# Math 322

Q's / 9.5 ($16^*$, $17^*$)

thm: digraph has Euler circuit

iff ① weakly connected
② $\forall v(\deg^-(v) = \deg^+(v))$

thm: digraph has Euler path (not circuit)

iff ① weakly connected
② there are two vertices $\{v_1, v_2\}$
$\deg^-(v_1) > \deg^+(v_1)$
$\deg^-(v_2) < \deg^+(v_2)$
③ for all other $\deg^+(v) = \deg^-(v)$

23) (look @ book)

$\deg^+(a) = 1$ $\deg^-(a) = 1$

$\deg^+(b) = 1$ $\deg^-(b) = 2$

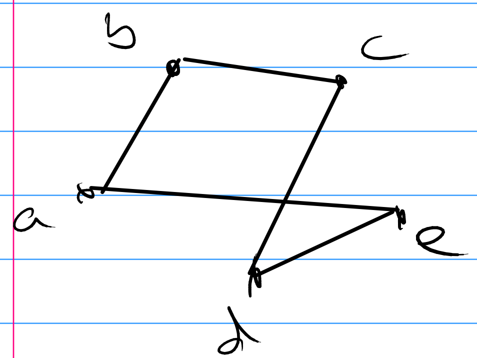
No Euler circuit.

$\deg^+(d) = 2$ $\deg^-(d) = 1$

$\deg^+(f) = 2$ $\deg^-(f) = 1$

no Euler path

# 9.6 Shortest Paths




path from a to e

a, b, c, d, e

a, e

path length = number of edges.

**Weighted Graph:** $G = (V, E)$

weight function $w(e)$ = edge e's 'weight'

length in a weighted graph

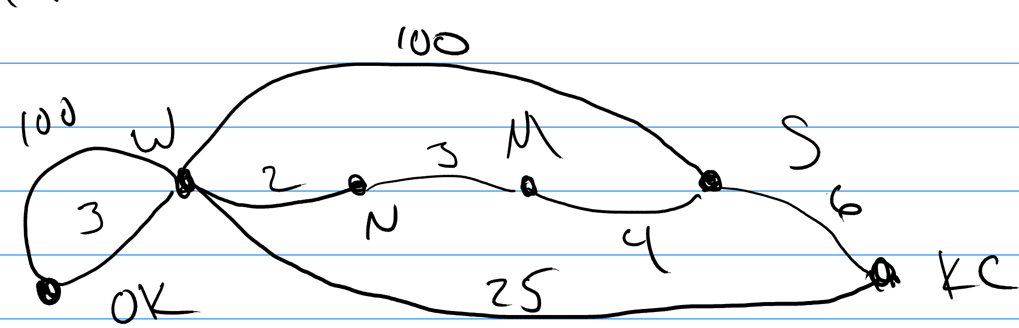
for a path: $e_1, e_2, e_3, \ldots, e_n$

Before: $n$

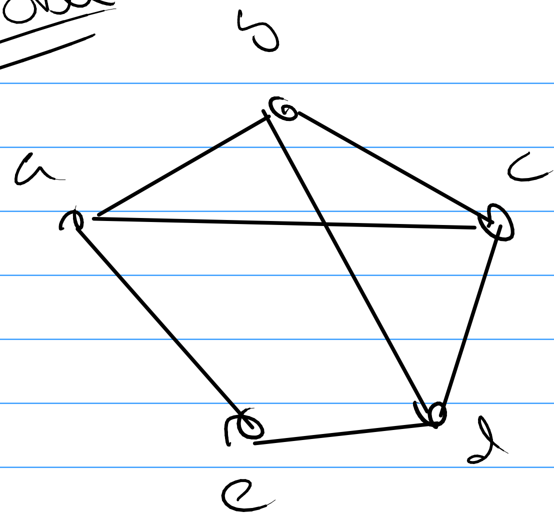
Now: $w(e_1) + w(e_2) + \cdots + w(e_n)$

if $w(e_i) = 1$ (same as before)

ex.



# Inverse problem.




a,b,c,d,e = 120

a,b,d,e = 90

,

,

etc.

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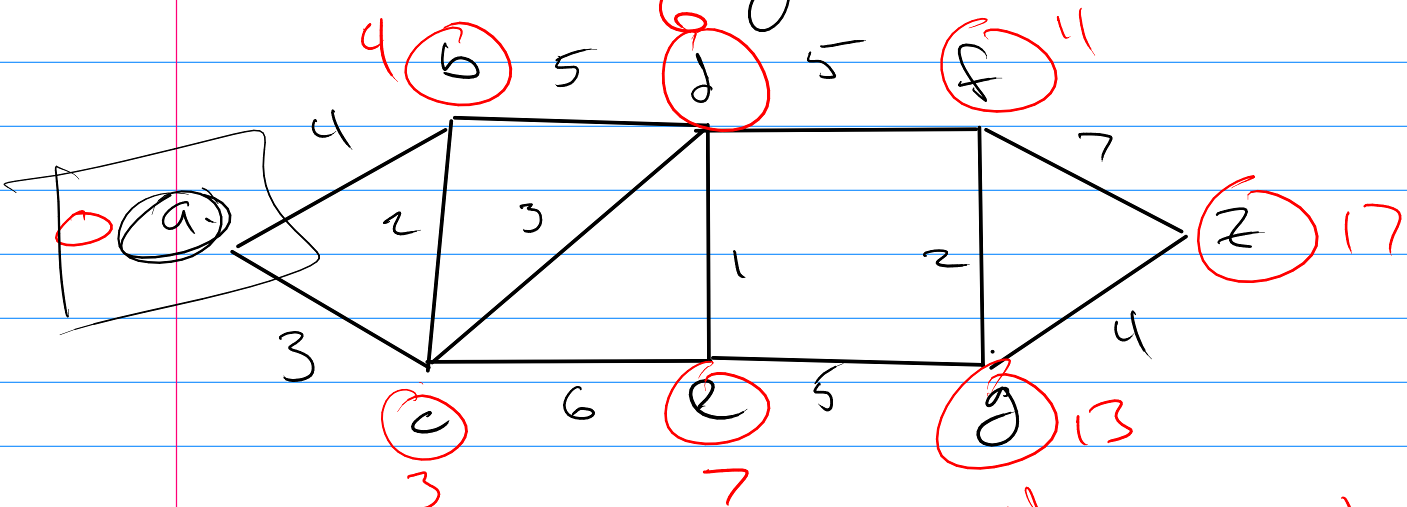
## Shortest path between two vertices

— Dijkstra's Algorithm



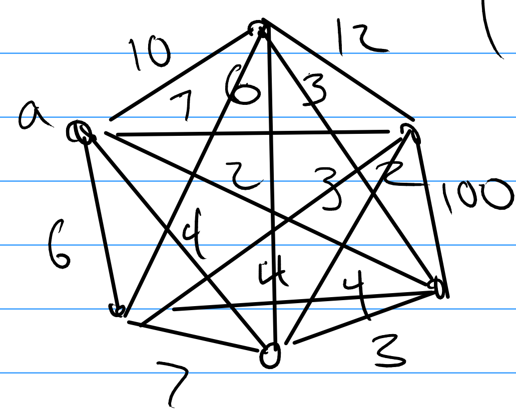

| Path | cost | path | cost |
|---|---|---|---|
| a | 0 | a,c,d,e | 7 |
| a,c | 3 | a,c,d,f | 11 |
| a,b | 4 | a,c,d,e,g | 13 |
| a,c,d | 6 | a,c,d,e,g,z | 17 |

# Traveling Salesman

Find cheapest Hamilton circuit.

typical: $K_n$

$K_6$




$|E| = 5+4+3+2+1$
$= \frac{5(6)}{2} = 15$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Hamilton Circuit visits each vertex.
By Dirac's th<sup></sup>m Hamilton circuit exists.

How many?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

Note: $w(e_1) + w(e_2) + \cdots + w(e_n)$

$= w(e_n) + \cdots + w(e_2) + w(e_1)$

$e_1, e_2, \ldots, e_n$ is different than $e_n, \ldots, e_2, e_1$

b/c weight is the same we call them the same. $5!/2$ unique Hamilton circ.

In general!

$K_n \longrightarrow$ check $\frac{(n-1)!}{2}$ paths.

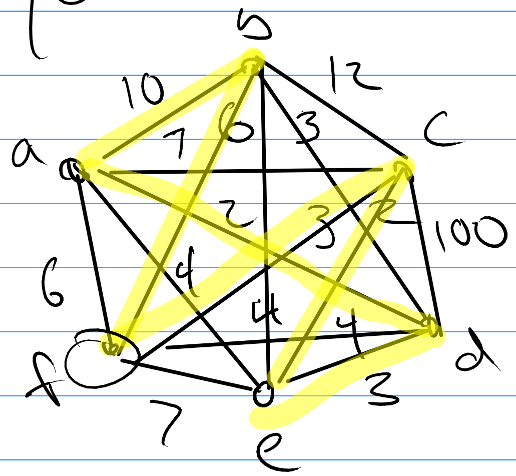
$K_6 \longrightarrow$ check $\frac{5!}{2} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2}$

$= \boxed{60}$

Task is to find least length of those 60 (in general $\frac{(n-1)!}{2}$)

Problem: there is no efficient alg.

You have to check them all.




$K_5$ $\frac{4!}{2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2}$

$= 12$

Cheap Neighbor Alg. ✓

a, d, e, c, f, b, a

$2 + 3 + 2 + 3 + 6 + 10 = \boxed{26}$

to check $\boxed{\dfrac{(n-1)!}{2}} \div \left(\underbrace{\text{check per second}}\cdot \underbrace{60\cdot 60\cdot 24\cdot 365}\right)$

Linear eqn

NP problems vs P problems

no polynomial — polynomial

Minesweeper
factoring.

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Extra Credit Proofs:

thm: $|A| = n$ $R$ is a relation on $A$.
If there is a path a to b of length at least 1 then there is a path of length $\leq n$. If the path is not a circuit $(a \neq b)$

then $\leq n-1$.

**Pf:** Case 1 $a = b$

Assume path $\overset{1}{a}, \overset{2}{x_1}, \overset{3}{x_2}, \ldots, \overset{m}{x_{m-1}}, a$ with length $m \geq 1$ exists.

this path has $m$ vertices listed.

If $m > n$ we can consider the vertices in the path to be pigeons and vertices in $A$ to be pigeon holes.

By pigeonhole principle there is at least one vertex in $A$ that has shown up two, or more, times in the path.

$\rightarrow$ the vertices between the same two vertices can be removed. You have a loop.

Visually $a, x_1, x_2, \ldots, \overbrace{x_i, \ldots, x_i}, \ldots, x_{m-1}, a$

remove the loop

or $\neg(m > n) \equiv m \leq n$

then pigeonhole principle doesn't apply.

So longest path is $n$.

## Case 2 $(a \neq b)$

$$\overset{1}{a}, \overset{2}{x_1}, \overset{3}{x_2}, \ldots, \overset{m}{x_{n-1}}, \overset{m+1}{b}$$

repeat argument $m+1 > n$

and the or statement

$$\neg(m+1 > n) \equiv m+1 \leq n$$

$$m \leq n-1$$

Exam Next thurs.