Elements of Probability Theory (Part III)

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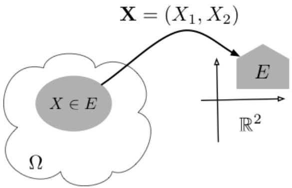
Probability in Dimension n

Random vector

Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. A random vector $\mathbf{X} = (X_1, \cdots, X_n)$ is a collection of n random variables $X_i : \Omega \to \mathbb{R}$ that together may be considered a (measurable) mapping $\mathbf{X} : \Omega \to \mathbb{R}^n$.



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A collection of event in Ω is mapped into a region on the Euclidean space under such mapping.

Joint probability distribution

The joint probability distribution of random vector $\mathbf{X} = (X_1, \cdots, X_n)$, denoted by $F_{\mathbf{X}}$, is defined as

$$F_{\mathbf{X}}(x_1, \cdots, x_n) = \mathcal{P}\left\{\{X_1 \leq x_1\} \cap \cdots \cap \{X_n \leq x_n\}\right\}.$$

Thus,

$$\mathcal{P}\left\{\mathbf{a} < \mathbf{X} \leq \mathbf{b}\right\} = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} dF_{\mathbf{X}}(x_1, \cdots, x_n),$$

in which $\{\mathbf{a} < \mathbf{X} \leq \mathbf{b}\} = \{a_1 < X_1 \leq b_1\} \cap \cdots \cap \{a_n < X_n \leq b_n\}$.

$F_{\mathbf{X}}$ is also known as joint cumulative distribution function.

Joint probability density function

If $p_{\mathbf{X}}(x_1, \cdots, x_n) = \partial^n F_{\mathbf{X}} / \partial x_1 \cdots \partial x_n$ exists, for any $x_1, \cdots, x_n$, then it is called joint probability density function of $\mathbf{X}$, and one has

$$F_{\mathbf{X}}(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p_{\mathbf{X}}(\xi_1, \cdots, \xi_n)\, d\xi_1 \cdots d\xi_n.$$

Note also that:

- $p_{\mathbf{X}}(x_1, \cdots, x_n) \geq 0$ for every $(x_1, \cdots, x_n) \in \mathbb{R}^n$
- $\displaystyle\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{X}}(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = 1$

Marginal probability density function

The marginal probability density function of X_i is defined as

$$p_{X_i}(x_i) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n-1 \, times} p_{\mathbf{X}}(x_1, \cdots, x_n) \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n,$$

for $i = 1, \cdots, n$.

Conditional distribution

Consider a pair of jointly distributed random variables X and Y. The conditional distribution of X, given the occurrence of the value y of Y, is defined as

$$F_{X|Y}(x \mid y) = \frac{F_{X,Y}(x,y)}{F_Y(y)}.$$

Thus

$$F_{X,Y}(x,y) = F_{X|Y}(x|y) \times F_Y(y),$$

and

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) \times p_Y(y).$$

Remark:
This definition extends naturally to the n-dimensional case.

Independence of distributions

The random variables X and Y are said to be independent if the realization of X does not affect the probability distribution of Y, i.e.,

$$F_{X|Y}(x|y) = F_X(x).$$

Therefore, for independent random variable one has

$$F_{X\,Y}(x,y) = F_X(x) \times F_Y(y),$$

and

$$p_{X\,Y}(x,y) = p_X(x) \times p_Y(y).$$

Remark:
This definition extends naturally to the n-dimensional case.

Statistics of random vectors

- second-order random vector

$$\mathbb{E}\left\{\| \mathbf{X} \|^2\right\} = \int_{\mathbb{R}^n} \| \mathbf{x} \|^2 \; dF_{\mathbf{X}}(\mathbf{x}) < +\infty$$

- mean vector

$$\mathbf{m}_{\mathbf{X}} = \mathbb{E}\left\{\mathbf{X}\right\} = \int_{\mathbb{R}^n} \mathbf{x} \, dF_{\mathbf{X}}(\mathbf{x}) \in \mathbb{R}^n$$

- correlation matrix

$$[R_{\mathbf{XY}}] = \mathbb{E}\left\{\mathbf{X}\mathbf{Y}^T\right\} = \int_{\mathbb{R}^n} \mathbf{x}\mathbf{x}^T \, dF_{\mathbf{XY}}(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n\times n}$$

- covariance matrix

$$\begin{aligned} [K_{\mathbf{XY}}] &= \mathbb{E}\left\{\left(\mathbf{X} - \mathbf{m}_{\mathbf{X}}\right)\left(\mathbf{Y} - \mathbf{m}_{\mathbf{Y}}\right)^T\right\} \in \mathbb{R}^{n\times n} \\ &= [R_{\mathbf{XY}}] - \mathbf{m}_{\mathbf{X}}\mathbf{m}_{\mathbf{Y}}^T \end{aligned}$$

Remark:

Matrices $[R_{\mathbf{XY}}]$ and $[K_{\mathbf{XY}}]$ are symetric positive semi-definite when $\mathbf{X} = \mathbf{Y}$.

Correlation of random variables

The random vectors $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ are said to be uncorrelated if covariance matrix $[K_{\mathbf{XY}}]$ is null, i.e.,

$$[R_{\mathbf{XY}}] = \mathbf{m}_{\mathbf{X}} \mathbf{m}_{\mathbf{Y}}^T.$$

If two random vectors are independent, then they are uncorrelated.

$$\text{independence} \Longrightarrow \text{uncorrelation}$$

But uncorrelated random vectors are not independent in general.

$$\text{uncorrelation} \not\Longrightarrow \text{independence}$$

Remark:
Uncorrelated random vectors which the joint distribution is Gaussian are independent.

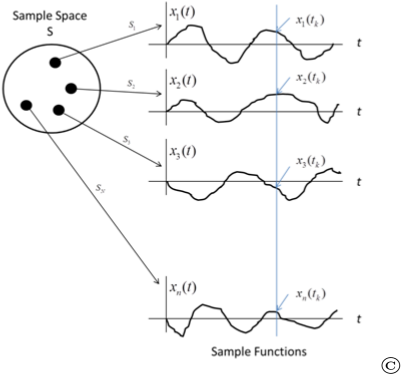
Notions of Random Processes

Random process

A real-valued random process (also called stochastic process) defined on probability space $(\Omega, \Sigma, \mathcal{P})$, indexed by $t \in \mathcal{T}$, is a mapping

$$(t, \omega) \in \mathcal{T} \times \Omega \to X(t, \omega) \in \mathbb{R},$$

such that, for fixed t, the output is a random variable $X(t, \cdot)$, while for fixed ω, $X(\cdot, \omega)$ is a function of t (sample function).



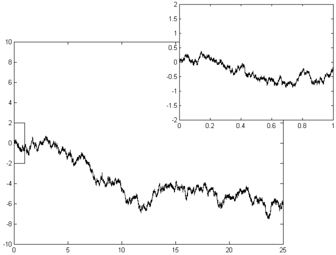
Interpretation and classification

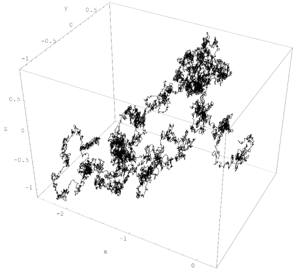
Note that, by definition, a real-valued random process $X(t, \omega)$ is a collection of real-valued random variables indexed by a parameter, and can be thought of as a time-dependent random variable.

According to the nature of the set of indices $\mathcal{T}$, a stochastic process is classified as:

- discrete – if $\mathcal{T}$ is countable
- continuous – if $\mathcal{T}$ is uncountable

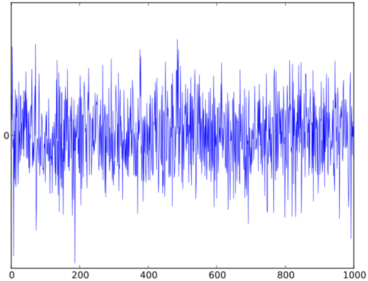
Wiener process (Brownian motion process)



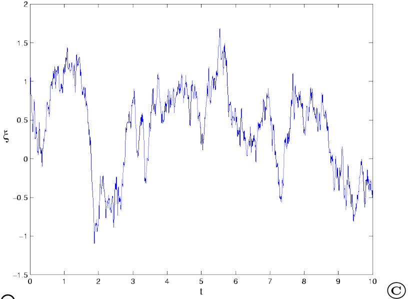


Wiener process — Wikipedia, The Free Encyclopedia, 2021.

White noise and colored noise



White noise



Colored noise

White noise — Wikipedia, The Free Encyclopedia, 2021.

Finite-dimensional distribution

A random process $X_t = X(t, \omega)$ is statistically defined to the order n by a set of its n-th order joint probability distribution, i.e.,

$$F_{X_t}(x_1, \cdots, x_n) = \mathcal{P}\left(X_{t_1} \leq x_1, \cdots, X_{t_n} \leq x_n\right),$$

which is the joint CDF of random variables $X_{t_1}, \cdots, X_{t_n}$.

Statistics of a random process

- second-order random process

$$\mathbb{E}\left\{X_t^2\right\} = \int_{\mathbb{R}} x^2 \, dF_{X_t}(x) < +\infty$$

- mean function

$$\mu_X(t) = \mathbb{E}\left\{X_t\right\} = \int_{\mathbb{R}} x \, dF_{X_t}(x)$$

- correlation function

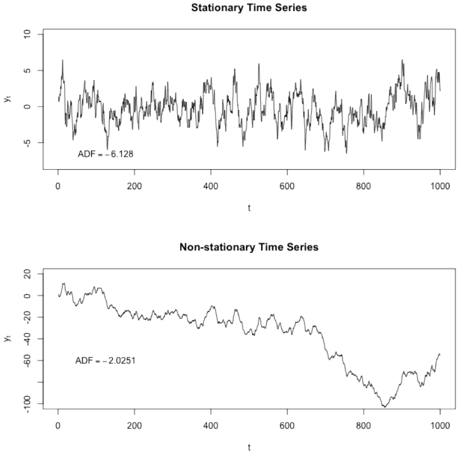
$$\texttt{corr}_X(t_1, t_2) = \mathbb{E}\left\{X_{t_1} X_{t_2}\right\} = \int_{\mathbb{R}} x_1 \, x_2 \, dF_{X_{t_1} X_{t_2}}(x_1, x_2)$$

- covariance function

$$\begin{aligned} \texttt{cov}_X(t_1, t_2) &= \mathbb{E}\left\{\left(X_{t_1} - \mu_X(t_1)\right)\left(X_{t_2} - \mu_X(t_2)\right)\right\} \\ &= \texttt{corr}_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned}$$

Stationary process

A stochastic process is stationary when all the random variables of that stochastic process are identically distributed.



Stationary process — Wikipedia, The Free Encyclopedia, 2021.

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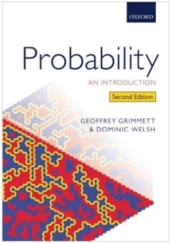

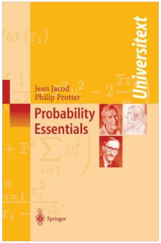

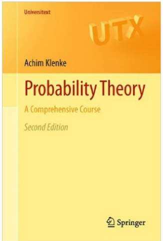

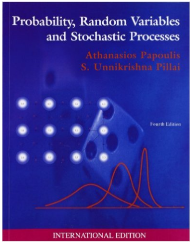

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