CS 517
Homework 1

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1 Question 1

1. Cantor's proof shows that if $f : X \to \mathcal{P}(X)$, then there exists at least one value $D \in \mathcal{P}(X) \setminus \text{range}(f)$. Prove that if X is infinite, then for any $f : X \to \mathcal{P}(X)$, there are infinitely many values in $\mathcal{P}(X) \setminus \text{range}(f)$.

Let $f : \mathbb{N} \to P(\mathbb{N})$ be a surjective function. This means that for every subset $S \subseteq \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $f(n) = S$.

Let us define our diagonal function D_i for each $i \in \mathbb{N}$ as follows:

$D_i = \{x \mid x \notin f(x) \wedge x \leq i\} \cup \{x+1 \mid x+1 \notin f(x) \wedge x > i+1\} \cup \{i+1 \mid i+1 \in f(i)\}$

We want to show that $D_i \neq f(i)$ for all $i, j, x \in \mathbb{N}$, and $D_i \neq D_j$ for all $i \neq j$.

Building the 2 two-dimensional table

Let M be a two-dimensional table where each cell $M(i,j)$ is defined as follows:

$$M(i,j) = \begin{cases} 1 & \text{if } i \in f(j), \\ 0 & \text{otherwise.} \end{cases}$$

This table represents the membership of the element i in the subset $f(j)$ for all $i, j \in \mathbb{N}$.

Proof

According to the construction of D_i, D_i will differ with $f(i)$ at at-least one position in the two dimensional table and therefore D_i will include at-least one set that is not a part of the subset $f(i)$ for all i. According to the construction of D_i there are infinitely many Diagonals like that, the proof for the uniqueness of each diagonal is given below.

Assume $k \in \mathbb{N}$ and consider the diagonals D_k and $D_k + 1$, we get the following cases:

- If $k + 2 \in f(k + 1)$, then $k + 2 \notin D_k$ by the definition of D_k.
- But $k + 2 \in D_k + 1$

This shows that the two diagonals D_k and D_{k+1} differ at the position $k + 2$. Therefore, $D_k \neq D_{k+1}$ which implies that $D_i \neq D_j$ for all $i \neq j$.

And by definition of D_i we can see that since D_i disagrees with $f(i)$ and $f(i+1)$ at at-least one position $D_i \neq f(i)$

Therefore, $D_i \neq f(i)$ for all $i \in \mathbb{N}$.

To further clarify the claim, we can see that the the diagonal D_i and the function $f(i)$ differ at least one position from the clause:

$$\{x \mid x \notin f(x) \wedge x \leq i\} \tag{1}$$

and the clause:

$$\{x + 1 \mid x + 1 \notin f(x) \wedge x > i + 1\} \tag{2}$$

Additionally, we can see that two diagonals D_k and $D_k + 1$ will differ at the position k through the clause:

$$\{i + 1 \mid i + 1 \in f(i)\} \tag{3}$$

We can see that if $i + 1$ is in $f(i)$ it is in D_i from (3) and at the same time if $i + 1$ is in $f(i + 1)$ it is not in D_{i+1} from (2) or (3). Therefore, the diagonals will differ at position $i + 1$.

Since, there are infinitely many natural numbers there are infinitely many diagonals D_i such that $D_i \neq D_j$ for all $i \neq j$, where $i, j \in \mathbb{N}$ and no surjective function f can map the set of natural numbers to its power-set.

2 Question 2

Prove that a language L is decidable if and only if both L and its complement $\overline{L}$ are recognizable.

You must prove both directions:

- Assuming L is decidable, show that L and its complement are recognizable.
- Assuming L and $\overline{L}$ are recognizable, show that L is decidable.

Proof

If L is decidable, then L and $\overline{L}$ are recognizable

Assume L is decidable. Then there exists a Turing machine M that decides L, halting and accepting on inputs $w \in L$ and halting and rejecting on inputs $w \notin L$. A language is recognizable if there exists a Turing machine that halts and accepts for all inputs in the language, and halts or runs indefinitely otherwise. Since M decides L, we can use M to construct Turing machines M' for L and M'' for $\overline{L}$.

Machine M' operates as follows:

1. Simulate M on input w.
2. If M accepts, then accept.
3. If M rejects, then halt.

Machine M'' operates as follows:

1. Simulate M on input w.
2. If M accepts, then halt.
3. If M rejects, then accept.

Since M halts on all inputs, both M' and M'' halt on all inputs where they accept, satisfying the condition for recognizability. Hence, L and $\overline{L}$ are recognizable.

If L and $\overline{L}$ are recognizable, then L is decidable

Now assume L and $\overline{L}$ are recognizable by Turing machines M and $\overline{M}$, respectively. We construct a decider Turing machine M^* for L that simulates M and $\overline{M}$ in parallel on input w.

Machine M^* operates as follows:

1. Create two simulation tapes: one for M and one for $\overline{M}$.

2. Alternately simulate a step of M on the first tape and a step of $\overline{M}$ on the second tape.

3. If the simulation of M accepts, halt and accept w.

4. If the simulation of $\overline{M}$ accepts, halt and reject w.

Since L and $\overline{L}$ are complements, every input w belongs to either L or $\overline{L}$, and either M or $\overline{M}$ will eventually accept. By alternating simulations, M^* ensures that neither machine is favored, which prevents the scenario where M^* could run indefinitely due to one of the machines running indefinitely on inputs not in its language. Thus, M^* is a decider for L, proving L is decidable.