Numbers of various sorts

Exercises

1. Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof. Let $n = 1$. Then $\frac{1(2)(3)}{6} = 1$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}
1^2 + \cdots + (k+1)^2 = 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{aligned}$$

□

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Proof. Let $n = 1$. Then $1^3 = 1^2$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}
1^3 + \cdots + (k+1)^3 &= (1^3 + \cdots + k^3) + (k+1)^3 = (1 + \cdots + k)^2 + (k+1)^3 \\
&= (1 + \cdots + k)^2 + (k+1)^2(k+1) = (1 + \cdots + k)^2 + k(k+1)^2 + (k+1)^2 \\
&= (1 + \cdots + k)^2 + 2\frac{k(k+1)}{2}(k+1) + (k+1)^2 \\
&= (1 + \cdots + k)^2 + 2(1 + \cdots + k)(k+1) + (k+1)^2 \\
&= (1 + \cdots + (k+1))^2
\end{aligned}$$

□

2. Find a formula for

(i) $\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + \cdots + (2n-1)$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n}(2i-1) = 1 + 2 + \cdots + 2n - 2(1 + 2 + \cdots + n) &= \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} \\
&= n(2n+1) - n(n+1) = 2n^2 + n - n^2 - n = n^2.
\end{aligned}$$

□

(ii) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$

Proof.

$$\sum_{i=1}^{n}(2i-1)^2 = 1^2+2^2+\cdots+(2n)^2-4(1^2+2^2+\cdots+n^2) = \frac{2n(2n+1)(4n+1)}{6} - 4\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{8n^3-2n}{6} = \frac{2n(2n-1)(2n+1)}{6}$$

□

3. If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}, \textbf{ if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1 \textbf{ (a special case of the first formula if we define 0! = 1)},$$

and for $k < 0$ or $k > n$ we just define the binomial coefficient to be 0.

(a) Prove that
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} = \frac{kn!}{k!(n+1-k)!} + \frac{(n+1-k)n!}{k!(n+1-k)!}$$
$$= \frac{n!(k+n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

□

(b) Notice that all the numbers in Pascal's triangle are natural numbers. Use part (a) to prove by induction that $\binom{n}{k}$ is always a natural number.

Proof. Let $n = 1$. Then $\binom{1}{0} = 1$ and $\binom{1}{1} = 1$, so the binomial coefficient is always a natural number. Next, suppose that for some number n, and $0 \le k \le n$, $\binom{n}{k}$ is always a natural number. Then if $k = 0$ or $k = n+1$, then $\binom{n+1}{k} = 1$, which is a natural number. Otherwise,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \text{ and } 1 \le k \le n, 0 \le k-1 \le n-1,$$

so $\binom{n+1}{k}$ is a sum of two natural numbers, and is therefore also a natural number. □

(c) Give another proof that $\binom{n}{k}$ is a natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers chosen from $1, \dots, n$.