Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

Quiz 1 - Solutions

Name: ______________________________

Honor pledge: "On my honor, as a student, I have neither given nor received unauthorized aid on this academic work."

1. (**15 points**) Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_5 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

Answer the following questions. **Justify and show your work in each case**.

Let's start by doing

```
>> v1=[2 1 4 -1]';
>> v2=[4 1 1 2]';
>> v3=[-3 1 -2 3]';
>> v4=[0 1 1 1]';
>> v5=[0 3 2 2]';
```

(i) (3 points) Is v_3 a linear combination of v_1, v_2, and v_4? If so, express it as such.
By doing

```
>> rref([v1 v2 v4 v3])
ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

we see (by interpreting what's going on along the 4-th row) that v_3 cannot be expressed as a linear combination of v_1, v_2, and v_4.

(ii) (3 points) Is the set $\{v_2, v_3, v_4, v_5\}$ linearly independent? Why?

By doing

```
>> z=[0 0 0 0]'

z =

     0
     0
     0
     0

>> rref([v2 v3 v4 v5 z])

ans =

     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
```

we see that the homogeneous system with matrix of coefficients $[v_2\, v_3\, v_4\, v_5]$ has only one solution, the trivial solution $[0\,0\,0\,0]'$. By definition, this tells us that the set $\{v_2, v_3, v_4, v_5\}$ is linearly independent.

Important: Notice that we introduced the vector z and tested v_2, v_3, v_4, and v_5 against z to form a homogeneous system and properly use the definition of linear independence. However, if a matrix has a zero column, that column will remain zero after finding the $rref$ of that matrix. Then, to save some time, we can *pretend* that we added the right-hand side vector z and just do

```
>> rref([v2 v3 v4 v5])

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

with the condition that at this point we *remember* that there should be a zero column at the end of the $rref$ matrix in order to correctly interpret the homogeneous system. As long as we are aware of this, we won't make mistakes. Doing just $rref([v_2\, v_3\, v_4\, v_5])$ (with no z or homogeneous system in the context) addresses the question of whether v_5 is a linear combination of v_2, v_3, and v_4.

(iii) (3 points) Does v_4 belong to $span\{v_1, v_3, v_5\}$? Why?

Let's do

```
>> rref([v1 v3 v5 v4])

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Then, v_4 does not belong to $span\{v_1, v_3, v_5\}$ as it cannot be written as a linear combination of v_1, v_3, and v_5.

(iv) (3 points) Does $v_1 + v_3$ belong to $span\{v_2 + v_1, v_3\}$? Why?
By doing

```
>> rref([v2+v1 v3 v1+v3])

ans =

     1     0     0
     0     1     0
     0     0     1
     0     0     0
```

we see that the answer is no. $v_1 + v_3$ cannot be written as a linear combination of $v_2 + v_1$ and v_3.

(v) (3 points) Are there real numbers x_1, x_2, x_3, x_4 that satisfy the equation

$$x_1 v_3 + x_2 v_2 + x_3 v_4 + x_4 v_1 = v_5?$$

If so, what are they?

Let's do

```
>> rref([v3 v2 v4 v1 v5])

ans =

    1.0000         0         0         0   -7.3333
         0    1.0000         0         0   -1.6667
         0         0    1.0000         0   19.6667
         0         0         0    1.0000   -7.6667
```

to find that the answer is yes and $x_1 = -7.3333$, $x_2 = -1.6667$, $x_3 = 19.6667$ and $x_4 = -7.6667$. That is,

$$-7.3333\, v_3 - 1.6667\, v_2 + 19.6667\, v_4 - 7.6667\, v_1 = v_5.$$

2. (**15 points**) True or False - **Circle the right one** (3 points each)

FALSE. There exist real numbers y_1, y_2, y_3, and y_4 such that

$$y_1 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix} + y_2 \begin{bmatrix} 4 \\ 1 \\ 4 \\ 4 \end{bmatrix} + y_3 \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \end{bmatrix} + y_4 \begin{bmatrix} 6 \\ 2 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

Just do $rref$.

TRUE. Consider the matrices

$$A = \begin{bmatrix} 2 & (1-\beta) & 25 & 1 \\ \gamma & x & -10 & \gamma \\ 5 & 5 & \gamma & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & \pi & (1-\beta) \\ \alpha & z & 2 \\ 5 & 5 & \theta \\ 9 & -1/2 & x \end{bmatrix}$$

Then the coefficient $(2,3)$ of the product AB equals $\gamma(1-\beta) + (2+\gamma)x - 10\theta$.

In order to determine the coefficient $(2,3)$ of the product AB we look at the 2-nd row of A and 3-rd column of B, multiply component-wise, and add those products.

TRUE. If $v \in span\{u_1, u_2, u_3\}$, then v can be expressed as a linear combination of u_1, u_2, and $u_1 + u_3$.

Indeed, if $v \in span\{u_1, u_2, u_3\}$, then there exist numbers x_1, x_2, x_3 such that

$$v = x_1 u_1 + x_2 u_2 + x_3 u_3.$$

But this can be written, for instance, as

$$v = x_1u_1 + x_2u_2 + x_3u_3 = x_1u_1 + x_2u_2 + x_3u_3 + x_3u_1 - x_3u_1 = (x_1 - x_3)u_1 + x_2u_2 + x_3(u_1 + u_3).$$

and we expressed v a linear combination of u_1, u_2, and $u_1 + u_3$.

TRUE. If the set of vectors $\{z_1, z_2, z_3\}$ is linearly independent, then they form a basis of $span\{z_1, z_2, z_3\}$.

Of course. The vectors $\{z_1, z_2, z_3\}$ span the linear subspace $span\{z_1, z_2, z_3\}$ and they are are linearly independent, hence they form a basis for $span\{z_1, z_2, z_3\}$.

TRUE. If the system $Mx = 0$ has exactly one solution (the trivial solution), then the columns of M are linearly independent.

Indeed, this is nothing but the definition of linear independence applied to the columns of M.