

1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y , $x + y = y + x$.
2. For all vectors x , y , and z , $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x .
4. For each vector x there is a vector y such that $x + y = 0$.
5. For each vector x , $1x = x$.
6. For each pair of real numbers a and b and each vector x , $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y , $a(x+y) = ax+ay$.
8. For each pair of real numbers a and b and each vector x , $(a+b)x = ax+bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. A *vector space* (or *linear space*) \mathbf{V} over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y , in \mathbf{V} there is a unique element $x + y$ in \mathbf{V} , and for each element a in F and each element x in \mathbf{V} there is a unique element ax in \mathbf{V} , such that the following conditions hold:

- (VS 1) For all x, y in \mathbf{V} , $x + y = y + x$ (commutativity of addition).
- (VS 2) For all x, y, z in \mathbf{V} , $(x+y)+z = x+(y+z)$ (associativity of addition).
- (VS 3) There exists an element in \mathbf{V} denoted by 0 such that $x + 0 = x$ for each x in \mathbf{V} .
- (VS 4) For each element x in \mathbf{V} there exists an element y in \mathbf{V} such that $x + y = 0$.
- (VS 5) For each element x in \mathbf{V} , $1x = x$.
- (VS 6) For each pair of elements a, b in F and each element x in \mathbf{V} , $(ab)x = a(bx)$.
- (VS 7) For each element a in F and each pair of elements x, y in \mathbf{V} , $a(x + y) = ax + ay$.
- (VS 8) For each pair of elements a, b in F and each element x in \mathbf{V} , $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x , respectively.

The elements of the field F are called **scalars** and the elements of the vector space \mathbf{V} are called **vectors**.

An object of the form (a_1, a_2, \dots, a_n) where the entries a_1, a_2, \dots, a_n are elements of a field F , is called an **n -tuple** with entries from F . The elements a_1, a_2, \dots, a_n are called the **entries** or **components** of the n -tuple. Two n -tuples (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are called **equal** if $a_i = b_i$ for $i = 1, 2, \dots, n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F . We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O .

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k , called the **coefficient** of x^k , is in F . If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1 ; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (Cancellation Law for Vector Addition). *If x , y , and z are vectors in a vector space \mathbf{V} such that $x + z = y + z$ then $x = y$.*

The vector 0 in (VS 3) is called the **zero vector** of \mathbf{V} , and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space \mathbf{V} , the following statements are true:*

- (a) $0x = 0$ for each $x \in \mathbf{V}$.
- (b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in \mathbf{V}$.
- (c) $a0 = 0$ for each $a \in F$.

Exercises

1. Label the following statements as true or false.

- (a) Every vector space contains a zero vector - **True**.
It is included in the definition of a vector space (VS 3).
- (b) A vector space may have more than one zero vector - **False**.
Suppose there were two such vectors, x and y , and one nonzero vector z .
Then $x + z = z = y + z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.
- (c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.
Consider $x = 0$ but $a \neq b$.
- (d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.
Consider $a = 0$ but $x \neq y$.
- (e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$ - **True**.
- (f) An $m \times n$ matrix has m columns and n rows - **False**.
An $m \times n$ matrix has m rows and n columns.
- (g) In $P(F)$, only polynomials of the same degree may be added - **False**.
Not true based on the definition of addition in $P(F)$.
- (h) If f and g are polynomials of degree n , then $f + g$ is a polynomial of degree n - **False**.
Consider x and $-x$.
- (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.
Follows from definition of scalar multiplication in $P(F)$.
- (j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.
If a is a nonzero scalar, it can be expressed as ax^0 .
- (k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S - **True**.
By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in S .

2. Write the zero vector of $M_{3 \times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$