Applied Matrix Theory - Math 551

Example of an open input-output economy model

Created by Prof. Diego Maldonado and Prof. Virginia Naibo

Kansas State University
Department of Mathematics

Problem. Suppose that we have an economy with labor (L), transportation (T), and food (F) industries. Suppose that \$1 in L requires 40 cents in T and 20 cents in F; while \$1 in T takes 50 cents in L and 30 cents in T; and \$1 in F uses 50 cents in L, 5 cents in T and 35 cents in F. Also, suppose that the external demand is of $\$10,000$ in L, $\$20,000$ in T, and $\$10,000$ in F. Find the production schedule, the consumption matrix, and determine what sector of the industry consumes the most transportation.

Solution. As in any other Math problem, the first step is to introduce variables. Let x_1, x_2, and x_3 represent the production of L, T, and F (in dollars), respectively. The key questions to be answered are: "How much L is consumed?", "How much T is consumed?", "How much F is consumed?". Let's answer the first one.

From the statement of the problem we read that the labor industry doesn't consume any labor. Also, we read that the each dollar of T takes 0.5 dollars of L, since x_2 dollars of T are being produced, they will consume $0.5x_2$ dollars worth of L. In addition, each dollar of F takes 0.5 dollars of L, since x_3 dollars of F are being produced, they will take $0.5x_3$ dollars of L. We also read that 10,000 dollars worth of L are required as an external demand. As discussed in class, this leads to the equation

$$0x_1 + 0.5x_2 + 0.5x_3 + 10000 = x_1$$

Reasoning along the same lines, we answer the remaining key questions (do it!) to obtain the system

$$(S)\left\{\begin{array}{rcrcrcrcl} 0x_1 & + & 0.5x_2 & + & 0.5x_3 & + & 10000 & = & x_1 \\ 0.4x_1 & + & 0.3x_2 & + & 0.05x_3 & + & 20000 & = & x_2 \\ 0.2x_1 & + & 0x_2 & + & 0.35x_3 & + & 10000 & = & x_3 \end{array}\right.$$

The resulting system (S) is known as **input-output model for the economy**. The matrix

$$C = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.05 \\ 0.2 & 0 & 0.35 \end{bmatrix}$$

is known as **the consumption matrix**. It must be noticed that the consumption matrix C *is not* the matrix of coefficients of (S). The vector

$$d = \begin{bmatrix} 10000 \\ 20000 \\ 10000 \end{bmatrix}$$

is known as **the external demand vector**. Since the external demand vector is not zero, we are dealing with **an open economy model**. And the vector,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is called **the production vector**. In this notation, the system (S) can be recast as

$$Cx + d = x. \tag{1}$$

If now I represents the 3×3 identity matrix, we have that (1) can be reordered as

$$(C - I)x = -d. \tag{2}$$

The convenience of doing this is that now it is apparent that the system (S) is equivalent to (2), and (2) has matrix of coefficients $C - I$ and right hand side vector $-d$. Hence, by using Matlab or our calculator, we obtain

```
>> C=[0 0.5 0.5; 0.4 0.3 0.05; 0.2 0 0.35]

C =

         0    0.5000    0.5000
    0.4000    0.3000    0.0500
    0.2000         0    0.3500

>> I = eye(3)

I =

     1     0     0
     0     1     0
     0     0     1

>> d=[10000 20000 10000]'

d =

       10000
       20000
       10000

>> rref([C - I, -d])
```

```
ans =

  1.0e+004 *

    0.0001         0         0    5.9200
         0    0.0001         0    6.4800
         0         0    0.0001    3.3600
```

and we immediately interpret that the system (S) has exactly one solution, and it is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 59200 \\ 64800 \\ 33600 \end{bmatrix}$$

That is, **the production schedule** should be \$59,200 worth of Labor, \$64,800 worth of Transportation, and \$33,600 worth of Food.

Next, in order to determine which industry consumes the most transportation, we come back to the answer to the second key question. Regarding transportation, we obtained that L consumes $0.4x_1$ of T, T consumes $0.3x_2$ of T, and F consumes $0.05x_3$ of T. Since, we now know the values of x_1, x_2, and x_3, we just compute

$$0.4x_1 = (0.4) \times 59,200 = 23,680,$$

$$0.3x_2 = (0.3) \times 64,800 = 19,440,$$

and

$$0.05x_3 = (0.05) \times 33,600 = 1,680.$$

Therefore, the Labor industry consumes the most Transportation.