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Application of Eigen Values and Eigen Vectors in Stress Analysis

(Diego: If you need clarification or discussion, please call or email me (532-2612, xin@ksu.edu).

Regards, Jack Xin)

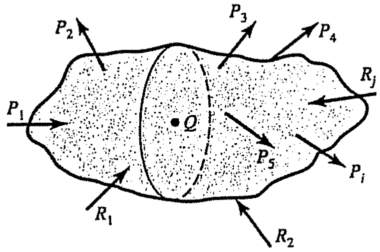
Concept of Stress

4–4 Stress

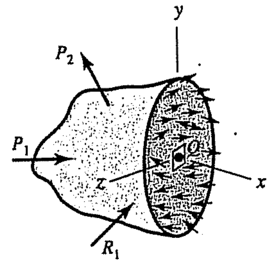
Consider a general solid body loaded as shown in Fig. 4–8*a*. P_i are applied forces and R_i are possible support forces. To determine the state of stress at point Q in the body, it is necessary to expose a surface containing point Q. This is done by making a planar slice, or break, through the body intersecting Q. The orientation of this slice is arbitrary, but it is generally made in a convenient plane where the state of stress can be determined easily or where certain geometric relations can be utilized. The slice, illustrated in Fig. 4–8*b*, is arbitrarily oriented by the surface normal x. This establishes the yz plane. The external forces on the portion of the remaining body are shown, as well as the internal force distribution across the exposed surface. The units of the force distribution are force per unit area. In general, the force distribution will not be uniform across the surface, and will be neither normal nor tangential to the surface at a given point.

Figure 4–8

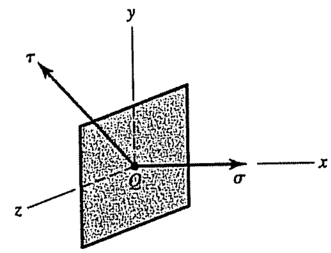
Internal force (stress) distributions.



(*a*) Structural member



(*b*) Isolated section



(*c*) Normal and shear stresses on an infinitesimal surface at Q

However, the force distribution at a point will have components in the normal and tangential directions giving rise to a *normal stress* and a *tangential shear stress*, respectively. Normal and shear stresses are labeled by the Greek symbols σ and τ, respectively, as shown in Fig. 4–8*c*. If the direction of σ is outward from the surface it is considered a *tensile stress* and is a positive normal stress. If σ is into the surface it is a *compressive stress* and commonly considered to be a negative quantity.

4–5 Cartesian Stress Components

The slice in Fig. 4–8*b* establishes the normal direction to the plane of the slice, the x direction, and from equilibrium the normal and shear stress distributions are established. Figure 4–9 shows the normal and net shear stresses on an infinitesimal surface area at point Q. The normal stress is labeled as σ_x. The symbol σ indicates a normal stress and the subscript x indicates the direction of the surface normal. For Cartesian components, it is necessary to establish the y direction, which again is arbitrary. Once this is established, the z direction follows directly for a right-handed rectangular Cartesian coordinate system. The net shear stress $(\tau_x)_{\text{net}}$ can then be resolved into components in the y and z directions labeled as τ_{xy} and τ_{xz}, respectively (see Fig. 4–9). Note that double subscripts are necessary for the shear. The first subscript indicates the direction of the surface normal whereas the second subscript is the direction of the stress.

To completely describe the state of stress at a point it would be necessary to examine all surfaces by making different planar slices through the point. Since different planar slices would necessitate different coordinates and different free-body diagrams, the stresses on each slice would be, in general, quite different. Then to understand the complete state of stress at a point, every possible surface intersecting the point should be examined. However, this would require an infinite number of slices intersecting the point. That is, if the point were to be completely isolated from the body, it would be described by an infinitesimal sphere. Naturally, this would be impossible to do, and it is also unnecessary since there is a simple method for accomplishing the same result, called

Figure 4–9

Stress components on surface normal to x direction.

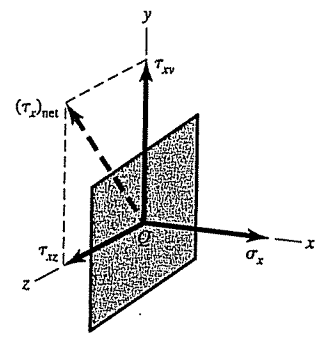
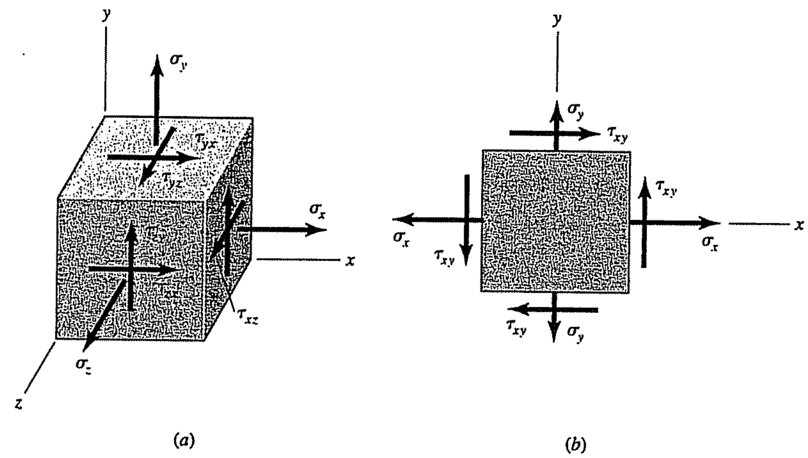


Figure 4–10

(*a*) General three-dimensional stress. (*b*) Plane stress with "cross-shears" equal.



coordinate transformation or *Mohr's circles.* The method is described in the next two sections.

Returning to the general body of Fig. 4–8*a*, a planar slice can be made through point Q perpendicular to the y direction. The procedure for describing the state of stress on the infinitesimal area normal to the y direction is identical to the technique used on Fig. 4–9. However, the stresses on the surface whose normal is in the y direction will be designated as σ_y, τ_{yx}, and τ_{yz} (see Fig. 4–10*a*). A third orthogonal slice is made perpendicular to the z direction, and the resulting stresses are σ_z, τ_{zx}, and τ_{zy}. Thus, the state of stress at a point described by three mutually perpendicular surfaces is shown in Fig. 4–10*a*. It can be shown through coordinate transformation that this is sufficient to determine the state of stress on *any* surface intersecting the point. As the dimensions of the cube in Fig. 4–10*a* approach zero, the stresses on the hidden faces become equal and opposite to those on the opposing visible faces.

In general, a complete state of stress is defined by nine stress components, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy}. For equilibrium, "cross-shears" are equal, hence

$$\tau_{yx} = \tau_{xy} \qquad \tau_{zy} = \tau_{yz} \qquad \tau_{xz} = \tau_{zx} \tag{4–7}$$

This reduces the number of stress components for most three-dimensional states of stress from nine to six quantities, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}.

A very common state of stress occurs when the stresses on one surface are zero. When this occurs the state of stress is called *plane stress.* Figure 4–10*b* shows a state of plane stress, arbitrarily assuming that the normal for the stress-free surface is the z direction such that $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. It is important to note that the element in Fig. 4–10*b* is still a three-dimensional cube. Also, here it is assumed that the cross-shears are equal such that $\tau_{yx} = \tau_{xy}$, and $\tau_{yz} = \tau_{zy} = \tau_{xz} = \tau_{zx} = 0$.

2D Stress State

For simplicity, let's focus on 2D stress state first. For a given coordinate system x-y, if the 2D stress state of point Q is characterized by (Fig.A)

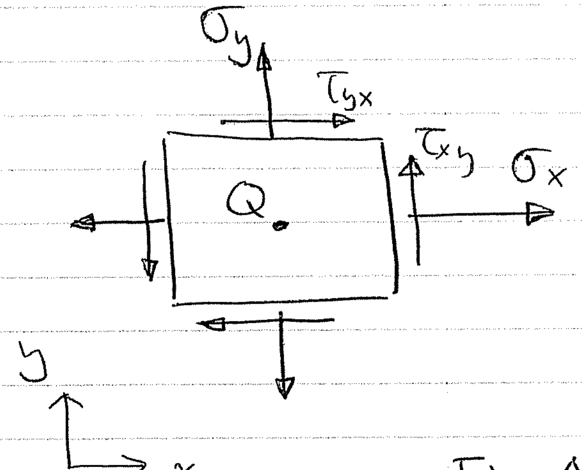


Fig. A

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \qquad (*1)$$

where τ_{yx} is always $= \tau_{xy}$.

we desire to find the stress $\vec{p}$ on an arbitrarily oriented plane MN that passes Q. The outward normal of MN is denoted by (Fig.B)

$$\vec{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$$

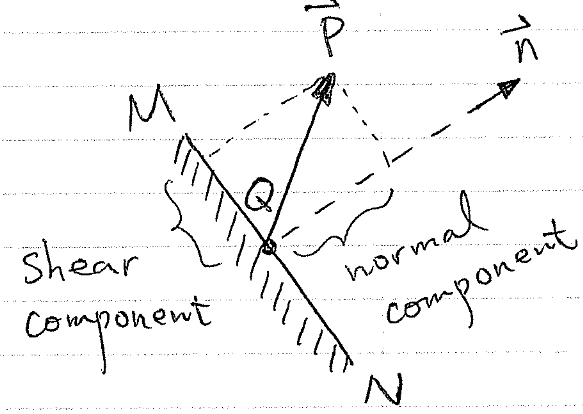


Fig. B

The direction of stress $\vec{p}$ is generally different from $\vec{n}$ or the tangent of the plane MN, therefore both normal and shear component may be present.

$\vec{p}$ in general has components in x and y, i.e.:

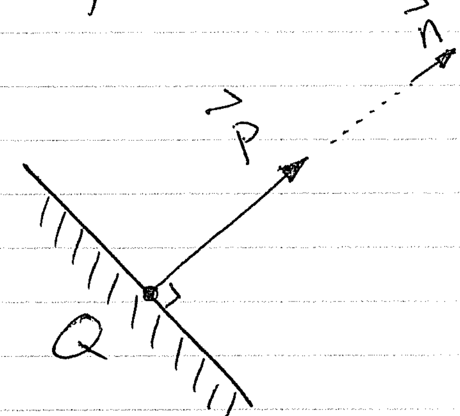
$$\vec{p} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

Using force equilibrium, it can be shown theoretically that $\vec{p}$ is linked to $[\sigma]$ and $\vec{n}$ by

$$\begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \qquad \times 2$$

Principal Direction and Principal Stress

If the orientation of MN is such that $\vec{p}$ is along $\vec{n}$, then the shear component on MN disappears, and only the normal stress remains. In such a case, $\vec{n}$ is called a principal direction, and the corresponding normal stress is called a principal stress, see Fig. C.



principal direction

Fig. C

It can be shown theoretically that for 2D stress state, there always exist at least two mutually perpendicular principal directions. (For 3D stress state, there always exist at least 3 mutually perpendicular principal directions.)

How to Find Principal Stresses and Principal Directions

Now, if $\vec{n}$ is a principal direction, by definition $\vec{p}$ is along $\vec{n}$, therefore we can write

$$\vec{p} = \lambda \vec{n} \qquad \text{where } \lambda \text{ is the magnitude}$$

or

$$\begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \lambda \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \qquad (*3)$$

Combining (2) and (3) gives

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \lambda \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \qquad (*4)$$

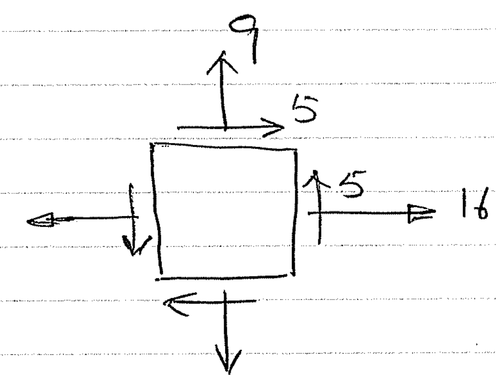
Eq. (*4) shows clearly that finding the principal stress (λ) and principal direction ($\vec{n}$) is fully equivalent to finding the eigen value and eigen vector of a matrix.

Example

The 2D stress state of a material point is

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 5 & 9 \end{bmatrix}$$

Find the principal directions and principal stresses.



Solution:

Eigen value eq: $\begin{vmatrix} 16-\lambda & 5 \\ 5 & 9-\lambda \end{vmatrix} = 0$

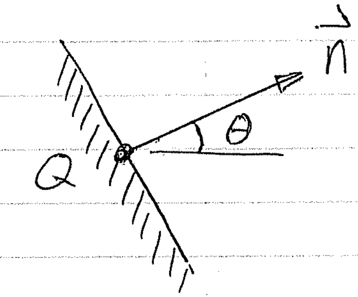
$$\Rightarrow \quad \lambda^2 - 25\lambda + 119 = 0$$

$$\lambda_1 = \boxed{18.6}, \qquad \lambda_2 = \boxed{6.4}$$

For principal stress 18.6, the following eqs determine the principal direction:

$$\begin{cases} \begin{bmatrix} 16-18.6 & 5 \\ 5 & 9-18.6 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = 0 \\ n_x^2 + n_y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0.887 \\ 0.461 \end{Bmatrix}$$



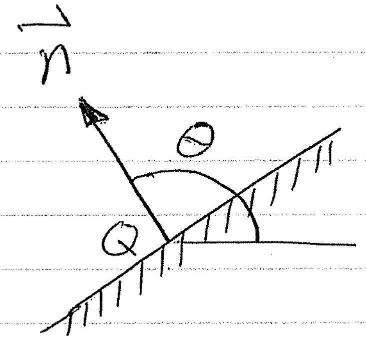
$$\theta = \cos^{-1} 0.887 = 27.5^\circ$$

$\therefore$ On the plane with $\theta = 27.5^\circ$,
the normal stress is 18.6 (which is the principal stress), while the shear stress is zero.

For principal stress 6.4

$$\begin{cases} \begin{bmatrix} 16-6.4 & 5 \\ 5 & 9-6.4 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = 0 \\ n_x^2 + n_y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} -0.462 \\ 0.887 \end{Bmatrix}$$



$\theta = 117.5°$

∴ On the plane with $\theta = 117.5°$
the normal stress is 6.4,
and the shear stress is zero.

3D Stress State

For 3D stress state, the procedure to find the principal stresses and corresponding directions are similar.

If the stress state is represented by a 3x3 matrix

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

and λ is a principal stress and $\vec{n}$ is the corresponding principal direction, then

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \lambda \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

Example:

A 3D stress state is

$$[\sigma] = \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix}$$

Find principal stresses and principal directions.

Solution: principal stress

$$\begin{vmatrix} 3-\lambda & 5 & 8 \\ 5 & 1-\lambda & 0 \\ 8 & 0 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 - 78\lambda + 108 = 0$$

$\lambda_1 = \boxed{11.83}$, $\lambda_2 = \boxed{1.28}$, $\lambda_3 = \boxed{-7.11}$

For $\lambda_1 = 11.83$

$$\begin{cases} \begin{bmatrix} 3-11.83 & 5 & 8 \\ 5 & 1-11.83 & 0 \\ 8 & 0 & 2-11.83 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = 0 \\ n_x^2 + n_y^2 + n_z^2 = 1 \end{cases}$$

$$\Rightarrow \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0.730 \\ 0.337 \\ 0.595 \end{Bmatrix}$$

Similarly, For $\lambda_2 = 1.28$

$$\begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0.047 \\ 0.841 \\ -0.528 \end{Bmatrix}$$

for $\lambda_3 = -7.11$

$$\begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0.685 \\ -0.422 \\ -0.601 \end{Bmatrix}$$