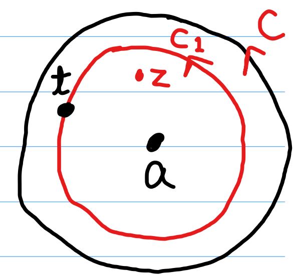
Applied Math : 2079-11-09

Taylor's Theorem

If $f(z)$ is analytic inside a circle with centre a and radius r, then for any point z inside the the circle C, the function $f(z)$ can be expanded in ascending power of $(z-a)$ as

$$f(z) = f(a) + \frac{z-a}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$



Proof: Consider a point z inside the circle C with centre at a and radius r. Draw another concentric circle C_1 inside C. Let t be any point on circle C_1. Then

$$|z-a| < |t-a| \quad \Rightarrow \quad \left|\frac{z-a}{t-a}\right| < 1$$

Therefore the series is uniformly convergent.

Now
$$\frac{1}{t-z} = \frac{1}{t-a+a-z} = \frac{1}{(t-a)-(z-a)}$$

$$= \frac{1}{(t-a)\left\{1 - \frac{z-a}{t-a}\right\}}$$

$$= \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

[If $|x|<1$
$(1-x)^{-1} = 1+x+x^2+\cdots$]

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \frac{(z-a)^2}{(t-a)^2} + \frac{(z-a)^3}{(t-a)^3} + \cdots\right]$$

So,
$$\frac{1}{t-z} = \frac{1}{t-a} + \frac{(z-a)}{(t-a)^2} + \frac{(z-a)^2}{(t-a)^3} + \frac{(z-a)^3}{(t-a)^4} + \cdots \quad \text{(1)}$$

Since the series is uniformly convergent, it is integrable in the that Region inside C and so it is integrable along C_1 also.

Multiply both sides of ① by $f(t)$ and integrate w.r.to t along C_1. So

$$\int_{C_1} \frac{f(t)}{t-z}\,dt = \int_{C_1} \frac{f(t)}{t-a}\,dt + \int_{C_1} \frac{(z-a)}{(t-a)^2} f(t)\,dt + \int_{C_1} \frac{(z-a)^2}{(t-a)^3} f(t)\,dt + \cdots \rightarrow ②$$

But by Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-z}\,dt, \quad f(a) = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-a}\,dt$$

$$f'(a) = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-a)^2}\,dt, \quad f''(a) = \frac{2!}{2\pi i}\int_{C_1} \frac{f(t)}{(t-a)^3}\,dt$$

and so on.

So, from ②,

$$\int_{C_1} \frac{f(t)}{t-z}\,dt = \int_{C_1} \frac{f(t)}{t-a}\,dt + (z-a)\int_{C_1} \frac{f(t)}{(t-a)^2}\,dt + (z-a)^2 \int_{C_1} \frac{f(t)}{(t-a)^3}\,dt + \cdots$$

$$\Rightarrow 2\pi i\, f(z) = 2\pi i\, f(a) + \frac{2\pi i}{1!}(z-a) f'(a) + \frac{2\pi i}{2!}(z-a)^2 f''(a) + \cdots$$

$$\therefore f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

which is required Taylor's series.

Note: If $a=0$, then the series is called Maclaurin's series.

Some special series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < \infty$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad |z| < \infty$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots \qquad |x| < 1$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad |z| < \infty$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad |z| < 1$$

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots \qquad |z| < 1$$

Laurent Series

If $f(z)$ is analytic in the region R (annulus or annular region) bounded by two concentric circles C_1 and C_2 with centre a and radii r_1 and r_2 $(r_1 > r_2)$ then for any z in R, Laurent series can be written as

analytic part

Principal part

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(t)}{(t-a)^{n+1}}\, dt, \quad n = 0, 1, 2, 3, \cdots$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} f(t)\,(t-a)^{n-1}\, dt, \quad n = 1, 2, 3, \cdots$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(t)}{(t-a)^{-n+1}}\, dt$$

Note The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the analytic part of the series and $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^{-n}}$ is called principal part

If principal part is zero, Laurent series reduces to Taylors series.

⑧ Find Laurent series of the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in the region $2 < |z| < 3$ (i.e. between $|z| = 2$ and $|z| = 3$)

Solution: Given $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$

$$z^2+5z+6 \overline{\big)\, z^2-1\,}\ \ 1$$
$$\frac{-z^2 \pm 5z \pm 6}{-5z-7}$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$5z+7 = A(z+3) + B(z+2)$$

when $z = -3$, $B = 8$

$z = -2$, $A = -3$

$$= 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right] \rightarrow ①$$

Since $2 < |z| < 3$

$$2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$$

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$$

From ①

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right] - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= 1 + \left(\frac{3}{z} - \frac{3\times 2}{z^2} + \frac{3\times 2^2}{z^3} - \frac{3\times 2^3}{z^4} + \cdots\right) - \frac{8}{3} + \frac{8z}{3^2} - \frac{8z^2}{3^3}$$

$$= 1 - \frac{8}{3} + \left(3 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^{n+1}}\right) + 8 \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{3^{n+2}}$$

(9) Find the Laurent Series for the function

$$f(z) = \frac{z}{(z+1)(z+2)} \quad \text{in } R: 1 < z < 2$$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

$$z = -2 \Rightarrow B = 2$$

$$z = -1 \Rightarrow A = -1$$

Solution

$$f(z) = \frac{z}{(z+1)(z+2)}$$

$$= \frac{-1}{z+1} + \frac{2}{z+2}$$

Region is $1 < z < 2$

so, $1 < z$ and $z < 2$

$\frac{1}{z} < 1$ and $\frac{z}{2} < 1$

so, $$f(z) = \frac{-1}{z(1+\frac{1}{z})} + \frac{2}{2(1+\frac{z}{2})}$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} + \left(1+\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) + \left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

(7) Find Laurent series for $f(z) = \dfrac{2z+1}{z^3+z^2-2z}$

in $0 < |z-1| < 1$

$$\frac{2z+1}{z(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

solution: $$f(z) = \frac{2z+1}{z(z^2+z-2)} = \frac{2z+1}{z(z-1)(z+2)}$$

$$= -\frac{1}{2z} + \frac{1}{z-1} - \frac{1}{2(z+2)}$$

$$= -\frac{1}{2} \frac{1}{1+(z-1)} + \frac{1}{z-1} - \frac{1}{2} \frac{1}{3+(z-1)}$$

$$= -\frac{1}{2}\left[1+(z-1)\right]^{-1} - \frac{1}{2}\,\frac{1}{3\left[1+\frac{z-1}{3}\right]} + \frac{1}{z-1}$$

$$= -\frac{1}{2}\left(1-(z-1)+(z-1)^2-\cdots\right)$$

$$-\frac{1}{6}\left(1+\frac{z-1}{3}\right)^{-1} + \frac{1}{z-1}$$

$$= -\frac{1}{2}\left(1-(z-1)+(z-1)^2-\cdots\right)$$

$$-\frac{1}{6}\left(1-\frac{z-1}{3}+\frac{(z-1)^2}{9}-\cdots\right)+\frac{1}{z-1}$$