1 Axioms for the natural numbers

Axiom. ***(Peano Postulates).*** *There exists a set* $\mathbb{N}$ *with an element* $1 \in \mathbb{N}$ *and a function* $s : \mathbb{N} \to \mathbb{N}$ *that satisfy the following three properties.*

a. There is no $n \in \mathbb{N}$ *such that* $s(n) = 1$*.*

b. The function is injective.

c. Let $G \subseteq \mathbb{N}$ *be a set. Suppose that* $1 \in G$ *and that if* $g \in G$ *then* $s(g) \in G$*. Then* $G = \mathbb{N}$*.*

Definition. *The set of* ***natural numbers****, denoted* $\mathbb{N}$*, is the set the existence of which is given in the Peano Postulates.*

Theorem 1. ***(Definition by Recursion).*** *Let* H *be a set, and let* $e \in H$ *and let* $k : H \to H$ *be a function. Then there is a unique function* $f : \mathbb{N} \to H$ *such that* $f(1) = e$ *and that* $f \circ s = k \circ f$*.*

Theorem 2. *There is a unique binary operation* $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *that satisfies the following two properties for all* $n, m \in \mathbb{N}$*.*

a. $n + 1 = s(n)$

b. $n + s(m) = s(n + m)$

Theorem 3. *There is a unique binary operation* $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *that satisfies the following two properties for all* $n, m \in \mathbb{N}$*.*

a. $n \cdot 1 = n$

b. $n \cdot s(m) = (n \cdot m) + n$

Theorem 4. *Let* $a, b, c \in \mathbb{N}$*.*

1. *If* $a + c = b + c$ *then* $a = b$ *(Cancellation Law for Addition).*
2. $(a + b) + c = a + (b + c)$ *(Associative Law for Addition).*
3. $1 + a = s(a) = a + 1$.
4. $a + b = b + a$ *(Commutative Law for Addition).*
5. $a + b \neq 1$.
6. $a + b \neq a$.
7. $a \cdot 1 = a = 1 \cdot a$ *(Identity Law for Multiplication).*
8. $(a + b)c = ac + bc$ *(Distributive Law).*
9. $ab = ba$ *(Commutative Law for Multiplication).*
10. $(ab)c = a(bc)$ *(Associative Law for Multiplication).*
11. *If* $ac = bc$ *then* $a = b$ *(Cancellation Law for Multiplication).*
12. $ab = 1$ *if and only if* $a = 1 = b$.

Definition. *The relation* $<$ *on* $\mathbb{N}$ *is defined by* $a < b$ *if and only if there is some* $p \in \mathbb{N}$ *such that* $a + p = b$ *for all* $a, b \in \mathbb{N}$. *The relation* $\leq$ *on* $\mathbb{N}$ *is defined by* $a \leq b$ *if and only if* $a < b$ *or* $a = b$, *for all* $a, b \in \mathbb{N}$.

Theorem 5. *Let* $a, b, c, d \in \mathbb{N}$.

1. $a \leq a$ *and* $a \nless a$, *and* $a < a + 1$.
2. $1 \leq a$.
3. *If* $a < b$ *and* $b < c$, *then* $a < c$; *if* $a \leq b$ *and* $b < c$ *then* $a < c$; *if* $a < b$ *and* $b \leq c$ *then* $a < c$; *if* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$.
4. $a < b$ *if and only if* $a + c < b + c$.
5. $a < b$ *if and only if* $ac < bc$.
6. *Precisely one of* $a < b$ *or* $a = b$ *or* $a > b$ *holds (Trichotomy Law).*
7. $a \leq b$ *or* $b \leq a$
8. *If* $a \leq b$ *and* $b \leq a$ *then* $a = b$.
9. *It cannot be that* $b < a < b + 1$.
10. $a \leq b$ *if and only if* $a < b + 1$.
11. $a < b$ *if and only if* $a + 1 \leq b$.

Theorem 6. ***(Well-Ordering Principle).*** *Let* $G \subseteq \mathbb{N}$ *be a non-empty set. Then there is some* $m \in G$ *such that* $m \leq g$ *for all* $g \in G$.

Exercises

1. Fill in the missing details in the proof of Theorem 1.2.6.