

**19.** The fact that  $a^2 \geq 0$  for all numbers  $a$ , elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

The three proofs of the Schwarz inequality outlined below have only one thing in common - their reliance on the fact that  $a^2 \geq 0$  for all  $a$ .

- (a) Prove that if  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$  for some number  $\lambda \geq 0$ , then equality holds in the Schwarz inequality. Prove the same thing if  $y_1 = y_2 = 0$ . Now suppose that  $y_1$  and  $y_2$  are not both 0, and that there is no number  $\lambda$  such that  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ . Then

$$0 < (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2)$$

Using problem 18, complete the proof of the Schwarz inequality.

*Proof.* Suppose that for some number  $\lambda \geq 0$ ,  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ . Then

$$\begin{aligned}\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} &= \sqrt{x_1^2 + x_2^2}\sqrt{(\lambda x_1)^2 + (\lambda x_2)^2} = \sqrt{x_1^2 + x_2^2}\sqrt{\lambda^2(x_1^2 + x_2^2)} \\ &= \lambda(\sqrt{x_1^2 + x_2^2})^2 = \lambda(x_1^2 + x_2^2) = \lambda x_1^2 + \lambda x_2^2 \\ &= x_1\lambda x_1 + x_2\lambda x_2 = x_1y_1 + x_2y_2,\end{aligned}$$

so equality holds.

Next, suppose that  $y_1 = y_2 = 0$ . Then

$$\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} = \sqrt{x_1^2 + x_2^2}\sqrt{0} = 0\sqrt{x_1^2 + x_2^2} = 0 = 0x_1 + 0x_2 = x_1y_1 + x_2y_2.$$

Finally, suppose that  $y_1$  and  $y_2$  are not both 0, and that there is no number  $\lambda$  such that  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ . Then the equation  $\lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) = 0$  has no solution  $\lambda$ . We can then divide by  $(y_1^2 + y_2^2)$  to obtain  $\lambda^2 + (\frac{-2(x_1y_1 + x_2y_2)}{(y_1^2 + y_2^2)})\lambda + \frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)} = 0$ .

Then if  $b = \frac{-2(x_1y_1 + x_2y_2)}{y_1^2 + y_2^2}$  and  $c = \frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}$ , from problem 18, we have  $b^2 - 4c < 0$ . Then

$$b^2 - 4c = \left(\frac{-2(x_1y_1 + x_2y_2)}{y_1^2 + y_2^2}\right)^2 - \frac{4(x_1^2 + x_2^2)}{y_1^2 + y_2^2} = \frac{4(x_1y_1 + x_2y_2)^2}{(y_1^2 + y_2^2)^2} - \frac{4(x_1^2 + x_2^2)}{y_1^2 + y_2^2},$$

so

$$(x_1y_1 + x_2y_2)^2 - (x_1^2 + x_2^2)(y_1^2 + y_2^2) < 0,$$

which finally results in

$$(x_1y_1 + x_2y_2) < \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

□

- (b) Prove the Schwarz inequality by using  $2xy \leq x^2 + y^2$  with  $x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}$  and  $y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$  first for  $i = 1$  and then for  $i = 2$ .

*Proof.* Let  $x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}$  and  $y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$  for  $i = 1$ . Then

$$2xy = 2 \frac{x_1 y_1}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \text{ and } x^2 + y^2 = \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}.$$

Next, for  $i = 2$

$$2xy = 2 \frac{x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \text{ and } x^2 + y^2 = \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}.$$

Knowing that  $2xy \leq x^2 + y^2$ , derived from  $(x - y)^2 \geq 0$ , we can sum the inequalities to obtain

$$2 \left( \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \right) \leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} = 2,$$

so

$$\frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \leq 1 \text{ and } x_1 y_1 + x_2 y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}.$$

□

- (c) Prove the Schwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2.$$

*Proof.* For some numbers  $x_1, x_2, y_1, y_2$ , we have

$$\begin{aligned} (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2 &= x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2 + x_1^2 y_2^2 - 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2 \\ &= x_1^2 y_1^2 + x_2^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_2^2 \\ &= (x_1^2 + x_2^2)(y_1^2 + y_2^2). \end{aligned}$$

Then since  $(x_1 y_2 - x_2 y_1)^2 \geq 0$ , we have

$$(x_1 y_1 + x_2 y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2) \text{ and } x_1 y_1 + x_2 y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}.$$

□

- (d) Deduce, from each of these three proofs, that equality holds only when  $y_1 = y_2 = 0$  or when there is a number  $\lambda \geq 0$  such that  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ .

*Proof.* For proof (a), it is already shown that if  $y_1$  and  $y_2$  are not both 0, and there is no number  $\lambda \geq 0$  such that  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ , then  $x_1 y_1 + x_2 y_2 < \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$ . For proof (b) we used the fact that  $(x - y)^2 \geq 0$ . Equality then holds when  $x = y$ , so

$$\frac{x_i}{\sqrt{x_1^2 + x_2^2}} = \frac{y_i}{\sqrt{y_1^2 + y_2^2}} \text{ and } x_i = y_i \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{y_1^2 + y_2^2}},$$

so  $x_i = \lambda y_i$ .

For proof (c), we've shown that equality holds when  $(x_1 y_2 - x_2 y_1)^2 = 0$ , so when  $x_1 y_2 = x_2 y_1$ . Then either  $y_1 = y_2 = 0$ , or if we assume without loss of generality that  $y_2 \neq 0$ , we can take  $\lambda = \frac{x_2}{y_2}$  so  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ . □

**20. Prove that if**

$$|x - x_0| < \frac{\epsilon}{2} \text{ and } |y - y_0| < \frac{\epsilon}{2},$$

**then**

$$|(x + y) - (x_0 + y_0)| < \epsilon \text{ and } |(x - y) - (x_0 - y_0)| < \epsilon.$$

*Proof.* Suppose that  $|x - x_0| < \frac{\epsilon}{2}$  and  $|y - y_0| < \frac{\epsilon}{2}$ . Then

$$|(x + y) - (x_0 + y_0)| = |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and

$$|(x - y) - (x_0 - y_0)| = |(x - x_0) + (y_0 - y)| \leq |x - x_0| + |y_0 - y| = |x - x_0| + |y - y_0| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

**21. Prove that if**

$$|x - x_0| < \min\left(\frac{\epsilon}{2(|y_0| + 1)}, 1\right) \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)},$$

**then**  $|xy - x_0y_0| < \epsilon$ .

*Proof.* We have  $|x| - |x_0| \leq |x - x_0| < 1$ , so  $|x| < 1 + |x_0|$ . Then

$$\begin{aligned} |xy - x_0y_0| &= |x(y - y_0) + y_0(x - x_0)| \leq |x||y - y_0| + |y_0||x - x_0| < (1 + |x_0|)|y - y_0| + |y_0||x - x_0| \\ &= \frac{\epsilon}{2} + |y_0||x - x_0|, \end{aligned}$$

and since  $|x - x_0| < \frac{\epsilon}{2(|y_0| + 1)}$ , then

$$|x - x_0|(|y_0| + 1) = |y_0||x - x_0| + |x - x_0| < \frac{\epsilon}{2},$$

so  $\frac{\epsilon}{2} + |y_0||x - x_0| < \epsilon$ , therefore  $|xy - x_0y_0| < \epsilon$ .

□

**22. Prove that if  $y_0 \neq 0$  and**

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right),$$

**then**  $y \neq 0$  and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon.$$

*Proof.* We have  $|y_0| = |(y_0 - y) + y| \leq |y - y_0| + |y|$ , so since  $|y - y_0| < \frac{|y_0|}{2}$ , then

$$|y| \geq |y_0| - |y - y_0| > |y_0| - \frac{|y_0|}{2} = \frac{|y_0|}{2}.$$

Then

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \left|\frac{y_0 - y}{yy_0}\right| = \frac{|y - y_0|}{|y||y_0|} < \frac{\frac{\epsilon|y_0|^2}{2}}{\frac{|y_0|}{2}|y_0|} < \frac{\frac{\epsilon|y_0|^2}{2}}{\frac{|y_0|}{2}|y_0|} = \epsilon$$

□

**23.** Replace the question marks in the following statement by expressions involving  $\lambda, x_0$  and  $y_0$ , so that the conclusion will be true:

If  $y_0 \neq 0$  and

$$|y - y_0| < ? \text{ and } |x - x_0| < ?$$

then  $y \neq 0$  and

$$\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \epsilon.$$

*Proof.* We've shown in problem 21, that if  $|x - x_0| < \min(\frac{\epsilon}{2(|y_0|+1)}, 1)$  and  $|y - y_0| < \frac{\epsilon}{2(|x_0|+1)}$ , then  $|xy - x_0y_0| < \epsilon$ . Then for  $\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \epsilon$  to be true, we want  $|x - x_0| < \min(\frac{\epsilon}{2(|y_0|+1)}, 1)$  and we need to find a condition for  $|y - y_0| < ?$  such that  $|\frac{1}{y} - \frac{1}{y_0}| < \frac{\epsilon}{2(|x_0|+1)}$ . We can then substitute  $\epsilon$  in problem 22 for  $\frac{\epsilon}{2(|x_0|+1)}$ , so

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{4(|x_0|+1)}\right).$$

□