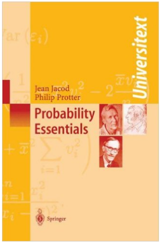

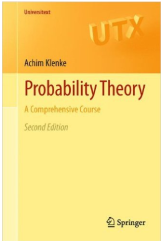

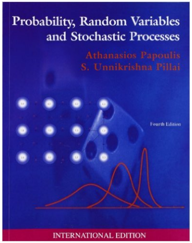

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