CIS 770: Formal Language Theory

Pavithra Prabhakar

Kansas State University

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- Inputs and outputs are complex objects, encoded as strings
- Examples of objects:
  - Matrices, graphs, geometric shapes, images, videos, ...
  - DFAs, NFAs, Turing Machines, Algorithms, other machines ...

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Encoding Complex Objects

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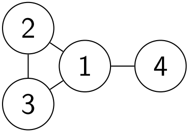
- Example: encoding a "graph."

  `(1,2,3,4)((1,2)(2,3)(3,1)(1,4))`

  encodes the graph



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- All these inputs can be encoded as strings and all these algorithms can be implemented as Turing Machines
- Some of these algorithms are for decision problems, while others are for computing more general functions
- All these algorithms terminate on all inputs

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Universal Turing Machine (a simple "Operating System"): Takes a TM M and a string w and simulates the execution of M on w

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- We just saw some example algorithms all of which terminate in a finite number of steps, and output yes or no (accept or reject). i.e., They decide the corresponding languages.

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  - $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ is undecidable

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Proposition

There are languages which are recognizable, but not decidable

Recognizing A_{TM}

Program U for recognizing A_{TM}:

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On input ⟨M, w⟩
    simulate M on w
    if simulated M accepts w, then accept
    else reject (by moving to q_rej)
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Deciding vs. Recognizing

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If L and $\overline{L}$ are recognizable, then L is decidable

Proof.

Program P for deciding L, given programs P_L and $P_{\overline{L}}$ for recognizing L and $\overline{L}$:

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- On input x, simulate P_L and $P_{\overline{L}}$ on input x. Whether $x \in L$ or $x \notin L$, one of P_L and $P_{\overline{L}}$ will halt in finite number of steps.
- Which one to simulate first?

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- On input x, simulate in parallel P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts

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- On input x, simulate in parallel P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts
- If P_L accepts, accept x and halt. If $P_{\overline{L}}$ accepts, reject x and halt.

Deciding vs. Recognizing

Proof (contd).

In more detail, P works as follows:

```
On input x
for i = 1, 2, 3, ...
    simulate P_L on input x for i steps
    simulate P_L̄ on input x for i steps
    if either simulation accepts, break
if P_L accepted, accept x (and halt)
if P_L̄ accepted, reject x (and halt)
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(Alternately, maintain configurations of P_L and $P_{\overline{L}}$, and in each iteration of the loop advance both their simulations by one step.) □

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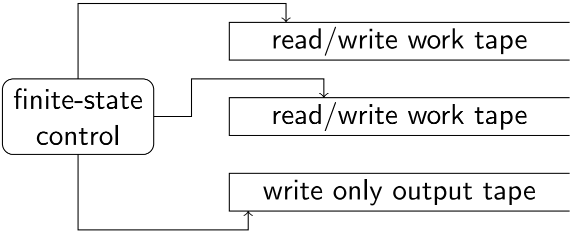
Note: Decidable languages are closed under complementation, but recognizable languages are not.

Enumerators



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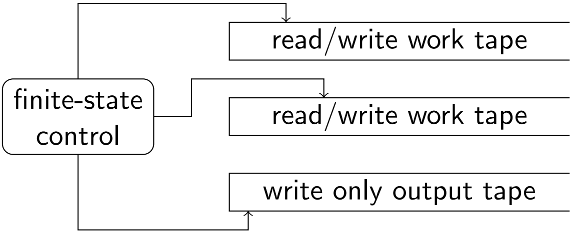

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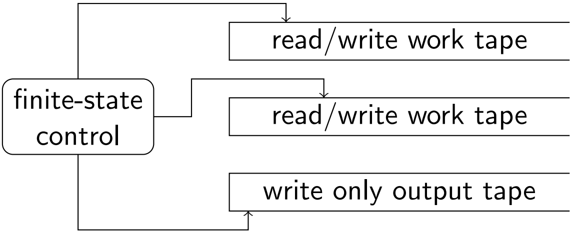

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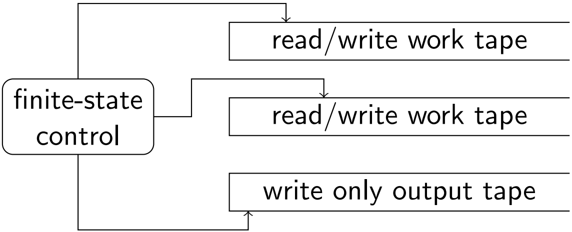

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  - Write-only means (a) symbol on output tape does not affect transitions, and (b) tape head only moves right.
- Intially all tapes blank (no input). During computation the machine adds symbols to the output tape. Output considered to be a list of words (separated by special symbol #)

Recursively Enumerable Languages

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Definition

L is recursively enumerable (r.e.) iff there is an enumerator M such that $L = E(M)$.

Recursively Enumerable Languages and TMs

Theorem

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Note

Hence, when we say a language L is recursively enumerable (r.e.) then

- there is a TM that accepts L, and
- there is an enumerator that enumerates L.

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Clearly, if $w \in L$, M accepts w, and if $w \notin L$ then M never halts.

Enumerators From Recognizers

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        lexicographic order)
    simulate M on w_1 for i steps, then on w_2 for i
        steps and ...simulate M on w_i for i steps
    if M accepts w_j within i steps then write w_j
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Observe that $w \in L(M)$ iff N will enumerates w.

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    simulate M on w_1 for i steps, then on w_2 for i
        steps and ...simulate M on w_i for i steps
    if M accepts w_j within i steps then write w_j
        (with separator) on output tape
```

Observe that $w \in L(M)$ iff N will enumerates w. N will enumerate strings many times! $\square$