Probabilistic Modeling of Uncertainties in Physical Systems

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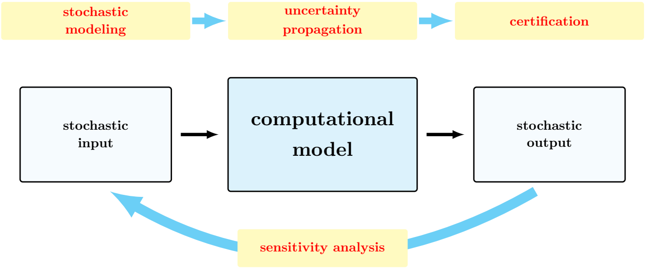
General framework for uncertainty quantification (UQ)



1. Stochastic Modeling: characterize inputs uncertainties
2. Uncertainty Propagation: quantify output uncertainties
3. Certification: establish acceptable levels of uncertainty
4. Sensitivity Analysis: explain the output variability

B. Sudret *A short review of computational methods for uncertainty quantification in engineering*, 2013.

How to model uncertainties in physical systems?

Several approaches are available:

- Probability theory
- Evidency theory
- Interval analysis
- Fuzzy logic

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Probability theory is the most used approach to describe uncertainties in physical systems (best suited for high-dimensions)

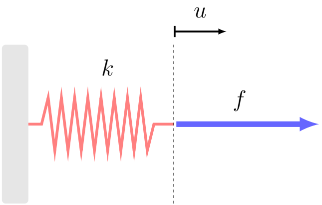
In general other approaches are used only when the probability theory can not be applied

Different probabilistic approaches

- Parametric probabilistic approach: model parameters as random objects (deal with data uncertainties)
- Nonparametric probabilistic approach: model operators as random operators (deal with data and model uncertainties)

In any approach the probability distribution of the random objects must be constructed, and not arbitrarily chosen.

A simple mechanical system



Parameter: k – spring stiffness
Input: f – external force
Response: u – displacement

- Mathematical model:

$$k\,u = f$$

- Model response:

$$u = \underbrace{k^{-1}\,f}_{g(k)}$$

(nonlinear mapping of k)

C. Soize, *Uncertainty Quantification: An Accelerated Course with Advanced Applications in Computational Engineering*, Springer, 2017.

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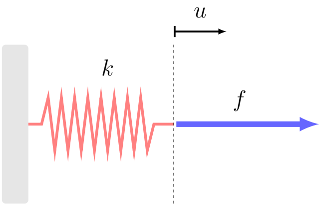

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What happens if the model parameter is random?

C. Soize, *Uncertainty Quantification: An Accelerated Course with Advanced Applications in Computational Engineering*, Springer, 2017.

Parametric probabilistic approach

Probability space: $(\Omega, \Sigma, \mathcal{P})$

Stiffness:

$$K : \Omega \to \mathbb{R}$$

Displacement:

$$U : \Omega \to \mathbb{R} \quad \text{such that} \quad K\,U = f$$

To ensure the consistency of the stochastic model:

- $\mathbb{E}\left\{K^2\right\} < +\infty$ "finite variance random variables"
- $\mathbb{E}\left\{U^2\right\} < +\infty$

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The importance of know parameters distribution

Hypotheses about random parameter K:

- finite variance – $\mathbb{E}\left\{K^2\right\} < +\infty$
- known mean – $\mathbb{E}\left\{K\right\} = \mu_K$
- unknown distribution – $p_K(k)$ is not known

Can we compute the model response mean value ?

$$\mathbb{E}\left\{U\right\} = \mathbb{E}\left\{K^{-1}\, f\right\}$$

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The input distribution is essential to obtain output statistics!

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Can we arbitrate distributions?

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Suppose $p_K(k)$ is arbitrarily chosen Gaussian

$$p_K(k) = \frac{1}{\sqrt{2\pi}\,\sigma_K} \exp\left\{ -\frac{(k-\mu_K)^2}{2\,\sigma_K^2} \right\}$$

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And make no sense from the physical point of view!

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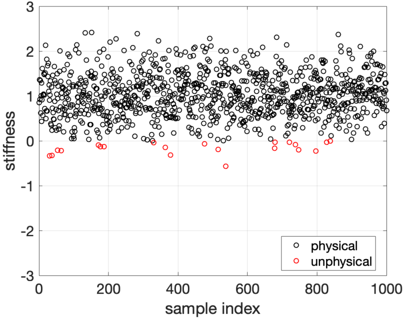
1st consequence of the unphysical distribution

$$f = 2 \qquad \mu_K = 1 \qquad \sigma_K = 1/2$$

$$K \sim \mathcal{N}(\mu_K, \sigma_K) \qquad\qquad U = K^{-1} f$$



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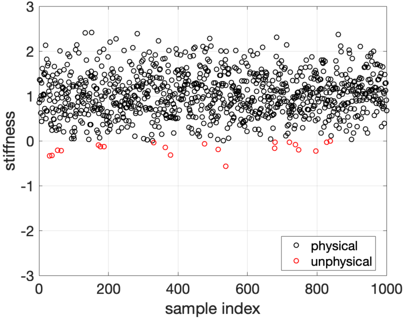
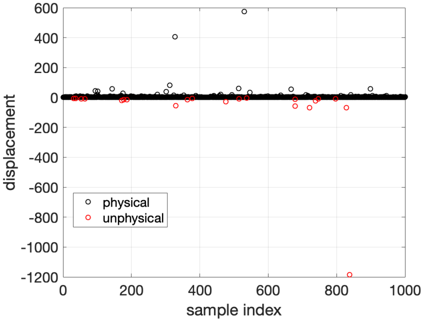


$U = K^{-1} f$



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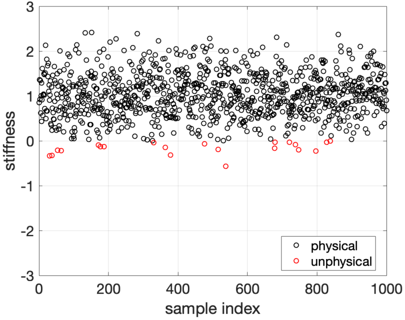

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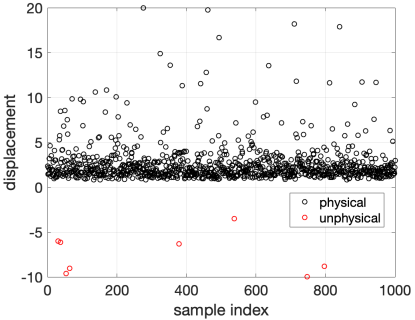

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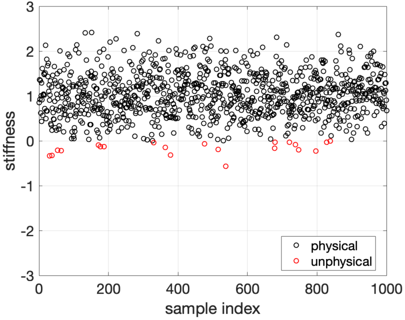

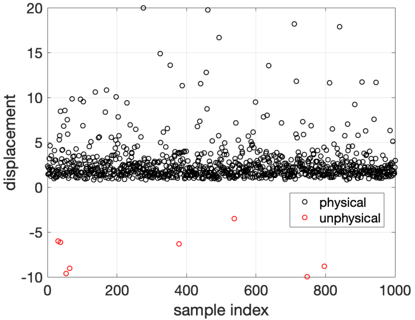

Scenarios that do not respect the
second law of thermodynamics may appear! ☹

2nd consequence of the unphysical distribution

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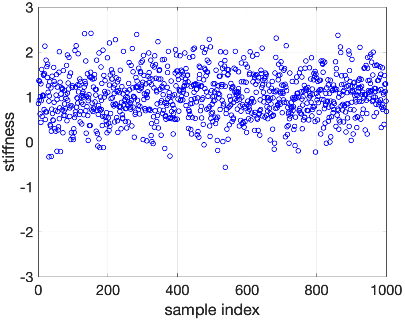

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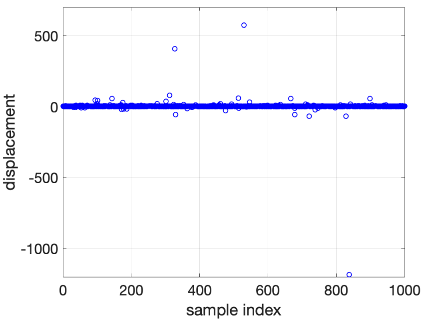

$$\widehat{\mu_U} = 2.3358$$

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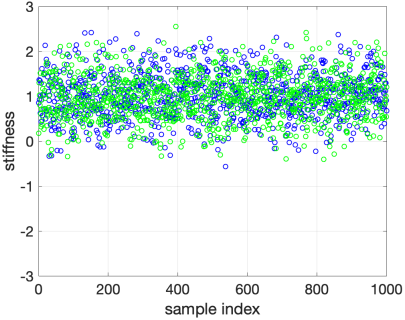

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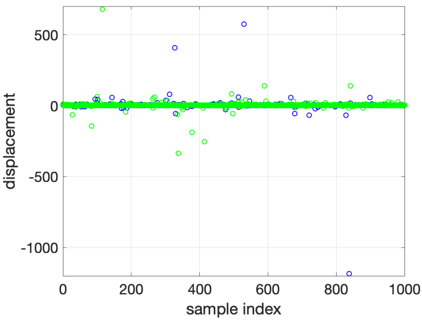

$$\widehat{\mu_U} = 2.6816$$

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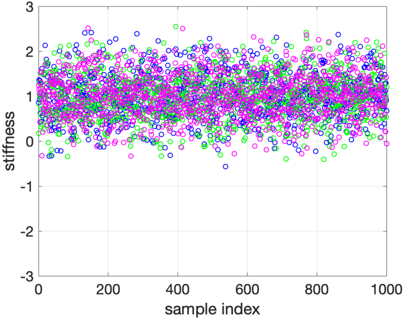

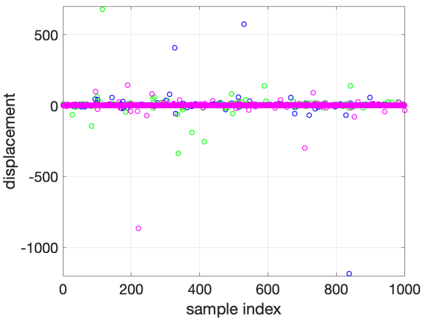

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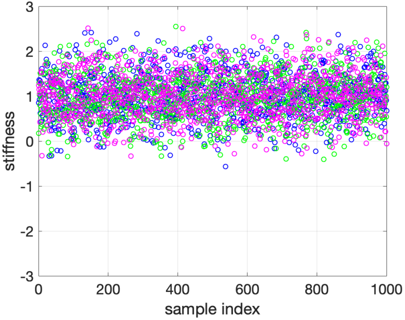

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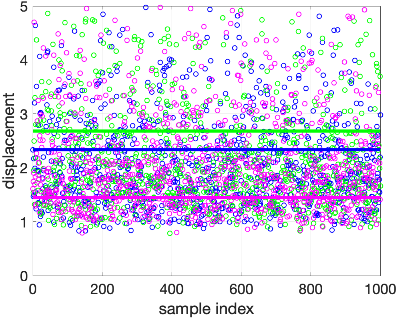

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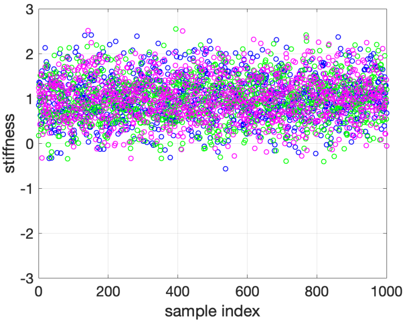

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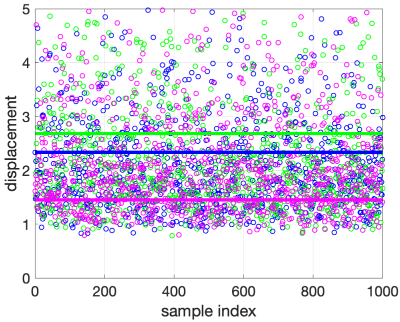

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Statistical estimates don't make sense! ☹

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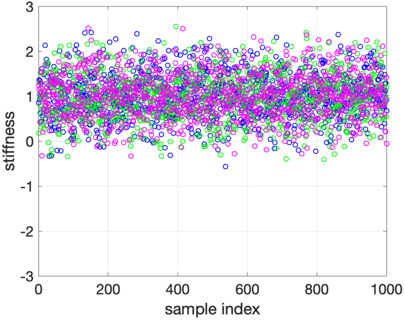

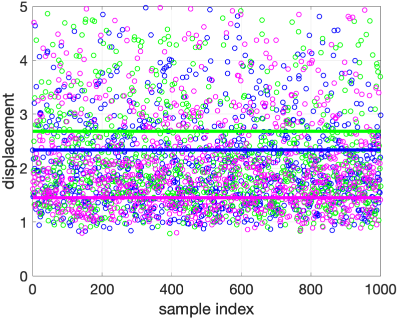

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What happens if the negative samples could be avoided ?

Trying to avoid the negative samples

$$f = 2 \quad \mu_K = 1 \quad \sigma_K = 3/10$$

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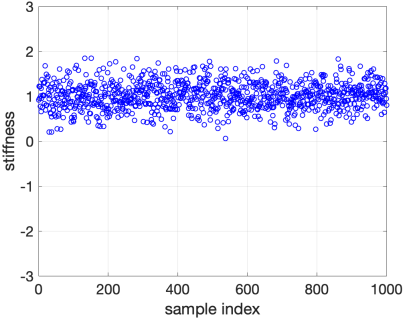

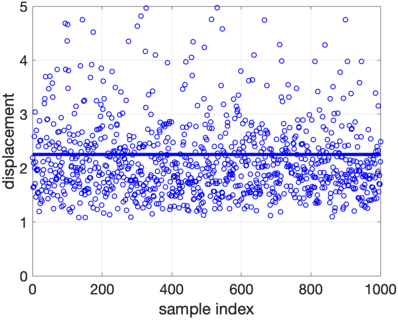

$$\widehat{\mu_U} = 2.2514$$

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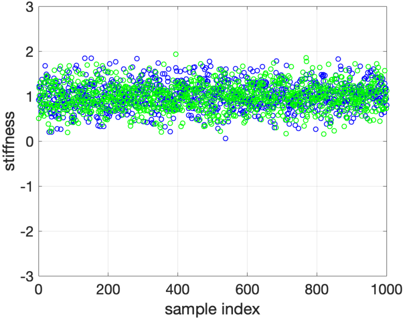

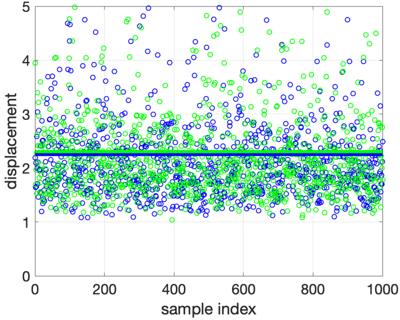

$$\widehat{\mu_U} = 2.3050$$

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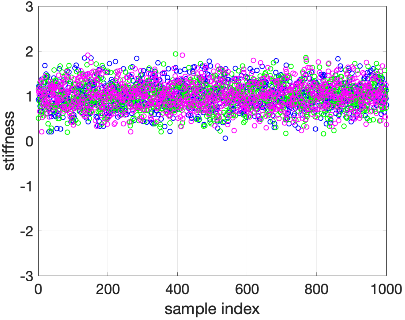

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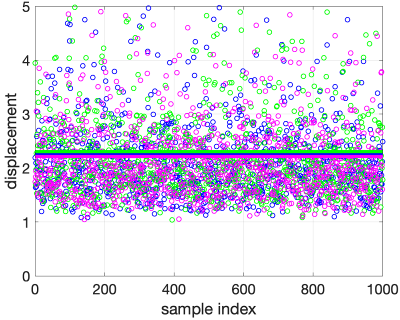

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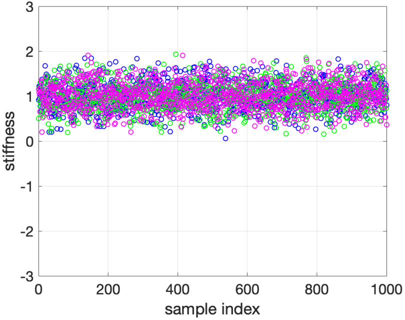

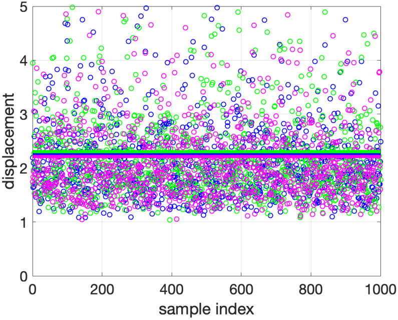

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Now the statistical estimates seem to make sense! ☺

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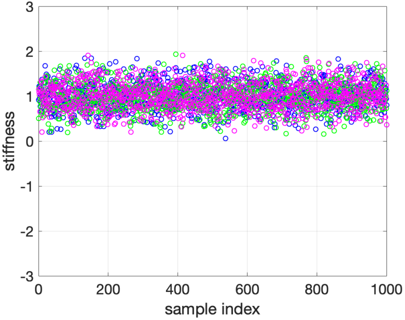

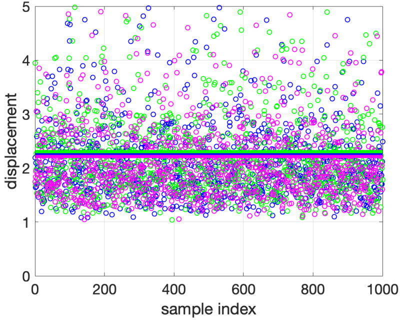

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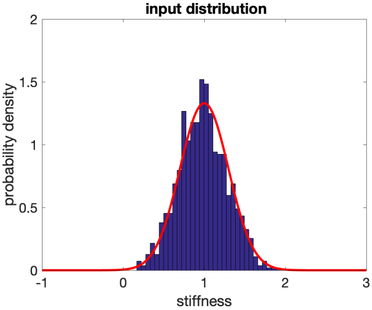
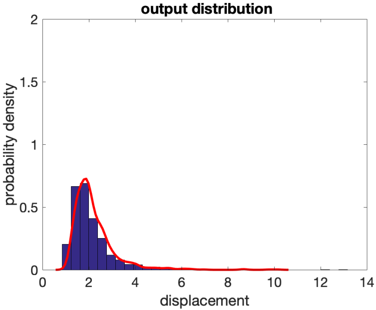
Is this really the case?

What is the big consequence of an infinite 2nd moment?





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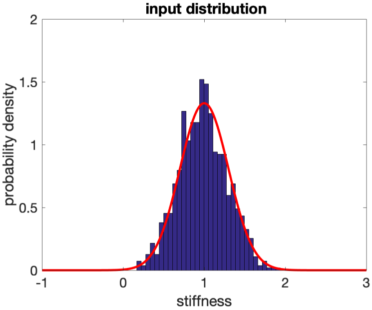

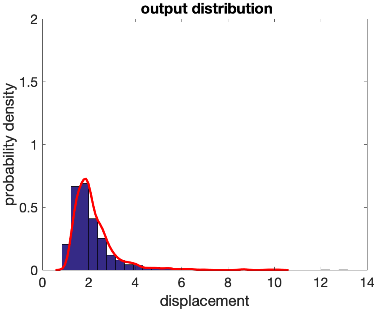

Apparently there is nothing wrong !

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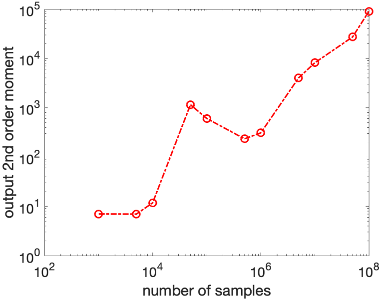
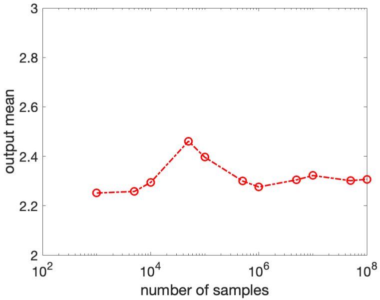
But note the convergence of the estimators





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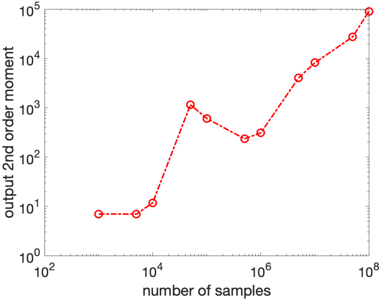

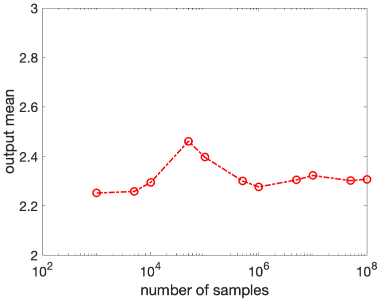

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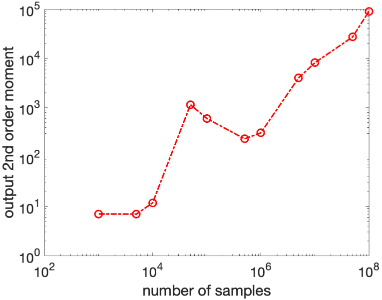

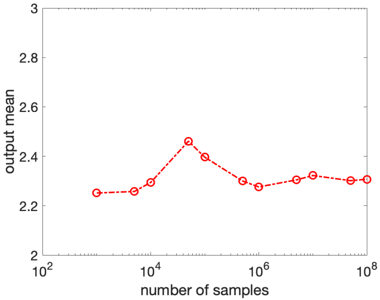

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The obtained response is not statistically significant ! ☹

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So we can not arbitrate distributions !

The Gaussian is a bad choice since K must be a positive-valued random variable.

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All these requirements are verified by the exponential distribution

$$p_K(k) = \mathbb{1}_{(0,+\infty)}(k)\frac{1}{\mu_K}\exp\left\{-\frac{k}{\mu_K}\right\}.$$

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In order to $\mathbb{E}\left\{U^2\right\} < +\infty$ **we must have** $\mathbb{E}\left\{K^{-2}\right\} < +\infty$

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An acceptable choice

With the following requirements:

- positive support – $\operatorname{Supp} p_K \subset (0, +\infty) \Longrightarrow K > 0 \quad a.s.$
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$$p_K(k) = \mathbb{1}_{(0,+\infty)}(k) \frac{1}{\mu_K} \frac{\delta_K^{-2\delta_K^{-2}}}{\Gamma(\delta_K^{-2})} \left(\frac{k}{\mu_K}\right)^{\delta_K^{-2}-1} \exp\left\{-\frac{k/\mu_K}{\delta_K^2}\right\}$$

The gamma distribution is an acceptable choice !

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How to safely specify a distribution?

Scenario 1: significant amount of experimental data is available

- Nonparametric statistical estimation

Scenario 2: few or none experimental data is available

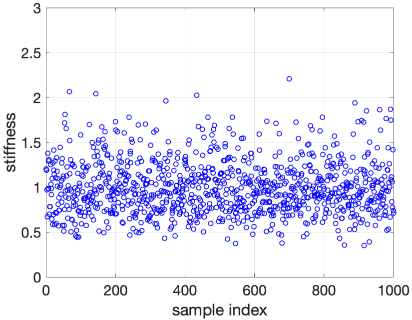
- Maximum Entropy Principle
  (a tool from information theory)

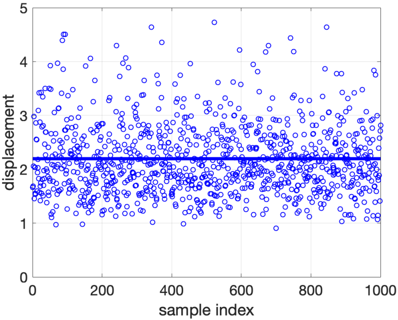
Physically consistent simulation

$$f = 2 \quad \mu_K = 1 \quad \sigma_K = 3/10$$

$K \sim \texttt{Gamma}(\mu_K, \sigma_K)$

$U = K^{-1} f$





$$\widehat{\mu_U} = 2.1963$$

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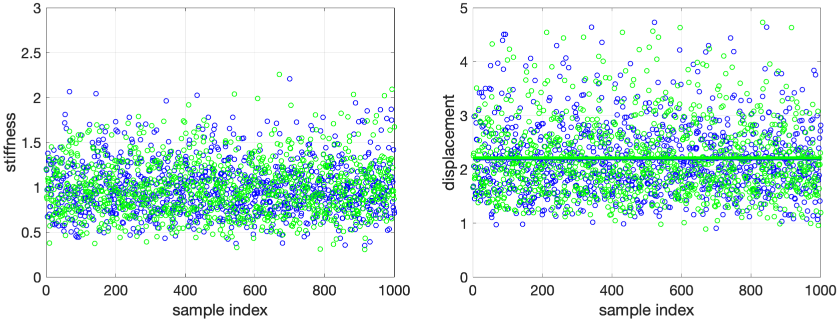

$$\widehat{\mu_U} = 2.2111$$

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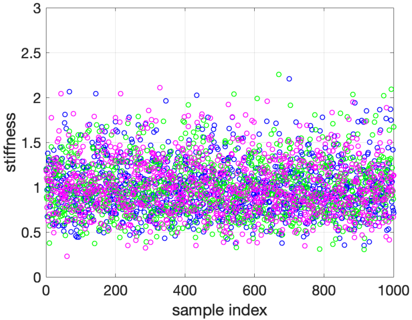

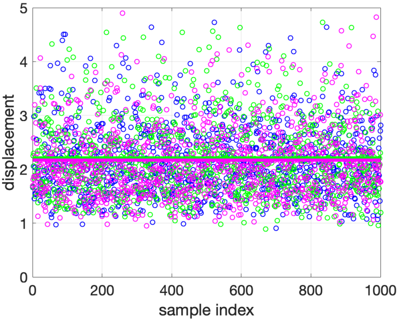

$$\widehat{\mu_U} = 2.1701$$

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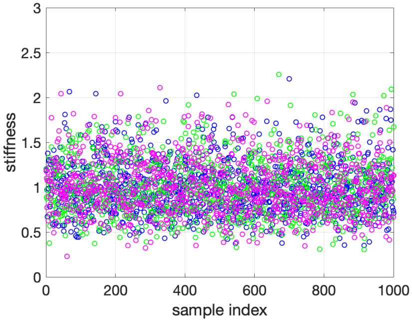

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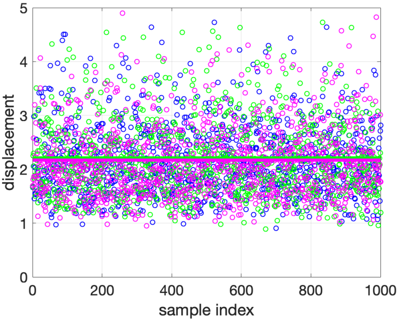

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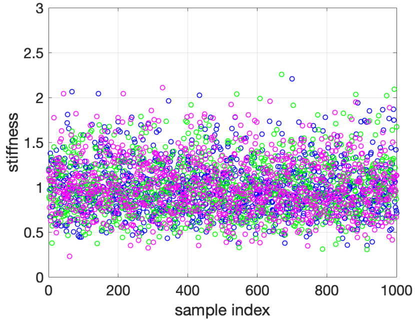

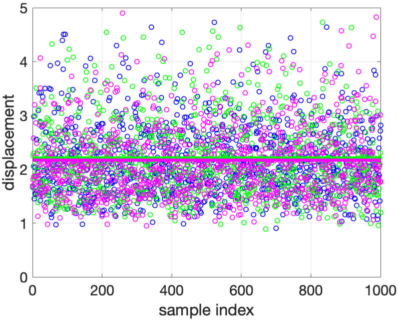

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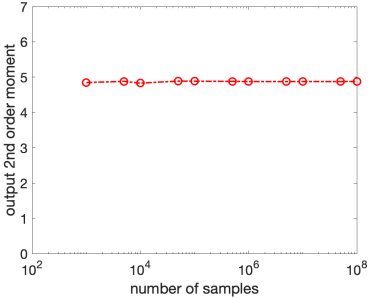
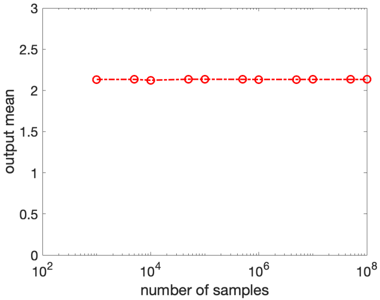
Is this really the case?

Physically consistent simulation





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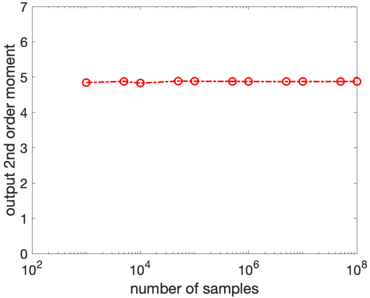

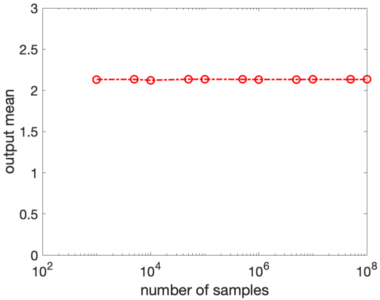

Now Monte Carlo simulation converges !

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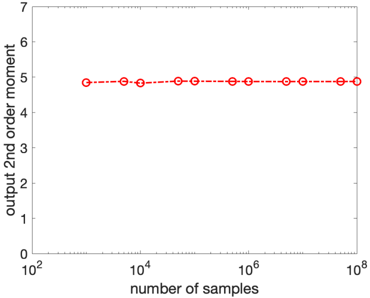

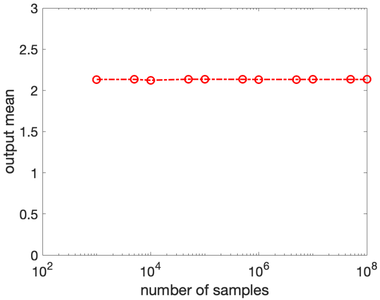

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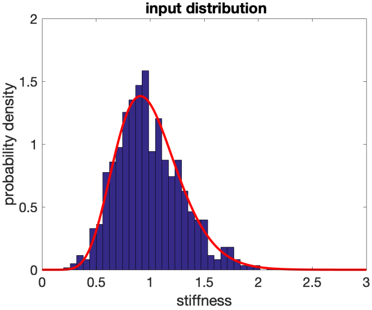

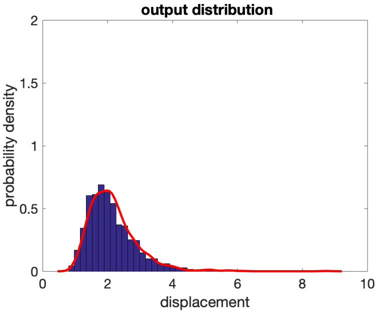

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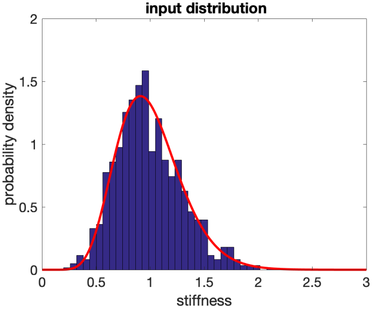

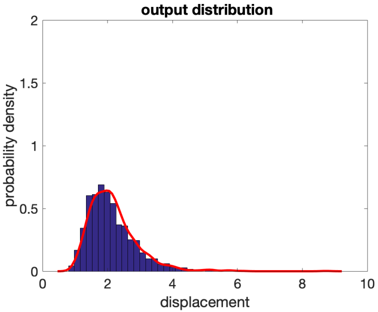

The gamma (input) distribution is mapped to an inverse-gamma (output) distribution !

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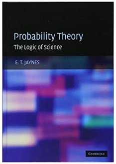

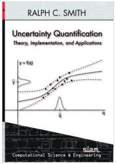

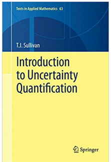

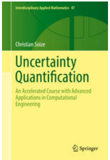

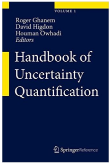

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