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Cardinal Interpolation and Spline Functions*

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INTRODUCTION

Let $\mathscr{S} = \{F(x)\}$ denote a linear space of functions from $\mathfrak{R}$ to $\mathfrak{C}$. Furthermore, let

$$y = (y_\nu), \qquad (-\infty < \nu < \infty) \tag{1}$$

be a sequence of numbers, the subscript ν ranging over all rational integers. We speak of a *cardinal interpolation problem* (C.I.P.) concerning the sequence y and the space $\mathscr{S}$, if we are to find functions F satisfying the two conditions

$$F(\nu) = y_\nu, \qquad \text{for all } \nu, \tag{2}$$

$$F \in \mathscr{S}. \tag{3}$$

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We shall denote this problem by the symbol

$$\text{(4)} \qquad \text{C.I.P.}\,(y;\mathcal{S}),$$

and use the same to denote the totality of its solutions, if any exist, so that (2) and (3) are described by writing

$$F \in \text{C.I.P.}\,(y;\mathcal{S}).$$

In the present paper we shall usually assume that

$$\mathcal{S} = L_p^m,$$

where m is a natural number and

$$\text{(5)} \qquad L_p^m = \{F(x); F^{(m-1)} \text{ absolutely continuous, } F^{(m)} \in L_p(\mathfrak{R})\},$$

while

$$\text{(6)} \qquad 1 \leqslant p \leqslant \infty.$$

We provide the space (5) with the seminorm

$$\text{(7)} \qquad \|F^{(m)}\|_p = \left(\int_{-\infty}^{\infty} |F^{(m)}(x)|^p\,dx\right)^{1/p} \qquad \text{if } p \text{ is finite,}$$

and

$$\text{(8)} \qquad \|F^{(m)}\|_\infty = \operatorname*{ess\,sup}_x |F^{(m)}(x)|.$$

A second choice for $\mathcal{S}$ will be

$$\mathcal{S} = L_{[1]}^m,$$

where we define this space as follows: We consider the sequence of consecutive unit intervals

$$\text{(9)} \qquad I_j^m = \left(j + \frac{m+1}{2} - 1, j + \frac{m+1}{2}\right) \qquad (-\infty < j < \infty)$$

and denote by $L_{[1]}^m$ the space of functions $F(x)$ such that $F^{(m-1)}$ is absolutely continuous and

$$\text{(10)} \qquad \|F^{(m)}\|_{[1]} = \sum_j \operatorname*{ess\,sup}_{x \in I_j^m} |F^{(m)}(x)| < \infty.$$

This formula also defines the seminorm for this space. Evidently

$$\text{(11)} \qquad L_{[1]} \subset L_1^m.$$

We also need the corresponding sequence spaces

$$\text{(12)} \qquad l_p^m = \{y; \|\Delta^m y\|_p < \infty\}$$

where

$$\|\Delta^m y\|_p = \begin{cases} \left(\sum_\nu |\Delta^m y_\nu|^p\right)^{1/p} & \text{if } 1 \leqslant p < \infty \\ \sup_\nu |\Delta^m y_\nu| & \text{if } p = \infty \end{cases} \tag{13}$$

The first problem concerning a C.I.P. (4) is the question of the existence of solutions. For our choice of spaces these problems are settled by

THEOREM 1. *The*

$$\text{C.I.P. } (y; L_p^m), \qquad (m \geqslant 1, 1 \leqslant p \leqslant \infty) \tag{14}$$

has solutions if and only if

$$y \in l_p^m. \tag{15}$$

The

$$\text{C.I.P. } (y; L_{[1]}^m), \qquad (m \geqslant 1), \tag{16}$$

has solutions if and only if

$$y \in l_1^m. \tag{17}$$

For the special case when $p = \infty$, Theorem 1 is due to Subbotin [7]. Theorem 1 will be established by means of the spline interpolants of degree m introduced by Schoenberg ([3], Theorem 8, pp. 79–80, for $t = 0$). Their existence and uniqueness for our C.I.P.'s is assured by

THEOREM 2. *If* (15) *holds then the problem* (14) *has a unique solution* $S_y(x)$ *which is a spline function of degree m having its knots at the points*

$$\nu + \tfrac{1}{2}(m+1), \qquad (\nu \text{ integer}, -\infty < \nu < \infty). \tag{18}$$

Likewise, if (17) *holds then problem* (16) *has a unique solution* $S_y(x)$ *of the same nature as above.*

Our spaces being provided with the seminorms (7), (8), and (10), respectively, the question of the existence of *optimal* solutions arises naturally. We say that F_* is an *optimal* solution of (14), provided that F_* is a solution of (14) that enjoys the extremum property

$$\|F_*^{(m)}\|_p \leqslant \|F^{(m)}\|_p \qquad \text{for all} \qquad F \in \text{C.I.P. } (y; L_p^m). \tag{19}$$

A similar definition, using (10), concerns the problem (16).

Let the sequence y satisfy one of the conditions of Theorem 1. Leaving aside, for the moment, the question of the existence of optimal solutions, we define the functionals

$$\mathscr{L}_p^m(y) = \inf_F \|F^{(m)}\|_p \qquad (y \in l_p^m) \tag{20}$$

and

(21) $$\mathscr{L}^m_{[1]}(y) = \inf_F \|F^{(m)}\|_{[1]} \qquad (y \in l_1{}^m),$$

where the infimum is formed for all solutions F of the corresponding C.I.P.

To describe our next results we need some notations, introduced in [3], pp. 79 and 114–116, that are to be discussed in greater detail in Subsections 1 and 2 below. We define

(22) $$\psi_k(u) = \left(\frac{2\sin(u/2)}{u}\right)^k, \qquad (k \text{ is a natural number}),$$

and

(23) $$\phi_k(u) = \sum_{j=-\infty}^{\infty} \psi_k(u + 2\pi j).$$

$\phi_k(u)$ is a cosine polynomial, positive for all real u (see Subsection 2 below). We may now state

THEOREM 3. *If $y \in l_p{}^m$ then*

(24) $$\mathscr{L}_p{}^m(y) \leqslant (\phi_{m+1}(\pi))^{-1}\|\Delta^m y\|_p \qquad (1 \leqslant p \leqslant \infty).$$

If $y \in l_1^{(m)}$ then

(25) $$\mathscr{L}^m_{[1]}(y) \leqslant (\phi_{m+1}(\pi))^{-1}\|\Delta^m y\|_1.$$

The explicit value of the constant is

(26) $$(\phi_{m+1}(\pi))^{-1} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{m+1}\left(\sum_{r=1}^{\infty}\frac{(-1)^{(r-1)(m+1)}}{(2r-1)^{m+1}}\right)^{-1}.$$

This theorem shows that the functionals $\mathscr{L}$ are *bounded* and the question arises concerning the values of the *best* constants in the formulae (24) and (25).

These are the *norms* of the functionals, defined by

(27) $$\|\mathscr{L}_p{}^m\| = \sup_y \mathscr{L}_p{}^m(y) \qquad \text{for } \|\Delta^m y\|_p \leqslant 1$$

and

(28) $$\|\mathscr{L}^m_{[1]}\| = \sup_y \mathscr{L}^m_{[1]}(y) \qquad \text{for } \|\Delta^m y\|_1 \leqslant 1.$$

We shall establish below partial results concerning the value of (27). The following is the main result of Subbotin's paper [7].

THEOREM 4 (Subbotin). *The constant appearing in* (24) *is the best constant if $p = \infty$, i.e.,*

(29) $$\|\mathscr{L}_\infty{}^m\| = (\phi_{m+1}(\pi))^{-1}.$$

We shall settle the case when $p = 2$:

THEOREM 5. *If $p = 2$ then*

(30) $$\|\mathscr{L}_2{}^m\| = (\phi_{2m}(\pi))^{-1/2} = \left(\frac{\pi}{2}\right)^m \left(2 \cdot \sum_1^\infty \frac{1}{(2r-1)^{2m}}\right)^{-1/2};$$

more precisely, if

(31) $$\|\Delta^m y\|_2 = 1$$

then

(32) $$1 < \mathscr{L}_2{}^m(y) < (\phi_{2m}(\pi))^{-1/2},$$

where both bounds are the best possible, but are not attained.

Denoting the values of the right-hand sides of (29) and (30) by A_m and B_m, respectively, it is easily seen that $A_1 = B_1 = 1$, while an obvious application of Cauchy's inequality shows that

$$A_m > B_m \qquad \text{if } m > 1.$$

This shows that the constant A_m in (24) is certainly not the best if $p = 2$ and $m > 1$.

Finally, we have

THEOREM 6. *The constant appearing in* (25) *is the best constant, hence*

(33) $$\|\mathscr{L}^m_{[1]}\| = (\phi_{m+1}(\pi))^{-1}.$$

The determination of the two norms (29) and (33) depends on the solutions of certain maximum–minimum problems. In order to sketch the method to be used for their determination, let L^m denote any of the spaces so far considered, while $\|F^{(m)}\|$ and $\|\Delta^m y\|$ are the corresponding seminorms. Finally, let

(34) $$B_1{}^m = \{y; \|\Delta^m y\| = 1\}.$$

We have defined above

(35) $$\mathscr{L}(y) = \inf_F \|F^{(m)}\| \qquad \text{for all } F \in \text{C.I.P. } (y; L^m)$$

and

(36) $$\|\mathscr{L}\| = \sup_y \mathscr{L}(y) \qquad \text{for } y \in B_1{}^m.$$

Finally, let S_y denote the spline solution of the C.I.P. $(y; L^m)$ according to Theorem 2. We describe the procedure in the form of a lemma.

LEMMA 1. *We assume that we have found a* $y^* \in B_1{}^m$ *such that*

(37) $$\|S_{y^*}^{(m)}\| = \max_y \|S_y^{(m)}\| \qquad \textit{for all } y \in B_1{}^m$$

and that

(38) S_{y^*} *is an optimal solution of the* C.I.P. (y^*, L^m).

Then

(39) $$\|\mathscr{L}\| = \|S_{y^*}^{(m)}\|.$$

A proof is immediate: By (35)

(40) $$\mathscr{L}(y) \leqslant \|S_y^{(m)}\| \qquad \text{for all } y \in B_1{}^m.$$

Taking suprema over all $y \in B_1{}^m$, we obtain, by (36) and (37), that

(41) $$\|\mathscr{L}\| \leqslant \|S_{y^*}^{(m)}\|.$$

On the other hand, (38) implies that

(42) $$\mathscr{L}(y^*) = \|S_{y^*}^{(m)}\|$$

and therefore

(43) $$\|\mathscr{L}\| \geqslant \|S_{y^*}^{(m)}\|,$$

by (36). Now (41) and (43) imply (39).

The method of Lemma 1 will be applied twice, yielding Theorems 4 and 6. The existence and nature of the spline solutions S_{y^*} which are optimal for these two C.I.P.'s, seem of sufficient interest to be stated here as theorems.

We start from the sequence y^* defined by

(44) $$y_j{}^* = (-1)^j \qquad \text{for all } j.$$

Let $E_m(x)$ denote the Euler polynomial, defined as the polynomial solution of the functional equation

$$\tfrac{1}{2}(f(x+1) + f(x)) = x^m.$$

Following Nörlund ([*1*], p. 24), we consider the extension $\bar{E}_m(x)$ defined by

(45) $$\bar{E}_m(x) = E_m(x) \qquad \text{if } 0 \leqslant x < 1$$

and

(46) $$\bar{E}_m(x+1) = -\bar{E}_m(x) \qquad \text{for all } x.$$

By (46), $\bar{E}_m(x)$ is a periodic function of period 2 that may also be described as a spline function of degree m having its knots at the integers. It now follows that

(47) $$S_{y^*}(x) = \bar{E}_m(x + \tfrac{1}{2}(m+1))/\bar{E}_m(\tfrac{1}{2}(m+1))$$

is identical with the spline solution of the

$$\text{C.I.P. } (y^*; L_\infty{}^m). \tag{48}$$

THEOREM 7 (Favard–Subbotin). *The periodic Euler function* (47) *is an optimal solution of the cardinal interpolation problem* (48).

This means that

$$\|S_{y^*}^{(m)}\|_\infty \leqslant \|F^{(m)}\|_\infty$$

for all other solutions F. Favard ([2], pp. 305–306) established this result under the restriction that only periodic solutions F, of period 2, are allowed to compete. Subbotin [7] removed this restriction without stating the result in the present form. For the L_2 extremum property of the Euler functions (47) of odd degrees, again within the family of periodic functions, see ([5], §8).

Concerning $L_{[1]}^m$, we have the following

THEOREM 8. *Let y be a real sequence satisfying the following two conditions:*

$$y \in l_1{}^m, \text{ which means } \sum_j |\Delta^m y_j| < \infty; \tag{49}$$

and

$$\text{the sequence } (-1)^j \Delta^m y_j \text{ has no change of sign.} \tag{50}$$

Then the spline solution S_y of the

$$\text{C.I.P. } (y; L_{[1]}^m) \tag{51}$$

is an optimal solution of (51).

The simplest possible nontrivial sequence is $y = \delta = (\delta_j)$, where

$$\delta_j = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases} \tag{52}$$

Since

$$(-1)^j \Delta^m \delta_j = (-1)^m \binom{m}{m+j},$$

we see that the sequence δ satisfies the two conditions of Theorem 8. Moreover, the spline interpolant S_δ admits the following explicit representation:

$$S_\delta(x) = L_{m+1}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_{m+1}(u)}{\phi_{m+1}(u)} e^{iux}\, du \qquad (-\infty < x < \infty) \tag{53}$$

(see [3], pp. 79–80; in particular, formula (9) for $t = 0$). We state the result as

COROLLARY 1. *The spline solution* (53) *of the "unit" interpolation problem*

$$F(\nu) = \delta_\nu \tag{54}$$

is an optimal solution of the

(55) $$\text{C.I.P. } (\delta; L^m_{[1]}).$$

Theorem 8 is no longer true if we replace the space $L^m_{[1]}$ by $L_1{}^m$. This is shown for $m = 2$ by observing that the quadratic spline function $L_3(x)$ is a solution of the

(56) $$\text{C.I.P. } (\delta; L_1{}^2),$$

but not an optimal solution of this problem. In fact, (56) has no optimal solutions. This is easily seen by showing that there are solutions F of (56), such that

$$\mathscr{L}_1{}^2(\delta) = 4 < \int_{-\infty}^{\infty} |F''(x)|\, dx < \int_{-\infty}^{\infty} |L_3''(x)|\, dx,$$

where $\|F''\|_1$ may come as close to the lower bound 4 as we wish, without being able to reach it.

As might be expected, the most complete results are available for

$$\text{C.I.P. } (y; L_2{}^m).$$

By Theorem 1 there are solutions if and only if

(57) $$y \in l_2{}^m, \qquad \text{i.e., } \sum_j |\Delta^m y_j|^2 < \infty,$$

which we assume to hold. Theorems 2 and 3 are, of course, applicable for $p = 2$. However, we have seen from our remark following the statement of Theorem 5, that (24) does not furnish the best bound for the functional $\mathscr{L}_2{}^m(y)$. In fact, the interpolating spline $S_y(x)$ of Theorem 2 is not able to produce it. A better substitute for Theorem 2, for our case $p = 2$, is as follows.

THEOREM 9. *The*

(58) $$\text{C.I.P. } (y; L_2{}^m), \qquad (y \in l_2{}^m),$$

admits a unique solution that is a spline function of degree $2m - 1$ *having its knots at the integers. We shall denote this solution by any one of the symbols*

(59) $$S = S(x) = S_{2,y} = S_{2,y}(x).$$

This spline solution is an optimal solution of the problem (58) *and we may therefore write*

(60) $$\mathscr{L}_2{}^m(y) = \|S_{2,y}^{(m)}\|_2.$$

This theorem was previously announced by the author (see [*6*], Theorem 7, p. 27). It is also contained as a special case in the recent general results of Golomb and Schoenberg [*9*]. We also mention here that this optimal solution of (58) is explicitly represented by the convergent series

(61) $$S_{2,y}(x) = \sum_j y_j L_{2m}(x - j),$$

where

$$L_{2m}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_{2m}(u)}{\phi_{2m}(u)} e^{iux}\, du, \qquad (-\infty < x < \infty). \tag{62}$$

(See [3], Theorem 8, pp. 79–80, for $t = 0$.)

The functional (60) also admits an explicit representation in terms of the sequence $\Delta^m y$. To describe it, we observe that (57) and the Riesz–Fischer theorem imply the existence of a function $g(u)$ defined by

$$\underset{n\to\infty}{\mathrm{l.i.m.}} \sum_{-n}^{n} \Delta^m y_j e^{iju} = g(u) \in L_2(-\pi, \pi). \tag{63}$$

The representation mentioned above is described by

THEOREM 10. *The square of the functional* (60) *can be expressed as follows*:

$$(\mathscr{L}_2{}^m(y))^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi_{2m}(u)} |g(u)|^2\, du. \tag{64}$$

Theorem 5, giving the value of the norm $\|\mathscr{L}_2{}^m\|$, will be an immediate corollary of Theorem 10.

The present paper is divided into three parts that are sufficiently described by the table of contents. The paper is self-contained and assumes no previous acquaintance with spline functions. To make it so, I have reproduced a few elementary facts from my previous paper [3]. The L_2-theory (Section III) is the more completely developed part of the subject. The reader who wishes to read only Section III, may omit Subsections 6, 7, and 8.

The basic ideas developed in this paper go back to 1963. (See, e.g., [6].) The reason for writing it at the present time was the appearance of the translation of Subbotin's interesting paper [7], which was kindly called to my attention by Mr. Blair Swartz. The main purpose of the present paper is to show that the tools developed in the 1946 paper [3] are well-suited to deal with the problems here discussed.

It is a pleasure to acknowledge helpful conversations with my colleague Louis B. Rall. I owe to him the convenient notations $L_p{}^m$ and $l_p{}^m$ used throughout the paper.

I. A Few Properties of Spline Functions with Equidistant Knots

We discuss here a few elementary facts, many of which are to be found in [3]. Precise references are given as we proceed.

1. *The B-Splines*

We consider the rectangular frequency function

$$M_1(x) = \begin{cases} 1 & \text{if } -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}, \\ 0 & \text{elsewhere,} \end{cases} \tag{1.1}$$

and form the frequency function $M_k(x)$ by convoluting $M_1(x)$ with itself k times:

$$M_k(x) = \overbrace{M_1 * \ldots * M_1}^{k}(x).$$

The Fourier transform of M_1 being

$$\int_{-\infty}^{\infty} M_1(x)\, e^{ixu}\, dx = \frac{2\sin(u/2)}{u},$$

we conclude that

$$\int_{-\infty}^{\infty} M_k(x)\, e^{ixu}\, dx = \psi_k(u), \tag{1.2}$$

where

$$\psi_k(u) = \left(\frac{2\sin(u/2)}{u}\right)^k. \tag{1.3}$$

Inverting (1.2), we find

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_k(u)\, e^{-inx}\, du. \tag{1.4}$$

In terms of the function

$$x_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

we easily obtain the explicit expression

$$M_k(x) = \frac{1}{(k-1)!}\,\delta^k x_+^{k-1},$$

which shows that $M_k(x)$ is a spline function of degree $k-1$ having as knots the points ν (ν integer), or $\nu + \frac{1}{2}$, depending on whether k is even or odd; moreover, $M_k(x)$ is positive in the interval $(-\frac{1}{2}k, \frac{1}{2}k)$ and vanishes everywhere in its complement. $M_k(x)$ is called a *central B-spline*. (See [3], pp. 67–71.)

We also define the so-called *forward B-spline* by

$$Q_k(x) = M_k\left(x - \frac{k}{2}\right) = \frac{1}{(k-1)!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(x-i)_+^{k-1}. \tag{1.5}$$

It has integer knots; it is positive in the interval $(0, k)$ and vanishes elsewhere.

The following two properties of B-splines are especially relevant:

LEMMA 2. *If $f(x)$ has an absolutely continuous $(k-1)$st derivative then*

$$\Delta^k f(j) = \int_j^{j+k} Q_k(x-j) f^{(k)}(x)\,dx. \tag{1.6}$$

LEMMA 3. *Every spline function $s(x)$ of degree $k-1$ $(-\infty < x < \infty)$, having integer knots, can be represented uniquely in the form*

$$s(x) = \sum_{j=-\infty}^{\infty} c_j Q_k(x-j), \tag{1.7}$$

and conversely, the series (1.7) *represents a spline function of this kind, no matter what values the coefficients may have.*

If $s(x)$ vanishes outside of the interval (r,t), where r, t are integers $(r \leqslant t-k)$, then the representation (1.7) *reduces to*

$$s(x) = \sum_{j=r}^{t-k} c_j Q_k(x-j). \tag{1.8}$$

(See [3], Theorem 5, p. 72, and [8], Theorem 4, p. 80.)

We shall also use the following convolution property of B-splines.

LEMMA 4. *If r, t are natural numbers and j, k are reals, then*

$$\int_{-\infty}^{\infty} M_r(x-j) M_t(x-k)\,dx = M_{r+t}(j-k). \tag{1.9}$$

Proof: From (1.2) we conclude that the Fourier transform of the convolution

$$\int M_r(y) M_t(x-y)\,dy$$

is $\psi_r(u)\psi_t(u) = \psi_{r+t}(u)$. This being also the transform of $M_{r+t}(x)$, we obtain

$$\int_{-\infty}^{\infty} M_r(y) M_t(x-y)\,dy = M_{r+t}(x),$$

whence (1.9) follows in view of the evenness of the functions $M_k(x)$. (1.5) and (1.9) evidently imply

$$\int_{-\infty}^{\infty} Q_r(x-j) Q_t(x-k)\,dx = M_{r+t}\left(j-k+\frac{r-t}{2}\right). \tag{1.10}$$

2. *The Cosine Polynomials $\phi_k(u)$ and Related Lemmas*

With $\psi_k(u)$ defined by (1.3) we define

$$\phi_k(u) = \sum_j \psi_k(u+2\pi j). \tag{2.1}$$

This is a function of period 2π. By a computation that is equivalent to Poisson's summation formula, we find the Fourier coefficients of $\phi_k(u)$ to be

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_k(u)\,e^{-iuv}\,du=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_j \psi_k(u+2\pi j)\right)e^{-iuv}\,du$$
$$=\sum_j \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_k(u+2\pi j)\,e^{-iuv}\,du$$
$$=\sum_j \frac{1}{2\pi}\int_{-\pi+2\pi j}^{\pi+2\pi j}\psi_k(u)\,e^{-iuv}\,du$$
$$=\frac{1}{2\pi}\int_{\infty}^{\infty}\psi_k(u)\,e^{-iuv}\,du=M_k(v)$$

and therefore

$$\phi_k(u)=\sum_v M_k(v)\,e^{ivu}=\sum_{|v|\leqslant k/2} M_k(v)\,e^{ivu}. \tag{2.2}$$

This shows that $\phi_k(u)$ is a cosine polynomial of the exact order $[(k+1)/2]-1$.
From (1.3) and (2.1) we find that

$$\phi_k(u)=\left(2\sin\frac{u}{2}\right)^k\sum_j \frac{(-1)^{jk}}{(u+2\pi j)^k}. \tag{2.3}$$

In terms of the two sequences of functions

$$\begin{aligned}\rho_k(u)&=\left(2\sin\frac{u}{2}\right)^k\sum_j \frac{1}{(u+2\pi j)^k},\\ \sigma_k(u)&=\left(2\sin\frac{u}{2}\right)^k\sum_j \frac{(-1)^j}{(u+2\pi j)^k},\end{aligned} \tag{2.4}$$

we obtain

$$\phi_k(u)=\begin{cases}\rho_k(u) & \text{if } k \text{ is even,}\\ \sigma_k(u) & \text{if } k \text{ is odd.}\end{cases} \tag{2.5}$$

The functions (2.4) can be obtained recursively by

$$\rho_{k+1}(u)=\cos\frac{u}{2}\,\rho_k(u)-\frac{2}{k}\sin\frac{u}{2}\,\rho_k{}'(u),$$

$$\sigma_{k+1}(u)=\cos\frac{u}{2}\,\sigma_k(u)-\frac{2}{k}\sin\frac{u}{2}\,\sigma_k{}'(u),$$

starting with the initial values

$$\rho_2(u)=1, \qquad \sigma_1(u)=1.$$

However, it is more convenient to pass to the new variable

$$x = \cos \tfrac{1}{2} u, \tag{2.6}$$

in terms of which we obtain the following: Defining two sequences of polynomials $U_k(x)$ and $V_k(x)$ by the recurrence relations

$$U_{k+1}(x) = xU_k(x) + \frac{1}{k+2}(1-x^2)\, U_k'(x), \qquad \text{with } U_0(x) = 1, \tag{2.7}$$

$$V_{k+1}(x) = xV_k(x) + \frac{1}{k+1}(1-x^2)\, V_k'(x), \qquad \text{with } V_0(x) = 1, \tag{2.8}$$

we may express the functions (2.4) by

$$\rho_k(u) = U_{k-2}(x), \qquad \sigma_k(u) = V_{k-1}(x). \tag{2.9}$$

Finally, we obtain by (2.5)

$$\phi_k(u) = \begin{cases} U_{k-2}(x) & \text{if } k \text{ is even,} \\ V_{k-1}(x) & \text{if } k \text{ is odd.} \end{cases} \tag{2.10}$$

See [3], pp. 114–116. From this reference we reproduce the proof of the following lemma.

LEMMA 5. *$U_k(x)$ and $V_k(x)$ are polynomials of exact degree k and are even or odd according as k is even or odd. The coefficients of their highest terms are positive. Also*

$$U_k(1) = V_k(1) = 1, \qquad U_k(-1) = V_k(-1) = (-1)^k. \tag{2.11}$$

Moreover, the zeros of the even polynomials $U_{2k}(x)$ and $V_{2k}(x)$ are simple and purely imaginary.

We omit proofs of the statements up to and including (2.11), as they follow by simple induction arguments. Concerning the nature of the zeros, we carry through the proof for $U_{2k}(x)$ only, since the proof for $V_{2k}(x)$ is entirely similar. In order to deal with real zeros, we define a new sequence of polynomials $u_k(x)$ by

$$u_k(x) = i^{-k}\, U_k(xi), \qquad (k = 0, 1, \ldots). \tag{2.12}$$

These polynomials are also real and satisfy a recurrence relation which, by (2.7), is seen to be

$$u_{k+1}(x) = xu_k(x) - \frac{1}{k+2}(1+x^2)\, u_k'(x). \tag{2.13}$$

From this we find

$$u_0(x) = 1, \qquad u_1(x) = x, \qquad u_2(x) = \tfrac{1}{3}(2x^2 - 1), \qquad u_3(x) = \tfrac{1}{3}(x^3 - 2x), \ldots.$$

In view of (2.12), it suffices to show that the zeros of $u_k(x)$ are real and simple, while those of $u_{2\nu}(x)$ are also different from zero. This is readily seen by induction as follows. Let $k = 2\nu$ be even and let us assume that the k zeros of $u_k(x)$ are

(2.14) $$-\xi_\nu, -\xi_{\nu-1}, \ldots, -\xi_1, \xi_1, \ldots, \xi_\nu \qquad (0 < \xi_1 < \ldots < \xi_\nu)$$

and are therefore simple. This, and the fact that $u_k'(x)$ has a highest term with a positive coefficient, imply that

$$u_k'(\xi_\nu) > 0,$$

and that the sequence of values of $u_k'(x)$ at the k zeros (2.14), alternate in sign. By (2.13) we find

$$u_{k+1}(\xi_\nu) < 0$$

and that the values of $u_{k+1}(x)$ at the k zeros (2.14), alternate in sign. Since $u_{k+1}(0) = 0$, we conclude that $u_{k+1}(x)$ has ν positive and ν negative zeros which must therefore be simple.

Let now $k = 2\nu + 1$ be odd, and let u_k have the simple zeros

(2.15) $$-\xi_\nu, \ldots, -\xi_1, 0, \xi_1, \ldots, \xi_\nu \qquad (0 < \xi_1 < \ldots < \xi_\nu).$$

We conclude as before that $u_{k+1}(\xi_\nu) < 0$, and that the values of $u_{k+1}(x)$, at the k points (2.15), alternate in sign. Again the conclusion is that $u_{k+1}(x)$ has simple real zeros, none of which vanishes. This establishes the desired result by mathematical induction.

We summarize our results as follows: By (2.10) and in terms of the variable $x = \cos(u/2)$, we may express the cosine polynomials $\phi_k(u)$ by

(2.16) $$\phi_{2m}(u) = U_{2m-2}(x) = c_1 \prod_{\nu=1}^{m-1} (x^2 + \alpha_\nu^2) \qquad (0 < \alpha_1 < \ldots < \alpha_{m-1}; c_1 > 0),$$

(2.17) $$\phi_{2m+1}(u) = V_{2m}(x) = c_2 \prod_{\nu=1}^{m} (x^2 + \beta_\nu^2) \qquad (0 < \beta_1 < \ldots < \beta_m; c_2 > 0).$$

Further properties of $\phi_k(u)$, not discussed in [*3*], are based on these representations. In particular,

$$\phi_1(u) = V_0(x) = 1, \qquad \phi_2(u) = U_0(x) = 1.$$

LEMMA 6. *If $k \geqslant 3$, then $\phi_k(u)$ is a cosine polynomial of degree $[(k+1)/2] - 1$ which is positive and strictly decreasing in the interval $0 \leqslant u \leqslant \pi$, assuming at the endpoints the values*

(2.18) $$\phi_k(0) = 1$$

(2.19) $$\phi_k(\pi) = 2\left(\frac{2}{\pi}\right)^k \sum_{r=1}^{\infty} \frac{(-1)^{(r-1)k}}{(2r-1)^k} > 0$$

It follows that

$$\max_u \phi_k(u) = \phi_k(0) = 1, \qquad \min_u \phi_k(u) = \phi_k(\pi), \tag{2.20}$$

and that these extreme values are assumed, mod 2π, *only for* $u = 0$ *and* $u = \pi$, *respectively.*

Proof. If k is even, say $k = 2m$, then by (2.16)

$$\phi_{2m}(u) = a_0 + a_1 x^2 + \ldots + a_{m-1} x^{2m-2} \tag{2.21}$$

where all a_j are positive. By (2.11), $\phi_{2m}(0) = U_{2m-2}(1) = 1$. As u increases from 0 to π, $x = \cos\frac{1}{2}u$ decreases from 1 to 0, while (2.21) and (2.3) show that $\phi_{2m}(u)$ decreases from 1 to the value

$$\phi_{2m}(\pi) = \left(\frac{2}{\pi}\right)^{2m} \sum_{-\infty}^{\infty} \frac{1}{(2j+1)^{2m}}.$$

If k is odd, $k = 2m+1$, we proceed similarly, using (2.17).

Let

$$z = e^{iu}, \tag{2.22}$$

in terms of which (2.2) becomes

$$\phi_k(u) = \sum_\nu M_k(\nu) z^\nu \qquad (|\nu| \leqslant k/2). \tag{2.23}$$

Our next objective is to obtain information on the zeros of the Laurent polynomial on the right-hand side that is easily derived from (2.16) and (2.17).

Let $k = 2m$ be even. Observe that (2.22) implies

$$4x^2 = (e^{iu/2} + e^{-iu/2})^2 = z + 2 + z^{-1}.$$

If $\alpha > 0$, then $4(x^2 + \alpha^2) = z + 2\delta + z^{-1}$, where $\delta > 1$. Factoring a quadratic we obtain

$$4(x^2 + \alpha^2) = z^{-1}(z^2 + 2\delta z + 1) = z^{-1}(z + \gamma)(z + \gamma^{-1}),$$

where $0 < \gamma = \delta - \sqrt{\delta^2 - 1} < 1$, and finally

$$x^2 + \alpha^2 = (4\gamma)^{-1}(1 + \gamma z)(1 + \gamma z^{-1}), \qquad 0 < \gamma < 1.$$

Applying this decomposition to each of the factors of the product (2.16), we obtain the identity

$$\phi_{2m}(u) = c_3 \prod_{\nu=1}^{m-1} \{(1 + \gamma_\nu z)(1 + \gamma_\nu z^{-1})\} \qquad (0 < \gamma_{m-1} < \ldots < \gamma_1 < 1). \tag{2.24}$$

From this we shall now draw two conclusions:

1. Denoting the right-hand side of (2.24) by $G(z)$ and expanding $(G(-z))^{-1}$ into a Laurent series convergent on the unit circle $|z|=1$, we find

$$\frac{1}{G(-z)} = c_3^{-1} \prod_1^{m-1} \frac{1}{(1-\gamma_\nu z)(1-\gamma_\nu z^{-1})} = \sum_{-\infty}^{\infty} \chi_\nu z^\nu \qquad (|z|=1),$$

where all coefficients χ_ν are *positive* (in fact, they form a *totally positive* sequence). Remembering the definition of $G(z)$, we obtain the expansion

$$\frac{1}{\phi_{2m}(u)} = \sum_{-\infty}^{\infty} (-1)^\nu \chi_\nu z^\nu \qquad (z = e^{iu}). \tag{2.25}$$

This establishes

LEMMA 7. *If $k \geqslant 3$ then the coefficients of the Fourier expansion*

$$\frac{1}{\phi_k(u)} = \sum_\nu \omega_\nu^{(k)} e^{i\nu u} \tag{2.26}$$

have the property that

$$(-1)^\nu \omega_\nu^{(k)} > 0 \qquad \textit{for all } \nu. \tag{2.27}$$

Proof. Indeed, by (2.25), $(-1)^\nu \omega_\nu^{(k)} = \chi_\nu > 0$. We have carried through the proof for $k = 2m$. However, it is clear that we obtain the same result for $k = 2m+1$ if we use the product representation (2.17).

2. Identifying the right side of (2.24) with the right side of (2.2) (for $k = 2m$), we obtain the identity

$$\sum_{-m+1}^{m-1} M_{2m}(\nu) z^\nu = c_3 \prod_{\nu=1}^{m-1} \{(1+\gamma_\nu z)(1+\gamma_\nu z^{-1})\},$$

and multiplying both sides by z^{m-1}, we have

$$\sum_0^{2m-2} M_{2m}(\nu - m + 1) z^\nu = c_3 \prod_1^{m-1} \{(1+\gamma_\nu z)(z+\gamma_\nu)\}. \tag{2.28}$$

This establishes

LEMMA 8. *The reciprocal polynomial*

$$\sum_0^{2m-2} M_{2m}(\nu - m + 1) z^\nu \tag{2.29}$$

has all its zeros negative and simple. Also the polynomial

$$\sum_0^{2m} M_{2m+1}(\nu - m) z^\nu \tag{2.30}$$

has only negative simple zeros.

Proof. The result concerning (2.29) is implied by (2.28) and the inequalities at the end of formula (2.24). Similarly, the product representation (2.17) allows to establish the statement concerning (2.30).

II. THE SPACES L_p^m AND $L_{[1]}^m$

3. *The Necessity of the Conditions of Theorem* 1

Let us begin with the space L_p^m $(1 \leqslant p < \infty)$ and let

$$F(x) \in \text{C.I.P.}\ (y; L_p^m). \tag{3.1}$$

By Lemma 2

$$\Delta^m y_j = \int_j^{j+m} F^{(m)}(x)\, Q_m(x-j)\, dx, \tag{3.2}$$

and Hölder's inequality $(p^{-1} + q^{-1} = 1)$ gives

$$|\Delta^m y_j| \leqslant \|Q_m\|_q \cdot \left(\int_j^{j+m} |F^{(m)}(x)|^p\, dx \right)^{1/p},$$

whence

$$|\Delta^m y_j|^p \leqslant (\|Q_m\|_q)^p \cdot \int_j^{j+m} |F^{(m)}(x)|^p\, dx.$$

Summing on j and raising the result to the power $1/p$,

$$\|\Delta^m y\|_p \leqslant \|Q_m\|_q (m+1)^{1/p} \|F^{(m)}\|_p, \tag{3.3}$$

which settles the matter for the present case.

If $p = \infty$, then (3.2) immediately gives

$$|\Delta^m y_j| \leqslant \|F^{(m)}\|_\infty \int_{-\infty}^{\infty} Q_m(x-j)\, dx = \|F^{(m)}\|_\infty,$$

whence

$$\|\Delta^m y\|_\infty \leqslant \|F^{(m)}\|_\infty, \tag{3.4}$$

and again we are through.

Finally, if

$$F(x) \in \text{C.I.P.}\ (y; L_{[1]}^m),$$

then F is also a solution of the C.I.P. $(y; L_1^m)$, in view of the inclusion relation (11). From the previous case, for $p = 1$, we conclude that (17) must hold, which completes our proof.

4. *The Problem of Determining the Spline Solution* S_y

Let us consider any one of the spaces L_p^m and $L_{[1]}^m$ and let us denote it by the symbol L^m, using the symbol l^m for the corresponding sequence space.

Let $\|F^{(m)}\|$ and $\|\Delta^m y\|$ denote the corresponding seminorms. Therefore the symbols

$$L^m, \qquad l^m, \qquad \|F^{(m)}\|, \qquad \|\Delta^m y\|,$$

stand either for

$$L_p{}^m, \qquad l_p{}^m, \qquad \|F^{(m)}\|_p, \qquad \|\Delta^m y\|_p \qquad (1 \leqslant p \leqslant \infty),$$

or else for

$$L_{[1]}^m, \qquad l_1{}^m, \qquad \|F^{(m)}\|_{[1]}, \qquad \|\Delta^m y\|_1,$$

respectively.

We assume that

$$\|\Delta^m y\| < \infty \tag{4.1}$$

and proceed to the construction of $S_y(x)$. From the description of S_y as given in Theorem 2, S_y is a spline function of degree m with the knots $\nu + \frac{1}{2}(m+1)$, and therefore

$$S^{(m)}(x) = s(x) \tag{4.2}$$

is a step function with discontinuities at the points $\nu + \frac{1}{2}(m+1)$. In terms of the function (1.1) we may therefore write

$$s(x) = \sum_k c_k M_1(x - k - \tfrac{1}{2}m), \tag{4.3}$$

with coefficients c_k yet to be determined.

The function $s(x)$ being a step function constant in each of the intervals (9), the norm of $s(x)$ turns out to be identical to the norm of the sequence $c = (c_k)$. Specifically, we have

$$\|s\|_p = \left(\int_{-\infty}^{\infty} |s|^p \, dx\right)^{1/p} = \left(\sum_k |c_k|^p\right)^{1/p} = \|c\|_p \qquad \text{if } p < \infty, \tag{4.4}$$

$$\|s\|_\infty = \sup_x |s(x)| = \sup_k |c_k| = \|c\|_\infty, \tag{4.5}$$

and

$$\|s\|_{[1]} = \|s\|_1 = \|c\|_1. \tag{4.6}$$

These relations evidently imply

LEMMA 9. *Let the spline function S_y be any solution of the differential equation* (4.2), *where $s(x)$ is defined by* (4.3). *Then*

$$S_y \in L^m \tag{4.7}$$

if and only if

$$(c_k) \in l. \tag{4.8}$$

We shall now attempt to satisfy the interpolatory conditions

$$S(j) = y_j \qquad \text{for all } j. \tag{4.9}$$

These evidently imply that

$$\Delta^m S(j) = \Delta^m y_j. \tag{4.10}$$

On the other hand,

$$\Delta^m S(j) = \int_{-\infty}^{\infty} S^{(m)}(x)\, Q_m(x-j)\, dx = \int_{-\infty}^{\infty} s(x)\, M_m(x - j - \tfrac{1}{2}m)\, dx; \tag{4.11}$$

by (1.6) and (1.5). Moreover, by (1.9) we have

$$\int_{-\infty}^{\infty} M_1(x - k - \tfrac{1}{2}m)\, M_m(x - j - \tfrac{1}{2}m)\, dx = M_{m+1}(k-j) = M_{m+1}(j-k).$$

By substituting the series (4.3) into the integral (4.11), we see that the interpolatory conditions (4.9) imply the relations

$$\sum_k M_{m+1}(j-k)\, c_k = \Delta^m y_j \qquad \text{for all } j. \tag{4.12}$$

Conversely, we may retrace our steps: The relations (4.12) imply the relations (4.10). These may be written as

$$\Delta^m(y_j - S(j)) = 0 \qquad \text{for all } j, \tag{4.13}$$

and imply the existence of a polynomial $P(x)$ of degree not exceeding $m-1$ such that

$$y_j - S(j) = P(j) \qquad \text{for all } j.$$

Therefore the spline function

$$S_y(x) = S(x) + P(x) \tag{4.14}$$

satisfies the relations (4.9).

The effective construction of S_y therefore hinges on our ability of showing the existence and unicity of a sequence $c = (c_k)$ satisfying the system (4.12) and also

$$c \in l. \tag{4.15}$$

5. *On Sequence Convolution Transformations*

The following results seem to be well known. We develop them in detail as we found no convenient reference.

LEMMA 10. *Let* $\alpha = (\alpha_j) \in l_1$; *hence*

$$A = \sum_\nu |\alpha_\nu| < \infty. \tag{5.1}$$

The convolution transformation

$$c_j = \sum_k \alpha_{j-k} d_k, \qquad d = (d_k) \in l_p \qquad (1 \leqslant p \leqslant \infty), \tag{5.2}$$

which transforms $d \to Td = c$, *is a bounded linear transformation of* l_p *into itself; in fact* (5.2) *implies*

$$\|c\|_p \leqslant A\|d\|_p. \tag{5.3}$$

The following beautiful proof is adapted from Krein ([*4*], pp. 227–228) who discusses the discrete Wiener–Hopf problem. We drop the subscript, writing $l_p = l$, $\|d\|_p = \|d\|$, etc.

Proof. Assuming $d \in l$, we may write (5.2) as

$$c_j = \sum_k \alpha_{-k} d_{k+j},$$

whence

$$|c_j| \leqslant \sum |\alpha_{-k}||d_{k+j}| < \infty, \tag{5.4}$$

by (5.1) and the boundedness of d_j. We consider the sequence of elements

$$z_k = (|d_{k+j}|) \qquad (-\infty < k < \infty) \tag{5.5}$$

which are evidently all in l and are such that

$$\|z_k\| = \|d\| \qquad \text{for all } k. \tag{5.6}$$

It follows that the series

$$t = \sum_k |\alpha_{-k}| z_k \tag{5.7}$$

converges in l, its sum t being an element

$$t = (t_j) \in l. \tag{5.8}$$

For the jth component of t we obtain from (5.7), (5.8), and (5.5)

$$t_j = \sum_k |\alpha_{-k}||d_{k+j}|,$$

and comparing with (5.4), we conclude that

$$|c_j| \leqslant t_j,$$

and therefore

$$\|c\| \leqslant \|t\|. \tag{5.9}$$

Thus $c \in l$. Moreover, by (5.9), (5.7), and (5.6),

$$\|Td\| = \|c\| \leqslant \|t\| \leqslant \sum_k |\alpha_{-k}|\|z_k\| = \sum_k |\alpha_{-k}|\|d\| = A\|d\|,$$

and (5.3) is established.

LEMMA 11. *Again, let* $\alpha = (\alpha_j) \in l_1$; *moreover, let*

$$f(z) = \sum_j \alpha_j z^j \neq 0 \qquad \text{if } |z| = 1. \tag{5.10}$$

Furthermore, let

$$(f(z))^{-1} = \sum_j \beta_j z^j \qquad \text{for } |z| = 1, \tag{5.11}$$

where

$$B = \sum_j |\beta_j| < \infty \tag{5.12}$$

by the Wiener–Lévy theorem. Then (5.2) *has an inverse in* l_p *which is explicitly given by*

$$d_j = \sum_k \beta_{j-k} c_k. \tag{5.13}$$

Proof. Our assumptions (5.10), (5.11) imply that

$$\sum_j \alpha_{r-j} \beta_{j-s} = \sum_j \beta_{r-j} \alpha_{j-s} = \delta_{rs}. \tag{5.14}$$

Multiplying (5.2) by β_{r-j} and summing on j, we obtain

$$\begin{aligned} \sum_j \beta_{r-j} c_j &= \sum_j \beta_{r-j} \sum_k \alpha_{j-k} d_k \\ &= \sum_k d_k \sum_j \beta_{r-j} \alpha_{j-k} = \sum_k d_k \delta_{rk} = d_r, \end{aligned}$$

which establishes (5.13). The interchange of the order of summation is permitted because

$$\sum_{j,k} |\beta_{r-j}| |\alpha_{j-k}| |d_k| \leqslant AB \sup_j |d_j| < \infty.$$

6. *Proofs of Theorems* 1, 2 *and* 3

In order to apply Lemma 11 to the inversion of the convolution transformation (4.12), we consider the positive cosine polynomial (Lemmas 6 and 7)

$$\phi_{m+1}(u) = \sum_\nu M_{m+1}(\nu) e^{i\nu u} \tag{6.1}$$

and expand its reciprocal in a Fourier series

$$\frac{1}{\phi_{m+1}(u)} = \sum_\nu \omega^{(m+1)} e^{i\nu u}. \tag{6.2}$$

For simplicity we drop the superscript and write $\omega_\nu^{(m+1)} = \omega_\nu$. Lemma 11 implies that

$$c_j = \sum_k \omega_{j-k} \Delta^m y_k \tag{6.3}$$

is a bounded linear transformation of l_p onto itself, whose inverse is

(6.4) $$\Delta^m y_j = \sum_k M_{m+1}(j-k)\, c_k.$$

Assuming (15) to hold, hence

(6.5) $$\Delta^m y = (\Delta^m y_j) \in l_p,$$

we conclude that the sequence (c_j), defined by (6.3), also belongs to l_p. It follows from Lemma 9 that the spline function $S_y(x)$ (of Lemma 9) satisfies

(6.6) $$S_y(x) \in L_p{}^m,$$

as well as the interpolatory conditions (4.9), provided that an appropriate polynomial is added to $S_y(x)$. Thereby we have completed a proof of Theorem 1 because from the assumption (6.5) we have derived the existence of a solution (6.6).

In fact, we have also established Theorem 2, except the unicity of S_y. That S_y is unique we see as follows: By Lemma 9, (6.6) implies that $(c_k) \in l_p$, while $(c_k) \in l_p$ is uniquely defined by the Eqs. (6.4).

We turn now to a proof of Theorem 3. With the same assumptions as in the previous paragraph, we know from (4.4), (4.5), and (4.6) that

(6.7) $$\|S_y^{(m)}\| = \|c\|.$$

From Lemma 10 [relation (5.3)] we know that (6.3) implies

(6.8) $$\|c\| \leqslant \left(\sum_j |\omega_j^{(m+1)}|\right) \|\Delta^m y\|,$$

while Lemma 7 shows that

(6.9) $$\sum_j |\omega_j^{(m+1)}| = \sum_j (-1)^j \omega_j^{(m+1)} = (\phi_{m+1}(\pi))^{-1}.$$

Now (6.7) and (6.8) imply that

$$\|S_y^{(m)}\| \leqslant (\phi_{m+1}(\pi))^{-1} \|\Delta^m y\|.$$

Since $\mathscr{L}^m(y) \leqslant \|S_y^{(m)}\|$, Theorem 3 is established.

7. *The Space* $L_\infty{}^m$: *Proofs of Theorems* 4 *and* 7

We shall now apply the procedure described by Lemma 1 (Introduction) to determine the norm $\|\mathscr{L}_\infty{}^m\|$. For the special sequence y^*, we select any sequence such that

(7.1) $$\Delta^m y_j{}^* = (-1)^j, \qquad (\text{all } j),$$

for instance the sequence

(7.2) $$y_j{}^* = (-1)^m 2^{-m} (-1)^j.$$

For any sequence y such that

$$\|\Delta^m y\|_\infty = 1,$$

we obtain from (6.3)

$$\|S_y^{(m)}\|_\infty = \|c\|_\infty = \sup_j \left|\sum_k \omega_{j-k}^{(m+1)} \Delta^m y_k\right|$$
$$\leqslant \sum_k |\omega_{j-k}^{(m+1)}| = (\phi_{m+1}(\pi))^{-1}.$$

However, for the special sequence y^*, (7.1) implies

$$\|S_{y^*}^{(m)}\| = \|c^*\|_\infty = \sum_k |\omega_{j-k}^{(m-1)}| = (\phi_{m+1}(\pi))^{-1}.$$

The sequence (7.2) is therefore seen to satisfy the condition (37) of Lemma 1. There remains to verify the second condition (38), to the effect that S_{y^*} is an optimal solution of the

(7.3) $$\text{C.I.P. } (y^*; L_\infty{}^m).$$

This requires a discussion of the function

(7.4) $$g_m(x) = \sum_j (-1)^j Q_m(x-j),$$

in particular, a precise description of $\operatorname{sgn} g_m(x)$. From Lemma 3 it is clear that $g_m(x)$ is a spline function of degree $m-1$, with integer knots. We need

LEMMA 12. *The function $g_m(x)$ vanishes at the points $\nu + \frac{1}{2}(m+1)$ and nowhere else. Moreover, it changes sign as we pass from one of the intervals*

(7.5) $$I_\nu{}^m = (\nu + \tfrac{1}{2}(m+1) - 1,\ \nu + \tfrac{1}{2}(m+1))$$

to the next.

Proof. For an integer k, by (1.5) and (2.2),

$$g_m(k + \tfrac{1}{2}m) = \sum_j (-1)^j Q_m(k + \tfrac{1}{2}m - j) = \sum_j (-1)^j M_m(k-j)$$
$$= (-1)^k \sum_j (-1)^j M_m(j) = (-1)^k \phi_m(\pi) \neq 0.$$

This shows that the function

(7.6) $$g_m(x + \tfrac{1}{2}m)/g_m(\tfrac{1}{2}m)$$

is a solution of the interpolation problem

$$F(\nu) = (-1)^\nu.$$

By (47) and (48), we know that the same interpolation problem is also solved by the periodic Euler function

$$\bar{E}_{m-1}(x+\tfrac{1}{2}m)/\bar{E}_{m-1}(\tfrac{1}{2}m). \tag{7.7}$$

Both (7.6) and (7.7) are splines of the same degree $m-1$, with the same knots $\nu+\frac{1}{2}m$, and they are solutions of the same

$$\text{C.I.P. } ((-1)^\nu; L_\infty^{m-1}).$$

From the unicity in Theorem 2 we conclude that the functions (7.6) and (7.7) are identical, and therefore

$$g_m(x) = c\cdot\bar{E}_{m-1}(x) \qquad (c \text{ constant} \neq 0). \tag{7.8}$$

Norlund's description of the Euler polynomials ([*1*], pp. 26–27) shows that $\bar{E}_{m-1}(x)$ has the properties ascribed to $g_m(x)$ by Lemma 12. This establishes the lemma.

We now return to the spline solution of the problem (7.3) and use the simpler notation

$$S_*(x) = S_{y*}(x).$$

Let $F(x)$ be any solution of the C.I.P. $(y^*, L_\infty^{(m)})$ and let us show that

$$\|F^{(m)}\|_\infty \geqslant \|S_*^{(m)}\|_\infty \ (= A_m). \tag{7.9}$$

From

$$\Delta^m y_j^* = \Delta^m F(j) = \int_{-\infty}^{\infty} F^{(m)}(x)\, Q_m(x-j)\, dx$$

we obtain

$$\sum_{-n}^{n} (-1)^j \Delta^m y_j^* = \int_{-\infty}^{\infty} F^{(m)}(x) \left(\sum_{-n}^{n} (-1)^j Q_m(x-j)\right) dx. \tag{7.10}$$

By (7.1), the left side $= 2n+1$, whence

$$2n+1 \leqslant \|F^{(m)}\|_\infty \int_{-\infty}^{\infty} \left|\sum_{-n}^{n} (-1)^j Q_m(x-j)\right| dx, \tag{7.11}$$

and therefore

$$\|F^{(m)}\|_\infty \geqslant (2n+1) \Big/ \int_{-\infty}^{\infty} \left|\sum_{-n}^{n} (-1)^j Q_m(x-j)\right| dx.$$

Comparing the integrand with the expansion (7.4), we obtain

$$\|F^{(m)}\|_\infty \geqslant (2n+1) \Big/ \left\{\int_{-n}^{n} |g_m(x)|\, dx + O(1)\right\}.$$

The function $|g_m(x)|$ having the period 1, letting $n \to \infty$ we obtain

$$\|F^{(m)}\|_\infty \geqslant \left(\int_0^1 |g_m(x)|\, dx\right)^{-1}. \tag{7.12}$$

We now use (7.10) again, this time for the spline solution

$$F(x) = S_*(x).$$

By (4.2) and (4.3),

$$S_*^{(m)}(x) = \sum_k c_k^* M_1(x - k - \tfrac{1}{2}m), \tag{7.13}$$

where, by (6.3), (7.1) and (6.9),

$$c_j^* = \sum \omega_{j-k}^{(m+1)}(-1)^k = (-1)^j(\phi_{m+1}(\pi))^{-1} = (-1)^j A_m. \tag{7.14}$$

The relations (7.13) and (7.14) show that

$$F^{(m)}(x) = S_*^{(m)}(x)$$

is a step function such that

$$|S_*^{(m)}(x)| = A_m \qquad \text{for all } x \neq \text{from the knots}, \tag{7.15}$$

and that it changes sign precisely at the points where we pass from one of the intervals (7.5) to the next. In other words, we obtain

$$S_*^{(m)}(x)\, g_m(x) = A_m |g_m(x)| \qquad \text{for all } x. \tag{7.16}$$

Now (7.10) implies

$$2n + 1 = A_m\left(\int_{-n}^{n} |g_m(x)|\, dx + O(1)\right),$$

whence

$$A_m = \left(\int_0^1 |g_m(x)|\, dx\right)^{-1}. \tag{7.17}$$

In view of (7.12) and (7.15), we obtain

$$\|F^{(m)}\|_\infty \geqslant A_m = \|S_*^{(m)}\|_\infty.$$

This establishes the second condition (38) of Lemma 1, and therefore also Theorem 4. It should be clear that we have also established Theorem 7.

8. *The Space $L_{[1]}^m$: Proofs of Theorems* 6 *and* 8

We consider now the space $L_{[1]}^m$, so that

$$y = (y_j) \in l_1^m, \tag{8.1}$$

which we assume to be on the boundary (34) of the unit ball, hence

$$\|\Delta^m y\|_1 = \sum_j |\Delta^m y_j| = 1. \tag{8.2}$$

For the interpolating spline S_y we have, by (4.6),

(8.3) $$\|S_y^{(m)}\|_{[1]} = \|c\|_1$$

and

(8.4) $$c_j = \sum_k \omega_{j-k}^{(m+1)} \Delta^m y_k.$$

Finally, let y^* be a real sequence satisfying the assumptions of Theorem 8:

(8.5) The sequence $(-1)^j \Delta^m y_j^*$ has no change of sign,

(8.6) $$\sum_j |\Delta^m y_j^*| = 1.$$

We can now argue as follows: On the one hand

$$\|S_y^{(m)}\|_{[1]} = \|c\|_1 = \sum_j |c_j| \leqslant \sum_{j,k} |\omega_{j-k}| |\Delta^m y_k|$$
$$= \left(\sum_j |\omega_j|\right)\left(\sum_k |\Delta^m y_k|\right) = \sum (-1)^j \omega_j^{(m+1)} = (\phi_{m+1}(\pi))^{-1},$$

while on the other hand

$$\|S_{y^*}^{(m)}\|_{[1]} = \|c^*\|_1 = \sum_j |c_j^*| = \sum_{j,k} |(-1)^j(-1)^{j-k}\omega_{j-k}(-1)^k \Delta^m y_k^*|$$
$$= \sum_{j,k} |\omega_{j-k}| |\Delta^m y_k^*| = (\phi_{m+1}(\pi))^{-1}.$$

These results show that *any* real sequence y^* satisfying (8.5) and (8.6) satisfies the condition (37) of Lemma 1. In order to establish Theorem 6 (by Lemma 1) and Theorem 8, we have still to show that the spline solution $S_{y^*}(x)$ is optimal for the

(8.7) $$\text{C.I.P. } (y^*; L_{[1]}^m).$$

Let $F(x)$ be a solution of the problem (8.7), and let us return to the relation (7.10). In terms of the function (7.4) and the intervals (7.5), we derive from (7.10)

$$\left|\sum_{-n}^{n} (-1)^j \Delta^m y_j^*\right| \leqslant \sum_j \operatorname*{ess\,sup}_{x \in I_j^m} |F^{(m)}(x)| \cdot \int_{I_j^m} \left|\sum_{-n}^{n} (-1)^j Q_m(x-j)\right| dx$$
$$= \int_0^1 |g_m(x)|\, dx \cdot \sum_j \operatorname*{ess\,sup}_{I_j^m} |F^{(m)}(x)| + o(1),$$

as $n \to \infty$. Letting $n \to \infty$, we obtain by (8.5) and (8.6),

$$\|\Delta^m y^*\|_1 = 1 \leqslant \|F^{(m)}\|_{[1]} \int_0^1 |g_m(x)|\, dx,$$

whence

(8.8) $$\|F^{(m)}\|_{[1]} \geqslant A_m = (\phi_{m+1}(\pi))^{-1},$$

by (7.17).

We finally use (7.10) again, this time with

$$F(x) = S_{y^*}(x). \tag{8.9}$$

For this function we know that

$$S_{y^*}^{(m)}(x) = c_j^* \qquad \text{if } x \in I_j^m$$

and that the c_j^* alternate in sign, because, for all j,

$$(-1)^j c_j^* = \sum_k (-1)^{j-k} \omega_{j-k} \cdot (-1)^k \Delta^m y_k^*$$

has the fixed sign of the quantities (8.5). For the solution (8.9), (7.10) therefore implies

$$\left| \sum_{-n}^{n} (-1)^j \Delta^m y_j^* \right| = \int_0^1 |g_m(x)|\, dx \cdot \sum_{-n}^{n} |c_k^*| + o(1).$$

Letting $n \to \infty$, we obtain

$$1 = \|c^*\|_1 \cdot \int_0^1 |g_m(x)|\, dx,$$

whence

$$\|S_{y^*}^{(m)}\|_{[1]} = \|c^*\|_1 = A_m. \tag{8.10}$$

Therefore by (8.8),

$$\|F^{(m)}\|_{[1]} \geqslant \|S_{y^*}^{(m)}\|_{[1]},$$

and the second condition (38) of Lemma 1 is satisfied. This establishes Theorems 6 and 8.

III. The Space L_2^m

The main purpose of this third part is to establish Theorems 5, 9, and 10, dealing with solutions in L_2^m. For the spline solution $S_y(x)$ of Theorem 2, the derivative $S_y^{(m)}(x)$ was the step function (4.3). Due to this fact, we had the very simple relations (4.4) and (4.5) between the norms $\|S_y^{(m)}\|_p$ and $\|c\|_p$, in fact the two were equal. Also now, the mth derivative, $S^{(m)}(x)$, is our main concern. However, now $S(x)$ is a spline function of degree $2m - 1$ with integer knots. It follows that $S^{(m)}(x)$ is a spline function of degree $m - 1$, and that it may therefore be written in the form

$$S^{(m)}(x) = \sum_j c_j Q_m(x - j).$$

Now the relationship between $\|S^{(m)}\|_p$ and $\|c\|_p$ is more complicated. Our main result in this direction is Theorem 12 of Subsection 10, its essential point being to show that the assumption

$$(*)\qquad \sum_j c_j\, Q_m(x-j) \in L_p$$

implies that

$$(**)\qquad c = (c_j) \in l_p.$$

The function $Q_m(x)$, being positive in its finite support $(0, m)$, would seem to make the implication from $(*)$ to $(**)$ heuristically clear, and a first reaction would be that a "direct" proof should be available. That there is no such direct proof is shown by the following:

Example. Let

$$\tilde{Q}(x) = M_2(x) + M_2(x-1) = \begin{cases} x+1 & \text{in } [-1, 0], \\ 1 & \text{in } [0, 1], \\ -x+2 & \text{in } [1, 2] \\ 0 & \text{elsewhere.} \end{cases}$$

We consider the function

$$\begin{aligned} s(x) &= \sum_j c_j\, \tilde{Q}(x-j) = \sum c_j M_2(x-j) + \sum c_j M_2(x-j-1) \\ &= \sum c_j M_2(x-j) + \sum c_{j-1} M_2(x-j) = \sum (c_j + c_{j-1}) M_2(x-j). \end{aligned}$$

Choosing $c_j = (-1)^j$, we have

$$s(x) = 0 \in L_p \qquad (p < \infty),$$

while the sequence c is *not* in l_p.

Our proof of $(*) \Rightarrow (**)$ is based on Theorem 11 (Subsection 9) and this, in turn, depends on the solutions of certain elementary eigenvalue problems. These matters are discussed in L_p, as no simplification results from a restriction to L_2.

9. *On Spline Functions Vanishing at all Integers* (*Theorem* 11)

Let k be a natural number and let

$$(9.1)\qquad S(x) = \sum_r c_r M_k(x-r)$$

be a spline function of degree $k-1$ having its knots at the points $\nu + \frac{1}{2}k$. Let the symbol

$$\Sigma_k^{\,0}$$

denote the subclass of functions of the above type such that

(9.2) $$S(j) = 0 \text{ for all integers } j.$$

Observe that the classes $\Sigma_1{}^0$ and $\Sigma_2{}^0$ are trivial, as they contain each only one element that also vanishes identically. Our aim is to establish the following theorem.

THEOREM 11. *Let $k \geqslant 3$, and let*

(9.3) $$S(x) \in \Sigma_k{}^0.$$

The assumption

(9.4) $$S(x) \in L_p{}^s, \qquad \textit{for some } s = 0, \ldots, k-1, \qquad (1 \leqslant p \leqslant \infty)$$

implies that

(9.5) $$S(x) = 0 \qquad \textit{for all } x.$$

Observe that the conditions (9.2) are equivalent, by (9.1), to the relations

(9.6) $$\sum_r M_k(j-r)\, c_r = 0 \qquad \text{for all } j.$$

This is a linear recurrence relation for the unknown sequence (c_r) that can be dealt with by classical methods. In order to adapt these to our particular situation, we introduce the family of spline functions

(9.7) $$S_k(x;\lambda) = \sum_r \lambda^r M_k(x-r) \qquad (\lambda \neq 0)$$

depending on the non-vanishing parameter λ. Let us first find out when these functions belong to $\Sigma_k{}^0$. Setting $x = n$, an integer, we obtain from (9.7)

$$S_k(n;\lambda) = \sum_r \lambda^r M(n-r) = \sum \lambda^r M(r-n)$$
$$= \sum \lambda^{r+n} M(r) = \lambda^n \sum_r \lambda^r M_k(r).$$

We conclude: If $S_k(x;\lambda)$ vanishes for *some* integer $x = n$, then it vanishes for *all* integers, and it does this if and only if λ is a root of the equation

(9.8) $$\sum_r M_k(r)\, \lambda^r = 0.$$

In order to fix the ideas, let us first deal with the case when *k is even, say*

(9.9) $$k = 2m.$$

In this case, (9.8) is equivalent to the equation

(9.10) $$\sum_0^{2m-2} M_{2m}(\nu - m + 1)\, \lambda^\nu = 0.$$

The left side is a reciprocal polynomial identical with the polynomial (2.29) of Lemma 8. By Lemma 8 we know that (9.10) has all its roots negative and simple. We may therefore assume that the roots $\lambda_1, \ldots, \lambda_{2m-2}$ satisfy the inequalities

$$\lambda_{2m-2} < \ldots < \lambda_m < -1 < \lambda_{m-1} < \ldots < \lambda_1 < 0 \tag{9.11}$$

and

$$\lambda_1 \lambda_{2m-2} = \lambda_2 \lambda_{2m-3} = \ldots = \lambda_{m-1} \lambda_m = 1. \tag{9.12}$$

Corresponding to these roots, we obtain $2m-2$ spline functions

$$S_l(x) = S_{2m}(x; \lambda_l) = \sum_r \lambda_l^r M_{2m}(x-r) \qquad (l = 1, \ldots, 2m-2), \tag{9.13}$$

all being elements of Σ_{2m}^0.

We claim that *every* $S(x) \in \Sigma_{2m}^0$ *may be uniquely represented in the form*

$$S(x) = \sum_{1}^{2m-2} a_l S_l(x) \tag{9.14}$$

for appropriate values of the coefficients a_l.

Indeed, the relations (9.6) (for $k = 2m$) show that we may choose arbitrarily the $2m-2$ coefficients

$$c_1, c_2, \ldots, c_{2m-2}, \tag{9.15}$$

all the others being determined recursively by (9.6) in terms of the data (9.15): c_0 is obtained from (9.6) for $j = m-1$, then c_{-1} for $j = m$, and so forth. To derive the representation (9.14), we only need to use the expansions (9.13) and determine the a_l as solutions of the non-singular system

$$\sum_{l=1}^{2m-2} a_l \lambda_l^r = c_r, \qquad (r = 1, \ldots, 2m-2).$$

This insures the agreement of the first $2m-2$ coefficients (9.15) of both sides of (9.14) [in their standard representations (9.1)], and therefore the identity (9.14), that is hereby established.

A proof of Theorem 11 requires a discussion of a few properties of the functions (9.13) which may be called the *eigensplines* of the class Σ_{2m}.

(i) Observe that (9.7) implies

$$S_k(x+1; \lambda) = \sum \lambda^r M_k(x+1-r) = \sum \lambda^{r+1} M_k(x-r),$$

whence the functional equation

$$S_k(x+1; \lambda) = \lambda S_k(x; \lambda). \tag{9.16}$$

In particular, *we obtain for our eigensplines* (9.13) *the identities*

$$S_l(x+1) = \lambda_l S_l(x) \qquad (l = 1, \ldots, 2m-2), \tag{9.17}$$

which imply that

(9.18) $$S_l(x+n) = \lambda_l^n S_l(x) \qquad (-\infty < n < \infty).$$

(ii) *Each of the eigensplines* $S_l(x)$ *is represented in the interval* $(0,1)$ *by a polynomial of exact degree* $2m-1$.

For if, for a certain l, we have $S_l(x) \in \pi_{2m-2}$ in $(0,1)$, (9.18) imply

(9.19) $$S_l(x) \in \pi_{2m-2} \text{ in every interval } (n, n+1).$$

On the other hand we know that

(9.20) $$S_l(x) \in C^{2m-2}(\Re).$$

Evidently, (9.19) and (9.20) show that $S_l(x)$ would reduce to a *single* polynomial of degree $2m-2$, for *all* real values of x, which is clearly absurd.

(iii) The relations (9.17) and the inequalities (9.11) show that the graph of $S_l(x)$ has roughly the same general behavior as the graph of $(\sin \pi x)e^{-x}$ if $l = 1, \ldots, m-1$, and that for $l = m, \ldots, 2m-2$, it behaves like $(\sin \pi x)e^{x}$.

(iv) There are symmetry relations between the two kinds of eigensplines discussed in (iii): From (9.13) and (9.12),

$$\begin{aligned} S_l(-x) &= \sum \lambda_l^r M(-x-r) = \sum \lambda_l^r M(x+r) \\ &= \sum \lambda_l^{-r} M(x-r) = \sum_r \lambda_{2m-1-l}^r M_{2m}(x-r), \end{aligned}$$

and therefore

(9.21) $$S_l(-x) = S_{2m-1-l}(x).$$

Proof of Theorem 11 *for* $k = 2m$. Let us consider the quantities

(9.22) $$\mu_{l,p} = \int_0^1 |S_l^{(s)}(x)|^p \, dx, \qquad (1 \leqslant p < \infty).$$

$S_l(x)$ being in $(0,1)$ a polynomial of exact degree $2m-1$, by Property (ii), while $0 \leqslant s \leqslant 2m-1$, we conclude that all these quantities are *positive*. Now (9.18) shows that

(9.23) $$\int_n^{n+1} |S_l^{(s)}(x)|^p \, dx = |\lambda_l|^{pn} \mu_{l,p}, \qquad (-\infty < n < \infty).$$

We assume now that

(9.24) $$S(x) \in \Sigma_{2m}^0$$

and

(9.25) $$S(x) \in L_p^s, \qquad (1 \leqslant p < \infty).$$

Furthermore, let

(9.26) $$S(x) = \sum_1^{2m-2} a_l S_l(x)$$

be the canonical representation (9.14).

We claim that

$$a_l = 0 \qquad \textit{for } l = m, m+1, \ldots, 2m-2. \tag{9.27}$$

For if

$$a_r \neq 0, \qquad a_{r+1} = \ldots = a_{2m-2} = 0 \qquad (r \geqslant m),$$

then (9.23) shows that

$$\int_n^{n+1} |S^{(s)}(x)|^p \sim |a_r|^p \mu_{r,p} |\lambda_r|^{np} \qquad \text{as } n \to +\infty,$$

and therefore by (9.11),

$$\int_0^\infty |S^{(s)}(x)|^p \, dx = \infty,$$

in contradiction with our assumption (9.25).

Moreover, *also*

$$a_l = 0 \qquad \textit{for } l = 1, \ldots, m-1 \tag{9.28}$$

must hold. For if

$$a_t \neq 0, \qquad a_1 = \ldots = a_{t-1} = 0 \qquad (t \leqslant m-1),$$

then again (9.23) implies the relation

$$\int_n^{n+1} |S^{(s)}(x)|^p \, dx \sim |a_t|^p \mu_{t,p} |\lambda_t|^{np} \qquad \text{as } n \to -\infty,$$

and in view of (9.11),

$$\int_{-\infty}^0 |S^{(s)}(x)|^p \, dx = +\infty,$$

contradicting (9.25). Evidently, (9.27) and (9.28) imply the conclusion (9.5) of Theorem 11.

The case $p = \infty$ is settled by a similar argument, which we omit, using the quantities

$$\mu_{l,\infty} = \sup_{n \leqslant n \leqslant x} |S_l^{(s)}(x)|$$

and the relations

$$\sup_{n \leqslant x \leqslant n+1} |S_l^{(s)}(x)| = |\lambda_l|^n \mu_{l,\infty} \qquad (-\infty < n < \infty).$$

This completes our discussion of the case when $k = 2m$.

Proof of Theorem 11 *if* $k = 2m+1$. The proof is much the same as before, with a few changes in minor details. The characteristic equation (9.8) now becomes

$$\sum_r M_{2m+1}(r) \lambda^r \equiv \lambda^{-m} \sum_{\nu=0}^{2m} M_{2m+1}(\nu - m) \lambda^\nu = 0.$$

Again, by Lemma 8 we have a reciprocal equation with only negative simple roots that may be labeled to satisfy the relations

$$\lambda_{2m} < \ldots < \lambda_{m+1} < -1 < \lambda_m < \ldots < \lambda_1 < 0,$$
$$\lambda_1 \lambda_{2m} = \lambda_2 \lambda_{2m-1} = \ldots < \lambda_m \lambda_{m+1} = 1.$$

The corresponding eigensplines are now

$$S_l(x) = \sum_r \lambda_l^r M_{2m+1}(x-r), \qquad (l = 1, \ldots, 2m).$$

Again an arbitrary element $S(x) \in \Sigma^0_{2m+1}$ may be represented uniquely in the form

$$S(x) = \sum_1^{2m} a_l S_l(x).$$

The analogue of property (ii) is that $S_l(x)$ is represented in the interval $(-\frac{1}{2}, \frac{1}{2})$ by a polynomial of exact degree $2m$. The remainder of the proof may be omitted.

10. *On Spline Functions in L_p (Theorem 12)*

As a first application of Theorem 11 we establish

THEOREM 12. *If*

$$s(x) = \sum_k c_k M_m(x-k), \tag{10.1}$$

then

$$s(x) \in L_p \tag{10.2}$$

if and only if

$$c = (c_k) \in l_p. \tag{10.3}$$

Proof. 1. *Proof of sufficiency.* We assume (10.3). From

$$|s(x)| \leqslant \sum_k |c_k| M_m(x-k)$$

it is clear that *without loss of generality we may assume all c_k to be real and nonnegative.* We assume $c_k \geqslant 0$ for all k and write $\mu = [m/2]$. Then

$$0 \leqslant s(x) = \sum_k c_k M_m(x-k) \leqslant M_m(0) \sum_{n-\mu}^{n+\mu} c_k, \qquad \text{if } n \leqslant x \leqslant n+1. \tag{10.4}$$

The monotonicity of the means $\mathfrak{M}_r(a)$ (see Hardy–Littlewood–Pólya, "Inequalities," 1934, pp. 26–27) implies that

$$\left(\sum_{n-\mu}^{n+\mu} c_k\right)^p \leqslant (2\mu+1)^{p-1} \sum_{n-\mu}^{n+\mu} c_k^{\,p},$$

and therefore by (10.4),

$$\int_n^{n+1} (s(x))^p \, dx \leqslant (M_m(0))^p (2\mu + 1)^{p-1} \sum_{n-\mu}^{n+\mu} c_k^p \qquad \text{for all } n.$$

Adding together all these inequalities, we obtain

$$\int_{-\infty}^{\infty} (s(x))^p \, dx \leqslant (M_m(0))^p (2\mu + 1)^p \sum_k c_k^p < \infty,$$

which concludes the proof of sufficiency.

2. *Proof of necessity.* We assume (10.2), and are to derive (10.3). Define y by $y_j = s(j)$; we first wish to show that

$$(y_j) \in l_p. \tag{10.5}$$

The case $p = \infty$ being clear, we assume $p < \infty$. Let m be even. Then

$$\int_0^1 |s(x)|^p \, dx = \int_0^1 \left| \sum_0^{m-1} a_\nu x^\nu \right|^p dx \geqslant |a_0|^p \min_b \int_0^1 \left| 1 + \sum_1^{m-2} b_\nu x^\nu \right|^p dx. \tag{10.6}$$

Denoting the last factor in (10.6) by $C_{m,p}$, we obtain

$$|y_0|^p = |a_0|^p \leqslant C_{m,p}^{-1} \int_0^1 |s(x)|^p \, dx, \tag{10.7}$$

whence, by shifting and adding the relations (10.7), $\|y\|_p \leqslant C_{m,p}^{-1/p} \|s\|_p$, and (10.5) is established. If m is odd we work similarly with the interval $(-\frac{1}{2}, \frac{1}{2})$. On the other hand, (10.5) and (10.1) imply the relations

$$\sum_k M_m(j-k)\, c_k = y_j. \tag{10.8}$$

To this system we can apply Lemma 11 and conclude the existence of a unique sequence $c^* = (c_k^*)$ such that

$$\sum_k M_m(j-k)\, c_k^* = y_j, \qquad (c_k^*) \in l_p. \tag{10.9}$$

By means of c^* we now define

$$s^*(x) = \sum_k c_k^* M_m(x-k). \tag{10.10}$$

The sufficiency part having been established, we conclude from the second relation (10.9) that

$$s^*(x) \in L_p. \tag{10.11}$$

Moreover, the first relations (10.9) show that

$$s^*(x) \in \text{C.I.P. } (y, L_p). \tag{10.12}$$

Since $s(x)$ and $s^*(x)$ are solutions of the same C.I.P. (y, L_p), we conclude that

$$s(x) - s^*(x) \in \Sigma_m^0.$$

Also, (10.2) and (10.11) imply that

$$s(x) - s^*(x) \in L_p.$$

From Theorem 11 we conclude that

$$s(x) = s^*(x) \qquad \text{for all } x.$$

Therefore $c_k = c_k{}^*$ for all k, and $c^* \in l_p$ now implies the desired conclusion (10.3).

11. *Two Lemmas on Spline Functions in L_2*

Let

$$s(x) = \sum_r^t c_j\, Q_m(x-j) \tag{11.1}$$

be a spline function of degree $m-1$ with knots at the integers and having its support in the interval $(r, t+m)$, $r \leqslant t$.

LEMMA 14. *The relation* (11.1) *implies the inequalities*

$$\phi_{2m}(\pi) \sum_r^t |c_j|^2 \leqslant \int_{-\infty}^{\infty} |s(x)|^2\, dx \leqslant \sum_r^t |c_j|^2, \tag{11.2}$$

where the constant on the left is given by (2.19) *for* $k = 2m$.

Proof. The relation (1.10) shows that

$$\int_{-\infty}^{\infty} |s(x)|^2\, dx = \int_{-\infty}^{\infty} \left| \sum_r^t c_j\, Q_m(x-j) \right|^2 dx = \sum_r^t M_{2m}(j-k)\, c_j\, \bar{c}_k. \tag{11.3}$$

From (2.2),

$$\phi_{2m}(u) = \sum_\nu M_{2m}(\nu)\, e^{i\nu u},$$

and the Hermitian form (11.3) is seen to be a section of the Toeplitz form associated with this Fourier series. Clearly

$$M_{2m}(j-k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{2m}(u)\, e^{-i(j-k)u}\, du,$$

and substituting into (11.3), we obtain

$$\int_{-\infty}^{\infty} |s(x)|^2\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{2m}(u) \left| \sum_r^t c_j\, e^{-iju} \right|^2 du. \tag{11.4}$$

On the other hand,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_r^t c_j\, e^{-iju} \right|^2 du = \sum_r^t |c_j|^2. \tag{11.5}$$

The information (2.20) of Lemma 6, concerning the extreme values of $\phi_{2m}(u)$, if applied to the integral on the right side of (11.4), immediately yields the desired inequalities (11.2).

LEMMA 15. *If*

$$s(x) = \sum_j c_j Q_m(x-j) \tag{11.6}$$

and

$$s(x) \in L_2, \tag{11.7}$$

then

$$\int_{-\infty}^{\infty} |s(x)|^2 \, dx = \sum_{j,k} M_{2m}(j-k) c_j \bar{c}_k. \tag{11.8}$$

Proof. From Theorem 12 we conclude that (11.7) implies that

$$\sum_j |c_j|^2 < \infty. \tag{11.9}$$

On the other hand, (11.3) shows that

$$\int_{-\infty}^{\infty} \left| \sum_{-n}^{n} c_j Q_m(x-j) \right|^2 dx = \sum_{j,k=-n}^{n} M_{2m}(j-k) c_j \bar{c}_k. \tag{11.10}$$

By (11.9) and the inequalities (11.2) of Lemma 14, it is clear that

$$\operatorname*{l.i.m.}_{n\to\infty} \sum_{-n}^{n} c_j Q_m(x-j) = s(x) \qquad \text{in } L_2(\mathfrak{R}),$$

and it follows that the integral (11.10) converges to the integral (11.8). The absolute convergence of the double series (11.8) is seen from

$$2 \sum_{j,k} M(j-k) |c_j| |\bar{c}_k| \leqslant \sum M(j-k) (|c_j|^2 + |c_k|^2)$$
$$= 2 \sum_j |c_j|^2 < \infty.$$

12. *Proof of Theorem* 9

We assume that

$$y \in l_2^m \tag{12.1}$$

and wish to construct the solution $S(x)$ of problem (58) as described in Theorem 9. We could now argue as follows: It is easy to see that our assumption (12.1) implies that

$$y \in l_2^{2m-1}.$$

We can now apply Theorem 2, for $2m-1$ rather than m, and conclude the existence of the interpolating spline S of degree $2m-1$ such that

$$S(x) \in L_2^{2m-1}.$$

However, it would not be easy now to show that $S(x) \in L_2{}^m$, as we must.

For this reason we prefer to proceed differently: As in Subsection 4 we again try to satisfy the relations (4.10), but this time by a function $S(x)$ whose mth derivative is

$$S^{(m)}(x) = s(x) = \sum_k c_k\, Q_m(x-k) \tag{12.2}$$

rather than (4.3). Using (1.10), for $r = t = m$, to implement the relation

$$\Delta^m S(j) = \int_{-\infty}^{\infty} s(x)\, Q_m(x-j)\, dx,$$

we see that the relations (4.10) are equivalent to the relations

$$\sum_k M_{2m}(j-k)\, c_k = \Delta^m y_j \qquad \text{for all } j. \tag{12.3}$$

Using Lemma 11 and the expansion (2.26), for $k = 2m$, we construct $S(x)$ as follows: the unique solution $(c_k) \in l_2$ of the system (12.3), also defined by the inverse system

$$c_j = \sum_k \omega_{j-k}^{(2m)}\, \Delta^m y_k, \tag{12.4}$$

is used to define the last member of the relations (12.2). By Theorem 12 we know that

$$S^{(m)}(x) = s(x) \in L_2. \tag{12.5}$$

Finally, an appropriate solution $S(x)$ of this differential equation produces the desired spline solution $S(x)$, as explained in connection with the relations (4.13) and (4.14).

The unicity of our spline solution $S(x)$ follows from Theorem 12 and the fact that the system (12.3) has a unique solution (c_k) in l_2.

Let us now establish the optimality property (60) of the spline solution $S(x)$ just obtained. If $F(x)$ is any solution of the

$$\text{C.I.P. } (y; L_2{}^m), \tag{12.6}$$

then

$$\int_{-\infty}^{\infty} |F^{(m)} - S^{(m)}|^2\, dx = \int_{-\infty}^{\infty} |F^{(m)}|^2\, dx + \int_{-\infty}^{\infty} S^{(m)}\, \bar{S}^{(m)}\, dx - 2\,\mathrm{Re} \int_{-\infty}^{\infty} S^{(m)}\, \bar{F}^{(m)}\, dx,$$

which we may also write as

$$\int_{-\infty}^{\infty} |F^{(m)} - S^{(m)}|^2\, dx = \int_{-\infty}^{\infty} |F^{(m)}|^2\, dx - \int_{-\infty}^{\infty} |S^{(m)}|^2\, dx + 2\,\mathrm{Re} \int_{-\infty}^{\infty} S^{(m)}(\bar{S}^{(m)} - \bar{F}^{(m)})\, dx. \tag{12.7}$$

Let us show that the last term on the right side vanishes: Writing $R(x) = S(x) - F(x)$, (12.2) implies

$$\int_{-\infty}^{\infty} \bar{R}^{(m)} S^{(m)}\,dx = \int_{-\infty}^{\infty} \bar{R}^{(m)}(x) \Big(\sum_j c_j\, Q_m(x-j)\Big)\,dx, \tag{12.8}$$

while

$$\int_{-\infty}^{\infty} \bar{R}^{(m)}(x) \Big(\sum_{-n}^{n} c_j\, Q_m(x-j)\Big)\,dx = \sum_{-n}^{n} c_j \int_{-\infty}^{\infty} \bar{R}^{(m)}(x)\, Q_m(x-j)\,dx. \tag{12.9}$$

We claim that *the left side of* (12.9) *converges to the right side of* (12.8) *as* $n \to \infty$. Indeed, the integrand on the left of (12.9) is dominated by the function (independent of n)

$$|\bar{R}^{(m)}(x)| \cdot \sum_{-\infty}^{\infty} |c_j|\, Q_m(x-j), \tag{12.10}$$

and this function is in $L_1(\mathfrak{R})$ for the following reason: from $\sum |c_j|^2 < \infty$ and Theorem 12, we conclude that the second factor of (12.10) is in L_2; Since $\bar{R}^{(m)} \in L_2$, we see that the function (12.10) is summable by Schwarz's inequality. This establishes our last italicized statement by the bounded convergence theorem. On the other hand, observe that

$$\int_{-\infty}^{\infty} \bar{R}^{(m)}(x)\, Q_m(x-j)\,dx = \Delta^m \bar{R}(j) = 0 \qquad \text{for all } j,$$

because $\bar{R} = S - F$ vanishes at all integers. It follows that the left side of (12.9) vanishes, for all n, and therefore also its limit (12.8). Thus (12.7) reduces to

$$\int_{-\infty}^{\infty} |F^{(m)}|^2\,dx = \int_{-\infty}^{\infty} |S^{(m)}|^2\,dx + \int_{-\infty}^{\infty} |F^{(m)} - S^{(m)}|^2\,dx,$$

which evidently implies that $S(x)$ is the unique optimal solution of (12.6). This completes our proof of Theorem 9.

13. *Proofs of Theorems* 10 *and* 5

Let (12.1) hold, and let $S(x)$ denote the spline solution of (12.6) constructed in Subsection 12. From (12.2), (12.3), (12.4), and Lemma 15 (Subsection 11), we obtain

$$\begin{aligned}
(\mathscr{L}_2{}^m(y))^2 &= \int_{-\infty}^{\infty} |S^{(m)}(x)|^2\,dx = \sum_{j,\,k} M_{2m}(j-k)\, c_j\, \bar{c}_k \\
&= \sum_j c_j \sum_k M_{2m}(j-k)\, \bar{c}_k = \sum_j c_j\, \overline{\Delta^m y_j} \\
&= \sum_j \Big(\sum_k \omega_{j-k}^{(2m)} \Delta^m y_k\Big) \overline{\Delta^m y_j},
\end{aligned}$$

and finally

$$(13.1) \qquad (\mathscr{L}_2^m(y))^2 = \sum_{j,k} \omega_{j-k}^{(2m)} \Delta^m y_j \overline{\Delta^m} y_k.$$

Observe that the double-series (13.1) is absolutely convergent: writing

$$d_j = |\Delta^m y_j| \qquad \text{and} \qquad \Omega = \sum_j |\omega^{(2m)}|,$$

we see that it is dominated by

$$\tfrac{1}{2} \sum_{j,k} |\omega_{j-k}^{(2m)}|(d_j^2 + d_k^2) = \Omega \sum_j d_j^2 < \infty.$$

Writing

$$g_n(u) = \sum_{-n}^{n} \Delta^m y_\nu e^{i\nu u},$$

we obtain from

$$\frac{1}{\phi_{2m}(u)} = \sum_\nu \omega_\nu^{(2m)} e^{i\nu u}$$

on the one hand

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi_{2m}(u)} |g_n(u)|^2 \, du = \sum_{j,k=-n}^{n} \omega_{j-k}^{(2m)} \Delta^m y_j \overline{\Delta^m} y_k,$$

and on the other, by (13.1),

$$(\mathscr{L}_2^m(y))^2 = \lim_{n\to\infty} \sum_{-n}^{n} \omega_{j-k}^{(2m)} \Delta^m y_j \overline{\Delta^m} y_k.$$

It follows that

$$(13.2) \qquad \begin{aligned} (\mathscr{L}_2^m(y))^2 &= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi_{2m}(u)} |g_n(u)|^2 \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\phi_{2m}(u)} |g(u)|^2 \, du, \end{aligned}$$

where $g(u)$ is defined by (63). This establishes Theorem 10.

Our last concern is a proof of Theorem 5. This will be an immediate consequence of (13.2) and the behavior of $\phi_{2m}(u)$ as described by Lemma 6 (Subsection 2). For if $g(u)$ is defined by (63), and if we assume that $\|\Delta^m y\|_2 > 0$, then (13.2) and Parseval's theorem show that

$$(13.3) \qquad \left(\frac{\mathscr{L}_2^m(y)}{\|\Delta^m y\|_2}\right)^2 = \frac{\int_{-\pi}^{\pi} (\phi_{2m}(u))^{-1} |g(u)|^2 \, du}{\int_{-\pi}^{\pi} |g(u)|^2 \, du}.$$

The factor $|g(u)|^2$ in the integrands of (13.3) may be thought of as an arbitrary nonnegative element of $L(-\pi,\pi)$, different from the zero element. From

Lemma 6 [relations (2.20)], and the fact that the right side of (13.3) is a weighted average of the function $(\phi_{2m}(u))^{-1}$, we obtain the inequalities

$$1 \cdot \|\Delta^m y\|_2 < \mathscr{L}_2{}^m(y) < B_m \|\Delta^m y\|_2, \tag{13.4}$$

where

$$B_m = (\phi_{2m}(\pi))^{-1/2}.$$

Also the constants 1 and B_m in (13.4) are best possible, and equality in (13.4) is excluded. This completes a proof of Theorem 5 and the paper.

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