# 1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors $x$ and $y$, $x + y = y + x$.
2. For all vectors $x$, $y$, and $z$, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted $0$ such that $x + 0 = x$ for each vector $x$.
4. For each vector $x$ there is a vector $y$ such that $x + y = 0$.
5. For each vector $x$, $1x = x$.
6. For each pair of real numbers $a$ and $b$ and each vector $x$, $(ab)x = a(bx)$.
7. For each real number $a$ and each pair of vectors $x$ and $y$, $a(x + y) = ax + ay$.
8. For each pair of real numbers $a$ and $b$ and each vector $x$, $(a + b)x = ax + bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

# 2 Vector Spaces

**Definition.** *A* ***vector space*** *(or* ***linear space****)* $\mathbf{V}$ *over a field $F$ consists of a set on which two operations (called* ***addition*** *and* ***scalar multiplication****, respectively) are defined so that for each pair of elements $x$, $y$, in $\mathbf{V}$ there is a unique element $x + y$ in $\mathbf{V}$, and for each element $a$ in $F$ and each element $x$ in $\mathbf{V}$ there is a unique element $ax$ in $\mathbf{V}$, such that the following conditions hold:*

*(VS 1) For all $x$, $y$ in $\mathbf{V}$, $x + y = y + x$ (commutativity of addition).*

*(VS 2) For all $x$, $y$, $z$ in $\mathbf{V}$, $(x + y) + z = x + (y + z)$ (associativity of addition).*

*(VS 3) There exists an element in $\mathbf{V}$ denoted by $0$ such that $x + 0 = x$ for each $x$ in $\mathbf{V}$.*

*(VS 4) For each element $x$ in $\mathbf{V}$ there exists and element $y$ in $\mathbf{V}$ such that $x + y = 0$.*

*(VS 5) For each element $x$ in $\mathbf{V}$, $1x = x$.*

*(VS 6) For each pair of elements $a$, $b$ in $F$ and each element $x$ in $\mathbf{V}$, $(ab)x = a(bx)$.*

*(VS 7) For each element $a$ in $F$ and each pair of elements $x$, $y$ in $\mathbf{V}$, $a(x + y) = ax + ay$.*

*(VS 8) For each pair of elements $a$, $b$ in $F$ and each element $x$ in $\mathbf{V}$, $(a + b)x = ax + bx$.*

*The elements $x + y$ and $ax$ are called the* ***sum*** *of $x$ and $y$ and the* ***product*** *of $a$ and $x$, respectively.*

The elements of the field $F$ are called **scalars** and the elements of the vector space $\mathbf{V}$ are called **vectors**.

An object of the form $(a_1, a_2, ..., a_n)$ where the entries $a_1, a_2, ..., a_n$ are elements of a field $F$, is called an $n$**-tuple** with entries from $F$. The elements $a_1, a_2, ..., a_n$ are called the **entries** or **components** of the $n$-tuple. Two $n$-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are called **equal** if $a_i = b_i$ for $i = 1, 2, ..., n$.

An $m \times n$ **matrix** with entries from a field $F$ is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry $a_{ij}$ is an element of $F$. We call the entries $a_{ij}$ with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by $O$.

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field $F$ is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where $n$ is a nonnegative integer and each $a_k$, called the **coefficient** of $x^k$, is in $F$. If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be $-1$; otherwise, the **degree** of a polynomial is defined to be the largest exponent of $x$ that appears in the representation with a nonzero coefficient.

**Theorem 1** (**Cancellation Law for Vector Addition**). *If $x$, $y$, and $z$ are vectors in a vector space $\mathbf{V}$ such that $x + z = y + z$ then $x = y$.*

The vector $\mathit{0}$ in (VS 3) is called the **zero vector** of $\mathbf{V}$, and the vector $y$ in (VS 4) is called the **additive inverse** of $x$ and is denoted by $-x$.

**Theorem 2.** *In any vector space $\mathbf{V}$, the following statements are true:*

*(a)* $0x = \mathit{0}$ *for each* $x \in \mathbf{V}$.

*(b)* $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in \mathbf{V}$.

*(c)* $a\mathit{0} = \mathit{0}$ *for each* $a \in F$.

## Exercises

**1. Label the following statements as true or false.**

(a) Every vector space contains a zero vector - **True**.

It is included in the definition of a vector space (VS 3).

(b) A vector space may have more than one zero vector - **False**.

Suppose there were two such vectors, $x$ and $y$, and one nonzero vector $z$. Then $x+z = z = y+z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.

(c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.

Consider $x = \mathit{0}$ but $a \neq b$.

(d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.

Consider $a = 0$ but $x \neq y$.

(e) A vector in $F^n$ may be regarded as a matrix in $M_{n\times 1}(F)$ - **True**.

(f) An $m \times n$ matrix has $m$ columns and $n$ rows - **False**.

An $m \times n$ matrix has $m$ rows and $n$ columns.

(g) In $P(F)$, only polynomials of the same degree may be added - **False**.

Not true based on the definition of addition in $P(F)$.

(h) If $f$ and $g$ are polynomials of degree $n$, then $f + g$ is a polynomial of degree $n$ - **False**.

Consider $x$ and $-x$.

(i) If $f$ is a polynomial of degree $n$ and $c$ is a nonzero scalar, then $cf$ is a polynomial of degree $n$ - **True**.

Follows from definition of scalar multiplication in $P(F)$.

(j) A nonzero scalar of $F$ may be considered to be a polynomial in $P(F)$ having degree zero - **True**.

If $a$ is a nonzero scalar, it can be expressed as $ax^0$.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if the have the same value at each element of $S$ - **True**.

By definition, two functions $f$, $g$ in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each $x$ in F.

**2. Write the zero vector of $M_{3\times 4}(F)$.**

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in$ V and any $a, b \in F$.**

$(a + b)(x + y) = (a + b)x + (a + b)y = ax + bx + ay + by$.

**9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).**

**Corollary.** *The vector 0 described in (VS 3) is unique.*

*Proof.* Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + z = y + z = z$. Then $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$. $\square$

**Corollary.** *The vector y described in (VS 4) is unique.*

*Proof.* Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x + y = x + z = 0$. Then $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$. $\square$

**11. Let $\mathbf{V} = \{\mathit{0}\}$ consist of a single vector $\mathit{0}$ and define $\mathit{0} + \mathit{0} = \mathit{0}$ nad $c\mathit{0} = \mathit{0}$ for each scalar $c$ in $F$. Prove that V is a vector space over $F$. (V is called the zero vector space.)**

*Proof.* For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x + y = 0 + 0 = y + x$ (VS 1)
2. $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$ (VS 2)
3. $x + 0 = 0 + 0 = 0 = x$ (VS 3)
4. $x + y = 0 + 0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore $\mathbf{V}$ satisfies all conditions necessary for it to be a vector space. $\square$

**13. Let V denote the set of ordered pairs of real numbers. If $(a_1, a_2)$ and $(b_1, b_2)$ are elements of V and $c \in R$, define**

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

**and**

$$c(a_1, a_2) = (ca_1, a_2).$$

**Is V a a vector space over $R$ with these operations?**

*Proof.* Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so $\mathbf{V}$ is not a vector space over $R$. □

**14. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in C$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over $C$ by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?**

*Proof.* Notice that any number $x \in R$ can be expressed as $x + 0i$ in $C$, so if $\mathbf{V}$ is a vector space over $C$, it is also a vector space over $R$. □

**15. Let $\mathbf{V} = \{(a_1, a_2, \ldots, a_n) : a_i \in R$ for $i = 1, 2, \ldots, n\}$; so V is a vector space over $R$ by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?**

*Proof.* Consider $c = x + yi$ and $a = (a_1)$ with $y, a_1 \neq 0$. Then $ca = (x + yi)(a_1) = (xa_1 + ya_1 i)$, so the entries of ca aren't in $R$, so $\mathbf{V}$ is not a vector space over $C$. □

**17. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$, where $F$ is a field. Define the addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathbf{V}$, define**

$$c(a_1, a_2) = (a_1, 0).$$

**Is V a vector space over $F$ with these operations?**

*Proof.* Consider $(a_1, a_2) \in \mathbf{V}$ with $a_2 \neq 0$. Then $1(a_1, a_2) = (a_1, 0)$, so $1(a_1, a_2) \neq (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over $F$. □

**18. Let $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in R$, define**

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \textbf{ and } c(a_1, a_2) = (ca_1, ca_2).$$

**Is V a vector space over $R$ with these operations?**

*Proof.* Consider $(a_1, a_2) = (1, 1)$ and $(b_1, b_2) = (2, 2)$. Then $(a_1, a_2) + (b_1, b_2) = (5, 7)$ and $(b_1, b_2) + (a_1, a_2) = (4, 5)$, so $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$, therefore $\mathbf{V}$ is not a vector space over $R$. □

**21. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field $F$. Let**

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \textbf{ and } w \in \mathbf{W}\}.$$

**Prove that $\mathbf{Z}$ is a vector space over $F$ with the operations**

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \textbf{ and } c(v_1, w_1) = (cv_1, cw_1).$$

*Proof.* For $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$ and $a, b \in F$

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ (VS 1)
2. $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$ (VS 2)
3. $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$ (VS 3)
4. $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$ (VS 4)
5. $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$ (VS 5)
6. $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$ (VS 6)
7. $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$ (VS 7)
8. $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$ (VS 8)

$\square$

# 3 Subspaces

**Definition.** *A subset $\mathbf{W}$ of a vector space $\mathbf{V}$ over a field $F$ is called a* ***subspace*** *of $\mathbf{V}$ if $\mathbf{W}$ is a vector space over $F$ with the operations of addition and scalar multiplication defined on $\mathbf{V}$.*

**Theorem 3.** *Let $\mathbf{V}$ be a vector space and $\mathbf{W}$ a subset of $\mathbf{V}$. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if the following three conditions hold for the operations defined in $\mathbf{V}$.*

1. $0 \in \mathbf{W}$.
2. $x + y \in \mathbf{W}$ *whenever* $x \in \mathbf{W}$ *and* $y \in \mathbf{W}$.
3. $cx \in \mathbf{W}$ *whenever* $c \in F$ *and* $x \in \mathbf{W}$.

The **transpose** $A^t$ of an $m \times n$ matrix $A$ is the $n \times m$ matrix obtained from $A$ by interchanging the rows with the columns; that is $(A^t)_{ij} = A_{ji}$.

A **symmetric** matrix is a matrix $A$ such that $A^t = A$.

An $m \times n$ matrix $A$ is called **upper triangular** if all its entries lying below the diagonal entries are zero, that is, $A_{ij} = 0$ whenever $i > j$. An $n \times n$ matrix $M$ is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero.

The **trace** of an $n \times n$ matrix $M$, denoted $\mathrm{tr}(M)$, is the sum of diagonal entries of $M$; that is

$$\mathrm{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$$

**Theorem 4.** *Any intersection of subspaces of a vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$.*

## Exercises

**1. Label the following statements as true or false.**

(a) If $\mathbf{V}$ is a vector space and $\mathbf{W}$ is a subset of $\mathbf{V}$ that is a vector space, then $\mathbf{W}$ is a subspace of $\mathbf{V}$ - **True**.

This is the definition of a subspace.

(b) The empty set is a subspace of every vector space - **False**.

The empty set does not contain 0, which is necessary for it to be a subspace.

(c) If $\mathbf{V}$ is a vector space other than the zero vector space, then $\mathbf{V}$ contains a subspace $\mathbf{W}$ such that $\mathbf{W} \neq \mathbf{V}$ - **True**.

The zero subspace fulfills this condition.

(d) The intersection of any two subsets of $\mathbf{V}$ is a subspace of $\mathbf{V}$ - **False**.

If neither subset contains 0, then their intersection can not be a subspace.

(e) An $n \times n$ diagonal matrix can never have more than $n$ nonzero entries - **True**.

All non-diagonal entries of a diagonal matrix are always 0, and an $n \times n$ matrix has $n$ diagonal entries.

(f) The trace of a square matrix is the product of its diagonal entries - **False**.

The trace of a square matrix is the sum of its diagonal entries.

(g) Let $\mathbf{W}$ be the $xy$-plane in $R^3$; that is, $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathbf{W} = R^2$ - **False**.
$R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$, so $\mathbf{W} \neq R^2$.

**3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathbf{M}_{n\times n}(F)$ and any $a, b \in F$.**

*Proof.*

$$(aA^t + bB^t)_{ij} = aA^t{}_{ij} + bB^t{}_{ij} = aA_{ji} + bB_{ji} = (aA + bB)_{ji} = (aA + bB)^t{}_{ij},$$
$$\text{so } (aA^t + bB^t) = (aA + bB)^t.$$

□

**4. Prove that $(A^t)^t = A$ for each $A \in \mathbf{M}_{m\times n}(F)$.**

*Proof.*

$$(A^t)^t{}_{ij} = A^t{}_{ji} = A_{ij} \text{ so } (A^t)^t = A.$$

□

**5. Prove that $A + A^t$ is symmetric for any square matrix $A$.**

*Proof.* Let $A$ be an $n \times n$ square matrix. Then

$$A_{ij} + A^t{}_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = A_{ji} + A^t{}_{ji},$$

so $(A + A^t)_{ij} = (A + A^t)_{ji} = (A + A^t)^t{}_{ij}$, therefore $A + A^t$ is symmetric. □

**6. Prove that $\mathbf{tr}(aA + bB) = a\mathbf{tr}(A) + b\mathbf{tr}(B)$ for any $A, B \in \mathbf{M}_{n\times n}(F)$.**

*Proof.* For any $i \leq n$, $(aA + bB)_{ii} = (aA)_{ii} + (bB)_{ii} = a(A_{ii}) + b(B_{ii})$, so $\mathrm{tr}(aA + bB) = a\mathrm{tr}(A) + b\mathrm{tr}(B)$. $\square$

**7. Prove that diagonal matrices are symmetric matrices.**

*Proof.* Let $A$ be an $n \times n$ diagonal matrix. Then for any $i, j \leq n$ such that $i \neq j$, $A_{ij} = 0 = A_{ji} = {A^t}_{ij}$. If $i = j$, then $A_{ij} = A_{ji} = {A^t}_{ij}$, so the matrix is symmetric. $\square$