CIS 770: Formal Language Theory

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Optimal Algorithms

Best Solutions

If a problem can be solved computationally, is there always a "best" method for solving it?

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Example

Merge Sort and Heap Sort achieve the assymptotically best time possible for sorting (in a certain model).

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Manuel Blum

Best Solutions

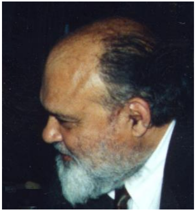
If a problem can be solved computationally, is there always a "best" method for solving it?

Example

Merge Sort and Heap Sort achieve the assymptotically best time possible for sorting (in a certain model).

Blum's Speedup Theorem

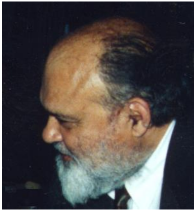
No! There are computationally solvable problems that have no optimal algorithms! In other words, there are problems for which any algorithm can always be made faster.



Anil Nerode

Myhill-Nerode Theorem

There is a "unique" "optimal" "algorithm" for every problem that can be solved using finite memory.



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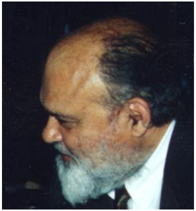

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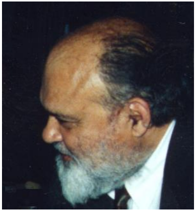

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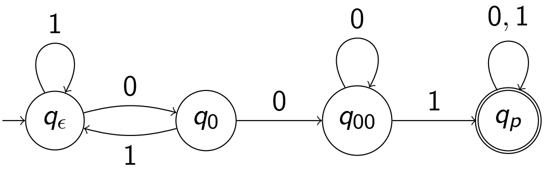
There is a "unique" "optimal" "algorithm" for every problem that can be solved using finite memory.

- "algorithm" here means a deterministic machine
- "optimal" means requires least memory, i.e., has fewest states
- "unique" means that any two DFAs with fewest states for a language are "isomorphic"

Suffix Language of a State



DFA M

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$,
$\text{suffix}(M, q) = \{w \in \Sigma^* \mid \hat{\delta}(q, w) \in F\}$.

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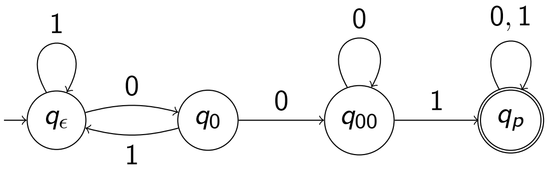

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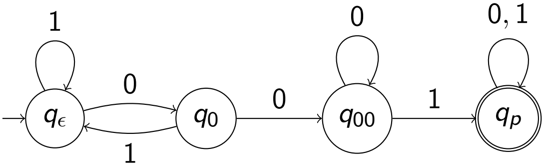

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For example, $\mathrm{suffix}(M, q_{00}) =$

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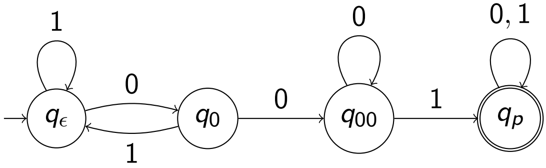

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For example, $\mathrm{suffix}(M, q_{00})$ $=0^*1(0 \cup 1)^*$.

Suffix Languages

Definition

For a language $L \subseteq \Sigma^*$, and a string $x \in \Sigma^*$, the suffix language of L with respect to x, is defined as

$$\mathrm{suffix}(L, x) = \{y \in \Sigma^* \mid xy \in L\}$$

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The class of suffix languages of L is

$$\mathcal{C}_{\mathrm{suf}}(L) = \{\mathrm{suffix}(L, x) \mid x \in \Sigma^*\}$$

Example: L_{odd}

Example

Consider $L_{\mathrm{odd}} = \{w \in \{0,1\}^* \mid w \text{ has an odd number of 1s}\}$

- $\mathrm{suffix}(L_{\mathrm{odd}}, \epsilon) =$

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Example: L_{odd}

Class of Suffix Languages

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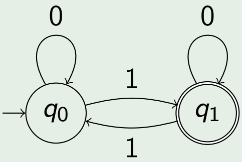
- For x that has an odd number of 1s, $\mathrm{suffix}(L_{\mathrm{odd}}, x) = L_{\mathrm{even}}$
- Thus, $\mathcal{C}_{\mathrm{suf}}(L_{\mathrm{odd}}) = \{L_{\mathrm{odd}}, L_{\mathrm{even}}\}$

Example

Recall, $\mathcal{C}_{\text{suf}}(L_{\text{odd}}) = \{L_{\text{odd}}, L_{\text{even}}\}$. A DFA for L_{odd} is



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DFA and Suffix Languages

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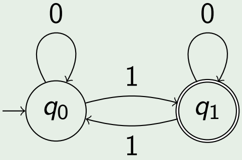

DFA for L_{odd}

Observe that $\mathrm{suffix}(M, q_0) = L_{\mathrm{odd}}$, and $\mathrm{suffix}(M, q_1) = L_{\mathrm{even}}$.

Example: L_{001}

Example

Consider $L_{001} = (0 \cup 1)^*001(0 \cup 1)^*$.

- $\mathrm{suffix}(L_{001}, \epsilon) =$

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- For $x \in L_{001}$, $\mathrm{suffix}(L_{001}, x) = (0 \cup 1)^*$

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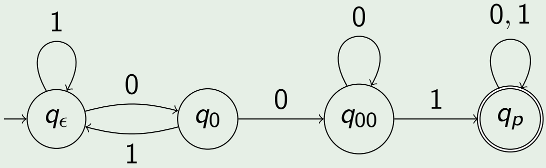
A DFA for L_{001} is



DFA for L_{odd}

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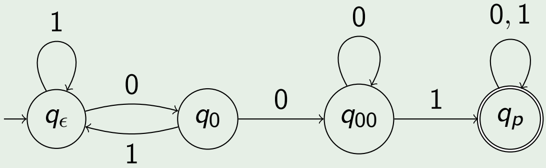

DFA for L_{odd}

Observe that the suffix languages of the states correspond to the class of suffix languages.

Example: L_{0n1n}

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Consider $L_{0n1n} = \{0^n 1^n \mid n \geq 0\}$.

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- $\mathrm{suffix}(L_{0n1n}, 0^i) = \{0^{n-i}1^n \mid n \geq i\}$

Proposition

$\mathcal{C}_{\mathrm{suf}}(L_{0n1n})$ *has infinitely many languages.*

Proof.

Observe that for $i \neq j$, $\mathrm{suffix}(L_{0n1n}, 0^i) \neq \mathrm{suffix}(L_{0n1n}, 0^j)$. $\square$

Recap . . .

Observations

In the previous examples,

Recap . . .

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- For regular L,

Recap . . .

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  - $\mathcal{C}_{\mathrm{suf}}(L)$ has only finitely many languages

Recap ...

Observations

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  - There is a DFA D for L, the suffix languages of the states of D, correspond to the languages in $\mathcal{C}_{\mathrm{suf}}(L)$

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- For non-regular L,
  - $\mathcal{C}_{\mathrm{suf}}(L)$ has infinitely many languages

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Observations

In the previous examples,

- For regular L,
  - $\mathcal{C}_{\mathrm{suf}}(L)$ has only finitely many languages
  - There is a DFA D for L, the suffix languages of the states of D, correspond to the languages in $\mathcal{C}_{\mathrm{suf}}(L)$
- For non-regular L,
  - $\mathcal{C}_{\mathrm{suf}}(L)$ has infinitely many languages

Are these observations true in general?

States and Suffix Languages

Proposition

Let $M = (Q, \Sigma, \delta, q_0, F)$ *and let* $L = L(M)$*. Then if* $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$ *(i.e., both* x *and* y *take* M *to the same state), then* $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$*.*

Proof.

$\square$

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Proof.

Consider $z \in \mathrm{suffix}(L, x)$. Then $xz \in L$.

$\square$

States and Suffix Languages

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Let $M = (Q, \Sigma, \delta, q_0, F)$ and let $L = L(M)$. Then if $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$ (i.e., both x and y take M to the same state), then $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$.

Proof.

Consider $z \in \mathrm{suffix}(L, x)$. Then $xz \in L$.

$$\hat{\delta}(q_0, xz)(\in F) \quad = \hat{\delta}(\hat{\delta}(q_0, x), z) \qquad \text{(prop. } \hat{\delta}(q, uv) = \hat{\delta}(\hat{\delta}(q, u), v)\text{)}$$

□

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&= \hat{\delta}(\hat{\delta}(q_0, y), z) && (\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y))
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$\square$

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\end{aligned}$$

Similarly, we can show that if $z \in \mathrm{suffix}(L, y)$ then $z \in \mathrm{suffix}(L, x)$. $\square$

Regularity and Suffix Languages

Corollary

If L is regular then $\mathcal{C}_{\mathrm{suf}}(L)$ is finite.

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If L is regular then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(M)$.

$\square$

Regularity and Suffix Languages

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If L is regular then $\mathcal{C}_{\rm suf}(L)$ is finite.

Proof.

If L is regular then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(M)$.

- We have shown that, if x, y reach the same state in M then $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$

$\square$

Regularity and Suffix Languages

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If L is regular then $\mathcal{C}_{\mathrm{suf}}(L)$ is finite.

Proof.

If L is regular then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(M)$.

- We have shown that, if x, y reach the same state in M then $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$
- Thus, $|\mathcal{C}_{\mathrm{suf}}(L)| \leq |Q|$, which is finite.

$\square$

Canonical DFAs for Regular Languages

Proposition

For a language $L \subseteq \Sigma^*$, if $\mathcal{C}_{\mathrm{suf}}(L)$ is finite then there is a DFA M^L such that $L(M^L) = L$.

Canonical DFAs for Regular Languages

Proposition

For a language $L \subseteq \Sigma^*$, if $\mathcal{C}_{\mathrm{suf}}(L)$ is finite then there is a DFA M^L such that $L(M^L) = L$.

Proof.

Define $M^L = (Q^L, \Sigma, \delta^L, q_0^L, F^L)$ as follows.

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Is δ well-defined?

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Is δ well-defined? Same state can have multiple names (i.e., x, y s.t. $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$). $\cdot\!\cdot\!\longrightarrow$

Canonical DFAs for Regular Languages

Transition function is well-defined

Proof (contd).

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- $\delta^L(\mathrm{suffix}(L, x), a) = \mathrm{suffix}(L, xa)$
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Canonical DFAs for Regular Languages

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$$\begin{aligned} z \in \mathrm{suffix}(L, xa) &\iff xaz \in L \\ &\iff az \in \mathrm{suffix}(L, x) = \mathrm{suffix}(L, y) \\ &\iff yaz \in L \iff z \in \mathrm{suffix}(L, ya) \end{aligned}$$

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  - Hence, $\mathrm{suffix}(L, xa) = \mathrm{suffix}(L, ya)$.

Canonical DFAs for Regular Languages

Proof (contd).

- For any string $x \in \Sigma^*$, $\hat{\delta}_{M^L}(q_0^L, x) = \mathrm{suffix}(L, x)$.

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- For any string $x \in \Sigma^*$, $\hat{\delta}_{M^L}(q_0^L, x) = \mathrm{suffix}(L, x)$.
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- $x \in L$ iff $\epsilon \in \mathrm{suffix}(L, x)$

Canonical DFAs for Regular Languages

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Proof (contd).

- For any string $x \in \Sigma^*$, $\hat{\delta}_{M^L}(q_0^L, x) = \text{suffix}(L, x)$.
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  - Formally, by induction on $|x|$
- $x \in L$ iff $\epsilon \in \text{suffix}(L, x)$ iff $\text{suffix}(L, x) \in F^L$
- Hence, $x \in L$ iff M^L accepts x. $\square$

Example of Canonical DFA

Example

Consider $L_{001} = (0 \cup 1)^* 001 (0 \cup 1)^*$. Recall that the suffix languges are

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L_1 &= L_{001} = \mathrm{suffix}(L_{001}, \epsilon) = \mathrm{suffix}(L_{001}, (0 \cup \epsilon)1) \\
L_2 &= 00^*1(0 \cup 1)^* \cup 1L_{001} = \mathrm{suffix}(L_{001}, 0) \\
L_3 &= 0^*1(0 \cup 1)^* = \mathrm{suffix}(L_{001}, 00) = \mathrm{suffix}(L_{001}, 000) \\
L_4 &= (0 \cup 1)^* = \mathrm{suffix}(L_{001}, 001) = \mathrm{suffix}(L_{001}, 001(0 \cup 1))
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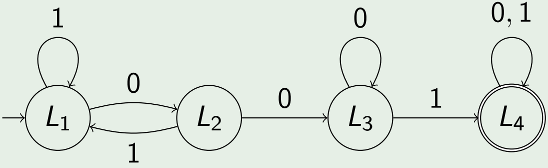

Canonical DFA is the smallest DFA

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For any regular language L, M^L is the unique smallest DFA that recognizes L upto isomorphism.

Isomorphism

Definition

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be two DFAs. A function $f : Q_1 \to Q_2$ is said to be isomorphism iff

- f is bijective, i.e., one-to-one and onto
- $f(q_1) = q_2$
- For every $p \in Q_1$ and $a \in \Sigma$, $f(\delta_1(p, a)) = \delta_2(f(p), a)$
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M_1 and M_2 are said to be isomorphic if there is an isomorphism f from M_1 to M_2.
Thus, if M_1 and M_2 are isomorphic then they are the "same" machine except for possibly renaming states.

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Recall $M^L = (Q^L, \Sigma, \delta^L, q_0^L, F^L)$. Let $M = (Q, \Sigma, \delta, q_0, F)$ be some DFA such that $L(M) = L$.

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  - f is well-defined, because we showed that if x, y both take M to q, then $\mathrm{suffix}(L, x) = \mathrm{suffix}(L, y)$.

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- Thus, $|Q| \geq |Q^L|$ $\cdots\!\longrightarrow$

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Canonical DFA is the smallest DFA

f preserves transitions

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- Suppose $q \in F$, and $f(q) = \mathrm{suffix}(L, x)$. Hence, $\hat{\delta}_M(q_0, x) = q$, and so $x \in L$, and $\epsilon \in \mathrm{suffix}(L, x)$.

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- Suppose $q \in F$, and $f(q) = \mathrm{suffix}(L, x)$. Hence, $\hat{\delta}_M(q_0, x) = q$, and so $x \in L$, and $\epsilon \in \mathrm{suffix}(L, x)$. Thus, $f(q) \in F^L$. The converse holds by reversing this argument, and so f is an isomorphism. $\square$

Myhill-Nerode Theorem

In summary . . .

Theorem

The following facts are true.

1. *If L is regular then $\mathcal{C}_{\mathrm{suf}}(L)$ is finite.*

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2. *If $\mathcal{C}_{\mathrm{suf}}(L)$ is finite then L is regular; more precisely, there is a DFA M^L, whose states are the languages in $\mathcal{C}_{\mathrm{suf}}(L)$, such that $L(M^L) = L$.*
3. *For any regular language L, M^L is the unique (upto isomorphism) DFA with fewest states that recognizes L.*

Minimization

Problem

Given a DFA M, construct the DFA with fewest states M' such that $L(M') = L(M)$.

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Applications

Algorithms using DFAs run in time directly related to the number of states of DFA. Implementation of the DFA itself takes memory proportional to log number of states. So constructing small DFAs is very critical.

Ideas

Let $M = (Q, \Sigma, \delta, q_0, F)$, and $L = L(M)$. Recall that from the proof of Myhill-Nerode Theorem

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Ideas

Let $M = (Q, \Sigma, \delta, q_0, F)$, and $L = L(M)$. Recall that from the proof of Myhill-Nerode Theorem

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Thus M can be "minimized" by collapsing states q_1 and q_2 if $\mathrm{suffix}(M, q_1) = \mathrm{suffix}(M, q_2)$.

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We will say that p and q are distinguishable when this happens.

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Distinguishability

Inductive Definition

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- If $p \in F$ and $q \notin F$ then p and q are distinuishable
- If for some a, $\delta(p, a) = p'$ and $\delta(q, a) = q'$, and p' and q' are distinguishable, then p and q are distinguishable

Let `distinct` be a table with an entry for each pair of states. Initially all entries are 0.

```
if p ∈ F and q ∉ F (or vice versa)
then distinct(p, q) := 1
repeat
for each pair (p,q) and symbol a
    if distinct(δ(p,a), δ(q,a)) = 1,
    then distinct(p, q) := 1
until no changes in table
```

Minimization Algorithm

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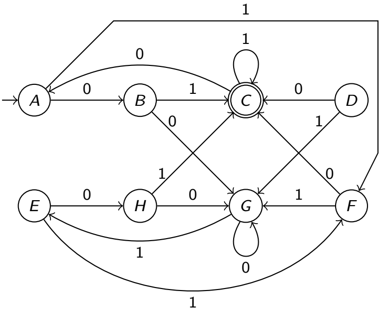
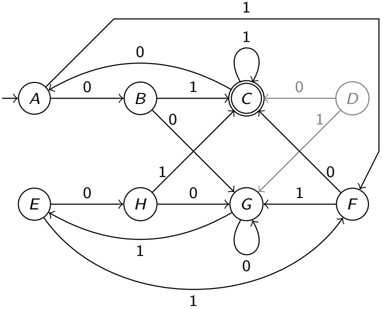
3. Collapse pairs that are not distinguishable

Example

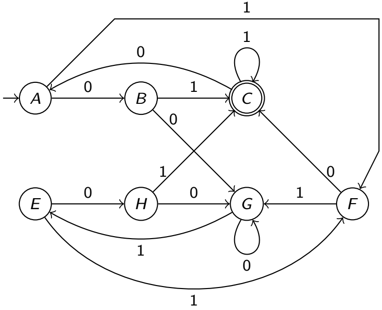


Example



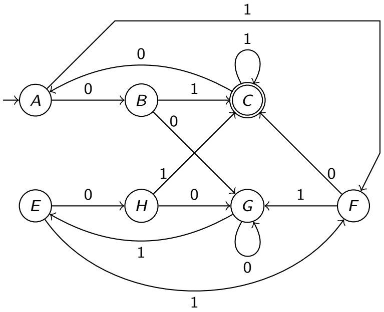
Example

	A	B	C	E	F	G
B						
C	⋆	⋆				
E			⋆			
F			⋆			
G			⋆			
H			⋆			



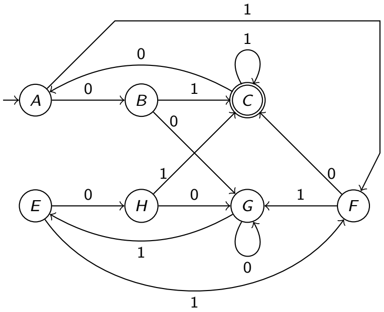
Example

	A	B	C	E	F	G
B	⋆					
C	⋆	⋆				
E		⋆	⋆			
F	⋆	⋆	⋆	⋆		
G		⋆	⋆		⋆	
H	⋆		⋆	⋆	⋆	⋆



Example

	A	B	C	E	F	G
B	⋆					
C	⋆	⋆				
E		⋆	⋆			
F	⋆	⋆	⋆	⋆		
G	⋆	⋆	⋆	⋆	⋆	
H	⋆		⋆	⋆	⋆	⋆



Example

	A	B	C	E	F	G
B	⋆					
C	⋆	⋆				
E		⋆	⋆			
F	⋆	⋆	⋆	⋆		
G	⋆	⋆	⋆	⋆	⋆	
H	⋆		⋆	⋆	⋆	⋆