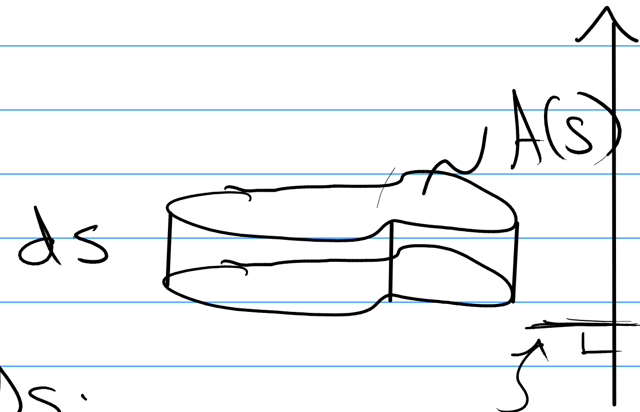
Math 243

Cylindrical Shells (Volume of rotation)

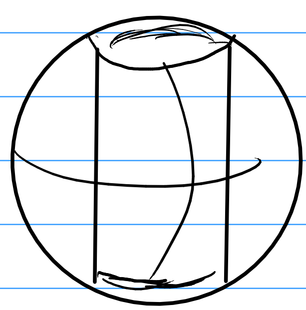
slicing, disks, washers

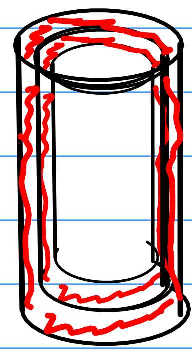
$\Rightarrow$ stacked cylinders

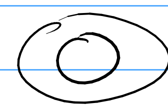


$$V_i = A(s_i)\,\Delta s_i$$

you slice perpendicular to variable

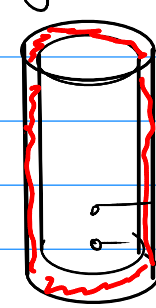






$$V \approx \sum_{i=1}^{n} V_i$$

Volume of one of the right cylindrical shell

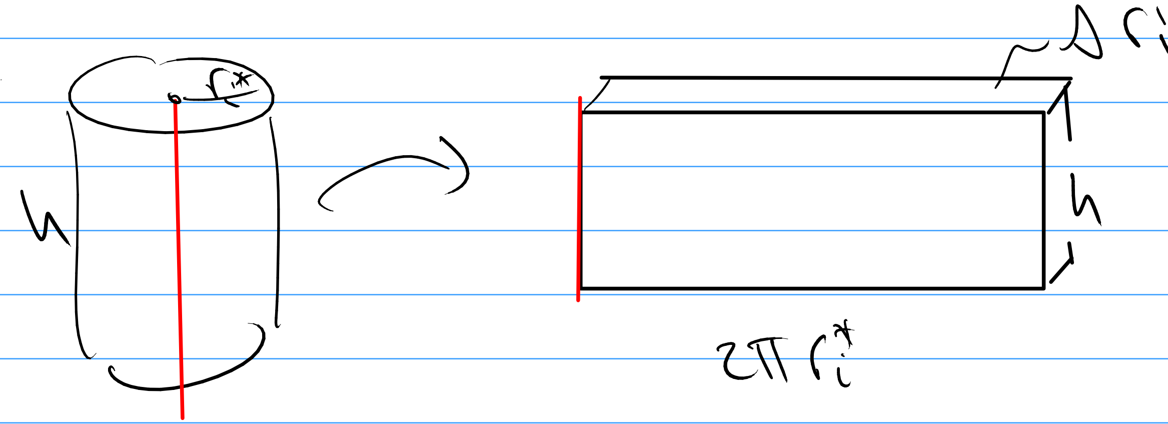


$$V_i = \pi r_{out}^2 h - \pi r_{in}^2 h$$

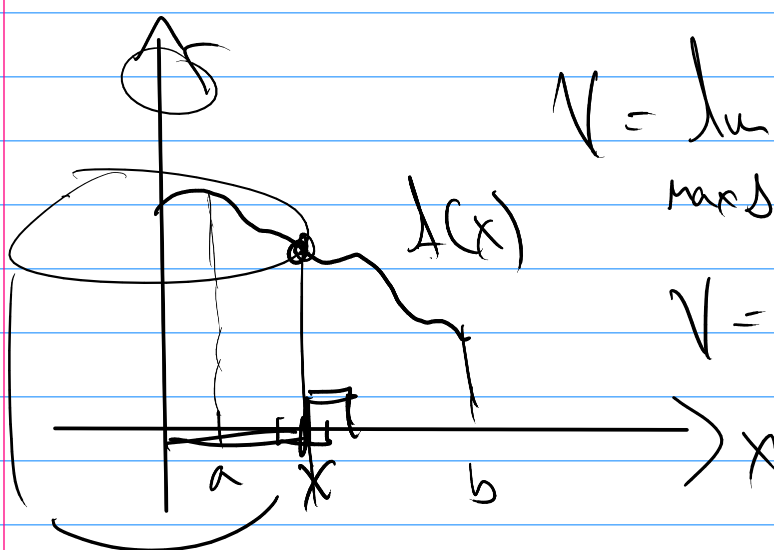
$$V_i = \pi h (r_{out}^2 - r_{in}^2) = \pi h (r_{out} + r_{in})(r_{out} - r_{in})$$

$$V_i = 2\pi h \left(\overbrace{\frac{r_{out}+r_{in}}{2}}^{r_i^*}\right)\Delta r_i$$

$$V_i = \underbrace{2\pi\, r_i^*}\; h \;\Delta r_i$$



$$V = \lim_{\max \Delta r_i \to 0} \sum_{i=1}^{n} 2\pi \underbrace{(\text{radius})}_{r_i}\underbrace{(\text{height})}_{?}\ \underbrace{\text{width}}_{\Delta r_i}$$



$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \underbrace{2\pi\, x_i\, f(x_i)}\,\Delta x_i$$

$$V = \int_a^b 2\pi x\, f(x)\,dx$$

<u>Shells:</u>

$$V = \int_{\text{slice start}}^{\text{slice end}} 2\pi(\text{radius})(\text{height})\, d(\underbrace{\text{variable}}_{\text{you are } \perp \text{ to.}})$$

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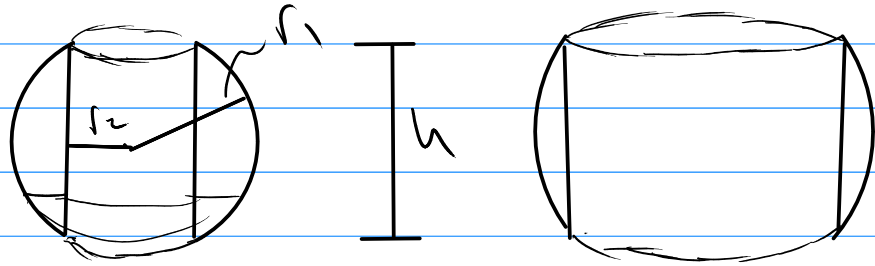
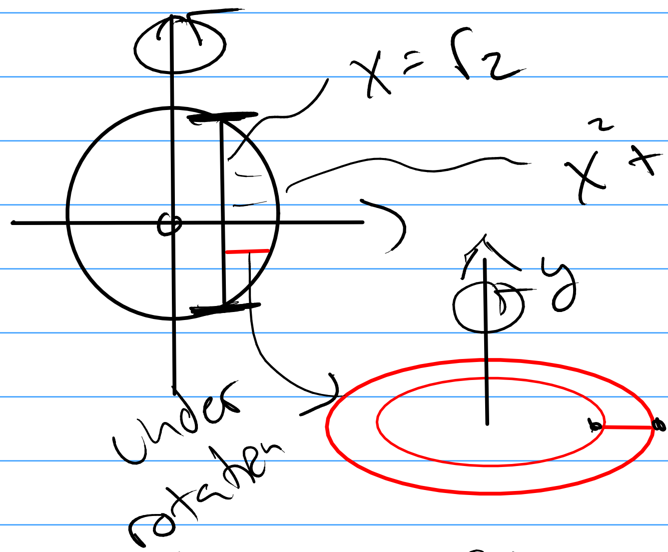




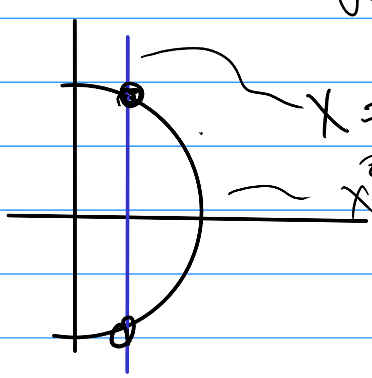
$x^2 + y^2 = r_1^2 \rightarrow x = \pm\sqrt{r_1^2 - y^2}$

$$V = \int_{?}^{?} \pi \left(\text{outer radius}^2 - \text{inner radius}^2 \right) dy$$

$$V = \int_{?}^{?} \pi \left((r_1^2 - y^2) - r_2^2 \right) dy$$

$$V = \int_{?}^{?} \pi \left(r_1^2 - r_2^2 - y^2 \right) dy$$

← where do they cross?



$\left.\begin{array}{l} x = r_2 \\ x^2 + y^2 = r_1^2 \end{array}\right] \rightarrow r_2^2 + y^2 = r_1^2$

$y^2 = r_1^2 - r_2^2$

$y = \pm\sqrt{r_1^2 - r_2^2}$

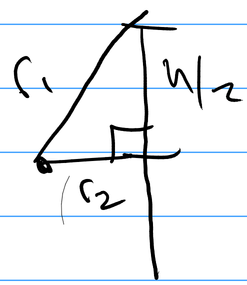
So $V = 2\int_0^{\sqrt{r_1^2-r_2^2}} \pi(r_1^2-r_2^2-y^2)\,dy$

$$V = 2\pi\left((r_1^2-r_2^2)y - \frac{1}{3}y^3\right)\Big|_0^{\sqrt{r_1^2-r_2^2}}$$

$$V = 2\pi\left((r_1^2-r_2^2)^{3/2} - \frac{1}{3}(r_1^2-r_2^2)^{3/2}\right)$$

$$V = \frac{4\pi}{3}(r_1^2-r_2^2)^{3/2}$$

given ball



$$r_2^2 = r_1^2 - (h/2)^2$$

So..

$$V = \frac{4\pi}{3}(r_1^2-r_2^2)^{3/2}$$

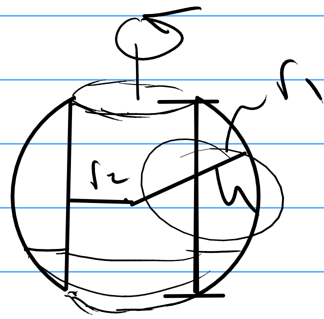
$$V = \frac{4\pi}{3}\left(\cancel{r_1^2} - \cancel{r_1^2} + \left(\frac{h}{2}\right)^2\right)^{3/2}$$

$$V = \frac{\cancel{4}\pi}{3}\,\frac{h^3}{\cancel{8}_2} = \boxed{\frac{\pi}{6}h^3}$$

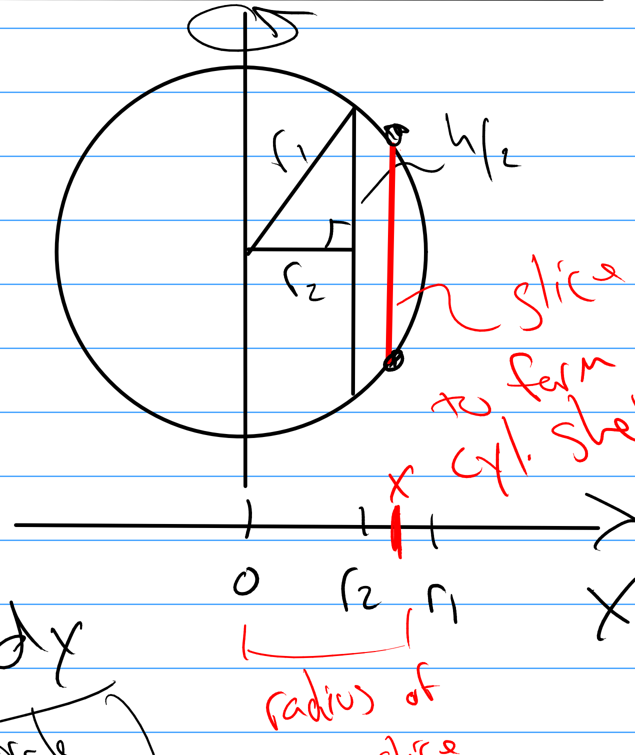
$h = 1\text{ cm}$ left after you drill

$$V = \frac{\pi}{6}(1)^3 = \frac{\pi}{6} \approx 0.52\text{ cm}^3$$

Same problem via shells

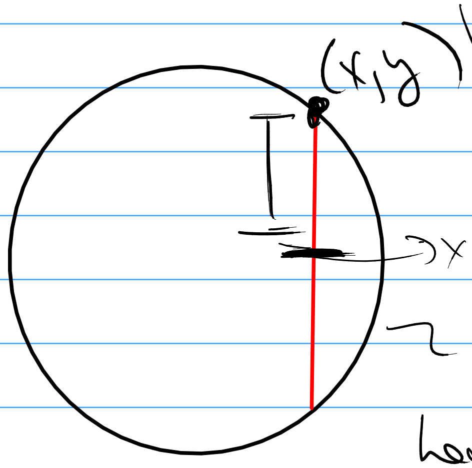


$$r_2^2 + \left(\frac{h}{2}\right)^2 = r_1^2$$



$$V = \int_{r_2}^{r_1} 2\pi x \,(\text{height})\, dx$$

top of circle to bottom in the slice.



$$x^2 + y^2 = r_1^2$$

$$\text{height} = 2\sqrt{r_1^2 - x^2}$$

$$V = \int_{r_2}^{r_1} 2\pi x \left(2\sqrt{r_1^2 - x^2}\right) dx$$

$$V = 4\pi \int_{r_2}^{r_1} x\sqrt{r_1^2 - x^2}\, dx$$

let $u = r_1^2 - x^2$

$du = -2x\, dx$

$x = r_1 \Rightarrow u = 0$

$$V = -2\pi \int u^{1/2}\, du$$

$x = r_2 \Rightarrow u = r_1^2 - r_2^2$

$$V = 2\pi \int_0^{r_1^2 - r_2^2} u^{1/2}\, du = \frac{4\pi}{3} u^{3/2} \Big|_0^{r_1^2 - r_2^2}$$

$$V = \frac{4\pi}{3}\left(r_1^2 - r_2^2\right)^{3/2} \quad (\text{same!})$$

use $r_2^2 + \left(\frac{h}{2}\right)^2 = r_1^2$ (do same as above to show only h matters).

$$V = \frac{\pi}{6} h^3$$

$$2\pi \int_0^2 \frac{y}{1+y^2}\, dy$$

$$\int_{\text{start}}^{\text{end}} 2\pi \text{ radius} \cdot \text{height } dy$$

$$\int_0^2 2\pi (y)\left(\frac{1}{1+y^2}\right) dy$$

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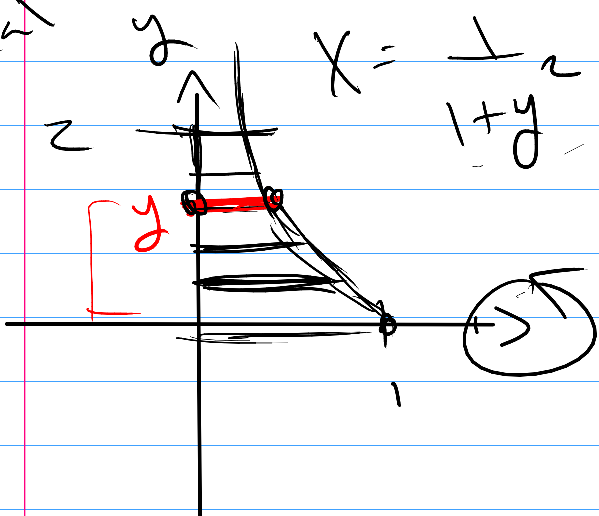
like a "real" prob.



$$x = \frac{1}{1+y^2}$$

$$V = \int_0^2 2\pi\, y\left(\frac{1}{1+y^2}\right) dy$$

$$V = 2\pi \int_0^2 y\left(\frac{1}{1+y^2}\, dy\right)$$

let $u = 1+y^2$

$du = 2y\, dy$

$$V = \pi \int_1^5 \frac{1}{u}\, du = \pi \ln|u| \Big|_1^5$$

$$V = \boxed{\pi \ln 5}$$

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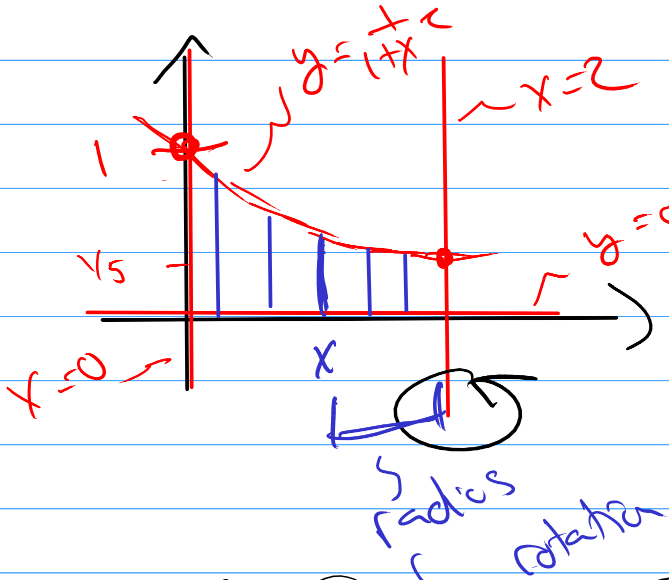
(24) $y = \frac{1}{1+x^2}$ $\quad y=0 \quad x=0 \quad x=2$

about $x=2$



$$V = \int_0^2 2\pi(2-x)\left(\frac{1}{1+x^2}\right)dx$$

$$V = 2\pi \int_0^2 (2-x)\,\frac{1}{1+x^2}\,dx$$

$$V = 2\pi \int_0^2 \left(\frac{2}{1+x^2}\right) - \left(\frac{x}{1+x^2}\right) dx$$

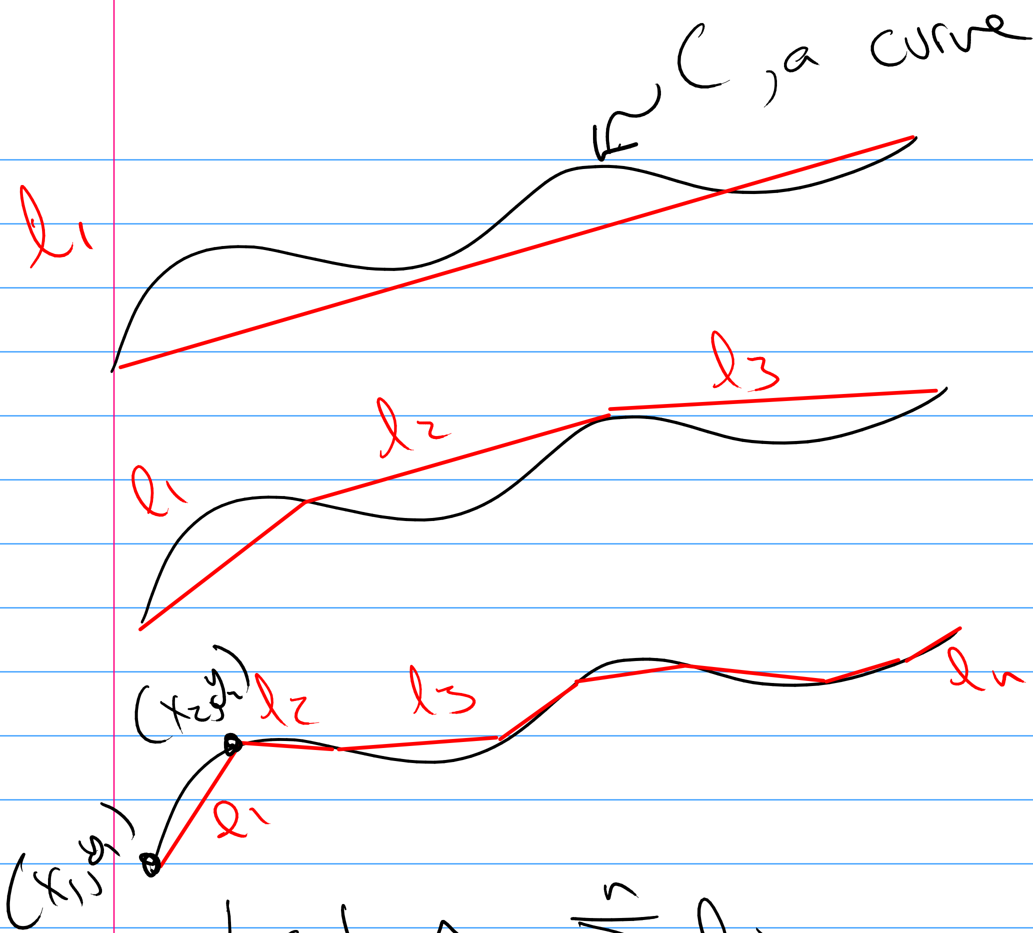
substitution

$$V = 2\pi\left[\, 2\tan^{-1}x \,\Big|_0^2 - \frac{1}{2}\ln(1+x^2)\Big|_0^2 \,\right]$$

$$V = 2\pi\left[\, 2\tan^{-1}2 - \frac{1}{2}\ln 5 \,\right]$$

$$\int_a^b f(x)\,dx$$

$$= \lim_{\max \Delta x_i \to 0} \boxed{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i}$$

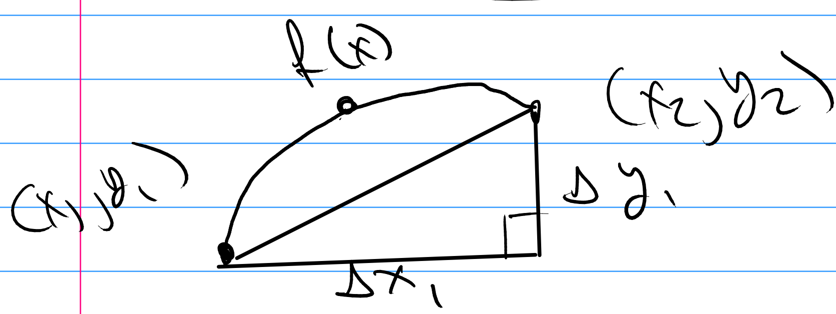


$$|C| \approx \sum_{i=1}^{n} l_i$$

$$|l_1| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

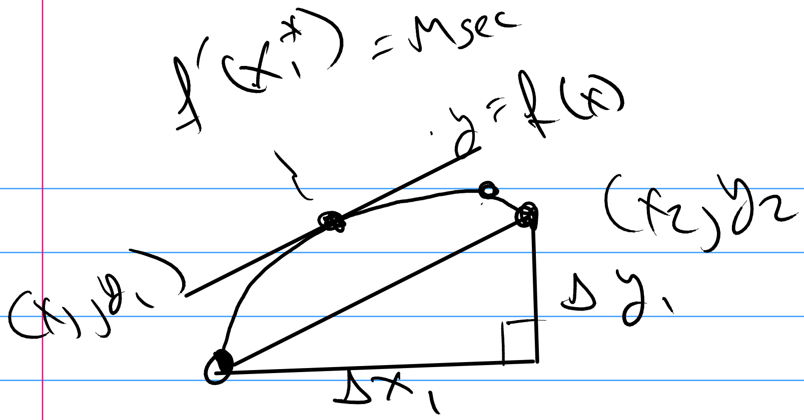
$$|l_i| = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

$$|C| \approx \boxed{\sum_{i=1}^{n} \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}}$$



$$\Delta x_1 = (x_2-x_1)$$

$$\Delta y_1 = (y_2-y_1)$$



$$m_{sec} = \frac{\Delta y_1}{\Delta x_1}$$

by Mean Value thm there is an $x_1^* \in [x_1, x_2]$

$$\rightarrow f'(x_1^*) = \frac{\Delta y_1}{\Delta x_1} \rightarrow \Delta y_1 = \Delta x_1 f'(x_1^*)$$

Now ...

$$|l_1| = \sqrt{(\underbrace{x_2 - x_1}_{\Delta x_1})^2 + (\underbrace{y_2 - y_1}_{\Delta y_1})^2}$$

$$|l_1| = \sqrt{\Delta x_1^2 + \Delta x_1^2 (f'(x_1^*))^2}$$

$$|l_1| = \sqrt{1 + (f'(x_1^*))^2}\, \Delta x_1$$

$$|l_i| = \sqrt{1 + (f'(x_i^*))^2}\, \Delta x_i$$

$$|\text{curve}| \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x_i$$

$$|\text{curve}| = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x_i$$

$$\text{Arc length} = \int_a^b \sqrt{1+(f')^2}\, dx$$

ex $\quad x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2}; \; y \in [1, 9]$

$$\text{A.L.} = \int_1^9 \sqrt{1 + \left(\tfrac{1}{2}y^{1/2} - \tfrac{1}{2}y^{-1/2}\right)^2}\, dy$$

$$\text{A.L.} = \int_1^9 \sqrt{1 + \tfrac{1}{4}y - \tfrac{1}{2} + \tfrac{1}{4}y^{-1}}\, dy$$

$$\text{A.L.} = \int_1^9 \sqrt{\tfrac{1}{4}y + \tfrac{1}{2} + \tfrac{1}{4}y^{-1}}\, dy$$

$$= \frac{1}{2}\int_1^9 \sqrt{\underset{\substack{\shortparallel \\ (y^{1/2})^2}}{y} + 2 + \underset{\substack{\shortparallel \\ (y^{-1/2})^2}}{y^{-1}}}\, dy$$

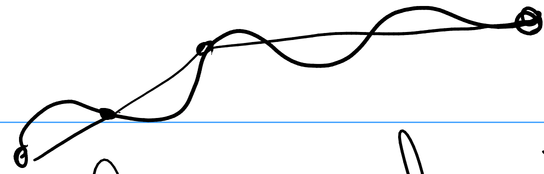
$$a^2 + 2ab + b^2$$

$$= \frac{1}{2}\int_1^9 \sqrt{(y^{1/2} + y^{-1/2})^2}\, dy$$

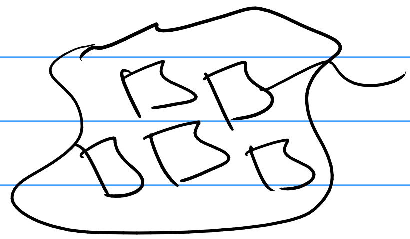
$$= \frac{1}{2}\int_1^9 y^{1/2} + y^{-1/2}\, dy$$

$$= \frac{1}{2}\left(\tfrac{2}{3}y^{3/2} + 2y^{1/2}\right)\Big|_1^9$$

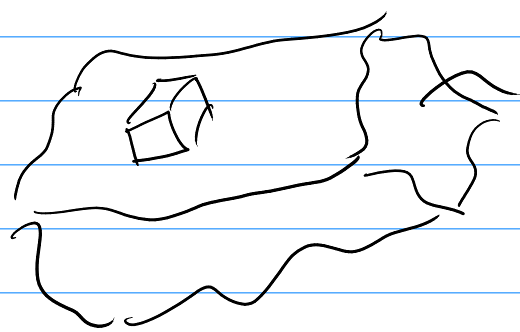
$$= \frac{1}{2}\left[(18 + 6) - \left(\tfrac{2}{3} + 2\right)\right] = \boxed{32/3}$$

1D 

$$\text{length} = \lim \sum l_i$$ Known 1D measure

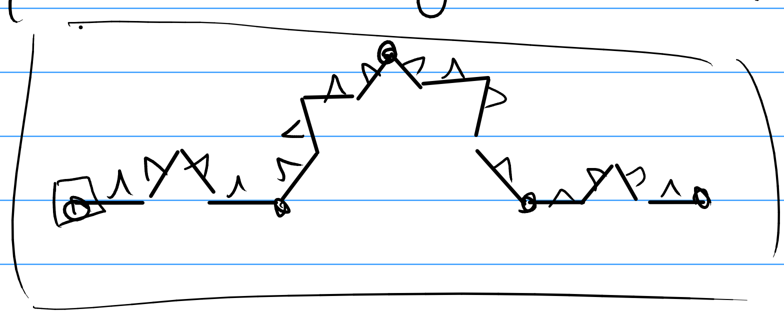
2D 

$$\text{Area} = \lim \sum A_i$$ 2D measure

3D 

$$\text{Volume} = \lim \sum V_i$$ Known 3D measure

If you measure a 1D object
you should get a 1D measure.



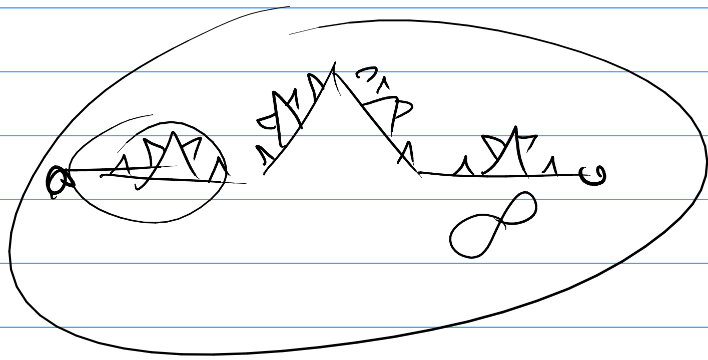
$$l = 1 = \left(\tfrac{4}{3}\right)^0$$
$$l = \tfrac{4}{3} = \left(\tfrac{4}{3}\right)^1$$
$$l = \tfrac{16}{9} = \left(\tfrac{4}{3}\right)^2$$

$$l = \infty$$

$$\text{Measure} = \lim \sum \underbrace{(d)^{p}}_{\text{tile in } p \text{ dimension}}$$

$p = 1, 2, 3, 4,$

1.67



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