Quiz 2

1. Consider the sequence defined inductively as follows: $a_0 = 0$, and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$. We claim that $a_n = n$ for all n. We prove this by induction. For the base case observe that $a_0 = 0$ by definition. Assume that for all $n < k$, we have $a_n = n$. Now $a_k = a_{\lceil k/2 \rceil} + a_{\lfloor k/2 \rfloor}$ by definition. From the induction hypothesis, we have $a_{\lceil k/2 \rceil} = \lceil k/2 \rceil$ and $a_{\lfloor k/2 \rfloor} = \lfloor k/2 \rfloor$. Thus, $a_k = a_{\lceil k/2 \rceil} + a_{\lfloor k/2 \rfloor} = \lceil k/2 \rceil + \lfloor k/2 \rfloor = k$. Thus, the claim is established by induction.

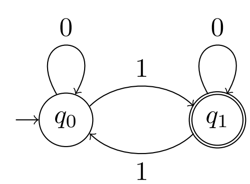
   (A) The proof is correct.

   (B) For some values of k, the induction hypothesis does not allow us to conclude $a_{\lceil k/2 \rceil} = \lceil k/2 \rceil$

   (C) For some values of k, the induction hypothesis does not allow us to conclude $a_{\lfloor k/2 \rfloor} = \lfloor k/2 \rfloor$.

   (D) For some values of k, $\lceil k/2 \rceil + \lfloor k/2 \rfloor \neq k$.

   The correct answer is (B). When $k = 1$ we get, $a_{\lceil k/2 \rceil} = a_{\lceil 1/2 \rceil} = a_1$, which is the same as a_k and not defined by the previous elements of the sequence.



2. Consider the automaton M_1 shown above. We will show that $w \in \mathbf{L}(M_1)$ iff w has an odd number of 1s. We will prove this by induction on the length of w. For the base case, observe that when $w = \epsilon$, w has an even number of 1s. Further, $q_0 \xrightarrow{\epsilon}_{M_1} q_0$, and so M_1 does not accept w. For the induction step, consider two cases. When $w = 0u$, w has an odd number of 1s iff u has an odd number of 1s, iff (by ind. hyp.) $q_0 \xrightarrow{u}_{M_1} q_1$ iff $q_0 \xrightarrow{w=0u}_{M_1} q_1$ (since $\delta(q_0, 0) = q_0$). When $w = 1u$, w has an odd number of 1s iff u has an even number of 1s iff (by ind. hyp.) $q_1 \xrightarrow{u}_{M_1} q_1$ iff $q_0 \xrightarrow{w=1u}_{M_1} q_1$ (since $\delta(q_0, 1) = q_1$). Thus, the claim is established by induction.

   (A) The proof is correct.

   (B) The proof is incorrect because not all possible strings w have been considered in the induction step.

   (C) The proof is incorrect because there is no basis for concluding that $q_1 \xrightarrow{u}_{M_1} q_1$ by induction hypothesis in the case when $w = 1u$.

   (D) The proof is incorrect because the base case has not been proved.

   Correct answer is (C). When we assume in our induction hypothesis that any string u such that $|u| < |w|$ is accepted by the automaton iff it has odd number of 1s, it doesn't tell us how the automaton might behave starting from a state other than the inital state.

3. Recall the language $L_2 \subseteq \{0,1\}^*$ defined as

$$L_2 = \{u10 \mid u \in \{0,1\}^*\} \cup \{u11 \mid u \in \{0,1\}^*\}$$

That is, L_2 contains all strings that have a 1 as the second last symbol. Consider the following proof that any DFA recognizing L_2 must have at least 4 states: Let M be a DFA with initial state q_0 recognizing L_2 having less than 4 states. Let $q_0 \xrightarrow{00}_M A$, $q_0 \xrightarrow{01}_M B$, $q_0 \xrightarrow{10}_M C$, $q_0 \xrightarrow{11}_M D$, where A, B, C, D are some states of M. Now, if M has less than 4 states, then two out of A, B, C, D must be the same state. If $A = B$ then M either accepts both 000 and 010 or neither; but only 010 should be accepted. If $A = C$ then M either accepts both 00 and 10 or neither; again only 10 must be accepted. If $A = D$ then M either accepts both 00 and 11 or neither; however, only 11 must be accepted. If $B = C$ then M either accepts both 01 and 10 or neither; again only 10 should be accepted. If $B = D$ then M either accepts both 01 and 11 or neither; only 11 should be accepted. Finally, if $C = D$ then M either accepts both 100 and 110 or neither; however, only 110 should be accepted. Thus, A, B, C, D must all be different, and M cannot have less than 4 states.

(A) The proof is correct.

(B) The proof is incorrect because there is a DFA with 3 states that recognizes L_2.

(C) The proof does not show that there is no DFA with less than 4 states accepting L_2. It only shows that a specific DFA M that has 5 states q_0, A, B, C, and D cannot accept L_2.

(D) The proof is incorrect because there is no basis for assuming that two out of A, B, C, D must be the same.

Correct answer is (A). Because this proof is correct, option (B) cannot be true. Option (C) is also not valid because in the proof we are only calling some states of M by q_0, A, B, C, D, and are not saying that M has exactly these states. Finally option (D) is not correct because the fact that two out of A, B, C, D must be the same follows from pigeon hole principle, under the assumption that M has less than 4 states.

4. Recall the language $L_2 \subseteq \{0,1\}^*$ defined as

$$L_2 = \{u10 \mid u \in \{0,1\}^*\} \cup \{u11 \mid u \in \{0,1\}^*\}$$

That is, L_2 contains all strings that have a 1 as the second last symbol. Consider the following proof that any DFA recognizing L_2 must have at least 4 states: Any DFA recognizing L_2 has to remember the last two symbols of the input string. Since there are 4 strings of length 2, it must have 4 states.

(A) The proof is correct.

(B) The proof is incorrect because there is a DFA with 3 states that recognizes L_2.

(C) The proof is incorrect because there is no basis for assuming that a DFA recognizing L_2 has to remember that last two symbols of the input string.

(D) The proof is incorrect because this statement has to be proved by induction.

Correct answer is (C). While the statement is correct (and proved in the previous question), the proof is not complete. The reason why the proof is incomplete is because the assumption that the last 2 symbols need to be remembered is only an intuition, and the argument fails establish this mathematicallt. Thus option (A) and (B) are not true. Also, option (D) cannot be true, because nothing “has to be proved by induction”.