Math 243

Ch 8 Exam (12 probs (+) 2 extra credit)

8.1 Seq. 1 prob

8.2 Series 1 prob conv? div?

$$\sum_{n=0}^{\infty} a_n$$

$$S_t = \sum_{n=0}^{t} a_n = a_0 + a_1 + \dots + a_t$$

$$S = \boxed{\lim_{t \to \infty} S_t}$$

8.3 2 probs. (Series)

① Integral Test

② Comparison

8.4 2 probs (Series)

① Alt. Series.

② Abs. Conv. Tests

8.5 1 prob

Find interval of conv. using abs. conv. tests.

ex $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{1/4}} x^n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}}{(n+1)^{1/4}} \frac{x^{n+1}}{1}}{\frac{\cancel{(-2)^n}}{n^{1/4}} \frac{\cancel{x^n}}{1}} \right|$$

$$= 2 \lim_{n\to\infty} |x| \frac{n^{1/4}}{(n+1)^{1/4}}$$

$$= 2 \lim_{n\to\infty} |x| \left(\frac{1}{1+\frac{1}{n}} \right)^{1/4}$$

$$= 2|x| < 1$$

abs. conv if.

Interval $|x| < \boxed{\frac{1}{2} = R}$

$x = -1/2$

$$\sum_{n=1}^{\infty} \frac{\cancel{(-2)^n}}{n^{1/4}} \cancel{\left(-\frac{1}{2}\right)^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \quad 1/4 < 1$$

div. p-series

$x = 1/2$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{1/4}} \left(\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$$

conv. alt. series

ex continued.

interval $R = \frac{1}{2}$ $\left(-\frac{1}{2}, \frac{1}{2}\right]$

8.6 2 probs (Power series)

① use power series for $\frac{1}{1-x}$

for .. example $\frac{3x}{1+x^3}$

② use power series for $\frac{1}{1-x}$ plus.

derivatives or integrals.

ex $\int \left(\frac{1}{1+x}\right) dx = \ln|1+x| + C$

ex $\frac{d}{dx}\left[\frac{1}{1+x^2}\right] = \boxed{\frac{2x}{(1+x^2)^2}}$

w power series

ex

power series of $\frac{2x}{(1+x^2)^2}$

b/c $\frac{d}{dx}\left[\frac{1}{1+x^2}\right] = \frac{2x}{(1+x^2)^2}$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\frac{2x}{(1+x^2)^2} = -2x + 4x^3 - 6x^5 + 8x^7 - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n 2n\, x^{2n-1}$$

8.7 2 probs

① form Maclaurin for $f(x)$

② form Taylor for $f(x)$

8.8 1 prob.

Find $T_n(x)$ for a given n.

(don't graph)

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extra credit. ① seq conv / div.

② series conv / div.

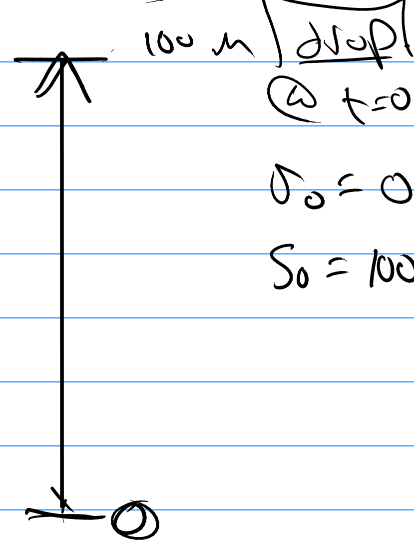
9.1B Parametric curves.

Parametric eqns. $s(t)$ = position @ time t.

$\frac{d}{dt} s(t) = v(t)$ velocity

$\frac{d}{dt}(v(t)) = \frac{d^2}{dt^2}(s(t)) = a(t)$ accel.

ex: **Freefall** (no air resist.)



@ $t=0$

$v_0 = 0$

$s_0 = 100$

$$a(t) = -(9.8\, m/s)/sec$$

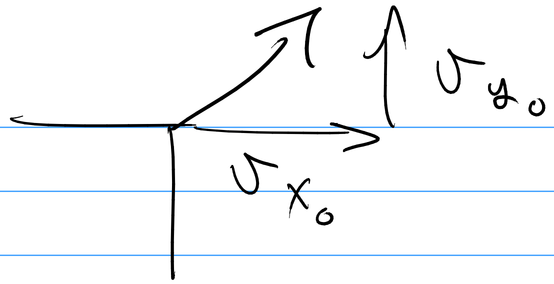
$$v(t) = -9.8t + v_0 = 0$$

$$v(t) = -9.8t$$

$$s(t) = -4.9t^2 + s_0 = 100$$

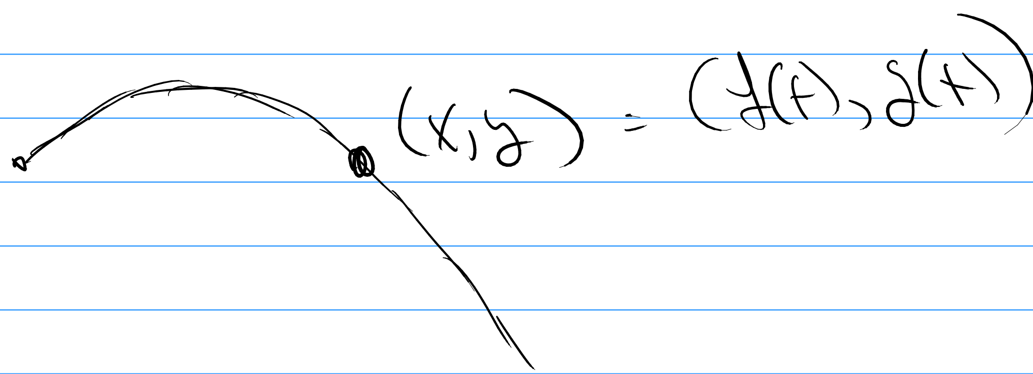
$$s(t) = -4.9t^2 + 100$$

$\boxed{\text{or}}$



gravity
↑ only works with the y's

$S_{y's}$ ← $f(t)$ $\qquad$ $S_{x's}$ ← $g(t)$

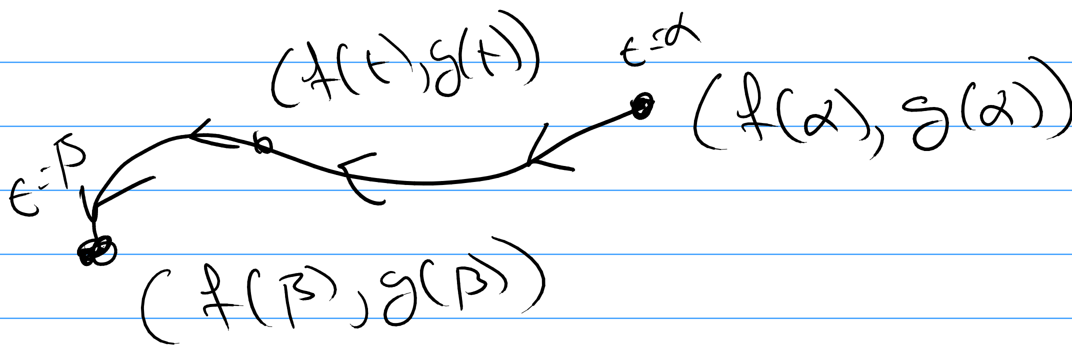


Parametric Curves

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\} \text{ \underline{parametric eqns}}$$

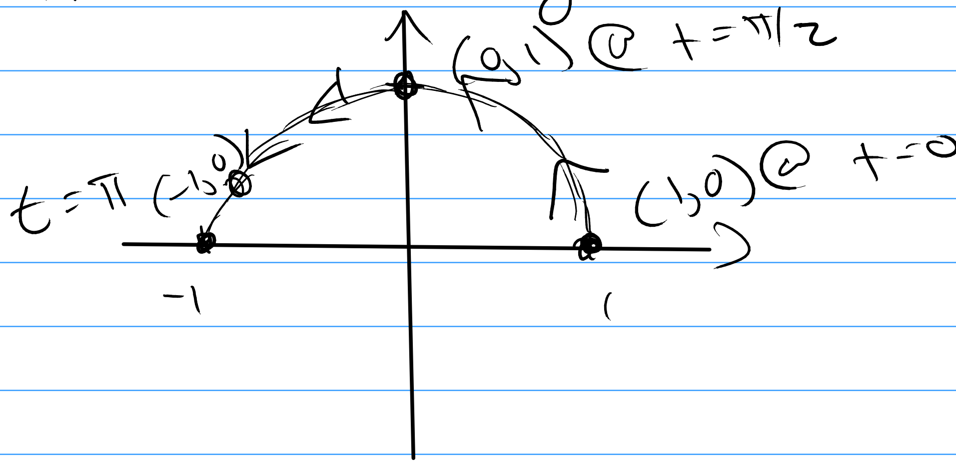
$t \equiv$ parameter

as t varies in $[\alpha, \beta]$ $\quad (x,y) = (f(t), g(t))$
varies and traces out a curve.



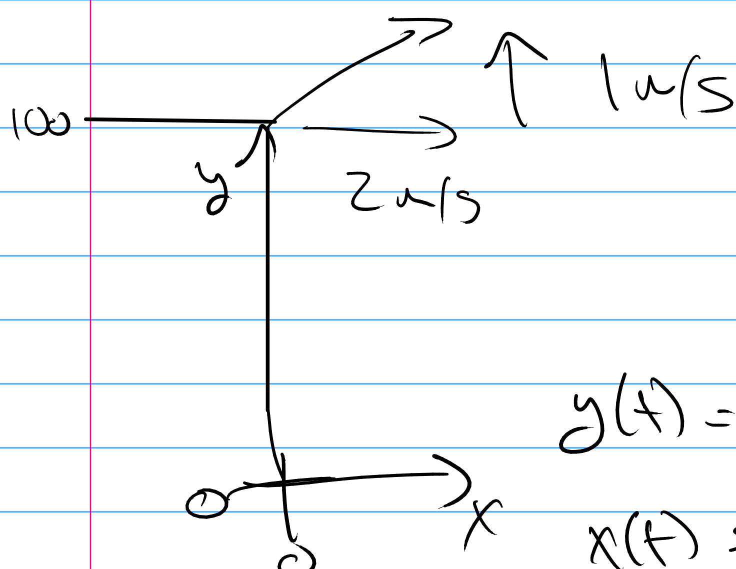
ex

$x = \cos(t)$ $\quad y = \sin(t)$ $\quad t \in [0, \pi]$



$$x^2 = \cos^2(t) \qquad y^2 = \sin^2(t)$$

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$



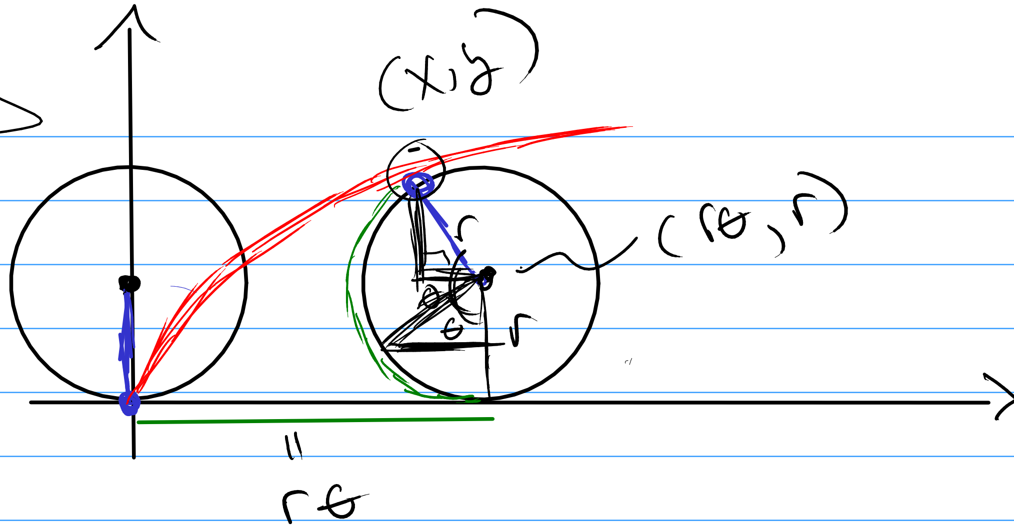
$$y(t) = -\frac{1}{2} g t^2 + v_{y_0} t + y_0$$

$$x(t) = v_{x_0} t + x_0$$

$$y(t) = -4.9 t^2 + t + 100$$

$$x(t) = 2t$$

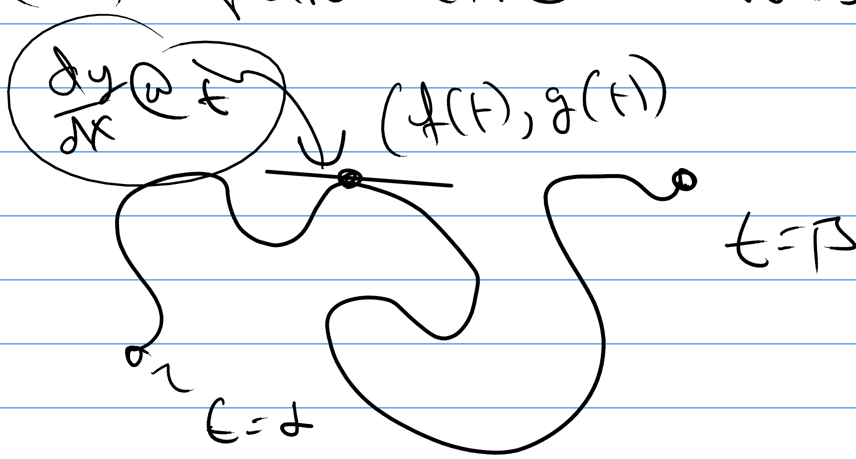
Classic



$$x = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = r - r\cos\theta = r(1 - \cos\theta)$$

Calculus (+) Parametric Curves.

Slope



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dx}{dt} \neq 0$$

$$y = f(t) \qquad x = g(t) \qquad \frac{dy}{dx} = \frac{f'(t)}{g'(t)}$$

ex $x = rt - r\sin t$ $y = r - r\cos t$

$$\frac{dy}{dx} = \frac{r\sin t}{r - r\cos t}$$ function of t.

concavity / curvature / 2nd deriv.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] \qquad \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

function of t

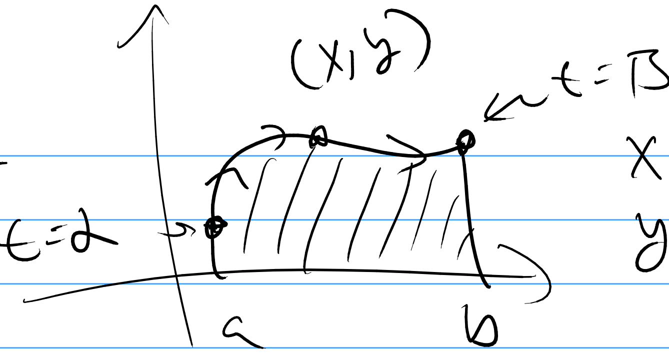
ex Cycloid.

$x = rt - r\sin t$ $y = r - r\cos t$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{r\sin t}{r - r\cos t}$$ function of t.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{r\sin t}{r - r\cos t}\right]}{\frac{d}{dt}\left[rt - r\sin t\right]} = \text{Finish!}$$

Integration:



$x = f(t)$

$y = g(t)$

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

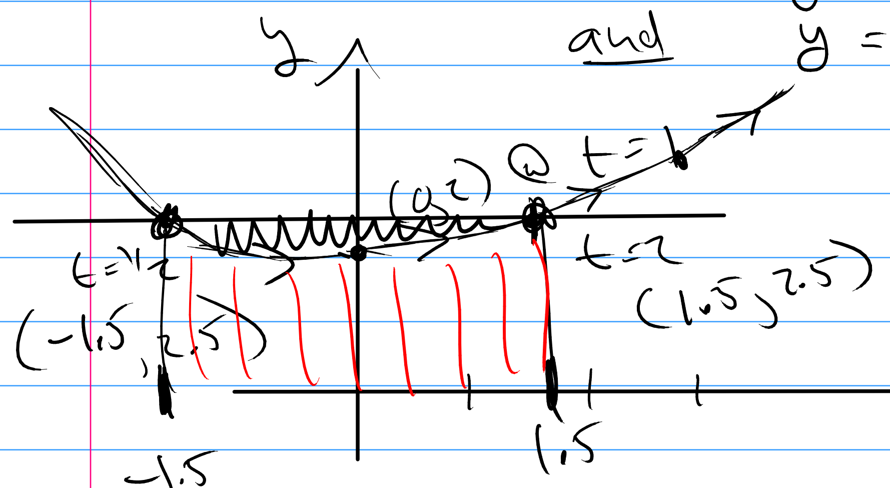
(+) curve is traversed once.

(+) $y > 0$

ex) Area between $x = t - \frac{1}{t}$, $y = t + 1/t$ and $y = 2.5$.

$t = 1$

$t = 2$



$y = 2.5$

$t + \frac{1}{t} = 2.5$

$t^2 + 1 = \frac{5}{2}t$

$2t^2 - 5t + 2 = 0$

$(2t-1)(t-2) = 0$

$t = 1/2 \quad t = 2$

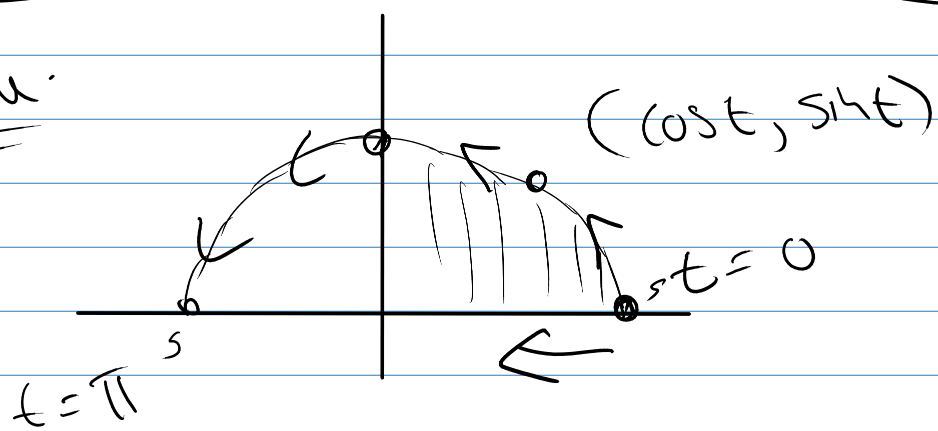
$$A = \int_a^b y\,dx$$

$$\text{Ans} = (3)(2.5) - \int_{1/2}^{2} \underbrace{\left(t+\frac{1}{t}\right)}_{y} \underbrace{\left(1+\frac{1}{t^2}\right)dt}_{dx = x'\,dt}$$

$$\text{Ans} = 7.5 - \int_{1/2}^{2} (t + 2t^{-1} + t^{-3})\,dt$$

$$= \underline{\underline{\text{Finish!}}}$$

ex Circle.



$$\text{Area Circle} = 2\int_0^{\pi} \sin t\,(-\sin t)\,dt$$

$$= -2\int_0^{\pi} \sin^2 t\,dt$$

$$1 - 2\sin^2 x = \cos 2x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= -2\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$$

$$= \int_0^{\pi} \cos 2x\,dx - \int_0^{\pi} dx$$

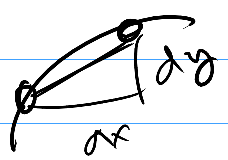
$= \frac{1}{2}\sin 2x \Big|_0^{\pi} - \pi$

$= -\pi$ (net signed area from right to left)

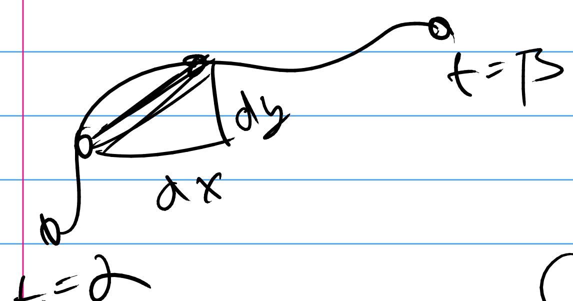
$\boxed{\text{area} = \pi}$

or just integrate $\int_{\pi}^{0} y\,dx$ (left to right)

$\boxed{\text{Arclength}}$ $\quad L = \boxed{\int_a^b \sqrt{dx^2 + dy^2}}$



$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$



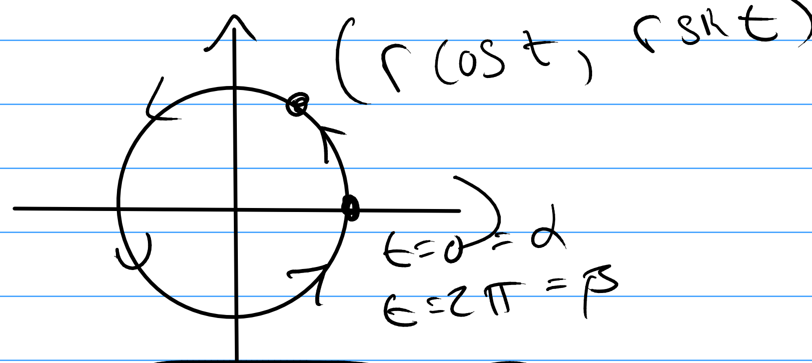
$\int_{\alpha}^{\beta} \sqrt{dx^2 + dy^2}$

$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$L = \int_\alpha^\beta \sqrt{(y')^2 + (x')^2}\, dt$$

(curve is traversed once)

ex **Circle**



$$L = \int_0^{2\pi} \sqrt{(r\cos t)^2 + (-r\sin t)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{r^2\cos^2 t + r^2\sin^2 t}\, dt$$

$$L = \int_0^{2\pi} r\, dt = r \cdot t \Big|_0^{2\pi}$$

$$= \boxed{2\pi r}$$

Find a length of a loop of

$$x = 3t - t^3 \qquad y = 3t^2$$

@ $x=0$ $\quad 0 = 3t - t^3$
$\quad 0 = t(3-t^2) \Rightarrow t = 0 \quad t = \pm\sqrt{3}$

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(6t)^2 + (3-3t^2)^2}\, dt$$

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{36t^2 + 9 - 18t^2 + 9t^4}\, dt$$

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9 + 18t^2 + 9t^4}\, dt$$

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \cancel{\sqrt{}}(3 + 3t^2)^{\cancel{2}}\, dt$$

$$L = 2\int_{0}^{\sqrt{3}} 3 + 3t^2\, dt$$

$$L = 2(3t + t^3)\Big|_0^{\sqrt{3}}$$

$$L = \boxed{6\sqrt{3} + 2 \cdot 3^{3/2}}$$