Numbers of various sorts

Exercises

1. Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof. Let $n = 1$. Then $\frac{1(2)(3)}{6} = 1$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}1^2 + \cdots + (k+1)^2 = 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6} \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}\end{aligned}$$

□

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Proof. Let $n = 1$. Then $1^3 = 1^2$, so the formula holds. Now assume that the formula is true for some $k \in N$. Then

$$\begin{aligned}1^3 + \cdots + (k+1)^3 &= (1^3 + \cdots + k^3) + (k+1)^3 = (1 + \cdots + k)^2 + (k+1)^3 \\ &= (1 + \cdots + k)^2 + (k+1)^2(k+1) = (1 + \cdots + k)^2 + k(k+1)^2 + (k+1)^2 \\ &= (1 + \cdots + k)^2 + 2\frac{k(k+1)}{2}(k+1) + (k+1)^2 \\ &= (1 + \cdots + k)^2 + 2(1 + \cdots + k)(k+1) + (k+1)^2 \\ &= (1 + \cdots + (k+1))^2\end{aligned}$$

□

2. Find a formula for

(i) $\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + \cdots + (2n-1)$

Proof.

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) = 1 + 2 + \cdots + 2n - 2(1 + 2 + \cdots + n) &= \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} \\ &= n(2n+1) - n(n+1) = 2n^2 + n - n^2 - n = n^2.\end{aligned}$$

□

(ii) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$

Proof.

$$\sum_{i=1}^{n}(2i-1)^2 = 1^2+2^2+\cdots+(2n)^2-4(1^2+2^2+\cdots+n^2) = \frac{2n(2n+1)(4n+1)}{6}-4\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{8n^3-2n}{6} = \frac{2n(2n-1)(2n+1)}{6}$$

□

3. If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!}, \textbf{ if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1 \textbf{ (a special case of the first formula if we define 0! = 1)},$$

and for $k < 0$ or $k > n$ we just define the binomial coefficient to be 0.

(a) Prove that
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof.

$$\binom{n}{k-1}+\binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!}+\frac{n!}{k!(n-k)!} = \frac{kn!}{k!(n+1-k)!}+\frac{(n+1-k)n!}{k!(n+1-k)!}$$
$$= \frac{n!(k+n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

□

(b) Notice that all the numbers in Pascal's triangle are natural numbers. Use part (a) to prove by induction that $\binom{n}{k}$ is always a natural number.

Proof. Let $n = 1$. Then $\binom{1}{0} = 1$ and $\binom{1}{1} = 1$, so the binomial coefficient is always a natural number. Next, suppose that for some number n, and $0 \le k \le n$, $\binom{n}{k}$ is always a natural number. Then if $k = 0$ or $k = n+1$, then $\binom{n+1}{k} = 1$, which is a natural number. Otherwise,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \text{ and } 1 \le k \le n, 0 \le k-1 \le n-1,$$

so $\binom{n+1}{k}$ is a sum of two natural numbers, and is therefore also a natural number. □

(c) Give another proof that $\binom{n}{k}$ is a natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers chosen from $1, \dots, n$.

Proof. The number of k-tuples of integers chosen from $1, \dots, n$ is $n(n-1)\dots(n-k+1)$, because there is n choices for the first element, $n-1$ choices for the second, etc. Now, for each k-tuple, it can be arranged in $k(k-1)\dots(1) = k!$ different ways, so to get the number of sets of size k, with elements chosen from $1, \dots, n$, we have $\frac{n(n-1)\dots(n-k+1)}{k!} = \binom{n}{k}$. $\square$

(d) Prove the "binomial theorem": If a and b are any numbers and n is a natural number, then

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\ &= \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j.\end{aligned}$$

Proof. Let $n = 1$. Then

$$(a+b)^1 = a + b = a^1 + b^1 = \sum_{j=0}^{1} \binom{n}{j} a^{n-j}b^j,$$

so the statement holds true.

Next, suppose that the statement is true for some $n \geq 1$. Then

$$\begin{aligned}(a+b)^{n+1} = (a+b)(a+b)^n &= (a+b)\sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j \\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n+1-j}b^j + \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^{j+1} \\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n+1-j}b^j + \sum_{j=1}^{n+1}\binom{n}{j-1}a^{n+1-j}b^j \\ &= a^{n+1} + \sum_{j=1}^{n}(\binom{n}{j} + \binom{n}{j-1})a^{n+1-j}b^j + b^{n+1} \\ &= a^{n+1} + \sum_{j=1}^{n}\binom{n+1}{j}a^{n+1-j}b^j + b^{n+1} \\ &= \sum_{j=0}^{n+1} a^{n+1-j}b^j.\end{aligned}$$

$\square$

(e) Prove that

(i)

$$\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n$$

Proof. Let $n = 1$. Then

$$\sum_{j=0}^{n} \binom{n}{j} = 1 + 1 = 2 = 2^1,$$

so the formula holds.

Next, suppose that for some $n \geq 1$, $\sum_{j=0}^{n} = 2^n$. Then

$$\sum_{j=0}^{n+1} \binom{n+1}{j} = \sum_{j=0}^{n+1} \left(\binom{n}{j} + \binom{n}{j-1} \right) = \binom{n}{0} + \sum_{j=1}^{n} \binom{n}{j} + \sum_{j=1}^{n} \binom{n}{j-1} + \binom{n}{n}$$
$$= \sum_{j=0}^{n} \binom{n}{j} + \sum_{j=0}^{n} \binom{n}{j} = 2^n + 2^n = 2^{n+1}.$$

□

Proof. (alternative)

$$2^n = (1+1)^n = \sum_{j=0}^{n} 1^{n-j} 1^j \binom{n}{j} = \sum_{j=0}^{n} \binom{n}{j}$$

□

(ii)

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = 0$$

Proof.

$$0 = (1 + (-1))^n = \sum_{j=0}^{n} 1^{n-j} (-1)^j \binom{n}{j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j}.$$

□

(iii)

$$\sum_{l \text{ odd}} \binom{n}{l} = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}$$

Proof.

$$0 = \sum_{l=0}^{n} (-1)^l \binom{n}{l} = \sum_{l \text{ even}} \binom{n}{l} - \sum_{l \text{ odd}} \binom{n}{l},$$

so

$$\sum_{l \text{ even}} \binom{n}{l} = \sum_{l \text{ odd}} \binom{n}{l}.$$

Then

$$2^n = \sum_{l=0}^{m} \binom{n}{l} = \sum_{l \text{ even}} \binom{n}{l} + \sum_{l \text{ odd}} \binom{n}{l} = 2 \sum_{l \text{ odd}} \binom{n}{l},$$

so

$$\sum_{l \text{ odd}} = \frac{2^n}{2} = 2^{n-1}$$

$\square$

(iv)

$$\sum_{l \text{ even}} \binom{n}{l} = \binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}$$

Proof. It follows from proof of (iii). $\square$

5.

(a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$.

Proof. Let $n = 1$. Then $1 + r = \frac{(1+r)(1-r)}{1-r} = \frac{1-r^2}{1-r}$, so the formula holds.
Next, suppose that the formula is true for some $n \geq 1$. Then

$$\begin{aligned} 1 + r + r^2 + \cdots + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + \frac{r^{n+1}(1 - r)}{1 - r} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} = \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$\square$

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r, and solving the two equations for S.

Proof.

$$S = 1 + r + \cdots + r^n \text{ and } Sr = r + r^2 + \cdots + r^{n+1}.$$

Then

$$S(1 - r) = 1 + r + \cdots + r^n - r - r^2 - \cdots - r^{n+1} = 1 - r^{n+1},$$

so

$$S = \frac{1 - r^{n+1}}{1 - r}$$

$\square$

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6. The formula for $1^2 + \cdots + n^2$ can be derived as follows. We begin with the formula

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \dots, n$ and adding, we obtain

$$\begin{array}{c} 2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ \vdots \\ (n+1)^3 - n^3 = 3n^2 + 3n + 1 \\ \hline (n+1)^3 - 1 = 4[1^2 + \cdots + n^2] + 3[1 + \cdots + n] + n. \end{array}$$

Thus we can find $\sum_{k=1}^n k^2$ if we already know $\sum_{k=1}^n k$. Use this method to find

(i) $1^3 + \cdots + n^3$.

Proof. We begin with

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Then we have

$$(n+1)^4 - 1 = 4\sum_{j=1}^n j^3 + 6\sum_{k=1}^n k^2 + 4\sum_{l=1}^n l + n$$

so

$$\begin{aligned} \sum_{j=1}^n j^3 &= \frac{(n+1)^4 - 1 - 6\sum_{k=1}^n k^2 - 4\sum_{l=1}^n l - n}{4} \\ &= \frac{n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} - n}{4} \\ &= \frac{n^4 + 4n^3 + 6n^2 + 3n - 2n^3 - 3n^2 - n - 2n^2 - 2n}{4} \\ &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \end{aligned}$$

□

(ii) $1^4 + \cdots + n^4$.

Proof. We begin with

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

Then we have

$$(n+1)^5 - 1 = 5\sum_{i=1}^n i^4 + 10\sum_{j=1}^n j^3 + 10\sum_{k=1}^n k^2 + 5\sum_{l=1}^n l + n$$

so

$$\begin{aligned}
\sum_{i=1}^{n} i^4 &= \frac{(n+1)^5 - 1 - 10\sum_{j=1}^{n} j^3 - 10\sum_{k=1}^{n} k^2 - 5\sum_{l=1}^{n} l - n}{5} \\
&= \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n - 5\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) - 10\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - n}{5} \\
&= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}
\end{aligned}$$

□

(iii) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$.

Proof. We begin with

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$$

Then we have

$$\sum_{j=1}^{n} \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

□

(iv) $\frac{3}{1^2\cdot 2^2} + \frac{5}{2^2\cdot 3^2} + \cdots + \frac{2n+1}{n^2(n+1)^2}$.

Proof. We begin with

$$\frac{1}{(k+1)^2} - \frac{1}{k^2} = \frac{2k+1}{k^2(k+1)^2}$$

Then we have

$$\sum_{j=1}^{n} \frac{2j+1}{j^2(j+1)^2} = \frac{1}{(n+1)^2} - 1$$

□

8. Prove that every natural number is either even or odd.

Proof. Since $1 = 2\cdot 0 + 1$, 1 is obviously an odd number. Now suppose that some natural number n is either odd or even. We consider two cases:

Case 1: n is odd. Then $n = 2k+1$ for some integer k, and $n+1 = 2k+2 = 2(k+1)$. Since $k+1$ is an integer, $n+1$ is even.

Case 2: n is even. Then $n = 2k$ for some integer k, and $n+1 = 2k+1$, so $n+1$ is odd. □

9. Prove that if a set A of natural numbers contains n_0 and contains $k+1$ whenever it contains k then A contains all natural number $\geq n_0$.

Proof. Suppose that a set A of natural numbers contains n_0 and contains $k+1$ whenever it contains k, and that there is some smallest element i, such that $i \notin A$ and $i > n_0$. Then $i-1 \in A$, or else i would not be the smallest such element. Since $i-1 \in A$, $(i-1)+1 \in A$, so $i \in A$, which is a contradiction. □

10. Prove the principle of mathematical induction from the well-ordering principle.

Proof. Suppose that for some statement P, A is the set of all natural numbers, for which P is true, and that if $k \in A$, then $k+1 \in A$. Then A has a least member n_0, and contains all natural numbers $\geq n_0$, so if $1 \in A$, then A is the set of all natural numbers, and therefore P is true for all natural numbers. $\square$

11. Prove the principle of complete induction from the ordinary principle of induction. Hint: If A contains 1 and A contains $n+1$ whenever it contains $1, \dots, n$, consider the set B of all k such that $1, \dots, k$ are all in A.

Proof. Suppose that for some statement P, A is the set of all natural numbers for which P is true, $1 \in A$, and whenever $1, \dots, n$ are all in A, then $n+1 \in A$. Let B be the set of all k such that $1, \dots, k$ are all in A. Then obviously $1 \in B$. Now suppose that for some element k, $1, \dots, k$ are all in A, and therefore $k \in B$. Since $1, \dots, k$ are all in A, then $k+1 \in A$, so $1, \dots, k+1$ are all in A, and therefore $k+1 \in B$. Then $B = N$, so $A = N$, and thus P is true for all natural numbers. $\square$

12.

(a) If a is rational and b is irrational, is $a+b$ necessarily irrational? What if a and b are both irrational?

Proof. Suppose that for some rational number a and some irrational b, $a+b$ is rational. Then $a+b = \frac{x}{y}$ with x, y being integers. Then $b = \frac{x}{y} - a = \frac{x-ay}{y}$ so b is rational, which is a contradiction. $\square$

(b) If a is rational and b is irrational, is ab necessarily irrational?

Proof. For some rational number a and irrational b, ab is rational if $a = 0$, since $0b = 0$ is rational. Suppose that $a \neq 0$ and ab is rational. Then $ab = \frac{x}{y}$ for some integers x, y, so $b = \frac{x}{ay}$, which would imply that b is rational, which is a contradiction. $\square$

(c) Is there a number a such that a^2 is irrational but a^4 is rational?

Proof. Consider $a = \sqrt{\sqrt{2}}$. Then $a^2 = \sqrt{2}$ but $a^4 = 2$. $\square$

(d) Are there two irrational numbers whose sum and product are both rational?

Proof. Consider consider $a + \sqrt{2}$ and $a - \sqrt{2}$ with a rational. Then $a + \sqrt{2} + a - \sqrt{2} = 2a$ which is rational, and $(a+\sqrt{2})(a-\sqrt{2}) = a^2 - 2$ which is rational. $\square$

13.

(a) Prove that $\sqrt{3}, \sqrt{5}$ and $\sqrt{6}$ are irrational. Hint: To treat $\sqrt{3}$ for example, use the fact that every integer is of the form $3n, 3n+1$ or $3n+2$. Why doesn't this proof work for $\sqrt{4}$.

Proof. Since

$$(3n+1)^2 = 3(3n^2+2n)+1$$
$$(3n+2)^2 = 3(3n^2+4n+1)+1$$

and

$$(3n)^2 = 3(3n)^2,$$

it follows that if k^2 is divisible by 3, then k is also divisible by 3. Suppose that $\sqrt{3}$ is rational, that is, there exist some integers $x, y,\ y \neq 0$ with no common divisor, such that $\sqrt{3} = \frac{x}{y}$. Then $(\frac{x}{y})^2 = 3$, so $x^2 = 3y^2$, so x must be divisible by 3. Then $x = 3z$ for some integer z, so $3y^2 = 9z^2$, and thus $y^2 = 3z^2$, which means that y is divisible by 3, which contradicts x and y having no common divisor, which means that $\sqrt{3}$ can not be rational.

$\sqrt{5}$ PROOF GOES HERE □