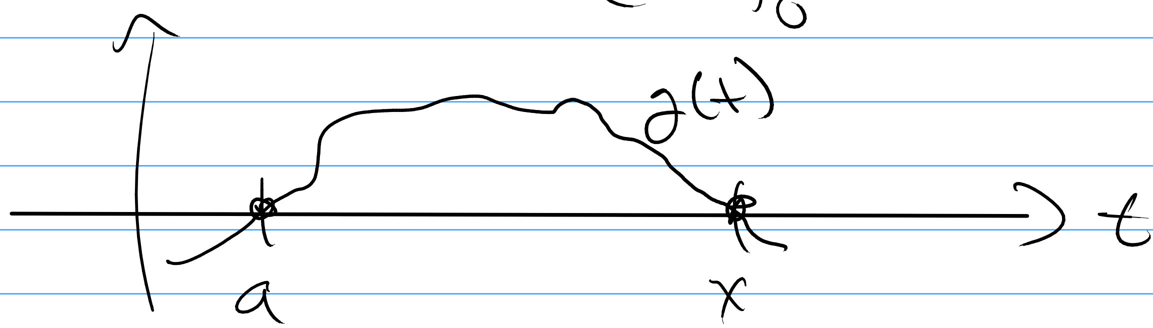
Math 243

Taylor's Formula: $f(x) = T_n(x) + R_n(x)$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$



Rolle's th^k: $g(a) = 0 \qquad g(x) = 0$

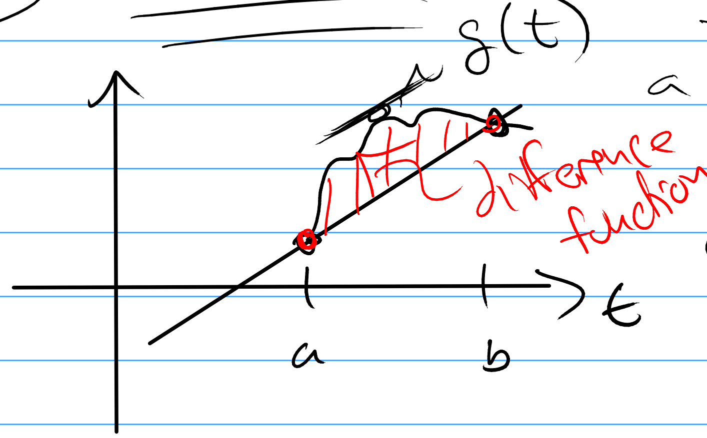
$\rightarrow$ there is a z between a and x

Such that $g'(z) = 0$

Tex3

Mean Value:

there is a c between a and b.



$$g'(c) = \frac{g(b) - g(a)}{b-a}$$

What about Taylor's formula?

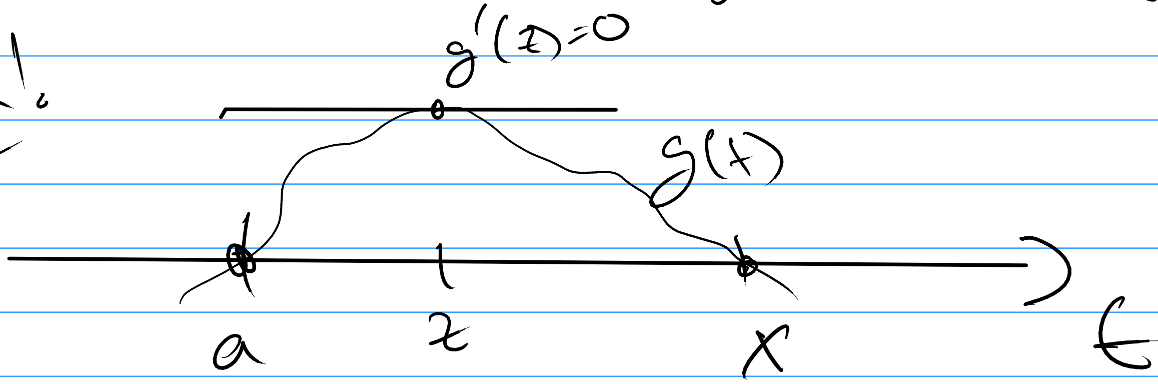
$$f(x) = T_n(x) + \boxed{R_n(x)}$$

Know: $R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$

$$R_n(x) = f(x) - \boxed{T_n(x)}$$

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Idea!



Have:

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f^{(2)}(a)}{2!}(x-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - \frac{R_n(x)}{(x-a)^{n+1}}(x-t)^{n+1}$$

Apply <u>Rolle's</u> th^{m}.

$$g(a) = f(x) - \underbrace{f(a) - f'(a)(x-a) - \dots - \frac{f^{(n)}(a)}{n!}(x-a)^n}_{} - R_n(x)$$

$$= \left| f(x) - T_n(x) \right| - R_n(x) = R_n(x) - R_n(x)$$

$$= 0$$

b/c $x - x = 0$

$$g(x) = f(x) - f(x) - 0 = 0$$

so there is a z between a and x

such that $\frac{d}{dt}[g(t)]\Big|_{t=z} = 0$

$$g(t) = f(x) - f(t) - f'(t)\underline{(x-t)} - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n - \frac{R_n(x)}{(x-a)^{n+1}}(x-t)^{n+1}$$

$$g'(t) = 0 - \cancel{f'(t)} + \cancel{f'(t)} - \cancel{f''(t)(x-t)}$$

2nd $\frac{d}{dt}\left[-\frac{f^{(2)}(t)}{2!}\boxed{(x-t)^2}\right]$ $+ \cancel{f^{(2)}(t)(x-t)} - \cancel{\frac{f^{(3)}(t)}{2!}}(x-t)^2$

3rd $\frac{d}{dt}\left[-\frac{f^{(3)}(t)}{3!}(x-t)^3\right]$ $+ \cancel{\frac{f^{(3)}(t)}{2!}(x-t)^2} - \frac{f^{(4)}(t)}{3!}(x-t)^3$

$+$ etc...

$$g'(t) = 0 - \underbrace{\frac{f^{(n+1)}(t)}{n!}(x-t)^n}_{\text{all the } T_n \text{ part left}} + \frac{(n+1)\boxed{R_n(x)}}{(x-a)^{n+1}}(x-t)^n$$

at $t = z$ $\quad g'(z) = 0$

$$0 = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n + \frac{(n+1)R_n(x)}{(x-a)^{n+1}}(x-z)^n$$

$$0 = \frac{-f^{(n+1)}(z)}{(n+1)!} + \frac{R_n(x)}{(x-a)^{n+1}}$$

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}}$$

Ex $\quad \int_0^1 x\cos(x)\,dx \quad$ (error within 3 decimal places)

$$\int_0^1 x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots\right)dx$$

$$\int_0^1 \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \frac{x^9}{8!} \cdots\right)dx$$

$$= \frac{1}{2}x^2\Big|_0^1 - \frac{x^4}{4\cdot 2!}\Big|_0^1 + \frac{x^6}{6\cdot 4!}\Big|_0^1 - \frac{x^8}{8\cdot 6!}\Big|_0^1 + \cdots$$

$$= \frac{1}{2} - \frac{1}{4\cdot 2!} + \frac{1}{6\cdot 4!} - \frac{1}{8\cdot 6!} + \frac{1}{10\cdot 8!} - \frac{1}{12\cdot 10!} + \cdots$$

$$= \boxed{\frac{1}{2} - \frac{1}{8} + \frac{1}{144}} - \frac{1}{5760} + \cdots$$

larger than true error

$$\frac{1}{5760} \approx 0{,}0001$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$R = 1$$

all series are about $x = 0$.

appox $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$ near $a = 9$

$\boxed{\text{Taylor Series}}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = x^{-1/2}$$
$$f'(x) = -\tfrac{1}{2}x^{-3/2}$$
$$f^{(2)}(x) = \tfrac{3}{4}x^{-5/2}$$
$$f^{(3)}(x) = -\frac{1\cdot 3\cdot 5}{2^3}x^{-7/2}$$
$$f^{(4)}(x) = \frac{1\cdot 3\cdot 5\cdot 7}{2^4}x^{-9/2}$$

$$f^{(n)}(x) = (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} x^{-\frac{(2n+1)}{2}}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} 9^{-\frac{(2n+1)}{2}} (x-9)^n$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots\boxed{(2n-1)}}{n!\,2^n\,3^{2n+1}}(x-9)^n$$

$$= \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1\cdot 3}{2!\,2^2\,3^5}(x-9)^2 - \cdots$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

OOPS

Can't $\div 0$

$$\begin{array}{r|l}
 & \frac{1}{x} - \frac{1}{3}x \\
\hline
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
 & 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \\
\hline
 & -\frac{1}{3}x^2 + \cdots
\end{array}$$

$(\sin x)(\cos x)$

$$\begin{array}{llll}
x & - \frac{x^3}{3!} & + \frac{x^5}{5!} & - \cdots \\
1 & - \frac{x^2}{2!} & + \frac{x^4}{4!} & - \cdots \\
\hline
x & - \frac{x^3}{3!} & + \frac{x^5}{5!} & - \cdots \\
 & - \frac{x^3}{2!} & + \frac{x^5}{3!\,2!} & \cdots \\
 & & + \frac{x^5}{4!} & - \cdots \\
\hline
x & -\left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 & + \left(\frac{1}{5!} + \frac{1}{3!\,2!} + \frac{1}{4!}\right)x^5 & - \cdots
\end{array}$$

Application

$$f(x) \approx T_n(x)$$

error $\leftrightarrow$ $R_n(x)$?

$\boxed{\text{use}}$ $f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ to approx.

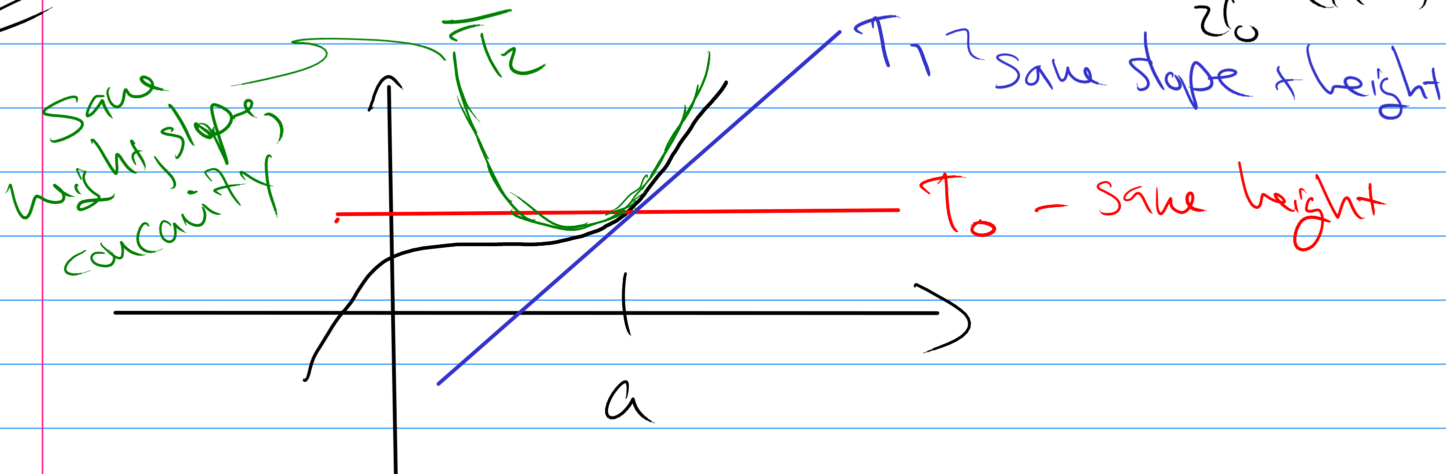
$\boxed{\text{use}}$ $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ how bad?

$$T_0(x) = f(a)$$

b/c $T_1(x) = f(a) + f'(a)(x-a)$

Linear Approx.

b/c $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$



How bad is the approximate?

(1) $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ ← worst case

or

(2) is it alt. series? Use Alt. Series estimation. (next term > error)

(3) graph

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ex $f(x) = \frac{\ln x}{x}$ $a=1$ $n=3$ $f(1)=0$

$f'(x) = \frac{1-\ln x}{x^2}$ $f'(1) = 1$

$f''(x) = \frac{-\cancel{x} - (1-\ln x)(2\cancel{x})}{x^{\cancel{4}3}} = \frac{-3+2\ln x}{x^3}$ $f''(1) = -3$

$f'''(x) = \frac{2\cancel{x} - (-3+2\ln x)(3\cancel{x^2})}{x^{\cancel{6}4}} = \frac{11-6\ln x}{x^4}$ $f'''(1) = 11$

$$f(x) \approx 0 + 1\cdot(x-1) - 3(x-1)^2 + 11(x-1)^3$$

$$\boxed{\frac{\ln x}{x} \approx (x-1) - 3(x-1)^2 + 11(x-1)^3 \quad \text{near } x=1}$$

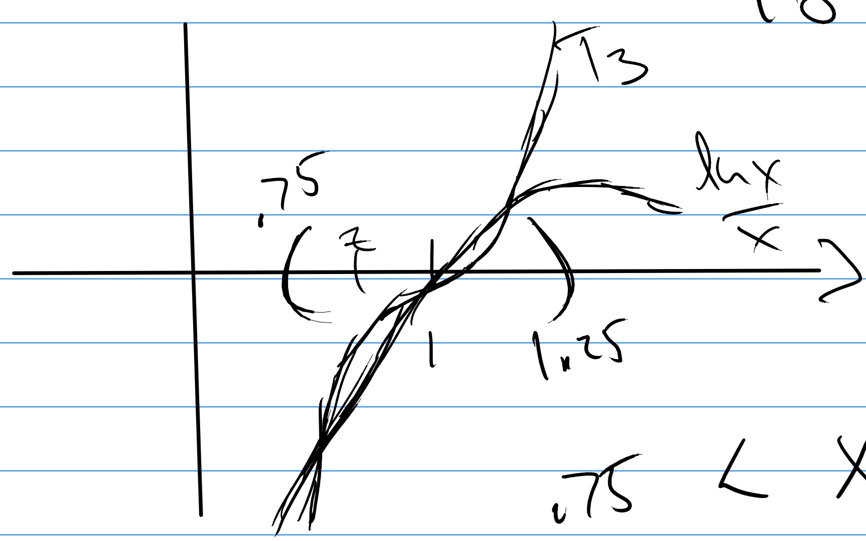
error?

$$f^{(3)}(x) = \frac{11 - 6\ln x}{x^4}$$

$$f^{(4)}(x) = \frac{-6\cancel{x^3} - (11 - 6\ln x)(4\cancel{x^3})}{\cancel{x^8}\, x^5}$$

$$f^{(4)}(x) = \frac{-50 + 24\ln x}{x^5}$$

$$R_3(x) = \frac{\frac{-50 + 24\ln z}{z^5}}{4!}(x-1)^4$$



$$\frac{\ln x}{x}$$

$$.75 < x < 1.25$$

$$|R_3(x)| = \left| \frac{\frac{-50+24\ln z}{z^5}}{4!} (x-1)^4 \right|$$

on $x \in [.75, 1.25]$ maximize!

$(x-1)^4$ has max $.75$ or 1.25

$$|R_3(x)| < \left| \frac{-50+24\ln z}{z^5} \cdot \frac{1}{256 \cdot 4!} \right|$$

$$|R_3(x)| < \left| \frac{-50+24\ln 1.25}{(.75)^5} \right| \left| \frac{1}{256 \cdot 4!} \right|$$

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estimate $e^{0.1}$ to 4 decimals?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x \approx \boxed{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$ z is between 0 and x

$$R_n(x) = \left| \frac{e^{z}}{(n+1)!} x^{n+1} \right| < 0.00001$$

@ $x = .1$

$$R_n(.1) = \left| \frac{e^{z}}{(n+1)!} (.1)^{n+1} \right| < 0.00001$$

z is between 0 and 0.1

$$e^{z} < e^{1} < 3$$

want to find n such that

$$\rightarrow \quad \frac{(.1)^{n+1}}{(n+1)!} < \frac{1}{300000}$$

$$\frac{1}{(n+1)!} \frac{1}{10^{n+1}} < \frac{1}{300000}$$

$\boxed{\text{if } n=4}$

$$\frac{1}{5!} \frac{1}{10^{5}} < \frac{1}{300000}$$

$$e^{.1} \approx 1 + .1 + \frac{(.1)^2}{2!} + \frac{(.1)^3}{3!} + \frac{(.1)^4}{4!}$$