Math 243

Q's / Exam ch7 (10 probs + 2)
(12 pts each)

tonight 10pm → thurs 7pm

105 mins

(1) (2) Area from 7.1

(3) (4) Vol. from 7.2

(5) Vol from 7.3

(6) Arc length from 7.4

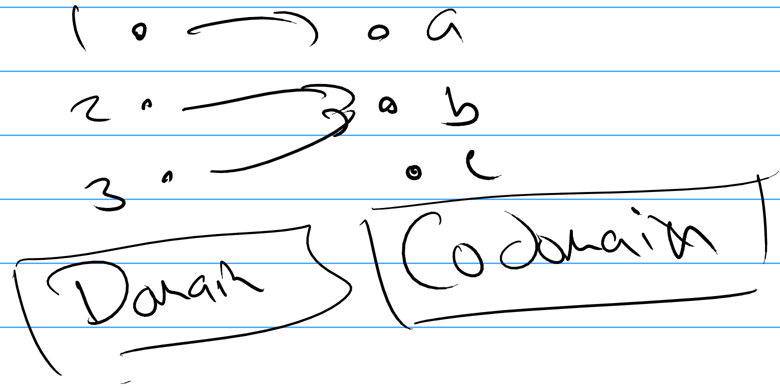
(7) Spring / work
(8) Fluid / work
(9) Cable / work
from 7.5

(10) Diff Eq from 7.6

(11) (12) * (extra credit) Diff Eq.
Center of Mass

8.1

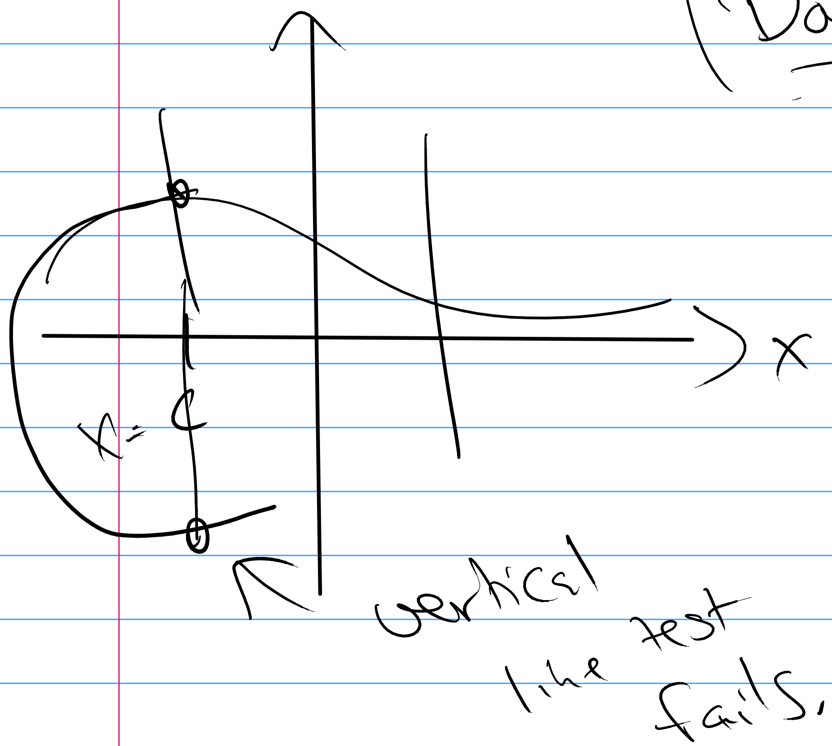
Functions



$f(x) = \frac{1}{x}$

if $x \neq 0$

f is a function



Functions are rules that map elements from domain to elements in the codomain with the following two restrictions

1. it must map <u>all</u> elements of domain
2. each element of domain gets <u>one</u> element from codomain

8.1 Sequences

function whose domain is a subset of the integers. Typically $\{1, 2, 3, \ldots\}$

ex 1 $f(n) = n^2 + 1 \quad n = 1, 2, 3, \ldots$

$f(1) = 2$
$f(2) = 5$
$f(3) = 10$
$\vdots$

ex 2 $f(n)$ = number of letters for the word representing 'n'

$n = 0, 1, 2, 3, \ldots$
$f(0) = 4$
$f(1) = 3$
$f(2) = 3$
$f(3) = 5$
$f(4) = 4$
$\vdots$

Notation: instead of $f(n)$ use a_n

$\{a_n\}_{n=i}^{j} \qquad a_i, a_{i+1}, a_{i+2}, \ldots, a_j$

ex $\left\{\frac{1}{n}\right\}_{n=1}^{4} \qquad \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

ex's $\{n\}_{n=1}^{\infty}$ if no bounds $n=1$ to ∞

$\{1, 2, 3, 4, \ldots, n, \ldots\}$

ones to know ..

$\{n^2\}$ $\{1, 4, 9, 16, 25, \ldots\}$

$\{n^3\}$ $\{1, 8, 27, 64, \ldots\}$

$\{an+b\}$

ex $\{3n+4\}$ $\{7, 10, 13, \ldots\}$

$\{n!\}_{n=0}^{\infty}$ $\{1, 1, 2 \overset{*3}{,} 6 \overset{*4}{,} 24 \overset{*5}{,} 120, \ldots$

Ideas:

seq (rule) $\longrightarrow$ numbers

or numbers $\longrightarrow$ seq (rule)

function!

Types of functions:

① Explicit functions (closed forms)

ex) $a_n = n'_0 + \frac{n^2-1}{n^3}$

$a_n = \frac{1}{n}$

$a_n = \frac{1}{n^2} + n^n$

② Inductive & recursive function

ex) basis: (start values)

$a_0 = 0 \quad a_1 = 1$

Inductive / recursive rule

$a_n = a_{n-1} + a_{n-2} \quad n = 2, 3, \ldots$

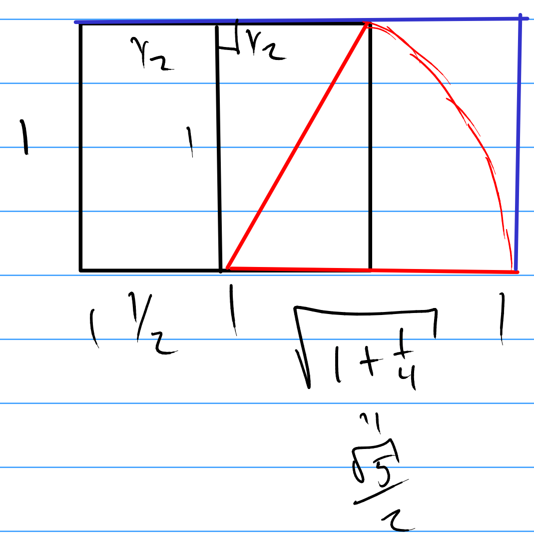
$a_0 = 0, \; a_1 = 1$
$a_2 = a_0 + a_1 = 0 + 1 = 1$
$a_3 = a_1 + a_2 = 1 + 1 = 2$

$\{a_n\} \qquad \{0, 1, 1, 2, 3, \ldots\}$

closed form given a recursive form?

ex $\{f_n\}_{n=0}^{\infty}$ $\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$



length $= \frac{1+\sqrt{2}}{2}$

width $= 1$

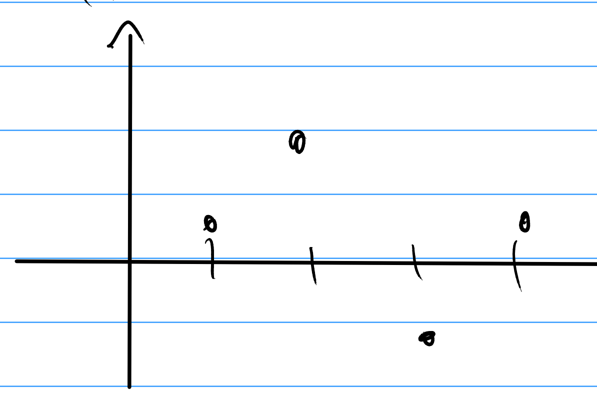
ratio $\boxed{\frac{1+\sqrt{5}}{2}}$

Q's | to now $f(x)$ was our typical function.

Domain : all $\mathbb{R}$

Range : all $\mathbb{R}$

Now: $f(n) = a_n$



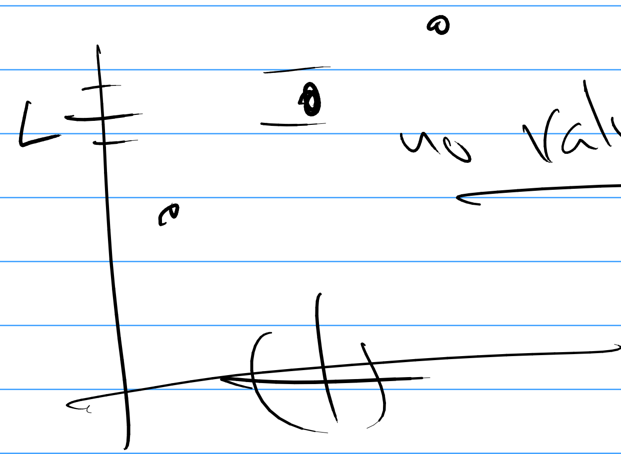
f(x) had limits.

$$\lim_{x \to c} f(x) = L$$

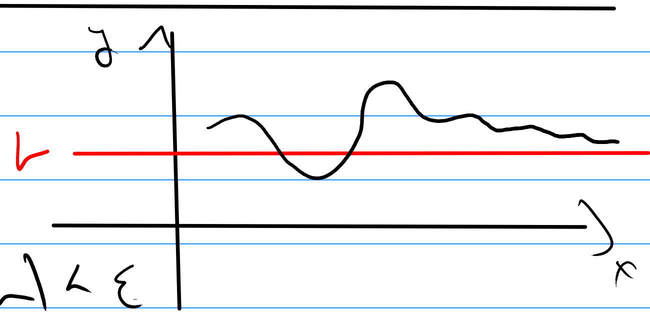
if $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

So does f(n) have a similar limit?

No!



$$\lim_{x \to \infty} f(x) = L$$



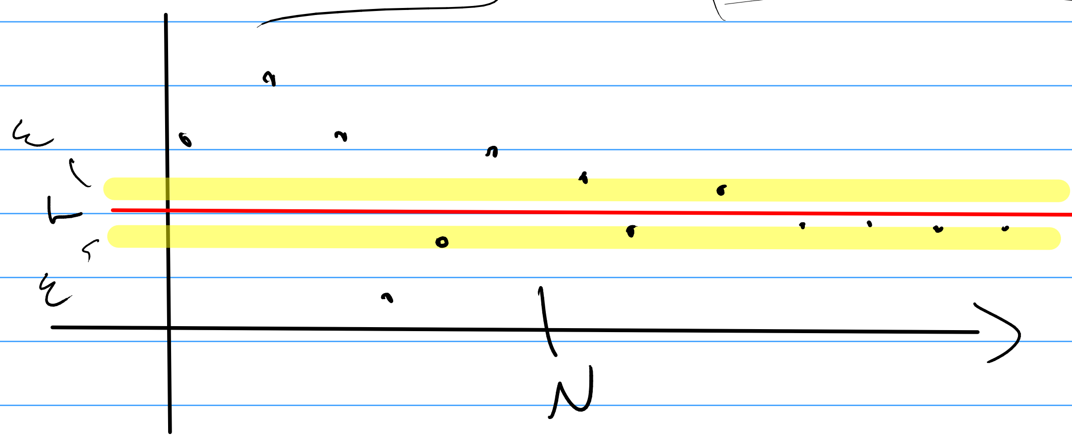
as $x > M \Rightarrow |f(x) - L| < \varepsilon$

for $f(n) = a_n$

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

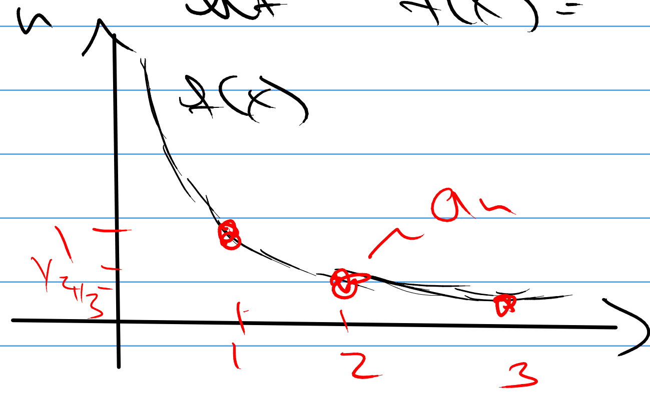
if for all $\varepsilon > 0$ there is a $N \in \mathbb{Z}$

such that if $\boxed{n > N} \to \boxed{|a_n - L| < \varepsilon}$



$\boxed{\text{Thm}}$ if $f(n) = a_n$ where $f(x)$ is a cont. real valued function.

Ex $a_n = \frac{1}{n}$ let $f(x) = \frac{1}{x}$



then if $\boxed{\lim_{x\to\infty} f(x) = L}$

$$\to \lim_{n\to\infty} a_n = L$$

$\lim_{n\to\infty} \frac{1}{n}$ b/c $f(x) = \frac{1}{x}$ is $\frac{1}{n}$ @ n

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\to \lim_{n\to\infty} \frac{1}{n} = 0$$

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① $\boxed{L \text{ is finite.}}$

$\lim_{n\to\infty} a_n = L$

if limit exists

→ convergent $\{a_n\}$

Otherwise

→ divergent $\{a_n\}$

② $\boxed{L \text{ is infinite}}$

$\lim_{n\to\infty} a_n = \infty$ for all $M \in \mathbb{Z}^+$ there is $N \in \mathbb{Z}$ such that

if $n > N \to a_n > M$.

if $\lim_{n\to\infty} a_n = \infty$ $\{a_n\}$ is divergent

$\boxed{\{a_n\}}$ $\{a_n\}$ and $\{b_n\}$ are convergent

① $\lim_{n\to\infty} \{c_1 a_n + c_2 b_n\}$

$$= c_1 \lim_{n\to\infty} a_n + c_2 \lim_{n\to\infty} b_n$$

② $\lim_{n\to\infty} \{c_1\} = c_1$

③ $\lim_{n\to\infty} \{a_n b_n\} = \lim_{n\to\infty} \{a_n\} \cdot \lim_{n\to\infty} \{b_n\}$

④ $\lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = \frac{\lim_{n\to\infty} \{a_n\}}{\lim_{n\to\infty} \{b_n\}}$

⑤ $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$

$p > 0$ $a_n > 0$

$\boxed{\text{Squeeze th}^{m}}$ $\quad a_n \le b_n \le c_n$

for $\quad n \ge n_0$

if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L \to \lim\limits_{n\to\infty} b_n = L$

$\boxed{\text{Corr}}$ if $\lim\limits_{n\to\infty} |a_n| = 0 \to \lim\limits_{n\to\infty} a_n = 0$

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ex's $\quad a_n = \dfrac{n+1}{3n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{3n-1} = \lim_{n\to\infty} \frac{\cancel{1}^{1} + \cancel{1/n}^{0}}{\cancel{3}^{3} - \cancel{1/n}^{0}}$$

$$= \boxed{\frac{1}{3}}$$

ex $\quad \displaystyle\lim_{n\to\infty} \frac{n^2+3n-1}{n^9-2n+1} = \lim_{n\to\infty} \frac{\cancel{1}^{1} + \cancel{3/n}^{0} - \cancel{1/n^2}^{0}}{\cancel{n^7}^{\infty} - \cancel{2/n}^{0} + \cancel{1/n^2}^{0}}$

$$= \lim_{n\to\infty} \frac{1}{n^7} = 0$$

ex $$\lim_{n\to\infty} \frac{(e^n+e^{-n})/e^{2n}}{(e^{2n}-1)/e^{2n}} = \lim_{n\to\infty} \frac{e^{-n}+e^{-3n}}{1-e^{-2n}}$$

Note: $\lim_{n\to\infty} r^n$ converges only if $-1 < r \le 1$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1<r<1 \\ 1 & r=1 \end{cases}$$

$$= \lim_{n\to\infty} \frac{\cancel{\left(\frac{1}{e}\right)^n}^{\,0} + \cancel{\left(\frac{1}{e^3}\right)^n}^{\,0}}{1-\cancel{\left(\frac{1}{e^2}\right)^n}^{\,0}} = \frac{0}{1} = 0$$

Def: ① $a_n < a_{n+1}$ $(n \ge 1)$

$\{a_n\}$ is increasing

② $a_n > a_{n+1}$ $(n \ge 1)$ Monotone

$\{a_n\}$ is decreasing

③ $a_n \le M$ $(n \ge 1)$ bounded above

④ $a_n \ge m$ $(n \ge 1)$ bounded below

bounded seq.

$\boxed{\text{Thm}}$ $\{a_n\}$ is montonic and bounded <u>then</u> it is convergent.

Ex $\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \right\}$

$\left\{ 2^{1/2}, 2^{3/4}, 2^{7/8}, 2^{15/16}, \ldots \right\}$

$$\left\{ 2^{\frac{2^n-1}{2^n}} \right\}$$

$$\lim_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = ?$$

$$\lim_{x\to\infty} 2^{\frac{2^x-1}{2^x}} = 2^{\lim_{x\to\infty} \frac{2^x-1}{2^x}} = 2^1$$

b/c $\frac{2^x-1}{2^x} = \frac{2^n-1}{2^n}$ on $n \in \mathbb{Z}$

$$\therefore \lim_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = \boxed{2}$$

Series – adding the terms of a sequence.

$\{a_n\}_{n=1}^{\infty}$ $\{a_1, a_2, a_3, \ldots\}$

infinite series. $a_1 + a_2 + a_3 + \cdots$

$$\sum_{n=1}^{\infty} a_n$$

How to add an ∞ number of terms?

We can do partial sums.

$$S_t = \sum_{n=1}^{t} a_n = a_1 + a_2 + \cdots + a_t$$

$\uparrow$ value of partial sum @ t

b/c S_t is function on integers

it is a seq. $\{S_t\}$

$\nwarrow$ seq. of partial sums

If $S_t \to S$ as $t \to \infty$ (Limit of a seq)

then $\sum_{n=1}^{\infty} a_n = S$ and series is convergent if $\{S_t\}$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

(ex) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4}$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

subtract

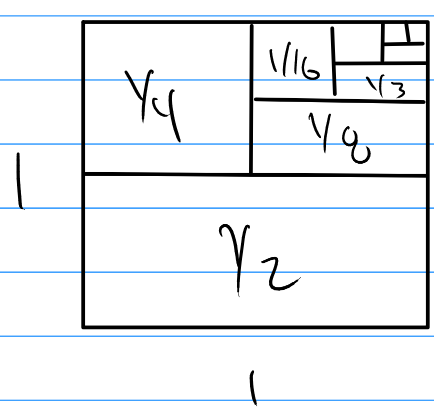
$$S_t = \frac{1}{2} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cancel{\cdots} + \cancel{\frac{1}{2^t}}$$
$$-\ \frac{1}{2}S_t = \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cancel{\frac{1}{16}} + \cdots + \cancel{\frac{1}{2^t}} + \frac{1}{2^{t+1}}$$

$$\frac{1}{2}S_t = \frac{1}{2} - \frac{1}{2^{t+1}}$$

mult. by 2

$$S_t = 1 - \frac{1}{2^t}$$

$$\lim_{t\to\infty} S_t = \lim_{t\to\infty} 1 - \cancel{\frac{1}{2^t}}^{\,\to 0} = 1$$



graphical

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

(*) $$\sum_{n=1}^{\infty} ar^{n+1} = \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \text{divergent} & |r| \geq 1 \end{cases}$$

$\boxed{\text{Geometric Series}}$

Telescoping Series.

(ex) $$\sum_{n=1}^{\infty} (a_{n+1} - a_n) = (\cancel{a_2} - a_1) + (\cancel{a_3} - \cancel{a_2}) + (\cancel{a_4} - \cancel{a_3}) + (\cancel{a_5} - \cancel{a_4}) + \cdots$$

$$\sum_{n=2}^{t} \frac{2}{n^2-1} = \sum_{n=2}^{t} \frac{2}{(n+1)(n-1)}$$

$$= \sum_{n=2}^{t} \frac{A}{n+1} + \frac{B}{n-1}$$

$$2 = A(n-1) + B(n+1)$$

$$A = -1 \qquad B = 1$$

$$\sum_{n=2}^{t} \frac{2}{n^2-1} = \sum_{n=2}^{t} \frac{-1}{n+1} + \frac{1}{n-1}$$

$$= \left[-\cancel{\frac{1}{3}} + 1 \right] + \left[-\cancel{\frac{1}{4}} + \frac{1}{2} \right] + \left[-\cancel{\frac{1}{5}} + \cancel{\frac{1}{3}} \right]$$

$$+ \left[-\cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} \right] + \left[-\cancel{\frac{1}{7}} + \cancel{\frac{1}{5}} \right] + \cdots$$

$$\cdots + \left[\frac{-1}{t} + \cancel{\frac{1}{t-2}} \right] + \left[\frac{-1}{t+1} + \cancel{\frac{1}{t-1}} \right]$$

$$S_t = 1 + \frac{1}{2} - \frac{1}{t} - \frac{1}{t+1}$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \boxed{\frac{3}{2}}$$

$$0.99\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^{n-1}\left(\frac{1}{10}\right)$$

geo. series $\displaystyle\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$

$$\sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{9/10}{1-1/10} = \frac{9/10}{9/10} = 1$$

Divergence

Harmonic $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$ $\qquad > \frac{1}{4} + \frac{1}{4} = 1/2$

$S_3 = 1 + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}}$

$> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$

$S_t = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{t-1}}$

$\rightarrow S_3 > 1 + \frac{1}{2} + \frac{1}{2}$

$S_4 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\vdots$

$S_t > 1 + \left(\frac{1}{2}\right)(t-1)$

$\lim_{t\to\infty} 1 + \frac{t-1}{2} = \infty$ divergent

$\rightarrow \lim_{t\to\infty} S_t = \infty$ divergent if $\sum a_n$ is convergent $\qquad p \to q$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \equiv$

$\to \lim_{n\to\infty} a_n = 0 \qquad \neg q \to \neg p$

$$\boxed{\text{th}^m \quad \text{if } \lim_{n\to\infty} a_n \neq 0 \text{ then } \sum a_n \text{ is divergent}}$$

divergence test

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$

use divergence test

$$\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges

ex $\displaystyle\sum_{n=1}^{\infty} \left(\frac{8}{10}\right)^{n-1} - \left(\frac{3}{10}\right)^{n}$

use divergence test $\displaystyle\lim_{n\to\infty} \left(\frac{8}{10}\right)^{n-1} - \left(\frac{3}{10}\right)^{n} = 0$

test fails. (don't know conv/div)

Laws of Conv. Series

$\sum a_n$ $\sum b_n$ are conv. $c_1 \in \mathbb{R}$
$c_2 \in \mathbb{R}$

$$\boxed{\sum c_1 a_n + c_2 b_n = c_1 \sum a_n + c_2 \sum a_n}$$

$$\sum_{n=1}^{\infty} (.8)^{n-1} - (.3)^n$$

$$= \sum_{n=1}^{\infty} \underbrace{1 \cdot (.8)^{n-1}}_{a \cdot r^{n-1}} - \underbrace{(.3) \cdot (.3)^{n-1}}_{ar^{n-1}}$$

$|r| < 1$ conv. geo. series.

$$= \boxed{\frac{1}{.2} - \frac{.3}{.7}}$$