Elements of Probability Theory (Part I)

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Probability in Dimension 1

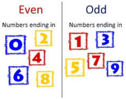
Random experiment

An experiment which repeated under same fixed conditions produce different results is called random experiment.

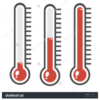
Examples:

1. Rolling a cube-shaped fare die

2. Choosing an integer even number randomly



3. Measuring temperature



Probability space

The mathematical framework in which a random experiment is described consists of a triplet $(\Omega, \Sigma, \mathcal{P})$, called probability space.

The elements of a probability space are:

- Ω: sample space
  (set with all possible events)
- Σ: σ-algebra on Ω
  (set with relevant events only)
- $\mathcal{P}$: probability measure
  (measure of expectation of an event occurrence)

Sample space

A non-empty set which contains all possible events for a certain random experiment is called sample space, being represented by Ω.

Examples:

1. Rolling a cube-shaped fare die (finite Ω)

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

2. Choosing an integer even number randomly (denumerable Ω)

$$\Omega = \{\cdots, -8, -6, -4, -2,\ 0,\ 2,\ 4,\ 6,\ 8, \cdots\}$$

3. Measuring temperature in Kelvin (non-denumerable Ω)

$$\Omega = [a, b] \subset [0, +\infty)$$

σ-algebra of events

In general, not all of the events in Ω are of interest.

Intuitively, a σ-algebra on Ω is the set of relevant outcomes for a random experiment. Formally, Σ is a σ-algebra on Ω if

- $\phi \in \Sigma$
  (contains the empty set)
- $\mathcal{A}^c \in \Sigma$ for any $\mathcal{A} \in \Sigma$
  (closed under complementation)
- $\displaystyle\bigcup_{i=1}^{\infty} \mathcal{A}_i \in \Sigma$ for any $\mathcal{A}_i \in \Sigma$
  (closed under denumerable unions)

Examples of σ-algebras

Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Are σ-algebras:

- $\Sigma = \big\{\phi, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\big\}$
- $\Sigma = \big\{\phi, \{1, 2, 3, 4\}, \{5, 6\}, \Omega\big\}$
- $\Sigma = 2^{\Omega}$ (set of all subsets)

Are not σ-algebras:

- $\Sigma = \big\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \Omega\big\}$
- $\Sigma = \big\{\phi, \{1, 2\}, \{3, 4\}, \{5, 6\}, \Omega\big\}$

G. Grimmett and D. Welsh, **Probability: An Introduction**. Oxford University Press, 2 edition, 2014.

Probability measure

A probability measure is a function $\mathcal{P} : \Sigma \to [0,1] \subset \mathbb{R}$ such that

- $\mathcal{P}\left\{\mathcal{A}\right\} \geq 0$ for any $\mathcal{A} \in \Sigma$
  (probability is nonnegative)
- $\mathcal{P}\left\{\Omega\right\} = 1$
  (entire space has probability one)
- $\mathcal{P}\left\{\bigcup_{i=1}^{\infty} \mathcal{A}_i\right\} = \sum_{i=1}^{\infty} \mathcal{P}\left\{\mathcal{A}_i\right\}$ for any $\mathcal{A}_i$ mutually disjoint
  (σ-additivity)

Remark:
$\mathcal{P}\left\{\phi\right\} = 0$ (empty set has probability zero)

An example in discrete probability

A fair coin is thrown twice.

The number of faces is of interest.

A. C. Morgado, J. B. Pitombeira, P. C. P. Carvalho, P. J. Fernandez, **Análise Combinatória e Probabilidade**, SBM, 2016

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Probability space 1:

$$\Omega_1 = \big\{(H,H),(H,T),(T,H),(T,T)\big\}$$

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$$\Omega_1 = \big\{(H,H),(H,T),(T,H),(T,T)\big\}$$

$$\mathcal{P}_1\big\{(H,H)\big\} = 1/4, \quad \mathcal{P}_1\big\{(H,T)\big\} = 1/4,$$
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Probability space 2:

$$\Omega_2 = \{0,1,2\}$$

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Probability space 2:

$$\Omega_2 = \big\{0,1,2\big\}$$

$$\mathcal{P}_2\left\{0\right\} = 1/4, \quad \mathcal{P}_2\left\{1\right\} = 1/2, \quad \mathcal{P}_2\left\{2\right\} = 1/4$$

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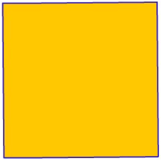
An example in continuous probability



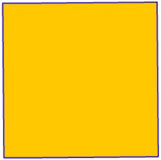
A point is randomly choosen in a square (side b).

Event A: point lie above main diagonal

Event B: point lie in main diagonal

Event C: point lie outside main diagonal

An example in continuous probability



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$$\mathcal{P}\left\{A\right\} = \frac{\text{area of upper triangle}}{\text{area of square}} = \frac{0.5\, b^2}{b^2} = \frac{1}{2}$$

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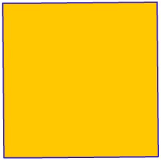

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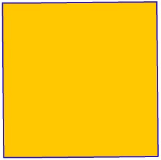

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$$\mathcal{P}\left\{B\right\} = \frac{\text{area of main diagonal}}{\text{area of square}} = \frac{0}{b^2} = 0$$

$$\mathcal{P}\left\{C\right\} = \frac{\text{area outside main diagonal}}{\text{area of square}} = \frac{b^2 - 0}{b^2} = 1$$

Remarks on probability

Probability zero

- An impossible event has probability zero
(e.g. roll a six faces dice, numbered from 1 to 6, and get 7)
- Not every event with probability zero is impossible
(e.g. randomly pick a point on the main diagonal of a square)

Probability one

- An event which occurrence is certain has probability one
(e.g. throw a coin and obtain head or tail)
- Not every event with probability one occurs
(e.g. randomly pick a point outside square's main diagonal)

Conditional probability

Consider a pair of random events A and B such that $\mathcal{P}\{B\} > 0$.

The conditional probability of A, given the occurrence of B, denoted as $\mathcal{P}\{A|B\}$, is defined as

$$\mathcal{P}\{A \mid B\} = \frac{\mathcal{P}\{A \cap B\}}{\mathcal{P}\{B\}}.$$

It follows that

$$\mathcal{P}\{A \cap B\} = \mathcal{P}\{A \mid B\} \times \mathcal{P}\{B\}.$$

An example in conditional probability

Somebody rolls a pair of six-sided dice.

$A =$ value rolled on die 1

$B =$ value rolled on die 2

What is the probability that $A = 2$ given that $A + B \leq 5$?

Conditional probability — Wikipedia, The Free Encyclopedia, 2017.

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A=2

+		B					
		1	**2**	**3**	**4**	**5**	**6**
A	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
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$$\mathcal{P}\left\{A\right\} = 6/36$$

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A+B≤5

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$A \,|\, A{+}B{\leq}5$

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$$\mathcal{P}\left\{A\right\} = 6/36$$

$A{+}B{\leq}5$

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$$\mathcal{P}\left\{A{+}B{\leq}5\right\} = 10/36$$

$A\,|\,A{+}B{\leq}5$

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$$\mathcal{P}\left\{A\,|\,A{+}B{\leq}5\right\} = 3/10$$

Conditional probability — Wikipedia, The Free Encyclopedia, 2017.

Independence of events

If the occurrence of an event B does not affect the occurrence of an event A one has

$$\mathcal{P}\left\{A \mid B\right\} = \mathcal{P}\left\{A\right\}.$$

In this way, once $\mathcal{P}\left\{A \cap B\right\} = \mathcal{P}\left\{A \mid B\right\} \times \mathcal{P}\left\{B\right\}$ it is true that

$$\mathcal{P}\left\{A \cap B\right\} = \mathcal{P}\left\{A\right\} \times \mathcal{P}\left\{B\right\}.$$

Events A and B in which the latter holds are said to be independent.

Remark:
This notion generalizes itself naturally to n events.

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

An example in events independence

A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: $\spadesuit$ "spade"

Are these events independent?

Probability space 1 (fair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit \end{array} \right\}$$

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$$\mathcal{P}_1 \{Q\} = 4/52$$

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$\mathcal{P}_1\left\{Q\right\} = 4/52$

$\mathcal{P}_1\left\{\spadesuit\right\} = 13/52$

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$$\mathcal{P}_1\{Q\} = 4/52$$
$$\mathcal{P}_1\{\spadesuit\} = 13/52$$
$$\mathcal{P}_1\{Q\spadesuit\} = 1/52$$

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$$\mathcal{P}_1\left\{Q\right\} = 4/52$$

$$\mathcal{P}_1\left\{\spadesuit\right\} = 13/52$$

$$\mathcal{P}_1\left\{Q\spadesuit\right\} = 1/52 = \underbrace{4/52}_{\mathcal{P}_1\{Q\}} \times \underbrace{13/52}_{\mathcal{P}_1\{\spadesuit\}}$$

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Are these events independent?

Probability space 1 (fair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit \end{array} \right\}$$

$$\mathcal{P}_1 \{Q\} = 4/52$$

$$\mathcal{P}_1 \{\spadesuit\} = 13/52$$

$$\mathcal{P}_1 \{Q\spadesuit\} = 1/52 = \underbrace{4/52}_{\mathcal{P}_1\{Q\}} \times \underbrace{13/52}_{\mathcal{P}_1\{\spadesuit\}}$$

Events are independent!

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

An example in events independence

A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: $\spadesuit$ "spade"

Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

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Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$$\mathcal{P}_2 \{Q\} = 28/52$$

An example in events independence

A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: $\spadesuit$ "spade"

Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$\mathcal{P}_2\left\{\mathrm{Q}\right\} = 28/52$
$\mathcal{P}_2\left\{\spadesuit\right\} = 1/2$

An example in events independence

A card is drawn from a deck with 52 unknown cards.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$\mathcal{P}_2\left\{Q\right\} = 28/52$

$\mathcal{P}_2\left\{\spadesuit\right\} = 1/2$

$\mathcal{P}_2\left\{Q\spadesuit\right\} = 1/2$

An example in events independence

A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: ♠ "spade"

Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$$\mathcal{P}_2\left\{Q\right\} = 28/52$$

$$\mathcal{P}_2\left\{\spadesuit\right\} = 1/2$$

$$\mathcal{P}_2\left\{Q\spadesuit\right\} = 1/2 \neq \underbrace{28/52}_{\mathcal{P}_2\{Q\}} \times \underbrace{1/2}_{\mathcal{P}_2\{\spadesuit\}}$$

An example in events independence

A card is drawn from a deck with <u>52 unknown cards</u>.

Event 1: Q "queen"

Event 2: $\spadesuit$ "spade"

Are these events independent?

Probability space 2 (unfair deck):

$$\text{cards} = \left\{ \begin{array}{c} 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, \\ 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \\ Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, Q\spadesuit, \end{array} \right\}$$

$$\mathcal{P}_2\{Q\} = 28/52$$

$$\mathcal{P}_2\{\spadesuit\} = 1/2$$

$$\mathcal{P}_2\{Q\spadesuit\} = 1/2 \neq \underbrace{28/52}_{\mathcal{P}_2\{Q\}} \times \underbrace{1/2}_{\mathcal{P}_2\{\spadesuit\}}$$

Events are not independent!

Further remarks on probability

The last example shows that:

- Different probability spaces, for the same random experiment, can produce different predictions
- A probability space that does not accurately describe a random event can produce completely erroneous predictions
- The notion of independence strongly depends on the probability measure employed

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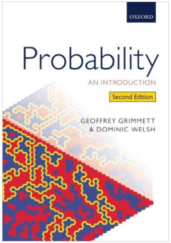

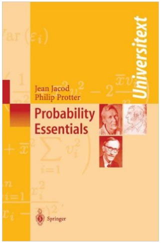

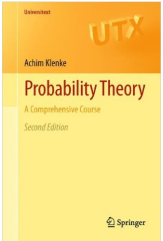

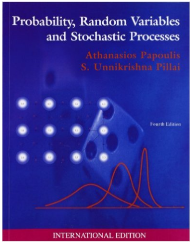

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