Quaternions and the Topology of $SO(3)$

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Introduction

Quaternions were first introduced by Hamilton in 1843. He had tried to develop a three-dimensional number system equipped with addition and multiplication, however was ultimately unable to do so. Hamilton's epiphany came famously whilst he was crossing the Broom Bridge in Dublin on the way to the Royal Irish Academy when he realised he needed *four* dimensions not three and he immediately carved the equation into the bridge.

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

where,

$$\mathbf{k} = \mathbf{ij}$$

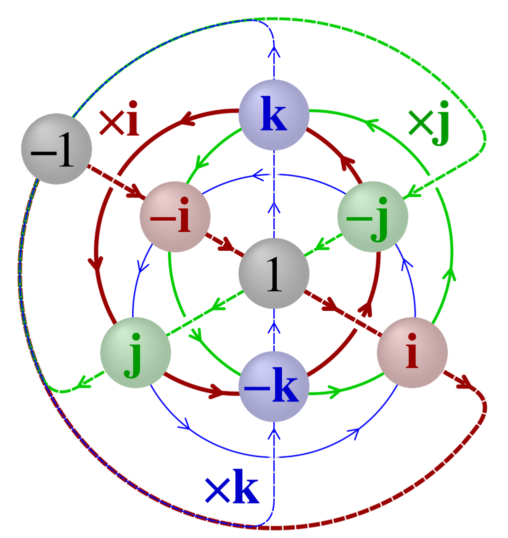
A key property is that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ **anti-commute** with one another, i.e $\mathbf{ij} = -\mathbf{ji}$. We then define the set of quaternions as,

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\}$$

For a general quaternion, $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we can define:

1. its **conjugate**, $\bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$
2. its **norm**, $N(q) = q\bar{q} = a^2 + b^2 + c^2 + d^2$
3. its **scalar part**, which is just given by a, and
4. its **vector part**, is given by $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$

We call a quaternion pure if it's scalar part is 0.



Quaternion multiplication table

Multiplying Quaternions

If $a, b \in \mathbb{H}$ are pure then,

$$ab = -(a \cdot b) + (a \times b) \qquad (1)$$

This is easy to verify, and for pure $a, b, c \in \mathbb{H}$ the scalar part of abc is given by,

$$-a \cdot (b \times c) \qquad (2)$$

Note for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det[\mathbf{u}, \mathbf{v}, \mathbf{w}] \qquad (3)$$

We often split quaternions into scalar and vector parts when multiplying so we can make use of these geometric identities.

Spatial Rotations

For $q \in S^3$, where $S^3 = \{q \in \mathbb{H} : N(q) = 1\}$ define a map, $\rho_q : \mathbb{H} \to \mathbb{H}$ by $\rho_q(x) = qx\bar{q}$. The first thing we notice about this mapping is that it is a linear transformation. Secondly, we note that this transformation preserves the space of pure quaternions.
Let $q = s + v$, where s is scalar and v is pure.

$$qx\bar{q} = (s + v)x(s - v) = s^2 x + s(vx - xv) - vxv$$

It can be easily verified that all three terms are pure and so we have shown it preserves the space. I now make two claims.

CLAIM 1: ρ_q is an orthogonal linear transformation
If we can show that ρ_q preserves length then we can deduce it is an orthogonal linear transformation. We can show this using the multiplicity of the norm.

$$N(qx\bar{q}) = N(q)N(x)N(\bar{q}) = N(x)$$

CLAIM 2: ρ_q has determinant 1
From linear algebra we know we can find the matrix of a linear transformation by finding the image of each basis vector so,

$$\det(\rho_q) = \det[q\mathbf{i}\bar{q}, q\mathbf{j}\bar{q}, q\mathbf{k}\bar{q}]$$

But from (2) and (3) we can see that this just the scalar part of,

$$-(q\mathbf{i}\bar{q})(q\mathbf{j}\bar{q})(q\mathbf{k}\bar{q}) = -q\mathbf{ijk}\bar{q} = q\bar{q} = 1$$

ρ_q is an orthogonal linear transformation with determinant 1, therefore it is a 3D rotation!

Can we can express all rotations in this form?

Lets introduce the homomorphism $\phi : S^3 \to SO(3)$ - the group of all 3D rotations - with $q \mapsto \rho_q$. Now consider it's kernel.

$$q \in \ker(\phi) \iff \forall x, qx\bar{q} = x \iff \forall x, qx = xq \iff q \in \mathbb{R}$$

So ϕ has kernel $\{\pm 1\}$. Since $SO(3)$ and S^3 have the same number of dimensions we can deduce using a simple dimension count to see that ϕ is surjective.

Some Topological Definitions

We will introduce some topological notions

- **Path Connected**
In a space X, two points, $x, y \in X$ are **path connected** if we can find a continuous '*path*' between the two points, or more formally, there exists a continuous function,

$$f : [0,1] \to X \ s.t \ f(0) = x \text{ and } f(1) = y$$

If the initial point, x, is the same as the terminal point, y, then we call the path a **loop** and if all pairs of points of X are path-connected we say the space X is path-connected.

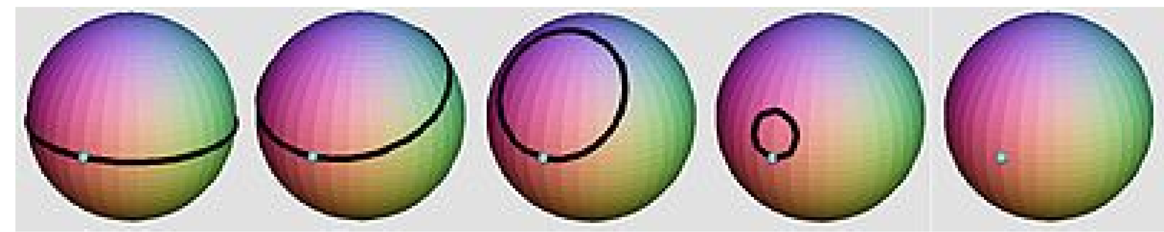
- **Homotopy**
We call two paths **homotopic** if one can '*continuously deform*' one into the other. Formally, two paths, f and g, are homotopic if there exists a continuous function,

$$H : [0,1] \times [0,1] \to X \ \ s.t \ \ H(x,0) = f(x) \text{ and } H(x,1) = g(x), \ \forall x \in [0,1]$$

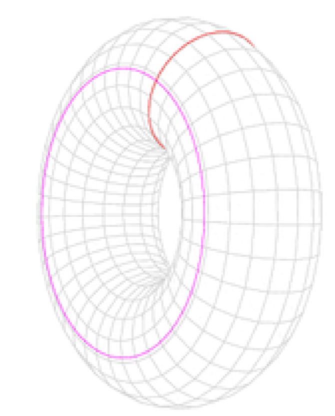
- **Simply Connected**
Finally, we say a space x is **simply connected** if every loop in X is homotopic to a constant map - or nulhomotopic. Intuitively this means we can continuously deform every loop in X down to a single point or to a loop with zero length.

Simply Connected Space Example

The boundary of a sphere - S^2 - is simply connected, graphic below explains why this is expected.



The boundary of a torus on the other hand is not simply connected as a loop which goes around one of the rings cannot not be deformed to a point, as illustrated below.



Topology of $SO(3)$

It is a well known result that S^n is simply connected for $n \geq 2$ so the domain of our homomorphism - S^3 - is simply connected. However ϕ is not an isomorphism as it is not injective so S^3 and $SO(3)$ are not homeomorphic.

Note that $q \in S^3$ gets mapped to the same rotation as it's antipode $-q$ under ϕ. So if we *join* each point in S^3 to it's antipode we would get a space which *is* isomorphic to $SO(3)$. Consider now a path γ in S^3 with initial point 1 and terminal point -1, when this path gets mapped to $SO(3)$ the initial point and terminal point get mapped to the same point so the path $\phi(\gamma)$ is a loop.

Since S^3 is not contractable, any path homotopic to γ must have non-zero length and so the loop in $SO(3)$, $\phi(\gamma)$, cannot be continuously deformed to a loop with zero length. Therefore $SO(3)$ is **not** simply connected.

Note on Dirac's String Trick

We considered a loop $\phi(\gamma)$ which isn't nulhomotopic in $SO(3)$ but interestingly if we compose the path with itself,

$$\gamma'(s) = \begin{cases} \gamma(2s) & ,0 \leq s \leq \frac{1}{2}; \\ \gamma(2s-1), & \frac{1}{2} < s \leq 1 \end{cases}$$

Then this loop *is* nulhomotopic in $SO(3)$. This fact is physically demonstrated in Dirac's 'string trick'.

References

[1] J. Voight. Quaternion Algebras.
[2] S. Altmann. Rotations, Quaternions, and Double Groups. Oxford University Press, 1986.