1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.

2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.

3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.

4. For each vector x there is a vector y such that $x + y = 0$.

5. For each vector x, $1x = x$.

6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.

7. For each real number a and each pair of vectors x and y, $a(x+y) = ax + ay$.

8. For each pair of real numbers a and b and each vector x, $(a+b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. *A **vector space** (or **linear space**) **V** over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in **V** there is a unique element x + y in **V**, and for each element a in F and each element x in **V** there is a unique element ax in **V**, such that the following conditions hold:*

*(VS 1) For all x, y in **V**, x + y = y + x (commutativity of addition).*

*(VS 2) For all x, y, z in **V**, (x+y)+z = x+(y+z) (associativity of addition).*

*(VS 3) There exists an element in **V** denoted by 0 such that x + 0 = x for each x in **V**.*

*(VS 4) For each element x in **V** there exists and element y in **V** such that x + y = 0.*

*(VS 5) For each element x in **V**, 1x = x.*

*(VS 6) For each pair of elements a, b in F and each element x in **V**, (ab)x = a(bx).*

*(VS 7) For each element a in F and each pair of elements x, y in **V**, a(x + y) = ax + ay.*

*(VS 8) For each pair of elements a, b in F and each element x in **V**, (a + b)x = ax + bx.*

*The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.*

The elements of the field F are called **scalars** and the elements of the vector space $\mathbf{V}$ are called **vectors**.

An object of the form $(a_1, a_2, ..., a_n)$ where the entries $a_1, a_2, ..., a_n$ are elements of a field F, is called an n-**tuple** with entries from F. The elements $a_1, a_2, ..., a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are called **equal** if $a_i = b_i$ for $i = 1, 2, ..., n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (Cancellation Law for Vector Addition). *If x, y, and z are vectors in a vector space $\mathbf{V}$ such that $x + z = y + z$ then $x = y$.*

The vector 0 in (VS 3) is called the **zero vector** of $\mathbf{V}$, and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space $\mathbf{V}$, the following statements are true:*

(a) $0x = 0$ for each $x \in \mathbf{V}$.

(b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in \mathbf{V}$.

(c) $a0 = 0$ for each $a \in F$.

Exercises

1. Label the following statements as true or false.

(a) Every vector space contains a zero vector - **True**.

(b) A vector space may have more than one zero vector - **False**.

(c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.

(d) In any vector space, $ax = ay$ implies that $x = y$ - **True**.

(e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$ - **True**.

(f) An $m \times n$ matrix has m columns and n rows - **False**.

(g) In $P(F)$, only polynomials of the same degree may be added - **False**.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n - **False**.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.

(j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if the have the same value at each element of S - **True**.

2. Write the zero vector of $M_{3 \times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$