Instantons and Symmetry Breaking

Amey Gaikwad

IIT Bombay

Mentor : Dr. Pallab Basu, ICTS-TIFR

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Transition amplitude

- The transition amplitude by Feynman's path integral approach:

$$\langle x_f, t_f | x_i, t_i \rangle = N \int_{t_i}^{t_f} D[x] e^{iS_M}$$

- After changing the time to imaginary time $t \rightarrow -i\tau$ the integral changes to

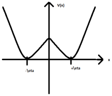
$$N \int_{t_i}^{t_f} D[x] e^{iS_M} \rightarrow N \int_{\tau_i}^{\tau_f} D[x] e^{-S_E}$$

- where S_M changes to S_E in the following manner

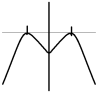
$$L_M = \frac{1}{2}(\frac{dx}{dt})^2 - V(x) \rightarrow L_E = \frac{1}{2}(\frac{dx}{d\tau})^2 + V(x)$$

Double Well Potential

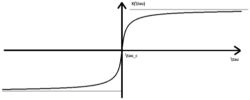
- Want to find $\langle \eta, \tau_0/2| - \eta, -\tau_0/2 \rangle$ with τ_0 going to ∞
- Move to imaginary time and use quasi-classical approximation.
- Instanton : $X(\tau) = \eta tanh(\omega(\tau - \tau_c)/2)$
- τ_c is known as the centre of the instanton



(a) Minkowskian time; (Credits: *NSVZ*)



(b) Euclidean Time;(Credits:*NSVZ*)



(c) Instanton

Instanton gas

- Many classical paths connecting the hills in Euclidean time
- String together the instanton and anti-instanton solutions.
- Total tunneling amplitude due to all possible classical paths ($d =$ instanton density)

$$\sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} sinh(\omega\tau_0 d)$$

$$d = \left(\sqrt{\frac{6}{\pi}}\sqrt{S_0} e^{-S_0}\right)$$

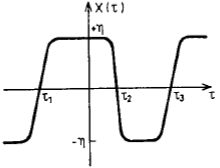


Fig. 5. The chain of n well-separated instantons (antiinstantons).

Figure: Instanton gas model; (Credits: *NSVZ*)

Yang Mills Lagrangian

- The lagrangian is invariant under the fundamental representation of the SU(2) group:

$$L = \frac{1}{4} Tr.[G_{\mu\nu} G^{\mu\nu}]$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Lagrangian invariant under the transformation ($\Omega \in SU(2)$)

$$A_\mu \to \Omega A_\mu \Omega^{-1} + \Omega \partial_\mu \Omega^{-1}$$

- Find the ground states $\implies A_0 = \mathbf{0}$
- For finite energy, need $A_i \to 0$ as $|x| \to \infty$
- Leads to mapping $\mathbb{R}^3$ (with infinity identified) to $S^3 \implies$ homotopy sectors $\implies$ countable sectors in which the vacua fall.
- **Winding number** of the vacuum.

Visualizing the instanton

- Analogy to a particle sitting at the bottom of a circle.
- Due to quantum fluctuations, the particle can tunnel through the potential barrier and come back to the ground state.
- In Yang Mills, all the ground states invariably sit at the bottom.
- The instanton refers to the tunneling through the potential barrier and completing the circle.
- This "circle" direction is given by the **Chern Simon's current**

$$K_\mu = 2\epsilon_{\mu\nu\alpha\beta}(A^a_\nu \partial_\alpha A^a_\beta + \frac{g}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta)$$

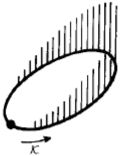


Figure: The direction of the circle given by the current; (Credits: *NSVZ*)

Winding number and instanton number

- For instanton, finite energy $\implies$ asymptotically vanishing energy
- Asymptotic behaviour + SU(2) $\implies$ countable sectors
- The instantons fall into denumerably many sectors characterised by the instanton number
- Instanton number = (Winding number of future vacuum) - (Winding number of the past vacuum)

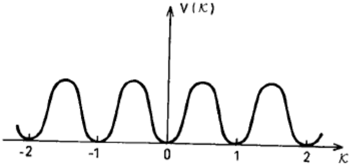


Figure: Open the circle shown before; (Credits: *NSVZ*)

Spontaneous symmetry breaking

- Consider the Lagrangian:

$$L = \frac{(\partial_\mu \varphi)(\partial^\mu \varphi)}{2} - \frac{m^2\varphi^2}{2} - \frac{\lambda}{4}\varphi^4$$

- Lagrangian and ground state invariant under symmetry $\varphi \to -\varphi$

$$L = \frac{(\partial_\mu \varphi)(\partial^\mu \varphi)}{2} + \frac{m^2\varphi^2}{2} - \frac{\lambda}{4}\varphi^4$$

- Lagrangian invariant but not the ground state

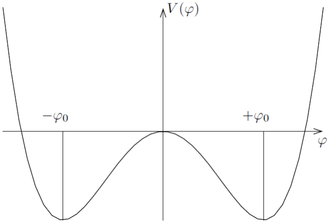


Figure: The two degenerate ground states; (Credits: *Rubakov*)

Nambu Goldstone bosons

$$L = \frac{(\partial_\mu \varphi)(\partial^\mu \varphi^\dagger)}{2} + \frac{\mu^2 \varphi^\dagger \varphi}{2} - \frac{\lambda}{4}(\varphi^\dagger \varphi)^2$$

- Completely broken U(1) symmetry
- Perturbative Lagrangian

$$L = \frac{(\partial_\mu \chi)^2}{2} + \frac{(\partial_\mu \theta)^2}{2} - \mu^2 \chi^2$$

- $\theta =$ Nambu Goldstone boson

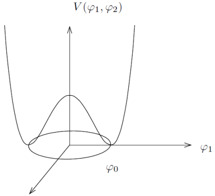


Figure: The degenerate ground states: the circle; (Credits: *Rubakov*)

Higgs mechanism (U1)

- Symmetry breaking for gauge fields.

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - [-\mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\varphi = \partial_\mu\varphi - iA_\mu\varphi$$

- Lagrangian invariant under

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$$

$$\varphi'(x) = e^{i\alpha(x)}\varphi(x)$$

- Completely broken symmetry.

$$A_\mu = \frac{1}{e}\partial_\mu\alpha \qquad \varphi = e^{i\alpha(x)}\varphi_0$$

Higgs boson

- Choosing the ground state $A_\mu = 0$ and $\varphi = \mu/\sqrt{2\lambda}$
- Perturbative Lagrangian after a convenient redefinition of variable:

$$L_{pert} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{e^2\varphi_0^2}{2}B_\mu B^\mu + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2$$

$$B_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta$$

- Massive vector field / gauge boson
- Massive Higgs scalar field $\rightarrow$ Higgs boson

Summary

- **Instantons**: Classical finite action solutions of the Euler Lagrange equations obtained from the Euclidean Lagrangian which is got from the Minkowskian Lagrangian by moving to imaginary time.
- Help in calculating the tunneling amplitude in case of topologically different degenerate vacua.
- The second part of the project dwelt on topics like symmetry breaking and the Higgs mechanism.
- Nambu Goldstone bosons occur in the case of a global symmetry group breaking.
- We say that the massless vector field 'eats' up a Goldstone boson and turns into a massive vector field.
- Future Work: Solitons, Polyakov monopole ($SU(2) \rightarrow U(1)$),Vortices and Confinement.

Acknowledgements

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Thank You!

Towards Monopoles

- Consider the lagrangian (a=1,2,3)

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2}(D_\mu\varphi)^a(D^\mu\varphi)^a - \frac{\lambda}{4}(\varphi^a\varphi^a - F^2)^2$$

$$D_\mu\varphi^a = \partial_\mu\varphi^a + g\epsilon^a_{bc}A^b_\mu\varphi^c$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^a_{bc}A^b_\mu A^c_\nu$$

- The Lagrangian is invariant under the following transformations:

$$\varphi^a \rightarrow (U)_{ab}\varphi^b$$

$$(L^a A^a_\mu)_{bc} \rightarrow U_{bd}[L^a A^a_\mu + (i/g)I\partial_\mu]_{de}(U^{-1})_{ec}$$

$$(U) = [exp(-iL^a\theta^a)]_{bc}$$

$$L^a_{bc} = i\epsilon_{abc}$$

Solving the field equation

- Static, finite energy solution
- Static energy:

$$E = \int \Big(\frac{1}{2} G^a_{ij} G^{aij} + \frac{1}{2}(D_i\varphi)^a (D^i\varphi)^a + \frac{\lambda}{4}(\varphi^a\varphi^a - F^2)^2$$

- $\varphi^a\varphi^a \to F^2$
- Asymptotic condition $\implies$ topological charge Q associated with solution.

$$k_\mu = (1/8\pi)\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\varphi}^a\partial^\rho\hat{\varphi}^b\partial^\sigma\hat{\varphi}^c$$

$$Q = \int d^3x k_0$$

Existence of Monopole

- Electromagnetic field associated with Abelian U(1) group.
- Possible to embed U(1) in SU(2)

$$F_{\mu\nu} = G^a_{\mu\nu}\hat{\varphi}^a - \frac{1}{g}\epsilon^{abc}\hat{\varphi}^a D_\mu \hat{\varphi}^b D_\nu \hat{\varphi}^c$$

- For $\hat{\varphi} = (0, 0, 1) \rightarrow F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$
- In EM, $D_\mu F^{\mu\nu} = 4\pi j^\nu$. In this case :

$$(1/2)\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = (4\pi/g)k_\mu \implies$$

$$\nabla \cdot \mathbf{B} = (4\pi k_0/g)$$

- Monopole charge $= Q/g$