Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

More on Null, Col, and Row

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Consider the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}$.

(a) Find a basis for the null space of A, null(A). Geometrically describe null(A).

(b) Find a basis for the column space of A, col(A). Geometrically describe col(A).

(c) Find a basis for the row space of A, row(A). Geometrically describe row(A).

(d) Give the dimension of each of the spaces above.

Solution:

(a) The null space of A, denoted by null(A), is the solution set of the linear system $Ax = 0$. Note that since A is a 3×3 matrix, then both x and 0 have three components. We have

$$\text{rref}\left(\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -5 & 1 & 0 \\ 1 & -4 & -7 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 13 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the solutions to the system

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for which $x_1 = -13\,t$, $x_2 = -5\,t$, $x_3 = t$, where t is any number. That is, all the solutions to the homogeneous system $Ax = 0$ can be written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13\,t \\ -5\,t \\ t \end{bmatrix} = t \begin{bmatrix} -13 \\ -5 \\ 1 \end{bmatrix},$$

where t is the free variable. In other words, $null(A)$ consists of all the multiples of the vector $[-13, -5, 1]'$, in symbols,

$$\text{null}(A) = \text{span}\left\{\begin{bmatrix} -13 \\ -5 \\ 1 \end{bmatrix}\right\},$$

and the set $\left\{ \begin{bmatrix} -13 \\ -5 \\ 1 \end{bmatrix} \right\}$ is a basis for null(A). Notice that any non-zero multiple of the vector $[-13, -5, 1]'$ will be form a basis for $null(A)$ as well.

Geometrically,

$$\begin{aligned} \text{null}(A) &= \text{ set of all vectors in } \mathbf{R}^3 \text{ that are parallel to } \begin{bmatrix} -13 \\ -5 \\ 1 \end{bmatrix} \\ &= \text{ line through the origin parallel to } \begin{bmatrix} -13 \\ -5 \\ 1 \end{bmatrix}. \end{aligned}$$

(b) The column space of A, denoted by col(A), is the subspace of all linear combinations of the columns of A. This is,

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} \right\}.$$

Since

$$\text{rref}\left(\begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix},$$

then the third column of A is a linear combination of the first and second columns of A. Also the first and second columns of A are linearly independent. Then

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix} \right\},$$

and therefore the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix} \right\}$ is a basis for col(A). Geometrically,

$$\text{col}(A) = \text{ plane through the origin generated by the vectors } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix}.$$

(c) The row space of A, denoted by row(A), is the subspace of all linear combinations of the rows of A, and therefore row(A)=col(A^T), where A^T is the transpose of A. We have

$$\text{rref}(A^T) = \text{rref}\left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & -5 & -4 \\ 3 & 1 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This tells that the last row of A is a linear combination of the first and second rows of A and that the first and second rows of A are linearly independent. Thern

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}.$$

The set $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}$ is a basis for row(A). Geometrically,

$$\text{row}(A) = \text{ plane through the origin generated by } \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}.$$

(d) $\dim(\text{null}(A)) = 1$ $\quad\quad$ $\dim(\text{col}(A)) = 2$ $\quad\quad$ $\dim(\text{row}(A)) = 2.$

Remark 1: In general one has: If A is $m \times n$, then

$$\begin{aligned} \text{rank}(A) &= \dim(\text{col}(A)) = \dim(\text{row}(A)), \\ n &= \text{rank}(A) + \text{nullity}(A). \end{aligned}$$

Recall that nullity(A) is just another name for dim(null(A)). Also, note that the first line says that rank(A) is the number of linearly independent columns of A and the number of linearly independent rows of A.

Remark 2: It turns out that the row space of a matrix A is also given by the span of all nonzero rows of rref(A). Then, for example, if A is the 3×3 matrix given above, one has also

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

Remark 3: If A is $m \times n$ note that the vectors of null(A) will have n components, the vectors of col(A) will have m components, and the vectors of row(A) will have n components.