Math 321

Q's / 5.4 #11

$c_k x^k$

$(x^2 + -x^{-1})^{100} = x^{200} - \cdots + x^{-100}$

$$(a+b)^{100} = \sum_{j=0}^{100} \binom{100}{j} a^{100-j} b^j$$

k^{th} term $\binom{100}{k} a^{100-k} b^k$

k^{th} term $\binom{100}{k} (x^2)^{100-k} (-x^{-1})^k$

$= (-1)^k \binom{100}{k} x^{200-2k} x^{-k}$

$= (-1)^k \binom{100}{k} x^{\boxed{200-3k}}$ k goes from 0 to 100

let $\alpha = 200 - 3k$

$\frac{200-\alpha}{3} = k$

α^{th} term $(-1)^{\frac{200-\alpha}{3}} \binom{100}{\boxed{\frac{200-\alpha}{3}}} x^\alpha$

$\alpha = 200, 197, 194, \ldots, -100$

$$\frac{1}{2}\binom{2n+2}{n+1} = \binom{2n}{n+1} + \binom{2n}{n}$$

pf:

$$\frac{1}{2}\,\frac{(2n+2)!}{(n+1)!\,(2n+2-n-1)!} = \frac{(2n)!}{(n+1)!\,(2n-n-1)!} + \frac{(2n)!}{n!\,(2n-n)!}$$

show equal by simplification

OR

pf: pascal's ID $\binom{\square}{\triangle} = \binom{\square-1}{\triangle-1} + \binom{\square-1}{\triangle}$

$$\binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n+1} \stackrel{?}{=} \frac{1}{2}\binom{2n+2}{n+1}$$

$$\frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right]$$

$$\binom{0}{0}$$

$$\boxed{2n+1} \atop n+1$$

$$\binom{2n+1}{n+1} = \binom{2n+1}{n}$$

$\binom{1}{0}$ $\binom{1}{1}$ 1 1

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$ 1 $\boxed{2}$ 1

$\binom{3}{0}$ 1 $\boxed{3}$ $\boxed{3}$ 1

1 4 $\boxed{6}$ 4 1

$$\frac{1}{2}\left[\binom{2n+1}{n+1}+\binom{2n+1}{n+1}\right]$$

$$=\frac{1}{2}\left[\binom{2n+1}{n}+\binom{2n+1}{n+1}\right]$$

$$=\frac{1}{2}\binom{2n+2}{n+1}$$

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$$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

let $x=1$ $y=-1$

$$0=\sum_{j=0}^{n}\binom{n}{j}(-1)^j$$

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots$$

$$\binom{n}{1}+\binom{n}{3}+\cdots=\binom{n}{0}+\binom{n}{2}+\cdots$$

Ch 7

$$\boxed{a_n = \text{expression of } a_{n-k}\text{'s}}$$

recurrence relation

$a_n = 2a_{n-1} + 1$

$\vdots$

$a_n = 2^n - 1$

ex takes \$1 coin, \$1 bill, \$5 bill

how many ways to take \$n?

\$n = \$(n-1) + \$1 coin

or

\$n = \$(n-1) + \$1 bill

or

\$n = \$(n-5) + \$5 bill

$a_n = a_{n-1} + a_{n-1} + a_{n-5}$

$$\boxed{a_n = 2a_{n-1} + a_{n-5}}$$ ✓

Bases:

$a_0 = 1$

$a_1 = 2$

$a_2 = 4$

$a_3 = 8$

$a_4 = 16$

$a_5 = 2a_4 + a_0$

$= 2 \cdot 16 + 1 = 33$

$a_6 = 2 \cdot 33 + 2 = 68$

7.2

$$a_n = \boxed{2} a_{n-1} + a_{n-5}$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$$

linear homogeneous recurrence relation of degree k with constant coeff.

$c_i \in \mathbb{R}$ $\quad c_k \neq 0$

linear $\equiv$ a_i's have a power of 1

homogeneous $\equiv$ all terms are multiples of a_i's

const. coeff $\equiv$ $\boxed{c_i} \cdot a_i$

constant!

$$\boxed{a_n = 2a_{n-1}} \qquad a_0 = 1$$

$a_1 = 2$

$a_2 = 4$

$a_3 = 8 \longrightarrow \boxed{a_n = 2^n}$

Hmm... let's guess that all solutions of these are $a_n = r^n$

So.. let's check!

$$a_n = r^n \qquad a_{\square} = r^{\square}$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$r^n = c_1 \frac{r^n}{r} + c_2 \frac{r^n}{r^2} + \cdots + c_k \frac{r^n}{r^k}$$

$$1 = \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots + \frac{c_k}{r^k}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k r^0$$

$$\boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0}$$

characteristic equation
solve?
→ characteristic roots.

ex $\quad f_n = 1 \cdot f_{n-1} + 1 \cdot f_{n-2}$

charac. eqn. → $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$f_n = f_{n-1} + f_{n-2}$$

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

const. const.

$$f_0 = 0 \qquad f_1 = 1$$

$$\boxed{f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n}$$