Open Quantum Systems Final Project

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Consider the following positive map for a qutrit system:

$$\Lambda_c(\mathrm{p}) = \begin{pmatrix} p11 + p22 & -p12 & -p13 \\ -p21 & p22 + p33 & -p23 \\ -p31 & -p32 & p33 + p11 \end{pmatrix}$$

where

$$\mathrm{p} = \begin{pmatrix} p11 & p12 & p13 \\ p21 & p22 & p23 \\ p31 & p32 & p33 \end{pmatrix}$$

Part 1

TASK: To show that map is not completely positive.
We use the method in Lecture notes: kraus_derive.pdf

We use $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, as the basis vectors. Hence, the maximally entangled state:

$$1/3 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

we use this matrix and to get the choi matrix,

$$1/3 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let us call this matrix $1/3 \cdot C$, and its eigen value be λ,
Hence, $|C - \lambda I| = 0$
-1 is one such λ which satisfies this equation.
Hence, C has an eigen value -1
As the choi matrix of the map has a negative eigen value, it is not a completely positive map.

Part 2

Now consider the modified map,
$\Lambda_{mc}(\rho)$ = p*ρ + (1-p)*$\Lambda_c(\rho)$
and find the threshold value of p, above which the modified map is completely positive
(p >= p_{ih})

Choi Matrix definition:

We calculate the choi matrix for the given map the same way we calculated the choi matrix for the previous map. The new choi matrix for a given value p is:

$$1/3 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 1 \end{pmatrix} \text{ where x is 2*p-1}$$

If we know that it is positive for a value of p, it will be positive for a value p2 > p. Hence we need to find the minimum p between 0-1 such that the map is positive

Let us define a function NEV(p) that returns true if for the given value of p, the map is positive
We use binary search to find the optimal p:

Algorithm 1: Binary search to find p_{ih}

```
max=1, min=0 ;
while max-min>0.000001 do
    mid=(max+min)/2;
    if NEV(mid) then
        max=mid;
    else
        min=mid;
return max;
```

Using binary search, we find out that the threshold value of p (p_{ih}) is 0.25

Part 3

TASK: Find the Kraus operators for the modified map with p = p_{ih}

The minimum value of p (p_{ih}) is 0.25 but the dot product for one of the vector set is not zero, hence they are non-orthoginal. Hence, we modify the NEV(p) function to also check and ensure that the dot product becomes 0, which gave us (p_{ih}) = 0.5.

The eigenvalues at p (p_{ih}) = 0.5 are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the corresponding eigenvectors are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We see that the eigen value is 1 for the 1st, 5th and 9th eigen value.
The Kraus Operators are:

$$k_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$k_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$k_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It can be observed from the result that $k_1^T.k_1 + k_5^T.k_5 + k_9^T.k_9$ is an Identity matrix.

The code corresponding to PART 1, PART 2 and PART 3 are in part1.py, part2.py and part3.py respectively.

Part 4

Extended problem statement to verify what was assumed in Part 2

We had assumed that there exists a threshold value p above which the map $\Lambda_{mc}(\rho)$ is completely positive. In this part, we attempt to prove this.

Let a chao matrix for a given value P be M

Let its minimum eigen value be λ

Then, we know that $|M - \lambda \cdot I| = 0$

$$\text{Here, } M - \lambda \cdot I = \begin{pmatrix} (1-\lambda) & 0 & 0 & 0 & 2p-1 & 0 & 0 & 0 & 2p-1 \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 2p-1 & 0 & 0 & 0 & (1-\lambda) & 0 & 0 & 0 & 2p-1 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \\ 2p-1 & 0 & 0 & 0 & 2p-1 & 0 & 0 & 0 & (1-\lambda) \end{pmatrix}$$

To simplify the calculation, we will substitute

1-λ/x = a

-λ/x = b

$$\text{We now get, } M - \lambda \cdot I = \begin{pmatrix} a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a \end{pmatrix}$$

We calculate the determinant of this matrix using row swapping property (swapping 2 rows will change the sign of the determinant) to get determinant as

$b^6(a-1)(a-2)$

Hence, we can see that the value of λ (minmimum eigen value) is either 0 or it is proportional to the value of p

Hence, we can conclude that there exists a threshold p_{ih} such that for all p > p_{ih}, the eigen values are not negative. Which implies that the map $\Lambda_{mc}(\rho)$ is completely positive

Link to code repository: github.com/OQ_PROJECT

References:

Choi, M. D. (1975). Completely positive linear maps on complex matrices. Linear algebra and its applications, 10(3), 285-290.

J. Preskill, Lecture Notes for Physics 229: Quantum Information and Computation, Chapter 3 (Updated), Section 3.3.1, http://www.theory.caltech.edu/people/preskill/ph219/chap3_15.pdf (2015).