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Instantons and Monopoles

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Abstract

Topological solutions of the Euclidean and the Minkowskian lagrangian equations hold special significance in the context of confinement.Monopoles and Instantons play an important role in understanding the confinement problem. The main aim of this review is to try to understand what instantons are topologically and how they appear in the Yang Mills theory. In the latter part of the review, the aim shifts to understanding what symmetry breaking is and what significance it plays in the context of monopoles. Two particular cases of monopoles are discussed - the Polyakov and the BPST. The entire review is done from a pure classical field theory approach and no technical details of QCD have been used.

Contents

Introduction

Protons, neutrons and electrons are baryons,i.e they are made up of three quarks. What keeps these things together are the strong force interactions between these quarks. Though, this fact is known, scientists have been unable to isolate these quarks. This is the problem of confinement. The strong force interactions between these particles are so large that even if we provide this amount of energy, the baryon will break up into two mesons, but never give a free quark. Many theories have been floated regarding this problem, but without proof. Instantons and monopoles are one of the few stepping stones

towards understanding how confinement actually works.

I start with instantons present in the double well potential, move to field theory, talk about instantons in the Yang Mills Lagrangian after a brief peep into Non Abelian gauge theories. In the latter part of the review, I discuss about spontaneous symmetry breaking and the Higgs mechanism. The review ends with a glance at monopoles.

1 Instantons

This section is devoted to the understanding of instantons in the double well potential. A brief look into the path integral formulation is made and the section ends with a discussion on the instanton gas model. The main aim is to understand what exactly are instantons and how they help in calculating the ground state and the first excited state energies of a particle in a double well potential.

1.1 Introduction

Instantons are classical solutions of the paths taken by a particle to move from one state to another "topologically different" state of the system. What we mean by "topologically different" will be clear by the end of this section. A brief peep into the Lagrangian formulation of quantum mechanics, which is needed for the study, has been mentioned.

1.2 Path Integral Formalism

In the particle case, we find that the Hamiltonian and the Lagrangian pictures are consistent with each other and equivalent to each other. The pictures can be transformed into each other through the Legendre transformations, namely.

$$\boldsymbol{H} = p\dot{q} - \boldsymbol{L} \tag{1.1}$$

where the symbols $\boldsymbol{H}$ and $\boldsymbol{L}$ denote the Hamiltonian and the Lagrangian respectively of the classical system being considered. A similar method and analogy can be followed in the case of quantum mechanics too. Most books follow the method of canonical quantization in quantum mechanics which hinges on the Hamiltonian picture and involves solving the Schrodinger's equation for the particle's wave function and gives a pretty local picture for getting the amplitude at each point. An analog to this picture is given by the path integral formulation of quantum mechanics which gyrates around the Lagrangian of the particle in question. We start by describing the transition

amplitude, which delves on the equivalence of the Schrodinger picture and the Heisenberg picture of quantum mechanics. The Lagrangian formalism also gives a better and a more physically intuitive way of looking at probabilities on finite time scales. This is important since solving the Schrodinger's equation for complicated potentials might not be feasible and the new method can give us a new direction of looking at the problem.

1.2.1 Transition amplitude

The transition amplitude gives us a measure of the probability of a particle moving from an initial spacetime point, say (x_i, t_i) to a final spacetime point, say (x_f, t_f). Then we define the transition amplitude as $\langle x_f, t_f | x_i, t_i \rangle$ where $|x_f, t_f\rangle$ and $|x_i, t_i\rangle$ represent the eigenkets in the Heisenberg picture and they are related to the Schrodinger's picture as given below:

$$|x_i, t_i\rangle = e^{iHt_i}|x_i\rangle \tag{1.2}$$

$$|x_f, t_f\rangle = e^{iHt_f}|x_f\rangle \tag{1.3}$$

where $|x_i\rangle$ and $|x_i\rangle$ represent the position eigenkets in the Schrodinger's picture. Therefore we get an expression for the transition amplitude as the following:

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle \tag{1.4}$$

where the exponential term represents the time evolution operator borrowed from elementary quantum mechanics. The most important feature about the transition amplitude is that it gives an easy outlet to calculate probabilities at finite time intervals, without solving the Schrodinger' equation.

1.2.2 Feynman's path integral

In this subsection,I will describe the formalism developed by Richard Feynman and how he related the transition amplitude to a very intuitive and non trivial integral in terms of the action of the trajectory of the particle.
The way to think about this is that to move from one spacetime point to another, the particle can take any path of the infinite number possibilities that it has. In classical mechanics, we would have minimized the classical action defined as

$$S = \int_{t_i}^{t_f} \boldsymbol{L}(q, \dot{q}, t) dt \tag{1.5}$$

This minimization would have led to an extremal path and that is the only path allowed classically. However quantum mechanically, each and every path is possible and the transition amplitude is obtained by summing over

the contributions from all the paths.

Consider the case of a double slit experiment where the particle has to cross a screen with two slits in it and reach the final point on the detector screen, with the initial point being the electron gun. The final amplitude or probability that we get on the screen is the sum of contributions of the particle wave coming from each of the two slits. Similar to this we consider the paths from an initial point to a final point with the screen being riddled with infinite number of holes and an infinite number of screens placed between the two points. Therefore the final transition amplitude is the sum of amplitudes from each possible path. Contribution to the amplitude from a path $\boldsymbol{x}(t)$ such that $\boldsymbol{x}(t_f) = \boldsymbol{x}_f$ and $\boldsymbol{x}(t_i) = \boldsymbol{x}_i$ is given by $e^{iS[x(t)]}$.Therefore we finally get

$$\langle x_f, t_f | x_i, t_i \rangle = \sum_{allpaths} e^{iS[x(t)]} \tag{1.6}$$

where the action is given by (1.5). Eq (1.6) can be refined and can be written as an integral as shown below:

$$\langle x_f, t_f | x_i, t_i \rangle = \sum_{allpaths} e^{iS[x(t)]} = \int \mathbf{D}[x(t)] e^{iS[x(t)]} \tag{1.7}$$

where the measure used takes into account all possible paths connecting the two points. Further from this formalism, the equivalence of Schrodinger's equation can be shown, and transition amplitudes for the free particle and the harmonic oscillator can be calculated, but not in this exposition. We will be using just the intuition and the definition of the path integral formulation and transition amplitude to move forward towards instantons.

1.2.3 The quasiclassical approximation

We can easily show that the classical path can be derived from the path integral formulation by assuming that the action of the path is large compared to Planck's constant (h). The transition amplitude in the case for free particle comes about to be the same when we consider the classical approximation and when we solve the integral properly.

Classical approximation : We take the contribution from just the classical path and write the path integral as:

$$\int \mathbf{D}[x(t)] e^{iS[x(t)]} \approx e^{iS_0} \tag{1.8}$$

The importance of the classical approximation comes because of the fact mentioned above for the case of the free particle. This just shows that for

complicated integrals, we can consider the classical approximation to give a major chunk of contribution to the transition amplitude.

Quasiclassical approximation : This approximation includes the classical contribution along with a error term upto a quadratic deviation. Usage of this approximation will be shown in the upcoming sections. The method is applied with the hope that along with the major chunk that the classical action contributes, the approximation which includes a small beam of paths around the classical path also contribute to such an extent such that the integral is almost equalled by the approximation.

The quasiclassical approximation is just the steepest descent method applied to the formula (1.10). Refer Appendix A for a brief peep into the saddle point integration method or the steepest descent method.

1.3 The single well potential

We consider a single well potential, however with a very general functional form $V(x)$. The Lagrangian is given by the following formula:

$$\mathbf{L} = \frac{1}{2}m\frac{dx}{dt}^2 - V(x). \tag{1.9}$$

We consider the initial point as (x_i,-$t_0/2$) and the final point as (x_f, $t_0/2$). Therefore the transition amplitude to move move from the initial point to the final point is given by (1.4) and (1.9):

$$\langle x_f|e^{-iH(t_0)}|x_i\rangle = N\int_{-t_0/2}^{t_0/2}\mathbf{D}[x(t)]e^{iS[x(t)]} \tag{1.10}$$

where N is the normalization constant and is fixed later.

Wick's rotation:

We now discuss the concept of Wick's rotation which involves shifting the contour over which we integrate. By changing the coordinate from t to τ with the relation between them being $t = -i\tau$. After doing this we define what we mean by the Euclidean Lagrangian and the Euclidean action corresponding to the Minkowskian Lagrangian and action. Therefore we get the following changes:

$$N\int_{-t_0/2}^{t_0/2}\mathbf{D}[x(t)]e^{iS[x(t)]} \longrightarrow N\int_{-\tau_0/2}^{\tau_0/2}\mathbf{D}[x(t)]e^{-S_E[x(t)]} \tag{1.11}$$

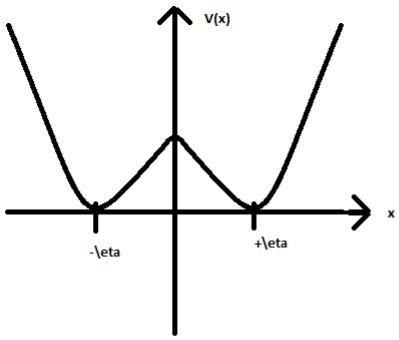

Figure 1: Minkowskian time; (Credits: *NSVZ*)

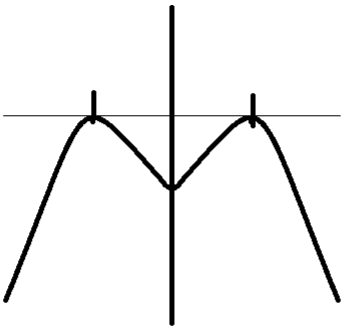

Figure 2: Euclidean Time;(Credits:*NSVZ*)

$$S[x(t)] \longrightarrow S_E[x(\tau)] \tag{1.12}$$

$$\mathbf{L}_E[x(\tau)] = \frac{1}{2}m\frac{dx}{d\tau}^2 + V(x). \tag{1.13}$$

The two pictures are equivalent and to show the conversion of one picture to another where we consider τ to be the real time via analytic continuation. What I mean by analytic continuation is that the change of variables from t to τ can be done seamlessly by altering the contour over which we want to integrate. A much better explanation can be found in (7) or (8).

We now consider the energy basis eigenkets of the Hamiltonian of the system.

$$H|n\rangle = E_n|n\rangle \tag{1.14}$$

and

$$\langle x_f|e^{-H\tau_0}|x_i\rangle = \sum_n e^{-E_n\tau_0}\langle x_f|n\rangle\langle n|x_i\rangle \tag{1.15}$$

We consider solutions with τ going to ∞ as we are concerned only with ground states and its energy. As can be seen from the above equation as ∞ is approached only the ground state term remains. For solving the integral we make the following observation. Any solution satisfying the end point boundary conditions can be written in the following manner:

$$x(\tau) = X(\tau) + \sum_n c_n x_n(\tau) \tag{1.16}$$

with X(τ) satisfying the end point boundary conditions and $x_n(\tau)$ satisfying the following condition:

$$x_n(\tau_0/2) = 0 \tag{1.17}$$

$$x_n(-\tau_0/2) = 0 \tag{1.18}$$

where $x_n(\tau)$ form an orthogonal function basis for all the functions satisfying the homogeneous boundary conditions. For the path integral we choose the measure as:

$$D[x] = \prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{1.19}$$

Now we move to the classical solution of the system, denoted by X(τ) but by minimising the Euclidean action. From Euler's Lagrange equation we get by minimising the action functional the following relation:

$$\frac{d^2X}{d\tau^2} - V''(X) = 0 \tag{1.20}$$

Now we use the quasiclassical approximation , introduced earlier to find the action upto a quadratic deviation. We get by using the Taylor's expansion in functional derivatives, the following relation:

$$S[X+\delta x] = S_0 + \int_{-\tau_0/2}^{\tau_0/2}[-\frac{1}{2}\frac{d^2\delta x}{d\tau^2} + \frac{1}{2}V''(X)\delta x]d\tau\delta x \tag{1.21}$$

Here S_0 refers to the action corresponding to the classical path and S refers to the action contributed by a narrow beam of paths around the classical path. The first functional derivative vanishes because the action is extremised by the classical path and the functional is being expanded around the classical path.

We now consider the solutions of the eigenvalue equation of the equation

in the integrand of the last equation and choose the eigenfunctions of those as our x_n mentioned earlier. Putting the boundary conditions, we get the eigenvalues. This will be done later and will depend on the nature of the potential and the final and the initial endpoints. For now, we see that after quasiclassical approximation, we get the following set of equations:

$$-\frac{d^2x_n}{d\tau^2} + V''(X(\tau))x_n = \epsilon_n x_n \tag{1.22}$$

$$S = S_0 + \frac{1}{2}\sum_n c_n \epsilon_n^2 \tag{1.23}$$

The integral (1.11) converts into a product of Gaussian integrals over the n variables and we get the transition amplitude as :

$$\langle x_f | e^{-H\tau_0} | x_i \rangle = e^{-S_0} N \prod_n \epsilon_n^{-\frac{1}{2}} \tag{1.24}$$

This will make sense only when the eigenvalues are positive, which is assumed to be so. The anomalous cases are present and the way to deal with them is to keep their variables unintegrated till the very end and integrate the ones that do make sense. We now consider the case of an approximate harmonic oscillator with $V''(0) = \omega^2$ and $x_i = x_f = 0$.Therefore we are interested in calculating

$$\langle 0 | e^{-H\tau_0} | 0 \rangle = N det\Big\{ -\frac{d^2}{d\tau^2} + \omega^2 \Big\}^{-\frac{1}{2}} \tag{1.25}$$

Using the boundary conditions given above for the x_n, we can calculate the eigenvalues to solve the above equation. The eigenvalues come out to be of the form:

$$\epsilon_n = \frac{\pi^2 n^2}{\tau_0^2} + \omega^2 \tag{1.26}$$

This leads to the splitting up of the determinant into two parts and solve each part separately to get :

$$\langle 0 | e^{-H\tau_0} | 0 \rangle = \frac{1}{\sqrt{(2\pi\tau_0)}} \Big(\frac{sinh(\omega\tau_0)}{\omega\tau_0} \Big)^{-\frac{1}{2}} \tag{1.27}$$

As $\tau_0 \to \infty$, we get the ground state wave function and energy as given by the formula mentioned earlier.

1.4 Double well potential

For the double well potential, due to quantum fluctuations, a particle sitting in one of the wells can tunnel through to the other well. This is the basis of the instanton. The classical path with finite action in the Euclidean time is the instanton and it gives the solution to tunneling in the Minkowski space by analytic continuation. It connects the two degenerate but non equivalent vacua of the double well potential. The ground state wave function of the double well potential can essentially be written as a linear combination of the eigen functions of the particle if supposed to be in the left hand well and the eigen functions of the particle if supposed to be in the left hand well. Here, we are considering to a very good approximation that the ground state wave function is just a linear combination of the two ground state wave functions ,if I assume the particle is inside just one of the two wells. Suppose the potential is given by the following formula;

$$V(x) = \lambda(x^2 - \eta^2)^2 \tag{1.28}$$

This potential mimics the double well potential with its minimas at η and $-\eta$. Suppose the particle is at $-\eta$ at time $-\tau_0/2$ and tunnels through to η at time $\tau_0/2$. We have to find this tunnelling amplitude with the time τ_0 going to infinity, since we are considering only the ground state of the wave function. Now the classical path for the particle moving from $-\eta$ to η from the respective times with τ_0 tending to infinity is given by:

$$X(\tau) = \eta \frac{tanh(\tau - \tau_c)}{2} \tag{1.29}$$

where τ_c denotes the centre of the instanton. This is the energy = 0 solution which is valid since we want the particle to start exactly from $-\eta$ and end exactly atη .Now the action corresponding to this classical path comes out to be:

$$S_{cl} = \frac{\omega^3}{12\lambda} \tag{1.30}$$

The action is completely independent of the τ_c term (the integration constant) , i.e the centre of the instanton doesn't matter. This gives rise to an additional degree of freedom in a direction along which the action does not change. So there exist a whole bunch of instantons with different centres that can tunnel through from one well to the other.

We now consider solving the double well potential problem in a manner similar to the one we used for the single well potential. We will calculate the determinant via a ratio of the determinant measures of the double well to

the single well. So there will be a term with the determinant in the double well case being normalized to the single well case and we get:

$$Ndet\Big\{-\frac{d^2}{d\tau^2}+V''(X)\Big\}^{-\frac{1}{2}} = Ndet\Big\{-\frac{d^2}{d\tau^2}+\omega^2\Big\}^{\frac{-1}{2}} \times \frac{det\Big\{-\frac{d^2}{d\tau^2}+V''(X)\Big\}^{\frac{-1}{2}}}{det\Big\{-\frac{d^2}{d\tau^2}+\omega^2\Big\}^{\frac{-1}{2}}} \tag{1.31}$$

We now solve the eigen value problem to get the eigenvalues so as to calculate the ratio of the determinants. We use an approximation and use the classical path that we got to replace in the equation and get the eigenvalues. The first two eigen values come out to be 0 and $\frac{3}{4}\omega^2$. The zero eigenvalue poses a problem for further analysis along the same lines as was done in the previous case. Refer eq (1.24). Now we cannot use the same method as was used earlier. The way to think about it is that any direction we take to go from one well to the other can be expressed as a particular solution, i.e the classical path and along with it the eigen functions. With regard to the eigen function corresponding to the zero eigenvalue, we can say that the contribution of this eigen function leads to no change in the value of the action corresponding to this path. Hence, we can say that the action remains invariant with the addition of some factor of the zero eigen function. This corresponds very closely to the relation the centre parameter holds with the action. Due to this, the zero mode is very closely related to the invariance in the action produced by the choice of the centre.Hence we can say that $S[X(\tau_c)] = S[X(\tau_c + \delta\tau_c)]$.

To deal with this problem, we leave the zero mode variable unintegrated. The eigen function corresponding to the zero eigenvalue is given by :

$$x_0(\tau) = \sqrt{\frac{3\omega}{8}}\frac{1}{cosh^2(\omega\tau/2)} \tag{1.32}$$

Since the invariance due to change in centre and the zero eigen function are related, the difference between $X(\tau_c)$ and $X(\tau_c+\delta\tau_c)$.Now ,we can relate the measure dc_0 with $d\tau_c$. We get the following relation :

$$dc_0 = \sqrt{S_0}d\tau_c \tag{1.33}$$

The detailed calculation is long and repeating it would take the physics out of the theory. So I will mention the final result and we will move to the instanton gas model.

$$\langle -\eta|e^{-H\tau_0}|\eta\rangle_{one\ \ instanton} = \Big(\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}\Big)\Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big)\omega d\tau_c \tag{1.34}$$

The first term can be identified to be the term also got in the case of the single well potential. This is important and consistent with the fact that our initial aim was to normalize the double well potential with respect to the single well potential which we have done. The second term in the brackets is the instanton density. What is important to note is that the term $\sqrt{S_0}$ is characteristic of the zero mode calculations. Also the term $\omega d\tau_c$ is known as the collective coordinate of the instanton.

This is a crude approximation since we have considered just a one instanton solution.The one instanton solution is only valid if a special condition connecting the variables and the constants used is satisfied. The total contribution to the tunnelling amplitude will also come from the multi-instanton solutions which will be discussed in the following section on the instanton gas model.

1.5 Instanton Gas

The Classical path we considered is not the only classical path that minimizes the action. Combine two instantons and one anti instanton and we get another tunneling possibility. This way there exist multi instanton solutions to the double well potential. We consider the instantons as non interacting with their centres separated by a sufficiently large distance. Thus for a valid solution, we can just add the contributions from the different instantons. Hence for an n instanton solution, we get:

$$\Big(\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}\Big)\Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big)^n\prod_n\omega_i d\tau_i \tag{1.35}$$

as the final tunneling amplitude, where τ_i represent the centres of the instantons. So if I want to integrate between $-\tau_0/2$ to $\tau_0/2$, I have to make sure the instanton centres lie within that range. This is important, since if we make a profile of how the instanton solution looks like, as shown in 1 we see that in order to move we need to cross the hump, which is present at the centre of the instanton, the hump has to be in the designated position for a physically realizable solution. Therefore for an instanton, I need to add up the the contributions from the odd number of instanton cases to get my final tunneling amplitude correct up-to the first order. The above equation can

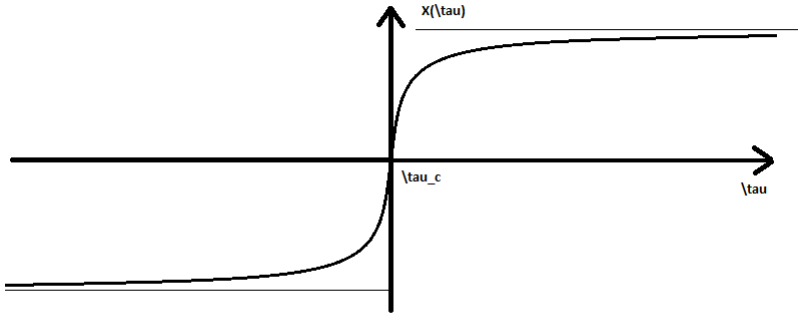

Figure 3: One instanton solution

rewritten as:

$$\Big(\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}\Big)\Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big)^n\int_{-\tau_0/2}^{\tau_0/2}\omega d\tau_n\int_{-\tau_0/2}^{\tau_n}\omega d\tau_{n-1}....\int_{-\tau_0/2}^{\tau_2}\omega d\tau_1 = \sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}\Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big)^n\frac{(\omega\tau_0)^n}{n!} \tag{1.36}$$

To get the final tunnelling amplitude we add the odd instanton contributions to get the following sum,

$$\langle-\eta|e^{-H\tau_0}|\eta\rangle = \sum_{n=1,3,5,7....}\sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}\Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big)^n\frac{(\omega\tau_0)^n}{n!} = \sqrt{\frac{\omega}{\pi}}e^{-\omega\tau_0/2}sinh(\omega\tau_0 d) \tag{1.37}$$

$$d = \Big(\sqrt{\frac{6}{\pi}}\sqrt{S_0}e^{-S_0}\Big) \tag{1.38}$$

where d is known as the instanton density. Now I show how the value of the ground state energy is approximated correct up-to a first order correction. At large time scales with τ_0 going to infinity, we write:

$$sinh(\omega\tau_0 d) = \frac{e^{\omega\tau_0 d} - e^{-\omega\tau_0 d}}{2} \tag{1.39}$$

Therefore

$$\langle-\eta|e^{-H\tau_0}|\eta\rangle = e^{-E_0\tau_0}\langle-\eta|0\rangle\langle 0|\eta\rangle \tag{1.40}$$

at large time scales and hence the above expression is equal to:

$$\sqrt{\frac{\omega}{\pi}}(e^{-\omega\tau_0/2})\frac{e^{\omega\tau_0 d} - e^{-\omega\tau_0 d}}{2} = \sqrt{\frac{\omega}{\pi}}(e^{-\frac{\omega\tau_0}{2}+\omega\tau_0 d})/2 \tag{1.41}$$

at large time scales. Hence after comparing coefficients, we get:

$$E_0 = \frac{\omega}{2} - \frac{\omega}{2}\sqrt{\frac{2\omega^3}{\lambda\pi}}e^{-\frac{\omega^3}{12\lambda}} \tag{1.42}$$

Now for the first excited state, we expand upto the 2nd energy state the expression for the tunneling amplitude as follows:

$$\langle -\eta|e^{-H\tau_0}|\eta\rangle = e^{-E_0\tau_0}\langle -\eta|0\rangle\langle 0|\eta\rangle + e^{-E_1\tau_0}\langle -\eta|1\rangle\langle 1|\eta\rangle \tag{1.43}$$

Now from the second term of the RHS we get by comparing the powers of e that the energy of the first excited state is given by:

$$E_1 = \frac{\omega}{2} + \frac{\omega}{2}\sqrt{\frac{2\omega^3}{\lambda\pi}}e^{-\frac{\omega^3}{12\lambda}} \tag{1.44}$$

Why can we get only energies only up-to the first excited state? Since all our calculations are made with respect to the first order approximations, starting from the quasi-classical approximation where we considered only upto a quadratic deviation the action to be integrated. Hence, only upto the first order do we get the energies.

In this whole instanton gas model, it is important to note that we have assumed that the centres of these instantons are assumed to be far apart so as to neglect the interactions between them so that while adding instantons and the anti - instantons, the nature of the combined solution looks as if the graphs have been superposed rather than interacted. This gives the value of the tunneling amplitude. This model of stringing instantons together without worrying about their interactions is known as the dilute instanton gas model, and though it is not totally correct, it gives very nice results as in this case, where the free parameter λ is used for consistency checks a posteriori, which can be best found in the reference mentioned. See 2

The concept of instantons can be extended to tunnelling considerations in QCD, especially in the regime of strong matter interactions manifested in Yang- Mills theory. Albeit, the case is much more complicated than this with an infinite number of degenerate vacua, the analysis will follow a similar route and idea.

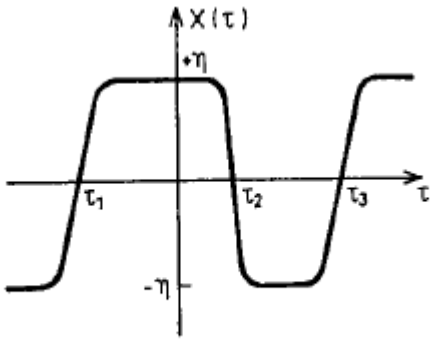

Fig. 5. The chain of n well-separated instantons (antiinstantons).

Figure 4: Stringing the instantons

2 Non Abelian Gauge theory

In the case of electrodynamics, the Lagrangian can be made invariant under the action of the elements of the group U(1) by coupling the electromagnetic field with a complex scalar field. Precisely, we get the following Lagrangian:

$$L_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - m^2(\varphi^\dagger\varphi) \tag{2.1}$$

with the following definitions:

$$D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi \tag{2.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.3}$$

where A is a real vector field and φ is a complex scalar field What I meant when I said that the Lagrangian was invariant under the action of the group elements of U(1), I meant the following:
The lagrangian is invariant under the following transformations:

$$\varphi(x) \to e^{i\alpha(x)}\varphi \tag{2.4}$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{2.5}$$

This invariance is not only under global transformations but also in the case of local transformations, albeit with changes occurring in the Lagrangian in order to make it covariant. In this section we take this generalisation one step further to the case of general groups and general representations of the group elements. Some particular examples are also mentioned for completeness and

complete comprehensibility.

We considered the combined interacting Lagrangian of the scalar field (matter field) and the vector "gauge field" (the EM field) . Why was the gauge field added? Because in the absence of the gauge field, the Lagrangian due to the scalar field $\boldsymbol{L}_\phi$ only would not have been invariant under local gauge transformations through the action of the elements of the group U(1). In order for the covariance of the scalar Lagrangian to hold, we introduce a vector field. Along with $\boldsymbol{L}_\phi$, we could also introduce a vector Lagrangian $\boldsymbol{L}_A$, which would also be invariant under the same transformation.

Keeping the spirit of the above method, we now consider a group G, in general Non Abelian. Consider an element $\omega \in G$. Consider a representation T of the group such that T(ω) is the representation of the group element ω . We again consider the original Lagrangian :

$$\boldsymbol{L} = (\partial_\mu \phi)^\dagger (\partial_\mu \phi) - m^2(\phi^\dagger \phi) + \lambda(\phi\phi^\dagger)^2 \tag{2.6}$$

In general we want to make this Lagrangian covariant under the following transformation:

$$\varphi'(x) = T(\omega)\varphi(x) \tag{2.7}$$

We note that we consider the fields ϕ and $\phi^\dagger$ as different and consider a them as rows of a single column vector,considered as a scalar field φ. We make the following definitions:

$$D_\mu \varphi = \partial_\mu \varphi + T(A_\mu)\varphi \tag{2.8}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.9}$$

where $T(A_\mu)$ represents the representation of the Lie algebra. We now define the structure constants which are always assumed to be real. Consider the generators of the Lie algebra AG, t_a and their respective representations $T(t_a) = T_a$:

$$[t_a, t_b] = C_{abc} t_c \tag{2.10}$$

where C_{abc} represents the real structure constants. The generators of the Lie algebra are the basis vectors for the algebra. Also,

$$A_\mu = g A^a_\mu t_a \tag{2.11}$$

where t_a are the Hermitian generators of the Lie algebra and g is the coupling constant. Similarly we can see that on performing the said non Abelian group transformations, we get

$$(D_u \varphi)' = T(\omega) D_\mu \varphi \tag{2.12}$$

Since (2.12) is the need to make the Lagrangian invariant, we conclude that the gauge field A_μ varies as :

$$A'_\mu = \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega^{-1} \tag{2.13}$$

$$(F_\mu \nu)' = T(\omega) F_{\mu\nu} T(\omega)^{-1} \tag{2.14}$$

Also:

$$F_{\mu\nu}{}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C^{abc} A^b_\mu A^c_\nu \tag{2.15}$$

Using the above equations, we can construct the most general Lagrangian which would be invariant under the transformations (2.7), (2.12), (2.13) and (2.14):

$$L = \frac{1}{4} Tr.[F_{\mu\nu} F^{\mu\nu}] + Tr.[(D_\mu \varphi)(D_\varphi)] - m^2 Tr[\varphi^2] - \lambda (Tr.[\varphi^2])^2 \tag{2.16}$$

The way the coupling constant appears in Non Abelian gauge theories is purely conventional. Also the way the equations of the gauge field look along with the arithmetic signs is also purely down to convention. Hence before using Non Abelian gauge theory, it is important to fix the conventions so as to avoid any discrepancies while solving the problem at hand. I will be sticking to the above convention always.

3 Yang Mills instantons

3.1 Introduction

The Yang Mills' theory and the Lagrangian poses a hotbed for combining quantum mechanics and deep topological nuances. This beautiful interplay is being explored in this monologue on discovering the instanton solutions of the Yang Mills theory and how they come about. In the double well case, we saw that in order to define instanton solutions, one needs the presence of degenerate vacua and a tunneling direction. In particle mechanics, as was the case before, this was apparent. However in field theory, it is a non trivial exercise to find these degenerate vacua (or even prove their existence), and isolate the tunneling direction from an infinite number of degrees of freedom. The Yang Mills Lagrangian consists of degenerate vacua, and instantons exist which keep tunneling between these topologically different vacua.The QCD Lagrangian consists of terms more than just the Yang Mills Lagrangian mentioned above. Also ,QCD obeys the non Abelian gauge invariance of the Lie group SU(3). However, we will be dealing with the SU(2) group. This

can be generalized to any group G, thanks to a theorem in homotopy, which states that any continuous mapping of S^3 into any general simple Lie group G can be continuously deformed into a mapping of S^3 into a subgroup of G. **Note** : Here S^3 refers to the hyper-sphere embedded in the 4 dimensional spacetime.

3.2 Setting the stage

The method followed here is inspired from (1) and (2) The first step to any instanton calculus is to move to Euclidean time, since instantons are classical finite action solutions to the Euclidean Lagrangian.Hence, they will satisfy for the field equations obtained from the same.The reason (specific to Yang-Mills theory) for why we move to Euclidean framework is given in Appendix C. However, converting to Euclidean time in case of Yang Mills case is trickier than in the case of a standard Lagrangian as was encountered in the case of the double well potential. The Yang- Mills Lagrangian is given by:

$$L = \frac{1}{4} Tr.[F_{\mu\nu} F^{\mu\nu}] \tag{3.1}$$

The Yang Mills Lagrangian is a non Abelian gauge theory with gauge group SU(2). So from the previous section, the first term of (2.16) is precisely what defines the Yang Mills theory.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.2}$$

$F_{\mu\nu}$ varies as (2.14) with the group element of SU(2) being in the fundamental representation where $F_{\mu\nu}$ is the field strength tensor and A_μ is the gauge field. Now, we move to Euclidean time via the substitution:

$$t = -i\tau \tag{3.3}$$

With only this substitution, we get the following expression for the F_{0i} component of the strength tensor.

$$F_{0i} = i\partial_\tau A_i - \partial_i A_0 + [A_0, A_i] \tag{3.4}$$

From the above equation, we find that with only one substitution, we get solutions of the gauge field which are complex and this truly contradictory since the observations that we get for the gauge fields are as real as they may get. Hence we need to do something more; precisely the following:

$$A_0 \to iA_0 \tag{3.5}$$

along with the transformation (3.3). Using these two changes, we get the following expression for the same component as was mentioned above:

$$F_{0i} = i\partial_\tau A_i - i\partial_i A_0 + i[A_0, A_i] \tag{3.6}$$

Hence the F_{0i} component is purely imaginary. The other components of the field strength tensor remain unaffected. Hence the action (in the Euclidean sense) after Wick's rotation changes to:

$$S = \int d^3x dt \Big[\frac{1}{2}(F^a_{0i})^2 - \frac{1}{4}(F^a_{ij})^2\Big] = iS_E \tag{3.7}$$

where

$$S_E = \int \frac{1}{4}(F_{\mu\nu})^2 d^4x \tag{3.8}$$

The field equations that we get by solving the Yang Mills Lagrangian are given by:

$$D_\mu F_{\mu\nu} = g j_\nu \tag{3.9}$$

$$\epsilon_{\mu\nu\rho\sigma}(D_\nu F_\rho \sigma)^a = 0 \tag{3.10}$$

where from (2.11) it follows that

$$D_\mu F_{\rho\sigma} = g(D_\mu F_{\rho\sigma})^a t^a \tag{3.11}$$

where t^a are the generators of the lie algebra of the lie group SU(2). Now we take a slight detour and consider the following: we fix the gauge $A_0 = 0$ and we get an expression for the static energy. this will be important for analyzing the gauge configurations of the degenerate vacua that are present in the Yang- Mills' theory. However, this detailed analysis will be done at later stage. What I wish to point out is the importance of the Gauss' constraint.

Gauss' constraint : Working in the above gauge does not lead to all the field equations. The one that is left out is the following:

$$D_i F^{0ia} = -g j^{0a} \tag{3.12}$$

Hence for the field equations to be satisfied at all times, we need to impose the above condition on the system over and top of the field equations we get after solving the Lagrangian. Hence for all permissible states, $|\psi\rangle$ that the particle might be in as times tend to positive and negative infinity, we need it to satisfy the above Gauss' constraint.

3.3 The Instantons of Yang Mills

From the approach taken in the double well case , we are always interested in the finite action solutions. There we needed the finite action constraint since we needed to calculate the energies. We may need to calculate the energies in this case too. So we put the finite action constraint. So this implies that the Lagrangian drops to zero at large radial distances.
$F_{\mu\nu} \to 0$ as $|x| \to \infty$. This necessity implies that at large distances, the gauge field assumes a pure gauge form, i.e:

$$A_\mu = \omega \partial_\mu \omega^{-1} \qquad |x| \to \infty \tag{3.13}$$

Strictly speaking $A_\mu = 0$. However, any gauge transformation of 0 is also a solution. Also it turns out that (3.10) does give zero when plugged into (3.8).
There are some very interesting things about the nature of the gauge transformations that are applicable and about the non trivial mapping between the group SU(2) and the 3 dimensional sphere.

1. Using a gauge transformation, we can make the group element ω to depend only on the three angle at infinity and not on the radial coordinate. This transformation is through a matrix known as the **path ordered integral** and defined as:

$$\tilde{U}(x) = P(exp(\int_0^r dr' A_{r'})) \tag{3.14}$$

The interpretation of the integral is important. the integral is carried along the radial direction joining the origin to the point. While considering the exponential, the integral is basically a product of tiny strips along the radial direction. They may not commute in general. Hence the ordering is defined as the product being in the increasing value of the radial coordinate in the radial direction. On performing the gauge transformation with this element, we get

$$A'_r = \tilde{U}\omega\tilde{U}^{-1} - \partial_r \tilde{U}\tilde{U}^{-1} = 0 \tag{3.15}$$

Hence, the resultant asymptotic value at spatial infinity leads to a pure gauge with a group element completely independent of the radial coordinate.

2. The group element is defined using 3 independent parameters and is defined on the 3 dimensional sphere. Refer eq (3.14) and (3.15). Since the group we are dealing with is SU(2), the three generators are the Pauli matrices. Along with the identity matrix, the Pauli matrices form a basis for

any 2x2 matrix and hence ω can be written as the following:

$$\omega = \sum_{0}^{3} a_\mu s^\mu \tag{3.16}$$

where s_μ represents the generators of the Lie group SU(2). Also, being in SU(2), it also satisfies the following property:

$$\sum_{0}^{3} a_\mu a^\mu = 1 \tag{3.17}$$

This implies again that we have 3 independent variables and there exist mappings from the 3 dimensional sphere (S^3_{phy}) to the SU(2) hyper-sphere (S^3_{int}). The set of continuous maps from S^3 to S^3 is disconnected into countably many infinite connected components. Each connected component is associated with an integer, i.e:

$$\Pi_3(SU(2)) = Z \tag{3.18}$$

though the above equality will hold true for any simple group. The integer associated to each such homotopy sector is given by the topological number/charge , also known as the instanton number , denoted by Q. It is given by the following formula:

$$Q = \frac{1}{24\pi^2} \int d\sigma_\mu \epsilon^{\mu\nu\alpha\beta} Tr.[\omega\partial_\nu\omega^{-1}\omega\partial_\alpha\omega^{-1}\omega\partial_\beta\omega^{-1}] \tag{3.19}$$

We note the following:

$$Tr.[F_{\mu\nu}\tilde{F}^{\mu\nu}] = \partial_\mu K_\mu \tag{3.20}$$

$$K_\mu = \epsilon^{\nu\alpha\beta}_\mu Tr.[F_{\nu\alpha}A_\beta - \frac{2}{3}A_\nu A_\alpha A_\beta] \tag{3.21}$$

where K_μ is known as the **Chern Simon's** current.

Note : The above formula always gives an integer for all maps between S^3_{phy} and S^3_{int} . In homotopy theory the above relation mimics the degree of freedom of the map between the two mentioned spaces. This Q is also known as the Pontryagin index.

3. When we talk about the the gauge field approaching a pure gauge form at large distances, keeping the above abstract in mind, we mean that ω need not be non singular on the whole 4 dimensional surface. Non singularity just on the surface of the 3 dimensional sphere will also do. It need not be the limit of some gauge configuration defined on the 3 dimensional sphere.

4. The transformations considered in point 1 and also, in general, when we talk about gauge transforming the asymptotic value of the gauge field, we mean through a non singular transformation which is non singular throughout the 4 dimensional space.

Distinction between points 3 and 4 is important and is also pointed out by Sidney Coleman. To summarize, the asymptotic value of the gauge field need not be non singular for the whole 4 dimensional space, however ,the gauge transformations that we talk about need to be non singular for the whole 4 dimensional space.

Action: The action of the instantons is next to calculate before moving to the actual solution of the 1- instanton. The following inequality holds as a relation between the action and the Pontryagin index :

$$S \geq \frac{8\pi^2}{g^2} \tag{3.22}$$

$$Q = -\frac{1}{16\pi^2}\int Tr.[F_{\mu\nu}\tilde{F}^{\mu\nu}]d^4x \tag{3.23}$$

$$S = \frac{1}{4}\int d^4x Tr.[F_{\mu\nu}F^{\mu\nu}] \tag{3.24}$$

Now, using the fact that SU(2) is a compact group, which implies the existence of a positive definite quadratic invariant , namely $(A, B) = -Tr.[AB]$, we get the inequality,:

$$-\int Tr.[(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2]d^4x \geq 0 \tag{3.25}$$

Using (3.25) , (3.22) can be proven, where,

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{3.26}$$

is known as the dual of he field strength tensor.
For inequality (3.22), equality holds when the self dual equation holds. For negative values of Q, we use (3.25) with the minus sign replaced by the plus sign. The action corresponding to the minimum of the action functional is the one that corresponds to the yang Mills instanton solution with the topological charge remaining fixed. Therefore,

$$S_{instanton} = \frac{8\pi^2}{g^2}|Q| \tag{3.27}$$

Also the equality in (3.25) implies that the self dual equation holds for the instanton.

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \tag{3.28}$$

Therefore solutions of the self dual and the anti-self dual solution gives rise to the instanton solution.

3.3.1 Analysis

As mentioned before, for the analysis part, we will work in the gauge $A_0 = 0$. Why we work in this gauge will be mentioned in Appendix D.

Therefore, the energy is given by (in this gauge):

$$E = \int \frac{1}{2}((\partial_0 A_i)^2 + \frac{1}{4}(F_{ij})^2)d^3x \tag{3.29}$$

The degenerate vacua are the states that occupy the minima of the energy functional. Thus we can gather from the above function that A_i must be of pure gauge form. Hence,

$$A_i|_{vac} = \tilde{\Omega}\partial_i\tilde{\Omega}^{-1} \tag{3.30}$$

Due to the gauge we are working in, $\tilde{\Omega}$ is independent of the time coordinate and hence only on the spatial coordinates. From the above equation, it is clear that as we approach spatial infinity, the value of the spatial components of the four vector should go to zero (for finite energy). Hence $A_i \to 0$ as $|x| \to \infty$. This implies that:

$$\tilde{\Omega}(x) \to 1 \qquad |x| \to \infty \tag{3.31}$$

The above equation holds for all degenerate vacua with no exceptions. We are interested in the following problem: At $\tau \to -\infty$ the particle is in one gauge configuration corresponding to a generate vacua and at $\tau \to \infty$, the particle must reach another degenerate vacua with a different gauge configuration. The following is a tricky concept:

According to the definition of degenerate vacua as mentioned above, we now define the topological number of the vacua, also known as the **Winding number** of the degenerate vacua. It is defined in the following manner.

$$n(\tilde{\Omega}(x)) = \frac{1}{24\pi^2}\int d^3x\epsilon^{ijk}Tr.[\tilde{\Omega}\partial_i\tilde{\Omega}^{-1}\tilde{\Omega}\partial_j\tilde{\Omega}^{-1}\tilde{\Omega}\partial_k\tilde{\Omega}^{-1}] \tag{3.32}$$

We now define the concept of static energy. The static energy refers to the energy when the time is frozen, i.e all the derivatives appearing in the energy

functional with respect to time drop off and what we are left is defined as the static energy and perceived as the potential energy of the system.

$$E_{stat} = -\frac{1}{2g^2}\int d^3x Tr.[F_{ij}F^{ij}] \tag{3.33}$$

This we define as the potential barrier and make the following statement: **A barrier does not exist between a pair of classical vacua if and only if the gauge function $\tilde{\Omega}$ can be continuously deformed from its value in the first vacuum of the pair to the value in the second vacuum (in such a way that the asymptotic boundary condition of it tending to 1 is not violated).** Now, similar as was the case in terms of the instanton, the above characterization also gives us a mapping from S^3_{phy} to S^3_{int} and hence we get homotopy sectors characterized by the winding number.A gauge configuration of one homotopy sector cannot be continuously deformed into a gauge configuration of another homotopy sector. This implies that the existence of the potential barrier will solely depend on whether the vacua between which we are tunneling belong to the same homotopy sector or not.If they do, then the potential barrier is zero and we can take a path through the classical vacua to move from one state to the other. However, if they belong to different homotopy sectors, to move from one to the other through the classical vacua without a singularity is not possible. Hence, the potential barrier is raised and it tunnels through between the two vacua. For the pair , for which no barrier exists they are known as topologically equivalent.
Note : If **we prove the existence** of a continuously deformable path from one gauge configuration to the other such that the energy remains zero during this transformation, we are done. The bold statement says that the tunneling solution is the only way to move between the two gauge configurations, i.e in other words, there exists no classical path connecting the two and tunneling has to be considered.

The relation between the Pontryagin index and the winding number:

The topological charge of the instanton solution depicts the number of wells through which the tunneling has occurred. We now show that the topological charge is the difference between the winding numbers of the gauge configurations of the degenerate vacua at an infinite time in the future and an infinite time in the past. We are working in the $A_0 = 0$ gauge. Therefore, we have $F_{\mu\nu} \to 0$, which implies that the the spatial components of the four vector A_μ are related to a pure gauge field. Also the spatial components go to zero for $|x| \to \infty$ according to the condition mentioned in the text above.

The case is a bit different here. Now, we are concerned with just the time going to infinity, and not at all interested with the spatial coordinates. So, we have two gauge configurations in the infinite past and in the infinite future with absolutely no variance with respect to the time coordinates. We concern ourselves with the following gauge configurations:

$$A_i|_{\tau\to-\infty} = \tilde{\Omega}_0 \partial_i \tilde{\Omega}_0^{-1} \tag{3.34}$$

$$A_i|_{\tau\to\infty} = \tilde{\Omega}_1 \partial_i \tilde{\Omega}_1^{-1} \tag{3.35}$$

Now the Pontryagin index is given by the volume integral over the whole phase space volume with the boundary conditions as the two aforementioned conditions. We consider the phase space volume as a cylinder since we are interested in only the time component going to infinity. For this reason and to mimic this boundary behaviour, we concern ourselves with the boundary of the phase space volume being a cylinder with the two opposite "flat" ends of the cylinder being the two configurations. Now we can write the following from (3.19) and (3.21)

$$Q = c \int \partial_\mu K^\mu \tag{3.36}$$

where c is a constant accompanying the integral. For the surface integral over this cylinder, the lateral surface has no contributions since the value of the gauge field at spatial infinity (where the lateral surface lies) is zero. Hence we are left with calculating the surface integrals at the two caps placed in the infinite past and in the infinite future. Hence, we get the result :

$$Q = n(\tilde{\Omega}_1) - n(\tilde{\Omega}_0) \tag{3.37}$$

Now that the relationship between the Pontryagin index and the winding number has been established along with a physical picture of what exactly these instantons are doing, we move to θ vacua.

θ Vacuum

The theta vacuum is constructed from the pre-vacua solutions using a theta parameter which is a global parameter and not a local parameter as the gauge transformations used in the Lagrangian gauge invariance. Now the pseudo-particle moves from one vacuum state to the other. The solution of the instanton is going to be immaterial of whether the degenerate vacua were chosen to be (2,3) or (n,n+1) for a one instanton problem. What this means is that there exists a translation symmetry in the problem related to the instanton solution. It does not care about the winding numbers of the degenerate vacua it is connecting as long as the difference in their winding

numbers is the same. Let the operator which shifts the vacua by one be denoted as T. Hence we see the following facts :

$$T|\Psi\rangle = \lambda|\Psi\rangle \tag{3.38}$$

$$H|\Psi\rangle = E|\Psi\rangle \tag{3.39}$$

Because of the symmetry [H,T] = 0. Hence they can be simultaneously be diagonalised. This leads to the conclusion that $|\Psi_n\rangle = \lambda|\Psi_0\rangle$ Since we want the nth degenerate pre-vacuum to be finite and hence the entire solution to be normalisable, we choose an oscillating $\lambda = e^{i\theta}$.

$$\Psi(\Theta) = e^{ix\theta}u(x) \tag{3.40}$$

Using the tight binding approximation that the pseudo-particle cannot lie in between the two gauge configurations leads us to believe that the u(x) can be written as a sum of Dirac delta functions with centres at n where n is any integer (positive, negative or zero)

$$|\Theta\rangle = \sum_n e^{in\theta}|n\rangle \tag{3.41}$$

We think of the operator T in the following sense :

$$T^{-1}A_iT = \Omega_1 A_i \Omega_1^{-1} + \Omega\partial_i\Omega_1$$

Using the expression for Ω_1 in the gauge $A_0 = 0$, using the definition mentioned above we get that

$$T(\Omega_1)A_{old} = A_{new} \tag{3.42}$$

Since this is independent of where you are starting from ,it implies that it is a symmetry of the system and hence commutes with the Hamiltonian.

Relation between the instanton gauge configuration living on S^3 and the group element corresponding to the gauge configuration of the vacua living on R^3 at some time in the infinite past.

When we fix the gauge, the instanton configuration moves into the S^3 with time going to infinity which i think is equivalent to the R^3 plane. The instanton is related to the gauge configurations between which it tunnels by the relation (obviously in the gauge $A_0 = 0$) by definition:

$$\Omega_1 = U_\infty U_{-\infty}^{-1} \tag{3.43}$$

Now consider the Pontryagin index. We have it defined on the 4 dimensional space by the 4-volume integral. Now we move to the 3-surface integral. Here the integrand consists of the instanton group element function which is required to be defined non singular only on the surface of S^3. Now we fix the gauge and hence using the above definition, we get that the Pontryagin index is the difference between the winding numbers of the two gauge configurations. By fixing the gauge we get rid of the extra index over the Levi Civita Tensor and since the gauge configurations are completely independent of time, everything seems consistent. In the end we get

$$Q = Q(U_\infty) + Q(U_{-\infty}^{-1}) = Q(U_\infty) - Q(U_{-\infty}) \tag{3.44}$$

3.3.2 An example

We give an example of how to arrive at the instanton solution for the $Q = 1$ case. The topological classification of the instanton solution is done on the basis of the pure gauge form that it adopts asymptotically at spatial infinity.

$$A_\mu = \omega \partial_\mu \omega \tag{3.45}$$

The above equation is the asymptotic behaviour. For a ω with the degree of mapping from S^3_{phy} to S^3_{int} being one, we take ω to be:

$$\omega = n_\alpha \sigma^\alpha \tag{3.46}$$

$$\sigma_\alpha = (I, -i\tau) \tag{3.47}$$

where n_α denotes the unit vector in the 4 dimensional space and σ_α represents the generators of the SU(2) group and τ refers to the Pauli matrices. Using the above expressions, we get:

$$A_\mu(r \to \infty) = \sigma_\alpha \sigma_\beta n_\alpha \frac{\partial_{\mu\beta} - n_\mu n_\beta}{r} \tag{3.48}$$

and

$$\sigma_\alpha \sigma_\beta^\dagger = \delta_{\alpha\beta} + i\eta_{\alpha\beta a}\tau^a \tag{3.49}$$

where $\eta_{\alpha\beta a}$ represents the **t'Hooft** symbols. Since instanton solutions are being looked at, we can conclude that (3.25) holds and hence (3.28) also holds.:

$$\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\eta_{\mu\nu a} = \eta_{\alpha\beta a} \tag{3.50}$$

and following are the non zero components:

$$\eta_{0ia} = -\eta_{0ia} = \delta_{ia} \tag{3.51}$$

$$\eta_{ija} = \epsilon_{ija} \tag{3.52}$$

From the above equations , we get a nice form for the asymptotic gauge field:

$$A_\mu = -i\eta_{\mu\alpha a}\frac{n_\alpha}{r}\tau^a \tag{3.53}$$

We generalize the above expression for all spacetime points as:

$$A_\mu = -i\eta_{\mu\alpha a}\frac{n_\alpha}{r}f(r)\tau^a \tag{3.54}$$

satisfying:

$$f(r) \to 1 \qquad r \to \infty \tag{3.55}$$

$$f(r) \to 1 \qquad r \to \infty \tag{3.56}$$

with the second equation holding for the non-singularity at the origin. The field strength tensor can be calculated and using the self-dual equation, we get the following differential form for $f(r)$:

$$f' = \frac{2}{r}f(1-f) \tag{3.57}$$

We get:

$$f(r) = \frac{r^2}{r^2+\rho^2} \tag{3.58}$$

We can generalize to the form:

$$f(r) = \frac{(r-a)^2}{(r-a)^2+\rho^2} \tag{3.59}$$

where a represents the centre of the instanton and ρ depicts the size of the instanton.

Note : We have chosen a specific form for the asymptotic pure gauge field, say U_1, which has a topological charge of 1. A matrix of the form $(U_1)^n$ has a topological charge of n and hence we get, in general an **n instanton** solution for the Yang Mills' theory.

4 Symmetry Breaking

After this discussion on instantons, I move to the latter part of the project - Symmetry breaking. The concept of spontaneous symmetry breaking is important to understand the existence of monopoles. In this section, I describe what exactly we mean by symmetry breaking. I describe just the complete symmetry breaking part. However, the concept of partial breaking of symmetry is mentioned in the section on Monopoles.

4.1 Nambu Goldstone bosons

These bosons appear when there is a case of spontaneous symmetry breaking. However, only in the context of Lagrangians which are invariant under global transformations, are they valid.

4.1.1 Discrete transformation

The Lagrangian chosen is:

$$L = \frac{(\partial_\mu \varphi)^2}{2} - V(\varphi) \tag{4.1}$$

$$V(\varphi) = \frac{m^2\varphi^2}{2} + \lambda\frac{\varphi^4}{4} \tag{4.2}$$

the above case corresponds to the ground state being the trivial vacuum solution for the field :

$$\varphi_0 = 0 \tag{4.3}$$

The Lagrangian chosen above gives all the information needed about the 'particles' involved in the theory.I will explain what we mean by a 'particle' her at the end of section (4.2) .Here the ground state is trivial and rewriting φ as

$$\varphi_0 + \chi \tag{4.4}$$

we get the same Lagrangian back. The ground state field configuration for the above field is symmetric under the discrete symmetry $\varphi \to -\varphi$ as is the Lagrangian. However this is not always the case. Consider the Lagrangian considered below:

$$L = \frac{(\partial_\mu \varphi)^2}{2} - V(\varphi) \tag{4.5}$$

$$V(\varphi) = -\frac{m^2\varphi^2}{2} + \lambda\frac{\varphi^4}{4} \tag{4.6}$$

For detecting the presence of a 'particle' at some field configuration , **we write the field as (4.4) about some field configuration, plug it in the original Lagrangian, if it conforms to the standards of a harmonic potential, then we say say the perturbative factor χ corresponds to a particle**.This field configuration is the ground state of the theory in consideration.Further elucidation at the end of section (4.2). The ground state field configuration for this Lagrangian is :

$$\varphi_0 = \pm\frac{\mu}{\sqrt{\lambda}} \tag{4.7}$$

The Lagrangian is invariant under the discrete symmetry, however the ground state field is not. This is a case of spontaneous symmetry breaking.

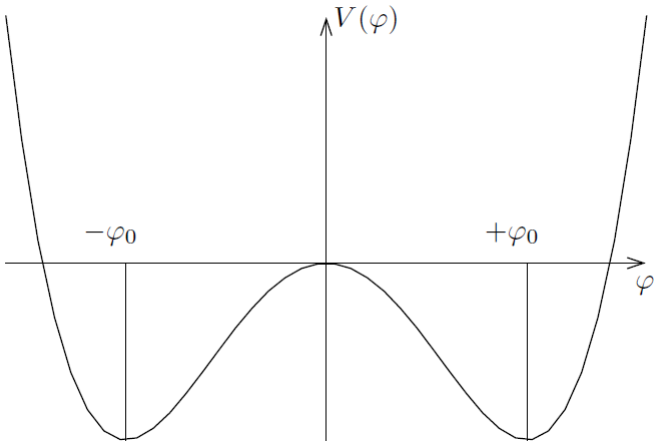

Figure 5: Discrete symmetry breaking

4.1.2 U(1) symmetry

Consider the Lagrangian :

$$L = \frac{(\partial_m u\varphi)^\dagger(\partial_\mu\varphi)}{2} - V(\varphi^\dagger\varphi) \tag{4.8}$$

$$V(\varphi^\dagger\varphi) = \frac{m^2\varphi^\dagger\varphi}{2} + \lambda\frac{(\varphi^\dagger\varphi)^2}{4} \tag{4.9}$$

The above Lagrangian is invariant under global U(1) symmetry and so is the ground state of the wave function which is again the trivial $\varphi = 0$. Again the information about the particles involved in the theory is observed directly from the global Lagrangian and even if we do a local analysis about the trivial ground state, we would get the same Lagrangian back and hence would also arrive at the same conclusions.Now moving on to the interesting case . Consider the Lagrangian:

$$L = \frac{(\partial_\mu\varphi)^\dagger(\partial_\mu\varphi)}{2} - V(\varphi^\dagger\varphi) \tag{4.10}$$

$$V(\varphi^\dagger\varphi) = -\frac{m^2\varphi^\dagger\varphi}{2} + \lambda\frac{(\varphi^\dagger\varphi)^2}{4} \tag{4.11}$$

The above Lagrangian is invariant under global U(1) symmetry. The ground state field configuration in the case of m being imaginary is given by :

$$\varphi(x) = e^{i\alpha}\frac{\varphi_0}{\sqrt{2}} \tag{4.12}$$

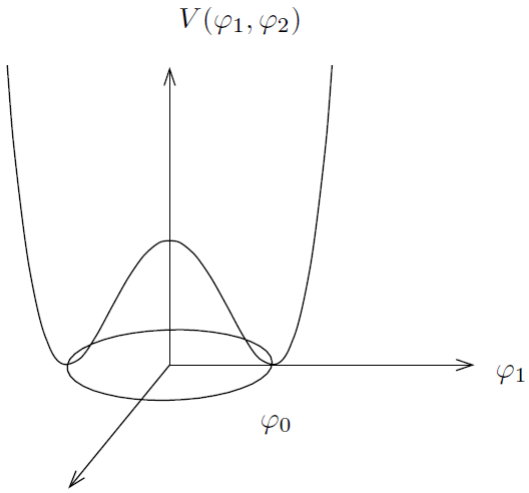

Figure 6: Mexican Hat - Global U(1) symmetry breaking

where φ_0 minimizes the potential.

$$\varphi_0 = \frac{\mu}{\sqrt{\lambda}} \tag{4.13}$$

This implies that the ground state wave function is not unique. Now the above Lagrangian does not signify the existence of any physical particle due to the mass being imaginary. To find out the particles involved in the theory , we are interested in the Lagrangian close to the ground state. Expanding perturbatively about the ground state , we get a local Lagrangian:

$$L_\chi = \frac{(\partial_\mu \chi)^2}{2} + \frac{(\partial_\mu \theta)^2}{2} - \mu^2 \chi^2 \tag{4.14}$$

As can be seen clearly from the above picture, we get the idea that we have one massive and one massless scalar field. The massless field is known as the Nambu Goldstone boson and the mode is known as the Nambu Goldstone massless mode. It is important to note that the deduction of the presence of the Nambu boson would not have been possible through the eyes of the global Lagrangian . As a result, in order to get an idea of the type of physical articles we are dealing with , it is important to find the Lagrangian in the locality of a ground state configuration and deduce the conclusions from there.

4.1.3 Goldstone's Theorem

The number of Nambu Goldstone modes that appear in the theory is equal to the number of broken generators of the theory. This theorem rises to

prominence to the case when there is a case of partial symmetry breaking instead of complete symmetry breaking. Partial symmetry breaking is the case where the ground state maintains some inherent symmetry of the original symmetry, i.e if the Lagrangian is invariant under some gauge group G, and if the ground states of this theory are invariant under the action of a subgroup H of G, we say that there is a case of partial symmetry breaking. The number of Nambu Goldstone bosons that appear in the perturbed Lagrangian is equal to the difference in the dimensions of the lie algebra of G and lie algebra of H. Refer (1) for further elucidation.

4.2 Higgs mechanism

There is an important distinction between the Higgs mechanism and the case we studied earlier. We studied the global U(1) symmetry breaking in section 4.1.2. Here we will extend it to U(1) gauge field, i.e local instead of global. Consider the Lagrangian :

$$L = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{(D_\mu\varphi)^\dagger(D_\mu\varphi)}{2} - V(\varphi) \tag{4.15}$$

$$V(\varphi^\dagger\varphi) = \frac{m^2\varphi^\dagger\varphi}{2} + \lambda\frac{(\varphi^\dagger\varphi)^2}{4} \tag{4.16}$$

As discussed in section 2, this is the Lagrangian corresponding to the electrodynamic field and a complex scalar field. The lagrangian is invariant under the Abelian group U(1). The energy functional for the Lagrangian mentioned above is given by:

$$E = \int \Big[\frac{1}{2}F_{0i}F_{0i} + \frac{1}{4}F_{ij}F_{ij} + \frac{(D_\mu\varphi)^\dagger(D_\mu\varphi)}{2} + V(\varphi)\Big]d^3x \tag{4.17}$$

Again the ground state field configuration is given by the trivial solution in this case.
However if m is imaginary then the first term in the potential expression is negative and the ground state field configuration is not trivial. We can easily say that for making the field strength tensor zero, we need the gauge field configuration to be a pure gauge, (analogous to the analysis done in section 3) i.e:

$$A_\mu = \frac{1}{e}\partial_\mu\alpha \tag{4.18}$$

Also, we get as the set of ground state fields as:

$$\varphi = e^{i\alpha(x)}\frac{\varphi_0}{\sqrt{2}} \tag{4.19}$$

where

$$\varphi_0 = \frac{\mu}{\sqrt{\lambda}} \tag{4.20}$$

is the value of the field which minimizes the potential function. Making a choice of the ground state with $\alpha = 0$, and writing a local Lagrangian about that ground state in order to get information about the particles involved in the theory, we make the following perturbation:

$$\varphi = \varphi_0 + \chi + i\theta \tag{4.21}$$

the above equation is analogous to the one we made in the case of the global U(1) symmetry case. The Lagrangian we get after using this as the value φ after suitable re-definitions is given by :

$$L_{local} = -\frac{1}{4}B_{\mu\nu}B_{\mu\nu} + \frac{e^2\varphi_0^2}{2}B_\mu B_\mu + \frac{(\partial_\mu\chi)^2}{2} - \mu^2\chi^2 \tag{4.22}$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{4.23}$$

$$B_\mu = A_\mu - \frac{1}{e\varphi_0}\partial_\mu\theta \tag{4.24}$$

The above Lagrangian shows the presence of a massive vector field and a massive scalar field. However the field θ is massless.

1. Since θ does not feature in the Lagrangian at all, there exist no field equations that it has to satisfy. This implies that it wont contribute to the degrees of freedom of the system.

2. Also important to note is the fact that the degrees of freedom in the case of the real mass term Lagrangian are also the same as in this case with the imaginary mass term.

3. What we are doing by analysing the perturbative Lagrangian is that we are basically probing the potential landscape and whenever we detect the 'particle', it means that we are in a simple harmonic potential well with the coefficient accompanying the square of the field being the frequency of the well. So, for example, if I were to write a perturbative Lagrangian about the top of the hill in the case of the Mexican Hat shape potential, it would give me an imaginary frequency well; this just means that the well is inverted instead of being in its natural shape.

5 Monopoles

Here I will be discussing a case of broken symmetry, i.e. the Lagrangian is invariant under the SU(2) symmetry, however it will turn out that the

symmetry will not be completely broken. When we find out the ground states , they would still have a residual U(1) symmetry.

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2}(D_\mu\varphi)^a(D^\mu\varphi)^a - \frac{\lambda}{4}(\varphi^a\varphi^a - F^2)^2 \tag{5.1}$$

The Lagrangian encodes a Non Abelian gauge theory invariant under the action of the SU(2) group with the elements of the group in the adjoint representation Following are the conventions under which the above theory is being used:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^a_{bc}A^b_\mu A^c_\nu \tag{5.2}$$

$$(D_\mu\varphi)^a = \partial_\mu\varphi^a + gC^a_{bc}A^b_\mu\varphi^c \tag{5.3}$$

In the matrix form itself, and removing the internal directions denoted by a we get:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{5.4}$$

$$D_\mu\varphi = \partial_\mu\varphi + [A_\mu, \varphi] \tag{5.5}$$

where C_{abc} are the structure constants of the theory The above equations have just been rewritten for convenience. They follow the same logic as was applied in section 2 for any general representation The field equations for the Lagrangian mentioned above are given by:

$$D_\mu G^{a\mu\nu} = gC^a_{bc}(D_\nu\varphi^b)\varphi^c \tag{5.6}$$

$$D_\mu D^\mu\varphi^a = -\lambda(\varphi^b\varphi^b)\varphi^a +^2 \varphi^a \tag{5.7}$$

Since we are talking about a gauge theory , we are dealing with a case of Higgs mechanism. So, the three scalar fields are the three Higgs fields. However this is a more complicated case. We discussed the Higgs mechanism in section 4.2 only for the Abelian group U(1). However the analysis could have been continued to the SU(2) group. Here, we change the representation to adjoint and U(1) to SU(2).

5.1 The static solutions

In this section, we look at the solitonic solutions,i.e we want the static, and finite energy localized solutions.For this purpose, we make one more restriction, we fix the gauge (turns out to be convenient):

$$A^a_0 = 0 \tag{5.8}$$

In this particular gauge the static energy is given by:

$$E = \int d^3x \Big[\frac{1}{4}G^a_{ij}G^{aij} + \frac{1}{2}(D_i\varphi)^a(D^i\varphi)^a + \frac{\lambda}{4}(\varphi^a\varphi^a - F^2)^2\Big] \tag{5.9}$$

This static energy needs to be localized for the type of solutions we are interested in. On applying the constraints under which the solution is expected to be in, we get the following field equations:

$$D_iG^{aij} = gC^a_{bc}(D^j\varphi^b)\varphi^c \tag{5.10}$$

$$D_iD^i\varphi^a = -\lambda(\varphi^b\varphi^b)\varphi^a +^2 \varphi^a \tag{5.11}$$

If we look at the ground states of the system, they will turn out to satisfy the following property:

$$\varphi^a\varphi^a = F^2 \tag{5.12}$$

What this condition means is that the vacua have a residual symmetry still unbroken ,i.e any ground state remains invariant under rotation. This is the case of partial symmetry breaking. Continuing this line further, we can conclude that $\varphi^a\varphi^a \to F^2$ for my energy to be finite. As, we have seen before in the case of the Yang Mills instantons, we had a similar boundary case creeping up there too, albeit in a dimension greater than the present case by one. There it led to the gauge group element to belong to the set of continuous functions from $S^3 \to S^3$. However in the present case , we have frozen the time so we would have the scalar field to belong the set of continuous functions from $S^2 \to S^2$. Now , we know that the set of continuous functions between these two spaces is disconnected and the disconnected space has countably many connected components. Each connected component is identified by a number known as the **Topological charge**.

Drawing an analogy with the instanton case, here also we need that as $|x| \to -\infty$ or ∞ , the static solution should tend to one of the vacua of the theory. This is with regard to the finiteness of the energy functional (5.9) . In fact, the asymptotics are with respect to the radial coordinate and whether one is talking about the negative infinity or the positive infinity only depends on the direction taken to reach infinite radius. Define the current similar to the Chern Simon's current as follows:

$$k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial_\nu\hat{\varphi}^a\partial_\rho\hat{\varphi}^b\partial_\sigma\hat{\varphi}^c \tag{5.13}$$

where

$$\hat{\varphi}^a = \frac{\varphi^a}{|\varphi|} \tag{5.14}$$

From just the definition of this current, we can observe due to the anti symmetry that

$$\partial_\mu k^\mu = 0 \tag{5.15}$$

Note : It is important to note a subtle difference between the Noether's current and charges and topological currents and charges respectively. The latter depend only on the topology ingrained within the theory and the Lagrangian and not on the dynamics of the field equations obtained from the Lagrangian. We define the topological charge as the following:

$$Q = \int d^3x k_0 \tag{5.16}$$

5.2 Polyakov Monopole

We note the following facts. The Electromagnetic field is associated with the Abelian U(1) group and is massless. (The photon is an interacting force and all interacting forces act through the gauge bosons. Since the photon is massless and as is clear from the electromagnetic Lagrangian, the Electromagnetic theory is associated with a massless vector field.) According to standard electromagnetism ,the Maxwell's equations are given by:

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu \tag{5.17}$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \tag{5.18}$$

Out of the above Maxwell's relations, the second equation corresponds to the conclusive proof that monopoles do not exist in the U(1) invariant group case.

U(1) is a subgroup of SU(2). Therefore it is possible to imbed an electromagnetic field in this non Abelian group. This is precisely what is done and what was shown by Polyakov was that such a gauge invariant form of the field strength tensor corresponding to the electromagnetic field exists in terms $G_{\mu\nu}$:

$$F_{\mu\nu} = \hat{\varphi}^a G^a_{\mu\nu} - \frac{1}{g}\hat{\varphi}^a D_\mu \hat{\varphi}^b D_\nu \hat{\varphi}^c \tag{5.19}$$

If we choose $\hat{\varphi} = (0, 0, 1)$, we get,

$$F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu \tag{5.20}$$

Writing an analog of the second Maxwell's equation (5.17) for the EM field strength tensor analog given by (5.18), we get

$$\frac{1}{2}\partial^\nu F^{\rho\sigma} = \frac{4\pi}{g} \tag{5.21}$$

Since $B_i = (1/2)\epsilon_{ijk}F^{jk}$, we get

$$\nabla \cdot \mathbf{B} = 4\pi \frac{k_0}{g} \tag{5.22}$$

Drawing analogy with the Maxwell's equation mentioned earlier, here we see the presence of the magnetic monopole. The monopole charge is given by :

$$m = \int d^3x \frac{k_0}{g} = \frac{Q}{g} \tag{5.23}$$

where Q is the topological charge mentioned earlier.

5.3 An example

A more detailed explanation of the particular example being discussed in this section is given in (2). (5.18) mentioned previously shows that the form of the electromagnetic field comes about only in the regions where $\hat{\varphi}$ is constant.Refer (5.19). I will be just mentioning an approach to arrive at the solution, as the exact derivation can be found in the references.

We work in the temporal gauge i.e $A_0^a = 0$ We choose an ansatz for $\hat{\varphi}$ and for the gauge field, apply the correct boundary conditions as mentioned above and solve the field equations with this ansatz. After following this process, we end up with the following expressions:

$$\varphi^a = \delta_{ia}(x^i/r)F(r) \tag{5.24}$$

$$A_i^a = \epsilon_{aij}(x^j/r)W(r) \tag{5.25}$$

$$r^2\frac{d^2K(r)}{dr^2} = K(r)(K(r)^2 - 1) + H(r)^2K(r) \tag{5.26}$$

$$r^2\frac{d^H(r)}{dr^2} = 2H(r)K(r)^2 + (\frac{H(r)^2}{g^2} - r^2F^2) \tag{5.27}$$

where

$$K(r) = 1 - grW(r) \tag{5.28}$$

$$H(r) = grF \tag{5.29}$$

The last two equations are known as a set of coupled non-autonomous differential equations. It is important to note that the ansatz chosen in the first two equations above is for a monopole with a topological charge equal to one. The boundary value of the scalar field vector points in a direction which is parallel to the coordinate at which it is being evaluated. Because of this, the ansatz is also known as the "**Hedgehog**".

5.4 BPS monopole

The solution of the equations (5.25) and (5.26) in the limit $\lambda \to 0$ was calculated by Bogomol'nyi, Prasad and Sommerfield. The solution we get in this limit is known as the BPS monopole. We get :

$$K(r) = \frac{rgF}{sinh(gFr)} \tag{5.30}$$

$$H(r) = \frac{rgF}{tanh(gFr)} - 1 \tag{5.31}$$

Functions (5.29) and (5.30) can be inserted into the ansatz equations (5.27) and (5.28) and we can obtain an exact solution in the limit $\lambda \to 0$ with $Q = 1$ and monopoloe charge $= \frac{1}{g}$ Specifically for the case of $\lambda = 0$, there exists an inequality similar to (3.22) or (3.25) that relates the static energy functional to the topological charge. This is the **Bogomol'nyi inequality**. Its given by:

$$E = \frac{4\pi Qf}{g} + \int d^3x \sum_{i,j,a} \frac{1}{4}(G^a_{ij} - \epsilon_{ijk} D^k \varphi^a)^2 \geq \frac{4\pi Qf}{g} \tag{5.32}$$

In any specific homotopy sector the energy is minimised only when the Bogomol'nyi condition is met, i.e:

$$G^a_{ij} = \epsilon_{ijk} D^k \varphi^a \tag{5.33}$$

If a field configuration satisfies this condition , then it represents a classical static solution. The BPS monopole solution (5.29) and (5.30) plugged into the ansatz (5.27), (5.28) ,(5.23) and (5.24) satisfy the condition (5.32).

This way we arrive at the solitonic solution of a 3+1 dimension system. It turns out that it is not possible to calculate the solitonic solutions for topological number greater than 1. The word "soliton" is being used naively here. It is not clear as yet whether the solution is actually a soliton or not (conforming to the strict rules laid out for the difference between the soliton and the solitary wave as laid down by (2)).

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References

[1] Classical Theory of Gauge Fields, *V. Rubakov*

[2] Solitons and Instantons, *R. Rajaraman*

[3] ABC of Instantons, ITEP Lectures on Physics, Chapter 3, *Novikov, Shifman, Vainshtein, Zakharov*

[4] Classical Solutions in Quantum field theory, *Erick.J.Weinberg*

[5] Monopoles, Instantons, and Confinement, *Gerard t'Hooft*

[6] Topological Solitons, *Manton, Sutcliffe*

[7] Methods of Theoretical Physics, *Morse, Feshbach*

[8] Mathematical Methods of Physics, *Mathews, Walker*

[9] Aspects of Symmetry, *Sidney Coleman*

A Saddle Point Integration

Consider the integral:

$$\int_C e^{zf(t)}dt$$

Due to the complex nature of the integrand, calculation of the integral would involve the sum of many terms which would destructively interfere with each other and hence cancel each other out. We would like to deform the contour in the complex space so that this contribution is down to a minimum and what we are left with is a logical integral without such interference that we can calculate.

For this purpose, the maximum contribution would come from the part

of the function $f(t)$ which extremizes its real part. Therefore deforming the original contour to a contour along which $Re(zf(t))$ is maximised along with the condition that the imaginary part remains constant , i.e :

$$Im(zf(t)) = c \tag{A.1}$$

Doing this would ensure that we get the maximum contribution to the integral and also that the oscillations would decrease since the imaginary part / oscillatory part is zero. Deforming the path is only possible feasible when there exist no zeros between the two contours. This is precisely what is done in the quasiclassical approximation. We deform the contour in the complex space and choose a path with the above features. This leads to the consideration of the classical path in Euclidean time.

B Pontryagin Index

The construction of the Pontryagin index has a mathematical structure which follows from the index theorem proved by Atiyah-Singer and always yields an integer on its evaluation. The Pontryagin index can be expressed as an integral over the group measure, the group being SU(2) and can be expressed using Gauss' divergence theorem to be equal to a surface integral of the Chern Simon's current over the 4 dimensional sphere at radius equal to infinity. The derivation is lengthy and can be found in (2):

$$Q = \frac{1}{24\pi^2} \int_{S^3 3} Tr[A_\nu A_\alpha A_\beta] \in^{\nu\ \alpha\beta} \tag{B.1}$$

The Pontryagin index behaves somewhat like a Jacobian over the 4 dimensional sphere,at least that is how it looks like, or is to be imagined for a mapping between S_1 and S_1. In the Yang-Mills case, it determines the number of times the 4 dimensional

The Pontryagin index is gauge invariant, again what we mean is that is the gauge transformation involved is smooth and well defined for the whole 4 dimensional space. Therefore, when we gauge transform the gauge field and take the limit of $|x| \to \infty$, we get :

$$A_\mu = U(\omega \partial_\mu \omega^{-1}) U^{-1} + U \partial_\mu U^{-1} \tag{B.2}$$

$$A_\mu = (U\omega)\partial_\mu (U\omega)^{-1} \tag{B.3}$$

For a smooth field, the homotopy sector is conserved as a consequence of the gauge invariance of the Pontryagin index. For any group element transformation U, the following argument holds: $U(|x|, \alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1, \alpha_2, \alpha_3$

refer to the three polar angles in 4 dimensional space. Through the transition among hyper-spheres, the smooth function U can be deformed to a radius of zero, where the function should not depend on the three polar angles in order to be non singular. Hence it converges to a constant matrix which can be continuously deformed to the identity matrix. This constant matrix holds for all values of |x|, which leads it to be an identity even at spatial infinity.

For a product of two transformations asymptotically, the Pontryagin index for the product is equal to the sum of the Pontryagin indices of each term in the product.

$$Q[U_1U_2] = Q[U_1] + Q[U_2] \tag{B.4}$$

C Euclidean over Minkowskian (YM)

The importance of Wick's rotation has been highlighted in this section.Solving the Euler Lagrange field equations along with the boundary conditions should be sufficient to get a solution for the gauge field. However, it is a trickier matter than it sounds. In particular the field equations are second order and hence difficult so solve. In particular to circumnavigate the situation, we concern ourselves with only a subset of solutions, which minimize the action - we consider the ones that bring the action to an absolute minimum. This condition is satisfied, on referring to the above equations to the case where the field strength tensor and its dual are equal.Refer eq 3.28.

This equation is known as the self- dual or the anti-self dual equations. The advantage of solving this subset of solutions is that it gives rise to classical solutions, called the instantons and also the fact that they result in us solving a first order differential equation rather than the second order general field equations.

Another use of Wick's rotation to Euclidean space is consistency. I show the importance of shifting to Euclidean time with an analogous example from electrodynamics. Suppose we want to solve the Electrodynamics field equation with the Lagrangian similar to the Yang- Mills lagrangian, but without the SU(2) gauge invariance. Therefore the field strength tensor is the same but with the commutator term going to zero. The dual equations again minimize the action, and hence we consider solutions of those type. We show that the solutions turn out to be inconsistent in the Minkowskian framework but consistent in the Euclidean framework.

From the self dual condition in Minkowskian time, we get, $\mathbf{B}_M = -\mathbf{E}_M$. From equations 1 and 4 in the first column of the table, we find that the condition is inconsistent and hence the method wont yield the solutions desired.

Table 1: Differences in the Minkowskian and Euclidean approach

Minkowskian	Euclidean
$\nabla \mathrm{x}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$	$\nabla \mathrm{x}\mathbf{B} = i\frac{\partial \mathbf{E}}{\partial t}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \cdot \mathbf{E} = 0$	$\nabla \cdot \mathbf{E} = 0$
$\nabla \mathrm{x}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \mathrm{x}\mathbf{E} = -i\frac{\partial \mathbf{B}}{\partial t}$

On the other hand, we know that

$$E_M = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{C.1}$$

In the Euclidean time τ as $t = -i\tau$, ϕ changes to $i\phi$ and hence we get,

$$E_E = -i\nabla\phi - i\frac{\partial \mathbf{A}}{\partial \tau} \tag{C.2}$$

and hence we get that $E_E = iE_M$ and hence $E_E = -iB_E$ since the vector potential doesn't change on moving to Euclidean time. Therefore the self dual equation in case of Euclidean time is $E_E = -iB_E$. We can see from the second column of the table that the equations are consistent with each other.

Hence, we can draw analogies from the electrodynamics case and extrapolate them to the case of Yang Mills equations, and say that while the Minkowskian picture doesn't yield consistent solutions in the case of the self dual solutions, the Euclidean solutions do. Hence it is imperative to move to Euclidean time to find solutions which cannot be replicated when one is working in the Minkowskian time. Hence performing Wick's rotation is something much more than a change of variables technique. It is a trick to change the picture in which we are working, so as to get the solution which is not attainable in the other picture.

D A_0 Gauge for YM vacua

To know the nature of the vacua, we need energy minimized and hence the Hamiltonian. So we move from the Lagrangian to the Hamiltonian.

The elctrodynamic Lagrangian :

$$L = \int [\frac{\in_o^2}{2}\mathbf{E}^2 - \frac{\mathbf{B}^2}{2\mu_0}] d^3x \tag{D.1}$$

Euler Lagrange equations for the above lagrangian are:

$$\frac{\partial L}{\partial(\partial_i \phi)} = \in_o E_i \tag{D.2}$$

$$\partial_i(\frac{\partial L}{\partial(\partial_i \phi)}) = \nabla \cdot \mathbf{E} \tag{D.3}$$

$$\frac{\partial L}{\partial \phi} = 0 \tag{D.4}$$

$$\partial_t(\frac{\partial L}{\partial(\partial_t \phi)}) = 0 \tag{D.5}$$

Combining the above equations, we get the Euler Lagrange equations with respect to the scalar field ϕ,

$$\nabla \cdot \mathbf{E} = 0 \tag{D.6}$$

Now with respect to A_i, :

$$\frac{\partial L}{\partial(\partial_i A_j)} = 0 \tag{D.7}$$

$$\frac{\partial L}{\partial(\partial_t A_i)} = - \in_o E_i \tag{D.8}$$

$$\frac{\partial L}{\partial A_j} = 0 \tag{D.9}$$

From the above Euler Lagrange equations , we get the second Maxwell's equation:

$$\nabla \times \mathbf{B} = \partial_t \mathbf{E} \tag{D.10}$$

We now define the canonical momenta in the similar way as was done in classical particle mechanics:

$$\Pi_{A_i} = - \in_o E_i \tag{D.11}$$

$$\Pi_\phi = 0 \tag{D.12}$$

Now we move from the Lagrangian $L(\phi, \partial_t\phi, A_i, \partial_t A_i)$ to the Hamiltonian $H(\phi, \partial_t\phi, A_i, \Pi_{A_i})$ using Legendre transformation and we get ,

$$H = (\Pi_{A_i})(\partial_t A_i) - L \tag{D.13}$$

$$H = \frac{\in_o^2}{2}\mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0} + \in_o \mathbf{E} \cdot \nabla\phi \tag{D.14}$$

Now, using Hamilton's equation, we only get back one Maxwell's equation, namely the second one mentioned above. Now, for a unique solution, we fix the gauge by using $\phi = 0$ and imposing the external condition that $\nabla \cdot \mathbf{E} = 0$, we get back the first Maxwell's equation too through the method of Lagrange multipliers with the above condition used as a constraint for the Hamiltonian formulation. Fixing $\varphi = 0$ is basically moving to the temporal gauge ($A_0 = 0$).

We now extrapolate these results obtained for the case of electrodynamics to the case of Yang-Mills' theory and say that:

$$H = \int [\frac{\in_o^2}{2}\mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0}] d^3x \tag{D.15}$$

The main motivation to fix the gauge is to move to the Hamiltonian form cleanly and gauge away the unnecessary degrees of freedom.