1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.
4. For each vector x there is a vector y such that $x + y = 0$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y, $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x, $(a + b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. *A **vector space** (or **linear space**) $\mathbf{V}$ over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in $\mathbf{V}$ there is a unique element $x + y$ in $\mathbf{V}$, and for each element a in F and each element x in $\mathbf{V}$ there is a unique element ax in $\mathbf{V}$, such that the following conditions hold:*

(VS 1) For all x, y in $\mathbf{V}$, $x + y = y + x$ (commutativity of addition).

(VS 2) For all x, y, z in $\mathbf{V}$, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in $\mathbf{V}$ denoted by 0 such that $x + 0 = x$ for each x in $\mathbf{V}$.

(VS 4) For each element x in $\mathbf{V}$ there exists and element y in $\mathbf{V}$ such that $x + y = 0$.

(VS 5) For each element x in $\mathbf{V}$, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in $\mathbf{V}$, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in $\mathbf{V}$, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in $\mathbf{V}$, $(a + b)x = ax + bx$.

*The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.*

The elements of the field F are called **scalars** and the elements of the vector space $\mathbf{V}$ are called **vectors**.

An object of the form $(a_1, a_2, ..., a_n)$ where the entries $a_1, a_2, ..., a_n$ are elements of a field F, is called an n**-tuple** with entries from F. The elements $a_1, a_2, ..., a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are called **equal** if $a_i = b_i$ for $i = 1, 2, ..., n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (**Cancellation Law for Vector Addition**). *If x, y, and z are vectors in a vector space $\mathbf{V}$ such that $x + z = y + z$ then $x = y$.*

The vector *0* in (VS 3) is called the **zero vector** of $\mathbf{V}$, and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space $\mathbf{V}$, the following statements are true:*

(a) $0x = 0$ *for each* $x \in \mathbf{V}$.

(b) $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in \mathbf{V}$.

(c) $a0 = 0$ *for each* $a \in F$.

Exercises

1. Label the following statements as true or false.

(a) Every vector space contains a zero vector - **True**.

It is included in the definition of a vector space (VS 3).

(b) A vector space may have more than one zero vector - **False**.

Suppose there were two such vectors, x and y, and one nonzero vector z. Then $x+z = z = y+z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.

(c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.

Consider $x = \mathit{0}$ but $a \neq b$.

(d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.

Consider $a = 0$ but $x \neq y$.

(e) A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$ - **True**.

(f) An $m \times n$ matrix has m columns and n rows - **False**.

An $m \times n$ matrix has m rows and n columns.

(g) In $P(F)$, only polynomials of the same degree may be added - **False**.

Not true based on the definition of addition in $P(F)$.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n - **False**.

Consider x and $-x$.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.

Follows from definition of scalar multiplication in $P(F)$.

(j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.

If a is a nonzero scalar, it can be expressed as ax^0.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if the have the same value at each element of S - **True**.

By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in F.

2. Write the zero vector of $M_{3\times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in$ V and any $a, b \in F$.

$(a + b)(x + y) = (a + b)x + (a + b)y = ax + bx + ay + by$.

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Corollary. *The vector 0 described in (VS 3) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + z = y + z = z$. Then $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$. $\square$

Corollary. *The vector y described in (VS 4) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + y = x + z = 0$. Then $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$. $\square$

11. Let $\mathbf{V} = \{\mathit{0}\}$ consist of a single vector $\mathit{0}$ and define $\mathit{0} + \mathit{0} = \mathit{0}$ nad $c\mathit{0} = \mathit{0}$ for each scalar c in F. Prove that $\mathbf{V}$ is a vector space over F. ($\mathbf{V}$ is called the zero vector space.)

Proof. For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x + y = 0 + 0 = y + x$ (VS 1)
2. $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$ (VS 2)
3. $x + 0 = 0 + 0 = 0 = x$ (VS 3)
4. $x + y = 0 + 0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore $\mathbf{V}$ satisfies all conditions necessary for it to be a vector space. $\square$

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is V a a vector space over R with these operations?

Proof. Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a+b)(x_1, x_2) = ((a+b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a+b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so $\mathbf{V}$ is not a vector space over R. □

14. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in C$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over C by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Proof. Notice that any number $x \in R$ can be expressed as $x + 0i$ in C, so if $\mathbf{V}$ is a vector space over C, it is also a vector space over R. □

15. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in R$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over R by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?

Proof. Consider $c = x + yi$ and $a = (a_1)$ with $y, a_1 \neq 0$. Then $ca = (x + yi)(a_1) = (xa_1 + ya_1 i)$, so the entries of ca aren't in R, so $\mathbf{V}$ is not a vector space over C. □

17. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define the addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathbf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations?

Proof. Consider $(a_1, a_2) \in \mathbf{V}$ with $a_2 \neq 0$. Then $1(a_1, a_2) = (a_1, 0)$, so $1(a_1, a_2) \neq (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over F. □

18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \textbf{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over R with these operations?

Proof. Consider $(a_1, a_2) = (1, 1)$ and $(b_1, b_2) = (2, 2)$. Then $(a_1, a_2) + (b_1, b_2) = (5, 7)$ and $(b_1, b_2) + (a_1, a_2) = (4, 5)$, so $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over R. □

21. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field F. Let

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

Prove that $\mathbf{Z}$ is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. For $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$ and $a, b \in F$

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ (VS 1)
2. $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$ (VS 2)
3. $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$ (VS 3)
4. $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$ (VS 4)
5. $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$ (VS 5)
6. $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$ (VS 6)
7. $a((v_1, w_1)+(v_2, w_2)) = a(v_1+v_2, w_1+w_2) = (av_1+av_2, aw_1+aw_2) = (av_1, aw_1)+(av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$ (VS 7)
8. $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$ (VS 8)

$\square$

3 Subspaces

Definition. *A subset $\mathbf{W}$ of a vector space $\mathbf{V}$ over a field F is called a* ***subspace*** *of $\mathbf{V}$ if $\mathbf{W}$ is a vector space over F with the operations of addition and scalar multiplication defined on $\mathbf{V}$.*

Theorem 3. *Let $\mathbf{V}$ be a vector space and $\mathbf{W}$ a subset of $\mathbf{V}$. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if the following three conditions hold for the operations defined in $\mathbf{V}$.*

1. $0 \in \mathbf{W}$.
2. $x + y \in \mathbf{W}$ *whenever* $x \in \mathbf{W}$ *and* $y \in \mathbf{W}$.
3. $cx \in \mathbf{W}$ *whenever* $c \in F$ *and* $x \in \mathbf{W}$.

The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is $(A^t)_{ij} = A_{ji}$.

A **symmetric** matrix is a matrix A such that $A^t = A$.

An $m \times n$ matrix A is called **upper triangular** if all its entries lying below the diagonal entries are zero, that is, $A_{ij} = 0$ whenever $i > j$. An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero.

The **trace** of an $n \times n$ matrix M, denoted $\mathrm{tr}(M)$, is the sum of diagonal entries of M; that is

$$\mathrm{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$$

Theorem 4. *Any intersection of subspaces of a vector space* $\mathbf{V}$ *is a subspace of* $\boldsymbol{V}$.

Definition. *If* S_1 *and* S_2 *are nonempty subsets of a vector space* V*, then the* ***sum*** *of* S_1 *and* S_2*, denoted* $S_1 + S_2$*, is the set* $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.

Definition. *A vector space* V *is called the* ***direct sum*** *of* W_1 *and* W_2 *if* W_1 *and* W_2 *are subspaces of* V *such that* $W_1 \cap W_2 = 0$ *and* $W_1 + W_2 = V$*. We denote that* V *is the direct sum of* W_1 *and* W_2 *by writing* $V = W_1 \bigoplus W_2$.

Exercises

1. Label the following statements as true or false.

(a) If $\mathbf{V}$ is a vector space and $\mathbf{W}$ is a subset of $\mathbf{V}$ that is a vector space, then $\mathbf{W}$ is a subspace of $\mathbf{V}$ - **True**.

This is the definition of a subspace.

(b) The empty set is a subspace of every vector space - **False**.

The empty set does not contain 0, which is necessary for it to be a subspace.

(c) If $\mathbf{V}$ is a vector space other than the zero vector space, then $\mathbf{V}$ contains a subspace $\mathbf{W}$ such that $\mathbf{W} \neq \mathbf{V}$ - **True**.

The zero subspace fulfills this condition.

(d) The intersection of any two subsets of $\mathbf{V}$ is a subspace of $\mathbf{V}$ - **False**.

If neither subset contains 0, then their intersection can not be a subspace.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries - **True**.

All non-diagonal entries of a diagonal matrix are always 0, and an $n \times n$ matrix has n diagonal entries.

(f) The trace of a square matrix is the product of its diagonal entries - **False**.

The trace of a square matrix is the sum of its diagonal entries.

(g) Let $\mathbf{W}$ be the xy-plane in R^3; that is, $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathbf{W} = R^2$ - **False**.
$R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$, so $\mathbf{W} \neq R^2$.

3. Prove that $(aA + bB)^t = aA^t + bB^t$ **for any** $A, B \in \mathbf{M}_{n\times n}(F)$ **and any** $a, b \in F$**.**

Proof.

$$(aA^t + bB^t)_{ij} = aA^t{}_{ij} + bB^t{}_{ij} = aA_{ji} + bB_{ji} = (aA + bB)_{ji} = (aA + bB)^t{}_{ij},$$
$$\text{so } (aA^t + bB^t) = (aA + bB)^t.$$

□

4. Prove that $(A^t)^t = A$ **for each** $A \in \mathbf{M}_{m\times n}(F)$**.**

Proof.

$$(A^t)^t{}_{ij} = A^t{}_{ji} = A_{ij} \text{ so } (A^t)^t = A.$$

□

5. Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. Let A be an $n \times n$ square matrix. Then

$$A_{ij} + {A^t}_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = A_{ji} + {A^t}_{ji},$$

so $(A + A^t)_{ij} = (A + A^t)_{ji} = {(A + A^t)^t}_{ij}$, therefore $A + A^t$ is symmetric. $\square$

6. Prove that $\mathbf{tr}(aA + bB) = a\mathbf{tr}(A) + b\mathbf{tr}(B)$ for any $A, B \in \mathbf{M}_{n\times n}(F)$.

Proof. For any $i \le n$, $(aA + bB)_{ii} = (aA)_{ii} + (bB)_{ii} = a(A_{ii}) + b(B_{ii})$, so $\mathrm{tr}(aA + bB) = a\mathrm{tr}(A) + b\mathrm{tr}(B)$. $\square$

7. Prove that diagonal matrices are symmetric matrices.

Proof. Let A be an $n \times n$ diagonal matrix. Then for any $i, j \le n$ such that $i \ne j$, $A_{ij} = 0 = A_{ji} = {A^t}_{ij}$. If $i = j$, then $A_{ij} = A_{ji} = {A^t}_{ij}$, so the matrix is symmetric. $\square$

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers

(a) $W_1 = \big\{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\big\}$.

0 is in W_1. Let $x = (3x_2, x_2, -x_2)$ and $y = (3y_2, y_2, -y_2)$. Then $x + y = (3x_2 + 3y_2, x_2 + y_2, -x_2 - y_2) = (3(x_2 + y_2), x_2 + y_2, -(x_2 + y_2))$, so $x + y \in W_1$. Finally let $c \in R$. Then $cx = (3cx_2, cx_2, -cx_2)$, so $cx \in W_1$, therefore W_1 is a subspace of R^3.

(b) $W_2 = \big\{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\big\}$.

$0 \notin W_2$, so W_2 is not a subspace of R^3.

(c) $W_3 = \big\{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0\big\}$

$0 \in W_3$. Let $x = (x_1, x_2, 7x_2 - 2x_1)$ and $y = (y_1, y_2, 7y_2 - 2y_1)$. Then $x + y = (x_1 + y_1, x_2 + y_2, 7x_2 - 2x_1 + 7y_2 - 2y_1) = (x_1 + y_1, x_2 + y_2, 7(x_2 + y_2) - 2(x_1 + y_1))$, so $x + y \in W_3$. Finally, let $c \in R$. Then $cx = (cx_1, cx_2, c(7x_2 - 2x_1)) = (cx_1, cx_2, 7cx_2 - 2cx_1) \in W_3$, so W_3 is a subspace of R^3.

(d) $W_4 = \big\{(a_1, a_2, a_3) \in R^3 : a_1 - 4a_2 - a_3 = 0\big\}$

$0 \in W_4$. Let $x = (4x_2 + x_3, x_2, x_3)$ and $y = (4y_2 + y_3, y_2, y_3)$. Then $x + y = (4x_2 + x_3 + 4y_2 + y_3, x_2 + y_2, x_3 + y_3) = (4(x_2 + y_2) + (x3 + y_3), x_2 + y_2, x_3 + y_3) \in W_4$. Finally let $c \in R$. Then $cx = (c(4x_2 + x_3), cx_2, cx_3) = (4cx_2 + cx_3, cx_2, cx_3) \in W_4$, so W_4 is a subspace of R^3.

(e) $W_5 = \big\{(a_1, a_2, a_3) \in R^3 : a_1 + 2a_2 - 3a_3 = 1\big\}$

$0 \notin W_5$, so W_5 is not a subspace of R^3.

(f) $W_6 = \big\{(a_1, a_2, a_3) \in R^3 : 5{a_1}^2 - 3{a_2}^2 + 6{a_3}^2 = 0\big\}$

Let $x = \left(\sqrt{\frac{3}{5}{x_2}^2 - \frac{6}{5}{x_3}^2}, x_2, x_3\right)$ and $c \in R$. Then $cx = \left(c\sqrt{\frac{3}{5}{x_2}^2 - \frac{6}{5}{x_3}^2}, cx_2, cx_3\right) = (\sqrt{\frac{3}{5}c^2{x_2}^2 - \frac{6}{5}c^2{x_3}^2}) \notin W_6$, so W_6 is not a subspace of R^3.

9. Let W_1, W_3 and W_4 be as in exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$, and observe that each is a subspace of R^3.

Proof. $W_1 \cap W_3 = \left\{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2, a_2 = \frac{1}{4}a_1, \text{ and } a_3 = 7a_2 - 2a_1\right\}$. We have $a_1 = 3a_2$ and $4a_2 = a_1$, so $a_1 = a_2 = 0$ always holds. Then $a_3 = 7a_2 - 2a_1 = 0$, so $W_1 \cap W_3$ is the zero subspace of R^3.

$W_1 \cap W_4 = \left\{(a_1, a_2, a_3) \in R^3 : a_1 = 4a_2 - a_3, a_2 = \frac{1}{3}a_1, \text{ and } a_3 = -a_2\right\}$. We have $a_2 = \frac{1}{3}a_1 = \frac{4}{3}a_2 - \frac{1}{3}a_3$, so $0 = \frac{1}{3}(a_2 - a_3)$, and thus $a_3 = a_2$. Since $a_3 = a_2$ and $a_3 = -a_2$, $a_3 = 0 = a_2$, and then $a_1 = 4a_2 - a_3 = 0$, so $W_1 \cap W_4$ is the zero subspace of R^3.

$W_3 \cap W_4 = \left\{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0 \text{ and } a_1 - 4a_2 - a_3 = 0\right\}$. We then have $a_1 = \frac{11}{3}a_2$ and $a_3 = 7a_2 - 2a_1 = 7a_2 - \frac{22}{3}a_2 = -\frac{1}{3}a_2$. Then $0 \in W_3 \cap W_4$. Let $x = (\frac{11}{2}x_2, x_2, -\frac{1}{3}x_2)$ and $y = (\frac{11}{3}y_2, y_2, -\frac{1}{3}y_2)$. Then $x + y = (\frac{11}{3}(x_2 + y_2), x_2 + y_2, -\frac{1}{3}(x_2 + y_2)) \in W_3 \cap W_4$. Let $c \in R$. Then $cx = (\frac{11}{3}cx_2, cx_2, -\frac{1}{3}cx_2) \in W_3 \cap W_4$, so $W_3 \cap W_4$ is a subspace of R^3. □

11. Is the set $W = \{f(x) \in P(F) : f(x) = 0$ or $f(x)$ has degree $n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Proof. Let $f(x) = x^2 + x$ and $g(x) = -x^2 + x$. Then $n = 2$, and $f(x) + g(x) = 2x$, so W is not closed under addition, and therefore it is not a subspace of $P(F)$. □

12. Prove that the set of $m \times n$ upper triangular matrices is a subspace of $M_{m\times n}(F)$.

Proof. Let W be the set of $m \times n$ upper triangular matrices. Then $0 \in W$. Let $x, y \in W$. Then $x + y \in W$, since for all i, j such that $i > j$, $(x + y)_{ij} = 0 + 0 = 0$. Let $c \in F$. Then $cx \in W$, since for all i, j such that $i > j$, $(cx)_{ij} = c \cdot 0 = 0$, so W is a subspace of $M_{m\times n}(F)$. □

17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. Suppose that W is a subset of a vector space V. If W is a subspace of V, then it must contain 0, so $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$ is true by theorem 1.3. For the converse, suppose that $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$. Then for $a = 0$, $ax = 0 \in W$, so W satisfies all conditions necessary for it to be a subspace of V. □

18. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Proof. Let W be a subset of a vector space V. If W is a subspace of V, then it has to contain 0, and for $a \in F$ and $x, y \in W$, $ax \in W$ and $x + y \in W$. If $z = ax$, then $z + y \in W$ has to be true.

For the converse, suppose that $0 \in W$ and for $a \in F$ and $x, y \in W$, $ax + y \in W$. If $y = 0$, then $ax + y = ax \in W$. Since $ax \in W$, let $z = ax$, so $z + y \in W$, which means that W is a subspace of V. □

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \ldots, a_n$.

Proof. Let $n = 1$. Then $a_1w_1 \in W$ by the definition of a subspace. Next, suppose that for some $n \geq 2$, $a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1} \in W$. Then

$$a_1w_1 + a_2w_2 + \ldots a_nw_n = (a_1w_1 + a_2w_2 + \ldots a_{n-1}w_{n-1}) + a_nw_n.$$

Since $(a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1}) \in W$ and $a_nw_n \in W$ and subspaces are closed under addition, $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$. $\square$

23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Assume, without loss of generality that x is a vector in W_1. Then, since W_2 contains 0, $x + 0 = x \in W_1 + W_2$.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Let W_3 be a subspace of V, such that it contains both W_1 and W_2. Since subspaces are closed under addition, for any $x \in W_1$ and $y \in W_2$, $x + y \in W_3$, so W_3 must contain $W_1 + W_2$.

24. Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Proof. Let $x = (x_1, x_2, \ldots, x_n) \in W_1$ and $y = (y_1, y_2, \ldots, y_n) \in W_2$. Then

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = (x_1 + 0, x_2 + 0, \ldots, x_{n-1} + 0, 0 + y_n) = (x_1, x_2, \ldots, y_n),$$

where $x_1, x_2, \ldots, x_{n-1}, y_n \in F$, so $F^n = W_1 + W_2$.

Next,

$$W_1 \cap W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 = a_2 = \cdots = a_n = 0\} = \{0\},$$

so $F^n = W_1 \bigoplus W_2$. $\square$

27. Let V denote the vector space of all upper triangular $n \times n$ matrices, and let W_1 denote the subspace of V consisting of all diagonal matrices. Define

$$W_2 = \{A \in V : A_{ij} = 0 \textbf{ whenever } i \geq j\}.$$

Show that $V = W_1 \bigoplus W_2$.

Proof. Let $A \in W_1$ and $B \in W_2$. Then

$$A + B = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & b_{12} & \ldots & b_{1n} \\ 0 & 0 & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & b_{12} & \ldots & b_{1n} \\ 0 & a_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix},$$

so $V = W_1 + W_2$.

Next,

$$W_1 \cap W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j \text{ or } i \neq j\} = \{0\},$$

so $V = W_1 \bigoplus W_2$. $\square$

28. A matrix M is called skew-symmetric if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic two, and let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \bigoplus W_2$.

Proof. $0 \in W_1$, since $0^t = 0 = -0$. Let $A, B \in W_1$. Then

$$(A+B)^t = A^t + B^t = -A + (-B) = -(A+B),$$

so $A + B \in W_1$. Now let $a \in F$. Then $(aA)^t = aA^t = -aA$, so $aA \in W_1$, and thus W_1 is a subspace of $M_{n\times n}(F)$.

Let A be any square $n \times n$ matrix. Then if B is an $n \times n$ square matrix, such that $B_{ij} = A_{ij}$ whenever $i = j$, $B_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij})$ whenever $i < j$ and $B_{ij} = B_{ji}$ whenever $i > j$, then B is a symmetric matrix, so $B \in W_2$, and if C is an $n \times n$ square matrix, such that $C_{ij} = 0$ whenever $i = j$, $C_{ij} = \frac{1}{2}(A_{ji} - A_{ij})$ whenever $i > j$ and $C_{ij} = -C_{ji}$ whenever $i < j$, then C is skew-symmetric, so $C \in W_1$. Then:

1. If $i = j$
$$B_{ij} + C_{ij} = A_{ij} + 0 = A_{ij}$$
2. If $i < j$
$$B_{ij} + C_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij}) - \frac{1}{2}(A_{ji} - A_{ij}) = A_{ij}$$
3. If $j > i$
$$B_{ij} + C_{ij} = A_{ji} + \frac{1}{2}(A_{ij} - A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) = A_{ji} + A_{ij} - A_{ji} = A_{ij},$$

so $A = B + C$, and thus $M_{n\times n}(F) = W_1 + W_2$.

$$W_1 \cap W_2 = \left\{A \in M_{n\times n}(F) : A^t = -A \text{ and } A^t = A\right\} = \{0\},$$

so $M_{n\times n}(F) = W_1 \bigoplus W_2$. □

30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

Proof. Suppose that V is the direct sum of W_1 and W_2, and that there exist some $x_1, x_1' \in W_1$ and $x_2, x_2' \in W_2$, such that $x_1 + x_2 = x_1' + x_2' \in V$. Then $x_1 - x_1' = x_2' - x_2 \in W_1 \cap W_2 = \{0\}$, so $x_1 - x_1' = 0 = x_2' - x_2$, and $x_1 = x_1'$ and $x_2 = x_2'$.

For the converse, suppose that any vector in V can be represented uniquely as $x_1 + x_2$, for $x_1 \in W_1$ and $x_2 \in W_2$. Then $V = W_1 + W_2$. Let $y \in W_1 \cap W_2$. Then $y = y + 0 = 0 + y$, and since it is unique, $y = 0$, so $W_1 \cap W_2 = \{0\}$. □

4 Linear combinations and systems of linear equations

Definition. *Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a* ***linear combination*** *of vectors of S if there exist a finite number of vectors $u_1, u_2, \dots, u_n$ in S and scalars $a_1, a_2, \dots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2, \dots, u_n$ and call $a_1, a_2, \dots, a_n$ the* ***coefficients*** *of the linear combination.*

Definition. *Let S be a nonempty subset of a vector space V. The* ***span*** *of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span($\emptyset$) $= \{0\}$.*

Theorem 5. *The span of any subset S of a vector space V is a subspace of V that contains S. Moreover, any subspace of V that contains S must also contain the span of S.*

Definition. *A subset S of a vector space V* ***generates*** *(or* ***spans****) V if span(S) $= V$. In this case, we also say that the vectors of S generate (or span) V.*

Exercises

1. Label the following statements as true or false

(a) The zero vector is a linear combination of any nonempty set of vectors - **True**.

For any vectors $v_1, v_2, \dots, v_n$, $0 = 0v_1 + 0v_2 + \cdots + 0v_n$.

(b) The span of $\emptyset$ is $\emptyset$ - **False**.

span($\emptyset$) is defined to be $\{0\}$.

(c) If S is a subset of a vector space V, then span(S) equals the intersection of all subspaces of V that contain S - **True**.

It follows from theorem 1.5.

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant - **False**.

It is permissible to multiply an equation by any nonzero constant.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another - **True**.

(f) Every system of linear equations has a solution - **False**.

Any system of linear equations that, while solving, produces a system containing $0 = c$, where c is nonzero, does not have a solution.

7. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $e_1, e_2, \dots, e_n$ generates F^n.

Proof. Let a be a vector in F^n, so $a = (a_1, a_2, \dots, a_n)$. Then

$$(a_1, a_2, \dots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n = (a_1, 0, \dots, 0) + (0, a_2, \dots, 0) + \cdots + (0, 0, \dots, a_n)$$

□

8. Show that $P_n(F)$ is generated by $\{1, x, \ldots, x^n\}$.

Proof. Let a be a vector in $P^n(F)$. Then $a = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 * 1$, so $P_n(F)$ is generated by $\{1, x, \ldots, x^n\}$ □

9. Show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \textbf{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

generate $M_{2\times 2}(F)$.

Proof. Let $A \in M_{2\times 2}(F)$. Then

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

10. Show that if

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \textbf{ and } M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Proof. Let A be a symmetric 2×2 matrix. Then

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = aM_1 + bM_3 + cM_2$$

. □

11. Prove that span($\{x\}$) $= \{ax : a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in R^3.

Proof. Let x be a vector in some vector space over a field F, and let $y \in \text{span}(\{x\})$. Then y is a linear combination of x, so $y = ax$ for some $a \in F$. Let $z \in \{ax : a \in F\}$. Then $z = ax$, so z is a linear combination of x, so $z \in \text{span}(\{x\})$. span($\{x\}$) in R^3 is a line going through x and the origin. □

12. Show that a subset W of a vector space V is a subspace of V if and only if span(W) $= W$.

Proof. Let W be a subset of a vector space V, and that span(W) $= W$. Then $0 \in W$, since $0x = 0$ for any $x \in W$. Let x and y be elements of W. Then $x + y \in \text{span}(W)$, since its a linear combination of elements of W, and therefore $x + y \in W$. For any scalar c, $cx \in \text{span}(W)$, so $cx \in W$, which means that W is a subspace of V.

Now suppose that W is a subspace of V. Then for vectors $x_1, x_2, \ldots, x_n \in W$, and scalars $a_1, a_2, \ldots, a_n$, $a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W$, so $W = \text{span}(W)$. □

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$ then span$(S_1) \subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span$(S_1) = V$, deduce that span$(S_2) = V$.

Proof. Let S_1 and S_2 be subsets of a vector space V, such that $S_1 \subseteq S_2$. Then for all elements $x_1, x_2, \ldots, x_n \in S_1$, we know that $x_1, x_2, \ldots, x_n \in S_2$, so $\text{span}(S_2)$ contains all linear combinations $a_1x_1, a_2x_2, \ldots, a_nx_n$, with $a_1, a_2, \ldots, a_n$ being scalars, that is $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Now suppose that $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$. Then $\text{span}(S_1) = V \subseteq \text{span}(S_2)$. Suppose that $\text{span}(S_2) \neq V$, that is, there exists some element $x \in \text{span}(S_2)$ such that $x \notin V$. Then x is a linear combination of elements of S_2, and since S_2 is a subset of V, x is a linear combination of some elements of V. Since V is a subspace, this implies that $x \in V$, which leads to a contradiction. $\square$

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then span$(S_1 \cup S_2) =$ span$(S_1) +$ span(S_2).

Proof. Let S_1 and S_2 be arbitrary subsets of a vector space V. For a vector $v \in \text{span}(S_1 \cup S_2)$, we have $v = a_1x_1 + a_2x_2 + \ldots a_nx_n + b_1y_1 + \ldots b_my_m$, with $x_1, x_2, \ldots, x_n \in S_1$, $y_1, y_2, \ldots, y_m \in S_2$ and scalars $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$. Then

$$a_1x_1 + a_2x_2 + \ldots a_nx_n \in \text{span}(S_1) \text{ and } b_1y_1 + b_2y_2 + \cdots + b_my_m \in \text{span}(S_2),$$

so for $x = a_1x_1 + a_2x_2 + \ldots a_nx_n$ and $y = b_1y_1 + b_2y_2 + \cdots + b_mx_m$, we have

$$v = x + y, \text{ where } x \in \text{span}(S_1), y \in \text{span}(S_2),$$

so $v \in \text{span}(S_1) + \text{span}(S_2)$.

Now let $v \in \text{span}(S_1) + \text{span}(S_2)$. Then $v = x + y$, with $x \in \text{span}(S_1)$ and $y \in \text{span}(S_2)$. Then

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n \text{ with } x_1, x_2, \ldots, x_n \in S_1$$

and

$$y = b_1y_1 + b_2y_2 + \cdots + b_my_m \text{ with } y_1, y_2, \ldots, y_m \in S_2.$$

Then v is a linear combination of vectors in S_1 and vectors in S_2, which means its a linear combination of vector in $S_1 \cup S_2$, so $v \in \text{span}(S_1 \cup S_2)$. $\square$

16. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be uniquely written as a linear combination of vectors of S.

Proof. Suppose that for a vector $x \in \text{span}(S)$, x can be written as two different linear combinations of vectors of S, that is, for $x_1, x_2, \ldots, x_n \in S$ and for scalars $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1x_1 + b_2x_2 + \cdots + b_nx_n \text{ and } a_i \neq b_i \text{ for } 1 \leq i \leq n.$$

Then

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n - b_1x_1 - b_2x_2 - \ldots b_nx_n = (a_1 - b_1)x_1 + (a_2 - b_2)x_2 + \ldots (a_n - b_n)x_n = 0,$$

so $(a_1 - b_1) = (a_2 - b_2) = \cdots = (a_n - b_n) = 0$, so $a_i = b_i$, for $1 \leq i \leq n$. $\square$

5 Linear Dependence and Linear Independence

Definition. *A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

In this case we also say that the vectors of S are linearly dependent.

If for any vectors $u_1, u_2, \ldots, u_n$ we have $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$ if $a_1 = a_2 = \cdots = a_n = 0$, we call this the **trivial representation** of 0 as a linear combination of $u_1, u_2, \ldots, u_n$.

Definition. *A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.*

Theorem 6. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*

Corollary. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.*

Theorem 7. *Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.*

Exercises

1. Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S - **False**.

At least one vector in S is a linear combination of other vectors in S.

(b) Any set containing the zero vector is linearly dependent - **True**.

$0 = 0v_1 + 0v_2 + \cdots + 0v_n$, so 0 is a linear combination of other vectors.

(c) The empty set is linearly dependent - **False**.

Linearly dependent sets must be nonempty.

(d) Subsets of linearly dependent sets are linearly dependent - **False**.

Consider $S_1 = \{(1,0),(0,1),(0,2)\}$ and $S_2 = \{(1,0),(0,1)\}$. Then S_1 is linearly dependent, $S_2 \subset S_1$, and S_2 is linearly independent.

(e) Subsets of linearly independent sets are linearly dependent - **True**.

Follows from theorem 1.6.

(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero - **True**.

4. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Proof.

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = (a_1, a_2, \ldots, a_n),$$

so it is equal 0 only when $a_1 = a_2 = \cdots = a_n = 0$, which means $\{e_1, e_2, \ldots, e_n\}$ is linearly independent. $\square$

5. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

Proof. If

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

for some scalars $a_0, a_1, \ldots, a_n$, then this is the unique 0 vector in $P_n(F)$, so $a_0 = a_1 = \cdots = a_n = 0$. $\square$

6. In $M_{3\times 2}(F)$, prove that the set

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

is linearly dependent.

Proof.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

so the set is linearly dependent. $\square$

8. Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = \mathbb{R}$, then S is linearly independent.

Proof. If

$$a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) = 0$$

then

$$a_1 + a_2 = 0$$
$$a_1 + a_3 = 0$$
$$a_2 + a_3 = 0$$

so $a_1 = a_2 = a_3 = 0$. $\square$

(b) Prove that if F has characteristic two, then S is linearly dependent.

Proof. If F has characteristic two, then

$$(1,1,0) + (1,0,1) + (0,1,1) = (1+1, 1+1, 1+1) = 0$$

so S is linearly dependent. $\square$

9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof. Suppose that $\{u, v\}$ is linearly dependent. Then for some scalars a_1, a_2 not both 0,

$$a_1 u + a_2 v = 0,$$

so $a_1 u = -a_2 v$. Suppose without loss of generality that $a_1 \neq 0$. Then

$$u = -\frac{a_2}{a_1} v,$$

so v is a multiple of u.

For the converse, suppose without loss of generality that v is a multiple of u. Then for some scalar a, $u = av$, so $u - av = 0$, which means that $\{u, v\}$ is linearly dependent. □

10. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

$(1, 0, 0), (0, 1, 0), (1, 1, 0)$

12. Prove Theorem 1.6 and its corollary.

Proof. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$, and suppose that S_1 is linearly dependent. That means, there exists a vector $x_0 \in S_1$ that can be represented as a linear combination of some other vectors in S_1: $x_1, x_2, \ldots, x_n$. So

$$x_0 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

Then since $x_0, x_1, \ldots, x_n$ are all in S_2, there exists a vector in S_2 that can be represented as a linear combination of some other vectors in S_2, so S_2 is linearly dependent. □

Corollary. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.*

Proof. This is just the contraposition of Theorem 1.6. □

14. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

Proof. Let $S = \{0\}$. Then S is linearly dependent, because $0 = a \cdot 0$ for any non-zero scalar a. Suppose that there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$, so

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Then

$$0 = a_1 u_1 + a_2 u_2 + \ldots a_n u_n - v,$$

so S is linearly dependent.

For the converse, suppose that $S \neq \{0\}$ and that there are no distinct vectors $v, u_1, u_2, \ldots, u_n$ such that v is a linear combination of $u_1, u_2, \ldots, u_n$. Suppose that S is linearly dependent. Then there exist scalars $a_0, a_1, \ldots, a_n$ not all 0 such that

$$0 = a_0 v + a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Then at least 1 scalar is not 0, say a_0. Then

$$-a_0 v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$

so

$$v = -\frac{1}{a_0}(a_1 u_1 + a_2 u_2 + \cdots + a_n u_n)$$

which contradicts the original assumption, so S has to be linearly independent. $\square$

15. Let $S = \{u_1.u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k < n)$.

Proof. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. If $u_1 = 0$ then

$$0 = au_1 + 0u_2 + 0u_3 + \cdots + 0u_n$$

with a nonzero, so S is linearly dependent. Suppose that for some k, $1 \leq k < n$, $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$. Then there exist some scalars $a_1, a_2, \ldots, a_k$ such that

$$u_{k+1} = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k,$$

so

$$0 = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k - u_{k+1}$$

so S is linearly dependent.

For the converse, suppose that $u_1 \neq 0$ and that there is no k such that $1 \leq k < n$ and $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$. Suppose that S is linearly dependent. Then

$$0 = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

with scalars $a_1, a_2, \ldots, a_n$ not all 0. Then there exists a nonzero scalar such that the scalars after it in the above sum are all 0, say that scalar is a_n. Then

$$u_n = -\frac{1}{a_n}(a_1 u_1 + a_2 u_2 + \cdots + a_{n-1} u_{n-1}).$$

So $u_n \in \text{span}(\{u_1, u_2, \ldots, u_{n-1}\})$ which contradicts the original assumption, so S has to be linearly independent. $\square$

16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Proof. Suppose that a set S of vectors is linearly independent. Then it follows from the corollary of Theorem 1.6 that any finite subset of S has to be linearly independent.

For the converse, suppose that each finite subset of S is linearly independent. Then there is no finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0,$$

which is the definition of linear independence. $\square$

20. Let $f, g \in F(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $F(\mathbb{R}, \mathbb{R})$.

Proof. Suppose that f and g are linearly dependent. Then $f = ig$ for some i. So $f(0) = i \cdot g(0)$, and $1 = i \cdot 1$, so $i = 1$. We then have $f(1) = g(1)$, so $e^r = e^s$, and $r = s$ which contradicts $r \neq s$. □

21. Let S_1 and S_2 be disjoint linearly independent subsets of V. Prove that $S_1 \cup S_2$ is linearly dependent if and only if span$(S_1) \cap$ span$(S_2) \neq \{0\}$.

Proof. Suppose that $S_1 \cup S_2$ is linearly dependent. Then there exist vectors $u_1, u_2, \ldots, u_n \in S_1$, and $v_1, v_2, \ldots, v_m \in S_2$ and scalars $a_1, \ldots, a_n, b_1, \ldots, b_m$ not all zero such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n + b_1v_1 + b_2v_2 + \cdots + b_mv_m = 0.$$

Then

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = -b_1v_1 - b_2v_2 - \cdots - b_mv_m,$$

and since S_1 and S_2 are linearly independent, we know these are not equal to 0, so it is a non-zero element of $\text{span}(S_1) \cap \text{span}(S_2)$.

For the converse, suppose that $\text{span}(S_1) \cap \text{span}(S_2) \neq \{0\}$. Then there exists a non-zero element

$$x = a_1u_1 + a_2u_2 + \cdots + a_nu_n = b_1v_1 + b_2v_2 + \cdots + b_mv_m$$

for some vectors $u_1, u_2, \ldots, u_n \in S_1$ and $v_1, v_2, \ldots, v_m \in S_2$ and vectors $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$. Then

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n - b_1v_1 - b_2v_2 - \cdots - b_mv_m = 0,$$

so $S_1 \cup S_2$ is linearly dependent. □

6 Bases and Dimension

Definition. *A **basis** β for a vector space $\boldsymbol{V}$ is a linearly independent subset of $\boldsymbol{V}$ that generates $\boldsymbol{V}$. If β is a basis for $\boldsymbol{V}$, we also say that the vectors of β form a basis for $\boldsymbol{V}$.*

Theorem 8. *Let $\boldsymbol{V}$ be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in $\boldsymbol{V}$. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for $\boldsymbol{V}$ if and only if each $v \in \boldsymbol{V}$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Theorem 9. *If a vector space $\boldsymbol{V}$ is generated by a finite set S, then some subset of S is a basis for $\boldsymbol{V}$. Hence $\boldsymbol{V}$ has a finite basis.*

Theorem 10. ***(Replacement Theorem.)*** *Let $\boldsymbol{V}$ be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of $\boldsymbol{V}$ containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates $\boldsymbol{V}$.*

Corollary. *Let $\boldsymbol{V}$ be a vector space having a finite basis. Then all bases for $\boldsymbol{V}$ are finite, and every basis for $\boldsymbol{V}$ contains the same number of vectors.*

Definition. *A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique integer n such that every basis for $\boldsymbol{V}$ contains exactly n elements is called the **dimension** of $\boldsymbol{V}$ and is denoted by $dim(\boldsymbol{V})$. A vector space that is not finite-dimensional is called **infinite-dimensional**.*

Corollary. *Let $\boldsymbol{V}$ be a vector space with dimension n.*

(a) Any finite generating set for $\boldsymbol{V}$ contains at least n vectors, and a generating set for $\boldsymbol{V}$ that contains exactly n vectors is a basis for $\boldsymbol{V}$.

(b) Any linearly independent subset of $\boldsymbol{V}$ that contains exactly n vectors is a basis for $\boldsymbol{V}$.

(c) Every linearly independent subset of $\boldsymbol{V}$ can be extended to a basis for $\boldsymbol{V}$, that is, if L is a linearly independent subset of $\boldsymbol{V}$, then there is a basis β of $\boldsymbol{V}$ such that $L \subseteq \beta$.

Theorem 11. *Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $dim(W) \leq dim(V)$. Moreover, if $dim(W) = dim(V)$, then $V = W$.*

Corollary. *If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.*

6.1 Exercises

1. Label the following statements as true or false

(a) The zero vector space has no basis - **False**.

The basis for the zero vector space is $\emptyset$.

(b) Every vector space that is generated by a finite set has a basis - **True**.

(c) Every vector space has a finite basis - **False**.

Consider $P(F)$ for which the basis is $\{1, x, x^2, \dots\}$.

(d) A vector space cannot have more than one basis - **False**.

Consider R where both $\{1\}$ and $\{-1\}$ are a basis.

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same - **True**.

It follows from corollary 1 of the replacement theorem.

(f) The dimension of $P_n(F)$ is n - **False**.

The dimension of $P_n(F)$ is $n + 1$.

(g) The dimension of $M_{m\times n}(F)$ is $m + n$ - **False**.

The dimension of $M_{m\times n}(F)$ is $m \times n$.

(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2 - **True**.

This is stated in the replacement theorem.

(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way - **False**.

This is only true if S is linearly independent.

(j) Every subspace of a finite-dimensional space is finite-dimensional - **True**.

It follows from theorem 1.11.

(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n - **True**.

It follows from theorem 1.11.

(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V - **True**.

It follows from corollary 2 of the replacement theorem.

6. Give three different bases for F^2 and $M_{2\times 2}(F)$

Let $a \in F$ and $a \neq 0$. Then $A = \{(a,0),(0,a)\}$, $B = \{(-a,0),(0,-a)\}$ and $C\,\{(a,0),(0,-a)\}$ are all bases for F^2.

Then $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right\}$, $E = \left\{ \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} \right\}$, and $F = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} \right\}$ are all bases for $M_{2\times 2}(F)$.

9. The vectors $u_1 = (1,1,1,1), u_2 = (0,1,1,1), u_3 = (0,0,1,1)$ and $u_4 = (0,0,0,1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1,a_2,a_3,a_4) in F^4 as a linear combination of u_1,u_2,u_3,u_4.

$$\begin{aligned} &a_1(1,1,1,1) + (a_2 - a_1)(0,1,1,1) + (a_3 - a_2)(0,0,1,1) + (a_4 - a_3)(0,0,0,1) \\ &\qquad = (a_1, a_1 + a_2 - a_1, a_1 + a_2 - a_1 + a_3 - a_2, a_1 + a_2 - a_1 + a_3 - a_2 + a_4 - a_3) \\ &\qquad = (a_1,a_2,a_3,a_4) \end{aligned}$$

11. Let u and v be distinct vectors of a vector space V. Show that if $\{u,v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.

Proof. Let w be a vector in V. Then $w = xu + yv$ for some scalars x and y. Then

$$xu + yv = yu + yv + xu - yu = y(u+v) + \frac{x-y}{a}(au),$$

so $A = \{u+v, au\}$ generates V.

Additionally,

$$xu + yv = \frac{x}{a}au + \frac{y}{b}bv,$$

so $B = \{au, bv\}$ also generates V.

Since $dim(V) = 2$ and both A and B are generating sets for V, each with 2 vectors, they are both a basis for V. □

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Proof. If $i, j > 0$ and $i, j \leq n$, and E_{ij} is a matrix with 1 in the (i, j)-th entry, and 0 for all other entries, then a basis for W is $\{E_{ij} | i \neq j\} \cup \{E_{11} - E_{ii} | 1 < i \leq n\}$. W is $n^2 - 1$ dimensional. □

16. The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Proof. If $i, j > 0$ and $i, j \leq n$, and E_{ij} is a matrix with 1 in the (i, j)-th entry, and 0 for all other entries, then a basis for W is $\{E_{ij} | i \leq j\}$. W has dimension $n + (n + 1) + \cdots + 1 = \frac{n(n+1)}{2}$. □

17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Proof. If $i, j > 0$ and $i, j \leq n$, then a basis for W is the set of matrices, such that the entry (i, j) is equal to 1, (j, i) is equal to -1, and $i < j$. W has dimension $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$. □

19. Complete the proof of Theorem 1.8.

Proof. In the chapter, it is already proven that if $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V, then each $v \in V$ can be uniquely expressed as a linear combination of vectors of β. Now it remains to prove that the converse of this statement is also true.

Suppose that each $v \in V$ can be uniquely expressed as a linear combination of vectors in β. Then $\text{span}(\beta) = V$. Since it is a unique combination, and $0 \in V$,

$$0 = 0u_1 + 0u_2 + \ldots 0u_n$$

is a unique combination, so β is linearly independent. Since β spans V and is linearly independent, it is a basis for V. □

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite)

Proof. We consider two cases: S is finite, or S is inifinite. If S is finite, then by theorem 1.9, we know that there exists a subset of S that is a basis for V, therefore it remains to prove the statement for when S is infinite.

If V has dimension 0, then $\beta = \emptyset$ is a basis for V, and $\beta \subset S$. Otherwise, S contains a non-zero element u_1, and $\{u_1\}$ is a linearly independent set. We can continue choosing elements $u_2, \ldots, u_k \in S$ such that $\{u_1, \ldots, u_k\}$ is an independent set, until there are no more elements that can be added, without making the set linearly dependent. Then by the replacement theorem, the process will have to terminate with a linearly independent set $\beta = \{u_1, \ldots, u_k\}$, such that $k \leq n$. Let v be a vector in S. If $v \in \beta$ then $v \in \text{span}(\beta)$. If $v \notin \beta$ then $\beta \cup \{v\}$ is a linearly dependent set, so $v \in \text{span}(\beta)$. Then $S \subset \text{span}(\beta)$, and therefore β generates V. Since it is also linearly independent, β is a basis for V. □

(b) Prove that S contains at least n vectors.

Proof. If V has dimension n, then by definition, any basis for V has exactly n vectors, and thus S has to contain at least n vectors. □

21. Prove that a vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

Lemma. *If a vector space V has no infinite linearly independent subset, then there exists a finite generating set for V.*

Suppose that a vector space V has no finite generating set. Then any generating set S for V has to be infinite. We can then create a subset B of S, by first taking any non-zero element from S, and then adding each element of S that can not be represented as a linear combination of all previous elements in B. This process can't terminate, because if it did, B would be a finite generating set for V, contradicting our initial assumption. Then B is an infinite linearly independent subset of V.

Proof. Let V be a vector space, and suppose that V does not contain an infinite linearly independent subset. Then there exists a finite generating set for V, and therefore a subset of that generating set would be a finite basis for V, so V is finite-dimensional.

Now suppose that V contains an infinite linearly independent subset. Then if it was finite-dimensional, it would contradict the replacement theorem by containing a linearly independent subset with more elements than its dimension, so V has to be infinite-dimensional. □

25. Let V, W, and Z be as in exercise 21 of Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

Proof. The dimension of Z is $m + n$. If β_V and β_W are a basis for V and W respectively, a basis β for Z could be constructed as

$$\beta = \{(v, 0) \mid v \in \beta_v\} \cup \{(0, w) \mid w \in \beta_w\}$$

□