

# 1 Introduction

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors  $x$  and  $y$ ,  $x + y = y + x$ .
2. For all vectors  $x$ ,  $y$ , and  $z$ ,  $(x + y) + z = x + (y + z)$ .
3. There exists a vector denoted  $0$  such that  $x + 0 = x$  for each vector  $x$ .
4. For each vector  $x$  there is a vector  $y$  such that  $x + y = 0$ .
5. For each vector  $x$ ,  $1x = x$ .
6. For each pair of real numbers  $a$  and  $b$  and each vector  $x$ ,  $(ab)x = a(bx)$ .
7. For each real number  $a$  and each pair of vectors  $x$  and  $y$ ,  $a(x + y) = ax + ay$ .
8. For each pair of real numbers  $a$  and  $b$  and each vector  $x$ ,  $(a + b)x = ax + bx$ .

Any mathematical structure possessing these eight properties is called a *vector space*.

## 2 Vector Spaces

**Definition.** A *vector space* (or *linear space*)  $\mathbf{V}$  over a field  $F$  consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements  $x$ ,  $y$ , in  $\mathbf{V}$  there is a unique element  $x + y$  in  $\mathbf{V}$ , and for each element  $a$  in  $F$  and each element  $x$  in  $\mathbf{V}$  there is a unique element  $ax$  in  $\mathbf{V}$ , such that the following conditions hold:

- (VS 1) For all  $x$ ,  $y$  in  $\mathbf{V}$ ,  $x + y = y + x$  (commutativity of addition).
- (VS 2) For all  $x$ ,  $y$ ,  $z$  in  $\mathbf{V}$ ,  $(x + y) + z = x + (y + z)$  (associativity of addition).
- (VS 3) There exists an element in  $\mathbf{V}$  denoted by  $0$  such that  $x + 0 = x$  for each  $x$  in  $\mathbf{V}$ .
- (VS 4) For each element  $x$  in  $\mathbf{V}$  there exists an element  $y$  in  $\mathbf{V}$  such that  $x + y = 0$ .
- (VS 5) For each element  $x$  in  $\mathbf{V}$ ,  $1x = x$ .
- (VS 6) For each pair of elements  $a$ ,  $b$  in  $F$  and each element  $x$  in  $\mathbf{V}$ ,  $(ab)x = a(bx)$ .
- (VS 7) For each element  $a$  in  $F$  and each pair of elements  $x$ ,  $y$  in  $\mathbf{V}$ ,  $a(x + y) = ax + ay$ .
- (VS 8) For each pair of elements  $a$ ,  $b$  in  $F$  and each element  $x$  in  $\mathbf{V}$ ,  $(a + b)x = ax + bx$ .

The elements  $x + y$  and  $ax$  are called the **sum** of  $x$  and  $y$  and the **product** of  $a$  and  $x$ , respectively.

The elements of the field  $F$  are called **scalars** and the elements of the vector space  $\mathbf{V}$  are called **vectors**.

An object of the form  $(a_1, a_2, \dots, a_n)$  where the entries  $a_1, a_2, \dots, a_n$  are elements of a field  $F$ , is called an  **$n$ -tuple** with entries from  $F$ . The elements  $a_1, a_2, \dots, a_n$  are called the **entries** or **components** of the  $n$ -tuple. Two  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are called **equal** if  $a_i = b_i$  for  $i = 1, 2, \dots, n$ .

An  $m \times n$  **matrix** with entries from a field  $F$  is a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry  $a_{ij}$  is an element of  $F$ . We call the entries  $a_{ij}$  with  $i = j$  the **diagonal entries** of the matrix.

The  $m \times n$  matrix in which each entry equals zero is called the **zero matrix** and is denoted by  $O$ .

If the number of rows and columns of a matrix are equal, the matrix is called **square**.

A **polynomial** with coefficients from a field  $F$  is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where  $n$  is a nonnegative integer and each  $a_k$ , called the **coefficient** of  $x^k$ , is in  $F$ . If  $f(x) = 0$ , then  $f(x)$  is called the **zero polynomial** and, for convenience, its degree is defined to be  $-1$ ; otherwise, the **degree** of a polynomial is defined to be the largest exponent of  $x$  that appears in the representation with a nonzero coefficient.

**Theorem 1 (Cancellation Law for Vector Addition).** *If  $x$ ,  $y$ , and  $z$  are vectors in a vector space  $\mathbf{V}$  such that  $x + z = y + z$  then  $x = y$ .*

The vector  $0$  in (VS 3) is called the **zero vector** of  $\mathbf{V}$ , and the vector  $y$  in (VS 4) is called the **additive inverse** of  $x$  and is denoted by  $-x$ .

**Theorem 2.** *In any vector space  $\mathbf{V}$ , the following statements are true:*

- (a)  $0x = 0$  for each  $x \in \mathbf{V}$ .
- (b)  $(-a)x = -(ax) = a(-x)$  for each  $a \in F$  and each  $x \in \mathbf{V}$ .
- (c)  $a0 = 0$  for each  $a \in F$ .

## Exercises

### 1. Label the following statements as true or false.

- (a) Every vector space contains a zero vector - **True**.  
It is included in the definition of a vector space (VS 3).
- (b) A vector space may have more than one zero vector - **False**.  
Suppose there were two such vectors,  $x$  and  $y$ , and one nonzero vector  $z$ . Then  $x+z = z = y+z$ , and  $x + (z + (-z)) = x = y + (z + (-z)) = y$ .
- (c) In any vector space,  $ax = bx$  implies that  $a = b$  - **False**.  
Consider  $x = 0$  but  $a \neq b$ .
- (d) In any vector space,  $ax = ay$  implies that  $x = y$  - **False**.  
Consider  $a = 0$  but  $x \neq y$ .
- (e) A vector in  $F^n$  may be regarded as a matrix in  $M_{n \times 1}(F)$  - **True**.
- (f) An  $m \times n$  matrix has  $m$  columns and  $n$  rows - **False**.  
An  $m \times n$  matrix has  $m$  rows and  $n$  columns.
- (g) In  $P(F)$ , only polynomials of the same degree may be added - **False**.  
Not true based on the definition of addition in  $P(F)$ .
- (h) If  $f$  and  $g$  are polynomials of degree  $n$ , then  $f + g$  is a polynomial of degree  $n$  - **False**.  
Consider  $x$  and  $-x$ .
- (i) If  $f$  is a polynomial of degree  $n$  and  $c$  is a nonzero scalar, then  $cf$  is a polynomial of degree  $n$  - **True**.  
Follows from definition of scalar multiplication in  $P(F)$ .
- (j) A nonzero scalar of  $F$  may be considered to be a polynomial in  $P(F)$  having degree zero - **True**.  
If  $a$  is a nonzero scalar, it can be expressed as  $ax^0$ .
- (k) Two functions in  $\mathcal{F}(S, F)$  are equal if and only if they have the same value at each element of  $S$  - **True**.  
By definition, two functions  $f, g$  in  $\mathcal{F}(S, F)$  are equal when  $f(x) = g(x)$  for each  $x$  in  $S$ .

### 2. Write the zero vector of $M_{3 \times 4}(F)$ .

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### 8. In any vector space $V$ , show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$ .

$$(a+b)(x+y) = (a+b)x + (a+b)y = ax + bx + ay + by.$$

**9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).**

**Corollary.** *The vector  $0$  described in (VS 3) is unique.*

*Proof.* Suppose that there are vectors  $x, y, z \in \mathbf{V}$  such that  $x + z = y + z = z$ . Then  $x = x + 0 = x + (z + (-z)) = (x + z) + (-z) = (y + z) + (-z) = y + (z + (-z)) = y + 0 = y$ .  $\square$

**Corollary.** *The vector  $y$  described in (VS 4) is unique.*

*Proof.* Suppose that there are vectors  $x, y, z \in \mathbf{V}$  such that  $x + y = x + z = 0$ . Then  $y = 0 + y = x + (-x) + y = (x + y) + (-x) = (x + z) + (-x) = x + (-x) + z = 0 + z = z$ .  $\square$

**11. Let  $\mathbf{V} = \{0\}$  consist of a single vector  $0$  and define  $0 + 0 = 0$  nad  $c0 = 0$  for each scalar  $c$  in  $F$ . Prove that  $\mathbf{V}$  is a vector space over  $F$ . ( $\mathbf{V}$  is called the zero vector space.)**

*Proof.* For any  $x, y, z \in \mathbf{V}$  and  $a, b \in F$ :

1.  $x + y = 0 + 0 = y + x$  (VS 1)
2.  $(x + y) + z = (0 + 0) + 0 = 0 + (0 + 0) = x + (y + z)$  (VS 2)
3.  $x + 0 = 0 + 0 = 0 = x$  (VS 3)
4.  $x + y = 0 + 0 = 0$  (VS 4)
5.  $1x = 1 \times 0 = 0 = x$  (VS 5)
6.  $(ab)x = (ab) \times 0 = 0 = a(b \times 0) = a(bx)$  (VS 6)
7.  $a(x + y) = a(0 + 0) = 0 + 0 = a \times 0 + a \times 0 = ax + ay$  (VS 7)
8.  $(a + b)x = (a + b) \times 0 = 0 = 0 + 0 = a \times 0 + b \times 0 = ax + bx$  (VS 8)

Therefore  $\mathbf{V}$  satisfies all conditions necessary for it to be a vector space.  $\square$

**13. Let  $\mathbf{V}$  denote the set of ordered pairs of real numbers. If  $(a_1, a_2)$  and  $(b_1, b_2)$  are elements of  $\mathbf{V}$  and  $c \in R$ , define**

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

**and**

$$c(a_1, a_2) = (ca_1, a_2).$$

**Is  $\mathbf{V}$  a vector space over  $R$  with these operations?**

*Proof.* Let  $(x_1, x_2) \in \mathbf{V}$  and  $a, b \in R$ . Then

$$(a + b)(x_1, x_2) = ((a + b)x_1, x_2) = (ax_1 + bx_1, x_2)$$

**and**

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = (ax_1 + bx_1, x_2^2)$$

so

$$(a + b)(x_1, x_2) \neq a(x_1, x_2) + b(x_1, x_2)$$

so  $\mathbf{V}$  is not a vector space over  $R$ . □

**14. Let  $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in C \text{ for } i = 1, 2, \dots, n\}$ ; so  $\mathbf{V}$  is a vector space over  $C$  by Example 1. Is  $\mathbf{V}$  a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?**

*Proof.* Notice that any number  $x \in R$  can be expressed as  $x + 0i$  in  $C$ , so if  $\mathbf{V}$  is a vector space over  $C$ , it is also a vector space over  $R$ . □

**15. Let  $\mathbf{V} = \{(a_1, a_2, \dots, a_n) : a_i \in R \text{ for } i = 1, 2, \dots, n\}$ ; so  $\mathbf{V}$  is a vector space over  $R$  by Example 1. Is  $\mathbf{V}$  a vector space over the field of complex numbers with the operations of coordinatewise addition and scalar multiplication?**

*Proof.* Consider  $c = x + yi$  and  $a = (a_1)$  with  $y, a_1 \neq 0$ . Then  $ca = (x + yi)(a_1) = (xa_1 + ya_1i)$ , so the entries of  $ca$  aren't in  $R$ , so  $\mathbf{V}$  is not a vector space over  $C$ . □

**17. Let  $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in F\}$ , where  $F$  is a field. Define the addition of elements of  $\mathbf{V}$  coordinatewise, and for  $c \in F$  and  $(a_1, a_2) \in \mathbf{V}$ , define**

$$c(a_1, a_2) = (a_1, 0).$$

**Is  $\mathbf{V}$  a vector space over  $F$  with these operations?**

*Proof.* Consider  $(a_1, a_2) \in \mathbf{V}$  with  $a_2 \neq 0$ . Then  $1(a_1, a_2) = (a_1, 0)$ , so  $1(a_1, a_2) \neq (a_1, a_2)$ , therefore  $\mathbf{V}$  is not a vector space over  $F$ . □

**18. Let  $\mathbf{V} = \{(a_1, a_2) : a_1, a_2 \in R\}$ . For  $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$  and  $c \in R$ , define**

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

**Is  $\mathbf{V}$  a vector space over  $R$  with these operations?**

*Proof.* Consider  $(a_1, a_2) = (1, 1)$  and  $(b_1, b_2) = (2, 2)$ . Then  $(a_1, a_2) + (b_1, b_2) = (5, 7)$  and  $(b_1, b_2) + (a_1, a_2) = (4, 5)$ , so  $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$ , therefore  $\mathbf{V}$  is not a vector space over  $R$ . □

**21. Let  $\mathbf{V}$  and  $\mathbf{W}$  be vector spaces over a field  $F$ . Let**

$$\mathbf{Z} = \{(v, w) : v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

**Prove that  $\mathbf{Z}$  is a vector space over  $F$  with the operations**

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

*Proof.* For  $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in \mathbf{Z}$  and  $a, b \in F$

1.  $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$  (VS 1)
2.  $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = (v_1 + v_2 + v_3, w_1 + w_2 + w_3) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$  (VS 2)
3.  $(v_1, w_1) + 0 = (v_1, w_1) + (0, 0) = (v_1 + 0, w_1 + 0) = (v_1, w_1)$  (VS 3)
4.  $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$  (VS 4)
5.  $1(v_1, w_1) = (1v_1, 1w_1) = (v_1, w_1)$  (VS 5)
6.  $(ab)(v_1, w_1) = (abv_1, abw_1) = a(bv_1, bw_1) = a(b(v_1, w_1))$  (VS 6)
7.  $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$  (VS 7)
8.  $(a + b)(v_1, w_1) = ((a + b)v_1, (a + b)w_1) = (av_1 + bv_1, aw_1 + bw_1) = (av_1, aw_1) + (bv_1, bw_1) = a(v_1, w_1) + b(v_1, w_1)$  (VS 8)

□

### 3 Subspaces

**Definition.** A subset  $\mathbf{W}$  of a vector space  $\mathbf{V}$  over a field  $F$  is called a **subspace** of  $\mathbf{V}$  if  $\mathbf{W}$  is a vector space over  $F$  with the operations of addition and scalar multiplication defined on  $\mathbf{V}$ .

**Theorem 3.** Let  $\mathbf{V}$  be a vector space and  $\mathbf{W}$  a subset of  $\mathbf{V}$ . Then  $\mathbf{W}$  is a subspace of  $\mathbf{V}$  if and only if the following three conditions hold for the operations defined in  $\mathbf{V}$ .

1.  $0 \in \mathbf{W}$ .
2.  $x + y \in \mathbf{W}$  whenever  $x \in \mathbf{W}$  and  $y \in \mathbf{W}$ .
3.  $cx \in \mathbf{W}$  whenever  $c \in F$  and  $x \in \mathbf{W}$ .

The **transpose**  $A^t$  of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix obtained from  $A$  by interchanging the rows with the columns; that is  $(A^t)_{ij} = A_{ji}$ .

A **symmetric** matrix is a matrix  $A$  such that  $A^t = A$ .

An  $m \times n$  matrix  $A$  is called **upper triangular** if all its entries lying below the diagonal entries are zero, that is,  $A_{ij} = 0$  whenever  $i > j$ . An  $n \times n$  matrix  $M$  is called a **diagonal matrix** if  $M_{ij} = 0$  whenever  $i \neq j$ , that is, if all its nondiagonal entries are zero.

The **trace** of an  $n \times n$  matrix  $M$ , denoted  $\text{tr}(M)$ , is the sum of diagonal entries of  $M$ ; that is

$$\text{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}$$

**Theorem 4.** Any intersection of subspaces of a vector space  $\mathbf{V}$  is a subspace of  $\mathbf{V}$ .

**Definition.** If  $S_1$  and  $S_2$  are nonempty subsets of a vector space  $V$ , then the **sum** of  $S_1$  and  $S_2$ , denoted  $S_1 + S_2$ , is the set  $\{x + y : x \in S_1 \text{ and } y \in S_2\}$ .

**Definition.** A vector space  $V$  is called the **direct sum** of  $W_1$  and  $W_2$  if  $W_1$  and  $W_2$  are subspaces of  $V$  such that  $W_1 \cap W_2 = 0$  and  $W_1 + W_2 = V$ . We denote that  $V$  is the direct sum of  $W_1$  and  $W_2$  by writing  $V = W_1 \oplus W_2$ .

## Exercises

### 1. Label the following statements as true or false.

- (a) If  $\mathbf{V}$  is a vector space and  $\mathbf{W}$  is a subset of  $\mathbf{V}$  that is a vector space, then  $\mathbf{W}$  is a subspace of  $\mathbf{V}$  - **True**.

This is the definition of a subspace.

- (b) The empty set is a subspace of every vector space - **False**.

The empty set does not contain 0, which is necessary for it to be a subspace.

- (c) If  $\mathbf{V}$  is a vector space other than the zero vector space, then  $\mathbf{V}$  contains a subspace  $\mathbf{W}$  such that  $\mathbf{W} \neq \mathbf{V}$  - **True**.

The zero subspace fulfills this condition.

- (d) The intersection of any two subsets of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  - **False**.

If neither subset contains 0, then their intersection can not be a subspace.

- (e) An  $n \times n$  diagonal matrix can never have more than  $n$  nonzero entries - **True**.

All non-diagonal entries of a diagonal matrix are always 0, and an  $n \times n$  matrix has  $n$  diagonal entries.

- (f) The trace of a square matrix is the product of its diagonal entries - **False**.

The trace of a square matrix is the sum of its diagonal entries.

- (g) Let  $\mathbf{W}$  be the  $xy$ -plane in  $R^3$ ; that is,  $\mathbf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$ . Then  $\mathbf{W} = R^2$  - **False**.

$R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$ , so  $\mathbf{W} \neq R^2$ .

### 3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathbf{M}_{n \times n}(F)$ and any $a, b \in F$ .

*Proof.*

$$\begin{aligned}(aA^t + bB^t)_{ij} &= aA^t_{ij} + bB^t_{ij} = aA_{ji} + bB_{ji} = (aA + bB)_{ji} = (aA + bB)^t_{ij}, \\ \text{so } (aA^t + bB^t) &= (aA + bB)^t.\end{aligned}$$

□

### 4. Prove that $(A^t)^t = A$ for each $A \in \mathbf{M}_{m \times n}(F)$ .

*Proof.*

$$(A^t)^t_{ij} = A^t_{ji} = A_{ij} \text{ so } (A^t)^t = A.$$

□

**5. Prove that  $A + A^t$  is symmetric for any square matrix  $A$ .**

*Proof.* Let  $A$  be an  $n \times n$  square matrix. Then

$$A_{ij} + A^t_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = A_{ji} + A^t_{ji},$$

so  $(A + A^t)_{ij} = (A + A^t)_{ji} = (A + A^t)^t_{ij}$ , therefore  $A + A^t$  is symmetric.  $\square$

**6. Prove that  $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$  for any  $A, B \in \mathbf{M}_{n \times n}(F)$ .**

*Proof.* For any  $i \leq n$ ,  $(aA + bB)_{ii} = (aA)_{ii} + (bB)_{ii} = a(A_{ii}) + b(B_{ii})$ , so  $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$ .  $\square$

**7. Prove that diagonal matrices are symmetric matrices.**

*Proof.* Let  $A$  be an  $n \times n$  diagonal matrix. Then for any  $i, j \leq n$  such that  $i \neq j$ ,  $A_{ij} = 0 = A_{ji} = A^t_{ij}$ . If  $i = j$ , then  $A_{ij} = A_{ji} = A^t_{ij}$ , so the matrix is symmetric.  $\square$

**8. Determine whether the following sets are subspaces of  $R^3$  under the operations of addition and scalar multiplication defined on  $R^3$ . Justify your answers**

(a)  $W_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$ .

$0$  is in  $W_1$ . Let  $x = (3x_2, x_2, -x_2)$  and  $y = (3y_2, y_2, -y_2)$ . Then  $x + y = (3x_2 + 3y_2, x_2 + y_2, -x_2 - y_2) = (3(x_2 + y_2), x_2 + y_2, -(x_2 + y_2))$ , so  $x + y \in W_1$ . Finally let  $c \in R$ . Then  $cx = (3cx_2, cx_2, -cx_2)$ , so  $cx \in W_1$ , therefore  $W_1$  is a subspace of  $R^3$ .

(b)  $W_2 = \{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\}$ .

$0 \notin W_2$ , so  $W_2$  is not a subspace of  $R^3$ .

(c)  $W_3 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0\}$

$0 \in W_3$ . Let  $x = (x_1, x_2, 7x_2 - 2x_1)$  and  $y = (y_1, y_2, 7y_2 - 2y_1)$ . Then  $x + y = (x_1 + y_1, x_2 + y_2, 7(x_2 + y_2) - 2(x_1 + y_1))$ , so  $x + y \in W_3$ . Finally, let  $c \in R$ . Then  $cx = (cx_1, cx_2, c(7x_2 - 2x_1)) = (cx_1, cx_2, 7cx_2 - 2cx_1) \in W_3$ , so  $W_3$  is a subspace of  $R^3$ .

(d)  $W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 - 4a_2 - a_3 = 0\}$

$0 \in W_4$ . Let  $x = (4x_2 + x_3, x_2, x_3)$  and  $y = (4y_2 + y_3, y_2, y_3)$ . Then  $x + y = (4x_2 + x_3 + 4y_2 + y_3, x_2 + y_2, x_3 + y_3) = (4(x_2 + y_2) + (x_3 + y_3), x_2 + y_2, x_3 + y_3) \in W_4$ . Finally let  $c \in R$ . Then  $cx = (c(4x_2 + x_3), cx_2, cx_3) = (4cx_2 + cx_3, cx_2, cx_3) \in W_4$ , so  $W_4$  is a subspace of  $R^3$ .

(e)  $W_5 = \{(a_1, a_2, a_3) \in R^3 : a_1 + 2a_2 - 3a_3 = 1\}$

$0 \notin W_5$ , so  $W_5$  is not a subspace of  $R^3$ .

(f)  $W_6 = \{(a_1, a_2, a_3) \in R^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Let  $x = \left(\sqrt{\frac{3}{5}x_2^2 - \frac{6}{5}x_3^2}, x_2, x_3\right)$  and  $c \in R$ . Then  $cx = \left(c\sqrt{\frac{3}{5}x_2^2 - \frac{6}{5}x_3^2}, cx_2, cx_3\right) = \left(\sqrt{\frac{3}{5}c^2x_2^2 - \frac{6}{5}c^2x_3^2}, cx_2, cx_3\right) \notin W_6$ , so  $W_6$  is not a subspace of  $R^3$ .



**9. Let  $W_1, W_3$  and  $W_4$  be as in exercise 8. Describe  $W_1 \cap W_3$ ,  $W_1 \cap W_4$ , and  $W_3 \cap W_4$ , and observe that each is a subspace of  $R^3$ .**

*Proof.*  $W_1 \cap W_3 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 3a_2, a_2 = \frac{1}{4}a_1, \text{ and } a_3 = 7a_2 - 2a_1\}$ . We have  $a_1 = 3a_2$  and  $4a_2 = a_1$ , so  $a_1 = a_2 = 0$  always holds. Then  $a_3 = 7a_2 - 2a_1 = 0$ , so  $W_1 \cap W_3$  is the zero subspace of  $R^3$ .

$W_1 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : a_1 = 4a_2 - a_3, a_2 = \frac{1}{3}a_1, \text{ and } a_3 = -a_2\}$ . We have  $a_2 = \frac{1}{3}a_1 = \frac{4}{3}a_2 - \frac{1}{3}a_3$ , so  $0 = \frac{1}{3}(a_2 - a_3)$ , and thus  $a_3 = a_2$ . Since  $a_3 = a_2$  and  $a_3 = -a_2$ ,  $a_3 = 0 = a_2$ , and then  $a_1 = 4a_2 - a_3 = 0$ , so  $W_1 \cap W_4$  is the zero subspace of  $R^3$ .

$W_3 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 : 2a_1 - 7a_2 + a_3 = 0 \text{ and } a_1 - 4a_2 - a_3 = 0\}$ . We then have  $a_1 = \frac{11}{3}a_2$  and  $a_3 = 7a_2 - 2a_1 = 7a_2 - \frac{22}{3}a_2 = -\frac{1}{3}a_2$ . Then  $0 \in W_3 \cap W_4$ . Let  $x = (\frac{11}{2}x_2, x_2, -\frac{1}{3}x_2)$  and  $y = (\frac{11}{3}y_2, y_2, -\frac{1}{3}y_2)$ . Then  $x + y = (\frac{11}{3}(x_2 + y_2), x_2 + y_2, -\frac{1}{3}(x_2 + y_2)) \in W_3 \cap W_4$ . Let  $c \in R$ . Then  $cx = (\frac{11}{3}cx_2, cx_2, -\frac{1}{3}cx_2) \in W_3 \cap W_4$ , so  $W_3 \cap W_4$  is a subspace of  $R^3$ .  $\square$

**11. Is the set  $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$  a subspace of  $P(F)$  if  $n \geq 1$ ? Justify your answer.**

*Proof.* Let  $f(x) = x^2 + x$  and  $g(x) = -x^2 + x$ . Then  $n = 2$ , and  $f(x) + g(x) = 2x$ , so  $W$  is not closed under addition, and therefore it is not a subspace of  $P(F)$ .  $\square$

**12. Prove that the set of  $m \times n$  upper triangular matrices is a subspace of  $M_{m \times n}(F)$ .**

*Proof.* Let  $W$  be the set of  $m \times n$  upper triangular matrices. Then  $0 \in W$ . Let  $x, y \in W$ . Then  $x + y \in W$ , since for all  $i, j$  such that  $i > j$ ,  $(x + y)_{ij} = 0 + 0 = 0$ . Let  $c \in F$ . Then  $cx \in W$ , since for all  $i, j$  such that  $i > j$ ,  $(cx)_{ij} = c \cdot 0 = 0$ , so  $W$  is a subspace of  $M_{m \times n}(F)$ .  $\square$

**17. Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $W \neq \emptyset$  and, whenever  $a \in F$  and  $x, y \in W$ , then  $ax \in W$  and  $x + y \in W$ .**

*Proof.* Suppose that  $W$  is a subset of a vector space  $V$ . If  $W$  is a subspace of  $V$ , then it must contain  $0$ , so  $W \neq \emptyset$ , and for  $a \in F$  and  $x, y \in W$ ,  $x + y \in W$  and  $ax \in W$  is true by theorem 1.3. For the converse, suppose that  $W \neq \emptyset$ , and for  $a \in F$  and  $x, y \in W$ ,  $x + y \in W$  and  $ax \in W$ . Then for  $a = 0$ ,  $ax = 0 \in W$ , so  $W$  satisfies all conditions necessary for it to be a subspace of  $V$ .  $\square$

**18. Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $0 \in W$  and  $ax + y \in W$  whenever  $a \in F$  and  $x, y \in W$ .**

*Proof.* Let  $W$  be a subset of a vector space  $V$ . If  $W$  is a subspace of  $V$ , then it has to contain  $0$ , and for  $a \in F$  and  $x, y \in W$ ,  $ax \in W$  and  $x + y \in W$ . If  $z = ax$ , then  $z + y \in W$  has to be true.

For the converse, suppose that  $0 \in W$  and for  $a \in F$  and  $x, y \in W$ ,  $ax + y \in W$ . If  $y = 0$ , then  $ax + y = ax \in W$ . Since  $ax \in W$ , let  $z = ax$ , so  $z + y \in W$ , which means that  $W$  is a subspace of  $V$ .  $\square$

**20. Prove that if  $W$  is a subspace of a vector space  $V$  and  $w_1, w_2, \dots, w_n$  are in  $W$ , then  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$  for any scalars  $a_1, a_2, \dots, a_n$ .**

*Proof.* Let  $n = 1$ . Then  $a_1w_1 \in W$  by the definition of a subspace. Next, suppose that for some  $n \geq 2$ ,  $a_1w_1 + a_2w_2 + \dots + a_{n-1}w_{n-1} \in W$ . Then

$$a_1w_1 + a_2w_2 + \dots + a_nw_n = (a_1w_1 + a_2w_2 + \dots + a_{n-1}w_{n-1}) + a_nw_n.$$

Since  $(a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1}) \in W$  and  $a_nw_n \in W$  and subspaces are closed under addition,  $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ .  $\square$

**23. Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ .**

(a) Prove that  $W_1 + W_2$  is a subspace of  $V$  that contains both  $W_1$  and  $W_2$ .

Assume, without loss of generality that  $x$  is a vector in  $W_1$ . Then, since  $W_2$  contains 0,  $x + 0 = x \in W_1 + W_2$ .

(b) Prove that any subspace of  $V$  that contains both  $W_1$  and  $W_2$  must also contain  $W_1 + W_2$ .

Let  $W_3$  be a subspace of  $V$ , such that it contains both  $W_1$  and  $W_2$ . Since subspaces are closed under addition, for any  $x \in W_1$  and  $y \in W_2$ ,  $x + y \in W_3$ , so  $W_3$  must contain  $W_1 + W_2$ .

**24. Show that  $F^n$  is the direct sum of the subspaces**

$$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$$

**and**

$$W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

*Proof.* Let  $x = (x_1, x_2, \dots, x_n) \in W_1$  and  $y = (y_1, y_2, \dots, y_n) \in W_2$ . Then

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (x_1 + 0, x_2 + 0, \dots, x_{n-1} + 0, 0 + y_n) = (x_1, x_2, \dots, y_n),$$

where  $x_1, x_2, \dots, x_{n-1}, y_n \in F$ , so  $F^n = W_1 + W_2$ .

Next,

$$W_1 \cap W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \cdots = a_n = 0\} = \{0\},$$

so  $F^n = W_1 \oplus W_2$ .  $\square$

**27. Let  $V$  denote the vector space of all upper triangular  $n \times n$  matrices, and let  $W_1$  denote the subspace of  $V$  consisting of all diagonal matrices. Define**

$$W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j\}.$$

**Show that  $V = W_1 \oplus W_2$ .**

*Proof.* Let  $A \in W_1$  and  $B \in W_2$ . Then

$$A + B = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ 0 & 0 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & b_{12} & \cdots & b_{1n} \\ 0 & a_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},$$

so  $V = W_1 + W_2$ .

Next,

$$W_1 \cap W_2 = \{A \in V : A_{ij} = 0 \text{ whenever } i \geq j \text{ or } i \neq j\} = \{0\},$$

so  $V = W_1 \oplus W_2$ .  $\square$

**28. A matrix  $M$  is called skew-symmetric if  $M^t = -M$ . Clearly, a skew-symmetric matrix is square. Let  $F$  be a field. Prove that the set  $W_1$  of all skew-symmetric  $n \times n$  matrices with entries from  $F$  is a subspace of  $M_{n \times n}(F)$ . Now assume that  $F$  is not of characteristic two, and let  $W_2$  be the subspace of  $M_{n \times n}(F)$  consisting of all symmetric  $n \times n$  matrices. Prove that  $M_{n \times n}(F) = W_1 \oplus W_2$ .**

*Proof.*  $0 \in W_1$ , since  $0^t = 0 = -0$ . Let  $A, B \in W_1$ . Then

$$(A + B)^t = A^t + B^t = -A + (-B) = -(A + B),$$

so  $A + B \in W_1$ . Now let  $a \in F$ . Then  $(aA)^t = aA^t = -aA$ , so  $aA \in W_1$ , and thus  $W_1$  is a subspace of  $M_{n \times n}(F)$ .

Let  $A$  be any square  $n \times n$  matrix. Then if  $B$  is an  $n \times n$  square matrix, such that  $B_{ij} = A_{ij}$  whenever  $i = j$ ,  $B_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij})$  whenever  $i < j$  and  $B_{ij} = B_{ji}$  whenever  $i > j$ , then  $B$  is a symmetric matrix, so  $B \in W_2$ , and if  $C$  is an  $n \times n$  square matrix, such that  $C_{ij} = 0$  whenever  $i = j$ ,  $C_{ij} = \frac{1}{2}(A_{ji} - A_{ij})$  whenever  $i > j$  and  $C_{ij} = -C_{ji}$  whenever  $i < j$ , then  $C$  is skew-symmetric, so  $C \in W_1$ . Then:

1. If  $i = j$

$$B_{ij} + C_{ij} = A_{ij} + 0 = A_{ij}$$

2. If  $i < j$

$$B_{ij} + C_{ij} = A_{ij} + \frac{1}{2}(A_{ji} - A_{ij}) - \frac{1}{2}(A_{ji} - A_{ij}) = A_{ij}$$

3. If  $j > i$

$$B_{ij} + C_{ij} = A_{ji} + \frac{1}{2}(A_{ij} - A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) = A_{ji} + A_{ij} - A_{ji} = A_{ij},$$

so  $A = B + C$ , and thus  $M_{n \times n}(F) = W_1 + W_2$ .

$$W_1 \cap W_2 = \{A \in M_{n \times n}(F) : A^t = -A \text{ and } A^t = A\} = \{0\},$$

so  $M_{n \times n}(F) = W_1 \oplus W_2$ . □

**30. Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if each vector in  $V$  can be uniquely written as  $x_1 + x_2$  where  $x_1 \in W_1$  and  $x_2 \in W_2$ .**

*Proof.* Suppose that  $V$  is the direct sum of  $W_1$  and  $W_2$ , and that there exist some  $x_1, x_1' \in W_1$  and  $x_2, x_2' \in W_2$ , such that  $x_1 + x_2 = x_1' + x_2' \in V$ . Then  $x_1 - x_1' = x_2' - x_2 \in W_1 \cap W_2 = \{0\}$ , so  $x_1 - x_1' = 0 = x_2' - x_2$ , and  $x_1 = x_1'$  and  $x_2 = x_2'$ .

For the converse, suppose that any vector in  $V$  can be represented uniquely as  $x_1 + x_2$ , for  $x_1 \in W_1$  and  $x_2 \in W_2$ . Then  $V = W_1 + W_2$ . Let  $y \in W_1 \cap W_2$ . Then  $y = y + 0 = 0 + y$ , and since it is unique,  $y = 0$ , so  $W_1 \cap W_2 = \{0\}$ . □

## 4 Linear combinations and systems of linear equations

**Definition.** Let  $V$  be a vector space and  $S$  a nonempty subset of  $V$ . A vector  $v \in V$  is called a **linear combination** of vectors of  $S$  if there exist a finite number of vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  in  $F$  such that  $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$ . In this case we also say that  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$  and call  $a_1, a_2, \dots, a_n$  the **coefficients** of the linear combination.

**Definition.** Let  $S$  be a nonempty subset of a vector space  $V$ . The **span** of  $S$ , denoted  $\text{span}(S)$ , is the set consisting of all linear combinations of the vectors in  $S$ . For convenience, we define  $\text{span}(\emptyset) = \{0\}$ .

**Theorem 5.** The span of any subset  $S$  of a vector space  $V$  is a subspace of  $V$  that contains  $S$ . Moreover, any subspace of  $V$  that contains  $S$  must also contain the span of  $S$ .

**Definition.** A subset  $S$  of a vector space  $V$  **generates** (or **spans**)  $V$  if  $\text{span}(S) = V$ . In this case, we also say that the vectors of  $S$  generate (or span)  $V$ .

### Exercises

#### 1. Label the following statements as true or false

- (a) The zero vector is a linear combination of any nonempty set of vectors - **True**.  
For any vectors  $v_1, v_2, \dots, v_n$ ,  $0 = 0v_1 + 0v_2 + \dots + 0v_n$ .
- (b) The span of  $\emptyset$  is  $\emptyset$  - **False**.  
 $\text{span}(\emptyset)$  is defined to be  $\{0\}$ .
- (c) If  $S$  is a subset of a vector space  $V$ , then  $\text{span}(S)$  equals the intersection of all subspaces of  $V$  that contain  $S$  - **True**.  
It follows from theorem 1.5.
- (d) In solving a system of linear equations, it is permissible to multiply an equation by any constant - **True**.
- (e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another - **True**.
- (f) Every system of linear equations has a solution - **False**.  
Any system of linear equations that, while solving, produces a system containing  $0 = c$ , where  $c$  is nonzero, does not have a solution.