CIS 575. Introduction to Algorithm Analysis
Remarks on Assignment #2, Spring 2014

Question 1 For $n \geq 1$ we have the calculation

$$\lg(n!) \leq \lg(n^n) = n \lg n$$

and thus $\lg(n!) \leq 1 \cdot n \lg n$ which shows $\lg(n!) \in O(n \lg n)$.

On the other hand, assume that for some $k \geq 0$ we have $n! \in O(n^k)$; we shall show that this leads to a contradiction. By the definition of big-O, there exists $c > 0$ and $n_0 \geq 0$ such that $n! \leq cn^k$ for $n \geq n_0$. But for $n > \max(2k, c2^{k+1}, n_0)$ we have (since $n - k > n - n/2 = n/2$) that

$$n! \geq n(n-1)..(n-k+1)(n-k) \geq (n-k)(n-k)^k \geq \frac{n}{2}(\frac{n}{2})^k = n(\frac{1}{2})^{k+1}n^k > cn^k \geq n!$$

which is a contradiction.

Question 2 By assumption, and definition of Θ, there exists $c_0 > 0$ and n_0 such that for $n \geq n_0$ we have $f(n) \leq c_0 n$, there exists $c_1 > 0$ and n_1 such that for $n \geq n_1$ we have $g(n) \geq c_1 n^2$, and there exists $c_2 > 0$ and n_2 such that for $n \geq n_2$ we have $g(n) \leq c_2 n^2$.

For $n \geq \max(n_0, n_1, n_2)$ we thus have

$$f(n) + g(n) \leq c_0 n + c_2 n^2 \leq c_0 n^2 + c_2 n^2 = (c_0 + c_2)n^2$$

which shows that $f + g \in O(n^2)$, and also (as f is non-negative)

$$f(n) + g(n) \geq g(n) \geq c_1 n^2$$

which shows that $f + g \in \Omega(n^2)$. Hence $f + g \in \Theta(n^2)$. (Observe that we don't need that $f \in \Omega(n)$.

Question 3

Ranking:	$f : n!$	$>$	$e : 3^n$	$>$	$d : 2^{n+7}$	$=$	$a : 2^n$	$>$	$i : n^2$	$>$	$g : n \lg n$	$>$	$j : n$	$>$	$b : \sqrt{n}$	$>$	$c : \lg n$	$>$	$h : 1$
Least n with $x(n) \geq 1,000,000$	10		13		13		20		$1,000$		$\approx 2^{16}$		10^6		10^{12}		$2^{1,000,000}$		never