CIS 770: Formal Language Theory

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Decision Problems and Languages

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- A decision problem is represented as a formal language consisting of those strings (inputs) on which the answer is "yes".

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- The language of a Turing Machine M, denoted as $L(M)$, is the set of all strings w on which M accepts.
- A language L is recursively enumerable/Turing recognizable if there is a Turing Machine M such that $L(M) = L$.

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- Thus, if L is decidable then L is recursively enumerable.

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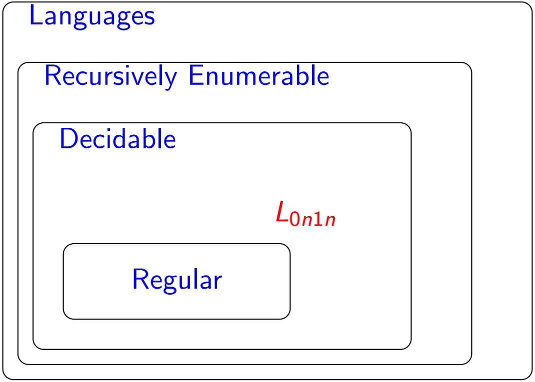
- This means that either L is not recursively enumerable. That is there is no turing machine M such that $L(M) = L$, or
- L is recursively enumerable but not decidable. That is, any Turing machine M such that $L(M) = L$, M does not halt on some inputs.

Big Picture



Relationship between classes of Languages

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- Any Turing Machine/program M can itself be encoded as a binary string. Moreover every binary string can be thought of as encoding a TM/program. (If not the correct format, considered to be the encoding of a default TM.)
- We will consider decision problems (language) whose inputs are Turing Machine (encoded as a binary string)

The Diagonal Language

Definition

Define $L_d = \{M \mid M \notin L(M)\}$.

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Definition

Define $L_d = \{M \mid M \not\in L(M)\}$. Thus, L_d is the collection of Turing machines (programs) M such that M does not halt and accept when given itself as input.

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Proposition

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- In what follows, we will denote the ith binary string (in lexicographic order) as the number i. Thus, we can say $j \in L(i)$, which means that the Turing machine corresponding to ith binary string accepts the jth binary string.

Proof (contd).

We can organize all programs and inputs as a (infinite) matrix, where the (i, j)th entry is Y if and only if $j \in L(i)$.

Inputs $\longrightarrow$

TMs $\downarrow$

	1	2	3	4	5	6	7	$\cdots$
1	N	N	N	N	N	N	N	
2	N	N	N	N	N	N	N	
3	Y	N	Y	N	Y	Y	Y	
4	N	Y	N	Y	Y	N	N	
5	N	Y	N	Y	Y	N	N	
6	N	N	Y	N	Y	N	Y	

Completing the proof

Diagonalization: Cantor

Proof (contd).

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Suppose L_d is recognized by a Turing machine, which is the jth binary string. i.e., $L_d = L(j)$.

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	1	$\boxed{\text{N}}$	N	N	N	N	N	N	
	2	N	$\boxed{\text{N}}$	N	N	N	N	N	
	3	Y	N	$\boxed{\text{Y}}$	N	Y	Y	Y	
	4	N	Y	N	$\boxed{\text{Y}}$	Y	N	N	
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Suppose L_d is recognized by a Turing machine, which is the jth binary string. i.e., $L_d = L(j)$. But $j \in L_d$ iff $j \notin L(j)$! $\square$

Acceptor for L_d?

Consider the following program

```
On input i
    Run program i on i
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Does the above program recognize L_d? No, because it may never output "yes" if i does not halt on i.

Models for Decidable Languages

Question

Is there a machine model such that

- all programs in the model halt on all inputs, and
- for each problem decidable by a TM, there is a program in the model that decides it?

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- Yes, $A_{\mathrm{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

The Universal Language

Proposition

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On input i
    Run M on input ⟨i,i⟩
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Observe that $L(D) = L_d$!

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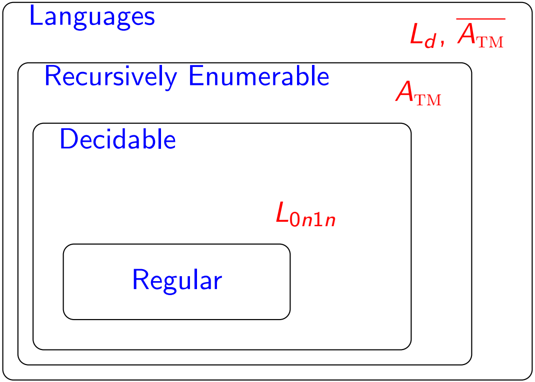
Observe that $L(D) = L_d$! But, L_d is not r.e. which gives us the contradiction. $\square$

A more complete Big Picture



Reductions

A reduction is a way of converting one problem into another problem such that a solution to the second problem can be used to solve the first problem. We say the first problem reduces to the second problem.

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- Informal Examples: Measuring the area of rectangle reduces to measuring the length of the sides; Solving a system of linear equations reduces to inverting a matrix
- The problem L_d reduces to the problem A_{TM} as follows: "To see if $w \in L_d$ check if $\langle w, w \rangle \in A_{\mathrm{TM}}$."

Undecidability using Reductions

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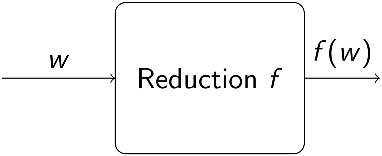
- On input w, apply reduction to transform w into an input w' for problem 2
- Run M on w', and use its answer.

Schematic View

$\xrightarrow{w}$

Reductions schematically

Schematic View



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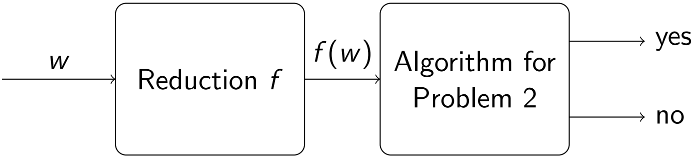

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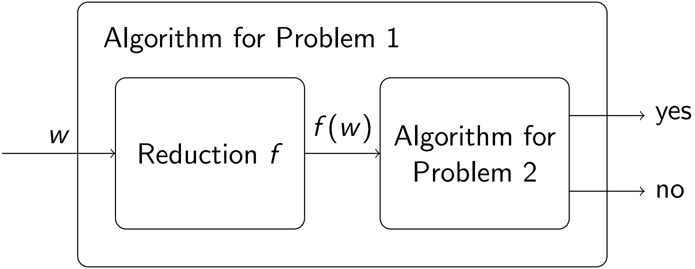

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Observe that $f(M)$ halts on input w if and only if M accepts w $\cdots\!\longrightarrow$

Proof (contd).

Suppose HALT is decidable. Then there is a Turing machine H that always halts and $L(H) = \mathrm{HALT}$.

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    Run H on ⟨f(M), w⟩
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T decides A_{TM}. But, A_{TM} is undecidable, which gives us the contradiction. □

Mapping Reductions

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Definition

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$$w \in A \text{ if and only if } f(w) \in B$$

In this case, we say A is mapping/many-one reducible to B, and we denote it by $A \leq_m B$.

Convention

In this course, we will drop the adjective “mapping” or “many-one”, and simply talk about reductions and reducibility.

Reductions and Recursive Enumerability

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Proof.

Let f be the reduction from A to B and let M_B be the Turing Machine recognizing B.

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If $A \leq_m B$ and B is recursively enumerable then A is recursively enumerable.

Proof.

Let f be the reduction from A to B and let M_B be the Turing Machine recognizing B. Then the Turing machine recognizing A is

```
On input w
    Compute f(w)
    Run M_B on f(w)
    Accept if M_B does and reject if M_B rejects
```

□

Reductions and non-r.e.

Corollary

If $A \leq_m B$ and A is not recursively enumerable then B is not recursively enumerable.

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Proof.

Let M_B be the Turing machine deciding B and let f be the reduction. Then the algorithm deciding A, on input w, computes $f(w)$ and runs M_B on $f(w)$. $\square$

Reductions and Decidability

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If $A \leq_m B$ and B is decidable then A is decidable.

Proof.

Let M_B be the Turing machine deciding B and let f be the reduction. Then the algorithm deciding A, on input w, computes $f(w)$ and runs M_B on $f(w)$. $\square$

Corollary

If $A \leq_m B$ and A is undecidable then B is undecidable.