KANSAS STATE UNIVERSITY | Department of Mathematics

Applied Matrix Theory - Math 551

Solutions to the problems in the Study Guide for Test 2

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The material for Test 2 includes the topics in HW5-HW8 and Lab5-Lab8 as well as the related notes posted on KSOL. Here are some practice problems.

1. Find a basis for the subspace $\mathcal{V} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : 2x_1 + x_2 = 0, x_1 - x_2 - x_3 = 0\}$.

Solution. $\mathcal{V} = null(A)$ where A is the matrix

```
A =

     2     1     0
     1    -1    -1
```

Next, (as explained, for instance, in the Notes on "Finding bases", on "More on Col, Row, Null", and Lab 5), a basis for $null(A)$ can be gotten by interpreting

```
>> rref([A, [0 0]'])

ans =

    1.0000         0   -0.3333         0
         0    1.0000    0.6667         0
```

or, using Matlab

```
>> null(A,'r')

ans =

    0.3333
   -0.6667
    1.0000
```

The vector above, or any multiple of it, forms a basis for $\mathcal{V}$. The dimension of $\mathcal{V}$ is 1.

2. Find a 2×2 matrix A that represents a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -9 \end{bmatrix}.$$

Solution. The desired matrix A must look like

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

where the entries x_1, x_2, x_3, and x_4 must satisfy the system

$$(S) \left\{ \begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & & & & & = & 1 \\ & & & & 2x_3 & + & x_4 & = & 4 \\ -x_1 & + & 3x_2 & & & & & = & 5 \\ & & & & -x_3 & + & 3x_4 & = & -9 \end{array} \right.$$

Next, we solve

```
>> rref([2 1 0 0 1; 0 0 2 1 4; -1 3 0 0 5; 0 0 -1 3 -9])

ans =

    1.0000         0         0         0   -0.2857
         0    1.0000         0         0    1.5714
         0         0    1.0000         0    3.0000
         0         0         0    1.0000   -2.0000
```

and find that

$$A = \begin{bmatrix} -0.2857 & 1.5714 \\ 3 & -2 \end{bmatrix}$$

3. Find bases for the column space, the row space, and the null space of the matrix

$$P = \begin{bmatrix} 5 & 3 & 1 & 0 \\ 4 & 4 & 6 & 6 \\ -1 & -9 & -1 & -7 \\ 0 & 2 & -4 & 1 \end{bmatrix}.$$

Indicate the corresponding dimensions of those subspaces.

Solution. See, for instance, the Notes on “Finding bases”, on “More on Col, Row, Null”, and Lab 5.

```
The null space of the matrix under consideration
has only the zero vector.
That is, the null space of the entered matrix has dimension 0.


The columns of the matrix under consideration
are linearly independent.
```

```
Therefore, they constitute a basis for its column space.
The dimension of its column space is 4.


The rows of the matrix under consideration are linearly independent.
Therefore, they constitute a basis for its row space.
The dimension of its row space is 4.
```

4. Find bases for the kernel and the range of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ represented by the matrix

$$A = \begin{bmatrix} -3 & 0 & -1 & 7 \\ 1 & 6 & 4 & 0 \\ 1 & 2 & 2 & 5 \end{bmatrix}.$$

Is T onto? Is T one-to-one?

Solution. See, for instance, the Notes on "Finding bases", on "More on Col, Row, Null", and Lab 5.

```
The dimension of the null space of A (i.e., the dimension
of the kernel of T) is 1.

A basis for the null space of A (i.e., the kernel of T)
is given by the columns of the following matrix:


BasisNullSpace =

    7.2500
    8.6250
  -14.7500
    1.0000

Hence, T is not one-to-one.

The dimension of the column space of the matrix A is 3. Hence, T is onto.

A basis for the column space of the matrix A (i.e., the range
of T) is given by the columns of the following matrix:


BasisColumnSpace =

    -3     0    -1
```

```
     1      6      4
     1      2      2
```

5. Find bases for the column space, the row space, and the null space of the matrix

$$Q = \begin{bmatrix} 4 & 3 & 1 & 0 & -1 \\ 4 & 4 & 0 & 2 & 1 \\ -1 & -3 & -1 & -1 & 1 \\ 2 & 1 & 0 & 3 & 0 \end{bmatrix}.$$

Indicate the corresponding dimensions of those subspaces.

Solution. See, for instance, the Notes on "Finding bases", on "More on Col, Row, Null", and Lab 5.

```
The dimension of the null space of the matrix under consideration is 1.

A basis for the null space of the matrix under consideration
is given by the columns of the following matrix:


BasisNullSpace =

    0.0294
   -0.3235
    1.8529
    0.0882
    1.0000


The dimension of the column space of the matrix is 4.

A basis for the column space of the matrix under consideration
is given by the columns of the following matrix:


BasisColumnSpace =

     4      3      1      0
     4      4      0      2
    -1     -3     -1     -1
     2      1      0      3
```

```
The rows of the matrix under consideration are linearly independent.
Therefore, they constitute a basis for its row space.
The dimension of its row space is 4.
```

6. Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ be a linear transformation and $\beta = \{u_1, u_2, u_3\}$ be a basis of $\mathbf{R}^3$ such that

$$u_1 = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

and

$$T(u_1) = \begin{bmatrix} 4 \\ 6 \\ -1 \\ 5 \end{bmatrix}, \quad T(u_2) = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad T(u_3) = \begin{bmatrix} 7 \\ 3 \\ 6 \\ -4 \end{bmatrix}.$$

Find $T(u)$ where

$$u = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

Solution. By doing

```
>> rref([u1 u2 u3 u])

ans =

    1.0000         0         0    1.2500
         0    1.0000         0    2.7500
         0         0    1.0000   -3.0000
```

we find that the coordinates of u in the basis β are given by 1.25, 2.75 and -3. That is,

$$u = 1.25u_1 + 2.75u_2 - 3u_3.$$

Since T is linear, we must have

$$T(u) = 1.25\,T(u_1) + 2.75\,T(u_2) - 3\,T(u_3) = 1.25 \begin{bmatrix} 4 \\ 6 \\ -1 \\ 5 \end{bmatrix} + 2.75 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 7 \\ 3 \\ 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -18.75 \\ 1.25 \\ -22.00 \\ 18.25 \end{bmatrix}.$$

7. Use *branching* to write a Matlab function that takes an arbitrary $n \times n$ matrix Q and returns a 1 if Q is orthogonal and a 0 otherwise.

Solution. There are many ways to do this. One possible option is the following

```
function out=checkortho(Q)
[m n]=size(Q);
I=eye(n);
if m==n & Q*Q'==I
    disp(' ')
    disp('The entered matrix is orthogonal.')
    out=1;
else
    disp(' ')
    disp('The entered matrix is not orthogonal.')
    out=0;
end
end
```

8. Consider the vectors

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -2 \\ 1 \\ 5 \end{bmatrix} \quad w_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad w_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 1 \\ 5 \\ -8 \end{bmatrix}.$$

Find a basis for the subspace of all the vectors in $\mathbf{R}^6$ which are orthogonal to w_1, w_2, and w_3.

Solution. Any collection of 3 linearly independent vectors in $\mathbf{R}^6$ which are all orthogonal to w_1, w_2, and w_3. One way to obtain these vectors (see your class notes) is to put the vectors w_1, w_2, and w_3 as the rows of a matrix A and then find a basis for $null(A)$ (here we are using that two vectors u and v are orthogonal whenever $u \cdot v = 0$). For finding bases of $null(A)$ see the notes on "Finding bases" and "More on Col, Row, Null".

As an illustration, the columns of the matrix

```
   -0.6667   -1.0000    1.5417
   -0.6667    1.0000   -3.0833
    1.0000         0   -1.2500
    1.0000         0         0
         0    1.0000         0
         0         0    1.0000
```

are 3 linearly independent vectors in $\mathbf{R}^6$ which are all orthogonal to w_1, w_2, and w_3.

9. Find a decomposition of the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ of the form

$$v = \bar{v} + w$$

such that $\bar{v}$ belongs to the subspace $\mathcal{U} = span(u_1, u_2)$, where $u_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$, and $w \in \mathcal{U}^{\perp}$.

Solution. See, for instance, the Notes on “Orthogonality”. The required decomposition

$$v = \bar{v} + w$$

is realized by the vectors

$$\bar{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

and

$$w = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

10. Write a Matlab function that takes an $m \times n$ matrix A and a vector $v \in \mathbf{R}^m$ and returns an $m \times n$ matrix B so that, for $1 \leq j \leq n$, the j-th column of B is the projection of v onto the subspace generated by the j-th column of A. The code should account for the case in which some of columns of A are the zero vector.

Solution. One way to achieve this would be

```
function B=projAv(A,v)

[m n]=size(A);

for j=1:n
    if norm(A(:,j))~=0
        u=A(:,j);
        B(:,j)=dot(u,v)/(dot(u,u))*u;
    else
        B(:,j)=zeros(m,1);
    end
end
```

11. Find bases for the kernel and the range of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ represented by the matrix

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Is T onto? Is T one-to-one?

Solution. See, for instance, the Notes on “Finding bases”, on “More on Col, Row, Null”, and Lab 5.

```
The dimension of the null space of A (i.e., the dimension
of the kernel of T) is 1.

A basis for the null space of A (i.e., the kernel of T)
is given by the columns of the following matrix:


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    7.2500
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    1.0000

Hence, T is not one-to-one.

The dimension of the column space of the matrix A is 3. Hence, T is onto.

A basis for the column space of the matrix A (i.e., the range
of T) is given by the columns of the following matrix:


BasisColumnSpace =

    -3     0    -1
     1     6     4
     1     2     2
```

12. Determine all the numbers x_1, x_2 and x_3 such that the vectors

$$u_1 = \begin{bmatrix} -3 \\ 1 \\ x_1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} x_3 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

are all mutually orthogonal.

Solution. The three orthogonality conditions yield

$$\begin{cases} -15 + 1 + x_1 + 3x_2 = 0 \\ -3x_3 + 3 + 2x_1 + 12 = 0 \\ 5x_3 + 3 + 2 + 4x_2 = 0 \end{cases}$$

That is,

$$\begin{cases} x_1 + 3x_2 = 14 \\ 2x_1 - 3x_3 = -15 \\ 4x_2 + 5x_3 = -5 \end{cases}$$

Now we solve

```
>> rref([1 3 0 14; 2 0 -3 -15; 0 4 5 -5])

ans =

    1.0000         0         0  -24.3333
         0    1.0000         0   12.7778
         0         0    1.0000  -11.2222
```

13. Find the distance between the vector

```
v =

     2
    -3
     5
     4
     1
```

and the column space of the matrix

```
A =

        0.5    2    0
         0    -2    2
         1    -1    0
         0     1   -2
         0     3    2
```

Solution. See, for instance, the Notes on "Orthogonality" and Lab 8.

```
The closest vector to the vector v among all the vectors
in the column space of the matrix A is given by

vbar =

    3.6632
```

```
   -3.2632
    4.1684
    2.6316
   -0.1053
```

```
The distance from v to the column space of A
equals 2.5731 (units of length).
```

14. Use a *for loop* to write a Matlab function that takes an arbitrary $m \times n$ matrix A and an $m \times 1$ column vector v and returns the sum of all the distances from v to the columns of A.

Solution.

```
function s=sumdistances(A,v)
[m n]=size(A);
s=0;
for k=1:n
    s=s+norm(v-A(:,k));
end
end
```