SLP PROJECT REPORT
Zero Dimensional Quantum Field Theory and Feynman Diagrams

Amey Gaikwad
Department of Physics, Indian Institute of Technology, Bombay

Supervisors:
Prof. P. Ramadevi (IIT Bombay)
Dr R. Loganayagam (ICTS-TIFR)

November 2017

Abstract

Feynman Diagrams were developed in order to bypass the analytical complexities of calculating scattering cross sections and terms of the perturbation series. To this end, this review aims to introduce Feynman diagrams in the simplest scenario - the zero dimensional quantum field theories. Though being the least complex, sticking to zero dimensional quantum field theories takes nothing away from the beauty and elegance of Feynman diagrams. The study is done in the Feynman path integral framework. After a brief introduction to the path integral formalism, followed by a section on Scattering amplitudes, we move to zero dimensional quantum field theories. Perturbation theories are studied for some specific models. The review ends by showing the equivalence of the analytical and the diagrammatic considerations for perturbation theory.

Contents

Chapter 1

Introduction

Through this review I aim to talk about zero dimensional quantum field theories. These are the simplest quantum field theories conceivable. The integrals are comparatively much easier to solve.While it blankets over the mathematical complexity that comes with moving to higher dimensions, it still conveys the same physical picture. Even when we talk about Feynman diagrams, studying them in scenarios where the complexity of mathematics does not obscure their elegance is an asset. Doing perturbative analysis while calculating cross sections, etc via Feynman diagrams provides a graphical and geometrical viewpoint of the whole construct. After having shown the equivalence of the analytical and the diagrammatic methods, we can finally get down to drawing diagrams without having to worry about the complicated integrals and numerical factors that these diagrams envelope. They provide an abstraction to the whole perturbative analysis construct.

The review starts with an introduction to the Feynman's path integral approach. The entire review works with this approach in mind. Most of the work done in this chapter has been studied from [3]. Here we talk about the advantages of the path integral approach over the conventional Hamiltonian method of quantum mechanics. After having defined the transition amplitude we move two very specific cases where we show how to calculate these transition amplitudes. The Schwinger action principle is stated and the transition amplitudes for the free particle and the harmonic oscillator are calculated by this principle. Most of this study has been done from [7]. Some very interesting comments about the transition amplitudes can be found in [4]. The Schwinger action principle turns out to be a very powerful tool as far as the calculation of transition amplitudes goes.

In the second chapter, we talk about Scattering matrix and Scattering amplitudes. The entire analysis has been done while referring to the excellent text [1]. The study has been done again in the path integral approach. We stick to quantum mechanics in the analysis, and hence only the one particle wave function has been taken into consideration. However, it can be easily extended to higher dimensions, albeit the fact that the analysis in higher dimensions also carries with it the added complexity and complex integrals. The definition of the Vacuum Persistence amplitude is given and a simple case of the forced harmonic oscillator has been discussed.Chapter 8 of [3] provides a very good read on harmonic oscillator.

In the following chapter, we shift gears to move into Zero dimensional quantum field theories. Chapter 4 offers a base to move into perturbation theory from chapter 5. The basic definitions like correlators and n-point functions are formulated. A very important theorem is stated: the Wick's theorem. This theorem provides the foundation for perturbation theory. The path integral approach makes the Wick's theorem appear in a very friendly form. This is another advantage of the path integral approach. The main texts followed were [5] and [6].

Here we start the analytical portion of the perturbation theory. The chapter begins with laying down the foundations for the the chapters to come. Green's functions and connected Green' functions are defined. Their respective generating functionals are defined and a methods to get the Green's functions from these generating functions is given. In the Perturbation theory section, three simple toy models are discussed: the φ^4, φ^3 and the $\varphi^{3|4}$ theories. Lower order Green's functions are calculated for φ^4 theory. The vacuum bubbles are also calculated for all three theories. The main books referred to were [5] and [10]

In Chapter 6, we discuss an important equation: the Schwinger Dyson equation. This is a differential equation which is satisfied by the path integral functional. This gives a method to form a recursive relation between Green's functions. The Schwinger Dyson equation is formulated for all the three aforementioned toy models. The main reference book would be [5].

The last chapter is the most important segment of the book. After laying down the Feynman rules, we show the diagrammatic Schwinger Dyson equations.The equivalence of the diagrammatic equation with the analytical equation is shown. As a result, some verification's are carried out. The 2nd green's function is calculated by diagrammatic techniques and results are matched with the those obtained in Chapter 5 and 6. The diagrammatic equation for connected Green's function is mentioned and the expression for the field functional is matched. This chapter provides a rough sketch of why the diagrammatic method is more powerful than the analytical method, even though they are exactly are equal. The main texts followed for this chapter were [2], [5], [8] and [9]. [2] provides a fantastic diagrammatic experience to quantum field theory and gives a real feel to why Feynman diagrams are so elegant.

A brief look into the future directions into which this work can be extended into is mentioned. This review hopes to be a nice stepping stone to further study in quantum field theory and covers the basic primers needed to do just that.

Chapter 2

Path integral Method

The path integral method in quantum mechanics was developed after the Hamiltonian approach has already been established. In order to make the Lagrangian outlook completely equivalent to its Hamiltonian counterpart, the path integral approach was conceived. Unlike the older version of quantum mechanics which just postulates the existence of the the wave function and the defining equation for it, namely the Schrodinger's equation, this newer version gives some semblance to the whole wave function construct.

2.1 The Formalism

The transition probability to move from an initial spacetime point (q', t') to a final spacetime point (q'', t'') is given by:

$$< q'', t''|q', t' > \tag{2.1.1}$$

Here $|q'', t'' >$ denotes the Heisenberg eigen-ket of the Heisenberg position operator. For a system governed by the Hamiltonian $H(P, Q)$, the above expression is given by:

$$< q'', t''|q', t' >= N \int Dq \int Dp \; \exp\Big(\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{q} - H(p, q)\Big) \tag{2.1.2}$$

where N is just a normalisation factor. The boundary conditions for the integral are given by:

$$\begin{aligned} q(t'') &= q'' \\ q(t') &= q' \end{aligned} \tag{2.1.3}$$

For particular Hamiltonians of the form given by :

$$H(p, q) = \frac{p^2}{2m} + V(q) \tag{2.1.4}$$

(**??**) transforms into

$$< q'', t''|q', t' >= N \int Dq \; \exp\Big(\frac{i}{\hbar}\int_{t'}^{t''} dtL(q, \dot{q})\Big) \tag{2.1.5}$$

What (2.1.5) signifies is much more than an equation. It highlights the true essence of quantum mechanics in the sense that the transition probability to move from one spacetime point to another is not given by just one path joining the two points. All the paths joining the two points has a contribution to the total probability. While some paths are more probable than the other, none of the paths joining the two points are excluded. The contribution of each path towards the total probability is related to the action corresponding to that path. Hence the classical path, which extremizes the action is the major contributor to the transition probability.

2.2 Evaluating the transition amplitude

The measure of the integral in question is the most important part. We divide the time interval into N parts of length ϵ each. Now

$$\begin{aligned} N\epsilon &= t'' - t' \\ \epsilon &= t_{i+1} - t_i \\ t_0 &= t' \\ t_N &= t'' \end{aligned} \tag{2.2.1}$$

The normalisation factor is A^{-N} where the value of A for Hamiltonians of the form (2.1.4) is given by:

$$A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \tag{2.2.2}$$

This results in (2.1.5) transform into:

$$< q'', t''|q', t' >= \lim_{\epsilon \to 0} \frac{1}{A} \int ... \int \int e^{i/\hbar S[q''t'', q't']} \frac{dx_1}{A} \frac{dx_2}{A} .... \frac{dx_{N-1}}{A} \tag{2.2.3}$$

The value of A is obtained by getting a consistency condition in order to get the Schrodinger's equation from the path integral approach. Also :

$$S[q(t)] = S[\overline{q}(t)] + S[y(t)] \tag{2.2.4}$$

where $\overline{q}(t)$ denotes the classical solution connecting the initial and the final spacetime points. $y(t)$ denotes the paths joining the origin with itself but at different times. Therefore, we get,

$$< q'', t''|q', t' >= \exp\left(\frac{i}{\hbar} S_{cl}\right) \int \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} \frac{1}{2} m\dot{y}^2 dt\right) D[y(t)] \tag{2.2.5}$$

with the boundary conditions for $y(t)$ defined as :

$$y(t') = y(t'') = 0 \tag{2.2.6}$$

2.2.1 Free particle

The Lagrangian for the free particle is given by:

$$L = \frac{1}{2} m\dot{q}^2 \tag{2.2.7}$$

Using (2.2.3), we get:

$$< 0, t''|0, t' >= \lim_{\epsilon\to 0}\left[\frac{m}{2\pi i\hbar\epsilon}\right]^{N/2}\int ...\int \exp\left(\frac{im}{2\pi i\hbar\epsilon}(2y_1^2-2y_1y_2+....2y_{N-1}^2)\right)dy_1...dy_{N-1} \tag{2.2.8}$$

with $y_0 = y_N = 0$ Now we can write (2.2.8) as :

$$< 0, t''|0, t' >= \lim_{\epsilon\to 0}\left[\frac{m}{2\pi i\hbar\epsilon N}\right]^{1/2} = \left[\frac{m}{2\pi i\hbar(t''-t')}\right]^{1/2} \tag{2.2.9}$$

Following (2.2.5), we get

$$< q'', t''|q', t' >= \exp\left[\frac{im}{2\hbar}\frac{(q''-q')^2}{(t''-t')}\right]\left[\frac{m}{2\pi i\hbar(t''-t')}\right]^{1/2} \tag{2.2.10}$$

2.2.2 Schwinger Action Principle

The Schwinger action principle is a consistency condition to check whether the transition amplitude calculated is correct or not.

$$\begin{aligned}\delta < q'', t''|q', t' > = &\frac{i}{h} < q'', t''|\left(\hat{p''}\delta q'' - \hat{H}''\delta t''\right)|q', t' > \\ &- \frac{i}{h} < q'', t''|\left(\hat{p'}\delta q' - \hat{H}'\delta t'\right)|q', t' >\end{aligned} \tag{2.2.11}$$

It is very easy to check that the transition amplitude that we had calculated in (2.2.10) satisfies the Schwinger action principle.

I now state the transition amplitude for a harmonic oscillator and mention why it satisfies the Schwinger action principle.

2.2.3 Harmonic Oscillator

The transition amplitude is given by:

$$\begin{aligned}< q'', t''|q', t' > = &\sqrt{\frac{m\omega_0}{2\pi i\hbar\sin(\omega_0(t''-t'))}}\times \\ &\exp\left(\frac{i}{\hbar}\left[\frac{m(q''-q')^2}{2(t''-t')}\left(\frac{\tan\frac{1}{2}\omega_0(t''-t')}{\frac{1}{2}\omega_0(t''-t')}\right)^{-1} - \overline{V}(t''-t')\right.\right. \\ &\left.\left.- \frac{\overline{F}^2}{2m\omega_0^2}\left(\frac{\tan\frac{1}{2}\omega_0(t''-t')}{\frac{1}{2}\omega_0(t''-t')} - 1\right)(t''-t')\right]\right)\end{aligned} \tag{2.2.12}$$

The equation is valid for a harmonic oscillator with the potential given by $V(x) = ax^2 + bx + c$, or in terms of the variables given in (2.2.12):

$$\begin{aligned}\overline{q} &= \frac{q''+q'}{2} \\ \overline{V} &= V(\overline{x}) \\ \overline{F} &= -V'(\overline{x}) \\ m\omega_0^2 &= V''(\overline{x}) = \text{constant}\end{aligned} \tag{2.2.13}$$

Computing the LHS of (2.2.12) by taking a partial derivative with respect to q'', we get :

$$\text{LHS} = m\omega_0(q''-q')\cot(\omega_0(t''-t')) - m\omega_0 q' \tan\left(\frac{1}{2}\omega_0(t''-t')\right) - \frac{b}{\omega_0}\tan\left(\frac{1}{2}\omega_0(t''-t')\right) \tag{2.2.14}$$

where b is defined as follows:

$$b = -\overline{F} - 2a\overline{q} \tag{2.2.15}$$

$$a = \frac{m\omega_0^2}{2} \tag{2.2.16}$$

Calculating the Heisenberg operator $p(t'')$ we get that RHS is also given by (2.2.14). This conclusion follows by equating the $\delta q''$ terms on the left and the right hand side of (2.2.12). A similar calculation results in the equality of LHS and RHS in the $\delta q'$, $\delta t'$ and $\delta t''$ terms. A slight complication occurs in the calculation of the $\delta t'$ and the $\delta t''$ terms in the sense that the following commutation relation needs to be kept in mind:

$$\begin{gathered}[q(t'), q(t'')] = i\hbar \sin(\omega_0(t''-t')) \\ [q(t''), p(t'')] = i\hbar \qquad [q(t'), p(t')] = i\hbar\end{gathered} \tag{2.2.17}$$

Chapter 3

Scattering Matrix

In this section, we introduce scattering amplitudes. This section can be generalised to field theory but for the purposes of this review, we will stick to one particle, its wave-function and its initial and final states. In the first part, we define the vacuum persistence amplitude and in the latter half, we extend this analysis to the actual calculation of the scattering matrix elements.

3.1 Vacuum Persistence Amplitude

We calculated the transition amplitude to move from one spacetime point to another in the previous section. In this section we calculate the transition amplitude between two spacetime points in the ground state. That is to say, we start from an initial spacetime point in the ground state and find the probability amplitude to find the particle at some final spacetime time again in the ground state.To this end, we define the transition amplitude in presence of a source term:

$$< q'', t''|q', t' >^J = N \int Dq \int Dp \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{q} - H(p,q) + Jq)\right] \tag{3.1.1}$$

Now

$$\begin{aligned}\Psi_n(q,t) = < q, t|n > = e^{-iE_n t/\hbar} < q|n > \\ H|n > = E_n|n >\end{aligned} \tag{3.1.2}$$

Using completeness relations we get

$$< q'', t''|q', t' >^J = \int dq_+ \int dq_- < q'', t''|q_+, t_+ >< q_+, t_+|q_-, t_- >^J < q_-, t_-|q', t' > \tag{3.1.3}$$

with the source being non zero only in the interval t_- and t_+. Now,

$$< q'', t''|q_+, t_+ > = \sum_n < q'', t''|n >< n|q_+, t_+ > \tag{3.1.4}$$

Using (3.1.4), and as $t_+ \to -i\infty$,we get:

$$< q'', t''|q_+, t_+ > = \Psi_0(q'', t'')\Psi_0^*(q_+, t_+) \tag{3.1.5}$$

A similar expression holds as $t' \to i\infty$:

$$< q_-, t_-|q', t' >= \Psi_0(q_-, t_-)\Psi_0^*(q', t') \tag{3.1.6}$$

Therefore using these two limits and from (3.1.3), we define the ground state to ground state transition amplitude or the **Vacuum persistence Amplitude** $W[J]$.

$$W[J] \propto \int Dq \int Dp \exp\Big[\frac{i}{\hbar}\int_{i\infty}^{-i\infty} dt(p\dot{q} - H(p,q) + Jq)\Big] \tag{3.1.7}$$

with the boundary conditions defined as follows:

$$\lim_{t'' \to \ -i\infty} q(t) = q''$$
$$\lim_{t' \to i\infty} q(t) = q' \tag{3.1.8}$$

3.2 An example

As an example we calculate the Vacuum persistence amplitude for a forced harmonic oscillator whose Lagrangian is given by :

$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2}{2}q^2 + f(t)q \tag{3.2.1}$$

The transition amplitude for the forced harmonic oscillator is given by:

$$< q'', t''|q', t' >= \Big(\frac{m\omega}{2\pi\hbar i\sin(\omega T)}\Big)^{1/2} e^{iS_{cl}/\hbar} \tag{3.2.2}$$

where S_{cl} is given by:

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\Big[(q'' + q')^2\cos(\omega T) - 2q''q' \tag{3.2.3}$$
$$+ \frac{2q''}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t - t_a)dt \tag{3.2.4}$$
$$+ \frac{2q'}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t_b - t)dt \tag{3.2.5}$$
$$- \frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t} f(t)f(s)\sin\omega(t_b - t)\sin\omega(s - t_a)ds\ dt\Big] \tag{3.2.6}$$

Now the eigenfunctions of the harmonic oscillator are given by :

$$\phi_n(q) = \frac{1}{(2^n n!)^{1/2}}\Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} H_n\Big(x\sqrt{\frac{m\omega}{\hbar}}\Big)e^{-(m\omega/2\hbar)x^2} \tag{3.2.7}$$

We introduce the notation G_{mn} to denote the amplitude that the the oscillator is initially at $t = 0$ in energy state n and is found in an energy state m at some later time, say $t = T$. Now G_{mn} is defined by:

$$G_{mn} = e^{(i/\hbar)E_m T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_m(q'') < q'', T|q', 0 > \phi_n(x_a)dq''\ dq' \tag{3.2.8}$$

In particular, for $m = n = 0$ the vacuum persistence amplitude can be computed to be :

$$G_{00} = \exp\Big(-\frac{1}{2\hbar m\omega}\int_0^\infty \int_0^t f(t)f(s)e^{-i\omega(t-s)}ds\ dt\Big) \tag{3.2.9}$$

We now move on to studying Scattering amplitudes in quantum mechanics.

3.3 Scattering Matrix Calculation - Quantum Mechanics

Scattering process involves particles coming in from infinite past, where they were free, interacting at some finite time and scattering off to the infinite future where they will be free again. In this section we consider just one particle, being in an initial state $|i>$ in the infinite past and being in a final state $|f>$ in the infinite future. The particle is free when it is in the states $|i>$ and $|f>$. Therefore the potential part of the Hamiltonian vanishes in the infinite past and in the infinite future so that the particle is free at those two time scales.

We talked about the vacuum persistence amplitude being the transition amplitude to move from the ground state to the ground state. This involved calculating the amplitude to traverse between two states that were eigen-states of the position operator. In this section, we generalise the result to finding the transition between two arbitrary states $|i>$ and $|f>$, which are not necessarily eigen-states of the position operator.

$$U_{fi}(t'', t') = \int\int dq'dq'' < f|q'', t'' >< q'', t''|q', t' >< q', t'|i > \tag{3.3.1}$$

Define

$$\Psi_a(q, t) =< q, t|a > \qquad\qquad a = i, f \tag{3.3.2}$$

$$\Psi_a(q, t) = \int dp < q, t|p >< p|a >= \int dp_S < q|e^{-i\hat{H}t/\hbar}|p >_S C_a(p)$$
$$< p|a >= C_a(p) \tag{3.3.3}$$

Assumption: V(q) has a short range and is negligible when $|q| > R_0$.

S_{fi} denotes the S matrix element and is defined as follows:

$$\begin{aligned} S_{fi} &= U_{fi}(+\infty, -\infty) \\ &\lim_{t''\to\infty t'\to-\infty} \int dq''dq'\Psi^*_f(q'', t'') < q'', t''|q', t' > \Psi_i(q', t') \end{aligned} \tag{3.3.4}$$

For Hamiltonian of the type (2.1.4), in the infinite past and in the infinite future, the potential can be assumed to be negligible due to the assumption made before.

$$\Psi_(q, t) \approx \int dpC_i(p) < q| \exp\Big(\frac{-i\hat{p}^2 t}{2\mu}\Big)|p >_S \tag{3.3.5}$$

$$\begin{aligned} \Psi_i(q, t) &\approx \int dpC_i(p)e^{-ip^2t/2\mu} < q|p > \\ &\approx \frac{1}{\sqrt{2\pi}}\int dpC_i(p)e^{-p^2t/2\mu}e^{ipq} \end{aligned} \tag{3.3.6}$$

Similarly we get an analogous expression for the wave function Ψ_f.

Now, we perform a change of variable

$$k = \Big(\frac{|t|}{2/mu}\Big)^{1/2}\Big(p - \frac{\mu q}{t}\Big) \tag{3.3.7}$$

Therefore we get the following wave-function:

$$\Psi_a(q,t) = \Big(\frac{\mu}{\pi|t|}\Big)^{1/2} e^{i\mu q^2/2t}\int dk C_a\Big[\frac{\mu q}{t} + \Big(\frac{2\mu}{|t|}\Big)^{1/2}k\Big]e^{-\epsilon(t)ik^2} \tag{3.3.8}$$

where

$$\begin{aligned} \epsilon(t) = 1 \qquad t > 0 \\ -1 \qquad t < 0 \end{aligned} \tag{3.3.9}$$

Using the approximation $\frac{q}{t} = O(1)$, we get:

$$\Psi_a(q,t) \sim \Big(\frac{\mu}{|t|}\Big)^{1/2} e^{i\mu q^2/2t} C_a(\frac{\mu q}{t}(1 + O(|t|^{-1/2})e^{-i\epsilon(t)\pi/4} \tag{3.3.10}$$

Perform another change of variables:

$$\begin{aligned} \frac{\mu q'}{t'} = p' \\ \frac{\mu q''}{t''} = p'' \end{aligned} \tag{3.3.11}$$

Now we get the expression for the S matrix element:

$$\begin{aligned} S_{fi} = \lim_{t''\to\infty t'\to-\infty}\int dp'dp''\mu^{-1}|t't''|^{1/2}C_f^*(p'')e^{-ip''^2t''/2\mu} \\ \Big\langle\frac{p''t''}{\mu}, t''\Big|\frac{p't'}{2\mu}, t'\Big\rangle e^{ip'^2t/2\mu}C_i(p') \end{aligned} \tag{3.3.12}$$

This is the S-matrix element in the momentum representation. To calculate the same, we need to calculate the transition amplitude as was done in Section 3.1. However in this case the boundary conditions are given by:

$$\begin{aligned} q(t) \sim \frac{p't}{\mu} \qquad t \to -\infty \\ q(t) \sim \frac{p''t}{\mu} \qquad t \to \infty \end{aligned} \tag{3.3.13}$$

In Section 3.1, the boundary conditions were not time dependent, however for the calculation of the scattering matrix elements , we need to calculate the transition probability amplitude with boundary conditions are time dependent.

Chapter 4

Gaussian Integrals and Wick's Theorem

Zero dimensional Quantum field theory gives a peep into what lies ahead in higher dimensional field theories. While the complication of higher dimensions remains nullified, it carries with it the same mathematical rigour that would be necessary for higher dimensions. The aim of this review is to give a glimpse into what lies in zero dimensional QFT. I start from the path the integral approach, and moving via the the definition and calculation of correlators explain the Feynman rules and Diagrams and explain its consistency through the Schwinger Dyson equation in the latter half of the review.

4.1 The First steps

In this section, I lay down the bare bones required for further development of zero dimensional Quantum field theory. In the first subsection the mathematical constructs of action, path integral and so on are defined. In the second subsection we move on to a brief look at Gaussian integrals which will prove to an important part for this review.

4.1.1 Basics of classical and quantum field theory

We are going to deal with a quantum field theory with dimension $d = 0$. Consider a manifold χ and a space of fields $\mathscr{F}$. We define the action $S : \mathscr{F} \rightarrow \mathscr{R}$ and the Lagrangian $L : \mathscr{F} \rightarrow \mathscr{R}$, through the following relation:

$$S(\varphi) = \int_{\chi} L(\varphi) dx \tag{4.1.1}$$

where $\varphi \in \mathscr{F}$ is a field. Now (4.1.1) acts like the definition of a field theory in any dimensions. The space of fields $\mathscr{F}$ is the space of functions which act on the manifold χ. In the case of a zero dimensional field theory, the manifold is just a point. Therefore a field is defined as $\phi : \text{point} \rightarrow R$. Also the space for fields will be the set of real numbers since different zero dimensional quantum field theories can map the manifold (or in this case a point) to any real number. Therefore in the case of the zero dimensional field theory, action and the Lagrangian are

the same.
The "theory" that we talk about is basically just the choice of the action we make. Everything is governed by the choice of this action functional.
We now define the partition function of a field theory $\mathscr{L}$.

$$Z = \int_{\mathscr{F}} e^{ikS(\varphi)} D\varphi \tag{4.1.2}$$

The integral is done over the space of fields defined on the spacetime manifold and is defined for a constant $k \in \mathscr{R}$. However, by performing a Wick rotation, we can define the partition function for a quantum field theory as follows:

$$Z = N \int_{\mathscr{F}} e^{-kS(\varphi)} D\varphi \tag{4.1.3}$$

The partition function or the path integral is normalised to be equal to one by the choice of the normalisation factor as explained in the following paragraph. This Euclidean partition function is better suited for the purposes of this review due to its convergence properties.It is important to note that the measure used in the integral $D\varphi$ is just basically an integral over the real line since we are dealing with zero dimensional quantum field theories and hence will from now on be replaced by $d\varphi$. We now define the probability density:

$$P(\varphi) = Ne^{-S(\varphi)} \tag{4.1.4}$$

where N is the normalisation factor. We fix it as follows:

$$N^{-1} = \int e^{-S(\varphi)} d\varphi \tag{4.1.5}$$

We can generalise this to K fields as follows:

$$P(\varphi_1, \varphi_2, ..., \varphi_K) = Ne^{-S(\varphi_1,\varphi_2,....,\varphi_K)} \tag{4.1.6}$$

with N defined as

$$N^{-1} = \int e^{-S(\varphi_1,\varphi_2,....,\varphi_K)} d\varphi_1 d\varphi_2 ... d\varphi_K \tag{4.1.7}$$

4.1.2 Gaussian Integrals

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{4.1.8}$$

We can generalise this result to d quantum fields, in which case we have

$$Z_0 = \int_{\mathscr{R}^d} d\varphi^d e^{-\frac{1}{2}<A\varphi,\varphi>} = \Big(\det \frac{A}{2\pi}\Big)^{-1/2} \tag{4.1.9}$$

where A is a real symmetric matrix. From equation (4.1.3), we choose the action for our system to be quadratic in the d variables x.

$$S(x) = \frac{1}{2} A\varphi^i \varphi^j =< A\varphi, \varphi > \tag{4.1.10}$$

and N is defined as given in equation (4.1.5). Since the positive definiteness of the matrix A is being assumed, we can perform a change of variables so that A is diagonalised. Now the inner product $< Ax, x >$ is just equal to $\sum_i^d \lambda_i^2$ where λ_is are the eigenvalues of the matrix A.

Now we define the partition function in the presence of a source term J.

$$Z_J = \frac{\int_{\mathscr{R}^d} e^{-\frac{1}{2}<A\varphi,\varphi>+<J,\varphi>} d\varphi^d}{\int_{\mathscr{R}^d} e^{-\frac{1}{2}<A\varphi,\varphi>} d\varphi^d} = \frac{1}{Z_0} \int_{\mathscr{R}^d} e^{-\frac{1}{2}<A\varphi,\varphi>+<J,\varphi>} d\varphi^d \tag{4.1.11}$$

Why it is normalised in the way it is will be clear in the following section. This turns out to be the generating functional for the correlators. To calculate this integral we perform a change of variables where

$$\varphi \to \varphi - A^{-1} J \tag{4.1.12}$$

to get the value of Z_J equal to:

$$Z_J = e^{\frac{1}{2}<J,A^{-1}J>} \tag{4.1.13}$$

4.2 Correlation functions and Wicks' Theorem

In this section, we look at correlators for zero dimensional quantum field theories and derive some important results related to them like Wicks' theorem.

4.2.1 Correlation functions

The correlators of m functions f_i defined as:

$$f_i : \mathscr{R} \to \mathscr{R} \tag{4.2.1}$$

is defined as:

$$< f_1 f_2 f_3 .... f_m >= \frac{1}{Z_0} \int d\varphi e^{-\frac{1}{2}<A\varphi,\varphi>} f_1(\varphi) f_2(\varphi) ... f_m(\varphi) \tag{4.2.2}$$

Now for evaluating the correlators for general functions as defined above, we need to find the correlators of the form $< \varphi_1 \varphi_2 .. \varphi_m >$. From these correlators we can extend to correlators of functions which are polynomials. We note the following:

$$\frac{\partial Z_J}{\partial J_i} = \frac{1}{Z_0} \int d\varphi \varphi^i e^{-\frac{1}{2}<A\varphi,\varphi>+<J,\varphi>} \tag{4.2.3}$$

Evaluating the above expression at $J = 0$ will give:

$$\left.\frac{\partial Z_J}{\partial J_i}\right|_{J=0} =< \varphi^i > \tag{4.2.4}$$

Thus we justify the comment made before regarding Z_J being the generating function for the correlators. We can extend equation (4.2.4) to an m point function (which is basically just $< \varphi^1 \varphi^2 ... \varphi^m >$) as follows:

$$< \varphi^{i_1} \varphi^{i_2} ... \varphi^{i_m} >= \frac{1}{Z_0} \partial_{i_1} \partial_{i_2} ... \partial_{i_m} Z_J \Big|_{J=0} = \partial_{i_1} \partial_{i_2} ... \partial_{i_m} e^{\frac{1}{2}<J,A^{-1}J>} \Big|_{J=0} \tag{4.2.5}$$

We calculate the two point function as it will hold to be important when we generalise to an m - point function by resorting to Wick's theorem. Therefore from equation (4.2.5), we get

$$< \varphi^i, \varphi^j >= \partial_i \partial_j e^{\frac{1}{2}<J, A^{-1}J>}\Big|_{J=0} = (A^{-1})_{ij} \tag{4.2.6}$$

We now generalise the statement by stating Wick's theorem.

4.2.2 Wick's Theorem

Theorem

$$\partial_{i_1}\partial_{i_2}...\partial_{i_m} e^{\frac{1}{2}<J, A^{-1}J>}\Big|_{J=0} = \sum (A^{-1})_{j_1 j_2}(A^{-1})_{j_3 j_4}...(A^{-1})_{j_{m-1} j_m} \tag{4.2.7}$$

if $m = 2n$ and zero otherwise. The sum is over all the partitions $(j_1, j_2), ..., (j_{m-1}, j_m)$ in pairs of the set $i_1, ..., i_m$ of indices.

What the theorem states is that odd correlators all evaluate to zero and the way to calculate the even correlators is to pair the indices up in all possible ways and sum the product of the "propagators" corresponding to each unordered pair considered among the indices. The **propagator** in this simplistic case of the zero dimensional QFT is just the element $(A^{-1})_{ij}$ which connects the indices x^i and x^j. The Wick's theorem can be extended to arbitrary linear functions of φ. We define the two point function $< f, g >$ by $< f, A^{-1}g >$.

Theorem

Let $f_1, f_2, ..f_m$ be arbitrary linear functions of the coordinates φ^i. Then all the m point functions vanish for odd m. For m=2n, one has

$$< f_1, f_2, ..., f_m >= \sum < f_{i_1}, f_{i_2} > ... < f_{i_{m-1}}, f_{i_m} >, \tag{4.2.8}$$

where the sum is over all the pairings $(i_1, i_2), ..., (i_{m-1}, i_m)$ of 1,...,m and the 2 point functions $< f_j, f_k >$ are defined as $< f_j, A^{-1}f_k >$.

As an example we calculate the 4 point function:

$$< f_1, f_2 f_3 f_4 >=< f_1, f_2 >< f_3, f_4 > + < f_2, f_3 >< f_1, f_4 > + < f_1, f_3 >< f_2, f_4 > \tag{4.2.9}$$

Also number of ways to join $2k$ points into pairs is given by $(2k)!/(2^k k!)$. This is because the first point is chosen in $2k$ ways. The second is given by $2k - 2$ ways and so on. This way, we get a total of $(2k)(2k-2)(2k-4)...(2)$ which is equivalent to $(2k)!/(2^k k!)$ Hence we get:

$$< (f \cdot \varphi)^{2k} >= \frac{(2k)!}{(2^k k!)} < f, f >^k \tag{4.2.10}$$

We now describe the perturbation series expansion which can be done more easily through Feynman diagrams.

Chapter 5

Green's Functions in Zero Dimensions

Perturbation theory in zero dimensional quantum field theories is comparatively easier to study than in higher dimensional quantum field theories. The main reason for this is because the measure concerning the integrals involved is simple. Also, the conditions on the matrices involved are simple. In higher dimensional quantum field theories the integrals are tricky and involve contour integration. In contrast, the analysis for zero dimensional quantum field theories offers a good starting point.

Greens' functions crop up every time in perturbation theory while we are studying through the path integral approach. We define what these are. We also define their variants : the connected Greens' functions.

5.1 Green's functions

We introduced correlators before. Green's functions is just another name for them. Here and in the following sections, the number of fields is just one and not d as it was in the previous section. We also talk of a general $S(\varphi)$ instead of the specialised one used before. Therefore, we summarise to get the following results:

$$G_n = < \varphi^n > = N \int e^{-S(\varphi)} \varphi^n d\varphi \tag{5.1.1}$$

We also make the following definition which will prove to be important in the following sections.

$$\mathscr{H}_n = \frac{\int \varphi^n e^{-S(\varphi)} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2} d\varphi} \tag{5.1.2}$$

Due to the choice of the normalisation constant from (4.1.5) we get that $G_0 = 1$ Now, we can express $Z(J)$ as a generalisation from what was defined as Z_J in (4.1.11) as the following:

$$Z(J) = N \int e^{-S(\varphi)+J\varphi} d\varphi = \frac{\int e^{-S(\varphi)+J\varphi} d\varphi}{\int e^{-S(\varphi)} d\varphi} \tag{5.1.3}$$

From the above expression if we Taylor expand the $e^{J\varphi}$ term we will get an expression for $Z(J)$ in terms of the Green's functions we described above.Therefore we get:

$$Z(J) = \sum \frac{1}{n!} J^n G_n \tag{5.1.4}$$

J is known as the source term. Now, we can also move to define the Green's functions in a manner analogous to what was done in (4.2.5) to get the following:

$$G_n = \frac{\partial^n}{\partial J^n} Z(J)\Big|_{J=0} \tag{5.1.5}$$

Now that the Green's functions have been defined, we move to explain what is meant by Connected Green's functions.

5.2 Connected green's Functions

We define the logarithm of the functional $Z(J)$ as the functional $W(J)$

$$W(J) = \log(Z(J)) = \sum_{n\geq 1} \frac{1}{n!} J^n C_n \tag{5.2.1}$$

where C_n are the connected Green's functions. Why they are called this way will only be clear after the Feynman rules and diagrams have been described. Right now , these connected Green's functions are the cumulants. This is because:

$$W(J) = \log(1 + JG_1 + \frac{J^2 G_2}{2} + O(J^3))$$

This implies that:

$$W(J) = JG_1 + \frac{J^2 G_2}{2} - \frac{(JG_1)^2}{2} + O(J^3) = JC_1 + \frac{J^2 C_2}{2}$$

implying that:

$$C_1 =< \varphi > \tag{5.2.2}$$

$$C_2 =< \varphi^2 > - < \varphi >^2 \tag{5.2.3}$$

So C_1 is equivalent to the mean and C_2 is equivalent to the variance of the 0 dimensional quantum field φ. We now define the **field functional** $\phi(J)$. :

$$\phi(J) = \frac{\partial}{\partial J} W(J) = \sum_{n\geq 0} \frac{1}{n!} J^n C_{n+1} \tag{5.2.4}$$

Also it can be checked that :

$$\phi(J) =< \varphi >_J \tag{5.2.5}$$

where $< \varphi >_J$ is the expectation value of φ corresponding to the partition function given by (5.1.3).

After introducing the free field theory, we move on to study perturbations in it. As a result we study perturbation theory for the φ^3 and φ^4 theories separately and together.

5.3 Free theory

The free field action is given by:

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 \tag{5.3.1}$$

Therefore we get:

$$Z(J) = \frac{\int e^{-\frac{1}{2}\mu\varphi^2 + J\varphi} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2} d\varphi} \tag{5.3.2}$$

$$= e^{\frac{J^2}{2\mu}} \tag{5.3.3}$$

For calculating the even ordered Greens functions we resort to Wick's Theorem (4.2.7). In the free field case, the matrix A is just a 1x1 matrix. Therefore the propagator "joining" any two fields, i.e $(A^{-1})_{11}$ is just $\frac{1}{\mu}$.Hence the Green's functions for the free field theory are given by:

$$G_{2n} = \frac{(2n)!}{2^n n!}\frac{1}{\mu^n}$$

$$G_{2n+1} = 0 \tag{5.3.4}$$

We now calculate the connected green's functions for the free field theory:

$$W(J) = log(Z(J))$$

$$= \frac{J^2}{2\mu} \tag{5.3.5}$$

This implies that the only non zero connected green's function is C_2

$$C_2 = \frac{1}{\mu}$$

$$C_n = 0 \qquad n \neq 2 \tag{5.3.6}$$

5.4 Perturbation Theory

The perturbation theory that will be discussed in this section will only pertain to a perturbation to the free field theory by $\frac{\lambda\varphi^4}{4!}$ or $\frac{\lambda\varphi^3}{3!}$. We calculate the expressions for $\mathscr{H}_0^{\text{theory}}$. Why these particular expressions ? They are the easiest to calculate and they represent the vacuum bubbles of the theory. When we come to Feynman diagrams, we will show that the vacuum bubbles obtained form the graphs will coincide with these expressions.

5.4.1 φ^4 theory

$$S_4(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{\lambda_4\varphi^4}{4!} \tag{5.4.1}$$

We define the "H functional" from (5.1.2) for this particular theory to be $\mathscr{H}_n^{(4)}$

$$\mathscr{H}_0^{(4)} = \frac{\int e^{-\frac{1}{2}\mu\varphi^2 - \frac{\lambda_4\varphi^4}{4!}} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2} d\varphi} \tag{5.4.2}$$

We can expand the perturbative term in the exponential so that we can write $\mathscr{H}_0^{(4)}$ as a sum of Greens' functions. Hence we get that:

$$\mathscr{H}_0^{(4)} = \sum_{n=0}^{\infty} \Big(-\frac{\lambda_4}{4!}\Big)^n \frac{1}{n!} G_{4n} \tag{5.4.3}$$

Thus we have expressed $\mathscr{H}_0^{(4)}$ of a φ^4 theory in terms of the green's functions of the free field theory. For calculating the green's functions of the φ^4 theory, we follow the definition of the Green's functions. Therefore the Green's functions are given by:

$$G_{2n} = \frac{\mathscr{H}_{2n}^{(4)}}{\mathscr{H}_0^{(4)}} \tag{5.4.4}$$

$$\mathscr{H}_{2n}^{(4)} = \frac{1}{\mu^n} \sum_{k\geq 0} \frac{(4k+2n)!}{2^{2k+n}(2k+n)!k!} \Big(-\frac{\lambda_4}{24}\Big)^k \tag{5.4.5}$$

We can simplify $\mathscr{H}_0^{(4)}$ to get the following expansion:

$$\mathscr{H}_0^{(4)} = 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{35}{384}\frac{\lambda_4^2}{\mu^4} + ... \tag{5.4.6}$$

For this particular theory, the second Green's function expression is given by:

$$G_2 = \frac{1}{\mu}\Big(1 - \frac{1}{2}\frac{\lambda_4}{\mu^2} + \frac{2}{3}\frac{\lambda_4^2}{\mu^4} + ...\Big) \tag{5.4.7}$$

5.4.2 φ^3 theory

Now consider the φ^3 theory, whose action is given by:

$$S_3(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{\lambda_3\varphi^3}{3!} \tag{5.4.8}$$

The "H functional for this theory is given by $\mathscr{H}_n^{(3)}$ For the theory which is given by $\mathscr{H}_0^{(3)}$:

$$\mathscr{H}_0^{(3)} = \frac{\int e^{-\frac{1}{2}\mu\varphi^2 - \frac{\lambda_3\varphi^3}{3!}} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2} d\varphi} \tag{5.4.9}$$

$$= 1 + \frac{5}{24}\frac{\lambda_3^2}{\mu^4} + \text{higher order terms in } \lambda_3 \tag{5.4.10}$$

5.4.3 $\varphi^{3|4}$ theory

"H functional for this theory is given by $\mathscr{H}_n^{3|4}$

Now we consider the combined field action:

$$S_{3|4}(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{\lambda_4\varphi^4}{4!} + \frac{\lambda_3\varphi^3}{3!} \tag{5.4.11}$$

$\mathscr{H}_0^{(3|4)}$ for this theory is given by:

$$\mathscr{H}_0^{(3|4)} = \frac{\int e^{-\frac{1}{2}\mu\varphi^2 - \frac{\lambda_4\varphi^4}{4!} - \frac{\lambda_3\varphi^3}{3!}} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2} d\varphi} \tag{5.4.12}$$

$$= 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{35}{384}\frac{\lambda_4^2}{\mu^4} + \frac{5}{24}\frac{\lambda_3^2}{\mu^4} - \frac{35}{64}\frac{\lambda_4\lambda_3^2}{\mu^6} + \frac{5005}{3072}\frac{\lambda_3^2\lambda_4^2}{\mu^8} + \text{higher order terms} \tag{5.4.13}$$

Following is the calculation for the path integral upto second order in $\lambda_{3|4}$. In the next chapter, we will show that it satisfies the Schwinger Dyson equation:

$$\begin{aligned} Z[J] &= \frac{\int e^{-\frac{1}{2}\mu\varphi^2 - \frac{\lambda_3}{3!}\varphi^3 - \frac{\lambda_4}{4!}\varphi^4 + J\varphi} d\varphi}{\int e^{-\frac{1}{2}\mu\varphi^2 - \frac{\lambda_3}{3!}\varphi^3 - \frac{\lambda_4}{4!}\varphi^4} d\varphi} \\ &= 1 + \Big(-\frac{\lambda_3}{2\mu} + 2\frac{\lambda_3\lambda_4}{3\mu^3}\Big)\frac{J}{\mu} + \Big(\mu - \frac{\lambda_4}{2\mu} + \Big(\frac{2\lambda_4^2}{3\mu^3} + \frac{5\lambda_3^2}{4\mu^2}\Big)\Big)\frac{J^2}{2\mu^2} \\ &+ \Big(-5\frac{\lambda_3}{2} + 25\frac{\lambda_3\lambda_4\mu}{4}\Big)\frac{J^3}{6\mu^3} + \Big(3\mu^2 - 4\lambda_4 + \Big(\frac{33\lambda_4^2}{4\mu^2} + \frac{25\lambda_3^2}{2\mu}\Big)\Big)\frac{J^4}{4!\mu^4} \end{aligned} \tag{5.4.14}$$

We will later show that these Green's functions can be expressed through Feynman graphs.

Chapter 6

The Schwinger Dyson equation

As was seen in the previous sections, we need to find the Green's functions correctly to find the perturbation series for a given theory. Keeping this in mind along with the fact that the Green's function can be generated by differentiating the action functional, finding an equation for $Z[J]$ would help us in relating recursive Green's functions.

6.1 Derivation

In this section, we show the method for getting the differential equation which would be satisfied by $Z(J)$.This differential equation is the Schwinger Dyson equation.

$$\begin{aligned} &\int d\varphi \frac{\partial}{\partial\varphi}\Big[-S(\varphi)+J\varphi\Big]e^{-S(\varphi)+J\varphi} \\ &= \int d\varphi \frac{\partial}{\partial\varphi}\Big[e^{-S(\varphi)+J\varphi}\Big] = 0 \end{aligned} \tag{6.1.1}$$

The last equality follows by assuming no boundary contribution and $S(\varphi)$ dies off quickly This implies that :

$$\begin{aligned} \int d\varphi \frac{\partial}{\partial\varphi}\Big[-S(\varphi)+J\varphi\Big]e^{-S(\varphi)+J\varphi} &= 0 \\ \int d\varphi\Big[-S'(\varphi)+J\Big]e^{-S(\varphi)+J\varphi} &= 0 \\ \Big[-S'(\frac{\partial}{\partial J})+J\Big]Z[J] &= 0 \end{aligned} \tag{6.1.2}$$

6.2 φ^4 Theory

For the φ^4 theory,

$$\begin{aligned} S(\varphi) &= \frac{\mu\varphi^2}{2} + \frac{\lambda_4\varphi^4}{4!} \\ S'(\varphi) &= \mu\varphi + \frac{\lambda_4\varphi^3}{3!} \end{aligned} \tag{6.2.1}$$

From (6.1):

$$\Big[\mu\frac{\partial}{\partial J} + \frac{\lambda_4}{3!}\Big(\frac{\partial}{\partial J}\Big)^3 - J\Big]Z[J] = 0 \tag{6.2.2}$$

We can find the recursion relations between the Green's functions of a theory using the Schwinger Dyson equation. For the φ^4 theory, using the series expansion from (4.3) , we get:

$$\begin{aligned} \mu G_{n+1} + \frac{\lambda_4}{3!}G_{n+3} - nG_{n-1} &= 0 & n \geq 1 \\ G_n = \frac{1}{\mu}\Big((n-1)G_{n-2} &- \frac{\lambda_4}{3!}G_{n+2}\Big) & n \geq 2 \end{aligned} \tag{6.2.3}$$

In a similar manner using (4.5) and (4.8), we can write the Schwinger Dyson equations for the $W[J]$ and $\phi[J]$. In particular for the φ^4 theory, the Schwinger Dyson equation for the connected functional is given by:

$$\mu\frac{\partial W}{\partial J} + \frac{\lambda_4}{3!}\Big[W''' + 3W'W'' + W'^3\Big] = J \tag{6.2.4}$$

6.3 φ^3 Theory

For the φ^3 theory on the other hand:

$$\begin{aligned} S(\varphi) &= \frac{\mu\varphi^2}{2} + \frac{\lambda_3\varphi^3}{3!} \\ S'(\varphi) &= \mu\varphi + \frac{\lambda_3\varphi^2}{2!} \end{aligned} \tag{6.3.1}$$

From (6.1):

$$\Big[\mu\frac{\partial}{\partial J} + \frac{\lambda_3}{2!}\Big(\frac{\partial}{\partial J}\Big)^2 - J\Big]Z[J] = 0 \tag{6.3.2}$$

Similarly for the φ^3 theory, the recursion relations are given as below:

$$\mu G_{n+1} + \frac{\lambda_3}{2!}G_{n+2} - nG_{n-1} = 0 \qquad n \geq 1 \tag{6.3.3}$$

6.4 $\varphi^{3|4}$ Theory

For the $\varphi^{3|4}$ theory,

$$\begin{aligned} S(\varphi) &= \frac{\mu\varphi^2}{2} + \frac{\lambda_3\varphi^3}{3!} + \frac{\lambda_4\varphi^4}{4!} \\ S'(\varphi) &= \mu\varphi + \frac{\lambda_3\varphi^2}{2} + \frac{\lambda_4\varphi^3}{3!} \end{aligned} \tag{6.4.1}$$

From (6.1), we get the Schwinger Dyson equation for the combined theory:

$$\Big[\mu\frac{\partial}{\partial J} + \frac{\lambda_4}{3!}\Big(\frac{\partial}{\partial J}\Big)^3 + \frac{\lambda_3}{2!}\Big(\frac{\partial}{\partial J}\Big)^2 - J\Big]Z[J] = 0 \tag{6.4.2}$$

Now, it can be checked that (6.4.2) conforms with the value of $Z[J]$ calculated in (5.4.14).

From (6.4.2), we can also obtain the following recursion relation between the green's functions of the theory:

$$\mu G_{n+1} + \frac{\lambda_3}{2!}G_{n+2} + \frac{\lambda_4}{3!}G_{n+3} = nG_{n-1} \qquad n \geq 1 \tag{6.4.3}$$

From (6.4.2), we obtain the Schwinger Dyson equation for the field functional for the $\varphi^{3|4}$ theory:

$$\begin{aligned} \phi(J) = \frac{J}{\mu} &- \frac{\lambda_3}{2\mu}\Big(\phi(J)^2 + \frac{\partial}{\partial J}\phi(J)\Big) \\ &- \frac{\lambda_4}{6\mu}\Big(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{\partial J^2}\phi(J)\Big) \end{aligned} \tag{6.4.4}$$

We will show in the next chapter the derivation of the above equation through Feynman diagrams.

Chapter 7

Feynman Diagrams

The method of formalising perturbation theory through Feynman diagrams is exactly equivalent to the one given through the analytic procedure mentioned earlier. Even the Schwinger Dyson equation can be expressed diagrammatically.

Now each diagram has internal and external lines. Internal lines start and end at vertices. On the other hand, external lines do not. We also have source terms present in the action. This is highlighted through the cross. Graphs can also be classified as connected or disconnected. This distinction will turn out to be very important as we will try look at the three functionals defined in the earlier sections through the eyes of the type of graphs describing them.

In this review, I will be presenting the perturbation results for just the ϕ^3/ϕ^4 theory. To this end, we define the Feynman rules as given in 7.1.

Now, that the Feynman rules have been defined, the combinatorial factors corresponding to the Feynman graphs need to be calculated.

7.1 Symmetry Factors

The Feynman diagrams correspond to analytical mathematical expressions. In addition to the Feynman rules defined in figure 7.1, the graphs also have some symmetry

7.1.1 Internal Lines

1. Permutation of k lines : $\frac{1}{k!}$

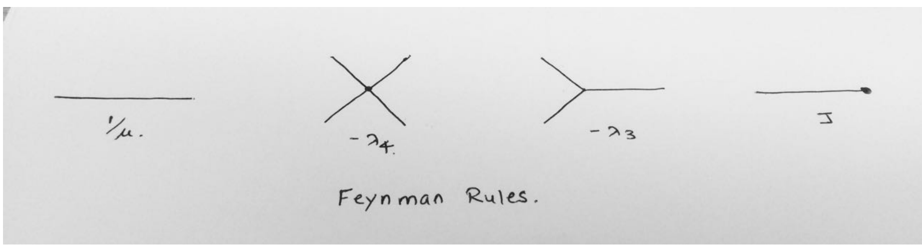

Figure 7.1: Feynman Rules for the φ^4 and φ^3 theories

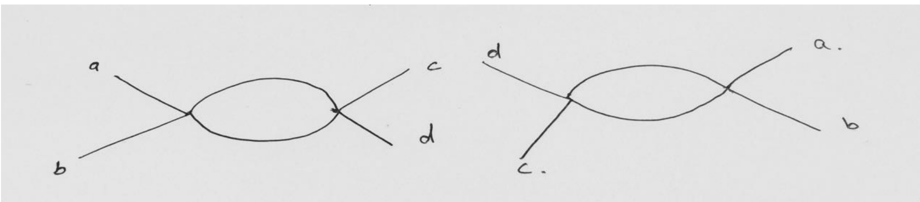

Figure 7.2: External Lines

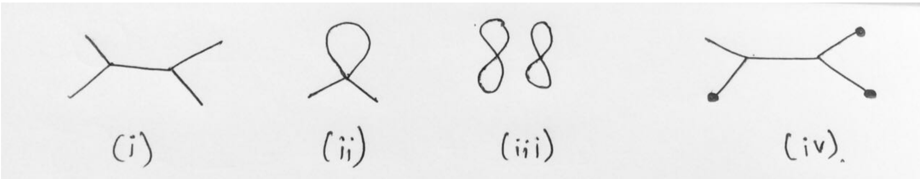

Figure 7.3: Particular examples for calculating symmetry factors

2. Permutation of m vertices: $\frac{1}{m!}$
3. Permutation of p dis-joint disconnected pieces : $\frac{1}{p!}$
4. q- fold Rotational Symmetry : $\frac{1}{p}$
5. Mirror Symmetry : $\frac{1}{2}$

7.1.2 External Lines

The two diagrams in 7.2 are the same. Hence due to a permutation of the external lines, we get the same graphs. This needs to be accounted for. Therefore in this case the multiplicity is 3.

NOTE : Also, if we replace p external lines with p sources, we get a factor of $\frac{1}{p!}$

7.2 Some Examples of Feynman Diagrams

Refer figure 7.3 for the following:

1. 5 lines and two 3-vertices: no symmetric factor $\frac{\lambda_3^2}{\mu^5}$
2. 3 lines and one 4 vertex : mirror symmetry: $-\frac{1}{2}\frac{\lambda_4}{\mu^3}$
3. 2 disconnected graphs gives a symmetry factor of $\frac{1}{2}$. Each connected component gives a factor of $\frac{1}{8}$. Therefore the diagram corresponds to : $\frac{1}{2}\left(\frac{1}{8}\frac{\lambda_4}{\mu^2}\right)^2$
4. The two sources can be interchanged, giving a symmetry factor of $\frac{1}{2}$. Therefore we evaluate the diagram to be: $\frac{1}{2}\frac{\lambda_3^2}{\mu^2}J^3$

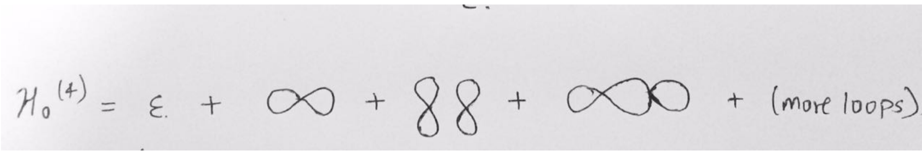

Figure 7.4: Vacuum Bubble expansion

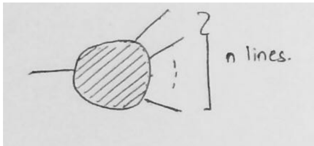

Figure 7.5: n-Legged Green's function

7.3 Vacuum Bubbles

Consider the Feynman graphs consisting of no external lines or source vertices.If we consider the φ^4 theory, we will be concerned with only the 4-vertices, and of no other type. Therefore, the set of vacuum bubbles denoted by $\mathscr{H}_0$ is given by figure 7.4. where ε denotes the empty graph. Since only internal lines are being considered here, we can resort to section 7.1.1 to find out the symmetry factors of each of the loops:

$$\mathscr{H} = 1 + \frac{-1}{8}\frac{\lambda_4}{\mu^2} + \frac{1}{2}\Big(-\frac{1}{8}\frac{\lambda_4}{\mu^2}\Big)^2 + (more\ \ loops)$$

$$\mathscr{H} = 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{35}{384}\frac{\lambda_4^2}{\mu^4} + ... \tag{7.3.1}$$

This is exactly the same as was obtained in (5.4.6). Thus, we have successfully expressed our first perturbative expansion for the φ^4 theory in terms of Feynman graphs.

7.4 Diagrammatic Schwinger Dyson equation

The Schwinger Dyson equation introduced in chapter 6 can also be expressed diagrammatically. We first define the Green's function diagrammatically in figure 7.5. This Green' function denotes one particle going in, interacting with the system and n particles coming out. Hence the Green's function epitomises the Scattering processes that were discussed in chapter 3. Now the diagrammatic Schwinger Dyson equation is given by 7.6. For the φ^4 theory, the Schwinger Dyson equation, only the 4-vertices occurs. The term in figure 7.5 corresponds to $\frac{\partial^{n+1}}{\partial J^{n+1}}\Big|_{J=0}$. Therefore for the φ^4 theory, from the diagrammatic equation in figure 7.7.

We evaluate the Schwinger Dyson equation in figure 7.7 to be equal to:

$$\frac{\partial^2 Z}{\partial J^2}\Big|_{J=0} = \frac{1}{\mu}Z[0] + \frac{-\lambda_4}{3!}\frac{\partial^4 Z}{\partial J^4}\Big|_{J=0} \tag{7.4.1}$$

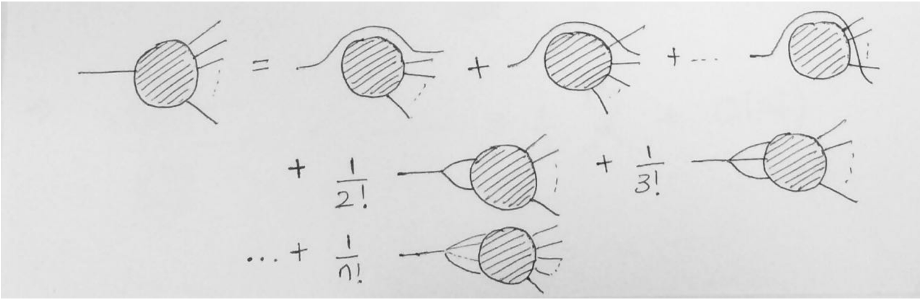

Figure 7.6: Diagrammatic Schwinger Dyson Equation

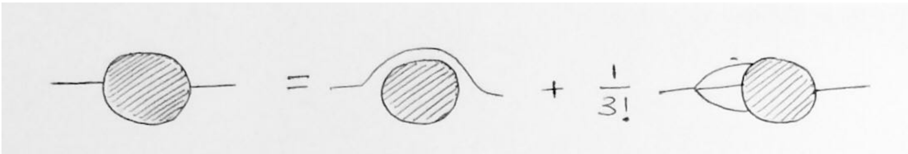

Figure 7.7: Diagrammatic Schwinger Dyson Equation for φ^4 theory

We can also define the Schwinger Dyson equation for the action functional including the source term as shown in figure 7.8.

The equation given in figure 7.8 is exactly what we got in (6.2.2). One can derive (7.4.1) from (6.2.2) by simply differentiating (6.2.2) and putting $J = 0$.

NOTE: The same notation has been used in figure figures 7.7 and 7.8 for two different mathematical constructs. However in the text that follows, the convention of figure 7.7 will be followed unless otherwise mentioned

Hence the diagrammatic Schwinger Dyson equation matches exactly the expression we got analytically in the earlier section. In figure 7.9, through the rules mentioned above, I calculate the second green's function or the two point correlator for the φ^4 theory. The calculation is done using the Diagrammatic Schwinger Dyson equation and will then be shown to be equal to the result obtained analytically. The calculation has been done only upto order λ_4. Higher order terms will require more complicated diagrammatic equations.

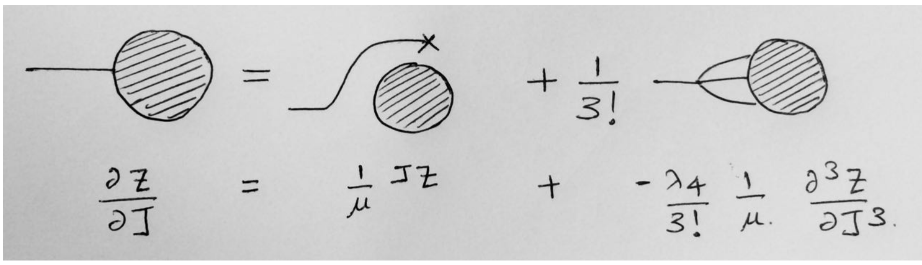

Figure 7.8: Diagrammatic Schwinger Dyson Equation for φ^4 theory in the presence of source J

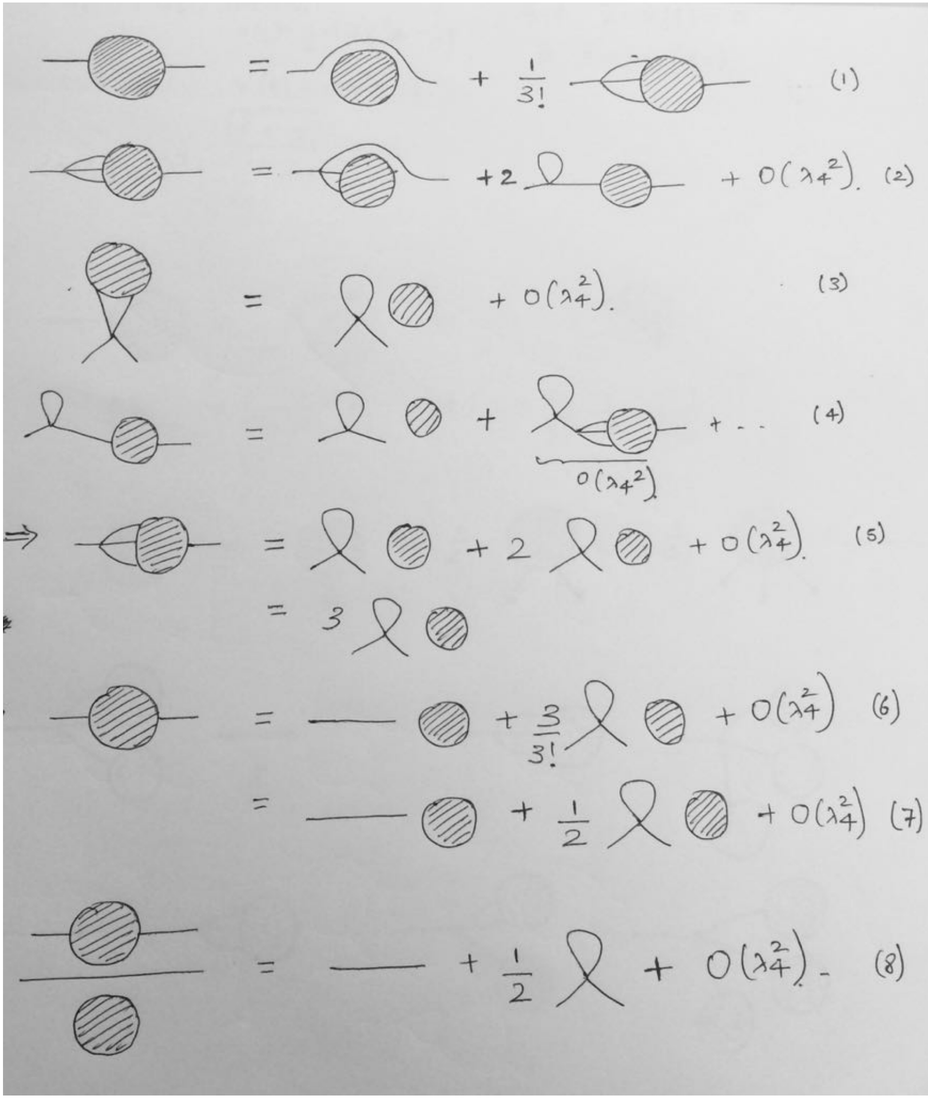

Figure 7.9: Expressing G_2 for the φ_4 theory through Feynman diagrams upto order λ_4

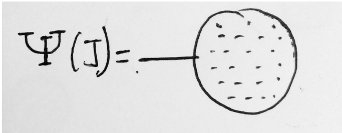

Figure 7.10:

Showing the equivalence of calculating the Green's function analytically and through the Diagrammatic Schwinger Dyson equation is not that hard. We recall (5.4.7) and (5.4.6):

$$\frac{1}{\mathscr{H}_0} = 1 + \frac{1}{8}\frac{\lambda_4}{\mu^2} - \frac{29}{384}\frac{\lambda_4^2}{\mu^4} + (\text{more loops}) \tag{7.4.2}$$

$$G_2 = \frac{\mathscr{H}_2}{\mathscr{H}_0} \tag{7.4.3}$$

$$G_2 = \frac{1}{\mu}\Big(1 - \frac{1}{2}\frac{\lambda_4}{\mu^2} + \frac{2}{3}\frac{\lambda_4^2}{\mu^4} + (\text{more loops})\Big) \tag{7.4.4}$$

Now we resort to the Feynman rules established in Figure 7.1, and (7.4.4). The first term in (7.4.4), i.e $\frac{1}{\mu}$ corresponds to the first Feynman rule that is a straight line. Move on to the second term of (7.4.4), namely $-\frac{1}{2}\frac{\lambda_4}{\mu^2}$. The second and the first Feynman rules of Figure 7.1 give rise to the Feynman diagram for the second term. The extra half factor arising is due to the symmetry factor for internal lines discussed in section 7.1.1 - mirror symmetry.

Hence, at least upto a single vertex, the analytical method and the diagrammatic method give the same interpretation and hence the same answers. Therefore they are equivalent. We have proved their equivalence in a non trivial case.For the free field theory, the exercise for proving the equivalence would not have been this difficult.

We now move on to show why the terms of the series expansion of $W[J]$ are known as the "connected Green's functions."

7.5 Connected Graphs - Field Functional

Let the graph in Figure 7.10 denote the set of all connected graphs with just one external line, but no restriction on the number of source vertices. $\mathscr{C}_n$ denote the the set of connected graphs with no sources and with n external lines. Now we look to establish a relation between these two constructs. If we refer to the note in section 7.1.2, we see that we can construct $\Psi(J)$ as was defined in figure 7.10 by the following equation.

$$\Psi(J) = \sum_{n\geq 0} \frac{1}{n!} J^n \mathscr{C}_{n+1} \tag{7.5.1}$$

We will show that $\Psi(J)$ is the same as $\phi(J)$ as defined in (6.4.4). We will also show that C_n and $\mathscr{C}_n$ are also the same.

Consider travelling along the propagator and entering the blob. One can have many choices regarding what we encounter in entering - a source or a bifurcation into 3 blobs (all equivalent) as shown in figure 7.11. Again following

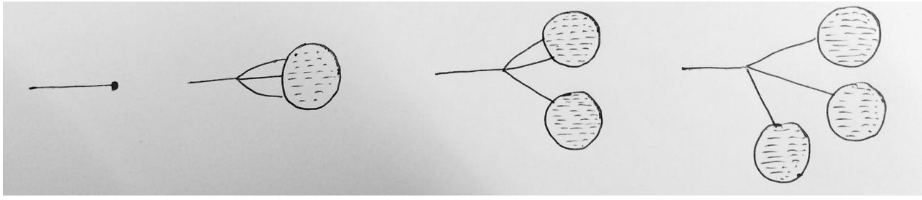

Figure 7.11: The types of environment one can find on entering the blob along the propagator

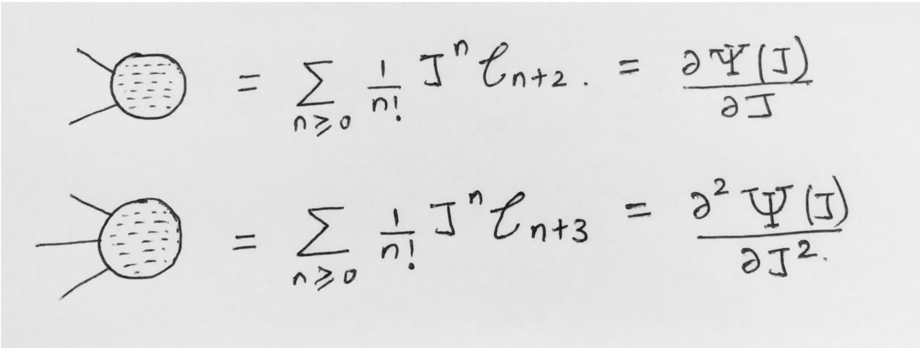

Figure 7.12: Some definitions

the note given in section 7.1.2 to get the definitions given in figure 7.12. Using these we can write the diagrammatic Schwinger Dyson equation for these connected blobs following the lead given by Figure 7.13 for the $\varphi^{3|4}$ theory. From the analytical relations given in Figure 7.12 , we get the following:

$$
\begin{aligned}
\text{(i)} &= \frac{J}{\mu} \\
\text{(ii)} &= -\frac{\lambda_3}{\mu}\frac{1}{2}\Psi(J)^2 \\
\text{(iii)} &= -\frac{1}{2}\frac{\lambda_3}{\mu}\frac{\partial}{\partial J}\Psi(J) \\
\text{(iv)} &= -\frac{1}{6}\frac{\lambda_4}{\mu}\Psi(J)^3 \\
\text{(v)} &= -\frac{\lambda_4}{\mu}\frac{1}{2}\Psi(J)\frac{\partial}{\partial J}\Psi(J) \\
\text{(vi)} &= -\frac{\lambda_4}{\mu}\frac{1}{6}\frac{\partial^2}{\partial J^2}\Psi(J)
\end{aligned} \tag{7.5.2}
$$

Therefore, we get:

$$
\begin{aligned}
\Psi(J) = \frac{J}{\mu} &- \frac{\lambda_3}{\mu}\frac{1}{2}\Psi(J)^2 - \frac{1}{2}\frac{\lambda_3}{\mu}\frac{\partial}{\partial J}\Psi(J) \\
&- \frac{1}{6}\frac{\lambda_4}{\mu}\Psi(J)^3 - \frac{\lambda_4}{\mu}\frac{1}{2}\Psi(J)\frac{\partial}{\partial J}\Psi(J) - \frac{\lambda_4}{\mu}\frac{1}{6}\frac{\partial^2}{\partial J^2}\Psi(J)
\end{aligned} \tag{7.5.3}
$$

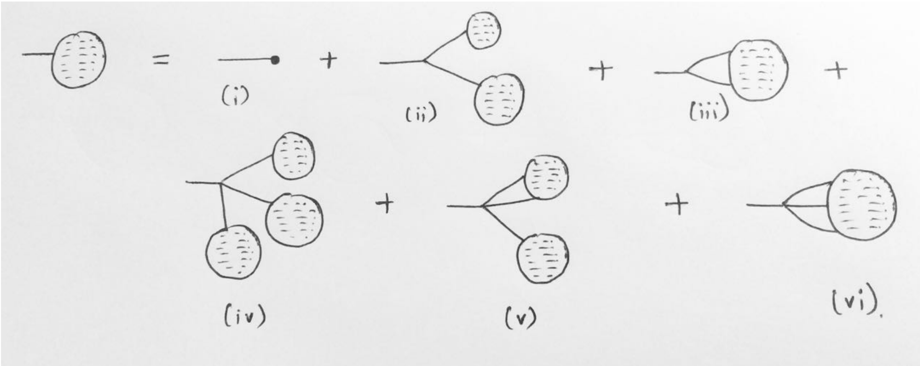

Figure 7.13: Diagrammatic Schwinger Dyson Equation for connected blobs in $\varphi^{3|4}$ theory

This is exactly the same equation that we had obtained in (6.4.4). Thus we get,

$$\Psi(J) = \phi(J) \tag{7.5.4}$$

and hence its derivatives are also equal which implies that:

$$C_n = \mathscr{C}_n \qquad\qquad n \geq 0 \tag{7.5.5}$$

Thus we now know why the Connected Green's functions are called the way they are. The connected Green's functions are related to the connected Feynman diagrams. Hence once again the diagrammatic Schwinger Dyson equation for connected blobs is equal to the the analytical answer that we get from eq (6.4.4).

7.6 Connected graphs- Part2

If we replace the external line in $\Psi(J)$ by an external source we get the connected functional $W(J)$. Thus we define $W(J)$ to be the sum of connected graphs with no external lines and at least one external source. This follows from the definition of $\psi(J)$ and the note in section 7.1.2. The additional source term adds an extra symmetry factor. Now, we defined the vacuum bubbles as having no external lines and no sources. Therefore the connected functional will not have any contribution from the vacuum bubbles. Refer Figure 7.14.

The first equation in Figure 7.14 comprises the equation for the connected functional. By squaring it we get the second equation. This way we can construct $Z(J)$ since we know that $Z(J) = e^{W(J)}$. It is important to note that the vacuum bubbles do not contribute to the path integral too. Also, while the connected functional consists of just the connected diagrams, the path integral consists of contributions from both kinds of Feynman diagrams - connected and disconnected.

The reason for the absence of the vacuum bubbles can be explained as follows: From Figure 7.9 we can see that for calculating the second Green's function we divided the final diagram by the vacuum bubbles. While normalising the vacuum bubbles get cancelled off as was shown through an example in Figure 7.9.

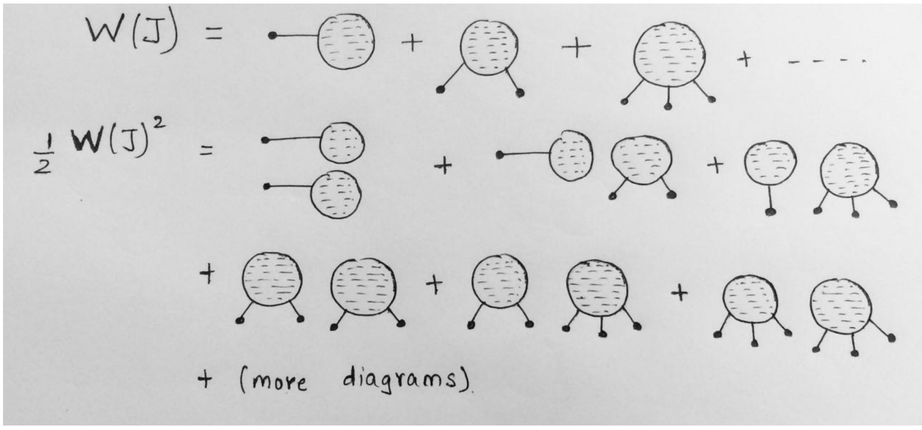

Figure 7.14: The equation for the connected functional and constructing the field functional

Chapter 8

Conclusion

After having shown the ease with which calculations can be made by the use of Feynman diagrams, we can extend the analysis to higher dimensions. Higher dimensions render a higher degree of complexity. In the zero dimensional quantum field theories, the propagators are just numbers, eq. in the toy models we discussed, the propagator was $\frac{1}{\mu}$. In higher dimensions, the Feynman propagator is actually a complex contour integral. The calculation of these integrals can be complicated. We don't worry about these though. In field theory, we are mainly interested in the Vacuum persistence amplitude which is defined in a manner analogous to the one defined in this review. This amplitude can be expressed in terms of the Feynman propagator. Hence, in the calculation of the perturbation series , the Feynman diagrams turn out to be exactly the same as are seen in zero dimensional quantum field theories. Hence the work in this review can be extended to higher dimensions.

Another direction for a follow up to this review would be learn effective field theories in zero dimensional quantum field theories, renormalisation methods and the one particle irreducible functionals. All these three concepts can be expressed through diagrammatics. This is the power of the method. It offers a new geometric method of thinking. After a while, just by drawing diagrams, we can form strong conclusions about a theory. There's no need to go into long calculations. Once the mathematical rigour of this method has been established we can freely extend these to other toy models.

On the other hand, as a follow up to the Scattering matrix amplitudes in quantum mechanics, we can move to Scattering in higher dimensional quantum field theories,i.e to fields. This can be further developed to calculate Scattering cross sections in a completely diagrammatic way. Further developments would include learning scattering in Quantum Electrodynamics and Non Abelian gauge theories like Yang-Mills theories. Diagrams in Quantum Chromodynamics turn out to be insightful as the analysis becomes tougher and tougher.

Feynman diagrams opens us to a host of areas of physics which can now be explored : Particle physics, Quantum Chromodynamics, String theory and so on. These can be generalised to world sheets in String Theory. In this case these world sheets imbibe the inherent symmetry properties of the system in consideration. Thus learning Feynman diagrams is a nice stepping stone to explore what lies ahead in other interesting areas of theoretical physics.

Chapter 9

Acknowledgements

I would like to thank my co-guide Dr. Loganayagam for all the helpful discussions we had over the course of four months. His insights gave a new dimension to the way I have started to look at quantum field theory. Thanks are also due to my guide Prof. Ramadevi for always being there for me, whenever I have needed help. Finally, I would also like to thanks IIT Bombay for giving such a nice platform for us to engage in research at such a young age. To explore and get a glimpse of Quantum Field theory was indeed an exciting experience.

Bibliography

[1] D. Bailin and A. Love. *Introduction to Gauge Field Theory Revised Edition*. Graduate Student Series in Physics. Taylor & Francis, 1993. ISBN: 9780750302814. URL: https://books.google.co.in/books?id=A9MU9pvcEGQC.

[2] P. Cvitanović, E. Gyldenkerne, and Nordisk institut for teoretisk atomfysik. *Field theory*. Nordita lecture notes. NORDITA, 1983. URL: https://books.google.co.in/books?id=m-QWAQAAMAAJ.

[3] R.P. Feynman, A.R. Hibbs, and D.F. Styer. *Quantum Mechanics and Path Integrals*. Dover Books on Physics. Dover Publications, 2010. ISBN: 9780486477220. URL: https://books.google.co.in/books?id=JkMuDAAAQBAJ.

[4] H. Kleinert. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*. EBL-Schweitzer. World Scientific, 2009. ISBN: 9789814273572. URL: https://books.google.co.in/books?id=VJ1qNz5xYzkC.

[5] Ronald Kleiss. *Pictures, Paths Particles Processes*. 2013, pp. 11–40.

[6] Michael Polyak. *Feynman Diagrams for Pedestrians and Mathematicians*. 2004, pp. 1–8.

[7] J. Schwinger and B.G. Englert. *Quantum Mechanics: Symbolism of Atomic Measurements*. Springer Berlin Heidelberg, 2013. ISBN: 9783662045893. URL: https://books.google.co.in/books?id=fDX6CAAAQBAJ.

[8] David Skinner. *Lectures on Quantum Field Theory*. Chapter 2.

[9] M. Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007. ISBN: 9781139462761. URL: https://books.google.co.in/books?id=5OepxIG42B4C.

[10] J. Zinn-Justin. *Path Integrals in Quantum Mechanics*. Oxford Graduate Texts. OUP Oxford, 2010. ISBN: 9780198566755. URL: https://books.google.co.in/books?id=MWQBAQAAQBAJ.