Elements of Probability Theory (Part IV)

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Convergence of Random Variables

Sure convergence

Let $\{X_1, X_2, X_3, \cdots, X_n \cdots\}$ be a sequence of random variables defined on probability space $(\Omega, \Sigma, \mathcal{P})$.

Such a sequence is said to converge surely towards the random variable X if

$$\lim_{n \to \infty} X_n(\omega) = X(\omega),$$

for all $\omega \in \Omega$.

Notation: $X_n \xrightarrow{s} X$

An example on sure convergence

Probability space:

$(\Omega, \Sigma, \mathcal{P})$ where $\Omega = [0,1)$ and $\mathcal{P}\left\{[0,\omega)\right\} = \omega$, for $\omega \in \Omega$.

Random variables:

$X_n(\omega) = \omega + \omega^n$ and $X(\omega) = \omega$.

For every $\omega \in [0,1)$ one has $\omega^n \to 0$ as $n \to \infty$, so that

$$X_n(\omega) \to X(\omega) = \omega.$$

Therefore,

$$X_n \xrightarrow{s} X.$$

Almost sure convergence

Let $\{X_1, X_2, X_3, \cdots, X_n \cdots\}$ be a sequence of random variables defined on probability space $(\Omega, \Sigma, \mathcal{P})$.

Such a sequence is said to converge almost surely towards the random variable X if

$$\mathcal{P}\left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} = 1.$$

Notation: $X_n \xrightarrow{a.s.} X$

An example on almost sure convergence

Probability space:

$(\Omega, \Sigma, \mathcal{P})$ where $\Omega = [0, 1]$ and $\mathcal{P}\left\{[0, \omega)\right\} = \omega$, for $\omega \in \Omega$.

Random variables:

$X_n(\omega) = \omega + \omega^n$ and $X(\omega) = \omega$.

For every $\omega \in [0, 1)$ one has $\omega^n \to 0$ as $n \to \infty$, so that

$$X_n(\omega) \to X(\omega) = \omega.$$

However, $X_n(1) = 2$ for every n, so that $X_n(1) \not\to X(1) = 1$.

But since $X_n \to X$ on $[0, 1)$ and $\mathcal{P}\left\{[0, 1)\right\} = 1$, one has

$$X_n \xrightarrow{a.s.} X.$$

Convergence in probability

Let $\{X_1, X_2, X_3, \cdots, X_n \cdots\}$ be a sequence of random variables defined on probability space $(\Omega, \Sigma, \mathcal{P})$.

Such a sequence is said to converge in probability towards the random variable X if

$$\lim_{n \to \infty} \mathcal{P}\left\{ |X_n - X| \geq \epsilon \right\} = 0,$$

or, equivalently,

$$\lim_{n \to \infty} \mathcal{P}\left\{ |X_n - X| < \epsilon \right\} = 1,$$

for all $\epsilon > 0$.

Notation: $X_n \xrightarrow{p} X$

An example on convergence in probability

Random variables:

X is the zero random variable, i.e., $X \equiv 0$
X_n is exponentialy distributed with $\lambda^{-1} = n$, i.e., $X_n \sim Exp(\lambda = 1/n)$

Distribution function:

$F_{X_n}(x) = 1 - e^{-n\,x}/n,\; x > 0$

Once

$$\mathcal{P}\left\{\, |X_n - X| \geq \epsilon \,\right\} = 1 - F_{X_n}(x) = e^{-n\,x}/n,$$

one has

$$\lim_{n \to \infty} \mathcal{P}\left\{\, |X_n - X| \geq \epsilon \,\right\} = 0.$$

Therefore,

$$X_n \xrightarrow{p} X.$$

Convergence in distribution

Let $\{X_1, X_2, X_3, \cdots, X_n \cdots\}$ be a sequence of random variables defined on probability space $(\Omega, \Sigma, \mathcal{P})$, and denote by F_n the distribution function of X_n

Such a sequence is said to converge in distribution towards the random variable X, with distribution function F, if

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for every $x \in \mathbb{R}$ where F is continuous.

Notation: $X_n \xrightarrow{d} X$

An example on convergence in distribution

Random variables:

X is defined on the support $\mathrm{Supp}\, X = (0, +\infty)$
X_n is defined on the support $\mathrm{Supp}\, X_n = (0, n]$

Distribution functions:

$$F_X(x) = 1 - e^{-x},\, x > 0 \iff X \sim Exp(\lambda = 1)$$
$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n,\, 0 < x \leq n$$

The limiting support of X_n is $\mathrm{Supp}\, X = (0, +\infty)$ and for all $x > 0$

$$\lim_{n \to \infty} F_n(x) = F(x) = 1 - e^{-x}.$$

Therefore,

$$X_n \xrightarrow{d} X, \text{ where } X \sim Exp(\lambda = 1).$$

Mean-square convergence

Let $\{X_1, X_2, X_3, \cdots, X_n \cdots\}$ be a sequence of random variables defined on probability space $(\Omega, \Sigma, \mathcal{P})$.

Such a sequence is said to converge in mean-square towards the random variable X if the moments $\mathbb{E}\left\{|X_n|^2\right\}$ and $\mathbb{E}\left\{|X|^2\right\}$ exists, and

$$\lim_{n \to \infty} \mathbb{E}\left\{|X_n - X|^2\right\} = 0.$$

Notation: $X_n \xrightarrow{m.s.} X$

An example on mean-square convergence

Random variables:

X is the zero random variable, i.e., $X \equiv 0$

X_n is uniform distributed over $(0, 1/n)$, i.e., $X_n \sim \mathcal{U}(0, 1/n)$

Density function:

$$p_{X_n}(x) = \begin{cases} n & \text{if } \; 0 \leq x \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

Once

$$\mathbb{E}\left\{ |X_n - X|^2 \right\} = \int_0^{1/n} x^2 \, n \, dx = \frac{1}{3\, n^2},$$

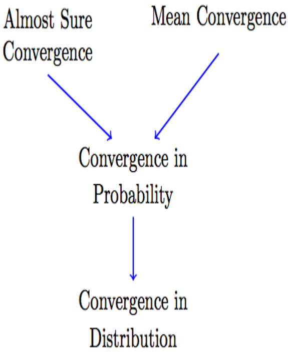
one has

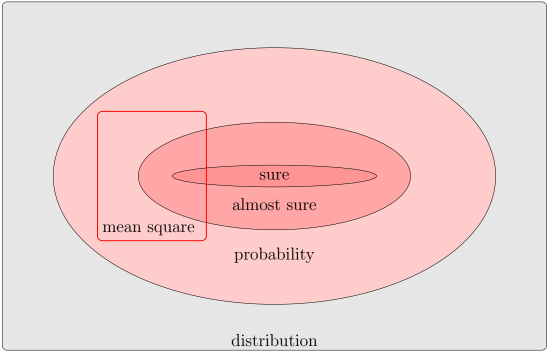
$$\lim_{n \to \infty} \mathbb{E}\left\{ |X_n - X|^2 \right\} = 0.$$

Therefore,

$$X_n \xrightarrow{m.s.} X.$$

Comparison of convergence notions





Important Theorems on Probability

Tchebysheff's (Chebyshev's) inequality

Let X be a random variable with finite mean value μ and non-zero finite variance σ^2. Let $\epsilon > 0$ be an arbitrary real number.

Tchebycheff inequality says that

$$\mathcal{P}\{|X-\mu| \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2},$$

or, equivalently,

$$\mathcal{P}\{|X-\mu| < \epsilon\} \geq 1 - \frac{\sigma^2}{\epsilon^2}.$$

Interpretation: *For random variables with μ and $\sigma^2 \neq 0$ finite, "nearly all" values are close to the mean.*

Law of large numbers (weak version)

Let $X_1, \cdots, X_n$ be sequence of independent and identically distributed (iid) random variables, with mean μ and variance σ^2 both finite.

The sample mean of this set of random variables, defined by

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n},$$

is also a random variable, with mean μ and variance σ^2/n.
Tchebycheff inequality says that

$$\mathcal{P}\{|\bar{X} - \mu| < \epsilon\} \geq 1 - \frac{\sigma^2}{\epsilon^2\, n},$$

so that sample mean converges in probability to the mean, i.e.,

$$\mathcal{P}\{|\bar{X} - \mu| < \epsilon\} \to 1, \quad \text{as well as } n \to \infty.$$

Interpretation: *The probability of sample mean assume values close to μ converge to 1 as $n \to \infty$.*

Central limit theorem

Let $X_1, \cdots, X_n$ be a sequence of random variables, with mean μ and variance σ^2, and define the normalized random variable

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

which has zero mean and unit variance.

This normalized random variable converges in distribution to the standard normal distribution, i.e.,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as well as } n \to \infty.$$

Interpretation: *The probability distribution of the sample mean tends to the Gaussian law with mean μ and variance σ^2 as $n \to \infty$.*

References

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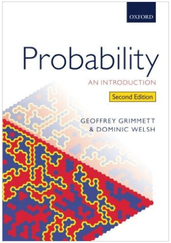

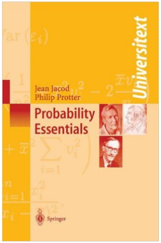

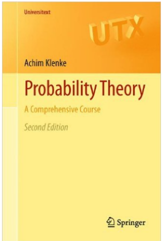

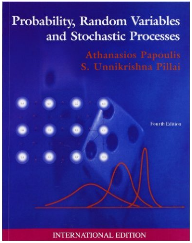

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