Unit I

1 Algorithms and Problem Solving

Syllabus

Algorithm : The Role of Algorithms in Computing - What are algorithms, Algorithms as technology, Evolution of Algorithms, Design of Algorithm, Need of Correctness of Algorithm, Confirming correctness of Algorithm - sample examples, Iterative algorithm design issues.
Problem solving Principles : Classification of problem, problem solving strategies, classification of time complexities (linear, logarithmic etc.)

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1.1 The Role of Algorithms in Computing

SPPU : April-18, 19, March-20, Marks 5

1.1.1 What are Algorithms ?

In this section we will first understand *"What is algorithm?"* and *"When it is required?"*

Definition of Algorithm : The algorithm is defined as a collection of unambiguous instructions occurring in some specific sequence and such an algorithm should produce output for given set of input in finite amount of time.

This definition of algorithm is represented in Fig. 1.1.1.

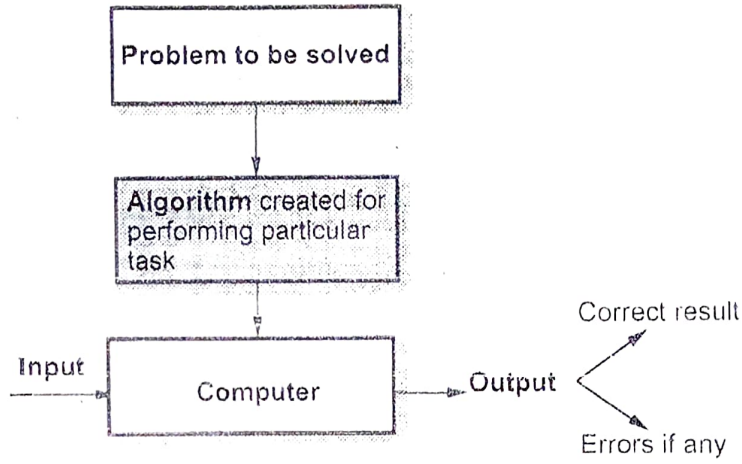


Fig. 1.1.1 Notion of algorithm

After understanding the problem statement we have to create an algorithm carefully for the given problem. The algorithm is then converted into some programming language and then given to some computing device (computer). The computer then executes this algorithm which is actually submitted in the form of source program. During the process of execution it requires certain set of input. With the help of algorithm (in the form of program) and input set, the result is produced as an output. If the given input is invalid then it should raise appropriate error message ; otherwise correct result will be produced as an output.

1.1.2 Properties of Algorithm

Simply writing the sequence of instructions as an algorithm is not sufficient to accomplish certain task. It is necessary to have following properties associated with an algorithm :

1. **Non-ambiguity :** Each step in an algorithm should be non-ambiguous. That means each instruction should be clear and precise. The instruction in an algorithm should not denote any conflicting meaning. This property also indicate the *effectiveness* of algorithm.
2. **Range of input :** The range of input should be specified. This is because normally the algorithm is input driven and if the range of the input is not been specified then algorithm can go in an infinite state.

3. **Multiplicity** : The same algorithm can be represented in several different ways. That means we can write in simple English the sequence of instructions or we can write it in the form of pseudo code. Similarly for solving the same problem we can write several different algorithms. For instance : for searching a number from the given list we can use sequential search or a binary search method. Here *"searching"* is a task and use of either a *"sequential search method"* or *"binary search method"* is an algorithm.
4. **Speed** : The algorithms are written using some specific ideas (which is popularly known as logic of algorithm). But such algorithms should be efficient and should produce the output with fast speed.
5. **Finiteness** : The algorithm should be finite. That means after performing required operations it should terminate.

1.1.3 Algorithms as Technology

- Algorithms are used as a tool to utilize reasonable amount of execution time and memory space.
- Different algorithms can be used to same problem. Each algorithm may take different amount of time.
- For example - For sorting the elements using bubble sort algorithm requires more time than that of merge sort or quick sort algorithm.
- Execution of algorithm depends upon two factors -
  1. Selection of **efficient algorithm** and
  2. **Fast hardware** upon which the algorithm is to be executed.
- For example if there are two algorithms **insertion sort**(less efficient sorting algorithm) and **merge sort**(more efficient sorting algorithm) and there are two computers - computer A and computer B. The computer B is more efficient than computer A. If the merge sort is executed on computer B then this execution is efficient than executing insertion sort on computer B. Thus hardware technology plays an important role in algorithm execution.
- Algorithms are core or fundamental factor for following technologies -
  - Advanced computer architectures
  - Object oriented systems
  - Fast networking applications that can be wired or wireless
  - Applications that use Graphical User Interfaces(GUI)
  - Integrated web technologies

- Ever **increasing storage space of computers** also affect the processing of algorithm. That means the larger problems, or the problems handling large amount of data can be executed efficiently on increased storage space of the computer.

Review Questions

1. *Explain characteristics of good algorithm ? List out the problems solved by algorithm.* **SPPU : April-18, Marks 5**
2. *Write a short note on algorithm as a technology with example.* **SPPU : April-19, Marks 5**
3. *What are algorithms ? Explain algorithm as technology with example.* **SPPU : March-20, Marks 5**

1.2 Evolution of Algorithms

SPPU : April-18, Marks 5

- **Introduction to Concept of Algorithm** : The first algorithm is developed by famous Indian sanskrit grammarian Panini in the text Maheshwar Sutra. This text consists of compact rules and algorithms for Phonetics and syntax of Sanskrit language. He proposed formal language theory almost parallel to modern theory and gave concepts parallel to modern mathematical functions.
- **Structured Programming** : In 1936 Alan Turing introduced the concept of **procedures** due to which the approach for systematic coding is developed. This led to concept of **structured programming**.
- **Correctness of Algorithm** : The concept of proving the **correctness** of **algorithm** is developed.
- **Analysis of Algorithm** : More and more efficient algorithms are then getting developed for solving the problems. Prof. Donald Knuth introduced the concept of **analysis of algorithms**.
- **Complexity Theory** : Later on the concept of **complexity theory** is introduced in which the problems were divided into two classes - **tractable and non-tractable**. The tractable problems is a class of problems which are solvable in reasonable amount of time while non-tractable problems cannot be solved in reasonable amount of time. The concept of **lower bound on running time** is emerged while solving the problems using complexity theory.

Parallel Algorithmic Development : Later on it is found that finding the proof of correctness is very difficult because some proofs can be lengthy or there is no guarantee that the proof will be correct. This led to two developments.

i) **Automatic proof assistants** - Programs which try to generate the detailed proof for a given algorithm with some assistance from a human analyst,

ii) **Parallel development** : The idea that an algorithm should be developed in parallel, along with its proof of correctness.

- **Formal Method of Specification** : A new format popularly known as **formal methods of specification** is introduced to define the proof of correctness.
- **Time Space-trade off** : While searching for effective algorithms researchers found the concept of **time-space trade-off**. The time space tradeoff is a concept which states that for making algorithm time efficient, the space is utilized more or for making the algorithm space efficient, the space is consumed a lot by the algorithm.
- **Approximation Algorithm** : There are lot of problems which are non-tractable. In such a case, the approximation algorithms are developed to solve such problems. Approximation algorithms are those algorithms which gives the expected result within a time limit which is a small multiple of the theoretical limit.
- **Randomness** : There are some algorithms that make use of random number generations for computing the result. For example genetic algorithms makes use of random number generation.

Review Question

1. *Write short note on evolution of algorithm.* **SPPU : April-18, Marks 5**

1.3 Design of Algorithm

SPPU : Dec.-09, Marks 6

Algorithm is basically a sequence of instructions written in simple English language. The algorithm is broadly divided into two sections as shown in Fig. 1.3.1.

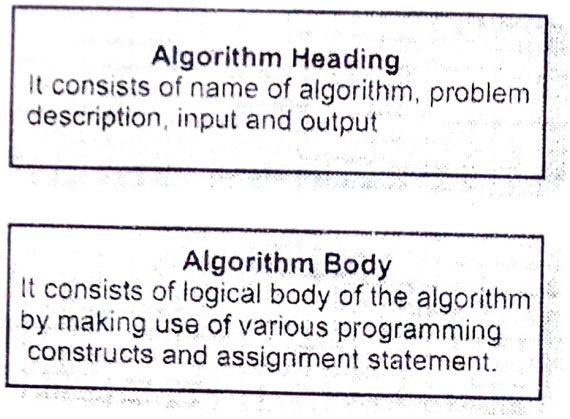
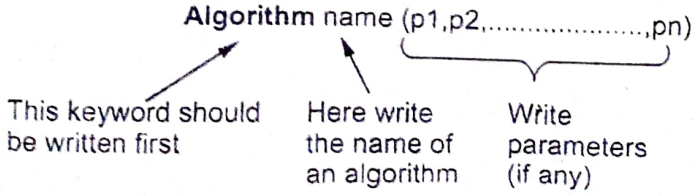


Fig. 1.3.1 Structure of algorithm

Let us understand some rules for writing the algorithm.

1. Algorithm is a procedure consisting of heading and body. The heading consists of keyword **Algorithm** and name of the algorithm and parameter list. The syntax is,



2. Then in the heading section we should write following things :

   //Problem Description :

   //Input :

   //Output :

3. Then body of an algorithm is written, in which various programming constructs like if, for, while or some assignment statements may be written.

4. The compound statements should be enclosed within { and } brackets.

5. Single line comments are written using // as beginning of comment.

6. The **identifier** should begin by letter and not by digit. An identifier can be a combination of alphanumeric string.

   It is not necessary to write data types explicitly for identifiers. It will be represented by the context itself. Basic data types used are integer, float, char, Boolean and so on. The pointer type is also used to point memory location. The compound data type such as structure or record can also be used.

7. Using assignment operator ← an assignment statement can be given.

   For instance : Variable ← expression

8. There are other types of operators such as Boolean operators such as true or false. Logical operators such as **and**, **or**, **not**. And relational operators such as <, <=, >, >=, =, ≠

9. The array indices are stored with in square brackets '[' ']'. The index of array usually start at zero. The multidimensional arrays can also be used in algorithm.

10. The inputting and outputting can be done using **read** and **write**.

    For example :

```
write("This message will be displayed on console");
read(val);
```

11. The conditional statements such as if-then or if-then-else are written in following form :

```
if (condition) then statement
if (condition) then statement else statement
```

    If the **if-then** statement is of compound type then { and } should be used for enclosing block.

12. **while** statement can be written as :

```
while (condition) do
{
        statement 1
        statement 2
            :
            :
        statement n
}
```

While the condition is true the block enclosed with { } gets executed otherwise statement after } will be executed.

13. The general form for writing for loop is :

```
for variable ← value1 to valuen do
{
        statement 1
        statement 2
            :
            :
        statement n
}
```

Here $value_1$ is initialization condition and $value_n$ is a terminating condition. The **step** indicates the increments or decrements in $value_1$ for executing the **for** loop.

Sometime a keyword **step** is used to denote increament or decreament the value of variable for example.

```
for i←1 to n step 1
{
    Write (i)
}
```

Here variable i is incremented by 1 at each iteration

14. The **repeat until** statement can be written as :

```
repeat
        statement 1
        statement 2
            :
            :
        statement n
until (condition)
```

15. The **break** statement is used to exit from inner loop. The **return** statement is used to return control from one point to another. Generally used while exiting from function.

Note that statements in an algorithm executes in sequential order i.e. in the same order as they appear-one after the other.

Some Examples

Example 1.3.1 *Write an algorithm to count the sum of n numbers.*

Solution :

Algorithm sum (1, n)

//Problem Description: This algorithm is for finding the

//sum of given n numbers

//Input: 1 to n numbers

```
//Output: The sum of n numbers
 result ← 0
 for i ← 1 to n do i ← i+1
     result ← result+i
 return result
```

Example 1.3.2 *Write an algorithm to check whether given number is even or odd.*

Solution :

```
Algorithm eventest (val)
//Problem Description: This algorithm test whether given
//number is even or odd
//Input: the number to be tested i.e. val
//Output: Appropriate messages indicating even or oddness
if (val%2=0) then
            write ("Given number is even")
else
            write("Given number is odd")
```

Example 1.3.3 *Write an algorithm for sorting the elements.*

Solution :

```
Algorithm sort (a,n)
//Problem Description: sorting the elements in ascending order
//Input:An array a in which the elements are stored and n
//is total number of elements in the array
//Output: The sorted array
for i ← 1 to n do
for j ← i+1 to n-1 do
{
 if(a[i]>a[j]) then
 {
            temp ← a[i]
            a[i] ← a[j]
            a[j] ← temp
 }
}
write ("List is sorted")
```

Example 1.3.4 *Write an algorithm to find factorial of n number.*

Solution :

```
Algorithm fact (n)
//Problem Description: This algorithm finds the factorial
//of given number n
//Input: The number n of which the factorial is to be
//calculated.
```

```
//Output:factorial value of given n number.
if(n ← 1) then
            return 1
else
            return n*fact(n-1)
```

Example 1.3.5 *Write an algorithm to perform multiplication of two matrices.*

Solution :

```
Algorithm Mul(A,B,n)
//Problem Description:This algorithm is for computing
//multiplication of two matrices
//Input:The two matrices A,B and order of them as n
//Output:The multiplication result will be in matrix C
for i ← 1 to n do
  for j ← 1 to n do
    C[i,j] ← 0
    for k ← 1 to n do
        C[i,j] ← C[i,j]+A[i,k]B[k,j]
```

Example 1.3.6 *Write an algorithm to compute GCD of two given integers.*

Solution :

```
Algorithm GCD_Euclid(a,b)
// Problem Description: This algorithm computes the GCD of
//two numbers a and b using Euclid's method.
//Input: two integers a and b.
//Output: GCD value of a and b.
while (b ≠ 0)do
{
 c ← a mod b
 a ← b
 b ← c
}
return a
```

Example 1.3.7 *Easter sunday is in principle the first sunday after the full moon after the spring equinox. Is this rule sufficiently precise to be called an algorithm ? Justify your answer.* **SPPU : Dec.-09, Marks 6**

Solution : There are two types of day calculations. By lunar age and by solar age. The lunar age starts at 1 and increases to 29 or 30 and then again start at 1. Each period of 29 or 30 makes a lunar month. Thus every alternate month is of 29 days. Hence in a year, if there are 12 months then.

12	×	29	+	6	= 354 days
↓		↓		↓	
(months)		(days)		(6 months have 30 days)	

The solar year is 11 days longer than the lunar age. Hence after 2 years there will be 22 more days in solar year than lunar age. To match both solar and lunar years we must add those many number of days to lunar calender. After 19 years there will be 209 days extra (As 19 X 11 = 209) in solar year than lunar year. If we perform 209 mod 30 then still we get 29 days exact. A cycle of lunar age is repeated after 30 days, hence by considering this issue, a sequence number called **Golden Number** is calculated as -

GN = Number of years mod 19+1.

Each year is associated with **Epact.**

Epact is nothing but the number of excess days in solar year. This epact is associated with the golden number as follows.

Epact = (11 x (GN – 1)) mod 30.

Thus there are some more calculations required to compute the epact of each year. The date of full moon is obtained for each epact and the first sunday following the full moon day of corresponding epact. This sunday is actually an **ester sunday**. Thus **computations** are involved in this ester sunday calculations. More over these calculations must be performed in some specific sequence. Hence this computation can be represented by **algorithm**.

1.4 Need of Correctness of Algorithm

SPPU Dec.-18, Marks 7

1. To ensure that the algorithm is developed to correctly **satisfy the functional requirements**, there is a need for correctness of algorithm.
2. To ensure that the **solution** for the given problem is **valid** we need to check the correctness of algorithm.
3. To check the given problem has **finite or infinite solution** it is essential to check the correctness of algorithm.
4. The need for correctness of algorithm is for **efficient execution** of given task on computers.
5. To ensure that **all the cases** in the problem statements are **covered**, it is necessary to check the correctness of algorithm.
6. For some application correctness of algorithm means safety of human life or very costly equipment. Hence it is necessary to ensure that the algorithm is correct.

Basic steps in algorithmic correctness

Following are the basic steps that need to be followed for checking the correctness of algorithm -

1. Identification of the properties of input data. These properties of data are called **preconditions**.
2. Identification of the properties which must be satisfied by the output data. These properties are called **postconditions**.

3. Proving that starting from the preconditions and executing each step specified in the algorithms one obtains the postconditions.

Example 1.4.1 *Compare between definiteness and effectiveness of an algorithm with example.*

SPPU : Dec.-18, Marks 7

Solution : Definiteness : 1) The algorithm must always terminate after some finite number of states. 2) Each step of algorithm must be precisely defined. Also, the actions must be precisely defined. These actions has to be carried out rigorously and unambiguously.

Effectiveness : 1) All operations specified in the algorithm must be sufficiently basic that they can be done exactly as per specification. 2) Each operation in the algorithmic steps must have finite length.

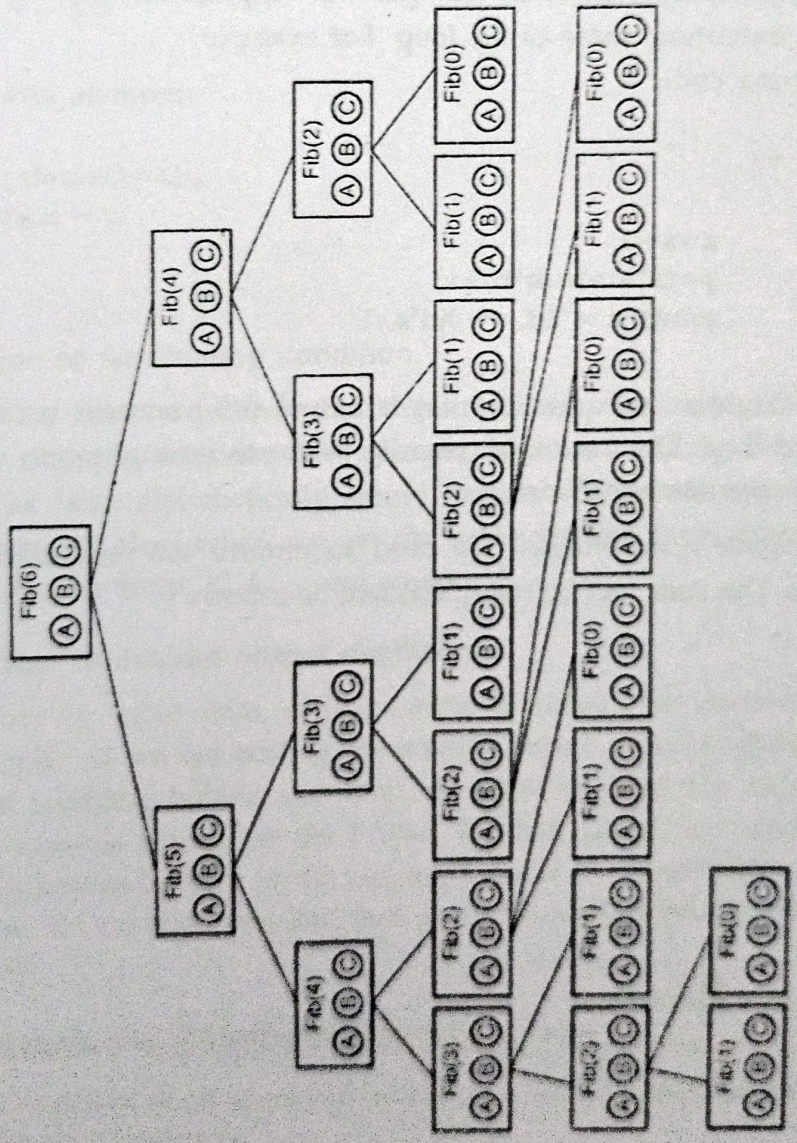


Fig. 1.4.1

1.5 Confirming Correctness of Algorithm - Sample Examples

SPPU : April-18, May-19, March-20, Marks 5

One simple method of confirming the correctness of algorithm is making use of **assertions**. An assertion is a logical statement or expression which is true. For example -

Example : Consider an algorithm for finding the smallest value among the three numbers. The assertions are denoted in curly brackets.

```
Algorithm Find_Min(int a,int b,int c)
{
    if a<b then                 //{a<b}
        if a<c then//{a<b, a<c}
            min<-a
        else            //{a<b, c<a}
            min<c
        endif
    else  //{b<a}
        if b<c then         //{b<a,b<c}
            min<-b
        else            //{b<a, c<b}
            min<-c
        endif
    endif
}
```

Finally we will get the minimum value from integers a,b and c in the variable min.

Example : Consider a problem of Traveling Salesman. This problem states that - " there is a traveling salesman who has to travel through different cities. He can visit each city one and only once. The total tour must be of minimum cost. That means total distance traveled by him must be least. "

For solving such problem using algorithm,

- Every tour is examined and cost of the tour is computed.
- The tour with minimum cost is examined and retained.

The algorithm will terminate because there are finite number of tours. But for large number of tours such algorithm need not be efficient as all the paths need to be examined.

This tells us that **algorithms correctness does not necessarily imply anything about its efficiency.**

Review Questions

1. *How to confirm the correctness of algorithm ? Explain with example.*

   SPPU : April-18, May-19, Marks 5

2. *What is need of correctness of algorithm ? What is loop invariant property ?*

   SPPU : March-20, Marks 5

1.6 Iterative Algorithm Design Issues

SPPU : May-19, March-20, Marks 6

- Iterative algorithm is an algorithm which has at least one iterative component or loop. Hence the part of algorithmic statements will be executed for n number of times.
- Even small amount of time spent on execution of loop will directly affect the efficiency of overall algorithm. This situation can be worst if nested loops (a loop within another loop) is present in the algorithm.
- There are various algorithmic design issues for iterative algorithms -
  i) Correct use of loops in the program.
  ii) Factors that affect the efficiency of algorithm.
  iii) Estimation and specification of execution time.
  iv) Order notations.

1.6.1 Use of Loops

To construct a loop three aspects need to be considered -

i) **Initial condition** : This condition must be true before execution of the loop.

ii) **Invariant relation** : This is a kind of relation that must hold before, during and after each iteration of the loop.

iii) **Terminate condition** : This is a condition under which the loop must terminate.

For example - Consider following loop

```
for(int i=1;i<=10; i++)
{
        printf("Hello");
}
```

Clearly the above loop will execute the printf statement for 10 times and we will get Hello to be printed 10 times on the console. Here

i=0 is **initial condition**

i++ means **incrementing** i on each iteration.

i<=10 is a **terminate condition**. That means when i value goes beyond 10 the loop will not be executed.

1.6.2 Efficiency of Algorithms

- The design and implementation of algorithm have deep influence on their efficiency. Every algorithm on its implementation makes use of CPU time and internal memory (RAM). The cost of utilization of these resources need to be considered while designing the algorithm.
- Today, designing the efficient algorithm is a major requirement.

- There is no standard method for designing efficient algorithm.
- There is a need to generate generalized algorithms as well as specific problem solving algorithm. It always depends upon the problem to be solved.
- There are some **optimization techniques** that improve the efficiency of execution of loop. These are

i. Removing redundant computations inside the loop

- Most of the times the algorithm becomes inefficient due to **redundant computations** or unnecessary storage.
- The effect of redundant instructions is serious when they are used within the loop. These loops repeatedly calculate the part of expression that remains constant throughout the execution phase of the loop. For example -

  Consider following code

```
a=5; b=10;x=1;
for(int i=0;i<10;i++)
{
        x=x+1;
        y=(a*a*a)+(b*b)+x;
        printf("x = %d, y= %d"x,y);
}
```

- In above code fragment computation of (a*a*a) and (b*b) remains unchanged through the execution of loop. The loop unnecessarily performs multiplication within the loop. This makes the algorithm inefficient.
- To make the above code efficient we need to remove the redundant computations inside the loop. The code can be made efficient as follows -

```
a=5; b=10;x=1;
int temp1=a*a*a;
int temp2=b*b;
for(int i=0;i<10;i++)
{
    x=x+1;
    y=temp1+temp2+x;
    printf("x = %d, y= %d"x,y);
}
```

ii. Referencing array element

- Accessing array element inside the loop, makes the algorithm inefficient. For example consider following code -

```
position = 0;
for (i = 1; i < n; i++)
{
```

```
        if (a [i] < a[position])
        {
                position = i;
        }
}
min_element = a[position];
```

In above code the array a[position] is accessed each time during the loop execution. If we denote it using some variable then the execution of above loop can be made efficient.

- The above code can be modified as follows to make it efficient -

```
position = 0;
min_element=a[position];
for (i = 1; i < n; i + +)
{
        if (a [i] < min_element)
        {
                min_element=a[i];
                position = i;
        }
}
```

iii. Late detection of terminating condition

Sometime more tests are conducted than required. This also makes the algorithm inefficient. For example consider the example of linear search. Suppose we want to search the names from alphabetically stored names and we are searching for the name starting with letter 'J'. Now when we get the name starting from letter 'J' we should not scan the letter starting from 'K','L'... and so on.

iv. Early detection of desired output condition

Due to nature of input data, it is possible to detect the desired output condition early. For example - if we are sorting the already sorted data using bubble sort method, then instead of applying bubble sort loop we must examine the input. How can one determine that input is already sorted ? Well, for that matter, we need to check whether there have been any exchanges in the current pass of the inner loop. If there have been no exchanges in the current pass, the data must be already sorted and the loop can be terminated early.

1.6.3 Estimating and Specifying Execution Time

The time complexity of an algorithm is the amount of computer time required by an algorithm to run to completion.

It is **difficult to compute** the **time complexity** in terms of **physically clocked time.** For instance in multiuser system, executing time depends on many **factors** such as -

- System load
- Number of other programs running
- Instruction set used
- Speed of underlying hardware

The time complexity is therefore given in terms of **frequency count.**

Frequency count is a count denoting number of times of execution of statement.

Definition : The frequency count is a count that denotes how many times particular statement is executed.

For Example : Consider following code for counting the frequency count

```
void fun()
{
 int a;
 a=10; ..............................................1
 printf("%d",a); ...................................1
}
```

The frequency count of above program is 2.

Example 1.6.1 *Obtain the frequency count for the following code.*

```
void fun()
{
 int a;
 a=0; ..............................................1
 for(i=0;i<n;i++) ..................................n+1
 {
    a = a+i; .......................................n
 }
 printf("%d",a); ...................................1
 }
```

Solution : The frequency count of above code is **2n + 3.**

The for loop in above given fragment of code is executed n times when the condition is true and one more time when the condition becomes false. Hence for the for loop the frequency count is n + 1. The statement inside the for loop will be executed only when the condition inside the for loop is true. Therefore this statement will be executed for n times. The last printf statement will be executed for once.

1.6.4 Algorithmic Strategies

Problem solving strategy is a general approach by which many problems can be solved algorithmically. These problems may belong to different areas of computing. These strategies are also called as **algorithmic techniques** or **algorithmic paradigms**.

1. **Brute Force :** This is straightforward technique with naïve approach.
2. **Divide and Conquer :** The problem is divided into smaller sub-instances and solution of these sub-instances help to solve the main problem.
3. **Dynamic Programming :** The results of smaller, reoccurring instances are obtained to solve the problem.
4. **Greedy Technique :** To solve the problem locally optimal decisions are made.
5. **Backtracking :** This method is based on trial and error. Each time the solution from the set of solutions is picked and checked if it is the accurate solution or not.

Review Question

1. *Explain issues related to iterative algorithm design.*

SPPU : May-19, Marks 6, March-20, Marks 5

1.7 Problem Solving Principles

The problem solving steps and principles are given below -

1. **Understand the Problem :** For understanding the problem statement, read the problem description carefully and ask the questions such as - What is unknown in the problem ? What is data ? What is the condition ? Then draw the figure and introduce the notations. Separate out various parts of conditions.
2. **Make Decisions :** After finding the required set of input for the given problem we have to analyze the input and need to decide certain issues such as **capabilities of computational devices, selection of problem solving approach, suitable data structures** and **algorithmic techniques**. These decisions serve as the base for the **design of algorithm**.
3. **Design or Specify the Problem :** The design or specification of the problem can be using natural language, flowchart or algorithms.
4. **Verify the Problem :** Problem verification involves checking the correctness of an algorithm. The proof of correctness of the problem can be complex sometimes.
5. **Analyze the Problem :** For analyzing the problem the time and space requirements are considered. The **amount of time** required by algorithm and **amount of space** required by the algorithm is computed and then the analysis is made.

6. **Implement :** The implementation of the problem can be done using some suitable **programming language**.

1.8 Classification of Problem

Generally any computing problem can be solved by an algorithm.There is large number of computing problems and some of them can be classified as,

1. Sorting
2. Searching
3. Numerical Problems
4. Geometric Problems
5. Combinatorial Problems
6. Graph Problems
7. String Processing Problems.

These important problem types can be discussed in detail as follows -

1. Sorting : **Sorting** means arranging the elements in increasing order or in decreasing order (also called as **ascending order** or **descending order** respectively). The sorting can be done on numbers, characters (alphabets), strings or employees record. For sorting any record we need to choose certain piece of information based on which sorting can be done. For instance: for keeping the employees record in sorting order we will arrange the employees record as per employee ID. Similarly one can arrange the library books according to title of the books. This piece of information which is required to sort the records is called **key**. The important property of this key is that **it should be unique**.

2. Searching : Searching is an activity by which we can find out the desired element from the list. The element which is to be searched is called **search key**. There are many searching algorithms such as sequential search, binary search, Fibonacci search and many more.

How to choose best searching algorithm ?

One cannot declare that particular algorithm is best searching algorithm because some algorithms work faster, but then they may require more memory. Some algorithms work better if the list is almost sorted. Thus efficiency of algorithm is varying at varying situations.

Searching in dynamic set of elements

There may be a set of elements in which repeated addition or deletion of elements occur. In such a situation searching an element is difficult. To handle such lists supporting data structures and algorithms are needed make the list balanced (organized).

3. Numerical Problems : The Numerical problems are based on mathematical equations, systems of equations, computing definite integrals, evaluating functions and so on. These mathematical problems are generally solved by approximate algorithms. These algorithms require manipulating of the real numbers hence we may get wrong output many times.

4. Geometric Problems : The geometric problems is one type of problem solving area in which various operations can be performed on geometric objects such as points, line, polygon. The geometric problems are solved mainly in applications to computer graphics, tomography or in robotics.

5. Combinatorial Problems : The combinatorial problems are related to the problems like computing permutations and combinations. The combinatorial problems are most difficult problems in computing area because of following causes -

- As problem size grows the combinatorial objects grow rapidly and reach to a huge value.
- There is no algorithm available which can solve these problems in finite amount of time.
- Many of these problems fall in the category of unsolvable problems. However there are certain problems that are solvable.

6. Graph Problems : Graph is a collection of vertices and edges. The graph problems involve graph traversal algorithms, shortest path algorithms and topological sorting and so on. Some graph problems are very hard to solve. For example Traveling salesman problem, graph coloring problems.

7. String Processing Problems : String is a collection of characters. In computer science the typical string processing algorithm is pattern matching algorithm. In these algorithms particular word is searched from the text. The algorithms lying in this category are simple to implement.

1.9 Problem Solving Strategies

Problem solving strategy is a general approach by which many problems can be solved algorithmically. These problems may belong to different areas of computing. These strategies are also called as **algorithmic techniques** or **algorithmic paradigms**.

1. **Brute Force :** This is straightforward technique with naïve approach.
2. **Divide and Conquer :** The problem is divided into smaller sub-instances and solution of these sub-instances help to solve the main problem.
3. **Dynamic Programming :** The results of smaller, reoccurring instances are obtained to solve the problem.

4. **Greedy Technique :** To solve the problem locally optimal decisions are made.
5. **Backtracking :** This method is based on trial and error. Each time the solution from the set of solutions is picked and checked if it is the accurate solution or not.

1.10 Classification of Time Complexities

We can classify the time complexity of the algorithm in various classes. These classes are also called as **basic efficiency classes**. Each class possessing certain characteristic. These classes are enlisted in the following table,

Name of efficiency class	Order of growth	Description	Example
Constant	1	As input size grows we get larger running time.	Scanning array elements.
Logarithmic	logn	When we get logarithmic running time then it is sure that the algorithm does not consider all its input rather the problem is divided into smaller parts on each iteration.	Performing binary search operation.
Linear	n	The running time of algorithm depends on the input size n.	Performing sequential search operation.
nlogn	nlogn	Some instance of input is considered for the list of size n.	Sorting the elements using merge sort or quick sort.
Quadratic	n^2	When the algorithm has two nested loops then this type of efficiency occurs.	Scanning matrix elements.
Cubic	n^3	When the algorithm has three nested loops then this type of efficiency occurs.	Performing matrix multiplication.
Exponential	2^n	When the algorithm has very faster rate of growth then this type of efficiency occurs.	Generating all subsets of n elements.
Factorial	n!	When an algorithm is computing all the permutations then this type of efficiency occurs.	Generating all permutations.

Example 1.10.1 *Compare orders of growth of $\log_2(n)$ and $\sqrt{n}$.*

Solution : For comparison, we will consider various values of n.

If n = 2

$$\log_2(n) = \log_2 2 = 1$$

$$\sqrt{n} = \sqrt{2} = 1.414$$

If n = 64

$$\log_2(64) = 6$$

$$\sqrt{64} = 8$$

If n = 256

$$\log_2(256) = 8$$

$$\sqrt{256} = 16$$

All these computations show that

$$\log_2(n) < \sqrt{n}$$

❑❑❑

Unit II

2 Analysis of Algorithms and Complexity Theory

Syllabus

Analysis : Input size, best case, worst case, average case
Counting Dominant operators, Growth rate, upper bounds, asymptotic growth, O, Ω, θ, o and ω notations, polynomial and non-polynomial problems, deterministic and non-deterministic algorithms, P- class problems, NP-class of problems, Polynomial problem reduction NP complete problems- vertex cover and 3-SAT and NP hard problem - Hamiltonian cycle.

Contents

2.1 Analysis of Algorithm ..... **Dec.-08,10,12, Aug.-15, May-09,** ..... Marks 8

2.2 Asymptotic Notations ..... **Dec.-06,08,09,10,12,14,19, May-08,09,12,13,14,17,18,19 Aug.-15,** ..... Marks 8

2.3 Best Case, Worst Case, Average Case ..... **Dec.-12, 18, Oct.-16, May-09,** ..... Marks 8

2.4 Growth Rate

2.5 Polynomial and Non-polynomial Problems

2.6 P-class Problems and NP-class of Problems .. **Dec.-12, May-18** ..... Marks 8

2.7 Deterministic and Non-deterministic Algorithms ..... **Dec.-06,10,15,16,18,19, May-07,08,10,11,14,17,** .... Marks 9

2.8 Polynomial Problem Reduction

2.9 NP Complete Problems ..... **May-14,16,17,18,19** ..... Marks 8

..... **Dec.-15,16,19** ..... Marks 8

2.10 NP Hard Problem ..... **May-18,** ..... Marks 8

2.11 Hamiltonian Cycle ..... **Dec.-19,** ..... Marks 8

Part I : Introduction to Analysis of Algorithm

2.1 Analysis of Algorithm

SPPU : Dec.-08,10,12, Aug.-15, May-09, Marks 8

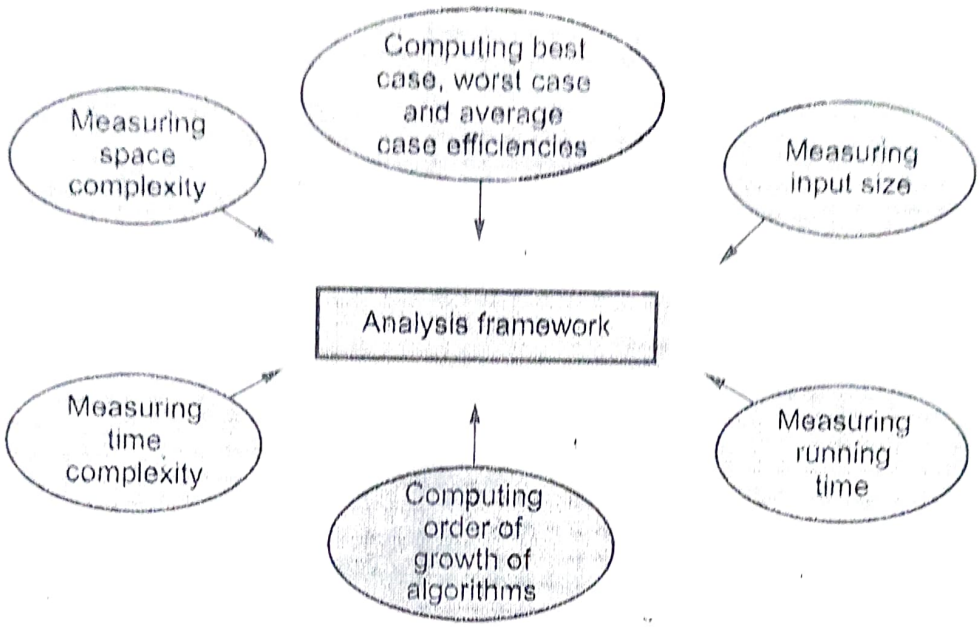


Fig. 2.1.1 Analysis framework

The efficiency of an algorithm can be decided by measuring the performance of an algorithm. We can measure the performance of an algorithm by computing two factors.

1. Amount of time required by an algorithm to execute.
2. Amount of storage required by an algorithm.

This is popularly known as **time complexity** and **space complexity** of an algorithm.

2.1.1 Space Complexity

The space complexity can be defined as amount of memory required by an algorithm to run.

To compute the space complexity we use two factors : **Constant and instance characteristics**. The space requirement S(p) can be given as

$$S(p) = C + Sp$$

where **C** is a constant i.e. fixed part and it denotes the space of inputs and outputs. This space is an amount of space taken by instruction, variables and identifiers. And **Sp** is a space dependent upon instance characteristics. This is a variable part whose space requirement depends on particular problem instance.

There are two types of components that contribute to the space complexity - Fixed part and variable part.

SPPU : May-09, Marks 4

The **fixed part** includes space for

- Instructions
- Variables
- Array size
- Space for constants

The **variable part** includes space for

- The variables whose size is dependent upon the particular problem instance being solved. The control statements (such as for, do, while, choice) are used to solve such instances.
- Recursion stack for handling recursive call.

Consider three examples of algorithms to compute the space complexity.

Example 1 : Compute the space required by following algorithm.

```
Algorithm Add (a,b,c)
//Problem Description : This algorithm computes the addition
//of three elements
//Input: a,b, and c are of floating type
//Output: The addition is returned
        return a+b+c
```

The space requirement for algorithm given in Example 1 is

$$S(p) = C \qquad \Theta(Sp) = 0$$

If we assume that a, b and c occupy one word size then total size comes to be 3.

Example 2 : Compute the space needed by the following algorithms justify your answer.

```
Algorithm Sum(a,n)
{
    s : = 0.0 ;
        For i : = 1 to n do
            s : = s + a[i] ;
            return s
}
```

SPPU : Dec.-08, Marks 8

In the given code we require space for

```
s : = 0          ←    O(1)
    For i : = 1 to n     ←    O(n)
        s : = s + a[i] ; ← O(n)
        returns ;        ← O(1)
```

Hence the space complexity of given algorithm can be denoted in terms of big-oh notation. It is **O(n).**

Example 3 : Compute the space required by following algorithm.

```
Algorithm Add (x,n)
//Problem Description: This is a recursive algorithm which
```

```
//computes addition of all the elements in an array x[]
//Input: x[i] is of floating type, total number of elements
//in an array
//Output: returns addition of n elements of an array
return Add (x,n-1)+x[n]
```

The space requirement is -

$$S(p) \geq 3(n + 1)$$

The internal stack used for recursion includes space for formal parameters, local variables and return address. The space required by each call to function Add requires atleast three words (space for n values + space for return address + pointer to x [ ]). The depth of recursion in n+1 (n times call to function and one return call). The recursion stack space will be $\geq 3(n+1)$.

2.1.2 Time Complexity

The time complexity of an algorithm is the **amount of computer time required** by an algorithm to run to completion.

It is **difficult to compute** the **time complexity** in terms of **physically clocked time**. For instance in multiuser system, executing time depends on many **factors** such as -

- System load
- Number of other programs running
- Instruction set used
- Speed of underlying hardware

The time complexity is therefore given in terms of frequency count.

Frequency count is a count denoting number of times of execution of statement.

Definition : The frequency count is a count that denotes how many times particular statement is executed.

For Example : Consider following code for counting the frequency count

```
void fun()
{
 int a;
 a=10; ..............................................1
 printf("%d",a); ..............................1
}
```

The frequency count of above program is 2.

Example 2.1.1 *Obtain the frequency count for the following code.*

Solution :

```
void fun()
{
 int a;
 a=0; ..............................................1
 for(i=0;i<n;i++) ..................................n+1
  {
     a  = a+i; ......................................n
  }
 printf("%d",a); ....................................1
}
```

The frequency count of above code is **2n + 3**.

The for loop in above given fragment of code is executed n times when the condition is true and one more time when the condition becomes false. Hence for the for loop the frequency count is n + 1. The statement inside the for loop will be executed only when the condition inside the for loop is true. Therefore this statement will be executed for n times. The last printf statement will be executed for once.

Example 2.1.2 *Obtain the frequency count for the following code.*

Solution :

```
void fun(int a[][],int b[][])
{
 int c[3][3];
 for(i=0;i<m;i++) ..................................m+1
 {
  for(j=0;j<n;j++) .................................m(n+1)
  {
   c[i][j]=a[i][j]+b[i][j]; .........................m.n
  }
 }
```

The frequency count = (m + 1) + m(n + 1) + mn = 2m + 2mn + 1 = **2m(1 + n) + 1**

Example 2.1.3 *Obtain the frequency count for the following code.*

```
for(i=1;i<=n;i++)
{
 for(j=1;j<=n;j++)
 {
   c[i][j]=0;
   for(k=1;k<=n;k++)
      c[i][j]=c[i][j]+a[i][k]*b[k][j];
 }
}
```

Solution :

Statement	Frequency count
for(i=1;i<=n;i++)	n + 1
for(j=1;j<=n;j++)	n.(n + 1)
c[i][j]=0;	n.(n)
for(k=1;k<=n;k++)	n.n(n + 1)
c[i][j]=c[i][j]+a[i][k]*b[k][j];	n.n.n
Total	$2n^3 + 3n^2 + 2n + 1$

After counting the frequency count, the constant terms can be neglected and only the order of magnitude is considered. The time complexity is denoted in terms of algorithmic notations. The Big oh notation is a most commonly used algorithmic notation. For the above frequency count all the constant terms are neglected and only the order of magnitude of the polynomial is considered. Hence the time complexity for the above code can be $O(n^3)$. The higher order is the polynomial is always considered.

2.1.3 Measuring an Input Size

It is observed that if the input size is longer, then usually algorithm runs for a longer time. Hence we can compute the efficiency of an algorithm as a function to which input size is passed as a parameter. Sometimes to implement an algorithm we require prior knowledge of input size. For example, while performing multiplication of two matrices we should know order of these matrices. Then only we can enter the elements of matrices. Sometimes the input size is taken as an approximate value. For example, in spell checking algorithms we can predict exact size of the input.

2.1.4 Measuring Running Time

SPPU : Dec.-10, Marks 8

We have already discussed that the time complexity is measured in terms of a unit called **frequency count**.

The time which is measured for analyzing an algorithm is generally running time.

From an algorithm :

- We first identify the important operation (core logic) of an algorithm. This operation is called the **basic operation**.
- It is not difficult to identify basic operation from an algorithm. Generally the operation which is more time consuming is a basic operation in the algorithm. Normally such basic operation is located in **inner loop**. For example in sorting algorithm the operation which is comparing the elements and then placing them at

appropriate locations is a basic operation. The concept of basic operations can be well understood with the help of following example.

Problem statement	Input size	Basic operation
Searching a key element from the list of n elements.	List of n elements.	Comparison of key with every element of list.
Performing matrix multiplication.	The two matrices with order $n \times n$.	Actual multiplication of the elements in the matrices.
Computing GCD of two numbers.	Two numbers.	Division.

Table 2.1.1 Basic operations from input

- Then we compute total number of time taken by this basic operation. We can compute the running time of basic operation by given formula.

$$T(n) \approx c_{op}\, C(n)$$

Running time of basic operation — Time taken by the basic operation to execute — Number of times the operation needs to be executed

- Using this formula the computing time can be obtained approximately.

Review Questions

1. *What are the basic components that contribute to the space complexity ?* **SPPU : May-09, Marks 4**
2. *Explain the different ways of measuring the running time of an algorithm.* **SPPU : Dec.-12, Marks 6**
3. *Explain time and space complexity with suitable examples.* **SPPU : Aug.-15, (In Sem.) Marks 4**

2.2 Asymptotic Notations

SPPU : Dec.-06,08,09,10,12,14, May-08,09,12,13,14,17, Aug.-15, Marks 8

To choose the best algorithm, we need to check efficiency of each algorithm. The efficiency can be measured by computing time complexity of each algorithm. Asymptotic notation is a shorthand way to represent the time complexity.

Using asymptotic notations we can give time complexity as "fastest possible", "slowest possible" or "average time".

Various notations such as Ω, Θ and O used are called **asymptotic notions.**

2.2.1 Big oh Notation

The **Big oh** notation is denoted by 'O'. It is a method of representing the **upper bound of** algorithm's running time. Using big oh notation we can give longest amount of time taken by the algorithm to complete.

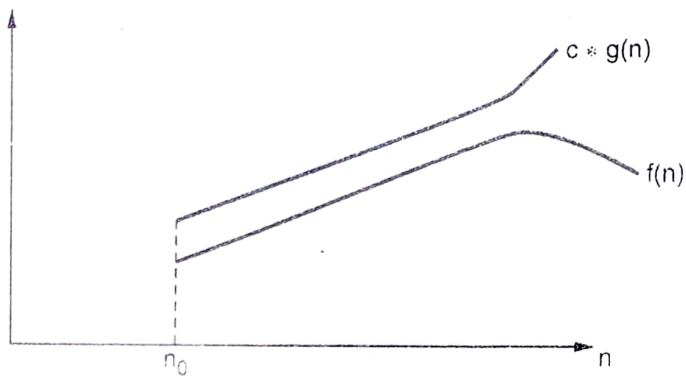
Definition

Let f(n) and g(n) be two non-negative functions.

Let n_0 and constant c are two integers such that n_0 denotes some value of input and $n > n_0$. Similarly c is some constant such that c > 0. We can write

$$f(n) \le c*g(n)$$

then f(n) is big oh of g(n). It is also denoted as **f(n) ∈ O (g(n))**. In other words f(n) is less than g(n) if g(n) is multiple of some constant c.



f(n) ∈ O(g(n))

Fig. 2.2.1 Big oh representation

Example : Consider function f(n) = 2n + 2 and g(n) = n^2. Then we have to find some constant c, so that f(n) ≤ c * g(n). As f(n) = 2n + 2 and g(n) = n^2 then we find c for n = 1 then

$$f(n) = 2n + 2$$
$$= 2(1) + 2$$
$$f(n) = 4$$

and $$g(n) = n^2$$
$$= (1)^2$$
$$g(n) = 1$$

i.e. $$f(n) > g(n)$$

If n = 2 then,

$$f(n) = 2(2) + 2$$
$$= 6$$
$$g(n) = (2)^2$$
$$g(n) = 4$$

i.e. $f(n) > g(n)$

If n = 3 then,

$$f(n) = 2(3) + 2$$
$$= 8$$
$$g(n) = (3)^2$$
$$g(n) = 9$$

i.e. $f(n) < g(n)$ is true.

Hence we can conclude that for n > 2, we obtain

$$f(n) < g(n)$$

Thus always upper bound of existing time is obtained by big oh notation.

2.2.2 Omega Notation

Omega notation is denoted by 'Ω'. This notation is used to represent the **lower bound** of algorithm's running time. Using omega notation we can denote shortest amount of time taken by algorithm.

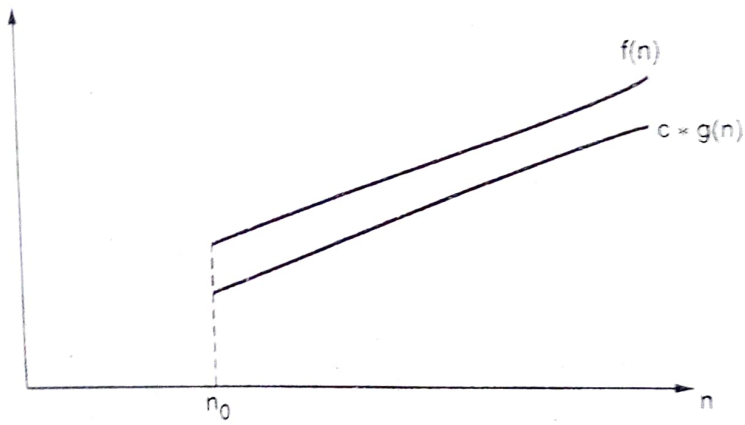
Definition

A function f(n) is said to be in Ω (g(n)) if f(n) is bounded below by some positive constant multiple of g(n) such that

$$f(n) \geq c * g(n) \qquad \text{For all } n \geq n_0$$

It is denoted as $f(n) \in \Omega(g(n))$. Following graph illustrates the curve for Ω notation.

Example :



$f(n) \in \Omega(g(n))$

Fig. 2.2.2 Omega representation

Consider $f(n) = 2n^2 + 5$ and $g(n) = 7n$

Then if $n = 0$

$$f(n) = 2(0)^2 + 5$$
$$= 5$$
$$g(n) = 7(0)$$
$$= 0 \qquad \text{i.e. } f(n) > g(n)$$

But if $n = 1$

$$f(n) = 2(1)^2 + 5$$
$$= 7$$
$$g(n) = 7(1)$$

7 i.e. $f(n) = g(n)$

If $n = 3$ then,

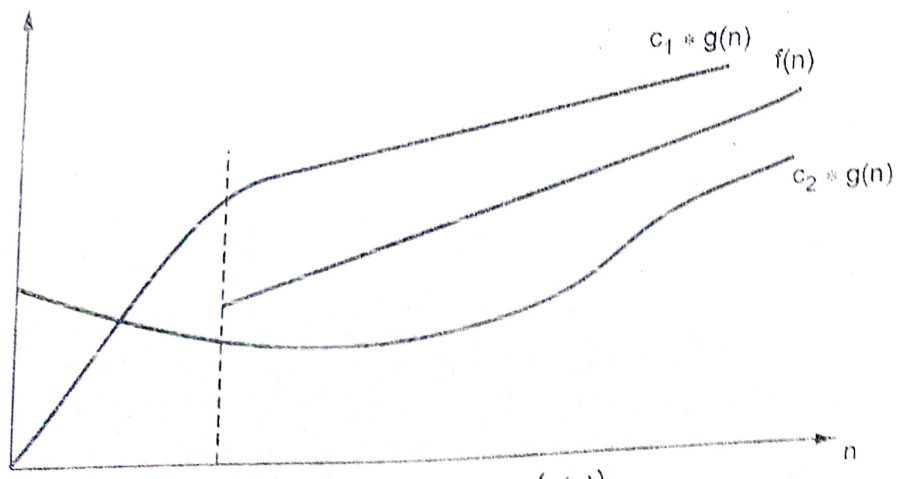
$$f(n) = 2(3)^2 + 5$$
$$= 18 + 5$$
$$= 23$$
$$g(n) = 7(3)$$
$$= 21$$

i.e. $f(n) > g(n)$

Thus for $n > 3$ we get $f(n) > c * g(n)$.

It can be represented as,

$$2n^2 + 5 \in \Omega(n)$$



Theta notation $f(n) \in \Theta(g(n))$

Fig. 2.2.3 Theta representation

2.2.3 Θ Notation

The theta notation is denoted by Θ. By this method the running time is between upper bound and lower bound.

Definition

Let f(n) and g(n) be two non negative functions. There are two positive constants namely c_1 and c_2 such that,

$$c_2 g(n) \leq f(n) \leq c_1 g(n)$$

Then we can say that,

$$f(n) \in \Theta(g(n))$$

Example :

If f(n) = 2n + 8 and g(n) = 5n.

where $n \geq 2$

Similarly $f(n) = 2n + 8$

$g(n) = 7n$

i.e. $5n < 2n + 8 < 7n$ For $n \geq 2$

Here $c_1 = 5$ and $c_2 = 7$ with $n_0 = 2$.

The theta notation is more precise with both big oh and omega notation.

Some examples of asymptotic order

1) $\log_2 n$ is f(n) then

$\log_2 n \in O(n)$ ∵ $\log_2 n \leq O(n)$, the order of growth of $\log_2 n$ is slower than n.

$\log_2 n \in O(n^2)$ ∵ $\log_2 n \leq O(n^2)$, the order of growth of $\log_2 n$ is slower than n^2 as well.

But

$\log_2 n \notin \Omega(n)$ ∵ $\log_2 n \leq \Omega(n)$ and if a certain function F(n) is belonging to Ω(n) it should satisfy the condition $f(n) \geq c * g(n)$

Similarly $\log_2 n \notin \Omega(n^2)$ or $\Omega(n^3)$

2) Let $f(n) = n(n-1)/2$

Then

$n(n-1)/2 \notin O(n)$ $\because$ $f(n) > O(n)$ we get $f(n) = n(n-1)/2 = \frac{n^2-1}{2}$

i.e. maximum order is n^2 which is $> O(n)$.

Hence $f(n) \notin O(n)$

But $n(n-1)/2 \in O(n^2)$ As $f(n) \leq O(n^2)$

and $n(n-1)/2 \in O(n^3)$

Similarly,

$n(n-1)/2 \in \Omega(n)$ $\because$ $f(n) \geq \Omega(n)$

$n(n-1)/2 \in \Omega(n^2)$ $\because$ $f(n) \geq \Omega(n^2)$

$n(n-1)/2 \notin \Omega(n^3)$ $\because$ $f(n) > \Omega(n^3)$

2.2.4 Properties of Order of Growth

1. If $f_1(n)$ is order of $g_1(n)$ and $f_2(n)$ is order of $g_2(n)$, then $f_1(n) + f_2(n) \in O(\max(g_1(n), g_2(n))$.
2. Polynomials of degree $m \in \Theta(n^m)$.

   That means maximum degree is considered from the polynomial.

   For example : $a_1n^3 + a_2n^2 + a_3n + c$ has the order of growth $\Theta(n^3)$.
3. $O(1) < O(\log n) < O(n) < O(n^2) < O(2^n)$.
4. Exponential functions a^n have different orders of growth for different values of a.

Key Points

i) $O(g(n))$ is a class of functions $f(n)$ that grows less fast than $g(n)$, that means $f(n)$ possess the time complexity which is always slower than the time complexities that $g(n)$ have.

ii) $\Theta(g(n))$ is a class of functions $f(n)$ that grows at same rate as $g(n)$.

iii) $\Omega(g(n))$ is a class of functions $f(n)$ that grows faster than or atleast as fast as $g(n)$. That means $f(n)$ is greater than $\Omega(g(n)$.

Example 2.2.1 *State whether the following functions are CORRECT or INCORRECT and justify your answer.*

i) $3n+2 = O(n)$ *ii)* $100n+6 = O(n)$ *iii)* $10n^2 + 4n + 2 = O(n^2)$

SPPU : Dec.-06, Marks 6, Dec.-09, 10, Marks 2

Solution : The function f(n) is said to be in O(g(n)) i.e.

$$f(n) = O(g(n) \text{ or } f(n) \in O(g(n)),$$

if it is bounded above by some constant multiple of g(n) for all large n.

In order words if,

$$f(n) \le c \cdot g(n) \quad \text{For } n \ge n_0$$

Consider

i) $f(n) = 3n + 2$

$g(n) = n$

This assertion can be CORRECT if $c = 5$ and $n \ge 1$

i.e.

$$f(n) = 3n_0 + 2$$

$$= 3(1) + 2 \qquad \text{When } n = 1$$

$$= 5$$

$$c \cdot g(n) = 5 \times 1$$

$$= 5$$

$\therefore$ $f(n) \le c \cdot g(n)$ is true.

If $n = 2$ and $c = 5$ then

$$f(n) = 3n + 2 = 3(2) + 2 = 8$$

$$g(n) = 2 \qquad \therefore \quad c \cdot g(n) = 5(2) = 10$$

Again $f(n) < g(n)$ is true

Thus we can find that $f(n) \le g(n)$ remains true for $n \ge 1$ when $c = 5$.

But $n = 0$ then

$$f(n) = 3n + 2 = 3(0) + 2 = 2$$

$$g(n) = 0 \; c \cdot g(n) = 5 \cdot (0) = 0$$

Then $f(n) > c \cdot g(n)$. Hence $3n + 2 = O(n)$ is correct for $n \ge 1$ and $c = 5$

$3n + 2 = O(n)$ is INCORRECT with $n = 0$ and any constant value.

ii) $f(n) = 100\,n + 6$

$g(n) = n$

$f(n) = O(n)$ is CORRECT if we consider $c = 106$ and $n \geq 1$.

Case 1 : If $n_0 = 1$ then

$$f(n) = 100n + 6$$
$$= 100(1) + 6$$
$$= 106$$
$$g(n) = 1$$
$$\therefore \quad c \cdot g(n) = 106 \cdot (1)$$
$$= 106$$

Thus $f(n) \leq c \cdot g(n)$ is true.

Case 2 : If $n_0 = 2$

$$f(n) = 100n + 6$$
$$= 100(2) + 6$$
$$= 206$$
$$g(n) = 2$$
$$\therefore \quad c \cdot g(n) = 106 \cdot (2)$$
$$= 212$$

Again $f(n) < c \cdot g(n)$

Thus if $f(n) \leq c \cdot g(n)$ for $n \geq 1$ then $100n + 6 = O(n)$ is CORRECT.

But for $n = 0$

$$f(n) = 100n + 6$$
$$= 100(0) + 6$$
$$= 6$$
$$g(n) = 0$$
$$c \cdot g(n) = c \cdot (0) = 0$$

Thus $f(n) > c.g(n)$

Hence $100n + 6 = O(n)$ is INCORRECT for $n = 0$.

iii) $f(n) = 100n^2 + 4n + 2$

$g(n) = n^2$

Consider $c = 106$. The values for n can be

Case 1 : If n = 1 then

$$f(n) = 100n^2 + 4n + 2$$
$$= 100(1)^2 + 4(1) + 2$$
$$= 100 + 4 + 2$$
$$= 106$$
$$g(n) = n^2$$
$$= (1)^2$$
$$= 1$$
$$c \cdot g(n) = 106(1)$$
$$= 106$$

$f(n) = c \cdot g(n)$ is true.

Case 2 : If n = 2 then

$$f(n) = 100n^2 + 4n + 2$$
$$= 100(2)^2 + 4(2) + 2$$
$$= 400 + 8 + 2$$
$$= 410$$
$$c \cdot g(n) = 106\,(2)^2$$
$$= 106(4)$$
$$= 424.$$

$f(n) < c \cdot g(n)$ is true.

From the above cases, $f(n) \le c \cdot g(n)$ remains true for $n \ge 1$.

Hence $100n^2 + 4n + 2 = 0(n^2)$ is CORRECT for $n \ge 1$ and $c = 106$.

But for n = 0, $100n^2 + 4n + 2 \ne 0(n^2)$

Example 2.2.2 *Prove that : $f(n) = a_m n^m + \ldots + a_1 n + a_0$, then*

$$f(n) = O\left(n^m\right)$$

SPPU : Dec.-06, 08, Marks 6, May-08, 13, Marks 8, Aug.-15 (In Sem.), Marks 4

Solution : Let,

$$f(n) = a_m n^m + a_{m-1} n^{m-1} + a_{m-2} n^{m-2} + \ldots + a_0 \quad \ldots(1)$$

If we treate a_m^i as a constant then the equation (1) becomes

$$f(n) = A \sum_{i=0}^{m} a^m \quad \ldots(2)$$

where A is $a_0^0 + a_1^1 + a_2^2 + \ldots$

If equation (2) is

$$f(n) = A \left[\sum_{i=0}^{n} 1\right] = A\,[1 + 1 + 1 + \ldots + 1]$$

$$= A\left(n^1\right) = A \cdot O\left(n^1\right)$$

If equation (2) is

$$f(n) = A \sum_{i=0}^{n} n = A\,\{1 + 2 + 3 + \ldots n\}$$

$$= A\left(n^2\right) = A \cdot O\left(n^2\right)$$

If equation (2) is

$$f(n) = A \sum_{i=0}^{n} n^2 = A\,O\left(n^3\right)$$

Thus $$f(n) = A \sum_{i=0}^{m} n = A\,O\left(n^m\right)$$

Thus neglecting the constant term we will have,

$$f(n) = O\left(n^m\right)$$

Example 2.2.3 *Interpret the following equations :*

i) $2n^2 + \Theta(n) = \Theta\left(n^2\right)$

ii) $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$

SPPU : May-09, Marks 4

Solution : i) $2n^2 + \theta(n) = \Theta(n^2)$

$f(n) = \Theta(g(n))$ is true when

$c_2 g(n) \le f(n) \le c_1 g(n)$ for all $n \ge n_0$ and where c_1 and c_2 are some positive constants.

Let, L.H.S $= 2^n + \Theta(n)$ and R.H.S $= \Theta(n^2)$

We can rewrite it as

$$2^n = \Theta(n^2) - \Theta(n)$$

$$2^n = \Theta(n^2 - n)$$

$$\therefore \quad f(n) = 2^n$$

$$g(n) = n^2 - n$$

Now if we have $c_2 = 1$ and $c_1 = 2$ then for $n >= 2$

We get $2^n = \Theta(n^2 - n)$ to be CORRECT. It is illustrated as follows -

$n = 2, c_2 = 1, c_1 = 2$

$$c_2 . g(n) \le f(n) \le c_1 g(n)$$

i.e.

$$1(n^2 - n) \le 2^n \le 2 \cdot (n^2 - n)$$

$$1(2^2 - 2) \le 2^1 \le 2(2^2 - 2)$$

$2 \le 2 \le 4$ is true.

$n = 3, c_2 = 1, c_1 = 2$

$$c_2(n^2 - n) \le 2^n \le c_1(n^2 - n)$$

$$1(3^2 - 3) \le 2^3 \le 2(3^2 - 3)$$

$6 \le 8 \le 12$ is true

Thus for $n \ge 2$ and $c_1 = 2$ and $c_2 = 1$ the $2^n + \Theta(n) = \Theta(n^2)$ is true.

ii) $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$

If we simplify above equation, we get

$$\cancel{2n^2} + 3n + 1 = \cancel{2n^2} + \Theta(n)$$

$$\therefore \quad 3n + 1 = \Theta(n)$$

Here $f(n) = 3n + 1$

$g(n) = n$

If we have $c_2 = 1$ and $c_1 = 4$ then for $n \geq 1$ We get $3n + 1 = \Theta(n)$ to be CORRECT. It is illustracted as follows -

If $n = 1, c_1 = 4\ c_2 = 1,$

Let, $c_2\, g(n) \leq f(n) \leq c_1 g(n)$

i.e. $c_2(n) \leq (3n+1) \leq c_1(n)$

$1(1) \leq (3 \times 1 + 1) \leq 4(1)$

$1 \leq 4 \leq 4$ is true

If $n = 2, c_1 = 4\ c_2 = 1,$

Let, $c_2 \cdot g(n) \leq f(n) \leq c_1 g(n)$

$1(2) \leq (3 \times 2 + 1) \leq 4(2)$

$2 \leq 7 \leq 8$ is true

Thus for $n \geq 1$ having $c_2 = 1$ and $c_1 = 4$ as constant values the given equation $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ is true.

2.2.5 Little Oh and Little Omega Notation

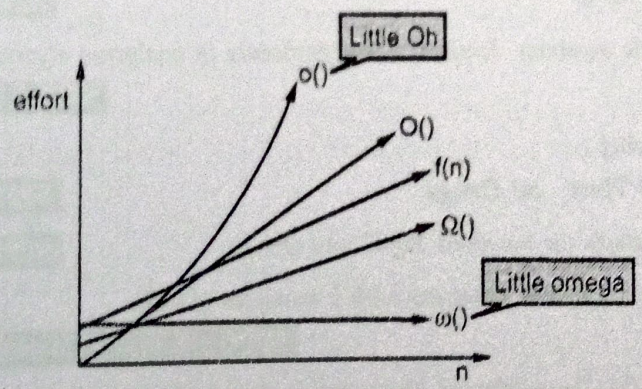


Fig. 2.2.4

$o(f) = O(f) - \Theta(f)$

$\omega(f) = \Omega(f) - \Theta(f)$

$$g \in o(f) \text{ if } \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$$

$$g \in \omega(f) \text{ if } \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$$

In other words

$o(f) \equiv \{g \mid \forall c \in N, \exists n0 \in N \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n0\}$

$o(f) \equiv \{g \mid \forall c \in N, \exists n0 \in N \text{ such that } 0 \leq cf(n) \leq g(n) \text{ for all } n \geq n0\}$

Relationship between all the asymptotic notations and small oh

"g(n) is O(f(n))" basically means that, f(n) describes the upper bound for g(n)

"g(n) is Ω (f(n))" basically means that f(n) describes the lower bound for g(n)

"g(n) is Θ (f(n))" basically means that f(n) describes the exact bound for g(n)

"g(n) is o(f(n))" basically means that g(n) is the upper bound for g(n) but that g(n) can never be equal to g(n)

Another way of saying

"g(n) is O(f(n))" basically means growth rate of g(n)<=growth rate of f(n)

"g(n) is Ω(f(n))" basically means growth rate of g(n)>=growth rate of f(n)

"g(n) is Θ(f(n))" basically means growth rate of g(n)==growth rate of f(n)

"g(n) is o(f(n))" basically means growth rate of g(n)< growth rate of f(n)

Review Questions

1. *Define the asymptotic notations :*
   i) Ω ii) ω iii) Θ **SPPU : May-09, Marks 6**
2. *Define asymptotic notations. Explain their significance in analyzing algorithms.* **SPPU : May-12, 14, Marks 5**
3. *Define the following :*
   i) Big "oh" ii) Theta iii) Omega. **SPPU : Dec.-12, Marks 6**
4. *Explain with example the notations Big-O and Omega.* **SPPU : Dec.-14, Marks 4**
5. *Explain all three asymptotic notations with 2 examples of each.* **SPPU : Aug.-15, Marks 6, Dec.-19, Marks 4**
6. *Explain Big Oh(O), Omega(Ω) and Theta(Θ) notation in detail along with suitable examples.* **SPPU : May-17, Marks 6**
7. *Explain asymptotic notations with example.* **SPPU : May-18, Marks 8**
8. *Define asymptotic notation. What is their significance in anlyzing algorithms ? Explain Big Oh, Omega and Theta notations.* **SPPU : May-19, Marks 8**

2.8 Best Case, Worst Case, Average Case

SPPU : Dec.-12, 15, Oct.-16, May-09, Marks 6

If an algorithm takes minimum amount of time to run to completion for a specific set of input then it is called **best case** time complexity.

For example : While searching a particular element by using sequential search we get the desired element at first place itself then it is called best case time complexity.

If an algorithm takes maximum amount of time to run to completion for a specific set of input then it is called **worst case** time complexity.

For example : While searching an element by using linear searching method if desired element is placed at the end of the list then we get worst time complexity.

The time complexity that we get for certain set of inputs is as a average same. Then for corresponding input such a time complexity is called **average case** time complexity.

Consider the following algorithm

```
Algorithm Seq_search(X[0 ... n-1 ],key)
// Problem Description: This algorithm is for searching the
//key element from an array X[0...n-1] sequentially.
//Input: An array X[0...n-1] and search key
//Output: Returns the index of X where key value is present
for i ← 0to n-1 do
 if(X[i]=key)then
  return i
```

Best case time complexity

Best case time complexity is a time complexity when an algorithm runs for short time. In above searching algorithm the element **key** is searched from the list of **n** elements. If the **key** element is present at first location in the list(X[0...n–1]) then algorithm run for a very short time and thereby we will get the best case time complexity. We can denote the best case time complexity as

$$C_{best} = 1$$

Worst case time complexity

Worst case time complexity is a time complexity when algorithm runs for a longest time. In above searching algorithm the element **key** is searched from the list of **n** elements. If the **key** element is present at n^{th} location then clearly the algorithm will run for longest time and thereby we will get the worst case time complexity. We can denote the worst case time complexity as

$$C_{worst} = n$$

The algorithm guarantees that for any instance of input which is of size n, the running time will not exceed $C_{worst}(n)$. Hence the worst case time complexity gives important information about the efficiency of algorithm.

Average case time complexity

This type of complexity gives information about the behaviour of an algorithm on specific or random input. Let us understand some terminologies that are required for computing average case time complexity.

Let the algorithm is for sequential search and P be a probability of getting successful search.

n is the total number of elements in the list.

The first match of the element will occur at i^{th} location. Hence probability of occurring first match is P/n for every i^{th} element.

The probability of getting unsuccessful search is (1 – P).

Now, we can find average case time complexity $C_{avg}(n)$ as -

$$C_{avg}(n) = \text{Probability of successful search (for elements 1 to n in the list)}$$
$$+ \text{Probability of unsuccessful search}$$

$$C_{avg}(n) = \left[1 \cdot \frac{P}{n} + 2 \cdot \frac{P}{n} + \ldots + i \cdot \frac{P}{n}\right] + n \cdot (1 - P)$$

There may be n elements at which chances of 'not getting element' are possible.

$$= \frac{P}{n}[1 + 2 + \ldots + i \ldots n] + n(1 - P)$$

$$= \frac{P}{n} \frac{n(n+1)}{2} + n(1 - P)$$

$$C_{avg}(n) = \frac{P(n+1)}{2} + n(1 - P)$$

Thus we can obtain the general formula for computing average case time complexity.

Suppose if P = 0 that means there is no successful search i.e. we have scanned the entire list of n elements and still we do not found the desired element in the list then in such a situation,

$$C_{avg}(n) = 0(n + 1)/2 + n(1 - 0)$$

$$C_{avg}(n) = n$$

Thus the average case running time complexity becomes equal to n.

Suppose if P = 1 i.e. we get a successful search then

$$C_{avg}(n) = 1\,(n+1)/2 + n\,(1-1)$$

$$C_{avg}(n) = (n+1)/2$$

That means the algorithm scans about half of the elements from the list.

For calculating average case time complexity we have to consider probability of getting success of the operation. And any operation in the algorithm is heavily dependent on input elements. Thus computing average case time complexity is difficult than computing worst case and best case time complexities.

Example 2.3.1 *Calculate the worst case time complexity of $f(n) = 6n(n^3 - n) + 9n$ using the running time complexity.*

Solution : While finding the worst case analysis we calculate **upper bound** on running time of an algorithm. The upper bound is calculated using big oh notation.

Let $f(n) = 6n(n^3 - n) + 9n$

i.e. $f(n) = 6n^4 - 6n^2 + 9n$

We will find $f(n) \le c * g(n)$.

For f(n), Let n = 1.

$6(1)^4 - 6(1)^2 + 9(1)$

$= 6 - 6 + 9$

$= 9$

For $c \ge 0$ and $g(n) = 9n^4$

We get $f(n) \le c * g(n)$.

i.e. $6n(n^3 - n) + 9n \le 9n^4$

$\therefore$ The worst case time complexity is $O(n^4)$.

Example 2.3.2 *To sort following set of the data, if we are having quick sort and merge sort available, which algorithm do you select for the following instance of the data to sort it in ascending order.*

Set-1 : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Set-2 : 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0

Set-3 : 0, 2, 4, 3, 5, 7, 8, 6, 9, 10, 1

Justify your answers with asymptotic notation.

SPPU : Oct.-16 (In Sem), Marks 6

Solution :

- The quicksort's behaviour changes on worst case and average case. If the set of data is already sorted in ascending order or descending order then this causes worst case behaviour of quicksort. This is because - the pivot element which we choose in this case is always a maximum element or minimum element. Due to which there will be an increase in number of comparisons.
- However, if data are in ascending or descending order then merge sort becomes faster.
- For set 1 and set 2 the merge sort algorithm will be chosen which results in O(n log n) performance.
- For set 3 the quicksort algorithm will be chosen which results in O(n log n) performance.

Review Questions

1. *What are the basic components that contribute to the space complexity ?* **SPPU : May-09, Marks 4**
2. *Explain the different ways of measuring the running time of an algorithm.* **SPPU : Dec.-12, Marks 6**
3. *Explain the best case, worst case and average case complexity of an algorithm with one example.* **SPPU : Dec.-18, Marks 8**

2.4 Growth Rate

Measuring the performance of an algorithm in relation with the input size n is called order of growth. For example, the order of growth for varying input size of n is as given below.

n	log n	n log n	n^2	2^n
1	0	0	1	2
2	1	2	4	4
4	2	8	16	16
8	3	24	64	256
16	4	64	256	65,536
32	5	160	1024	4,294,967,296

Table 2.4.1 Order of growth

We will plot the graph for these values. (Refer Fig. 2.4.1 on next page)

From the above drawn graph it is clear that the logarithmic function is the slowest growing function. And the exponential function 2^n is fastest and grows rapidly with varying input size n. The exponential function gives huge values even for small input n. For instance : For the value of n=16 we get 2^{16} = 65536.

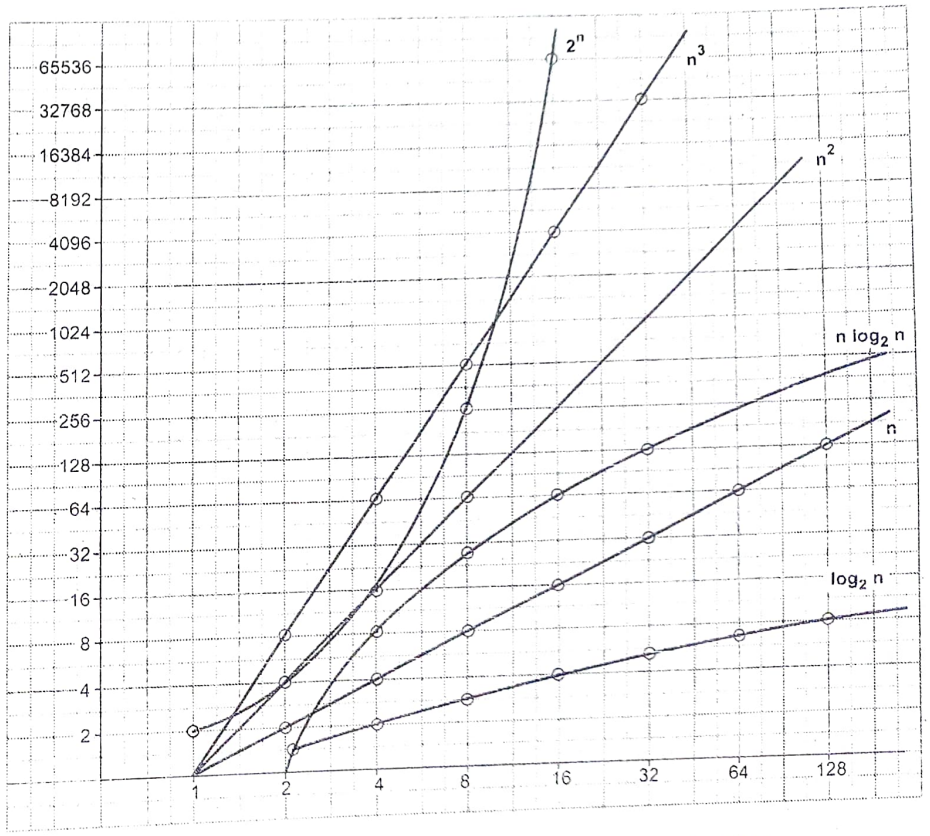


Fig. 2.4.1 Rate of growth of common computing time function

For example - Consider an example of sudo-ku game. In order to solve this entire puzzle, the algorithm would have to check each 3 x 3 matrix to see which numbers are missing, then each row, then each column and then make sure there are no repetitions of any digit from 0 - 9. This becomes more complex because the number of digits that are missing is inconsistent in each row, column and matrix. Solving this problem is not possible in polynomial time. But once we get the solution, then verifying this solution is possible within very less time. Thus verification of solution of this problem is possible in polynomial time.

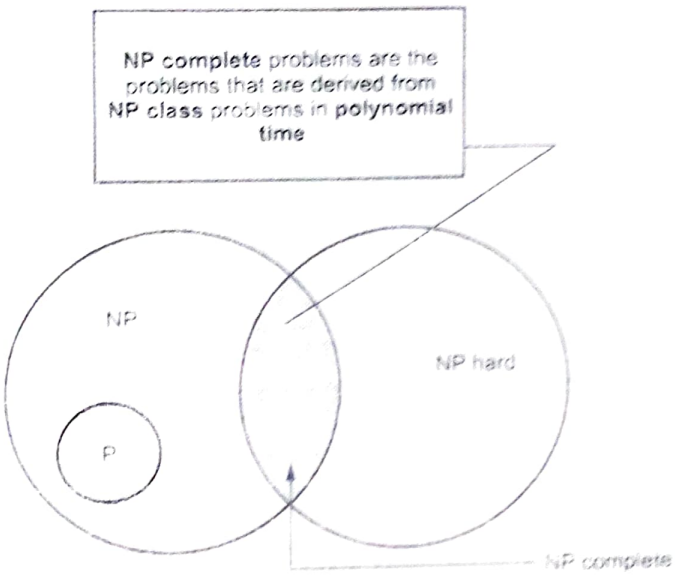


Fig. 2.6.2 P, NP, NP complete and NP hard class problems

- The NP class problems can be further categorized into NP-complete and NP hard problems.
- A problem D is called NP-complete if -

  i) It belongs to class NP.

  ii) Every problem in NP can also be solved in polynomial time.
- If an NP-hard problem can be solved in polynomial time then all NP-complete problems can also be solved in polynomial time.
- All NP-complete problems are NP-hard but all NP-hard problems cannot be NP-complete.

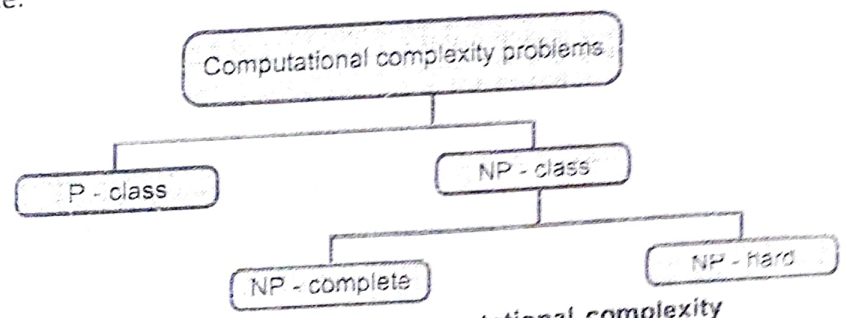


Fig. 2.6.3 Class of computational complexity

- The NP complete problems are intersection of NP class problems and NP hard problems.
- Normally the NP complete problems are decision problems(For example - is a graph Euler graph or not).
- The NP hard problems are optimization problems (For example - shortest path problem).
- **Computational complexity** - The computational problems is an infinite collection of instances with a solution for every instance.
  In computational complexity the solution to the problem can be "yes" or "no" type. Such type of problems are called **decision problem.** The computational problems can be **function problem**. The function problem is a computational problem where single output is expected for every input. The output of such type of problems is more complex than the decision problem.
  The computational problems also consists of a class of problems, whose output can be obtained in polynomial time.
- **Complexity classes** - The complexity classes is a set of problems of related complexity. It includes function problems, P classes, NP classes, optimization problem.
- **Intractability** - Problems that can be solved by taking a long time for their solutions is known as intractable problems.
  If NP is not same as P then NP complete problems are called intractable problems.
- The P class problems are considered as tractable problems.

2.6.1.1 Example of P Class Problem

Kruskal's algorithm : In Kruskal's algorithm the minimum weight is obtained. In this algorithm also the circuit should not be formed. Each time the edge of minimum weight has to be selected, from the graph. It is not necessary in this algorithm to have edges of minimum weights to be adjacent. Let us solve one example by Kruskal's algorithm.

Example :

Find the minimum spanning tree for the following figure using Kruskal's algorithm.

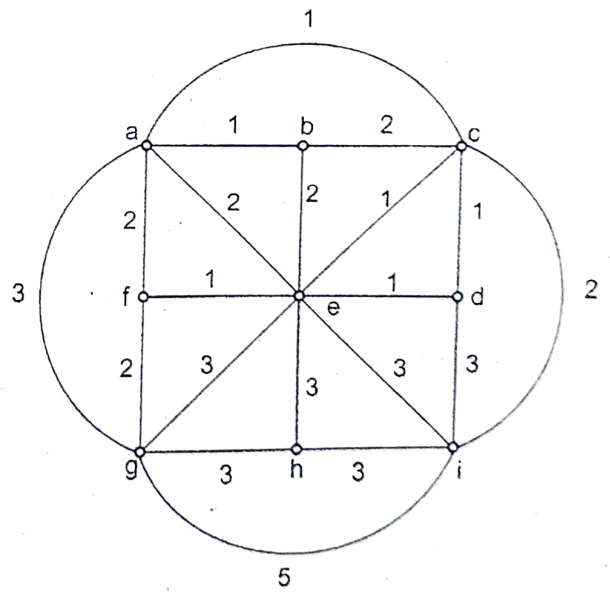


Fig. 2.6.4

In Kruskal's algorithm, we will start with some vertex and will cover all the vertices with minimum weight. The vertices need not be adjacent.

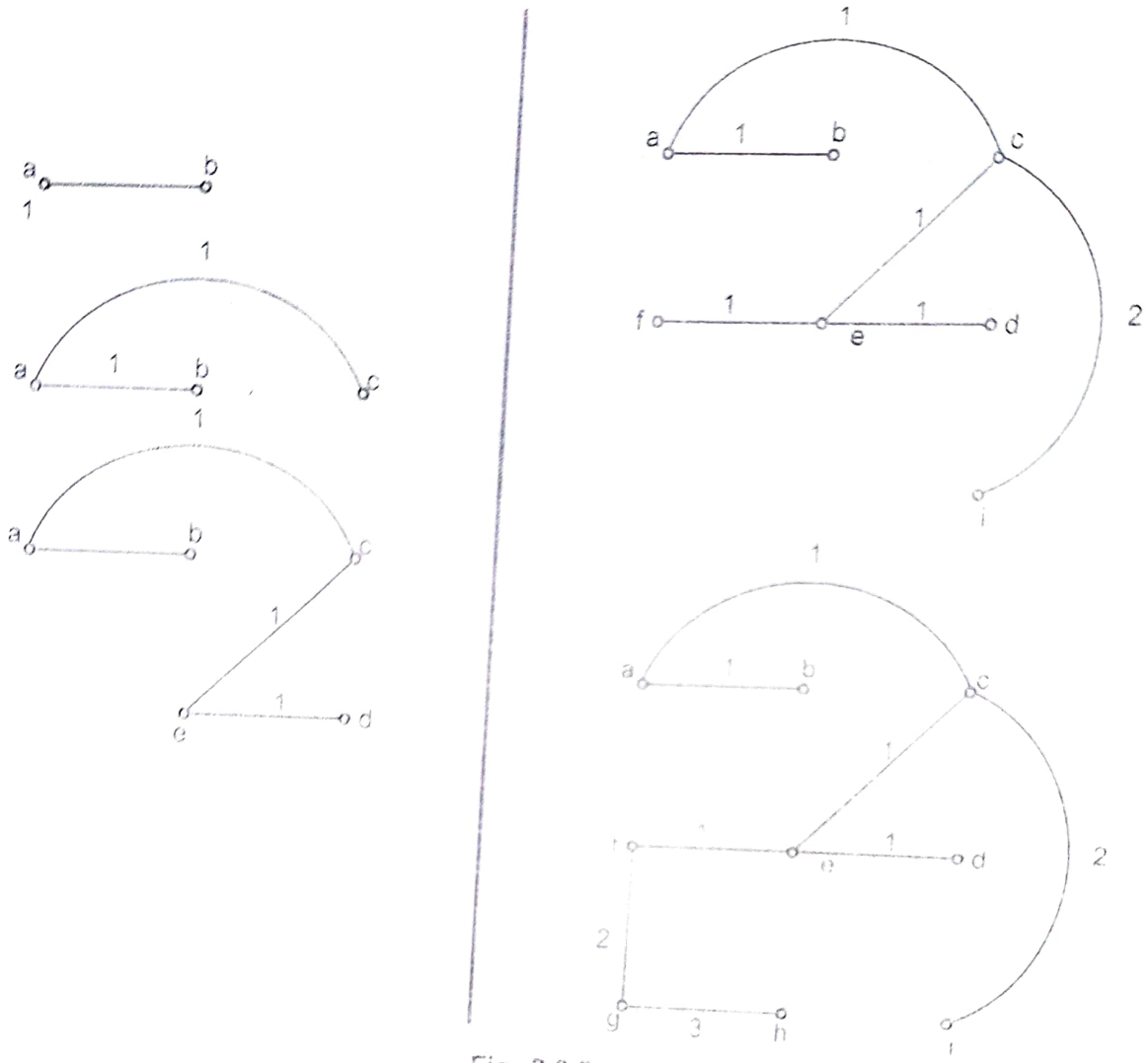


Fig. 2.6.5

2.6.1.2 Example of NP Class Problem

Travelling Salesman's Problem (TSP) : This problem can be stated as " Given a set of cities and cost to travel between each pair of cities, determine whether there is a path that visits every city once and returns to the first city. Such that the cost travelled is less".

For example : Refer Fig. 2.6.6.

The tour path will be a-b-d-e-c-a and total cost of tour will be 16.

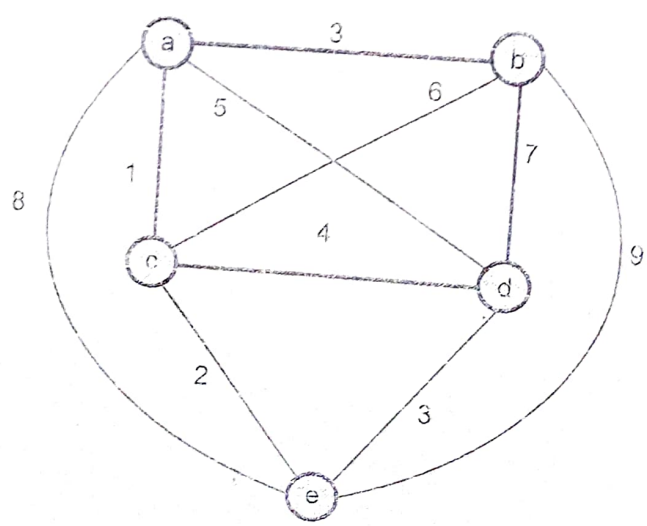


Fig. 2.6.6

This problem is NP problem as there may exist some path with shortest distance between the cities. If you get the solution by applying certain algorithm then Travelling Salesman problem is **NP Complete Problem**. If we get no solution at all by applying an algorithm then the travelling salesman problem belongs to **NP hard class**.

2.6.13 Difference between P and NP Class Problems

P class problems	NP class problems
An algorithm in which for given input the definite output gets generated is called Polynomial time algorithm(P class)	An algorithm is called non deterministically polynomial time algorithm (NP class) when for given input there are more than one paths that the algorithm can follow. Due to which one can not determine which path is to be followed after particular stage.
All the P class problems are basically deterministic	All the NP class problems are basically non deterministic.
Every problem which is a P class is also in NP class.	Every problem which is in NP is not the P class problem.
P class problems can be solved efficiently.	NP class problems can not be solved efficiently as efficiently as P class problems.
Examples : Binary search, bubble sort	**Examples** : Knapsack problem, traveling salesperson problem.

Example 2.6.1 *List three problems that have polynomial time algorithms. Justify your answer.*

Solution : The problems that can be solved in polynomial time are called P - class problems. For example -

1. **Binary search** - In searching an element using binary search method, the list is simply divided at the mid and either left or right sublist is searched for key element. This process is carried out in O(logn).
2. **Evaluation of polynomial** - In a polynomial evaluation we make out the summation of each term of polynomial. The evaluated result is simply an integer. This process is carried out in O(n) time.
3. **Sorting a list** - The elements in a list can be arranged either in ascending or descending order. This procedure is carried out in O(nlogn) time.

This shows that all the above problems can be solved in polynomial time.

Example 2.6.2 *Show that both P and NP are closed under the operation union, intersection, concatenation and kleen closure (*).}* **SPPU : Dec.-12, Marks 10**

Solution : Assume $L_1, L_2 \in NP$. That means there exists machines M_1 and M_2 that M_1 decides L_1 in nondeterministic time $O(n^p)$ and M_2 decides L_2 in nondeterministic time $O(n^q)$.

Solution :

```
Algorithm Non_DSort(A,n)
// A[1:n] is an array that stores n elements which are positive integers B[1:n] be an
auxiliary
// array in which elements are put at appropriate positions
{
// guessing stage
    for i←1 to n do
        B[i]←0;//initialize B to 0
    for i←1 to n do
    {
        j←choose(1,n)//select any element from the input set
        if(B[j]!=0) then
            fail();
        B[j]←A[i];
    }
//verification stage
    for i←1 to n–1 do
        if(B[i]>B[i+1]) then
            fail();// not sorted elements
        write(B[1:n]);// print the sorted list of elements
        success();
}
```

The time required by a non-deterministic algorithm with some input set is minimum number of steps needed to reach to a successful completion if there are multiple choices from input set leading to successful completion. In short, a non-deterministic algorithm is of complexity O(f(n)) where n is the total size of input.

2.7.1 Nondeterministic Algorithm for 0/1 Knapsack Problem

The *0/1 Knapsack Decision Problem* is to determine, for a given profit P, whether it is possible to load the knapsack so as to keep the total weight no greater than m, while making the total profit at least equal to Pt. This problem has the same parameters as the 0-1 Knapsack Optimization Problem plus the additional parameter Pt.

The non deterministic algorithm for 0/1 knapsack problem is as given below -

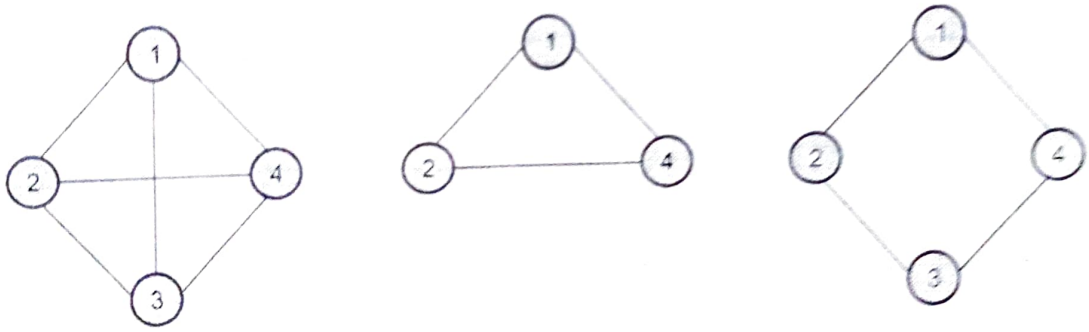
```
Algorithm Non_DKnaps()
//p[1:n] is a profit each object i where 1 ≤ i ≤ n
//w[1:n] is the weight of each object i where 1 ≤ i ≤ n
//Profit and Wt represent the current total profit and weight
// m is the capacity of knapsack
{
//initially no item is selected
```

```
Wt:=0;
Profit:=0;

// guessing stage
    for i:=1 to n do
    {
        x[i]:=choose(0,1);
        Wt:=Wt+x[i]*w[i];
        Profit:=Profit+x[i]*p[i];
    }
//verification stage
// selected items exceed the capacity or less profit is earned

    if((Wt>m) or (Profit<pt)) then
        fail();
    else
        success();//maximum profit earned
}
```

2.7.2 Clique Problem

Clique is nothing but a maximal complete subgraph of a graph G. The graph G is a set of (V, E) where V is a set of vertices and E is a set of edges. The size of Clique is number of vertices present in it.

For example :



Graph G Clique size = 3 Clique size = 4

Note that Clique of size = 3 and 4 are complete subgraph of graph G.

Max Clique Problem - It is an optimization problem in which the largest size Clique is to be obtained.

In the decision problem of Clique, we have to determine whether G has a Clique of size at least k for given k.

Let, DClique (G, k) is a deterministic decision algorithm for determining whether graph G has Clique or not of size atleast k. Then we have to apply DClique for k = n, n – 1, n – 2,... until the output is 1. The max Clique can be obtained in time $\leq n \cdot F(n)$ where F(n) is the time complexity of DClique. If Clique decision problem is solved in **polynomial time** then **max Clique** problem can be solved in **polynomial time.**

Non Deterministic algorithm for Clique

```
Algorithm NonDClique(G,n,k)
{
    S:=0
    for i:=1 to k do
    {
        t:=Choice(1,n);
        if t is from set S
            Failure();
        Add t to set S
    }
//Now S contains k distinct vertex indices
for all pairs(i,j) i and j are from set S and i!=j do
    if (i,j) is not an edge of graph G then
        Failure();
Success();
}
```

Review Questions

1. *Consider the following search algorithm :*

   j = any value between 1 to n

   If (a[j] = x) then

   print "Success";

   else

   print "Fails"

   Is this algorithm non-deterministic ? Justify your answer.

   SPPU : Dec.-06, Marks 6; Dec.-10, Marks 8

2. *What is the basic difference between deterministic and non-deterministic algorithms ?*

   SPPU : May-07, Marks 2; May-08, Marks 4

3. *Write a non-deterministic Knapsack algorithm.* **SPPU : May-07, 10, Marks 8**

4. *Comment on - "The time required by non-deterministic algorithm performing on any given input is the minimum number of steps needed to reach a successful completion if there exists a sequence of choices leading to such a completion".* **SPPU : May-07, Marks 2**

5. *Write non-deterministic algorithm for searching an item in an array. What is its complexity ?*

   SPPU : May-11, Marks 8

6. *What is a deterministic and non-deterministic algorithm ? Write a non-deterministic algorithm for searching element.* **SPPU : May-14, Marks 8**

7. *Write non deterministic algorithm for Clique decision problem.* **SPPU : Dec.-15 (End Sem.), Marks 8**

8. *What is 0-1 Knapsack problem ? Explain the algorithm as deterministic and non-deterministic versions.* **SPPU : Dec.-16, (End Sem), Marks 10**

9. *Explain non-determininstic clique problem along iwth alogorithm.* **SPPU : May-17, (End Sem), Marks 8**

10. *Give and explain non-deterministic sorting algorithm.* **SPPU : May-17, (End Sem), Marks 8**

11. *What are deterministic and non-deterministic algorithms ? Explain with example.* **SPPU : Dec.-19, Marks 8**

2.8 Polynomial Problem Reduction

To prove whether particular problem is NP complete or not we use polynomial time reducibility. That means if

$A \xrightarrow{\text{Poly}} B$ and $B \xrightarrow{\text{Poly}} C$ then $A \xrightarrow{\text{Poly}} C$.

The **reduction** is an important task in NP completeness proofs. This can be illustrated by Fig. 2.8.1. (See Fig. 2.8.1 on next page).

Various types of reductions are

Local replacement - In this reduction A→B by dividing input to A in the form of components and then these components can be converted to components of B.

Component design - In this reduction A→B by building special component for input B that enforce properties required by A.

- If L_1 and L_2 are two problems, then problem L_1 reduces to L_2 which can be denoted as $L_1 \propto L_2$ if and only if there is a way to solve L_1 by deterministic polynomial time algorithm using a deterministic algorithm that solves L_2 in polynomial time. It means that - if we have polynomial time algorithm for L_2, then we can solve L_1 in polynomial time.
- We can also state that if

  $L_1 \propto L_2$ and $L_2 \propto L_3$

  then $L_1 \propto L_3$
- Two problems P and Q are said polynomially equivalent if and only if $P \propto Q$ and $Q \propto P$.

  If problem P_1 is NP-complete and there is polynomial time reduction of P_1 to P_2 then P_2 is NP-complete.

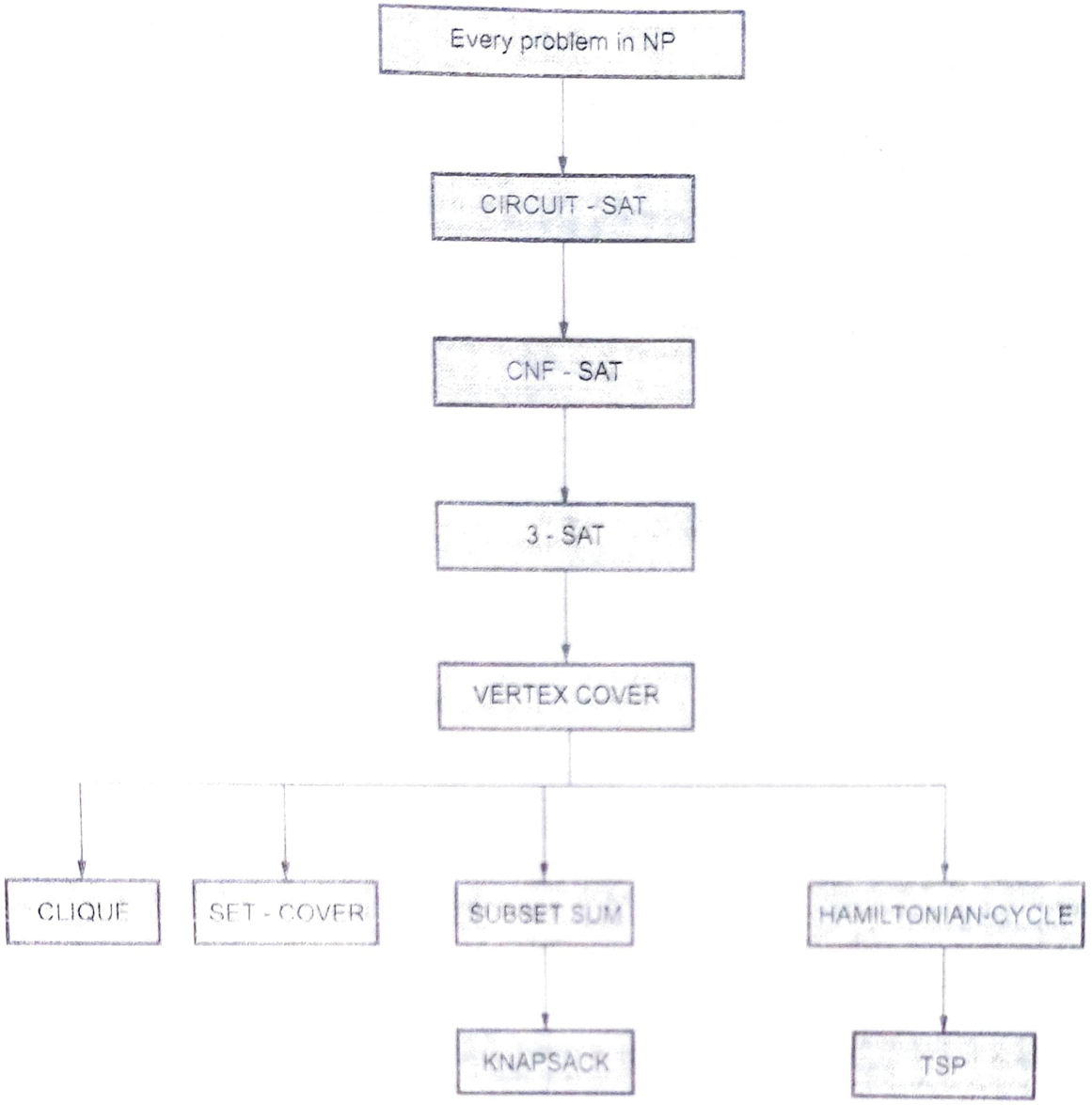


Fig. 2.8.1 Reduction in NP completeness

2.9 NP Complete Problems

SPPU : May-10,14,16,17, Dec.-15,16, Marks 8

In this section we will discuss two problems namely Vertex cover and the 3SAT problem which are actually NP Complete problems. Their proof of NP completeness is based on **reduction technique**. That means there are some problems which are already proved as NP Complete problems and using these problems we will prove that the vertex cover and the 3SAT problems are NP Complete problems.

Steps for proving NP - complete :

Consider that, we have to prove that B is in NP -

Step 1 : Select an NP complete language say A.

Step 2 : Construct a function f that maps the members of A to members of B.

Step 3 : Show that x is in A if and only if f(x) is in B.

Step 4 : Now show that the function f can be computed in **polynomial time.**

Step 5 : This if A is NP complete and it can be reduced to B in polynomial time, then B comes out to be NP complete.

Prove that SAT problem is NP complete

Proof :

1) SAT is NP.

2) Circuit SAT or (C-SAT) reduces to SAT. The reduction function f is as follows -

i) For every input wire add a new variable.

ii) For every output wire add a new variable.

iii) An equation is prepared for each gate.

iv) These sets of equations are separated by $\wedge$ values and adding final **output variable** at the end.

This transformation can be done in linear time or **polynomial time.** For instance - Consider following circuit for C-SAT problem.

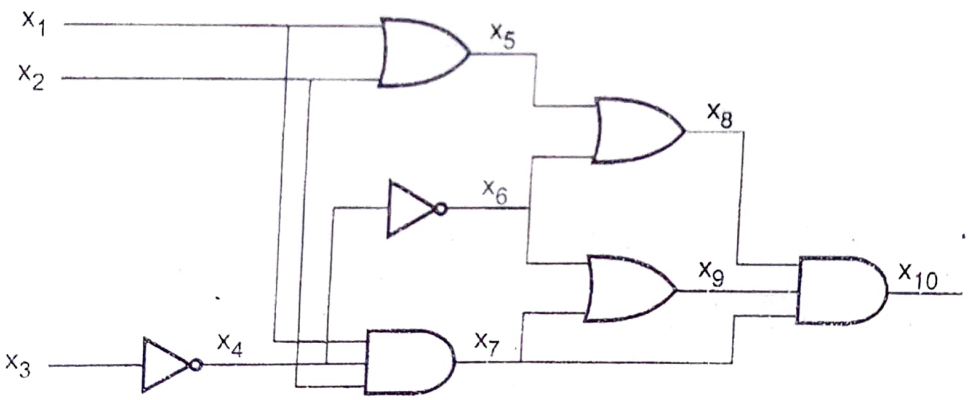


Fig. 2.9.1

$$x_5 = x_1 \vee x_2$$

$$x_4 = \neg x_3$$

$$x_6 = \neg x_4$$

$$x_7 = x_1 \wedge x_2 \wedge x_4$$

$$x_8 = x_5 \vee x_6$$

$$x_9 = x_6 \vee x_7$$

$$x_{10} = x_7 \wedge x_8 \wedge x_9$$

$$\therefore \quad f = x_5 \wedge x_4 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10}$$

This shows that we can reduce arbitrary instance of circuit-SAT problem to a specialized instance of SAT in **polynomial time.** And as we know that circuit-SAT is NP complete and reduction of circuit-SAT to SAT is in polynomial time, we must say that SAT is also an NP complete problem.

The 3-SAT problem

A 3-SAT problem is a problem which takes a Boolean formula S with each clause having exactly **three literals** and check is S is satisfied or not.

Prove that 3-SAT is NP complete

Proof : The language 3-SAT is a restriction of SAT. We replace each clause C that represents the SAT problem to a function **f** by family of D_C of clauses that represent satisfiability.

For example say

$$C = a \vee b \vee c \vee d \vee e$$

One can simulate this by

$$D_C = (a \vee b \vee x) \wedge (\bar{x} \vee c \vee y) \wedge (\bar{y} \vee d \vee e)$$

where x and y are new variables.

Need to verify :

1) If C is FALSE, then D_C is FALSE; and

2) If C is TRUE, then one can make D_C TRUE.

If f is satisfiable then there is assignment where each clause C is TRUE. This can be extended to make D_C TRUE.

Further if f is evaluated to FALSE, then some clauses say C' must be FALSE and thus corresponding family $D_{C'}$ evaluates to FALSE.

This conversion process can be done in **polynomial time.** Thus we have shown that SAT reduces to 3-SAT in polynomial time. As we know that SAT is a NP complete problem, so we must say that 3-SAT is also NP complete problem.

2.9.1 The 3 SAT Problems

Before understanding the 3 SAT problem we will get introduced with the satisfiability problem.

1. CNF - SAT problem

This problem is based on Boolean formula. The Boolean formula has various Boolean operations such as OR(+), AND (·) and NOT. There are some notations such as → (means implies) and ↔ (means if and only if).

A Boolean formula is in **Conjuctive Normal Form (CNF)** if it is formed as collection of subexpressions. These subexpressions are called **clauses**.

For example

$(\bar{a} + b + d + \bar{g})\ (c + \bar{e})\ (\bar{b} + d + \bar{f} + h)\ (a + c + e + \bar{h})$

This formula evaluates to 1 if b, c, d are 1.

The CNF-SAT is a problem which takes Boolean formula in CNF form and checks whether any assignment is there to Boolean values so that formula evaluates to 1.

2.9.2 Vertex Cover Problem

The vertex cover problem is to find vertex cover of minimum size in a given graph. The word vertex cover means each vertex covers its incident edges. Thus by vertex cover we expect to choose the set of vertices which cover all the edges in a graph.

For example : Consider a graph G as given below.

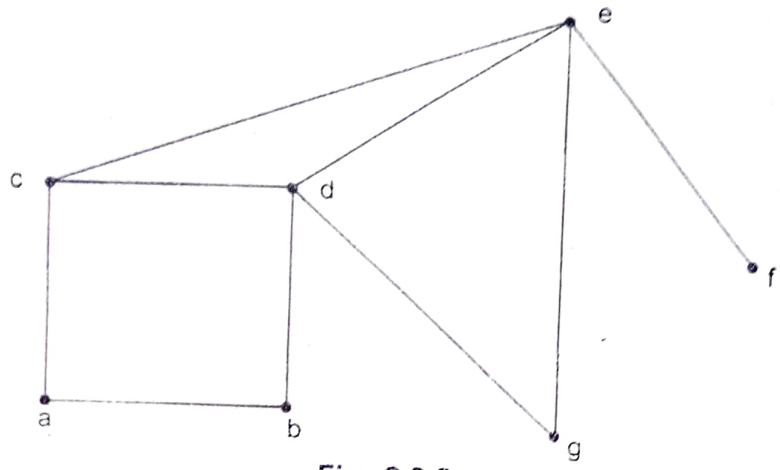


Fig. 2.9.2

Now we will select some arbitrary edge and delete all the incident edges. Repeat this process until all the incident edges get deleted.

Step 1 : Select an edge a-b and delete all the incident edges to vertex a or b.

Step 2 : Select an edge c-e and delete all the incident edges to vertex c or e.

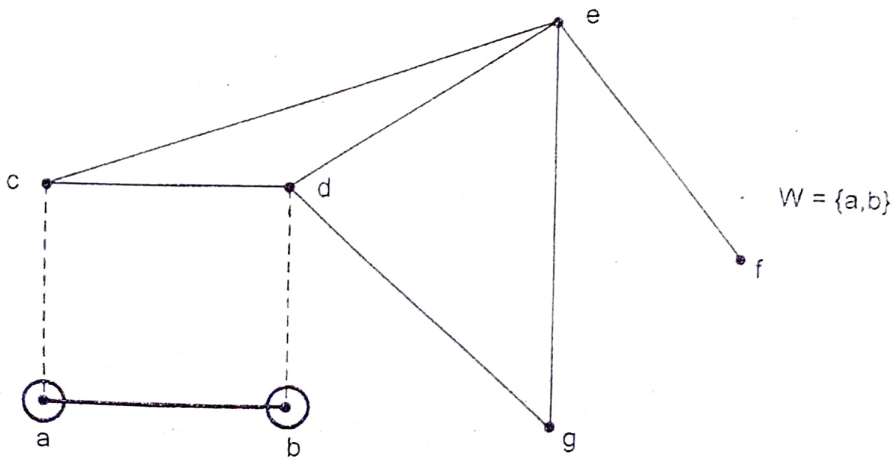


Fig. 2.9.3

Step 3 : Select an edge d-g. All the incident edges are already deleted.

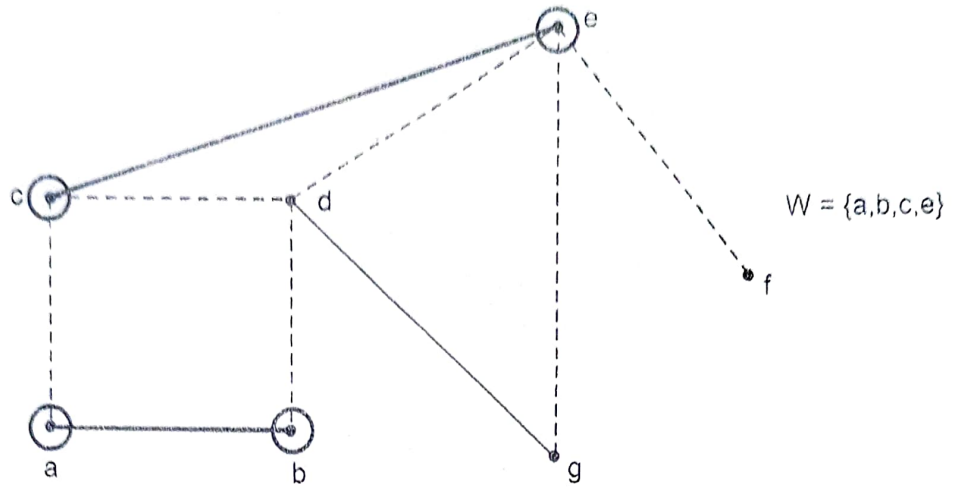


Fig. 2.9.4

Thus we obtain vertex cover as {a, b, c, d, e, g}.

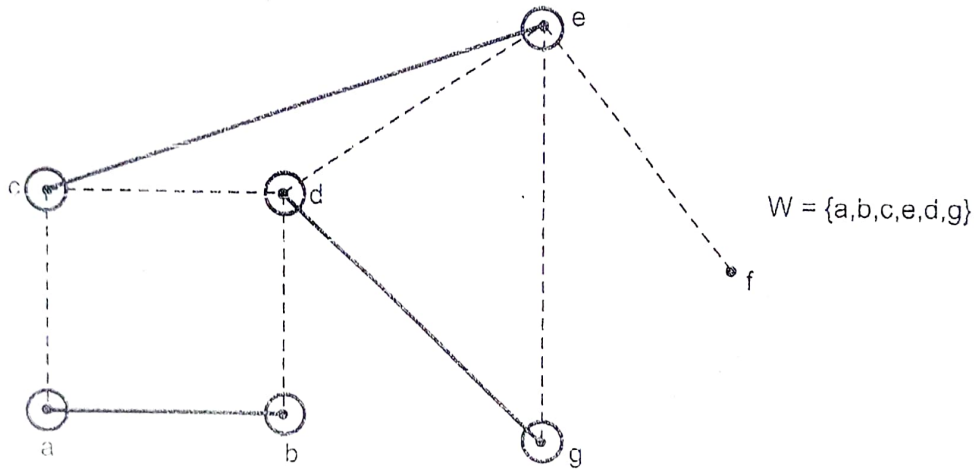


Fig. 2.9.5

Theorem : The vertex cover is NP-complete.

Proof :

To prove that vertex cover is NP-complete we will apply reduction technique. That means reduce 3-SAT to vertex cover. As we know 3-SAT to basically a Boolean expression s containing 3 variables in each clause and value of S is true.

To prove this theorem consider S be some Boolean formula in CNF (conjunctive Normal Form) having 3 variables in each clause no for each variable a we will create,

Fig. 2.9.6

If the clause is a + b + c we will create a triangle (as there are 3 - variables)

Using 3-SAT we will build a graph as follows for given expression.

$(a+b+c)\ (a+b+\bar{c})\ (\bar{a}+c+\bar{b})$

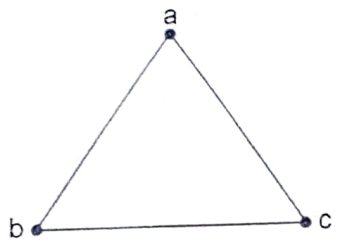


Fig. 2.9.7

The graph has vertex cover of size K = n + 2m, where n is number of variables in 3-SAT, m is number of clauses in CNF, K is total number of vertices that are present in the set of vertex cover.

For a clause (a + b + c) we can build a graph as :

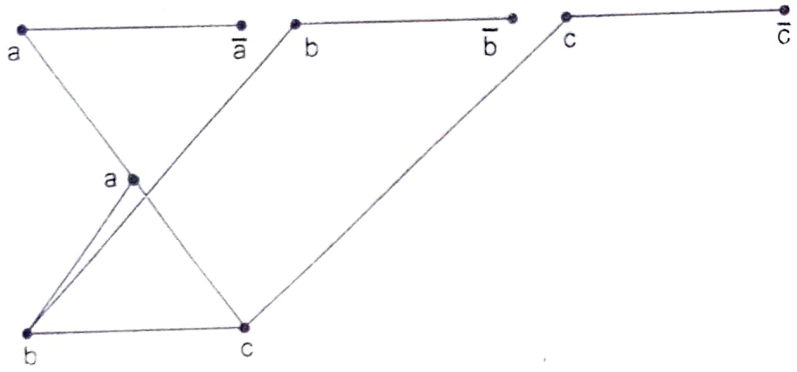


Fig. 2.9.8

Thus for given expression $(a+b+c)\ (a+b+\bar{c})\ (\bar{a}+c+\bar{d})$ we get following graph.

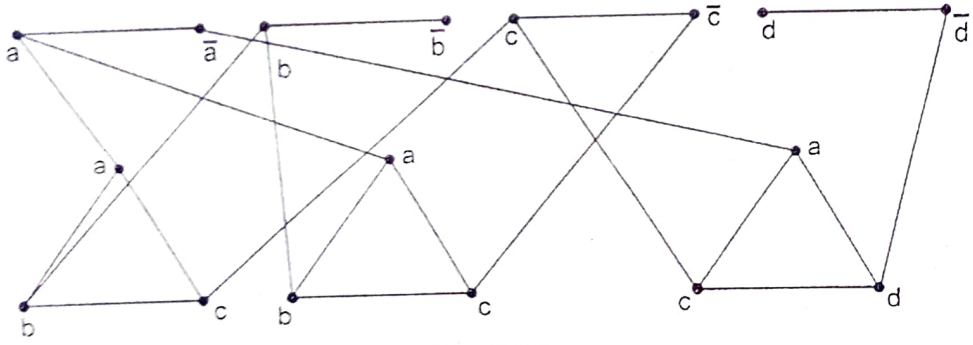


Fig. 2.9.9

Here n = Number of variable = 4

m = Number of clauses = 3

$\therefore$ K = n + 2m = 4 + 2(3) = 10

In vertex cover we get K = 10 vertices which cover all the vertices. The vertex cover is shown by following graph in which the covering vertices are rounded.

Thus K = n + 2m is proved as K = 10. This is how 3-SAT can be reduced to vertex cover. The 3-SAT is NP-complete. Hence vertex is NP-complete is proved.

Thus K = n + 2m is proved as K = 10. This is how 3-SAT can be reduced to vertex cover. The 3-SAT is NP-complete. Hence vertex is NP-complete is proved.

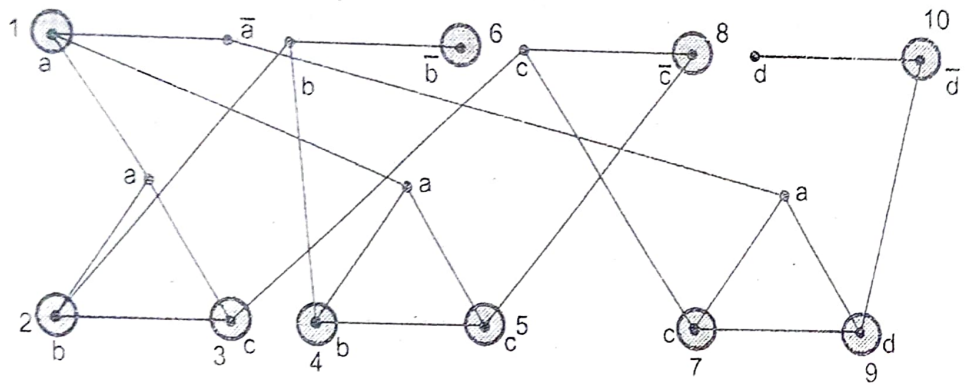


Fig. 2.9.10

Review Questions

1. *Prove that vertex cover problem is NP complete.* **SPPU : May-14, Marks 9**
2. *Prove that vertex cover problem is NP Complete.* **SPPU : Dec.-15 (End Sem.), Marks 8**
3. *What is SAT and 3-SAT problem ? Prove that the 3-SAT problem is NP complete.* **SPPU : May-16 (End Sem), 18, Marks 8**
4. *What NP-complete algorithm ? How do we prove that algorithm is NP complete ? (Give example)* **SPPU : Dec.-16, (End Sem), Marks 6**
5. *Prove that vertex cover problem in NP-complete.* **SPPU : May-17, (End Sem), Marks 8**
6. *What are steps to prove NP completeness of problem ? Prove that vertex cover problem is NP complete.* **SPPU : May-19, Marks 8**
7. *What is Boolean satisfiability problem ? Explain 3SAT problem. Prove 3-SAT is NP complete.* **SPPU : May-19, Marks 8**
8. *State vertex cover problem and prove that vertex cover problem is NP complete.* **SPPU : Dec.-19, Marks 8**

2.10 NP Hard Problem

SPPU : May-18, Marks 8

A problem A is NP-hard if there is an NP-complete problem B, such that B is reducible to A in polynomial time. NP-hard problems are as hard as NP-complete problems. NP-hard problem need not be in NP class.

- A NP problem such that, if it is in P, then NP = P. If a (not necessarily NP) problem has this same property then it is called **"NP-hard"**. Thus the class of NP-complete problem is the intersection of the NP and NP-hard classes.

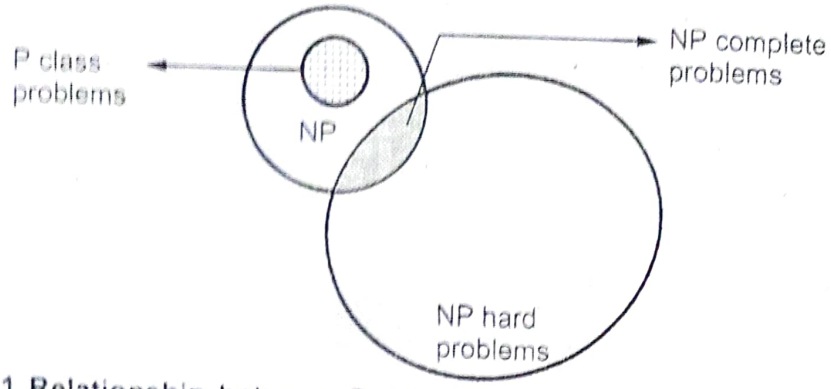


Fig. 2.10.1 Relationship between P, NP, NP-complete and NP-hard problems

- Normally the decision problems are NP-complete but optimization problems are NP-hard. However if problem L_1 is a decision problem and L_2 is optimization problem then it is possible that $L_1 \propto L_2$. For instance the Knapsack decision problem can be Knapsack optimization problem.
- There are some NP-hard problems that are not NP-complete. For example *halting problem*. The halting problem states that : "Is it possible to determine whether an algorithm will ever halt or enter in a loop on certain input ?"

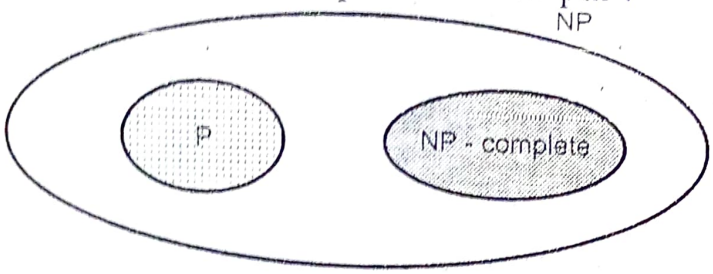


Fig. 2.10.2

Difference between NP hard and NP complete problem

Sr. No.	NP hard	NP complete
1.	NP hard problem (Say A) can be solved if and only if there is NP complete problem (say B) can be reducible into A in **polynomial time.**	NP complete problems can be solved by deterministic algorithm in polynomial time.
2.	To solve the NP hard problem, it must be in NP class.	To solve NP complete problem, it must be in both NP and NP hard problems.
3.	It is not a decision problem.	It is a decision problem.
4.	For example - Halting problem, Hamiltonian cycle problem, vertex cover problem.	For example - Determine whether a graph has a Hamiltonian cycle, determine whether a Boolean formula is satisfiable or not, circuit-satisfiability problem.

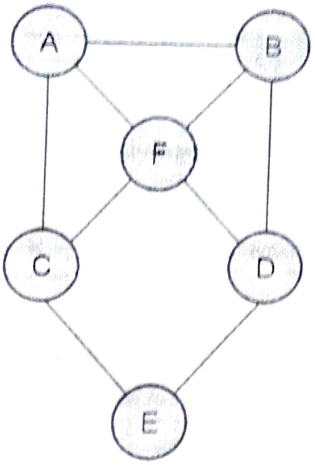
Review Question

1. *Write a short note on NP completeness of algorithm and NP hard.* **SPPU : May-18, Marks 8**

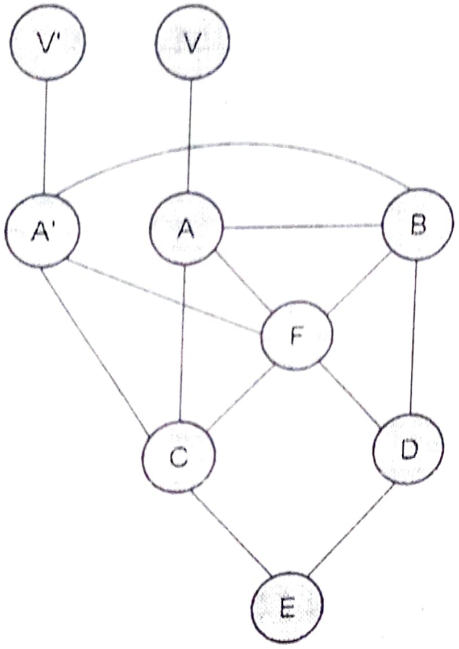
2.11 Hamiltonian Cycle

SPPU : Dec.-19, Marks 8

- Hamiltonian path is a simple open path that contains each vertex in a graph exactly once and if the source vertex and the destination vertex of this Hamiltonian path is the same then it is called Hamiltonian cycle.
- The Hamiltonian path problem is the problem to determine whether a given graph contains a Hamiltonian path.
- To show that this problem is NP-complete we first need to show that it actually belongs to the class NP and then find a known NP-complete problem that can be reduced to Hamiltonian path.
- For a given graph G we can solve Hamiltonian Path by deterministically choosing edges from G that are to be included in the path. Then we traverse the path and make sure that we visit each vertex exactly once. This obviously can be done in polynomial time and hence the problem belongs to NP.
- Now we have to find out an NP complete problem that can be reduced to Hamiltonian path. One such closely related problem is the problem to determine whether the graph contains Hamiltonian cycle (Hamiltonian cycle is basically an Hamiltonian path that begin and end in the same vertex). Hamiltonian cycle is NP complete so we try to reduce this problem to Hamiltonian path.
- For example - Consider a graph G, from which we can construct a graph G'' such that G contains a Hamiltonian cycle if and only if G'' contains Hamiltonian path.
- For instance in Graph G the Hamiltonian cycle is A-B-D-E-C-F-A and G'' contains the Hamiltonian path V- A-B-D-E-C-F-A'-V'.
- Conversely if G'' contains Hamiltonian Path then we can convert it to the Hamiltonian cycle present in G by removing the end points V and V' and mapping A' to A.
- Thus we can show that G contains Hamiltonian cycle if and only if G'' contains Hamiltonian path which concludes the proof that Hamiltonian path is NP complete.
- As every NP complete problem is NP hard problem as well, we conclude that Hamiltonian path problem is also NP hard.



Graph G



Graph G"

Fig. 2.11.1

Review Question

SPPU : Dec.-19, Marks 8

1. *Explain NP hard Hamiltonian cycle problem.*