Math 243

Power series $\sum_{n=0}^{\infty} c_n x^n$ about $x=0$

general power series $\sum_{n=0}^{\infty} c_n (x-a)^n$
about $x=a$

ex $\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$

radius of conv. $R=1$
interval of conv. $(-1, 1)$

an geometric series $\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$

$\to \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x|<1$

ex $\frac{dy}{dx} = y \to \int \frac{1}{y}\, dy = \int dx$

$\to \ln|y| = x + c$ given $y(0)=1$

$|y| = e^{x+c}$

$|y| = c e^x$

initial value prob. $y = e^x$

So: $e^x = \sum_{x=0}^{\infty} \frac{1}{n!} x^n$ $R = \infty$

$(-\infty, \infty)$

function power series

$\boxed{\text{Q:}}$ any other $f(x)$ functions that have a power series representation?

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ conv. on $|x|<1$

$$\sum_{n=0}^{\infty} \boxed{}^n = \frac{1}{1-\boxed{}} \quad \text{on } |\boxed{}|<1$$

Ex

$$\sum_{n=0}^{\infty} \boxed{x^2}^n = \frac{1}{1-\boxed{x^2}} \quad \text{on } |\boxed{x^2}|<1$$

$$\sum_{n=0}^{\infty} x^{2n} = \boxed{\frac{1}{1-x^2}} \quad \text{on } |x \cdot x|<1$$

$-1<x<1$

So if $-1<x<1$ $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$

ex $\quad \dfrac{3}{1+x^3} = 3\cdot\left(\dfrac{1}{1-(-x^3)}\right)$

$$\frac{1}{1-\boxed{-x^3}} = \sum_{n=0}^{\infty} \boxed{-x^3}^n \quad \text{on } \left|\boxed{-x^3}\right| < 1$$

$$\frac{3}{1+x^3} = 3\sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{on } |x^3|<1$$

$$-1<x<1$$

$$\frac{3}{1+x^3} = 3 - 3x^3 + 3x^6 - 3x^9 + \cdots$$

on $-1<x<1$

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ex $\quad f(x) = \dfrac{x^2}{a^3-x^3} = \dfrac{x^2}{a^3}\left(\dfrac{1}{1-\left(\frac{x}{a}\right)^3}\right)$

using

$$\frac{1}{1-\boxed{\left(\frac{x}{a}\right)^3}} = \sum_{n=0}^{\infty} \boxed{\left(x/a\right)^3}^n \quad \text{on } \left|\boxed{\left(\frac{x}{a}\right)^3}\right| < 1$$

$$\frac{x^2}{a^3-x^3} = \frac{x^2}{a^3}\left(\sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^{3n}\right) \quad \text{on } \left|\left(\frac{x}{a}\right)^3\right| < 1$$

$$\frac{x^2}{a^3-x^3} = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}} \quad \text{on } -1 < \frac{x^3}{a^3} < 1$$

$a>0$ $\qquad -a^3 < x^3 < a^3$

$$\frac{x^2}{a^3-x^3} = \sum_{n=0}^{\infty} \frac{1}{a^{3n+3}} x^{3n+2} \qquad -a < x < a$$

$$= \frac{1}{a^3}x^2 + \frac{1}{a^6}x^5 + \frac{1}{a^9}x^8 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \boxed{\text{conv}} \ (a-R, a+R)$$

term by term differentiation / integration.

$$\frac{d}{dx}[f(x)] = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$\rightarrow \frac{d}{dx}[f(x)] = \sum_{n=0}^{\infty} \frac{d}{dx}\{c_n (x-a)^n\}$$

$$\boxed{f'(x) = \sum_{n=0}^{\infty} (n \cdot c_n)(x-a)^{n-1}} \quad \text{on } (a-R, a+R)$$

ex $$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right]$$

$$= -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

b/c $$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \cancel{0} + 1x^0 + 2x^1 + 3x^2 + 4x^3 + \dots$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)\,x^n}$$

Integration

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \left[\int c_n (x-a)^n\,dx\right]$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$

conv on $(a-R, a+R)$

ex $\int \frac{1}{1-x}\,dx = -\ln(1-x)$

$\rightarrow \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$

$$-\ln(1-x) = C + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \dots$$

$x=0 \qquad \ln(1-0) = \ln(1) = 0 \rightarrow C = 0$

$$-\ln(1-x) = \tfrac{1}{1}x^1 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \dots$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n \quad \text{on } |x|<1$$

$$\ln(1-\boxed{\tfrac{1}{2}}) = -\left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4} + \cdots\right)$$

$$\ln\left(\frac{1}{2}\right) = -\left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\,\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4} + \cdots\right)$$

$$\ln(2) = \frac{1}{2} + \frac{1}{2}\,\frac{1}{2^2} + \frac{1}{3}\,\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4} + \cdots$$

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ex $$\int \boxed{\frac{1}{1+x^2}}\,dx = \tan^{-1}x + C$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\tan^{-1}x = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$\tan^{-1}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

@ $x=0$ $\quad \tan^{-1}(0)=0 \rightarrow C=0$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{on } |-x^2|<1$$

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \quad |x|<1$$

8.7 Suppose we have $f(x)$ a known function. When does it have a power series?

assume

$$f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} c_n x^n \text{ on } |x|<R$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad |x|<R$$

Note: $f(0) = c_0 + 0 + 0 + \cdots$

$$\rightarrow \boxed{c_0 = f(0)}$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

Note: $\boxed{f'(0) = c_1}$

$$f''(x) = 2c_2 + 3\cdot 2\cdot c_3 x + 4\cdot 3\cdot c_4 x^2 + \cdots$$

Note: $f''(0) = 2c_2 \rightarrow c_2 = \frac{1}{2} f''(0)$

$$f'''(x) = 3\cdot 2\cdot 1\cdot c_3 + 4\cdot 3\cdot 2\cdot c_4 x + \cdots$$

Note: $f'''(0) = 3!\, c_3$

$$\rightarrow c_3 = \frac{1}{3!} f^{(3)}(0)$$

If you continue: $C_n = \frac{1}{n!} f^{(n)}(0)$

Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{on } |x|<R$$

In general Series about $x=a$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Taylor Series about $x=a$

e^x e^x as a Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = e^x$

$f^{(1)}(x) = e^x$

$f^{(2)}(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

$f^{(n)}(0) = e^0 = 1$

$$\to \quad e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n} \quad |x| < R$$

?

use abs. conv. test (ratio test)

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x|$$

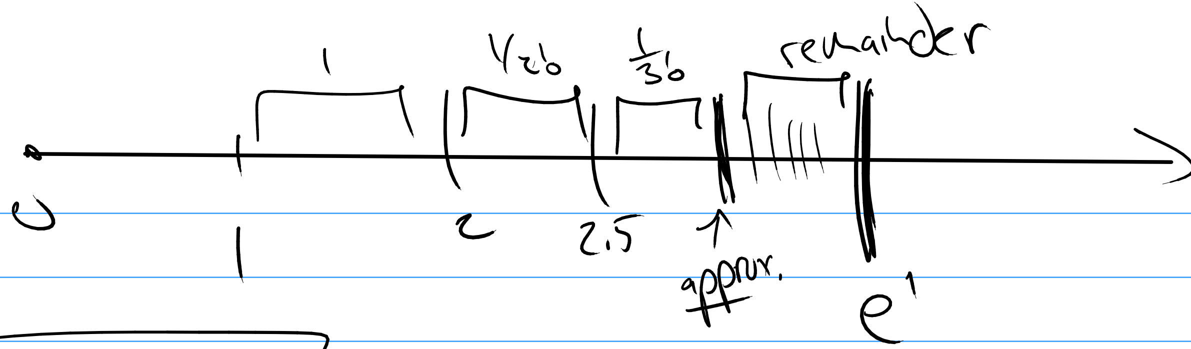
$$= \lim_{n\to\infty} \cancel{\frac{1}{n+1}}^{\to 0} \cancel{|x|}^{\to |x|} = 0 \cdot |x| = 0 < 1 \quad \underline{\text{always}}$$

so $R = \infty$

$$\therefore \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

ex $\quad e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} (1)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$

$$e = \underbrace{1 + 1 + \frac{1}{2!} + \frac{1}{3!}}_{\text{approx to } e} + \underbrace{\frac{1}{4!} + \cdots}_{\text{error = remainder}}$$



existence?

Taylor Series about $x=a$ conv. $|x-a|<R$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

nth degree taylor poly.

$$f(x)=\boxed{\frac{f^{(0)}(a)}{0!}(x-a)^0+\frac{f^{(1)}(a)}{1!}(x-a)+\dots+\frac{f^{n}(a)}{n!}(x-a)^n}$$

$$+\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}+\dots\dots$$

remainder

$$T_n(x)=f(a)+f'(a)(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$R_n(x)$ = remainder

Note: $\lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$f(x) = \lim_{n\to\infty} T_n(x) \quad \text{on } |x-a| < R$$

If you stop...

$$R_n(x) = f(x) - T_n(x)$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} f(x) - \overset{\to f(x)}{\cancel{T_n(x)}} = 0$$

$$\boxed{\text{to show } f(x) \text{ has a power series means to show } \lim_{n\to\infty} R_n(x) = 0}$$

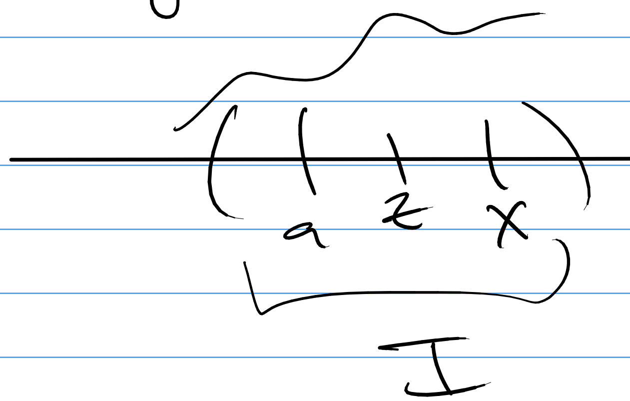
need to show existence

and $|x-a| < R$

Problem: $R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \ldots$

$$\lim_{n\to\infty}\left(\text{that mess}\right) = \overset{?}{0}$$

$\boxed{\text{Taylor's Formula}}$ if f is $(n+1)$ times differentiable on an interval I containing $x=a$, then there is a number z between x and a such that



$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

to show existence is to show

$$\lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} = 0$$

ex $\quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

to show existence is to show

Show: $\displaystyle \lim_{n\to\infty} \frac{e^z}{(n+1)!} x^{n+1} = 0$

$$\lim_{n\to\infty} \frac{e^z}{(n+1)!} x^{n+1} = e^z \lim_{n\to\infty} \frac{x^n}{n!} \to 0$$

$$= e^z \cdot 0 = 0$$

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p. 466 Maclaurin Series

Show: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad R = 1$

$$f(x) = (1-x)^{-1}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{3!}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{4!}{(1-x)^5}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{\cancel{n!}}{\cancel{n!}} x^n$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} (x)^n}$$

$$\frac{1}{1+x} = f(x)$$

$$f'(x) = -(1+x)^{-2}$$

$$f^{(2)}(x) = 2(1+x)^{-3}$$

$$f^{(3)}(x) = -3!(1+x)^{-4}$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

if $x = -x$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \cancel{(-1)^n} \cancel{(-1)^n} x^n$$

Same as $\quad \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\boxed{e^x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $\qquad$ $\boxed{\text{p. 466}}$

$\boxed{\sin x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad$ table.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\boxed{\sin x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\boxed{\sin x \approx x - \frac{1}{3!}x^3} + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$