Math 322

11.2 Representing $f: B^n \to B$

$f(x_1, x_2, \dots, x_n) = 0$ or 1

table

2^n rows

x_1	x_2	$\dots$	x_n	f
1	1	$\dots$	1	0 or 1
				$\vdots$
0	0	$\dots$	0	0 or 1

total number of tables $= (2^{2^n})$

How to represent these tables?

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x	y	$\bar{x}$
1	1	0
1	0	0
0	1	1
0	0	1

$f(x,y) = \bar{x}$

$f_1(x,y) = \bar{x}y + \bar{x}\bar{y}$

$f_2(x,y) = (\bar{x} + \bar{y})(\bar{x} + y)$

x	y	$\bar{x}$	$\bar{y}$	$\bar{x}y$	$\bar{x}\bar{y}$	$\bar{x}y + \bar{x}\bar{y}$
1	1	0	0	0	0	0
1	0	0	1	0	0	0
0	1	1	0	1	0	1
0	0	1	1	0	1	1

$$\boxed{\overline{x} = \overline{x}y + \overline{x}\,\overline{y}}$$

x	y	$\overline{x}$	$\overline{y}$	$(\overline{x}+\overline{y})$	$(\overline{x}+y)$	$(\overline{x}+\overline{y})(\overline{x}+y)$
1	1	0	0	0	1	0
1	0	0	1	1	0	0
0	1	1	0	1	1	1
0	0	1	1	1	1	1

$$\boxed{\overline{x} = (\overline{x}+\overline{y})(\overline{x}+y)}$$

x	y	f
1	1	0
1	0	0
0	1	1
0	0	1

product (AND)

$1 \cdot 1 \cdot 1 = 1$

only way to get a 1 is to have all one's

$\boxed{\overline{x} \cdot y = 1}$ only when $x=0$ $y=1$

$\boxed{\overline{x} \cdot \overline{y} = 1}$ only when $x=0$ $y=0$

$\overline{x} \cdot y + \overline{x} \cdot \overline{y}$

Sum of products.

to find an expression for f.

(1) Focus on the 1's of f.

Sum-of-products

Def: literal = variable or $\overline{\text{variable}}$

minterm = product of literals, one for **each** variable.

ex: $f(x, y, z)$

example minterm $= \bar{x}yz$ or $= xy\bar{z}$

Sum-of-products $\Leftarrow$ |Sum of minterms|

ex

x	y	z	minterms	f
1	1	1	xyz	1
1	1	0	$xy\bar{z}$	0
1	0	1	$x\bar{y}z$	1
1	0	0	$x\bar{y}\bar{z}$	0
0	1	1	$\bar{x}yz$	0
0	1	0	$\bar{x}y\bar{z}$	0
0	0	1	$\bar{x}\bar{y}z$	0
0	0	0	$\bar{x}\bar{y}\bar{z}$	0

$f = xyz + x\bar{y}z$

- techniques to find sum-of-products
  1. Table $\longrightarrow$ sum-of-products
  2. Boolean Algebra

$$f(x,y) = \overline{x} \cdot 1 = \overline{x} \cdot (y + \overline{y})$$
$$= \boxed{\overline{x}y + \overline{x}\,\overline{y}}$$

ex) $f(x,y,z) = \overline{x} + y = \overline{x} \cdot 1 + y \cdot 1$

$$= \overline{x}(y + \overline{y}) + y \cdot (x + \overline{x})$$
$$= \overline{x} \cdot y + \overline{x} \cdot \overline{y} + y \cdot x + y \cdot \overline{x}$$
$$= \overline{x} \cdot y + \overline{x}\,\overline{y} + x \cdot y$$
$$= \overline{x}y(z + \overline{z}) + \overline{x}\,\overline{y}(z + \overline{z}) + xy(z + \overline{z})$$
$$= \boxed{\overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z} + xyz + xy\overline{z}}$$

x	y	z	$f(x,y,z) = \bar{x} + y$	Minterm
1	1	1	1	xyz
1	1	0	1	$xy\bar{z}$
1	0	1	0	
1	0	0	0	
0	1	1	1	$\bar{x}yz$
0	1	0	1	$\bar{x}y\bar{z}$
0	0	1	1	$\bar{x}\bar{y}z$
0	0	0	1	$\bar{x}\bar{y}\bar{z}$

$$\underset{5}{\bar{x}yz} + \underset{6}{\bar{x}y\bar{z}} + \underset{7}{\bar{x}\bar{y}z} + \underset{8}{\bar{x}\bar{y}\bar{z}} + \underset{1}{xyz} + \underset{2}{xy\bar{z}}$$

Focus on zero?

$$\boxed{\bar{x} = (\bar{x}+\bar{y})(\bar{x}+y)}$$

x	y	f
1	1	0
1	0	0
0	1	1
0	0	1

under sum (OR

$\bar{x}+\bar{y}$ — 0 is unique

$\bar{x}+y$ — $0+0+0=0$

↑ Maxterms

Def: Maxterm ≡ sum of literals <u>one</u> for <u>each</u> variable.

Product - of - Sums $\rightarrow$ product of max terms.

ex $\bar{x} = (\bar{x} + \bar{y}) \cdot (\bar{x} + y)$

So...

1 Sum of Products $\rightarrow$ focus? having 1's making 1's
(min terms)

2 Product of Sums $\rightarrow$ focus? having 0's making 0's
(Max terms)

x	y	z	$f(x,y,z) = \bar{x} + y$	maxterms
1	1	1	1	
1	1	0	1	
1	0	1	0	$\bar{x} + y + \bar{z}$
1	0	0	0	$\bar{x} + y + z$
0	1	1	1	
0	1	0	1	
0	0	1	1	
0	0	0	1	

$f(x,y,z) = (\bar{x} + y + \bar{z})(\bar{x} + y + z)$

by Boolean Algebra?

$$f(x,y,z) = \overline{x} + y + 0 = (\overline{x} + y) + (z \cdot \overline{z})$$

$$= (\overline{x} + y + z) \cdot (\overline{x} + y + \overline{z})$$

<u>all</u> $f : B^n \to B$

Sum-of-Products <u>or</u> Product-of-Sums

can represent <u>all</u> $f : B^n \to B$.

$\{\cdot, +, \overline{}\}$ is functionall complete.

<u>DeMorgans</u> $\overline{x + y} = \overline{x} \cdot \overline{y}$

$$\overline{\overline{x + y}} = \overline{\overline{x} \cdot \overline{y}}$$

$$\boxed{x + y = \overline{\overline{x} \cdot \overline{y}}}$$

So $\{\cdot, \overline{}\}$ is functionally complete.

a

$$\overline{\overline{x \cdot y}} = \overline{\overline{x} + \overline{y}}$$

$$x \cdot y = \left[\overline{\overline{x} + \overline{y}}\right]$$

$\{+, \overline{\ \ }\}$ is also functionally complete.

Consider

x	y	NAND $x \mid y$	NOR $x \downarrow y$
1	1	0	0
1	0	1	0
0	1	1	0
0	0	1	1

$\{\cdot, \overline{\ \ }\}$ is functionally complete.

① $\overline{x} = x \mid x$

show:

x	$\overline{x}$	$x \mid x$
1	0	0
0	1	1

② $x \cdot y = (x \mid y) \mid (x \mid y)$

x	y	xy	$x \mid y$	$(x \mid y) \mid (x \mid y)$
1	1	1	0	1
1	0	0	1	0
0	1	0	1	0
0	0	0	1	0

So $\{\uparrow\}$ is functionally complete.

and using same type of argument

$\{\downarrow\}$ is functionally complete.

$$\begin{cases} \overline{x} = x \downarrow x \\ x+y = (x \downarrow y) \downarrow (x \downarrow y) \end{cases}$$

$$xy = (x \downarrow x) \downarrow (y \downarrow y)$$