

1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y , $x + y = y + x$.
2. For all vectors x , y , and z , $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x .
4. For each vector x there is a vector y such that $x + y = 0$.
5. For each vector x , $1x = x$.
6. For each pair of real numbers a and b and each vector x , $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y , $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x , $(a + b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. A *vector space* (or *linear space*) \mathbf{V} over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x , y , in \mathbf{V} there is a unique element $x + y$ in \mathbf{V} , and for each element a in F and each element x in \mathbf{V} there is a unique element ax in \mathbf{V} , such that the following conditions hold:

- (VS 1) For all x , y in \mathbf{V} , $x + y = y + x$ (commutativity of addition).
- (VS 2) For all x , y , z in \mathbf{V} , $(x + y) + z = x + (y + z)$ (associativity of addition).
- (VS 3) There exists an element in \mathbf{V} denoted by 0 such that $x + 0 = x$ for each x in \mathbf{V} .
- (VS 4) For each element x in \mathbf{V} there exists an element y in \mathbf{V} such that $x + y = 0$.
- (VS 5) For each element x in \mathbf{V} , $1x = x$.
- (VS 6) For each pair of elements a , b in F and each element x in \mathbf{V} , $(ab)x = a(bx)$.
- (VS 7) For each element a in F and each pair of elements x , y in \mathbf{V} , $a(x + y) = ax + ay$.
- (VS 8) For each pair of elements a , b in F and each element x in \mathbf{V} , $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x , respectively.

The elements of the field F are called **scalars** and the elements of the vector space \mathbf{V} are called **vectors**.

An object of the form (a_1, a_2, \dots, a_n) where the entries a_1, a_2, \dots, a_n are elements of a field F , is called an **n -tuple** with entries from F . The elements a_1, a_2, \dots, a_n are called the **entries** or **components** of the n -tuple. Two n -tuples (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are called **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F . We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O .

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k , called the **coefficient** of x^k , is in F . If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1 ; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (Cancellation Law for Vector Addition). *If x , y , and z are vectors in a vector space \mathbf{V} such that $x + z = y + z$ then $x = y$.*

The vector 0 in (VS 3) is called the **zero vector** of \mathbf{V} , and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space \mathbf{V} , the following statements are true:*

- (a) $0x = 0$ for each $x \in \mathbf{V}$.
- (b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in \mathbf{V}$.
- (c) $a0 = 0$ for each $a \in F$.

Exercises

1. Label the following statements as true or false.

- (a) Every vector space contains a zero vector - **True**.
It is included in the definition of a vector space (VS 3).
- (b) A vector space may have more than one zero vector - **False**.
Suppose there were two such vectors, x and y , and one nonzero vector z . Then $x+z = z = y+z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.
- (c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.
Consider $x = 0$ but $a \neq b$.
- (d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.
Consider $a = 0$ but $x \neq y$.
- (e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$ - **True**.
- (f) An $m \times n$ matrix has m columns and n rows - **False**.
An $m \times n$ matrix has m rows and n columns.
- (g) In $P(F)$, only polynomials of the same degree may be added - **False**.
Not true based on the definition of addition in $P(F)$.
- (h) If f and g are polynomials of degree n , then $f + g$ is a polynomial of degree n - **False**.
Consider x and $-x$.
- (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.
Follows from definition of scalar multiplication in $P(F)$.
- (j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.
If a is a nonzero scalar, it can be expressed as ax^0 .
- (k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S - **True**.
By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in S .

2. Write the zero vector of $M_{3 \times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. In any vector space V , show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.

$$(a+b)(x+y) = (a+b)x + (a+b)y = ax + bx + ay + by.$$

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Corollary. *The vector 0 described in (VS 3) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + z = y + z = z$. Then $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$. \square

Corollary. *The vector y described in (VS 4) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + y = x + z = 0$. Then $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$. \square

11. Let $\mathbf{V} = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ nad $c0 = 0$ for each scalar c in F . Prove that \mathbf{V} is a vector space over F . (\mathbf{V} is called the zero vector space.)

Proof. For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x + y = 0 + 0 = y + x$ (VS 1)
2. $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$ (VS 2)
3. $x + 0 = 0 + 0 = 0 = x$ (VS 3)
4. $x + y = 0 + 0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore \mathbf{V} satisfies all conditions necessary for it to be a vector space. \square

13. Let \mathbf{V} denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of \mathbf{V} and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is \mathbf{V} a vector space over R with these operations?

Proof. Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so \mathbf{V} is not a vector space over R . □

14. Let $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in C \text{ for } i = 1, 2, \dots, n\}$; so \mathbf{V} is a vector space over C by Example 1. Is \mathbf{V} a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Proof. Notice that any number $x \in R$ can be expressed as $x + 0i$ in C , so if \mathbf{V} is a vector space over C , it is also a vector space over R . □

15. Let $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in R \text{ for } i = 1, 2, \dots, n\}$; so \mathbf{V} is a vector space over R by Example 1. Is \mathbf{V} a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?

Proof. Consider $c = x + yi$ and $a = (a_1)$ with $y, a_1 \neq 0$. Then $ca = (x + yi)(a_1) = (xa_1 + ya_1i)$, so the entries of ca aren't in R , so \mathbf{V} is not a vector space over C . □

17. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define the addition of elements of \mathbf{V} coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathbf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is \mathbf{V} a vector space over F with these operations?

Proof. Consider $(a_1, a_2) \in \mathbf{V}$ with $a_2 \neq 0$. Then $1(a_1, a_2) = (a_1, 0)$, so $1(a_1, a_2) \neq (a_1, a_2)$, therefore \mathbf{V} is not a vector space over F . □

18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is \mathbf{V} a vector space over R with these operations?

Proof. Consider $(a_1, a_2) = (1, 1)$ and $(b_1, b_2) = (2, 2)$. Then $(a_1, a_2) + (b_1, b_2) = (5, 7)$ and $(b_1, b_2) + (a_1, a_2) = (4, 5)$, so $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$, therefore \mathbf{V} is not a vector space over R . □

21. Let \mathbf{V} and \mathbf{W} be vector spaces over a field F . Let

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

Prove that \mathbf{Z} is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. For $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$ and $a, b \in F$

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ (VS 1)
2. $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$ (VS 2)
3. $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$ (VS 3)
4. $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$ (VS 4)
5. $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$ (VS 5)
6. $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$ (VS 6)
7. $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$ (VS 7)
8. $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw_1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$ (VS 8)

□