Numbers of various sorts

Exercises

1. Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof. Let $n = 1$. Then $\frac{1(2)(3)}{6} = 1$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}
1^2 + \cdots + (k+1)^2 = 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{aligned}$$

□

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Proof. Let $n = 1$. Then $1^3 = 1^2$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}
1^3 + \cdots + (k+1)^3 &= (1^3 + \cdots + k^3) + (k+1)^3 = (1 + \cdots + k)^2 + (k+1)^3 \\
&= (1 + \cdots + k)^2 + (k+1)^2(k+1) = (1 + \cdots + k)^2 + k(k+1)^2 + (k+1)^2 \\
&= (1 + \cdots + k)^2 + 2\frac{k(k+1)}{2}(k+1) + (k+1)^2 \\
&= (1 + \cdots + k)^2 + 2(1 + \cdots + k)(k+1) + (k+1)^2 \\
&= (1 + \cdots + (k+1))^2
\end{aligned}$$

□

2. Find a formula for

(i) $\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + \cdots + (2n-1)$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n}(2i-1) = 1 + 2 + \cdots + 2n - 2(1 + 2 + \cdots + n) &= \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} \\
&= n(2n+1) - n(n+1) = 2n^2 + n - n^2 - n = n^2.
\end{aligned}$$

□

(ii) $\sum_{i=1}^{n}(2i-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$

Proof.

$$\sum_{i=1}^{n}(2i-1)^2 = 1^2 + 2^2 + \cdots + (2n)^2 - 4(1^2 + 2^2 + \cdots + n^2) = \frac{2n(2n+1)(4n+1)}{6} - 4\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{8n^3 - 2n}{6} = \frac{2n(2n-1)(2n+1)}{6}$$

□