Elliptic-curve and isogeny-based cryptography

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University of Bristol

SLMath Summer School
Isogeny-based cryptography
Day 2

Recall: Diffie–Hellman key exchange ’76

Public parameters:

- a prime p (experts: uses $\mathbb{F}_p^*$, today also elliptic curves)
- a number $g \pmod{p}$ (nonexperts: think of an integer less than p)

Alice	Eve	Bob
$\mathrm{sk}_A \xleftarrow{\text{random}} \{0 \ldots p-1\}$		$\mathrm{sk}_B \xleftarrow{\text{random}} \{0 \ldots p-1\}$
$\mathrm{pk}_A = g^{\mathrm{sk}_A}$		$\mathrm{pk}_B = g^{\mathrm{sk}_B}$
$ss := (g^{\mathrm{sk}_B})^{\mathrm{sk}_A}$		$ss := (g^{\mathrm{sk}_A})^{\mathrm{sk}_B}$

- Alice and Bob agree on a shared secret key ss, then they can use that to encrypt their messages.
- Eve sees $\mathrm{pk}_A = g^{\mathrm{sk}_A}$, $\mathrm{pk}_B = g^{\mathrm{sk}_B}$; can’t find sk_A, sk_B, ss.

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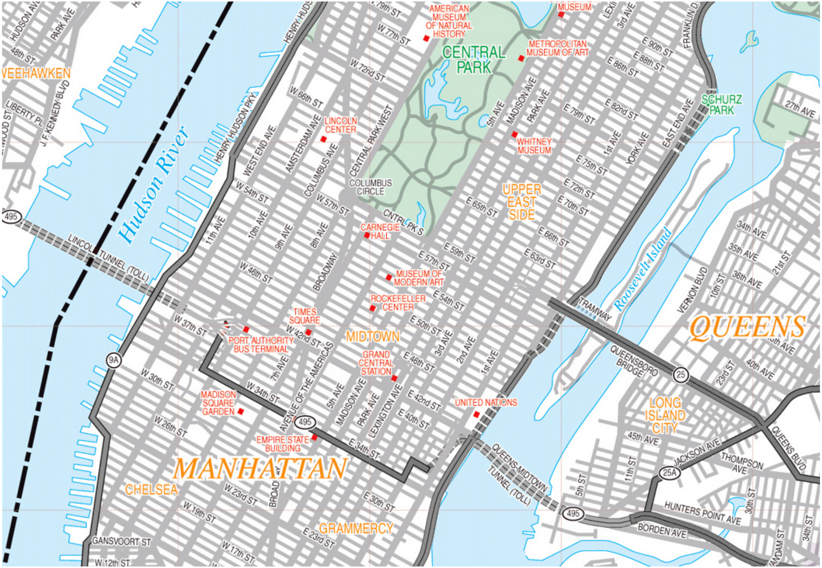
Graph walking Diffie–Hellman?



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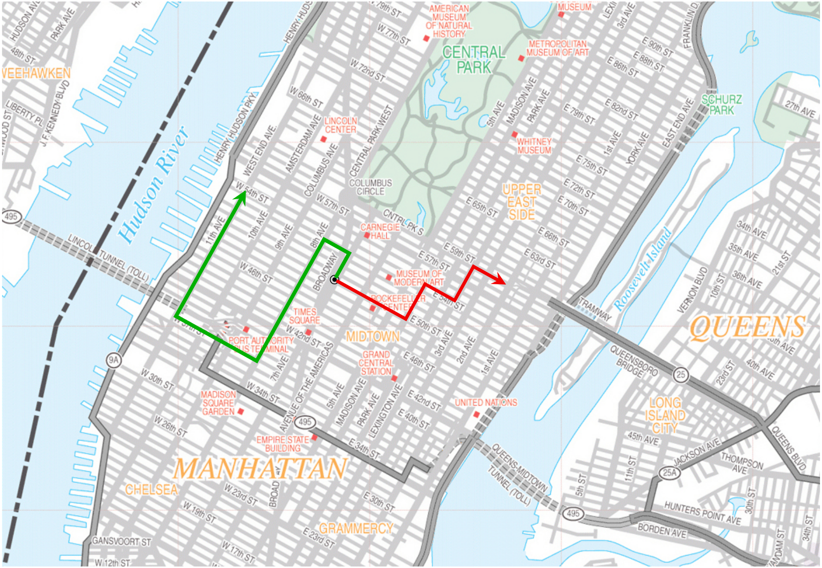

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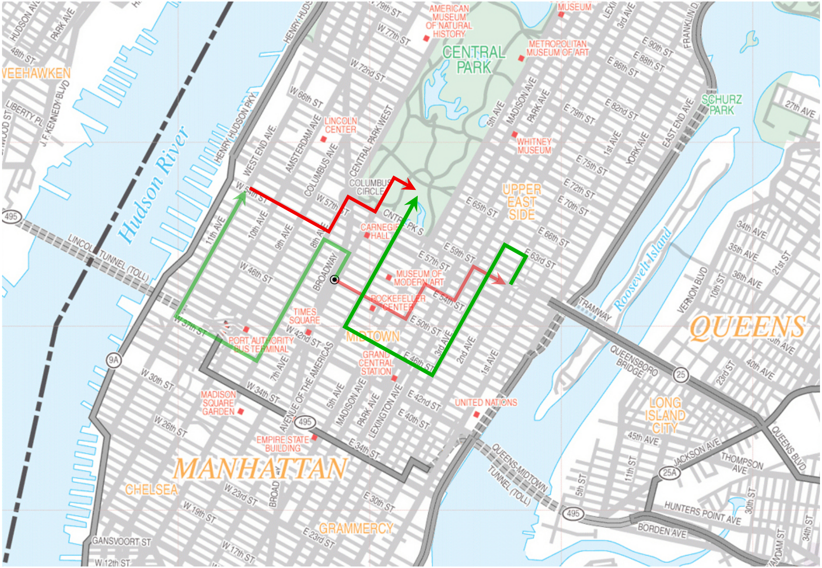

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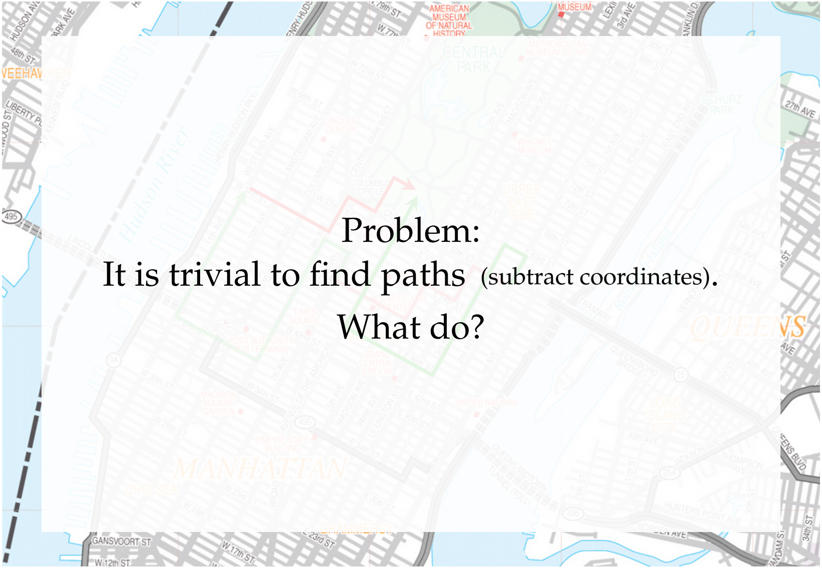

Problem:

It is trivial to find paths (subtract coordinates).

What do?

Big picture

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It is easy to construct graphs that satisfy *almost* all of these — not enough for crypto!

Stand back!

We’re going to do maths.

Maths background #1/3: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
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Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = \varphi \circ \widehat{\varphi} = [\deg \varphi]$.

Maths background #2/3: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

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Vélu operates in the field where the points in G live.

$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$

$\rightsquigarrow$ this is why we use supersingular curves!

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Math slide #3/3: Supersingular isogeny graphs

Let p be a prime, q a power of p, and ℓ a positive integer $\notin p\mathbb{Z}$.

An elliptic curve $E/\mathbb{F}_q$ is *supersingular* if $p \mid (q + 1 - \#E(\mathbb{F}_q))$.

We care about the cases $\#E(\mathbb{F}_p) = p + 1$ and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

$\rightsquigarrow$ easy way to control the group structure by choosing p!

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Let $S \not\ni p$ denote a set of prime numbers.

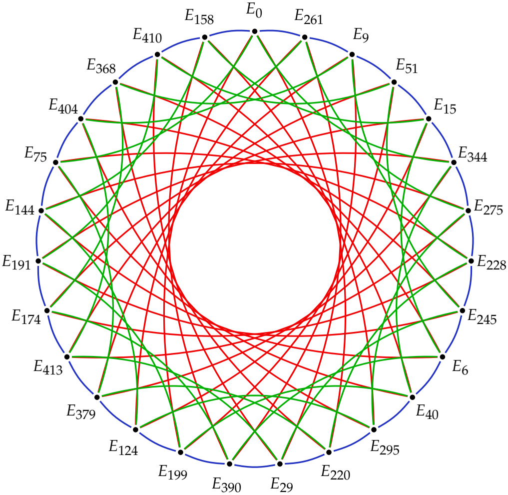
The supersingular S-isogeny graph over $\mathbb{F}_q$ consists of:

- vertices given by isomorphism classes of supersingular elliptic curves,
- edges given by equivalence classes[1] of ℓ-isogenies ($\ell \in S$),

both defined over $\mathbb{F}_q$.

[1] Two isogenies $\varphi\colon E \to E'$ and $\psi\colon E \to E''$ are identified if $\psi = \iota \circ \varphi$ for some isomorphism $\iota\colon E' \to E''$.

Graphs of elliptic curves



Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies

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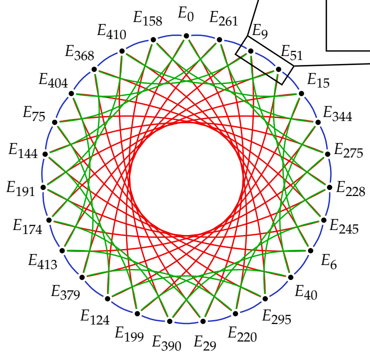
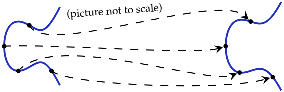


A 3-isogeny



$$E_{51}\colon y^2 = x^3 + 51x^2 + x \longrightarrow E_9\colon y^2 = x^3 + 9x^2 + x$$

$$(x, y) \longmapsto \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97},\; y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$

[ˈsiːˌsaɪd]

CRS or CSIDH

Traditionally, Diffie-Hellman works in a group G via the map

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$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Quantumifying Exponentiation

- We want to replace the exponentiation map

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- The action of a well-chosen $\mathfrak{l} \in \mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ on S moves the elliptic curves one step around one of the cycles.

$$\begin{array}{ccc} \mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ (\mathfrak{l}_3, E) & \mapsto & \mathfrak{l}_3 * E. \end{array}$$

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Diffie and Hellman go to the CSIDH

Alice

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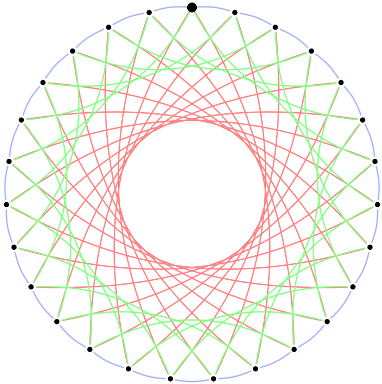

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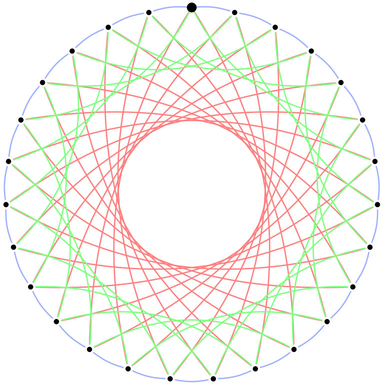

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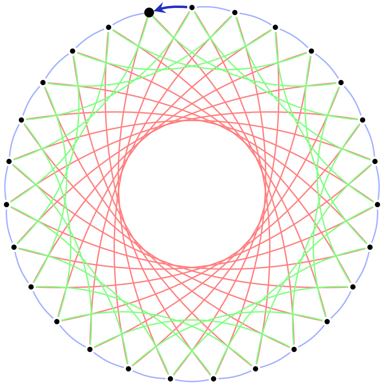

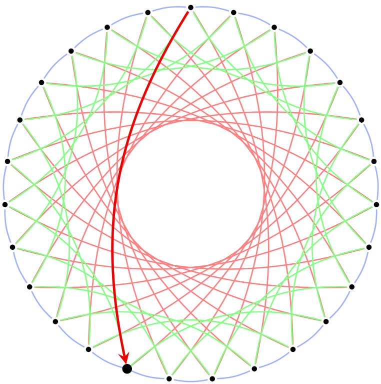

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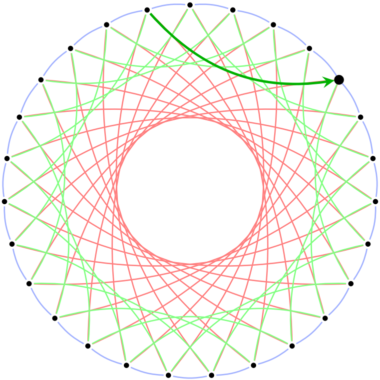

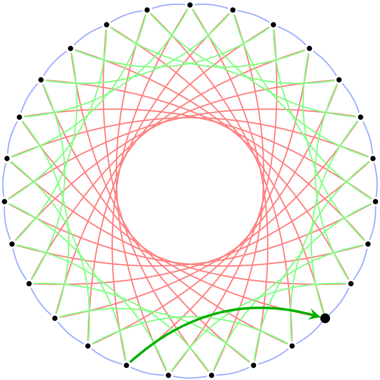

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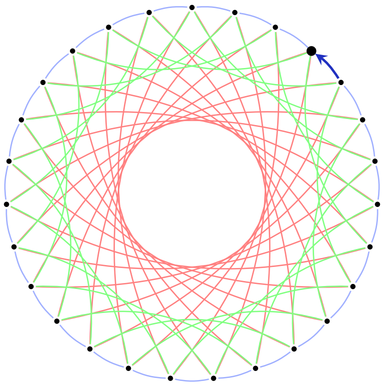

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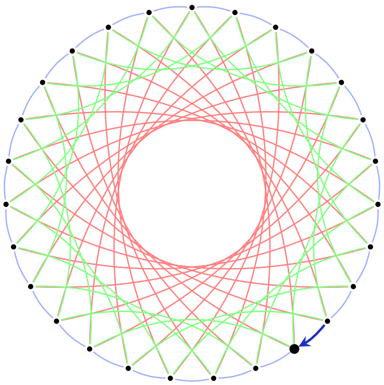

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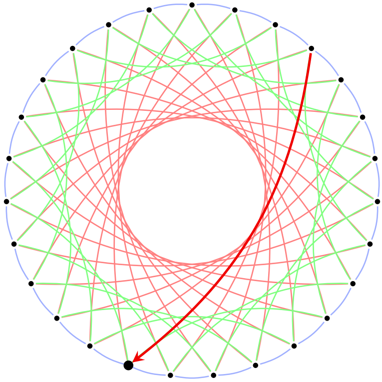

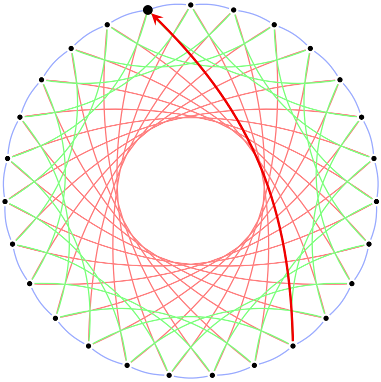

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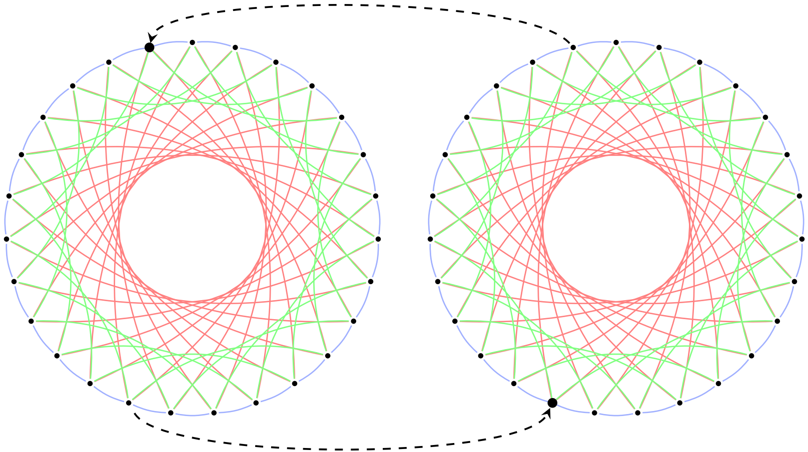

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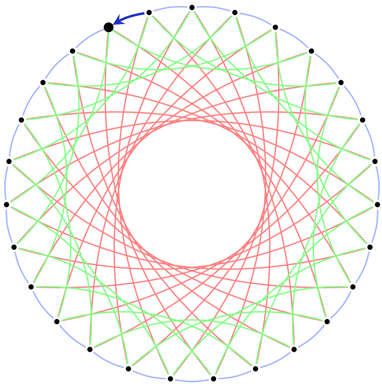

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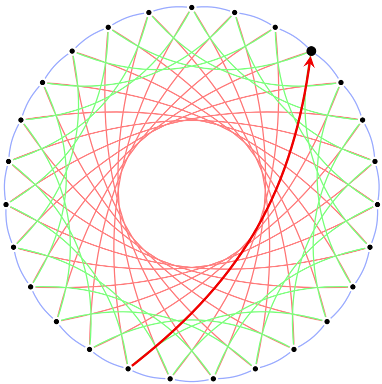

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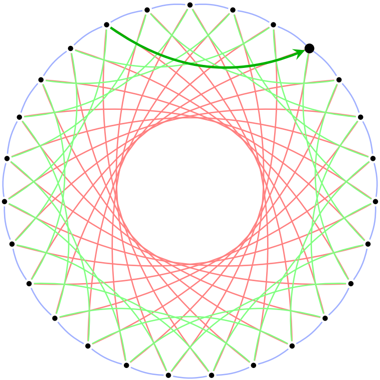

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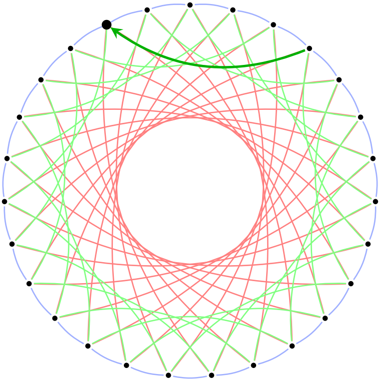

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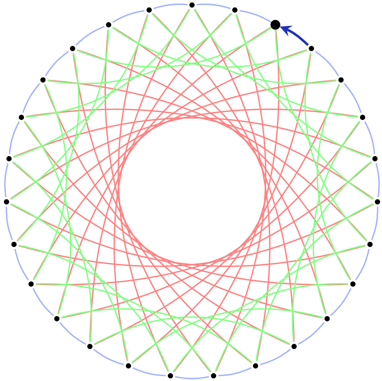

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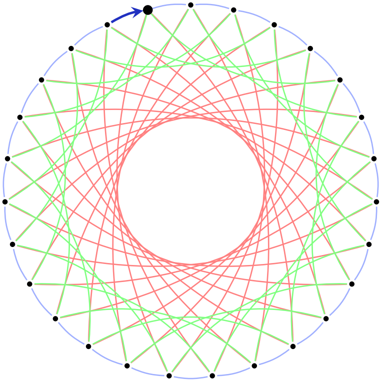

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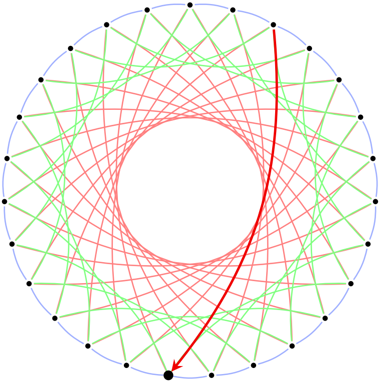

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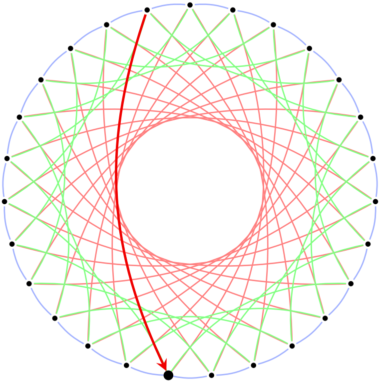

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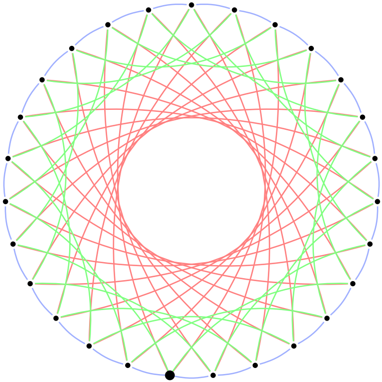
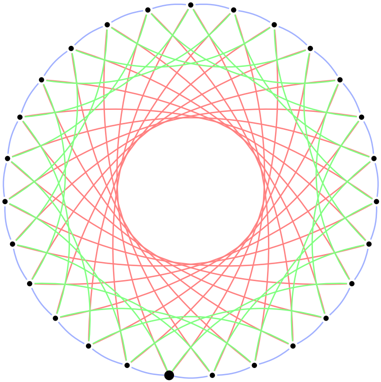


Bob

$[\mathfrak{l}_5, \mathfrak{l}_7, \mathfrak{l}_3{}^{-1}, \mathfrak{l}_5]$



Choosing parameters

In [CLMPR18], parameters are chosen as follows:

- $\ell_1, \ldots, \ell_{n-1}$ the first $n-1$ odd primes.
- $\ell_n > \ell_{n-1}$ the smallest prime such that $p = 4\ell_1 \cdots \ell_n - 1$ is prime.

Then:

- $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ correspond to kernels of $\mathbb{F}_p$-rational isogenies (see next slide) — fast.
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*Any $I \in \mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ can be written as $\prod \mathfrak{l}_i^{e_i}$ with $e_i \in [-5, \ldots, 5]$.

Compute neighbours in the graph

To compute a neighbour of E, we have to compute an ℓ-isogeny from E. To do this:

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Representing nodes of the graph

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$\Rightarrow$ Tiny keys!

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p + 1]P = \infty$.[1]

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- Part of CJS attack computes many paths in superposition.

Quantum Security

Original proposal in 2018 paper: $\mathbb{F}_p \approx 512$ bits.

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
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- Overheads from error correction, high quantum memory etc., not yet understood.

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- Not a subgroup $\rightsquigarrow$ Kuperberg has to use huge group

Better parameters - CSURF

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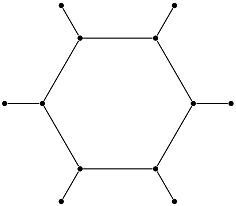
- The 2-isogeny graph looks like this:



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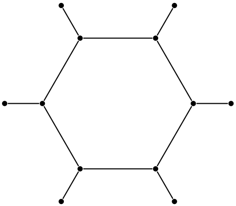

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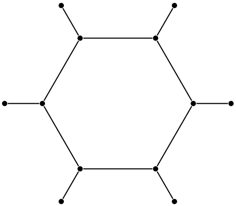

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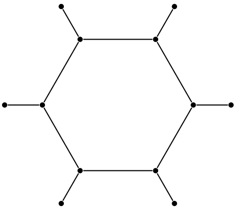

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$\rightsquigarrow$ How to compute 'on the surface'?

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- For any curve on the surface, the 2-isogeny with kernel $\langle(0, 0)\rangle$ is horizontal.

Venturing further beyond the CSIDH

A selection of more advances since original publication (2018):

- sqrtVelu [BDLS20]: square-root speed-up on computation of large-degree isogenies.
- Radical isogenies [CDV20]: significant speed-up on isogenies of small-ish degree.
- Some work on different curve forms (e.g. Edwards).
- Knowledge of $\mathrm{End}(E_0)$ and $\mathrm{End}(E_A)$ breaks CSIDH in classical polynomial time [Wes21].
- CTIDH [$\mathrm{B}^2\mathrm{C}^2\mathrm{LMS}^2$]: Efficient constant-time CSIDH-style construction.

References

[$B^2C^2LMS^2$]	`ctidh.isogeny.org`
[BD17]	`ia.cr/2017/334`
[BDLS20]	`velusqrt.isogeny.org`
[BEG19]	`ia.cr/2019/485`
[BLMP19]	`quantum.isogeny.org`
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[CD19]	`ia.cr/2019/1404`
[CDV20]	`ia.cr/2020/1108`
[FM19]	`ia.cr/2019/555`
[GMT19]	`ia.cr/2019/431`
[Wes21]	`ia.cr/2021/1583`