1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted *0* such that $x + 0 = x$ for each vector x.
4. For each vector x there is a vector y such that $x + y = 0$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y, $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x, $(a + b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. *A* ***vector space*** *(or* ***linear space****)* $\mathbf{V}$ *over a field F consists of a set on which two operations (called* ***addition*** *and* ***scalar multiplication****, respectively) are defined so that for each pair of elements x, y, in $\mathbf{V}$ there is a unique element $x + y$ in $\mathbf{V}$, and for each element a in F and each element x in $\mathbf{V}$ there is a unique element ax in $\mathbf{V}$, such that the following conditions hold:*

(VS 1) For all x, y in $\mathbf{V}$, $x + y = y + x$ (commutativity of addition).

(VS 2) For all x, y, z in $\mathbf{V}$, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in $\mathbf{V}$ denoted by 0 such that $x + 0 = x$ for each x in $\mathbf{V}$.

(VS 4) For each element x in $\mathbf{V}$ there exists and element y in $\mathbf{V}$ such that $x + y = 0$.

(VS 5) For each element x in $\mathbf{V}$, $1x = x$.

(VS 6) For each pair of elements a, b in F and each element x in $\mathbf{V}$, $(ab)x = a(bx)$.

(VS 7) For each element a in F and each pair of elements x, y in $\mathbf{V}$, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in F and each element x in $\mathbf{V}$, $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the ***sum*** *of x and y and the* ***product*** *of a and x, respectively.*

The elements of the field F are called **scalars** and the elements of the vector space $\mathbf{V}$ are called **vectors**.

An object of the form $(a_1, a_2, ..., a_n)$ where the entries $a_1, a_2, ..., a_n$ are elements of a field F, is called an n**-tuple** with entries from F. The elements $a_1, a_2, ..., a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are called **equal** if $a_i = b_i$ for $i = 1, 2, ..., n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (**Cancellation Law for Vector Addition**). *If x, y, and z are vectors in a vector space $\mathbf{V}$ such that $x + z = y + z$ then $x = y$.*

The vector $\mathit{0}$ in (VS 3) is called the **zero vector** of $\mathbf{V}$, and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space $\mathbf{V}$, the following statements are true:*

(a) $0x = \mathit{0}$ *for each* $x \in \mathbf{V}$.

(b) $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in \mathbf{V}$.

(c) $a\mathit{0} = \mathit{0}$ *for each* $a \in F$.

Exercises

1. Label the following statements as true or false.

(a) Every vector space contains a zero vector - **True**.

It is included in the definition of a vector space (VS 3).

(b) A vector space may have more than one zero vector - **False**.

Suppose there were two such vectors, x and y, and one nonzero vector z. Then $x+z = z = y+z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.

(c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.

Consider $x = \mathit{0}$ but $a \neq b$.

(d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.

Consider $a = 0$ but $x \neq y$.

(e) A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$ - **True**.

(f) An $m \times n$ matrix has m columns and n rows - **False**.

An $m \times n$ matrix has m rows and n columns.

(g) In $P(F)$, only polynomials of the same degree may be added - **False**.

Not true based on the definition of addition in $P(F)$.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n - **False**.

Consider x and $-x$.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.

Follows from definition of scalar multiplication in $P(F)$.

(j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.

If a is a nonzero scalar, it can be expressed as ax^0.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if the have the same value at each element of S - **True**.

By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in F.

2. Write the zero vector of $M_{3\times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in$ V and any $a, b \in F$.

$(a + b)(x + y) = (a + b)x + (a + b)y = ax + bx + ay + by$.

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Corollary. *The vector 0 described in (VS 3) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + z = y + z = z$. Then $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$. $\square$

Corollary. *The vector y described in (VS 4) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + y = x + z = 0$. Then $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$. $\square$

11. Let $\mathbf{V} = \{\mathit{0}\}$ consist of a single vector $\mathit{0}$ and define $\mathit{0} + \mathit{0} = \mathit{0}$ nad $c\mathit{0} = \mathit{0}$ for each scalar c in F. Prove that $\mathbf{V}$ is a vector space over F. ($\mathbf{V}$ is called the zero vector space.)

Proof. For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x + y = 0 + 0 = y + x$ (VS 1)
2. $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$ (VS 2)
3. $x + 0 = 0 + 0 = 0 = x$ (VS 3)
4. $x + y = 0 + 0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore $\mathbf{V}$ satisfies all conditions necessary for it to be a vector space. $\square$

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is V a a vector space over R with these operations?

Proof. Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so $\mathbf{V}$ is not a vector space over R. □

14. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in C$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over C by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Proof. Notice that any number $x \in R$ can be expressed as $x + 0i$ in C, so if $\mathbf{V}$ is a vector space over C, it is also a vector space over R. □

15. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in R$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over R by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?

Proof. Consider $c = x + yi$ and $a = (a_1)$ with $y, a_1 \neq 0$. Then $ca = (x + yi)(a_1) = (xa_1 + ya_1 i)$, so the entries of ca aren't in R, so $\mathbf{V}$ is not a vector space over C. □

17. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define the addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathbf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations?

Proof. Consider $(a_1, a_2) \in \mathbf{V}$ with $a_2 \neq 0$. Then $1(a_1, a_2) = (a_1, 0)$, so $1(a_1, a_2) \neq (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over F. □

18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over R with these operations?

Proof. Consider $(a_1, a_2) = (1, 1)$ and $(b_1, b_2) = (2, 2)$. Then $(a_1, a_2) + (b_1, b_2) = (5, 7)$ and $(b_1, b_2) + (a_1, a_2) = (4, 5)$, so $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over R. □

21. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field F. Let

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

Prove that $\mathbf{Z}$ is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. For $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$ and $a, b \in F$

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ (VS 1)
2. $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$ (VS 2)
3. $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$ (VS 3)
4. $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$ (VS 4)
5. $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$ (VS 5)
6. $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$ (VS 6)
7. $a((v_1, w_1)+(v_2, w_2)) = a(v_1+v_2, w_1+w_2) = (av_1+av_2, aw_1+aw_2) = (av_1, aw_1)+(av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$ (VS 7)
8. $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$ (VS 8)

$\square$

3 Subspaces

Definition. *A subset $\mathbf{W}$ of a vector space $\mathbf{V}$ over a field F is called a **subspace** of $\mathbf{V}$ if $\mathbf{W}$ is a vector space over F with the operations of addition and scalar multiplication defined on $\mathbf{V}$.*

Theorem 3. *Let $\mathbf{V}$ be a vector space and $\mathbf{W}$ a subset of $\mathbf{V}$. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if the following three conditions hold for the operations defined in $\mathbf{V}$.*

1. $0 \in \mathbf{W}$.
2. $x + y \in \mathbf{W}$ *whenever* $x \in \mathbf{W}$ *and* $y \in \mathbf{W}$.
3. $cx \in \mathbf{W}$ *whenever* $c \in F$ *and* $x \in \mathbf{W}$.

The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is $(A^t)_{ij} = A_{ji}$.

A **symmetric** matrix is a matrix A such that $A^t = A$.

An $m \times n$ matrix A is called **upper triangular** if all its entries lying below the diagonal entries are zero, that is, $A_{ij} = 0$ whenever $i > j$. An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero.

The **trace** of an $n \times n$ matrix M, denoted $\text{tr}(M)$, is the sum of diagonal entries of M; that is

$$\text{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$$

Theorem 4. *Any intersection of subspaces of a vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$.*

Exercises

1. Label the following statements as true or false.

(a) If **V** is a vector space and **W** is a subset of **V** that is a vector space, then **W** is a subspace of **V** - **True**.

This is the definition of a subspace.

(b) The empty set is a subspace of every vector space - **False**.

The empty set does not contain 0, which is necessary for it to be a subspace.

(c) If **V** is a vector space other than the zero vector space, then **V** contains a subspace **W** such that $\mathbf{W} \neq \mathbf{V}$ - **True**.

The zero subspace fulfills this condition.

(d) The intersection of any two subsets of **V** is a subspace of **V** - **False**.

If neither subset contains 0, then their intersection can not be a subspace.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries - **True**.

All non-diagonal entries of a diagonal matrix are always 0, and an $n \times n$ matrix has n diagonal entries.

(f) The trace of a square matrix is the product of its diagonal entries - **False**.

The trace of a square matrix is the sum of its diagonal entries.

(g) Let **W** be the xy-plane in R^3; that is, $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathbf{W} = R^2$ - **False**.

$R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$, so $\mathbf{W} \neq R^2$.

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathbf{M}_{n\times n}(F)$ and any $a, b \in F$.

Proof.

$$(aA^t + bB^t)_{ij} = aA^t{}_{ij} + bB^t{}_{ij} = aA_{ji} + bB_{ji} = (aA + bB)_{ji} = (aA + bB)^t{}_{ij},$$
$$\text{so } (aA^t + bB^t) = (aA + bB)^t.$$

□

4. Prove that $(A^t)^t = A$ for each $A \in \mathbf{M}_{m\times n}(F)$.

Proof.

$$(A^t)^t{}_{ij} = A^t{}_{ji} = A_{ij} \text{ so } (A^t)^t = A.$$

□

5. Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. Let A be an $n \times n$ square matrix. Then

$$A_{ij} + A^t{}_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = A_{ji} + A^t{}_{ji},$$

so $(A + A^t)_{ij} = (A + A^t)_{ji} = (A + A^t)^t{}_{ij}$, therefore $A + A^t$ is symmetric. □

6. Prove that $\mathbf{tr}(aA + bB) = a\mathbf{tr}(A) + b\mathbf{tr}(B)$ for any $A, B \in \mathbf{M}_{n\times n}(F)$.

Proof. For any $i \le n$, $(aA + bB)_{ii} = (aA)_{ii} + (bB)_{ii} = a(A_{ii}) + b(B_{ii})$, so $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$. $\square$

7. Prove that diagonal matrices are symmetric matrices.

Proof. Let A be an $n \times n$ diagonal matrix. Then for any $i, j \le n$ such that $i \ne j$, $A_{ij} = 0 = A_{ji} = {A^t}_{ij}$. If $i = j$, then $A_{ij} = A_{ji} = {A^t}_{ij}$, so the matrix is symmetric. $\square$

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers

(a) $W_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$.

0 is in W_1. Let $x = (3x_2, x_2, -x_2)$ and $y = (3y_2, y_2, -y_2)$. Then $x + y = (3x_2 + 3y_2, x_2 + y_2, -x_2 - y_2) = (3(x_2 + y_2), x_2 + y_2, -(x_2 + y_2))$, so $x + y \in W_1$. Finally let $c \in R$. Then $cx = (3cx_2, cx_2, -cx_2)$, so $cx \in W_1$, therefore W_1 is a subspace of R^3.

(b) $W_2 = \{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\}$.

$0 \notin W_2$, so W_2 is not a subspace of R^3.

(c) $W_3 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0\}$

$0 \in W_3$. Let $x = (x_1, x_2, 7x_2 - 2x_1)$ and $y = (y_1, y_2, 7y_2 - 2y_1)$. Then $x + y = (x_1 + y_1, x_2 + y_2, 7x_2 - 2x_1 + 7y_2 - 2y_1) = (x_1 + y_1, x_2 + y_2, 7(x_2 + y_2) - 2(x_1 + y_1))$, so $x + y \in W_3$. Finally, let $c \in R$. Then $cx = (cx_1, cx_2, c(7x_2 - 2x_1)) = (cx_1, cx_2, 7cx_2 - 2cx_1) \in W_3$, so W_3 is a subspace of R^3.

(d) $W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 - 4a_2 - a_3 = 0\}$

$0 \in W_4$. Let $x = (4x_2 + x_3, x_2, x_3)$ and $y = (4y_2 + y_3, y_2, y_3)$. Then $x + y = (4x_2 + x_3 + 4y_2 + y_3, x_2 + y_2, x_3 + y_3) = (4(x_2 + y_2) + (x3 + y_3), x_2 + y_2, x_3 + y_3) \in W_4$. Finally let $c \in R$. Then $cx = (c(4x_2 + x_3), cx_2, cx_3) = (4cx_2 + cx_3, cx_2, cx_3) \in W_4$, so W_4 is a subspace of R^3.

(e) $W_5 = \{(a_1, a_2, a_3) \in R^3 : a_1 + 2a_2 - 3a_3 = 1\}$

$0 \notin W_5$, so W_5 is not a subspace of R^3.

(f) $W_6 = \{(a_1, a_2, a_3) \in R^3 : 5{a_1}^2 - 3{a_2}^2 + 6{a_3}^2 = 0\}$

Let $x = \left(\sqrt{\frac{3}{5}{x_2}^2 - \frac{6}{5}{x_3}^2}, x_2, x_3\right)$ and $c \in R$. Then $cx = \left(c\sqrt{\frac{3}{5}{x_2}^2 - \frac{6}{5}{x_3}^2}, cx_2, cx_3\right) = (\sqrt{\frac{3}{5}c^2{x_2}^2 - \frac{6}{5}c^2{x_3}^2}) \notin W_6$, so W_6 is not a subspace of R^3.

9. Let W_1, W_3 and W_4 be as in exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$, and observe that each is a subspace of R^3.

Proof. $W_1 \cap W_3 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2, a_2 = \frac{1}{4}a_1, \text{ and } a_3 = 7a_2 - 2a_1\}$. We have $a_1 = 3a_2$ and $4a_2 = a_1$, so $a_1 = a_2 = 0$ always holds. Then $a_3 = 7a_2 - 2a_1 = 0$, so $W_1 \cap W_3$ is the zero subspace of R^3.

$W_1 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 4a_2 - a_3, a_2 = \frac{1}{3}a_1, \text{ and } a_3 = -a_2\}$. We have $a_2 = \frac{1}{3}a_1 = \frac{4}{3}a_2 - \frac{1}{3}a_3$, so $0 = \frac{1}{3}(a_2 - a_3)$, and thus $a_3 = a_2$. Since $a_3 = a_2$ and $a_3 = -a_2$, $a_3 = 0 = a_2$, and then $a_1 = 4a_2 - a_3 = 0$, so $W_1 \cap W_4$ is the zero subspace of R^3.

$W_3 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0 \text{ and } a_1 - 4a_2 - a_3 = 0\}$. We then have $a_1 = \frac{11}{3}a_2$ and $a_3 = 7a_2 - 2a_1 = 7a_2 - \frac{22}{3}a_2 = -\frac{1}{3}a_2$. Then $0 \in W_3 \cap W_4$. Let $x = (\frac{11}{2}x_2, x_2, -\frac{1}{3}x_2)$ and $y = (\frac{11}{3}y_2, y_2, -\frac{1}{3}y_2)$. Then $x + y = (\frac{11}{3}(x_2 + y_2), x_2 + y_2, -\frac{1}{3}(x_2 + y_2)) \in W_3 \cap W_4$. Let $c \in R$. Then $cx = (\frac{11}{3}cx_2, cx_2, -\frac{1}{3}cx_2) \in W_3 \cap W_4$, so $W_3 \cap W_4$ is a subspace of R^3. □

11. Is the set $W = \{f(x) \in P(F) : f(x) = 0$ or $f(x)$ has degree $n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Proof. Let $f(x) = x^2 + x$ and $g(x) = -x^2 + x$. Then $n = 2$, and $f(x) + g(x) = 2x$, so W is not closed under addition, and therefore it is not a subspace of $P(F)$. □

12. Prove that the set of $m \times n$ upper triangular matrices is a subspace of $M_{m\times n}(F)$.

Proof. Let W be the set of $m \times n$ upper triangular matrices. Then $0 \in W$. Let $x, y \in W$. Then $x + y \in W$, since for all i, j such that $i > j$, $(x + y)_{ij} = 0 + 0 = 0$. Let $c \in F$. Then $cx \in W$, since for all i, j such that $i > j$, $(cx)_{ij} = c \cdot 0 = 0$, so W is a subspace of $M_{m\times n}(F)$. □

17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. Suppose that W is a subset of a vector space V. If W is a subspace of V, then it must contain 0, so $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$ is true by theorem 1.3. For the converse, suppose that $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$. Then for $a = 0$, $ax = 0 \in W$, so W satisfies all conditions necessary for it to be a subspace of V. □

18. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Proof. Let W be a subset of a vector space V. If W is a subspace of V, then it has to contain 0, and for $a \in F$ and $x, y \in W$, $ax \in W$ and $x + y \in W$. If $z = ax$, then $z + y \in W$ has to be true.

For the converse, suppose that $0 \in W$ and for $a \in F$ and $x, y \in W$, $ax + y \in W$. If $y = 0$, then $ax + y = ax \in W$. Since $ax \in W$, let $z = ax$, so $z + y \in W$, which means that W is a subspace of V. □

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \ldots, a_n$.