**19. The fact that $a^2 \geq 0$ for all numbers $a$, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:**

$$x_1y_1 + x_2y_2 \leq \sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}.$$

**The three proofs of the Schwarz inequality outlined below have only one thing in common - their reliance on the fact that $a^2 \geq 0$ for all $a$.**

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number $\lambda \geq 0$, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that $y_1$ and $y_2$ are not both 0, and that there is no number $\lambda$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$0 < (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2)$$

Using problem 18, complete the proof of the Schwarz inequality.

*Proof.* Suppose that for some number $\lambda \geq 0$, $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned}\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} &= \sqrt{x_1^2 + x_2^2}\sqrt{(\lambda x_1)^2 + (\lambda x_2)^2} = \sqrt{x_1^2 + x_2^2}\sqrt{\lambda^2(x_1^2 + x_2^2)} \\ &= \lambda(\sqrt{x_1^2 + x_2^2})^2 = \lambda(x_1^2 + x_2^2) = \lambda x_1^2 + \lambda x_2^2 \\ &= x_1\lambda x_1 + x_2\lambda x_2 = x_1y_1 + x_2y_2,\end{aligned}$$

so equality holds.
Next, suppose that $y_1 = y_2 = 0$. Then

$$\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} = \sqrt{x_1^2 + x_2^2}\sqrt{0} = 0\sqrt{x_1^2 + x_2^2} = 0 = 0x_1 + 0x_2 = x_1y_1 + x_2y_2.$$

Finally, suppose that $y_1$ and $y_2$ are not both 0, and that there is no number $\lambda$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then the equation $\lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) = 0$ has no solution $\lambda$. We can then divide by $(y_1^2 + y_2^2)$ to obtain $\lambda^2 + (\frac{-2(x_1y_1 + x_2y_2)}{(y_1^2 + y_2^2)})\lambda + \frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)} = 0$. Then if $b = \frac{-2(x_1y_1 + x_2y_2)}{y_1^2 + y_2^2}$ and $c = \frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}$, from problem 18, we have $b^2 - 4c < 0$. Then

$$b^2 - 4c = (\frac{-2(x_1y_1 + x_2y_2)}{y_1^2 + y_2^2})^2 - \frac{4(x_1^2 + x_2^2)}{y_1^2 + y_2^2} = \frac{4(x_1y_1 + x_2y_2)^2}{(y_1^2 + y_2^2)^2} - \frac{4(x_1^2 + x_2^2)}{y_1^2 + y_2^2},$$

so

$$(x_1y_1 + x_2y_2)^2 - (x_1^2 + x_2^2)(y_1^2 + y_2^2) < 0,$$

which finally results in

$$(x_1y_1 + x_2y_2) < \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

□

(b) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ with $x = \frac{x_i}{\sqrt{{x_1}^2+{x_2}^2}}$ and $y = \frac{y_i}{\sqrt{{y_1}^2+{y_2}^2}}$ first for $i = 1$ and then for $i = 2$.

*Proof.* Let $x = \frac{x_i}{\sqrt{{x_1}^2+{x_2}^2}}$ and $y = \frac{y_i}{\sqrt{{y_1}^2+{y_2}^2}}$ for $i = 1$. Then

$$2xy = 2\frac{x_1y_1}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+y_2^2}} \text{ and } x^2 + y^2 = \frac{{x_1}^2}{{x_1}^2+{x_2}^2} + \frac{{y_1}^2}{{y_1}^2+{y_2}^2}.$$

Next, for $i = 2$

$$2xy = 2\frac{x_2y_2}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+y_2^2}} \text{ and } x^2 + y^2 = \frac{{x_2}^2}{{x_1}^2+{x_2}^2} + \frac{{y_2}^2}{{y_1}^2+{y_2}^2}.$$

Knowing that $2xy \leq x^2 + y^2$, derived from $(x - y)^2 \geq 0$, we can sum the inequalities to obtain

$$2\left(\frac{x_1y_1 + x_2y_2}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\right) \leq \frac{{x_1}^2+{x_2}^2}{{x_1}^2+{x_2}^2} + \frac{{y_1}^2+{y_2}^2}{{y_1}^2+{y_2}^2} = 2,$$

so

$$\frac{x_1y_1 + x_2y_2}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}} \leq 1 \text{ and } x_1y_1 + x_2y_2 \leq \sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}.$$

□

(c) Prove the Schwarz inequality by first proving that

$$({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

*Proof.* For some numbers $x_1, x_2, y_1, y_2$, we have

$$\begin{aligned}(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 &= {x_1}^2{y_1}^2 + 2x_1x_2y_1y_2 + {x_2}^2{y_2}^2 + {x_1}^2{y_2}^2 - 2x_1x_2y_1y_2 + {x_2}^2{y_1}^2 \\ &= {x_1}^2{y_1}^2 + {x_2}^2{y_1}^2 + {x_1}^2{y_2}^2 + {x_2}^2{y_2}^2 \\ &= ({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2).\end{aligned}$$

Then since $(x_1y_2 - x_2y_1)^2 \geq 0$, we have

$$(x_1y_1 + x_2y_2)^2 \leq ({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2) \text{ and } x_1y_2 + x_2y_2 \leq \sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}.$$

□

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

*Proof.* For proof (a), it is already shown that if $y_1$ and $y_2$ are not both 0, and there is no number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, then $x_1y_1 + x_2y_2 < \sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}$.
For proof (b) we used the fact that $(x - y)^2 \geq 0$. Equality then holds when $x = y$, so

$$\frac{x_i}{\sqrt{{x_1}^2+{x_2}^2}} = \frac{y_i}{\sqrt{{y_1}^2+{y_2}^2}} \text{ and } x_i = y_i\frac{\sqrt{{x_1}^2+{x_2}^2}}{\sqrt{{y_1}^2+{y_2}^2}},$$

so $x_i = \lambda y_i$.
For proof (c), we've shown that equality holds when $(x_1y_2 - x_2y_1)^2 = 0$, so when $x_1y_2 = x_2y_1$. Then either $y_1 = y_2 = 0$, or if we assume without loss of generality that $y_2 \neq 0$, we can take $\lambda = \frac{x_2}{y_2}$ so $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. □

**20. Prove that if**

$$|x - x_0| < \frac{\epsilon}{2} \textbf{ and } |y - y_0| < \frac{\epsilon}{2},$$

**then**

$$|(x + y) - (x_0 + y_0)| < \epsilon \textbf{ and } |(x - y) - (x_0 - y_0)| < \epsilon.$$

*Proof.* Suppose that $|x - x_0| < \frac{\epsilon}{2}$ and $|y - y_0| < \frac{\epsilon}{2}$. Then

$$|(x + y) - (x_0 + y_0)| = |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and

$$|(x - y) - (x_0 - y_0)| = |(x - x_0) + (y_0 - y)| \leq |x - x_0| + |y_0 - y| = |x - x_0| + |y - y_0| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

**21. Prove that if**

$$|x - x_0| < \mathbf{min}(\frac{\epsilon}{2(|y_0| + 1)}, 1) \textbf{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)},$$

**then** $|xy - x_0 y_0| < \epsilon$**.**

*Proof.* We have $|x| - |x_0| \leq |x - x_0| < 1$, so $|x| < 1 + |x_0|$. Then

$$\begin{aligned} |xy - x_0 y_0| = |x(y - y_0) + y_0(x - x_0)| \leq |x||y - y_0| + |y_0||x - x_0| &< (1 + |x_0|)|y - y_0| + |y_0||x - x_0| \\ &= \frac{\epsilon}{2} + |y_0||x - x_0|, \end{aligned}$$

and since $|x - x_0| < \frac{\epsilon}{2(|y_0|+1)}$, then

$$|x - x_0|(|y_0| + 1) = |y_0||x - x_0| + |x - x_0| < \frac{\epsilon}{2},$$

so $\frac{\epsilon}{2} + |y_0||x - x_0| < \epsilon$, therefore $|xy - x_0 y_0| < \epsilon$. □

**22. Prove that if** $y_0 \neq 0$ **and**

$$|y - y_0| < \mathbf{min}(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{2}),$$

**then** $y \neq 0$ **and**

$$|\frac{1}{y} - \frac{1}{y_0}| < \epsilon.$$

*Proof.* We have $|y_0| = |(y_0 - y) + y| \leq |y - y_0| + |y|$, so since $|y - y_0| < \frac{|y_0|}{2}$, then

$$|y| \geq |y_0| - |y - y_0| > |y_0| - \frac{|y_0|}{2} = \frac{|y_0|}{2}.$$

Then

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \left|\frac{y_0 - y}{y y_0}\right| = \frac{|y - y_0|}{|y||y_0|} < \frac{\frac{\epsilon |y_0|^2}{2}}{|y||y_0|} < \frac{\frac{\epsilon |y_0|^2}{2}}{\frac{|y_0|}{2}|y_0|} = \epsilon$$

□

**23. Replace the question marks in the following statement by expressions involving $\epsilon, x_0$ and $y_0$, so that the conclusion will be true:**

If $y_0 \neq 0$ and

$$|y - y_0| <? \text{ and } |x - x_0| <?$$

then $y \neq 0$ and

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \epsilon.$$

*Proof.* We've shown in problem 21, that if $|x - x_0| < \min(\frac{\epsilon}{2(|y_0|+1)}, 1)$ and $|y - y_0| < \frac{\epsilon}{2(|x_0|+1)}$, then $|xy - x_0 y_0| < \epsilon$. Then for $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \epsilon$ to be true, we want $|x - x_0| < \min(\frac{\epsilon}{2(|y_0|+1)}, 1)$ and we need to find a condition for $|y - y_0| <?$ such that $|\frac{1}{y} - \frac{1}{y_0}| < \frac{\epsilon}{2(|x_0|+1)}$. We can then substitute $\epsilon$ in problem 22 for $\frac{\epsilon}{2(|x_0|+1)}$, so

$$|y - y_0| < min(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{4(|x_0| + 1)}).$$

□