

Homework 1

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Deadline: Feb 02, 2026.

The solution to every problem should appear on a new page. Every solution must be concise and should not take more than an A4 page. **You may receive extra credit for exceptionally well-written arguments.** Contact your TA/instructor in case of an issue.

Asymptotic Analysis

Question 1:

[5]

Prove or Disprove the following statements*:

1. $f(n) + g(n) = \Theta(\min(f(n), g(n)))$
2. $f(n) = O(g(n)) \rightarrow \log(f(n)) = O(\log(g(n)))$
3. $f(n) = O(g(n)) \rightarrow g(n) = \Omega(f(n))$
4. $f(n) = o(g(n)) \rightarrow \log(f(n)) = o(\log(g(n)))$

Question 2:

[5]

Choose whether $f(n)$ belongs to one or more of the following: $O(g(n)), \Theta(g(n)), \Omega(g(n))$

#	$f(n)$	$g(n)$	O	Ω	Θ
1	n^{-2}	$\frac{1}{\sqrt[2]{n}}$			
2	2^n	$n^{n/2}$			
3	$\sqrt[3]{n^n}$	$n^{n/3}$			
4	$\log(2n^2)$	$\log(4n^3) - \log(2n)$			
5	n^{-3}	2^{50}			
6	$\log(n)$	\sqrt{n}			
7	$n!$	n^n			
8	$n \log(n!)$	$\log(n!)$			

*Note that we say $f(n) = o(g(n))$ if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

Equivalently, for every constant $c > 0$, there exists n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$. Informally: f grows strictly slower than g .

Order Statistics

The selection problem is to find the i th order statistic of a list. In class, we studied the deterministic linear-time selection algorithm SELECT (also known as Median of Medians).

Question 3: [5]

Show how to use SELECT as a subroutine to make Quicksort run in $O(n \log(n))$ time in the worst case, assuming that all elements are distinct.

(* Refer to relevant parts of CLRS book for a description of Quicksort).

Question 4: [5]

The SELECT algorithm divides the list into subgroups of size c , where c is typically set to 5. What effect would there be on the algorithm's worst-case run-time if c is chosen to be a) less than 5, b) more than 5, and c) if c becomes a function of n , i.e., $\log n, \sqrt{n}, n/2, n$?

Question 5: [5]

Design a linear time algorithm to find all of the $\log n$ smallest integers using the SELECT problem as a subroutine. For example, if we have a list of $n = 2^{100}$ integers, then we are interested in the 100 smallest integers in the given list.

Randomized Algorithms

RANDOMIZED-SELECT is a selection algorithm where pivots are chosen *uniformly* at random, unlike the Median of Medians problem which is a deterministic-linear algorithm.

Question 6: [5]

Show an example where RANDOMIZED-SELECT has a worst-case run-time of $O(n^2)$

Question 7: [5]

Compute the probability that a randomly chosen pivot splits the array into two parts each of size at least $\frac{3n}{10}$. Use this to show that the expected run-time of RANDOMIZED-SELECT is $\Theta(n)$.

Question 8: [5]

Suppose we want RANDOMIZED-SELECT to succeed in finding the k -th smallest element *without hitting a worst-case path*. If the probability of a good split is p , what is the probability that after t independent pivot choices, we always hit good splits?