Computing Assignment 4: Generic Description
Developing a State Estimator from Input-Output Perturbation Data

Consider a MISO ARX model of the form

$$A(q^{-1})y(k) = B_1(q^{-1})u_1(k) + B_2(q^{-1})u_2(k) + e(k) \tag{1}$$

$$A(q^{-1}) = 1 + a_1q^{-1} + a_2q^{-2} + a_3q^{-3} \tag{2}$$

$$B(q^{-1}) = b_{11}q^{-1} + b_{12}q^{-2} + b_{13}q^{-3} \tag{3}$$

$$B(q^{-1}) = b_{21}q^{-1} + b_{22}q^{-2} + b_{23}q^{-3} \tag{4}$$

This transfer function model is equivalent to the following difference equation

$$\begin{aligned} y(k) &= -a_1y(k-1) - a_2y(k-2) - a_3y(k-3) + b_{11}u_1(k-1) + b_{12}u_1(k-2) + b_{13}u_1(k-3) \\ &\quad +b_{21}u_2(k-1) + b_{22}u_2(k-2) + b_{23}u_2(k-3) + e(k) \end{aligned} \tag{5}$$

Assume that we have generated simulation/experimental data set

$$\mathcal{S} = \{(y(k), u_1(k), u_2(k)) : k = 0, 1, 2, ..., N\} \tag{6}$$

$$y(k) = Y(k) - Y_s \quad ; \quad u_1(k) = U_1(k) - U_{1s}; \quad u_2(k) = U_2(k) - U_{2s}$$

by injecting perturbations in open loop condition and it is desired to estimate the model parameter using this dynamic data.

1 Model Parameter Estimation

The above difference equation can be rearranged as follows

$$\begin{aligned} y(k) &= -a_3y(k-3) - a_2y(k-2) - a_1y(k-1) + b_{13}u_1(k-3) + b_{12}u_1(k-2) + b_{11}u_1(k-1) \\ &\quad +b_{23}u_2(k-3) + b_{22}u_2(k-2) + b_{21}u_2(k-1) + e(k) \end{aligned} \tag{7}$$

Defining stacked vectors

$$\mathcal{Y}(k) = \begin{bmatrix} y(k-3) \\ y(k-2) \\ y(k-1) \end{bmatrix} \quad ; \quad \boldsymbol{\alpha} = \begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix} \tag{8}$$

$$\mathcal{U}_1(k) = \begin{bmatrix} u_1(k-3) \\ u_1(k-2) \\ u_1(k-1) \end{bmatrix} \quad ; \quad \boldsymbol{\beta}_1 = \begin{bmatrix} b_{13} \\ b_{12} \\ b_{11} \end{bmatrix} \tag{9}$$

$$\mathcal{U}_2(k) = \begin{bmatrix} u_2(k-3) \\ u_2(k-2) \\ u_2(k-1) \end{bmatrix} \quad ; \quad \boldsymbol{\beta}_2 = \begin{bmatrix} b_{23} \\ b_{22} \\ b_{21} \end{bmatrix} \tag{10}$$

we have

$$y(k) = -\mathcal{Y}(k)^T\boldsymbol{\alpha} + \mathcal{U}_1(k)^T\boldsymbol{\beta}_1 + \mathcal{U}_2(k)^T\boldsymbol{\beta}_2 + e(k) \tag{11}$$

$$= \begin{bmatrix} -\mathcal{Y}(k)^T & \mathcal{U}_1(k)^T & \mathcal{U}_2(k)^T \end{bmatrix} \boldsymbol{\theta} \tag{12}$$

where

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} \tag{13}$$

Using this data, we get the following set of equations

$$\begin{aligned} y(3) &= \begin{bmatrix} -\mathcal{Y}(3)^T & \mathcal{U}_1(3)^T & \mathcal{U}_2(3)^T \end{bmatrix} \boldsymbol{\theta} + e(3) && (14) \\ y(4) &= \begin{bmatrix} -\mathcal{Y}(4)^T & \mathcal{U}_1(4)^T & \mathcal{U}_2(4)^T \end{bmatrix} \boldsymbol{\theta} + e(4) && (15) \\ ... &= .... \\ y(N) &= \begin{bmatrix} -\mathcal{Y}(N)^T & \mathcal{U}_1(N)^T & \mathcal{U}_2(N)^T \end{bmatrix} \boldsymbol{\theta} + e(N) && (16) \end{aligned}$$

or in matrix form

$$\mathbf{Y} = \boldsymbol{\Omega\theta} + \mathbf{E} \tag{17}$$

where

$$\Omega = \begin{bmatrix} -\mathcal{Y}(3)^T & \mathcal{U}_1(3)^T & \mathcal{U}_2(3)^T \\ -\mathcal{Y}(4)^T & \mathcal{U}_1(4)^T & \mathcal{U}_2(4)^T \\ ... & ... & ... \\ -\mathcal{Y}(N)^T & \mathcal{U}_1(N)^T & \mathcal{U}_2(N)^T \end{bmatrix} \; ; \; \mathbf{Y} = \begin{bmatrix} y(3) \\ y(4) \\ .... \\ y(N) \end{bmatrix} \text{ and } \mathbf{E} = \begin{bmatrix} e(3) \\ e(4) \\ .... \\ e(N) \end{bmatrix} \tag{18}$$

and the least squares estimate of $\boldsymbol{\theta}$ can be obtained as follows

$$\begin{aligned} \widehat{\boldsymbol{\theta}} &= \underset{\boldsymbol{\theta}}{Min}\ \mathbf{E}^T\mathbf{E} = \underset{\boldsymbol{\theta}}{Min}\ (\mathbf{Y} - \boldsymbol{\Omega\theta})^T (\mathbf{Y} - \boldsymbol{\Omega\theta}) \\ &= \left(\boldsymbol{\Omega}^T\boldsymbol{\Omega}\right)^{-1} \boldsymbol{\Omega}^T \mathbf{Y} && (19) \end{aligned}$$

2 State Space Realization

To implement state feedback controller or MPC, we can convert the identified model into a state space model.

$$y(k) = \frac{B_1(q^{-1})}{A(q^{-1})} u_1(k) + \frac{B_2(q^{-1})}{A(q^{-1})} u_2(k) + \frac{1}{A(q^{-1})} e(k) \tag{20}$$

This transfer function model can be easily converted into the observable canonical form as follows.

$$\begin{aligned} \frac{1}{A(q^{-1})} &= \frac{1}{A(q^{-1})} - 1 + 1 = \frac{1 - A(q^{-1})}{A(q^{-1})} + 1 \\ &= \frac{-\left(a_1 q^{-1} + a_2 q^{-1} + a_3 q^{-1}\right)}{1 + a_1 q^{-1} + a_2 q^{-1} + a_3 q^{-1}} + 1 && (21) \end{aligned}$$

Using above rearrangement, we can write

$$\begin{aligned} yk) &= \frac{b_{11}q^{-1} + b_{12}q^{-2} + b_{13}q^{-3}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}} u_1(k) + \frac{b_{21}q^{-1} + b_{22}q^{-2} + b_{23}q^{-3}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}} u_2(k) \\ &\quad + \frac{-\left(a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}\right)}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}} e(k) + e(k) && (22) \end{aligned}$$

Examining the coefficients of the above equation, state realization in observable canonical form can be written as follows

$$\begin{aligned} \mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma\mathbf{u}(k) + Le(k) && (23) \\ y(k) &= C\mathbf{x}(k) + e(k) && (24) \end{aligned}$$

$$\begin{aligned} \Phi &= \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \; ; \; \Gamma = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix} \; ; \; L = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} && (25) \\ C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} && (26) \end{aligned}$$

Note that the resulting model is a state estimator and can be used on-line as follows

$$\begin{aligned} e(k) &= y(k) - C\widehat{\mathbf{x}}(k) \\ \widehat{\mathbf{x}}(k+1) &= \Phi\widehat{\mathbf{x}}(k) + \Gamma\mathbf{u}(k) + Le(k) \end{aligned}$$

for k = 0,1,2,...starting with $\widehat{\mathbf{x}}(k) = \overline{\mathbf{0}}$

3 Observer for a MIMO System and Controller Synthesis

For the purpose of illustration, let us consider a 2×2 MIMO system. This MIMO system can be viewed as 2 MISO systems. Thus, the steps involved in developing a state estimator for the MIMO system are as follows

1. Using data set $\mathcal{S}_1$

$$\mathcal{S}_1 = \{(y_1(k), u_1(k), u_2(k)) : k = 0, 1, 2, ..., N\} \tag{27}$$

identify ARX model and construct state space model

$$\begin{aligned} \mathbf{x}^{(1)}(k+1) &= \Phi^{(1)}\mathbf{x}^{(1)}(k) + \Gamma^{(1)}\mathbf{u}(k) + L^{(1)}e_1(k) && (28) \\ y_1(k) &= C^{(1)}\mathbf{x}(k) + e_1(k) && (29) \end{aligned}$$

2. Using data set $\mathcal{S}_2$

$$\mathcal{S}_2 = \{(y_2(k), u_1(k), u_2(k)) : k = 0, 1, 2, ..., N\} \tag{30}$$

identify ARX model and construct state space model

$$\begin{aligned} \mathbf{x}^{(2)}(k+1) &= \Phi^{(2)}\mathbf{x}^{(2)}(k) + \Gamma^{(2)}\mathbf{u}(k) + L^{(2)}e_2(k) && (31) \\ y_2(k) &= C^{(2)}\mathbf{x}(k) + e_2(k) && (32) \end{aligned}$$

3. Define vectors

$$\mathbf{x}(k) = \begin{bmatrix} \mathbf{x}^{(1)}(k) \\ \mathbf{x}^{(2)}(k) \end{bmatrix} \; ; \; \mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \text{ and } \mathbf{e}(k) = \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} \tag{33}$$

and matrices

$$\begin{aligned} \mathbf{\Phi} &= \begin{bmatrix} \Phi^{(1)} & [0] \\ [0] & \Phi^{(2)} \end{bmatrix} \; ; \; \mathbf{\Gamma} = \begin{bmatrix} \Gamma^{(1)}(k) \\ \Gamma^{(2)}(k) \end{bmatrix} \; ; \; \mathbf{L} = \begin{bmatrix} L^{(1)} & \overline{\mathbf{0}} \\ \overline{\mathbf{0}} & L^{(2)} \end{bmatrix} && (34) \\ \mathbf{C} &= \begin{bmatrix} C^{(1)} & [0] \\ [0] & C^{(2)} \end{bmatrix} && (35) \end{aligned}$$

we can construct MIMO state space model as follows

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{u}(k) + \mathbf{L}\mathbf{e}(k) && (36) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{e}(k) && (37) \end{aligned}$$

Note that the resulting model can be used to develop an unconstrained state feedback controller as follows

- Using matrices (Φ, Γ), develop a state feedback control law

$$\mathbf{u}(k) = -\mathbf{G}\mathbf{x}(k) \tag{38}$$

- Use the identified model to construct the state sequence as follows

$$\begin{aligned} \mathbf{e}(k) &= \mathbf{y}(k) - C\widehat{\mathbf{x}}(k) && (39) \\ \widehat{\mathbf{x}}(k+1) &= \Phi\widehat{\mathbf{x}}(k) + \Gamma\mathbf{u}(k) + \mathbf{L}\mathbf{e}(k) && (40) \end{aligned}$$

for k = 0,1,2,...starting with $\widehat{\mathbf{x}}(k) = \overline{\mathbf{0}}$

- Implement the state feedback control law as

$$\mathbf{u}(k) = \mathbf{u}_s(k) - \mathbf{G}\left[\widehat{\mathbf{x}}(k) - \mathbf{x}_s(k)\right] \tag{41}$$

where $\mathbf{x}_s(k)$ and $\mathbf{u}_s(k)$ are target state and target input.

4 Programming Hints

- To program the parameter estimation algorithm, you need to create matrix Ω of dimension $(N-2) \times 9$ in MATLAB and use a for loop to make the following assignment

$$\Omega(1, \quad : \quad) = \left[\begin{array}{ccc} -\mathcal{Y}(3)^T & \mathcal{U}_1(3)^T & \mathcal{U}_2(3)^T \end{array} \right] \tag{42}$$
$$\Omega(2, \quad : \quad) = \left[\begin{array}{ccc} -\mathcal{Y}(4)^T & \mathcal{U}_1(4)^T & \mathcal{U}_2(4)^T \end{array} \right] \tag{43}$$
$$... = ...$$
$$\Omega(N-2, \quad : \quad) = \left[\begin{array}{ccc} -\mathcal{Y}(N)^T & \mathcal{U}_1(N)^T & \mathcal{U}_2(N)^T \end{array} \right] \tag{44}$$

Vector $\mathbf{Y}$ can be created in the same for loop.

- Since $\boldsymbol{\theta}$ is defined as (13), we have

$$\widehat{\boldsymbol{\alpha}} = \begin{bmatrix} \widehat{a}_3 \\ \widehat{a}_2 \\ \widehat{a}_1 \end{bmatrix} = \boldsymbol{\theta}(1:3) \quad ; \quad \widehat{\boldsymbol{\beta}}_1 = \begin{bmatrix} \widehat{b}_{13} \\ \widehat{b}_{12} \\ \widehat{b}_{11} \end{bmatrix} = \boldsymbol{\theta}(4:6) \quad ; \quad \widehat{\boldsymbol{\beta}}_2 = \begin{bmatrix} \widehat{b}_{23} \\ \widehat{b}_{22} \\ \widehat{b}_{21} \end{bmatrix} = \boldsymbol{\theta}(7:9) \tag{45}$$

To construct (Φ, Γ) matrices from these vectors we need to reverse the order of elements in these vectors, which can be done using *flip* command of MATLAB. If you have a vector $\mathbf{v} = \left[\begin{array}{ccc} 3 & 2 & 1 \end{array} \right]$ then *flip*($\mathbf{v}$)= $\left[\begin{array}{ccc} 1 & 2 & 3 \end{array} \right]$. Thus, the following commands can be used for constructing (Φ, Γ) matrices

$$\Phi(:,1) = flip(-\boldsymbol{\theta}(1:3)) \quad \text{and} \quad L = \Phi(:,1) \tag{46}$$

$$\Gamma(:,1) = flip(\boldsymbol{\theta}(4:6)) \quad ; \quad \Gamma(:,2) = flip(\boldsymbol{\theta}(7:9)) \tag{47}$$

- **Steps in algorithm to construct the MIMO state space model from perturbation data**

  1. Construct matrix $\Omega^{(1)}$ and vector $\mathbf{Y}^{(1)}$ using set $\mathcal{S}_1$
  2. Find $\widehat{\boldsymbol{\theta}}^{(1)}$ using equation (19)
  3. Construct $(\Phi^{(1)}, \Gamma^{(1)}, C^{(1)})$ using $\widehat{\boldsymbol{\theta}}^{(1)}$
  4. Construct matrix $\Omega^{(2)}$ and vector $\mathbf{Y}^{(2)}$ using set $\mathcal{S}_2$
  5. Find $\widehat{\boldsymbol{\theta}}^{(2)}$ using equation (19)
  6. Construct $(\Phi^{(2)}, \Gamma^{(2)}, C^{(2)})$ using $\widehat{\boldsymbol{\theta}}^{(2)}$
  7. Combine two MISO state space models by constructing matrices $(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \mathbf{C})$ using equations (34) and (35).