Quiz 6

Name: **Time:** March 22 , 2016

Instructions: Please fill in the solutions in the space provided for the questions highlighted in red.

Consider the language $L_{01} = \{0^k 1^k \mid k \geq 0\}$. Consider the following grammar $G_{01} = (V, \Sigma, R, S)$, where $V = \{S\}$, $\Sigma = \{0, 1\}$, the rules $R = \{S \to \epsilon \mid 00S11 \mid 000S111\}$.

We will prove that all the words derived from the grammar belong to the language L_{01}. We will prove that by showing that the statement $S(i)$ below holds for all i.

$S(i)$: If $S \Rightarrow^* w$ in i steps, then $w \in L_{01}$.

If $S(i)$ holds for all i, then note that we have shown that all words derived by the grammar are in L_{01}, because if a word is derivable in the grammar then it is derivable in some i steps, hence, by $S(i)$, $w \in L_{01}$.

We will show that $S(i)$ holds for all i by induction on i.

1. Base Case $i = 1$: What is $S(1)$?

   $S(1)$ is the following statement:

   If $S \Rightarrow^* w$ in 1 step, then $w \in L_{01}$.

   Show that $S(1)$ holds. (Hint: What ws are derivable in 1 step? Do they belong to L_{01}?)

   Let $S \Rightarrow^* w$ in 1 step. Note that the only word with terminals that can be derived in one step is derived by applying the rule $S \to \epsilon$. This means w is ϵ. We need to show that $\epsilon \in L_{01}$. But $\epsilon = 0^k 1^k$, when $k = 0$, therefore it belongs to L_{01}.

2. Induction step: Prove that if $S(i)$ holds for $1 \leq i \leq n$, then $S(n+1)$ holds.

   (Hint: Let $S \Rightarrow^* w$ in $n+1$ steps. What are the different ways in which you can split the derivation $S \Rightarrow^* w$ into sub-derivations of length less than or equal to n? Note that since n is at least 1, $S \Rightarrow \epsilon$ is not a derivation of $n+1$ steps.)

   Let $S(i)$ hold for all $1 \leq i \leq n$, that is, if $S \Rightarrow^* w$ in i steps, then $w \in L_{01}$.

   Consider $S \Rightarrow^* w$ in $n+1$ steps. The $n+1$ step derivation will have $S \Rightarrow 00S11$ as the first step or $S \Rightarrow 000S111$ as the first step.

   Case $S \Rightarrow^* w$ is of the form $S \Rightarrow 00S11 \Rightarrow^* w$: Then w has to be of the form $00u11$, where $S \Rightarrow^* u$ is $\leq n$ steps. Hence from induction hypothesis $u \in L_{01}$, that is, $u = 0^k 1^k$ for some k. Then $w = 00u11 = 0^{k+2} 1^{k+2}$ which is a string in L_{01}.

   Case $S \Rightarrow^* w$ is of the form $S \Rightarrow 00S11 \Rightarrow^* w$: Then w has to be of the form $000u111$, where $S \Rightarrow^* u$ is $\leq n$ steps. Hence from induction hypothesis $u \in L_{01}$, that is, $u = 0^k 1^k$ for some k. Then $w = 000u111 = 0^{k+3} 1^{k+3}$ which is string in L_{01}.