

## Homework 1

Professor: Dr. Mudassir Shabbir

Deadline: Feb 02, 2026.

The solution to every problem should appear on a new page. Every solution must be concise and should not take more than an A4 page. **You may receive extra credit for exceptionally well-written arguments.** Contact your TA/instructor in case of an issue.

## Asymptotic Analysis

### Question 1:

[5]

Prove or Disprove the following statements\*:

1.  $f(n) + g(n) = \Theta(\min(f(n), g(n)))$
2.  $f(n) = O(g(n)) \rightarrow \log(f(n)) = O(\log(g(n)))$
3.  $f(n) = O(g(n)) \rightarrow g(n) = \Omega(f(n))$
4.  $f(n) = o(g(n)) \rightarrow \log(f(n)) = o(\log(g(n)))$

### Question 2:

[5]

Choose whether  $f(n)$  belongs to one or more of the following:  $O(g(n)), \Theta(g(n)), \Omega(g(n))$

#	$f(n)$	$g(n)$	$O$	$\Omega$	$\Theta$
1	$n^{-2}$	$\frac{1}{\sqrt[2]{n}}$			
2	$2^n$	$n^{n/2}$			
3	$\sqrt[3]{n^n}$	$n^{n/3}$			
4	$\log(2n^2)$	$\log(4n^3) - \log(2n)$			
5	$n^{-3}$	$2^{50}$			
6	$\log(n)$	$\sqrt{n}$			
7	$n!$	$n^n$			
8	$n \log(n!)$	$\log(n!)$			

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\*Note that we say  $f(n) = o(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

Equivalently, for every constant  $c > 0$ , there exists  $n_0$  such that  $f(n) \leq c g(n)$  for all  $n \geq n_0$ . Informally:  $f$  grows strictly slower than  $g$ .

## Order Statistics

The selection problem is to find the  $i$ th order statistic of a list. In class, we studied the deterministic linear-time selection algorithm SELECT (also known as Median of Medians).

### Question 3:

[5]

Show how to use SELECT as a subroutine to make Quicksort run in  $O(n \log(n))$  time in the worst case, assuming that all elements are distinct.

(\* Refer to relevant parts of CLRS book for a description of Quicksort).

### Question 4:

[5]

The SELECT algorithm divides the list into subgroups of size  $c$ , where  $c$  is typically set to 5. What effect would there be on the algorithm's worst-case run-time if  $c$  is chosen to be a) less than 5, b) more than 5, and c) if  $c$  becomes a function of  $n$ , i.e.,  $\log n, \sqrt{n}, n/2, n$ ?

### Question 5:

[5]

Design a linear time algorithm to find all of the  $\log n$  smallest integers using the SELECT problem as a subroutine. For example, if we have a list of  $n = 2^{100}$  integers, then we are interested in the 100 smallest integers in the given list.

## Randomized Algorithms

RANDOMIZED-SELECT is a selection algorithm where pivots are chosen *uniformly* at random, unlike the Median of Medians problem which is a deterministic-linear algorithm.

### Question 6:

[5]

Show an example where RANDOMIZED-SELECT has a worst-case run-time of  $O(n^2)$

### Question 7:

[5]

Compute the probability that a randomly chosen pivot splits the array into two parts each of size at least  $\frac{3n}{10}$ . Use this to show that the expected run-time of RANDOMIZED-SELECT is  $\Theta(n)$ .

### Question 8:

[5]

Suppose we want RANDOMIZED-SELECT to succeed in finding the  $k$ -th smallest element *without hitting a worst-case path*. If the probability of a good split is  $p$ , what is the probability that after  $t$  independent pivot choices, we always hit good splits?