CIS 770: Formal Language Theory

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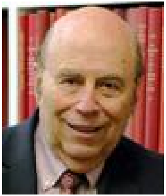
Nondeterminism

Michael Rabin and Dana Scott (1959)



Michael Rabin

Dana Scott

Nondeterminism

Given a current state of the machine and input symbol to be read, the next state is not uniquely determined.

Comparison to DFAs

Nondeterministic Finite Automata (NFA)

NFAs have 3 features when compared with DFAs.

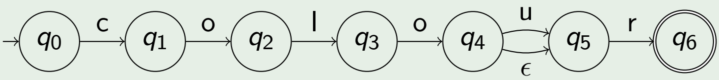
1. Ability to take a step without reading any input symbol
2. A state may have no transition on a particular symbol
3. Ability to transition to more than one state on a given symbol

ϵ-Transitions

Transitions without reading input symbols

Example

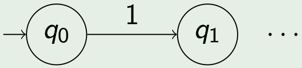
The British spelling of "color" is "colour". In a web search application, you may want to recognize both variants.



NFA with ϵ-transitions

No transitions

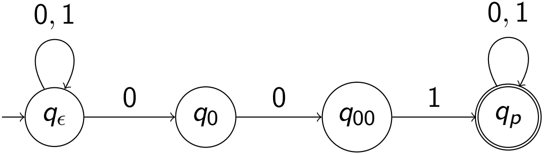
Example



No 0-transition out of initial state

In the above automaton, if the string starts with a 0 then the string has no computation (i.e., rejected).

Multiple Transitions



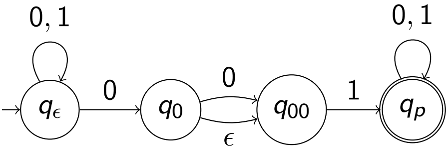
q_ϵ has two 0-transitions

Parallel Computation View

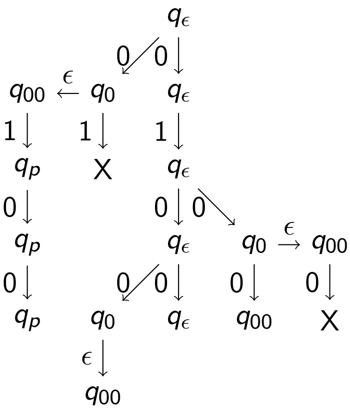
At each step, the machine "forks" a thread corresponding to one of the possible next states.

- If a state has an ϵ-transition, then you fork a new process for each of the possible ϵ-transitions, without reading any input symbol
- If the state has multiple transitions on the current input symbol read, then fork a process for each possibility
- If from current state of a thread, there is no transition on the current input symbol then the thread dies

Parallel Computation View: An Example



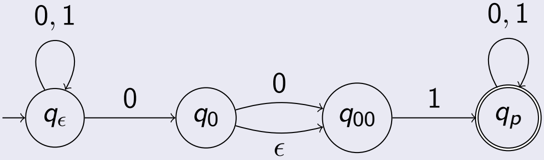
Example NFA



Computation on 0100

Nondeterministic Acceptance
Parallel Computation View

Input is accepted if after reading all the symbols, one of the live threads of the automaton is in a final/accepting state. If none of the live threads are in a final/accepting state, the input is rejected.

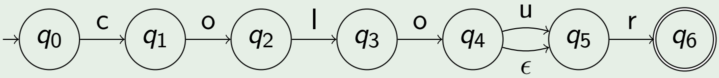


0100 is accepted because one thread of computation is
$q_\epsilon \xrightarrow{0} q_0 \xrightarrow{\epsilon} q_{00} \xrightarrow{1} q_p \xrightarrow{0} q_p \xrightarrow{0} q_p$

Computation: Guessing View

The machine magically guesses the choices that lead to acceptance

Example



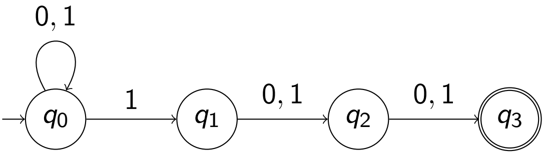
NFA M_{color}

After seeing "colo" the automaton guesses if it will see the british or the american spelling. If it guesses american then it moves without reading the next input symbol.

Observations: Guessing View

- If there is a sequence of choices that will lead to the automaton (not "dying" and) ending up in an accept state, then those choices will be magically guessed
- On the other hand, if the input will not be accepted then no guess will lead the to automaton being in an accept state
  - On the input "colobr", whether automaton M_{color} guesses british or american, it will not proceed when it reads 'b'.

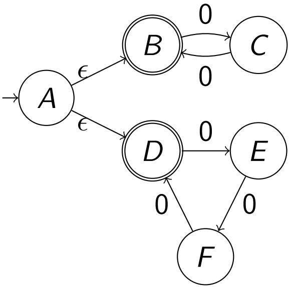
Example I



Automaton accepts strings having a 1 two positions from end of input

The automaton "guesses" at some point that the 1 it is seeing is 2 positions from end of input.

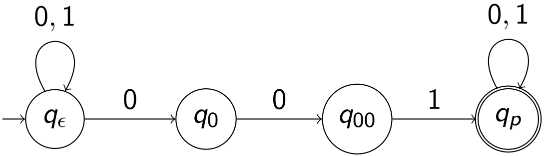
Example II



NFA accepting strings where the length is either a multiple 2 or 3

The NFA "guesses" at the begining whether it will see a multiple of 2 or 3, and then confirms that the guess was correct.

Example III



NFA accepting strings with 001 as substring

At some point the NFA "guesses" that the pattern 001 is starting and then checks to confirm the guess.

Nondeterministic Finite Automata (NFA)

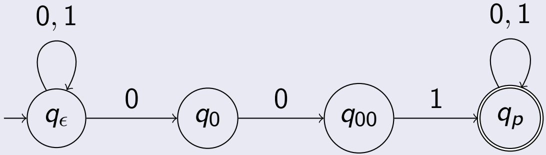
Formal Definition

Definition

A nondeterministic finite automaton (NFA) is $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the powerset of Q
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

Example of NFA



Transition Diagram of NFA

Formally, the NFA is $M_{001} = (\{q_\epsilon, q_0, q_{00}, q_p\}, \{0, 1\}, \delta, q_\epsilon, \{q_p\})$ where δ is given by

$$\begin{array}{lll} \delta(q_\epsilon, 0) = \{q_\epsilon, q_0\} & \delta(q_\epsilon, 1) = \{q_\epsilon\} & \delta(q_0, 0) = \{q_{00}\} \\ \delta(q_{00}, 1) = \{q_p\} & \delta(q_p, 0) = \{q_p\} & \delta(q_p, 1) = \{q_p\} \end{array}$$

δ is $\emptyset$ in all other cases.

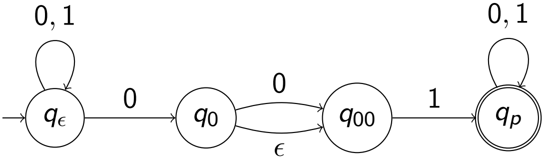
Computation

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and states $q_1, q_2 \in Q$, we say $q_1 \xrightarrow{w}_M q_2$ if there is one thread of computation on input w from state q_1 that ends in q_2. Formally, $q_1 \xrightarrow{w}_M q_2$ if there is a sequence of states $r_0, r_1, \ldots r_k$ and a sequence $x_1, x_2, \ldots x_k$, where for each i, $x_i \in \Sigma \cup \{\epsilon\}$, such that

- $r_0 = q_1$,
- for each i, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = q_2$, and
- $w = x_1 x_2 x_3 \cdots x_k$

Example Computation



$q_\epsilon \xrightarrow{0100}_M q_p$ because taking $r_0 = q_\epsilon$, $r_1 = q_0$, $r_2 = q_{00}$, $r_3 = q_p$, $r_4 = q_p$, $r_5 = q_p$, and $x_1 = 0$, $x_2 = \epsilon$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, we have

- $x_1 x_2 \cdots x_5 = 0\epsilon100 = 0100$
- $r_{i+1} \in \delta(r_i, x_{i+1})$

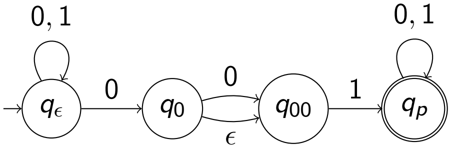
Defining $\hat{\Delta}$

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and state $q_1 \in Q$, we say $\hat{\Delta}(q_1, w)$ to denote states of all the active threads of computation on input w from q_1. Formally,

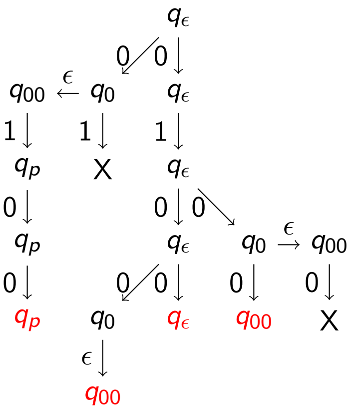
$$\hat{\Delta}(q_1, w) = \{q \in Q \mid q_1 \xrightarrow{w}_M q\}$$

Example



Example NFA

$$\hat{\Delta}(q_\epsilon, 0100) = \{q_p, q_{00}, q_\epsilon\}$$



Computation on 0100

Acceptance/Recognition

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say M accepts w iff $\hat{\Delta}(q_0, w) \cap F \neq \emptyset$

Definition

The language accepted or recognized by NFA M over alphabet Σ is $L(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$. A language L is said to be accepted/recognized by M if $L = L(M)$.

Observations about NFAs

Observation 1

For NFA M, string w and state q_1 it could be that

- $\hat{\Delta}(q_1, w) = \emptyset$
- $\hat{\Delta}(q_1, w)$ has more than one element

Observation 2

However, the following proposition about DFAs continues to hold for NFAs

- For NFA M, strings u and v, and state q,

$$\hat{\Delta}(q, uv) = \bigcup_{q' \in \hat{\Delta}(q,u)} \hat{\Delta}(q', v)$$

Using Nondeterminism

When designing an NFA for a language

- You follow the same methodology as for DFAs, like identifying what needs to be remembered
- But now, the machine can "guess" at certain steps

Back to the Future

Problem

For $\Sigma = \{0, 1, 2\}$, let

$$L = \{w\#c \mid w \in \Sigma^*,\ c \in \Sigma,\ \text{and } c \text{ occurs in } w\}$$

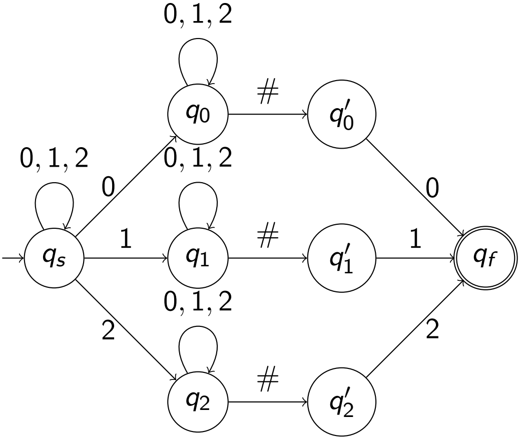
So $1011\#0 \in L$ but $1011\#2 \notin L$. Design an NFA recognizing L.

Solution

- Read symbols of w, i.e., portion of input before $\#$ is seen
- Guess at some point that current symbol in w is going to be the same as 'c'; store this symbol in the state
- Read the rest of w
- On reading $\#$, check that the symbol immediately after is the one stored, and that the input ends immediately after that.

Back to the Future

The Automaton



$$L(M) = \{w\#c \mid c \text{ occurs in } w\}$$

Halving a Language

Definition

For a language L, define $\frac{1}{2}L$ as follows.

$$\frac{1}{2}L = \{x \mid \exists y.\ |x| = |y| \text{ and } xy \in L\}$$

In other words, $\frac{1}{2}L$ consists of the first halves of strings in L

Example

If $L = \{001, 0000, 01, 110010\}$ then $\frac{1}{2}L = \{00, 0, 110\}$.

Recognizing Halves of Regular Languages

Proposition

If L is recognized by a DFA M then there is a NFA N such that $L(N) = \frac{1}{2}L$.

Proof Idea

On input x, need to check if x is the first half of some string $w = xy$ that is accepted by M.

- "Run" M on input x; let M be in state q_i after reading all of x
- Guess a string y such that $|y| = |x|$
- Check if M reaches a final state on reading y from q_i

How do you guess a string y of equal length to x using finite memory? Seems to require remembering the length of x!

Fixing the Idea

Problem and Fix(?)

- How do you guess a string y of equal length to x using finite memory? Guess one symbol of y as you read one symbol of x!
- How do you "run" M on y from q_i, if you cannot store all the symbols of y? Run M on y as you guess each symbol, without waiting to finish the execution on x!
- If we don't first execute M on x, how do we know the state q_i from which we have to execute y from? Guess it! And then check that running M on x does indeed end in q_i, your guessed state.

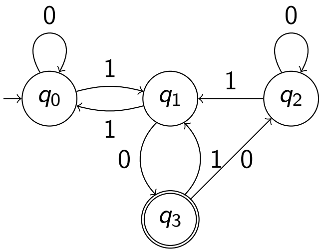
New Algorithm

On input x, NFA N

1. Guess state q_i and place "left finger" on (initial state of M) q_0 and "right finger" on q_i
2. As characters of x are read, N moves the left finger along transitions dictated by x and simultaneously moves the right finger along nondeterministically chosen transitions labelled by some symbol
3. Accept if after reading x, left finger is at q_i (state initially guessed for right finger) and right finger is at an accepting state

Things to remember: initial guess for right finger, and positions of left and right finger.

Algorithm on Example



DFA M

$100010 \in L$ and so $x = 100 \in \frac{1}{2}L$
NFA N execution on $x = 100$ is

String Read	Left Finger		Right Finger
ϵ	q_0		$\boxed{q_2}$
1	q_1		q_2
10	q_3	$=?$	q_1
100	$\boxed{q_2}$		$\boxed{q_3}$

$\uparrow$
accept?

Formal Construction of NFA N

States and Initial State

Given $M = (Q, \Sigma, \delta, q_0, F)$ recognizing L define $N = (Q', \Sigma, \delta', q_0', F')$ that recognizes $\frac{1}{2}L$

- $Q' = Q \times Q \times Q \cup \{s\}$, where $s \notin Q$
  - s is a new start state
  - Other states are of the form $\langle$left finger, initial guess, right finger$\rangle$; "initial guess" records the initial guess for the right finger
- $q_0' = s$

Formal Construction of NFA N

Transitions and Final States

- Transitions

  $$\delta'(s, \epsilon) = \{\langle q_0, q_i, q_i\rangle \mid q_i \in Q\}$$

  "Guess" the state q_i that the input will lead to

  $$\delta'(\langle q_i, q_j, q_k\rangle, a) = \{\langle q_l, q_j, q_m\rangle \mid \delta(q_i, a) = q_l, \exists b \in \Sigma.\, \delta(q_k, b) = q_m\}$$

  b is the guess for the next symbol of y and initial guess does not change

- $F' = \{\langle q_i, q_i, q_j\rangle \mid q_i \in Q, q_j \in F\}$