Q.1. Show that if n is a positive integer, then $1 + 2 + \cdots + n = n(n + 1)/2$

Solution: Let P(n) be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = n(n + 1)/2$. We must do two things to prove that P(n) is true for n = 1, 2, 3, ... Namely, we must show that P(1) is true and that the conditional statement P(k) implies P(k + 1) is true for k = 1, 2, 3, ...

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

Q. 2. Use mathematical induction to show that

$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ◀

Q. 3 Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term *a* and common ratio *r*:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n + 1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\begin{aligned} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} &= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1} \\ &= \frac{ar^{k+2} - a}{r - 1}. \end{aligned}$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

Q.4

Prove that $2 - 2.7 + 2.7^2 - \ldots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

Proof:

Let P(n) be $2\sum_{i=0}^{n}(-7)^i = (1-(-7)^{n+1})/4$, then

Basis Step: $2.(-7)^0 = 2$ and

$(1 - (-7)^{0+1})/4 = (1+7)/4 = 2$,

so P(0) is true.

Inductive Hypothesis: Assume that P(n) is true.

Inductive Step: if P(n+1) is true then prove is done. So P(n+1) is $2\sum_{i=0}^{n+1}(-7)^i$ = $2\sum_{i=0}^{n}(-7)^i + 2.(-7)^{n+1}$ so Using the assumption from the induction hypothesis we have

$$\begin{aligned} P(n+1) &= (1 - (-7)^{n+1})/4 + 2(-7)^{n+1} \\ &= (1 - (-7)^{n+1} + 8(-7)^{n+1})/4 \\ &= (1 + 7(-7)^{n+1})/4 \\ &= (1 - (-7)^{n+2})/4. \end{aligned}$$

Hence, P(n) is true for all nonnegative integers.

Q.5

Prove that $1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$, whenever n is a positive integer.

Proof:

Let $P(n) = 1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$, then

Basis Step: for n = 1, we have $P(1) = 1.1! = 1$, Similarly $P(1) = (1+1)! - 1 = 2-1 = 1$

Hence P(1) is true.

Inductive Hypothesis: Assume that P(n) is true, i.e. $1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$.

Inductive Step: if we are able to prove that P(n+1) is true then we are done. So we have

$$\begin{aligned} P(n+1) &= 1.1! + 2.2! + \ldots + n.n! + (n+1)(n+1)! \\ &= (n+1)! - 1 + (n+1)(n+1)! \text{ (using induction hypothesis)} \\ &= (n+1)n! + (n+1)(n+1)! - 1 \quad = (n+1)(n! + (n+1)!) - 1 \\ &= (n+1)(n!\,(1 + (n+1)) - 1 = (n+1)n!\,(n+2) - 1 \\ &= (n+2)! - 1 \end{aligned}$$

P(n+1) is true

Hence P(n) is true for all positive integers.

Q.6

Prove using mathematical induction that for all $n \geq 1$,

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Solution.

For any integer $n \geq 1$, let P_n be the statement that

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Base Case. The statement P_1 says that

$$1 = \frac{1(3-1)}{2},$$

which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}.$$

It remains to show that P_{k+1} holds, that is,

$$1 + 4 + 7 + \cdots + (3(k+1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}.$$

$$\begin{aligned}
1 + 4 + 7 + \cdots + (3(k+1) - 2) &= 1 + 4 + 7 + \cdots + (3(k+1) - 2) \\
&= 1 + 4 + 7 + \cdots + (3k + 1) \\
&= 1 + 4 + 7 + \cdots + (3k - 2) + (3k + 1) \\
&= \frac{k(3k-1)}{2} + (3k+1) \\
&= \frac{k(3k-1) + 2(3k+1)}{2} \\
&= \frac{3k^2 - k + 6k + 2)}{2} \\
&= \frac{3k^2 + 5k + 2)}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Q.7 Verify the given formula,

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Solution.

For any integer $n \geq 1$, let P_n be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}.$$

Base Case. The statement P_1 says that

$$1^2 + 2^2 = \frac{(1)(2(1)+1)(4(1)+1)}{3} = \frac{3(5)}{3} = 5,$$

which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}.$$

It remains to show that P_{k+1} holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}.$$

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 &= 1^2 + 2^2 + 3^2 + \cdots + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 && \text{(by } P_k\text{)} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(8k^2+6k+1) + 3(4k^2+4k+1) + 3(4k^2+8k+4)}{3} \\
&= \frac{(8k^3+6k^2+k) + (12k^2+12k+3) + (12k^2+24k+12)}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3}
\end{aligned}$$

On the other side of P_{k+1},

$$\frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3} = \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$$

$$= \frac{(k+1)(2k+3)(4k+5)}{3}$$
$$= \frac{(2k^2+5k+3)(4k+5)}{3}$$
$$= \frac{8k^3+30k^2+37k+15}{3}.$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Strong Induction (Second Principle of Mathematical Induction)

This method uses different inductive step than the first principle. Here we assume that P(k) is true for k = n0, n0 + 1, …., k and show that P(k+1) is true based on the assumption.
The steps in this method are:
Basis Step: Show P(n0) is true.
Inductive Hypothesis (Strong): Assume P(k) is true for all n0 <= k <= n.
Inductive Step: Show based on the assumption that P(k+1) is true.

Q. 8 Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer *n*.

Let *P(n)* denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."
BASIS STEP: P(0) is true because $7^{0+2} + 8^{2.0+1} = 7^2 + 8^1 = 57$ is divisible by 57, which is true.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}
7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\
&= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\
&= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.
\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Q.9 Use the Principle of Mathematical Induction to verify that, for n any positive integer, $6^n - 1$ is divisible by 5.

Solution.

For any $n \geq 1$, let P_n be the statement that $6^n - 1$ is divisible by 5.

Base Case. The statement P_1 says that

$$6^1 - 1 = 6 - 1 = 5$$

is divisible by 5, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is, $6^k - 1$ is divisible by 5.

It remains to show that P_{k+1} holds, that is, that $6^{k+1} - 1$ is divisible by 5.

$$\begin{aligned} 6^{k+1} - 1 &= 6(6^k) - 1 \\ &= 6(6^k - 1) - 1 + 6 \\ &= 6(6^k - 1) + 5. \end{aligned}$$

By P_k, the first term $6(6^k - 1)$ is divisible by 5, the second term is clearly divisible by 5. Therefore the left hand side is also divisible by 5. Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Q.10 Show that n! > 3 n for n ≥ 7.

Solution.

For any $n \geq 7$, let P_n be the statement that $n! > 3^n$.

Base Case. The statement P_7 says that $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 > 3^7 = 2187$, which is true.

Inductive Step. Fix $k \geq 7$, and suppose that P_k holds, that is, $k! > 3^k$.

It remains to show that P_{k+1} holds, that is, that $(k+1)! > 3^{k+1}$.

$$\begin{aligned} (k+1)! &= (k+1)k! \\ &> (k+1)3^k \\ &\geq (7+1)3^k \\ &= 8 \times 3^k \\ &> 3 \times 3^k \\ &= 3^{k+1}. \end{aligned}$$

Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Q.11

$$\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}\text{, for all natural numbers, } n \geq 2.$$

Solution Let the given statement be P(n), i.e.,

$$P(n):\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}\text{, for all natural numbers, } n \geq 2$$

We, observe that P (2) is true, since

$$\left(1-\frac{1}{2^2}\right)=1-\frac{1}{4}=\frac{4-1}{4}=\frac{3}{4}=\frac{2+1}{2\times 2}$$

Assume that P(n) is true for some $k \in \mathbf{N}$, i.e.,

$$P(k): \quad 1-\frac{1}{2^2} \;.\; 1-\frac{1}{3^2} \;...\; 1-\frac{1}{k^2} \;=\frac{k+1}{2k}$$

Now, to prove that P (k + 1) is true, we have

$$1-\frac{1}{2^2} \;.\; 1-\frac{1}{3^2} \;...\; 1-\frac{1}{k^2} \;.\; 1-\frac{1}{(k+1)^2}$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)=\frac{k^2+2k}{2k\,(k+1)}=\frac{(k+1)+1}{2(k+1)}$$

Thus, P (k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers, $n \geq 2$.

Q.12 $2^{2n} - 1$ is divisible by 3

Solution Let the statement P(n) given as

P(n) : $2^{2n} - 1$ is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by 3.}$$

Assume that P(n) is true for some natural number k, i.e.,

P(k): $2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in \mathbf{N}$

Now, to prove that P(k + 1) is true, we have

$$P(k+1): 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k}\,.\,2^2 - 1$$

$$= 2^{2k}\,.\,4 - 1 = 3.2^{2k} + (2^{2k} - 1)$$

$= 3.2^{2k} + 3q$

$= 3\ (2^{2k} + q) = 3m$, where $m \in \mathbf{N}$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Q.13 $2n + 1 < 2^n$, for all natual numbers $n \geq 3$.

Solution Let P(n) be the given statement, i.e., P(n) : $(2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that P(3) is true, since

$$2.3 + 1 = 7 < 8 = 2^3$$

Assume that P(n) is true for some natural number k, i.e., $2k + 1 < 2^k$

To prove P(k + 1) is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$2(k + 1) + 1 = 2\ k + 3$

$= 2k + 1 + 2 < 2^k + 2 < 2^k\ .\ 2 = 2^{k+1}$.

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, $n \geq 3$.

Q.14 **Using the principle of mathematical induction, prove that**

$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = (1/6)\ \{n(n + 1)(2n + 1\}$ for all $n \in$ N.

Let the given statement be P(n). Then,

P(n): $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = (1/6)\{n(n + 1)(2n + 1)\}$.

Putting n =1 in the given statement, we get

LHS = $1^2 = 1$ and RHS = $(1/6) \times 1 \times 2 \times (2 \times 1 + 1) = 1$.

Therefore LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

P(k): $1^2 + 2^2 + 3^2 + \ldots.. + k^2 = (1/6)\{k(k + 1)(2k + 1)\}$.

Now, $1^2 + 2^2 + 3^2 + \ldots\ldots... + k^2 + (k + 1)^2$

$= (1/6)\ \{k(k + 1)(2k + 1) + (k + 1)^2$

$= (1/6)\{(k + 1).(k(2k + 1)+6(k + 1))\}$

$= (1/6)\{(k + 1)(2k^2 + 7k + 6\})$

$= (1/6)\{(k + 1)(k + 2)(2k + 3)\}$

$= 1/6\{(k + 1)(k + 1 + 1)[2(k + 1) + 1]\}$

$\Rightarrow$ P(k + 1): $1^2 + 2^2 + 3^2 + \ldots.. + k^2 + (k+1)^2$

$= (1/6)\{(k + 1)(k + 1 + 1)[2(k + 1) + 1]\}$

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

Q.15 **By using mathematical induction prove that the given equation is true for all positive integers**

Solution:

From the statement formula

When n = 1,

LHS =1 x 2 = 2

RHS = $\frac{1(1+1)(4x1-1)}{3} = \frac{6}{3} = 2$

Hence it is proved that P (1) is true for the equation.

Now we assume that P (k) is true or 1 x 2 + 3 x 4 + 5 x 6 + ... + (2k - 1) x 2k = $\frac{k(k+1)(4k-1)}{3}$.

For P(k + 1)

LHS = 1 x 2 + 3 x 4 + 5 x 6 + ... + (2k - 1) x 2k + (2(k + 1) - 1) x 2(k + 1)

$= \frac{k(k+1)(4k-1)}{3}$ + (2(k + 1) - 1) x 2(k + 1)

$= \frac{(k+1)}{3}$ ($4k^2$ - k + 12 k + 6)

$= \frac{(k+1)(4k^2+8k+3k+6)}{3}$

$= \frac{(k+1)(k+2)(4k+3)}{3}$

$= \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3}$ = RHS for P (k+1)

Now it is proved that P (k + 1) is also true for the equation.

Q.16 **Using the principle of mathematical induction, prove that**

1 · 2 + 2 · 3 + 3 · 4 + ... + n(n + 1) = (1/3){n(n + 1)(n + 2)}.

Solution:

Let the given statement be P(n). Then,

P(n): 1 · 2 + 2 · 3 + 3 · 4 + ... + n(n + 1) = (1/3){n(n + 1)(n + 2)}.

Thus, the given statement is true for n = 1, i.e., P(1) is true.

Let P(k) be true. Then,

P(k): 1 · 2 + 2 · 3 + 3 · 4 + ... + k(k + 1) = (1/3){k(k + 1)(k + 2)}.

Now, 1 · 2 + 2 · 3 + 3 · 4 +...+ k(k + 1) + (k + 1)(k + 2)

= (1 · 2 + 2 · 3 + 3 · 4 + ... + k(k + 1)) + (k + 1)(k + 2)

= (1/3) k(k + 1)(k + 2) + (k + 1)(k + 2) [using (i)]

= (1/3) [k(k + 1)(k + 2) + 3(k + 1)(k + 2)

= (1/3){(k + 1)(k + 2)(k + 3)}

$\Rightarrow P(k + 1): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots\ldots + (k + 1)(k + 2)$

$= (1/3)\{k + 1)(k + 2)(k + 3)\}$

$\Rightarrow P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all values of $\in$ N.

Q.17 **4. By using mathematical induction prove that the given equation is true for all positive integers.**

$2 + 4 + 6 + \ldots + 2n = n(n+1)$

Solution:

From the statement formula

When $n = 1$ or $P(1)$,

LHS = 2

RHS $= 1 \times 2 = 2$

So $P(1)$ is true.

Now we assume that $P(k)$ is true or $2 + 4 + 6 + \ldots + 2k = k(k + 1)$.

For $P(k + 1)$,

LHS $= 2 + 4 + 6 + \ldots + 2k + 2(k + 1)$

$= k(k + 1) + 2(k + 1)$

$= (k + 1)(k + 2)$

$= (k + 1)((k + 1) + 1) =$ RHS for $P(k+1)$

Now it is proved that $P(k+1)$ is also true for the equation.

So the given statement is true for all positive integers.

Q.18 **Using the principle of mathematical induction, prove that**

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots .. + (2n - 1)(2n + 1) = (1/3)\{n(4n^2 + 6n - 1)$.

Solution:

Let the given statement be P(n). Then,

$P(n): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots\ldots + (2n - 1)(2n + 1) = (1/3)n(4n^2 + 6n - 1)$.

When n = 1, LHS = $1 \cdot 3 = 3$ and RHS = $(1/3) \times 1 \times (4 \times 1^2 + 6 \times 1 - 1)$

$= \{(1/3) \times 1 \times 9\} = 3$.

LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

$P(k): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots .. + (2k - 1)(2k + 1) = (1/3)\{k(4k^2 + 6k - 1)$ ......(i)

Now,

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots\ldots.. + (2k - 1)(2k + 1) + \{2k(k + 1) - 1\}\{2(k + 1) + 1\}$

$= \{1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots\ldots\ldots\ldots + (2k - 1)(2k + 1)\} + (2k + 1)(2k + 3)$

$= (1/3)\, k(4k^2 + 6k - 1) + (2k + 1)(2k + 3)$ [using (i)]

$= (1/3)\, [(4k^3 + 6k^2 - k) + 3(4k^2 + 8k + 3)]$

$= (1/3)(4k^3 + 18k^2 + 23k + 9)$

$= (1/3)\{(k + 1)(4k^2 + 14k + 9)\}$

$= (1/3)[k + 1)\{4k(k + 1)^2 + 6(k + 1) - 1\}]$

$\Rightarrow P(k + 1): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots .. + (2k + 1)(2k + 3)$

$= (1/3)[(k + 1)\{4(k + 1)^2 + 6(k + 1) - 1)\}]$

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

Q. 19 **By using mathematical induction prove that the given equation is true for all positive integers.**

$2 + 6 + 10 + \ldots.. + (4n - 2) = 2n^2$

From the statement formula

When n = 1 or P(1),

LHS = 2

RHS = $2 \times 1^2 = 2$

So P(1) is true.

Now we assume that P (k) is true or $2 + 6 + 10 + \ldots.. + (4k - 2) = 2k^2$

For P (k + 1),

LHS = $2 + 6 + 10 + \ldots.. + (4k - 2) + (4(k + 1) - 2)$

$= 2k^2 + (4k + 4 - 2)$

$= 2k^2 + 4k + 2$

$= (k+1)^2$

= RHS for P(k+1)

Now it is proved that P(k+1) is also true for the equation.

So the given statement is true for all positive integers.

Q.20 **By induction prove that $3^n - 1$ is divisible by 2 is true for all positive integers.**

Solution:

When n = 1, $P(1) = 3^1 - 1 = 2$ which is divisible by 2.

So P(1) is true.

Now we assume that P(k) is true or $3^k - 1$ is divisible by 2.

When P(k + 1),

$3^{k+1} - 1 = 3^k \times 3 - 1 = 3^k \times 3 - 3 + 2 = 3(3^k - 1) + 2$

As $(3^k - 1)$ and 2 both are divisible by 2, it is proved that divisible by 2 is true for all positive integers.

Q.21 By induction prove that $n^2 - 3n + 4$ is even and it is true for all positive integers.

Solution:

When n = 1, P (1) = 1 - 3 + 4 = 2 which is an even number.

So P (1) is true.

Now we assume that P (k) is true or $k^2 - 3k + 4$ is an even number.

When P (k + 1),

$(k + 1)^2 - 3(k + 1) + 4$

$= k^2 + 2k + 1 - 3k + 3 + 4$

$= k^2 - 3k + 4 + 2(k + 2)$

As $k^2 - 3k + 4$ and $2(k + 2)$ both are even, there sum also will be an even number.

So it is proved that $n^2 - 3n + 4$ is even is true for all positive integers.