# Math 293

$$\int (x^2+a^2)^n\,dx$$

let $u=(x^2+a^2)^n \rightarrow du = 2nx(x^2+a^2)^{n-1}\,dx$

$dv = dx \longrightarrow v = x$

$$= x(x^2+a^2)^n - 2n\underbrace{\boxed{\int x^2(x^2+a^2)^{n-1}\,dx}}$$

?

Stuck?

(or)

$$\int (x^2+a^2)^n\,dx = \int (x^2+a^2)(x^2+a^2)^{n-1}\,dx$$

$$\int (x^2+a^2)^n\,dx = \int x^2(x^2+a^2)^{n-1}\,dx + a^2\int (x^2+a^2)^{n-1}\,dx$$

$$\int (x^2+a^2)^n\,dx - a^2\int (x^2+a^2)^{n-1}\,dx = \int x^2(x^2+a^2)^{n-1}\,dx$$

take pt 1

$$\int (x^2+a^2)^n\,dx = x(x^2+a^2)^n - 2n\underbrace{\int x^2(x^2+a^2)^{n-1}\,dx}_{\text{by pt 2 } \updownarrow =}$$

$$\overline{\int (x^2+a^2)^n\,dx - a^2\int (x^2+a^2)^{n-1}\,dx}$$

$$\int (x^2+a^2)^n\,dx = x(x^2+a^2)^n - 2n\left[\int (x^2+a^2)^n\,dx - a^2\int (x^2+a^2)^{n-1}\,dx\right]$$

$$\int (x^2+a^2)^n dx = x(x^2+a^2)^n - 2n\int (x^2+a^2)^n dx + 2a^2 n\int (x^2+a^2)^{n-1} dx$$

$$\int (x^2+a^2)^n dx + 2n\int (x^2+a^2)^n dx = x(x^2+a^2)^n + 2a^2 n\int (x^2+a^2)^{n-1} dx$$

$$(1+2n)\int (x^2+a^2)^n dx = x(x^2+a^2)^n + 2a^2 n\int (x^2+a^2)^{n-1} dx$$

$$\int (x^2+a^2)^n dx = \frac{x(x^2+a^2)^n}{(1+2n)} + \frac{2a^2 n}{(1+2n)}\int (x^2+a^2)^{n-1} dx$$

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6.3

① When to use Substitution?

$$\int f(\underbrace{g(x)}) \cdot \underbrace{g'(x)} dx$$

You see a function (inside?)

<u>and</u> a product of its derivative.

② When to use "by Parts"?

$$\int f(x) \cdot g'(x) dx$$

You see a product of two functions

<u>and</u> ① one is differentiable (simplifies?)

② one can be integrated (simplifies?)

③ **Trig**:

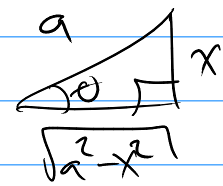
A) Use trig identities & substitution.

B) $\sqrt{a^2+x^2}$ or $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$

use a trig substitution for this one

ex $x = a\sin\theta$

$\sin\theta = \frac{x}{a}$

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6.3 Partial Fraction Decomposition

**Idea**: $\int \frac{\text{polynomial}}{\text{polynomial}}\,dx = ?$

$$\int \frac{b+a}{ab}\,dx = \int \frac{1}{a}\,dx + \int \frac{1}{b}\,dx$$

b/c $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{a \cdot b}$

So can you decompose $\frac{P(x)}{Q(x)}$ into a sum of partial fractions based on the factors of $Q(x)$?

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x^1 + a_0$$

1. $n$-roots in $\mathbb{C}$
2. factors are $(ax+b)$ or $(ax^2+bx+c)$

How to decompose?

1. degree $P(x) \geq$ degree $Q(x)$

$\rightarrow$ divide them

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \qquad \deg(R) < \deg(Q)$$

$$\frac{x^3+3x+1}{x^3-4}$$

$$\begin{array}{r} 1 \\ x^3+0x^2+0x-4 \,\big)\, \overline{x^3+0x^2+3x+1} \\ -\underline{\left(x^3+0x^2+0x-4\right)} \\ 3x+5 \end{array}$$

$$\frac{x^3+3x+1}{x^3-4} = 1 + \boxed{\frac{3x+5}{x^3-4}}$$

2. $\deg(P(x)) < \deg(Q(x))$

a) Factor $Q(x)$

ex: $Q(x) = x^3 - \left(\sqrt[3]{4}\right)^3 = \underbrace{\left(x - \sqrt[3]{4}\right)}\underbrace{\left(x^2 + \sqrt[3]{4}\,x + 4^{2/3}\right)}$

$x^3 - a^3$

$$
\begin{array}{r}
x^2 + ax + a^2 \\
(x-a)\,\overline{)\,x^3 + 0x^2 + 0x - a^3} \\
\underline{x^3 - ax^2} \\
ax^2 + 0x \\
\underline{ax^2 - a^2x} \\
a^2x - a^3 \\
\underline{a^2x - a^3} \\
0
\end{array}
$$

$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

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a) continued

If you do factor $Q(x)$ you get

① Linear factors $(ax+b)^n$

② Quadratic factors $(ax^2+bx+c)^n$

irreducible

b) $\dfrac{P(x)}{Q(x)}$

if $(ax+b)^n$ is a factor

partial fractions: $\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$

if $(ax^2+bx+c)^n$ is a factor

<u>partial fractions</u>: $\dfrac{A_1x+B_1}{(ax^2+bx+c)} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

with <u>all</u> $A_i, B_i$ being constants you need to find.

③ Solve for the constants

ex $$\frac{x^2+1}{(x+1)(x-2)^2} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$\frac{x^2+1}{(x+1)(x-2)^2} = \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$

$$x^2+1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

→ let $x=2$ $\qquad C = 5/3$

→ let $x=-1$ $\qquad A = 2/9$

$$x^2+1 = 2/9(x^2-4x+4) + B(x^2-x-2) + 5/3(x+1)$$

→ let $x=0$ $\quad 1 = 8/9 - 2B + 5/3 \qquad B = 7/9$

$$\frac{x^2+1}{(x+1)(x-2)^2} = \frac{2/9}{(x+1)} + \frac{7/9}{(x-2)} + \frac{5/3}{(x-2)^2}$$

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$$x^2+1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

as a system

$$x^2+1 = Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$$

$$\begin{array}{l} \underline{\underline{x^2}} \\ \underline{\underline{x}} \\ \text{const} \end{array} \begin{bmatrix} 1 = A + B \\ 0 = -4A - B + C \\ 1 = 4A - 2B + C \end{bmatrix} \leftarrow \text{system of linear eqns.}$$

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ex $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{a}{x-b}\,dx \quad (x \neq 0)$

$$x^2 - bx = (x)(x-b)$$

$$\frac{ax}{x^2-bx} = \frac{A}{x} + \frac{B}{x-b} = \frac{0}{x} + \frac{a}{x-b}$$

$$\frac{ax}{x^2-bx} = \frac{A(x-b)+Bx}{x(x-b)}$$

$$ax = A(x-b) + Bx$$

let $x=0$ $\quad 0 = -Ab \quad A = 0$

let $x=b$ $\quad ab = Bb \rightarrow B = a$

$$\frac{ax}{x^2-bx} = \frac{a\cancel{x}}{\cancel{x}(x-b)} = \frac{a}{x-b}$$

$$\int \frac{a}{x-b}\,dx = a\int \frac{1}{u}\,du = a\ln|u|+C$$

$$= \boxed{a\ln|x-b|+C}$$

let $u = x-b$

$du = dx$

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$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \frac{x^2+2x-1}{x(x+1)(x-1)}\,dx$$

$$\frac{x^2+2x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$x^2+2x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

let $x=0$ $\quad A=1$

let $x=-1$ $\quad B=-1$

let $x=1$ $\quad C=1$

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{x-1}\,dx$$

$$= \ln|x| - \ln|x+1| + \ln|x-1| + C$$

$$= \ln\left(\frac{|x|\,|x-1|}{|x+1|}\right) + C$$

$$= \ln\left|\frac{x(x-1)}{(x+1)}\right| + C$$

verify:

$$\left[\frac{\cancel{x+1}}{x(x-1)}\right] \cdot \frac{[(x-1)+x](x+1) - x(x-1)(1)}{(x+1)^{\cancel{2}}}$$

$$= \frac{(2x-1)(x+1) - x^2 + x}{x(x^2-1)}$$

$$= \frac{2x^2 + x - 1 - x^2 + x}{x^3 - x}$$

$$= \frac{x^2 + 2x - 1}{x^3 - x} \quad \checkmark \quad \text{ok!}$$

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$$f'(x) = \frac{e^x}{x} \qquad f(x) \leftarrow \text{Not elementary}$$

Anti Derivative

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Can't do it by hand?

① Tables of Integrals.

ex: $\int u e^{au}\, du = \frac{1}{a^2}(au-1)\, e^{au} + C$

For us this is usually substitution or algebraic manipulation to make your problem look like a "known" one.

ex $\displaystyle\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \frac{1}{2}\int_2^3 \frac{dx}{x^2\sqrt{x^2-7/4}}$

Basic Forms $\int x^n dx =$ etc.

$\sqrt{a^2+u^2}$ forms

$\sqrt{a^2-u^2}$ forms

$\sqrt{u^2-a^2}$ forms

#45 $\displaystyle\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u}+C$

our problem $\displaystyle\frac{1}{2}\int_2^3 \frac{dx}{x^2\sqrt{x^2-\left(\frac{\sqrt{7}}{2}\right)^2}} = \frac{1}{2}\frac{\sqrt{x^2-7/4}}{7/4\,x}\Bigg|_2^3$

$\displaystyle = \frac{1}{2}\frac{\sqrt{9-7/4}}{21/4} - \frac{1}{2}\frac{\sqrt{4-7/4}}{7/2} = ?$

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ex $\displaystyle\int \sin^2 x \cos x \ln(\sin x)\,dx = \int u^2 \ln u\, du$ ✯

let $u=\sin x$

$du = \cos x\, dx$

by #101 $\displaystyle\int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}\left[(n+1)\ln u - 1\right]+C$

✯ $\displaystyle= \frac{u^3}{9}\left[3\ln u - 1\right]+C = \frac{1}{9}\sin^3 x\left[3\ln(\sin x)-1\right]+C$

Formula:

$$\int u^n \ln u \, du = \frac{1}{n+1} u^{n+1} \ln u - \int \frac{1}{n+1} u^n \, du$$

$$\text{let } s = \ln u \longrightarrow ds = \frac{1}{u} du$$

$$dt = u^n du \longrightarrow t = \frac{1}{n+1} u^{n+1}$$

$$= \frac{1}{n+1} u^{n+1} \ln u - \frac{1}{(n+1)^2} u^{n+1} + C$$

$$= \frac{u^{n+1}}{(n+1)^{-2}} \left[ (n+1) \ln u - 1 \right] + C$$

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Symbolic Engines.

Maxima, Maple, Mathematica, Mupad, Sage Math, sympy, yacas, Geogebra.org