1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted $\mathit{0}$ such that $x + \mathit{0} = x$ for each vector x.
4. For each vector x there is a vector y such that $x + y = \mathit{0}$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y, $a(x+y) = ax+ay$.
8. For each pair of real numbers a and b and each vector x, $(a+b)x = ax+bx$.

Any mathematical structure possessing these eight properties is called a *vector space*.

2 Vector Spaces

Definition. *A **vector space** (or **linear space**)* $\mathbf{V}$ *over a field* F *consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements* x, y, *in* $\mathbf{V}$ *there is a unique element* $x + y$ *in* $\mathbf{V}$, *and for each element* a *in* F *and each element* x *in* $\mathbf{V}$ *there is a unique element* ax *in* $\mathbf{V}$, *such that the following conditions hold:*

(VS 1) For all x, y *in* $\mathbf{V}$, $x + y = y + x$ *(commutativity of addition).*

(VS 2) For all x, y, z *in* $\mathbf{V}$, $(x+y)+z = x+(y+z)$ *(associativity of addition).*

(VS 3) There exists an element in $\mathbf{V}$ *denoted by* $\mathit{0}$ *such that* $x + \mathit{0} = x$ *for each* x *in* $\mathbf{V}$.

(VS 4) For each element x *in* $\mathbf{V}$ *there exists and element* y *in* $\mathbf{V}$ *such that* $x + y = \mathit{0}$.

(VS 5) For each element x *in* $\mathbf{V}$, $1x = x$.

(VS 6) For each pair of elements a, b *in* F *and each element* x *in* $\mathbf{V}$, $(ab)x = a(bx)$.

(VS 7) For each element a *in* F *and each pair of elements* x, y *in* $\mathbf{V}$, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b *in* F *and each element* x *in* $\mathbf{V}$, $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the ***sum*** *of x and y and the* ***product*** *of a and x, respectively.*

The elements of the field F are called **scalars** and the elements of the vector space $\mathbf{V}$ are called **vectors**.

An object of the form $(a_1, a_2, ..., a_n)$ where the entries $a_1, a_2, ..., a_n$ are elements of a field F, is called an n**-tuple** with entries from F. The elements $a_1, a_2, ..., a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are called **equal** if $a_i = b_i$ for $i = 1, 2, ..., n$.

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix.

The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation with a nonzero coefficient.

Theorem 1 (**Cancellation Law for Vector Addition**)**.** *If x, y, and z are vectors in a vector space $\mathbf{V}$ such that $x + z = y + z$ then $x = y$.*

The vector *0* in (VS 3) is called the **zero vector** of $\mathbf{V}$, and the vector y in (VS 4) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 2. *In any vector space $\mathbf{V}$, the following statements are true:*

(a) $0x = 0$ *for each* $x \in \mathbf{V}$.

(b) $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in \mathbf{V}$.

(c) $a0 = 0$ *for each* $a \in F$.

Exercises

1. Label the following statements as true or false.

(a) Every vector space contains a zero vector - **True**.

It is included in the definition of a vector space (VS 3).

(b) A vector space may have more than one zero vector - **False**.

Suppose there were two such vectors, x and y, and one nonzero vector z. Then $x + z = z = y + z$, and $x + (z + (-z)) = x = y + (z + (-z)) = y$.

(c) In any vector space, $ax = bx$ implies that $a = b$ - **False**.

Consider $x = \mathit{0}$ but $a \neq b$.

(d) In any vector space, $ax = ay$ implies that $x = y$ - **False**.

Consider $a = 0$ but $x \neq y$.

(e) A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$ - **True**.

(f) An $m \times n$ matrix has m columns and n rows - **False**.

An $m \times n$ matrix has m rows and n columns.

(g) In $P(F)$, only polynomials of the same degree may be added - **False**.

Not true based on the definition of addition in $P(F)$.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n - **False**.

Consider x and $-x$.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n - **True**.

Follows from definition of scalar multiplication in $P(F)$.

(j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero - **True**.

If a is a nonzero scalar, it can be expressed as ax^0.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if the have the same value at each element of S - **True**.

By definition, two functions f, g in $\mathcal{F}(S, F)$ are equal when $f(x) = g(x)$ for each x in F.

2. Write the zero vector of $M_{3\times 4}(F)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. In any vector space V, show that $(a+b)(x+y) = ax+ay+bx+by$ for any $x, y \in \mathbf{V}$ and any $a, b \in F$.

$(a+b)(x+y) = (a+b)x + (a+b)y = ax + bx + ay + by.$

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Corollary. *The vector 0 described in (VS 3) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x+z = y+z = z$. Then $x = x+0 = x+(z+(-z)) = (x+z)+(-z) = (y+z)+(-z) = y+(z+(-z)) = y+0 = y$. □

Corollary. *The vector y described in (VS 4) is unique.*

Proof. Suppose that there are vectors $x, y, z \in \mathbf{V}$ such that $x+y = x+z = 0$. Then $y = 0+y = x+(-x)+y = (x+y)+(-x) = (x+z)+(-x) = x+(-x)+z = 0+z = z$. □

11. Let $\mathbf{V} = \{\mathit{0}\}$ consist of a single vector $\mathit{0}$ and define $\mathit{0} + \mathit{0} = \mathit{0}$ nad $c\mathit{0} = \mathit{0}$ for each scalar c in F. Prove that V is a vector space over F. (V is called the zero vector space.)

Proof. For any $x, y, z \in \mathbf{V}$ and $a, b \in F$:

1. $x+y = 0+0 = y+x$ (VS 1)
2. $(x+y)+z = (0+0)+0 = 0+(0+0) = x+(y+z)$ (VS 2)
3. $x+0 = 0+0 = 0 = x$ (VS 3)
4. $x+y = 0+0 = 0$ (VS 4)
5. $1x = 1 \times 0 = 0 = x$ (VS 5)
6. $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$ (VS 6)
7. $a(x+y) = a(0+0) = 0+0 = a \times 0 + a \times 0 = ax + ay$ (VS 7)
8. $(a+b)x = (a+b) \times 0 = 0 = 0+0 = a \times 0 + b \times 0 = ax + bx$ (VS 8)

Therefore $\mathbf{V}$ satisfies all conditions necessary for it to be a vector space. □

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is V a a vector space over R with these operations?

Proof. Let $(x_1, x_2) \in \mathbf{V}$ and $a, b \in R$. Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

and

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so $\mathbf{V}$ is not a vector space over R. □