

1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y , $x + y = y + x$.
2. For all vectors x , y , and z , $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x .
4. For each vector x there is a vector y such that $x + y = 0$.
5. For each vector x , $1x = x$.
6. For each pair of real numbers a and b and each vector x , $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y , $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x , $(a + b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. A *vector space* (or *linear space*) \mathbf{V} over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x , y , in \mathbf{V} there is a unique element $x + y$ in \mathbf{V} , and for each element a in F and each element x in \mathbf{V} there is a unique element ax in \mathbf{V} , such that the following conditions hold:

- (VS 1) For all x , y in \mathbf{V} , $x + y = y + x$ (commutativity of addition).
- (VS 2) For all x , y , z in \mathbf{V} , $(x + y) + z = x + (y + z)$ (associativity of addition).
- (VS 3) There exists an element in \mathbf{V} denoted by 0 such that $x + 0 = x$ for each x in \mathbf{V} .
- (VS 4) For each element x in \mathbf{V} there exists an element y in \mathbf{V} such that $x + y = 0$.
- (VS 5) For each element x in \mathbf{V} , $1x = x$.
- (VS 6) For each pair of elements a , b in F and each element x in \mathbf{V} , $(ab)x = a(bx)$.
- (VS 7) For each element a in F and each pair of elements x , y in \mathbf{V} , $a(x + y) = ax + ay$.
- (VS 8) For each pair of elements a , b in F and each element x in \mathbf{V} , $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x , respectively.

The elements of the field F are called **scalars** and the elements of the vector space \mathbf{V} are called **vectors**.

An object of the form (a_1, a_2, \dots, a_n) where the entries a_1, a_2, \dots, a_n are elements of a field F , is called an **n -tuple** with entries from F . The elements a_1, a_2, \dots, a_n are called the **entries** or **components** of the n -tuple. Two n -tuples (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are called **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F . We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O .

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k , called the **coefficient** of x^k , is in F . If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1 ; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (Cancellation Law for Vector Addition). *If x , y , and z are vectors in a vector space \mathbf{V} such that $x + z = y + z$ then $x = y$.*

The vector 0 in (VS 3) is called the **zero vector** of \mathbf{V} , and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space \mathbf{V} , the following statements are true:*

- (a) $0x = 0$ for each $x \in \mathbf{V}$.
- (b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in \mathbf{V}$.
- (c) $a0 = 0$ for each $a \in F$.

Exercises

1. Label the following statements as true or false.

- (a) Every vector space contains a zero vector - **True**.
It is included in the definition of a vector space (VS 3).
- (b) A vector space may have more than one zero vector - **False**.
Suppose there were two such vectors, x and y , and one nonzero vector z . Then $x+z = z = y+z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.
- (c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.
Consider $x = 0$ but $a \neq b$.
- (d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.
Consider $a = 0$ but $x \neq y$.
- (e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$ - **True**.
- (f) An $m \times n$ matrix has m columns and n rows - **False**.
An $m \times n$ matrix has m rows and n columns.
- (g) In $P(F)$, only polynomials of the same degree may be added - **False**.
Not true based on the definition of addition in $P(F)$.
- (h) If f and g are polynomials of degree n , then $f + g$ is a polynomial of degree n - **False**.
Consider x and $-x$.
- (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.
Follows from definition of scalar multiplication in $P(F)$.
- (j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.
If a is a nonzero scalar, it can be expressed as ax^0 .
- (k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S - **True**.
By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in S .

2. Write the zero vector of $M_{3 \times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. In any vector space V , show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.

$$(a+b)(x+y) = (a+b)x + (a+b)y = ax + bx + ay + by.$$

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Corollary. *The vector 0 described in (VS 3) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + z = y + z = z$. Then $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$. \square

Corollary. *The vector y described in (VS 4) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + y = x + z = 0$. Then $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$. \square

11. Let $\mathbf{V} = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ nad $c0 = 0$ for each scalar c in F . Prove that \mathbf{V} is a vector space over F . (\mathbf{V} is called the zero vector space.)

Proof. For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x + y = 0 + 0 = y + x$ (VS 1)
2. $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$ (VS 2)
3. $x + 0 = 0 + 0 = 0 = x$ (VS 3)
4. $x + y = 0 + 0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore \mathbf{V} satisfies all conditions necessary for it to be a vector space. \square

13. Let \mathbf{V} denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of \mathbf{V} and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is \mathbf{V} a vector space over R with these operations?

Proof. Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so \mathbf{V} is not a vector space over R . □

14. Let $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in C \text{ for } i = 1, 2, \dots, n\}$; so \mathbf{V} is a vector space over C by Example 1. Is \mathbf{V} a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Proof. Notice that any number $x \in R$ can be expressed as $x + 0i$ in C , so if \mathbf{V} is a vector space over C , it is also a vector space over R . □

15. Let $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in R \text{ for } i = 1, 2, \dots, n\}$; so \mathbf{V} is a vector space over R by Example 1. Is \mathbf{V} a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?

Proof. Consider $c = x + yi$ and $a = (a_1)$ with $y, a_1 \neq 0$. Then $ca = (x + yi)(a_1) = (xa_1 + ya_1i)$, so the entries of ca aren't in R , so \mathbf{V} is not a vector space over C . □

17. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define the addition of elements of \mathbf{V} coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathbf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is \mathbf{V} a vector space over F with these operations?

Proof. Consider $(a_1, a_2) \in \mathbf{V}$ with $a_2 \neq 0$. Then $1(a_1, a_2) = (a_1, 0)$, so $1(a_1, a_2) \neq (a_1, a_2)$, therefore \mathbf{V} is not a vector space over F . □

18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is \mathbf{V} a vector space over R with these operations?

Proof. Consider $(a_1, a_2) = (1, 1)$ and $(b_1, b_2) = (2, 2)$. Then $(a_1, a_2) + (b_1, b_2) = (5, 7)$ and $(b_1, b_2) + (a_1, a_2) = (4, 5)$, so $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$, therefore \mathbf{V} is not a vector space over R . □

21. Let \mathbf{V} and \mathbf{W} be vector spaces over a field F . Let

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

Prove that \mathbf{Z} is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Proof. For $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$ and $a, b \in F$

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ (VS 1)
2. $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$ (VS 2)
3. $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$ (VS 3)
4. $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$ (VS 4)
5. $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$ (VS 5)
6. $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$ (VS 6)
7. $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$ (VS 7)
8. $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw_1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$ (VS 8)

□

3 Subspaces

Definition. A subset \mathbf{W} of a vector space \mathbf{V} over a field F is called a **subspace** of \mathbf{V} if \mathbf{W} is a vector space over F with the operations of addition and scalar multiplication defined on \mathbf{V} .

Theorem 3. Let \mathbf{V} be a vector space and \mathbf{W} a subset of \mathbf{V} . Then \mathbf{W} is a subspace of \mathbf{V} if and only if the following three conditions hold for the operations defined in \mathbf{V} .

1. $0 \in \mathbf{W}$.
2. $x + y \in \mathbf{W}$ whenever $x \in \mathbf{W}$ and $y \in \mathbf{W}$.
3. $cx \in \mathbf{W}$ whenever $c \in F$ and $x \in \mathbf{W}$.

The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is $(A^t)_{ij} = A_{ji}$.

A **symmetric** matrix is a matrix A such that $A^t = A$.

An $m \times n$ matrix A is called **upper triangular** if all its entries lying below the diagonal entries are zero, that is, $A_{ij} = 0$ whenever $i > j$. An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero.

The **trace** of an $n \times n$ matrix M , denoted $\text{tr}(M)$, is the sum of diagonal entries of M ; that is

$$\text{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$$

Theorem 4. Any intersection of subspaces of a vector space \mathbf{V} is a subspace of \mathbf{V} .

Definition. If S_1 and S_2 are nonempty subsets of a vector space V , then the **sum** of S_1 and S_2 , denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.

Definition. A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = 0$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Exercises

1. Label the following statements as true or false.

- (a) If \mathbf{V} is a vector space and \mathbf{W} is a subset of \mathbf{V} that is a vector space, then \mathbf{W} is a subspace of \mathbf{V} - **True**.

This is the definition of a subspace.

- (b) The empty set is a subspace of every vector space - **False**.

The empty set does not contain 0, which is necessary for it to be a subspace.

- (c) If \mathbf{V} is a vector space other than the zero vector space, then \mathbf{V} contains a subspace \mathbf{W} such that $\mathbf{W} \neq \mathbf{V}$ - **True**.

The zero subspace fulfills this condition.

- (d) The intersection of any two subsets of \mathbf{V} is a subspace of \mathbf{V} - **False**.

If neither subset contains 0, then their intersection can not be a subspace.

- (e) An $n \times n$ diagonal matrix can never have more than n nonzero entries - **True**.

All non-diagonal entries of a diagonal matrix are always 0, and an $n \times n$ matrix has n diagonal entries.

- (f) The trace of a square matrix is the product of its diagonal entries - **False**.

The trace of a square matrix is the sum of its diagonal entries.

- (g) Let \mathbf{W} be the xy -plane in R^3 ; that is, $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathbf{W} = R^2$ - **False**.

$R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$, so $\mathbf{W} \neq R^2$.

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathbf{M}_{n \times n}(F)$ and any $a, b \in F$.

Proof.

$$\begin{aligned}(aA^t + bB^t)_{ij} &= aA^t_{ij} + bB^t_{ij} = aA_{ji} + bB_{ji} = (aA + bB)_{ji} = (aA + bB)^t_{ij}, \\ \text{so } (aA^t + bB^t) &= (aA + bB)^t.\end{aligned}$$

□

4. Prove that $(A^t)^t = A$ for each $A \in \mathbf{M}_{m \times n}(F)$.

Proof.

$$(A^t)^t_{ij} = A^t_{ji} = A_{ij} \text{ so } (A^t)^t = A.$$

□

5. Prove that $A + A^t$ is symmetric for any square matrix A .

Proof. Let A be an $n \times n$ square matrix. Then

$$A_{ij} + A^t_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = A_{ji} + A^t_{ji},$$

so $(A + A^t)_{ij} = (A + A^t)_{ji} = (A + A^t)^t_{ij}$, therefore $A + A^t$ is symmetric. \square

6. Prove that $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$ for any $A, B \in \mathbf{M}_{n \times n}(F)$.

Proof. For any $i \leq n$, $(aA + bB)_{ii} = (aA)_{ii} + (bB)_{ii} = a(A_{ii}) + b(B_{ii})$, so $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$. \square

7. Prove that diagonal matrices are symmetric matrices.

Proof. Let A be an $n \times n$ diagonal matrix. Then for any $i, j \leq n$ such that $i \neq j$, $A_{ij} = 0 = A_{ji} = A^t_{ij}$. If $i = j$, then $A_{ij} = A_{ji} = A^t_{ij}$, so the matrix is symmetric. \square

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3 . Justify your answers

(a) $W_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$.

0 is in W_1 . Let $x = (3x_2, x_2, -x_2)$ and $y = (3y_2, y_2, -y_2)$. Then $x + y = (3x_2 + 3y_2, x_2 + y_2, -x_2 - y_2) = (3(x_2 + y_2), x_2 + y_2, -(x_2 + y_2))$, so $x + y \in W_1$. Finally let $c \in R$. Then $cx = (3cx_2, cx_2, -cx_2)$, so $cx \in W_1$, therefore W_1 is a subspace of R^3 .

(b) $W_2 = \{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\}$.

$0 \notin W_2$, so W_2 is not a subspace of R^3 .

(c) $W_3 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0\}$

$0 \in W_3$. Let $x = (x_1, x_2, 7x_2 - 2x_1)$ and $y = (y_1, y_2, 7y_2 - 2y_1)$. Then $x + y = (x_1 + y_1, x_2 + y_2, 7(x_2 + y_2) - 2(x_1 + y_1))$, so $x + y \in W_3$. Finally, let $c \in R$. Then $cx = (cx_1, cx_2, c(7x_2 - 2x_1)) = (cx_1, cx_2, 7cx_2 - 2cx_1) \in W_3$, so W_3 is a subspace of R^3 .

(d) $W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 - 4a_2 - a_3 = 0\}$

$0 \in W_4$. Let $x = (4x_2 + x_3, x_2, x_3)$ and $y = (4y_2 + y_3, y_2, y_3)$. Then $x + y = (4x_2 + x_3 + 4y_2 + y_3, x_2 + y_2, x_3 + y_3) = (4(x_2 + y_2) + (x_3 + y_3), x_2 + y_2, x_3 + y_3) \in W_4$. Finally let $c \in R$. Then $cx = (c(4x_2 + x_3), cx_2, cx_3) = (4cx_2 + cx_3, cx_2, cx_3) \in W_4$, so W_4 is a subspace of R^3 .

(e) $W_5 = \{(a_1, a_2, a_3) \in R^3 : a_1 + 2a_2 - 3a_3 = 1\}$

$0 \notin W_5$, so W_5 is not a subspace of R^3 .

(f) $W_6 = \{(a_1, a_2, a_3) \in R^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Let $x = \left(\sqrt{\frac{3}{5}x_2^2 - \frac{6}{5}x_3^2}, x_2, x_3\right)$ and $c \in R$. Then $cx = \left(c\sqrt{\frac{3}{5}x_2^2 - \frac{6}{5}x_3^2}, cx_2, cx_3\right) = \left(\sqrt{\frac{3}{5}c^2x_2^2 - \frac{6}{5}c^2x_3^2}, cx_2, cx_3\right) \notin W_6$, so W_6 is not a subspace of R^3 .

9. Let W_1, W_3 and W_4 be as in exercise 8. Describe $W_1 \cap W_3$, $W_1 \cap W_4$, and $W_3 \cap W_4$, and observe that each is a subspace of R^3 .

Proof. $W_1 \cap W_3 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2, a_2 = \frac{1}{4}a_1, \text{ and } a_3 = 7a_2 - 2a_1\}$. We have $a_1 = 3a_2$ and $4a_2 = a_1$, so $a_1 = a_2 = 0$ always holds. Then $a_3 = 7a_2 - 2a_1 = 0$, so $W_1 \cap W_3$ is the zero subspace of R^3 .

$W_1 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 4a_2 - a_3, a_2 = \frac{1}{3}a_1, \text{ and } a_3 = -a_2\}$. We have $a_2 = \frac{1}{3}a_1 = \frac{4}{3}a_2 - \frac{1}{3}a_3$, so $0 = \frac{1}{3}(a_2 - a_3)$, and thus $a_3 = a_2$. Since $a_3 = a_2$ and $a_3 = -a_2$, $a_3 = 0 = a_2$, and then $a_1 = 4a_2 - a_3 = 0$, so $W_1 \cap W_4$ is the zero subspace of R^3 .

$W_3 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0 \text{ and } a_1 - 4a_2 - a_3 = 0\}$. We then have $a_1 = \frac{11}{3}a_2$ and $a_3 = 7a_2 - 2a_1 = 7a_2 - \frac{22}{3}a_2 = -\frac{1}{3}a_2$. Then $0 \in W_3 \cap W_4$. Let $x = (\frac{11}{2}x_2, x_2, -\frac{1}{3}x_2)$ and $y = (\frac{11}{3}y_2, y_2, -\frac{1}{3}y_2)$. Then $x + y = (\frac{11}{3}(x_2 + y_2), x_2 + y_2, -\frac{1}{3}(x_2 + y_2)) \in W_3 \cap W_4$. Let $c \in R$. Then $cx = (\frac{11}{3}cx_2, cx_2, -\frac{1}{3}cx_2) \in W_3 \cap W_4$, so $W_3 \cap W_4$ is a subspace of R^3 . \square

11. Is the set $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Proof. Let $f(x) = x^2 + x$ and $g(x) = -x^2 + x$. Then $n = 2$, and $f(x) + g(x) = 2x$, so W is not closed under addition, and therefore it is not a subspace of $P(F)$. \square

12. Prove that the set of $m \times n$ upper triangular matrices is a subspace of $M_{m \times n}(F)$.

Proof. Let W be the set of $m \times n$ upper triangular matrices. Then $0 \in W$. Let $x, y \in W$. Then $x + y \in W$, since for all i, j such that $i > j$, $(x + y)_{ij} = 0 + 0 = 0$. Let $c \in F$. Then $cx \in W$, since for all i, j such that $i > j$, $(cx)_{ij} = c \cdot 0 = 0$, so W is a subspace of $M_{m \times n}(F)$. \square

17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

Proof. Suppose that W is a subset of a vector space V . If W is a subspace of V , then it must contain 0 , so $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$ is true by theorem 1.3. For the converse, suppose that $W \neq \emptyset$, and for $a \in F$ and $x, y \in W$, $x + y \in W$ and $ax \in W$. Then for $a = 0$, $ax = 0 \in W$, so W satisfies all conditions necessary for it to be a subspace of V . \square

18. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Proof. Let W be a subset of a vector space V . If W is a subspace of V , then it has to contain 0 , and for $a \in F$ and $x, y \in W$, $ax \in W$ and $x + y \in W$. If $z = ax$, then $z + y \in W$ has to be true.

For the converse, suppose that $0 \in W$ and for $a \in F$ and $x, y \in W$, $ax + y \in W$. If $y = 0$, then $ax + y = ax \in W$. Since $ax \in W$, let $z = ax$, so $z + y \in W$, which means that W is a subspace of V . \square

20. Prove that if W is a subspace of a vector space V and w_1, w_2, \dots, w_n are in W , then $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$ for any scalars a_1, a_2, \dots, a_n .

Proof. Let $n = 1$. Then $a_1w_1 \in W$ by the definition of a subspace. Next, suppose that for some $n \geq 2$, $a_1w_1 + a_2w_2 + \dots + a_{n-1}w_{n-1} \in W$. Then

$$a_1w_1 + a_2w_2 + \dots + a_nw_n = (a_1w_1 + a_2w_2 + \dots + a_{n-1}w_{n-1}) + a_nw_n.$$

Since $(a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1}) \in W$ and $a_nw_n \in W$ and subspaces are closed under addition, $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$. \square

23. Let W_1 and W_2 be subspaces of a vector space V .

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2 .

Assume, without loss of generality that x is a vector in W_1 . Then, since W_2 contains 0, $x + 0 = x \in W_1 + W_2$.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Let W_3 be a subspace of V , such that it contains both W_1 and W_2 . Since subspaces are closed under addition, for any $x \in W_1$ and $y \in W_2$, $x + y \in W_3$, so W_3 must contain $W_1 + W_2$.

24. Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

and

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Proof. Let $x = (x_1, x_2, \dots, x_n) \in W_1$ and $y = (y_1, y_2, \dots, y_n) \in W_2$. Then

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (x_1 + 0, x_2 + 0, \dots, x_{n-1} + 0, 0 + y_n) = (x_1, x_2, \dots, y_n),$$

where $x_1, x_2, \dots, x_{n-1}, y_n \in F$, so $F^n = W_1 + W_2$.

Next,

$$W_1 \cap W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \cdots = a_n = 0\} = \{0\},$$

so $F^n = W_1 \oplus W_2$. \square

27. Let V denote the vector space of all upper triangular $n \times n$ matrices, and let W_1 denote the subspace of V consisting of all diagonal matrices. Define

$$W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j\}.$$

Show that $V = W_1 \oplus W_2$.

Proof. Let $A \in W_1$ and $B \in W_2$. Then

$$A + B = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ 0 & 0 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & b_{12} & \cdots & b_{1n} \\ 0 & a_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},$$

so $V = W_1 + W_2$.

Next,

$$W_1 \cap W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j \text{ or } i \neq j\} = \{0\},$$

so $V = W_1 \oplus W_2$. \square

28. A matrix M is called skew-symmetric if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n \times n}(F)$. Now assume that F is not of characteristic two, and let W_2 be the subspace of $M_{n \times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n \times n}(F) = W_1 \oplus W_2$.

Proof. $0 \in W_1$, since $0^t = 0 = -0$. Let $A, B \in W_1$. Then

$$(A + B)^t = A^t + B^t = -A + (-B) = -(A + B),$$

so $A + B \in W_1$. Now let $a \in F$. Then $(aA)^t = aA^t = -aA$, so $aA \in W_1$, and thus W_1 is a subspace of $M_{n \times n}(F)$.

Let A be any square $n \times n$ matrix. Then if B is an $n \times n$ square matrix, such that $B_{ij} = A_{ij}$ whenever $i = j$, $B_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij})$ whenever $i < j$ and $B_{ij} = B_{ji}$ whenever $i > j$, then B is a symmetric matrix, so $B \in W_2$, and if C is an $n \times n$ square matrix, such that $C_{ij} = 0$ whenever $i = j$, $C_{ij} = \frac{1}{2}(A_{ji} - A_{ij})$ whenever $i > j$ and $C_{ij} = -C_{ji}$ whenever $i < j$, then C is skew-symmetric, so $C \in W_1$. Then:

1. If $i = j$

$$B_{ij} + C_{ij} = A_{ij} + 0 = A_{ij}$$

2. If $i < j$

$$B_{ij} + C_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij}) - \frac{1}{2}(A_{ji} - A_{ij}) = A_{ij}$$

3. If $j > i$

$$B_{ij} + C_{ij} = A_{ji} + \frac{1}{2}(A_{ij} - A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) = A_{ji} + A_{ij} - A_{ji} = A_{ij},$$

so $A = B + C$, and thus $M_{n \times n}(F) = W_1 + W_2$.

$$W_1 \cap W_2 = \{A \in M_{n \times n}(F) : A^t = -A \text{ and } A^t = A\} = \{0\},$$

so $M_{n \times n}(F) = W_1 \oplus W_2$. □

30. Let W_1 and W_2 be subspaces of a vector space V . Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

Proof. Suppose that V is the direct sum of W_1 and W_2 , and that there exist some $x_1, x_1' \in W_1$ and $x_2, x_2' \in W_2$, such that $x_1 + x_2 = x_1' + x_2' \in V$. Then $x_1 - x_1' = x_2' - x_2 \in W_1 \cap W_2 = \{0\}$, so $x_1 - x_1' = 0 = x_2' - x_2$, and $x_1 = x_1'$ and $x_2 = x_2'$.

For the converse, suppose that any vector in V can be represented uniquely as $x_1 + x_2$, for $x_1 \in W_1$ and $x_2 \in W_2$. Then $V = W_1 + W_2$. Let $y \in W_1 \cap W_2$. Then $y = y + 0 = 0 + y$, and since it is unique, $y = 0$, so $W_1 \cap W_2 = \{0\}$. □

4 Linear combinations and systems of linear equations

Definition. Let V be a vector space and S a nonempty subset of V . A vector $v \in V$ is called a **linear combination** of vectors of S if there exist a finite number of vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n in F such that $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$. In this case we also say that v is a linear combination of u_1, u_2, \dots, u_n and call a_1, a_2, \dots, a_n the **coefficients** of the linear combination.

Definition. Let S be a nonempty subset of a vector space V . The **span** of S , denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S . For convenience, we define $\text{span}(\emptyset) = \{0\}$.

Theorem 5. The span of any subset S of a vector space V is a subspace of V that contains S . Moreover, any subspace of V that contains S must also contain the span of S .

Definition. A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V .

Exercises

1. Label the following statements as true or false

- (a) The zero vector is a linear combination of any nonempty set of vectors - **True**.
For any vectors v_1, v_2, \dots, v_n , $0 = 0v_1 + 0v_2 + \dots + 0v_n$.
- (b) The span of \emptyset is \emptyset - **False**.
 $\text{span}(\emptyset)$ is defined to be $\{0\}$.
- (c) If S is a subset of a vector space V , then $\text{span}(S)$ equals the intersection of all subspaces of V that contain S - **True**.
It follows from theorem 1.5.
- (d) In solving a system of linear equations, it is permissible to multiply an equation by any constant - **False**.
It is permissible to multiply an equation by any nonzero constant.
- (e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another - **True**.
- (f) Every system of linear equations has a solution - **False**.
Any system of linear equations that, while solving, produces a system containing $0 = c$, where c is nonzero, does not have a solution.

7. In F^n , let e_j denote the vector whose j th coordinate is 1 and whose other coordinates are 0. Prove that e_1, e_2, \dots, e_n generates F^n .

Proof. Let a be a vector in F^n , so $a = (a_1, a_2, \dots, a_n)$. Then

$$(a_1, a_2, \dots, a_n) = a_1e_1 + a_2e_2 + \dots + a_ne_n = (a_1, 0, \dots, 0) + (0, a_2, \dots, 0) + \dots + (0, 0, \dots, a_n)$$

□

8. Show that $P_n(F)$ is generated by $\{1, x, \dots, x^n\}$.

Proof. Let a be a vector in $P_n(F)$. Then $a = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \cdot 1$, so $P_n(F)$ is generated by $\{1, x, \dots, x^n\}$ \square

9. Show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

generate $M_{2 \times 2}(F)$.

Proof. Let $A \in M_{2 \times 2}(F)$. Then

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

\square

10. Show that if

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Proof. Let A be a symmetric 2×2 matrix. Then

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = aM_1 + bM_3 + cM_2$$

\square

11. Prove that $\text{span}(\{x\}) = \{ax : a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in R^3 .

Proof. Let x be a vector in some vector space over a field F , and let $y \in \text{span}(\{x\})$. Then y is a linear combination of x , so $y = ax$ for some $a \in F$. Let $z \in \{ax : a \in F\}$. Then $z = ax$, so z is a linear combination of x , so $z \in \text{span}(\{x\})$. $\text{span}(\{x\})$ in R^3 is a line going through x and the origin. \square

12. Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$.

Proof. Let W be a subset of a vector space V , and that $\text{span}(W) = W$. Then $0 \in W$, since $0x = 0$ for any $x \in W$. Let x and y be elements of W . Then $x + y \in \text{span}(W)$, since it's a linear combination of elements of W , and therefore $x + y \in W$. For any scalar c , $cx \in \text{span}(W)$, so $cx \in W$, which means that W is a subspace of V .

Now suppose that W is a subspace of V . Then for vectors $x_1, x_2, \dots, x_n \in W$, and scalars a_1, a_2, \dots, a_n , $a_1 x_1 + a_2 x_2 + \dots + a_n x_n \in W$, so $W = \text{span}(W)$. \square

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$ then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

Proof. Let S_1 and S_2 be subsets of a vector space V , such that $S_1 \subseteq S_2$. Then for all elements $x_1, x_2, \dots, x_n \in S_1$, we know that $x_1, x_2, \dots, x_n \in S_2$, so $\text{span}(S_2)$ contains all linear combinations $a_1x_1, a_2x_2, \dots, a_nx_n$, with a_1, a_2, \dots, a_n being scalars, that is $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Now suppose that $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$. Then $\text{span}(S_1) = V \subseteq \text{span}(S_2)$. Suppose that $\text{span}(S_2) \neq V$, that is, there exists some element $x \in \text{span}(S_2)$ such that $x \notin V$. Then x is a linear combination of elements of S_2 , and since S_2 is a subset of V , x is a linear combination of some elements of V . Since V is a subspace, this implies that $x \in V$, which leads to a contradiction. \square

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V , then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

Proof. Let S_1 and S_2 be arbitrary subsets of a vector space V . For a vector $v \in \text{span}(S_1 \cup S_2)$, we have $v = a_1x_1 + a_2x_2 + \dots + a_nx_n + b_1y_1 + \dots + b_my_m$, with $x_1, x_2, \dots, x_n \in S_1$, $y_1, y_2, \dots, y_m \in S_2$ and scalars a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_m . Then

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \in \text{span}(S_1) \text{ and } b_1y_1 + b_2y_2 + \dots + b_my_m \in \text{span}(S_2),$$

so for $x = a_1x_1 + a_2x_2 + \dots + a_nx_n$ and $y = b_1y_1 + b_2y_2 + \dots + b_my_m$, we have

$$v = x + y, \text{ where } x \in \text{span}(S_1), y \in \text{span}(S_2),$$

so $v \in \text{span}(S_1) + \text{span}(S_2)$. \square

16. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \dots, v_n \in S$ and $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$, then $a_1 = a_2 = \dots = a_n = 0$. Prove that every vector in the span of S can be uniquely written as a linear combination of vectors of S .

Proof. Suppose that for a vector $x \in \text{span}(S)$, x can be written as two different linear combinations of vectors of S , that is, for $x_1, x_2, \dots, x_n \in S$ and for scalars $a_1, b_1, a_2, b_2, \dots, a_n, b_n$,

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b_1x_1 + b_2x_2 + \dots + b_nx_n \text{ and } a_i \neq b_i \text{ for } 1 \leq i \leq n.$$

Then

$$a_1x_1 + a_2x_2 + \dots + a_nx_n - b_1x_1 - b_2x_2 - \dots - b_nx_n = (a_1 - b_1)x_1 + (a_2 - b_2)x_2 + \dots + (a_n - b_n)x_n = 0,$$

so $(a_1 - b_1) = (a_2 - b_2) = \dots = (a_n - b_n) = 0$, so $a_i = b_i$, for $1 \leq i \leq n$. \square