Better Estimation of the Expected Value (Continued)

In this lesson, we are going to stick to the restriction to distributions with support over 0.0 to 1.0 for pedagogic reasons, but our aim is to find a technique that gets us back to sampling over arbitrary distributions.

WE'LL COVER THE FOLLOWING

In the previous lesson, we implemented a better technique for estimating the expected value of a function `f` applied to samples from a distribution `p`:

1. Compute the total area (including negative areas) under the function `x => f(x) * p.Weight(x)`.
2. Compute the total area under `x => p.Weight(x)`.
    - This is 1.0 for a normalized PDF or the normalizing constant of a non-normalized PDF; if we already know it, we don't have to compute it.
3. The quotient of these areas is the expected value

Draw Backs of Using Quadrature to get an Approximate Numerical Solution

Essentially our technique was to use quadrature to get an approximate numerical solution to an integral calculus problem.

However, we also noted that it seems like there might still be room for improvement, in two main areas:

- This technique only works when we have a good bound on the support of the distribution; for our contrived example, we chose a “profit function” and a distribution where we said that we were only interested in the region from 0.0 to 1.0.
- Our initial intuition that implementing an estimate of “the average of many samples” by averaging many samples, seems correct; can we get back there?

The argument that we are going to make here (several times!) is: *two things that are both equal to the same third thing are also equal to each other*.

Recall that we arrived at our quadrature implementation by estimating that our continuous distribution’s expected value is close to the expected value of a very similar discrete distribution. We are going to make our argument a little bit more general here by removing the assumption that `p` is a normalized distribution. That means that we’ll need to know the normalizing factor `np`, which as we’ve noted is `Area(p.Weight)`.

We said that we could estimate the expected value like this:

1. Imagine that we create a 1000 sided “unfair die” discrete distribution.
2. Each side corresponds to a 0.001 wide slice from the range 0.0 to 1.0; let’s say that we have a variable x that takes on values 0.000, 0.001, 0.002, and so on, corresponding to the 1000 sides.
3. The weight of each side is the probability of choosing this slice: `p.Weight(x) / 1000 / np`
4. The *value* of each side is the “profit function” `f(x)`
5. The expected value of “rolling this die” is *the sum of (value times weight)*: the sum of `f(x) * (p.Weight(x) / 1000 / np)` over our thousand values of x.

Here’s the trick:

- Consider the standard continuous uniform distribution `u`. That’s a perfectly good distribution with support 0.0 to 1.0.
- Consider the function `w(x)` which is `x => f(x) * p.Weight(x) / np`. That’s a perfectly good function from `double` to `double`.

> **Question:** What is an estimate of the expected value of `w` over samples from `u`?

Show Hint

Things equal to the same are equal to each other, which implies this conclusion:

If we can compute an estimate of the expected value of `w` applied to samples from `u` by any technique then we have also computed an estimate of the expected value of `f` applied to samples from `p`.

Why is this important?

The problem with the naïve algorithm in our original case was that there was a "black swan" — a region of a large (negative) area that is sampled only one time in 1400 samples. But that region is sampled one time in about 14 samples when we sample from a uniform distribution, so we will get a much better and more consistent estimate of the expected value if we use the naïve technique over the uniform distribution.

To get 100x times as many samples in the black swan region, we do not have to do 100x times as many samples overall. We can just sample from a helper distribution that targets that important region more often.

Let's try it!

Modifications in Code of `ExpectedValue`

To not get confused here, we are going to rename some of our methods so that they're not all called `ExpectedValue`. The one that just takes any distribution and averages a bunch of samples is now `ExpectedValueBySampling` and the one that computes two areas and takes their quotient is `ExpectedValueByQuadrature`.

```
var p = Normal.Distribution(0.75, 0.09);
double f(double x) => Atan(1000 * (x - .45)) * 20 - 31.2;
var u = StandardContinuousUniform.Distribution;
double np = 1.0; // p is normalized
double w(double x) => f(x) * p.Weight(x) / np;
double w(double x) => f(x) * p.Weight(x) / np;
for (int i = 0; i < 100; ++i)
  Console.WriteLine($"{u.ExpectedValueBySampling(w):0.###}");
```

Remember, the correct answer that we computed by quadrature is 0.113. When sampling p directly we got values ranging from 0.7 to 0.14. But now we get:

```
0.114, 0.109, 0.109, 0.118, 0.111, 0.107, 0.113, 0.112, ...
```

So much better!

This is awesome, but wait, it gets more awesome. What is so special about the uniform distribution? Nothing, that's what. We are going to do this argument one more time.

Importance Sampling

Suppose we have distribution `q`, any distribution whatsoever, so long as its support is the same as `p` — in this case, 0.0 to 1.0. In particular, suppose that `q` is not necessarily a normalized distribution, but that again, we know its normalization factor. Call it `nq`. Recall that the normalization factor can be computed by `nq = Area(q.Weight)`.

Our special function `g(x)` is this oddity:

```
x => (f(x) * (p.Weight(x) / q.Weight(x)) * (nq / np)
```

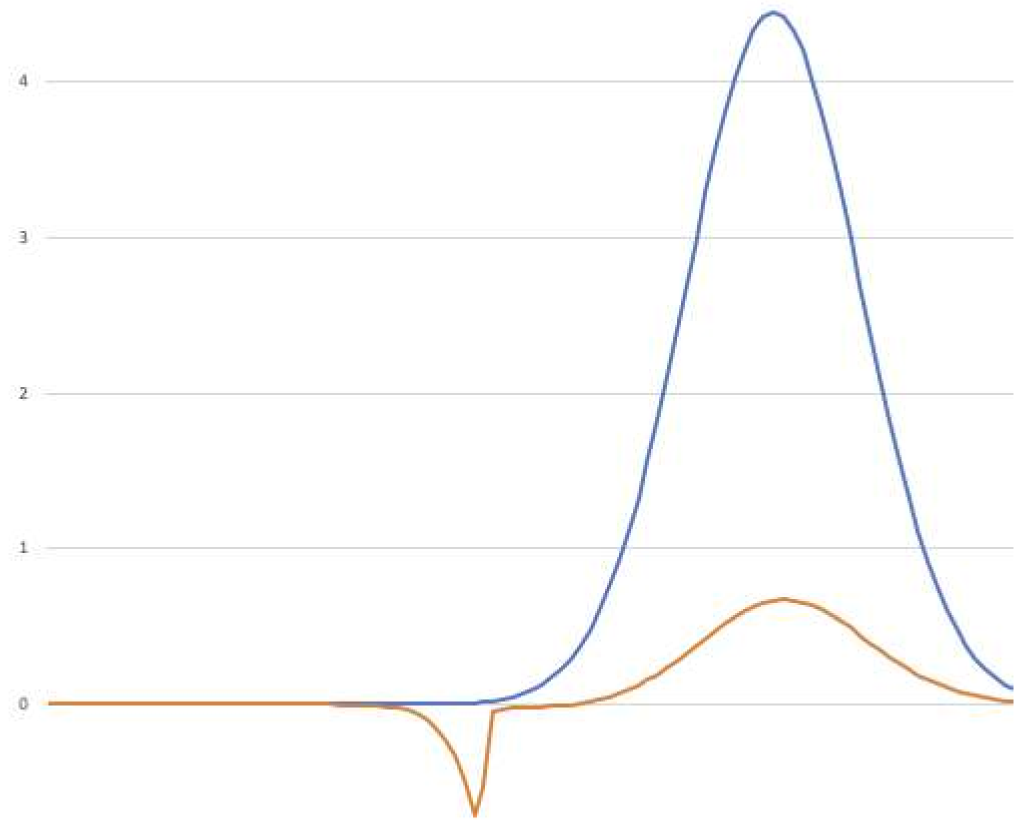
What is the expected value of `g` over distribution `q`? One more time:

- Create a 1000-sided unfair die, x as before.
- The *weight* of each side is the probability of choosing that side, which is `(q.Weight(x) / 1000) / nq`.
- The *value* of each side is `g(x)`.
- The expected value is the sum of `g(x) * (q.Weight(x) / 1000) / nq` but if you work that out, of course that is the sum of `f(x) * p.Weight(x) / np / 1000`

And once again, we've gotten back to the same sum by a clever choice of function. **If we can compute the expected value of `g` evaluated on samples from `q`, then we know the expected value of `f` evaluated on samples**

from `p`!

This means that we can choose our helper distribution `q` so that it is highly likely to pick values in regions we consider important. Let’s look at our graph of `p.Weight` and `f * p.Weight` again:



There are segments of the graph where the area under the blue line is very small but the area under the orange line is large, and that’s our black swan; **What we want is a distribution that samples from regions where the orange area is large, and if possible skips regions where it is small**. That is, we consider the large-area regions important contributors to the expected value, and the small-area regions unimportant contributors; we wish to target our samples so that no important regions are ignored. That’s why **this technique for computing expected value is called “importance sampling”**.

The uniform distribution is pretty good on the key requirement that it never be small when the area under the orange line is large because it is always the same size from 0.0 to 1.0; that’s why it is the uniform distribution, after all. It’s not great on our second measure; it spends around 30% of its time sampling from regions where the orange line has essentially no area under it.

> **Exercise:** Write some code that tries different distributions. For example, implement the distribution that has weight function `x => (0 <= x && x <= 1) ? x : 0`.
> (Remember that this is not a normalized distribution, so you'll have to compute `nq`.)

Does that give us an even better estimate of the expected value of f?

> **Exercise:** What would be the ideal choice of distribution?

The best possible distribution would be a singleton whose value is the expected value, but of course we do not know that. Is there a weight function for that we do know the value of, such that it is large in regions where the area under the orange line is large, and small everywhere else? We will suggest such a distribution in an upcoming lesson.

Show Hint

Summing Up

- Suppose we have a weighted distribution of doubles `p` and a function from `double` to `double` `f`.
- We wish to accurately estimate the average value of `f` when it is applied to a large set of samples from `p`; this is the *expected value*.
- However, there may be "black swan" regions where the value of `f` is important to the average, but the probability of sampling from that region is low, so our average could require a huge number of samples to get an accurate average.
- We can fix the problem by choosing any weighted distribution `q` that has the same support as `p` but is more likely to sample from important regions.
- The expected value of `g` (given above) over samples drawn from `q` is the

The expected value of `g` (given above) over samples drawn from `q` is the same as the expected value of `f` over samples from `p`.

This is great. Notice, we removed any restriction that `p` or `q` be distributions only on 0.0 to 1.0; this technique works for weighted distributions of doubles over any support!

> We can weaken our restriction that `q` have the same support as `p`; if we decide that `q` can have zero weight for particularly unimportant regions, where, say, we know that `f(x) * p.Weight(x)` is very small, then that’s still going to produce a good estimate.

> Something we probably should have mentioned before is that all of the techniques we are describing in this course for estimating expected values require that the expected value *exists*! Not all functions applied to probability distributions have an expected value because the average value of the function computed on a group of samples might not converge as the size of the group gets larger. An easy example is, suppose we have a standard normal distribution as our `p` and `x => 1.0 / p.Weight(x)` as our `f`. The more samples from `p` we take, the more likely it is that the average value of `f` gets *larger*!

However, it’s not all sunshine and roses. We still have two problems, and they’re pretty big ones:

1. How do we find a good-quality `q` distribution?
2. **We need to know the normalization constants for both distributions.** If we do not know them ahead of time (because, say, we have special knowledge that the continuous uniform distribution is normalized) then how are we going to compute them? `Area(p.Weight)` or `Area(q.Weight)` might be expensive or difficult to compute. It seems like in the general case we still have to solve the calculus problem. 😦

> **Exercise:** The **bold** sentence in our last bullet point contains a small but important error. What is it? The answer will be in an upcoming lesson.

We are not going to implement a general-purpose importance sampling algorithm until we've made at least some headway on these remaining problems.

Implementation

Let's have a look at the changes we have made in our code in this lesson:

Program.cs

Bernoulli.cs

Beta.cs

BetterRandom.cs

Distribution.cs

DistributionBuilder.cs

Empty.cs

Episode34.cs

Extensions.cs

Flip.cs

Gamma.cs

IDiscreteDistribution.cs

IDistribution.cs

IWeightedDistribution.cs

Markov.cs

```
using System;
using static System.Math;
namespace Probability
{
    static class Episode34
    {
        public static void DoIt()
        {
            Console.WriteLine("Episode 34 -- Importance sampling with SCUD");

            var p = Normal.Distribution(0.75, 0.09);
            Func<double, double> f = x => Atan(1000 * (x - .45)) * 20 - 31.2;
            var u = StandardContinuousUniform.Distribution;
            double np = 1.0; // p is normalized
            Func<double, double> w = x => f(x) * p.Weight(x) / np;
            for (int i = 0; i < 100; ++i)
                Console.WriteLine($"{u.ExpectedValueBySampling(w):0.###}");
        }
    }
}
```

In the next lesson, we will revise this lesson again; we’re going to make a similar argument — things equal to the same are equal to each other — but starting from a different place. We’ll end up with the same result, and deduce that importance sampling works.