Math 243

Q3) $\displaystyle\sum_{n=1}^{\infty} \frac{10^{n-1}}{(n-1)!} x^n$

$$\lim_{n\to\infty} \left| \frac{\frac{10^n}{n!} x^{n+1}}{\frac{10^{n-1}}{(n-1)!} \cancel{x^n}} \right| = \lim_{n\to\infty} \frac{10^{\cancel{n}}}{\cancel{10^{n-1}}} \frac{\cancel{(n-1)!}}{n!} |x|$$

$$= \lim_{n\to\infty} \frac{10}{n} |x| = 10|x| \left(\lim_{n\to\infty} \frac{1}{n} \right)^{\to 0}$$

$= 0 < 1$ always abs. conv.

$R = \infty$ Interval $(-\infty, \infty)$

$$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{(1)}}{n\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{(1)}} = \frac{1}{n}$$

or

$$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)!}}{n \cdot \cancel{(n-1)!}} = \frac{1}{n}$$

ex $\frac{(n-3)!_0}{(n+1)!_0} = \frac{\cancel{(n-3)!_0}}{(n+1)\,n\,(n-1)(n-2)\cancel{(n-3)!_0}}$

$$= \frac{1}{(n+1)(n)(n-1)(n-2)}$$

Given $\left(x(t),\ y(t)\right)$

Derivatives $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ $\qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)}$

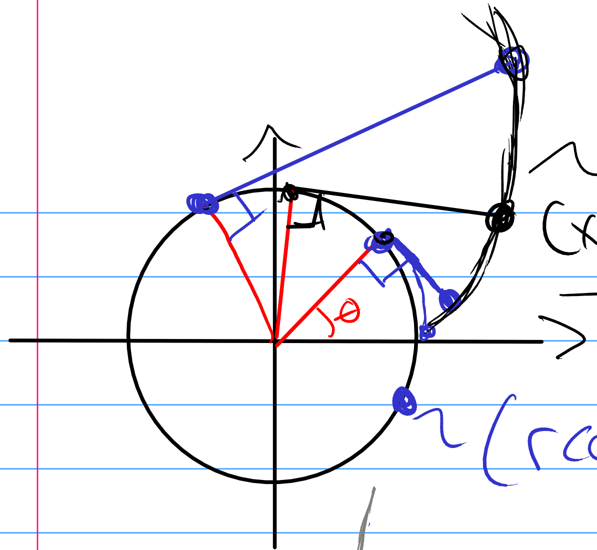
Def Integral $\int_{x=a}^{x=b} y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\,x'(t)\,dt$

@ $t=\alpha$ you are at $x=a$ (left)

@ $t=\beta$ you are at $x=b$ (right)

$\rightarrow$ $x(\alpha)=a$ $\qquad x(\beta)=b$

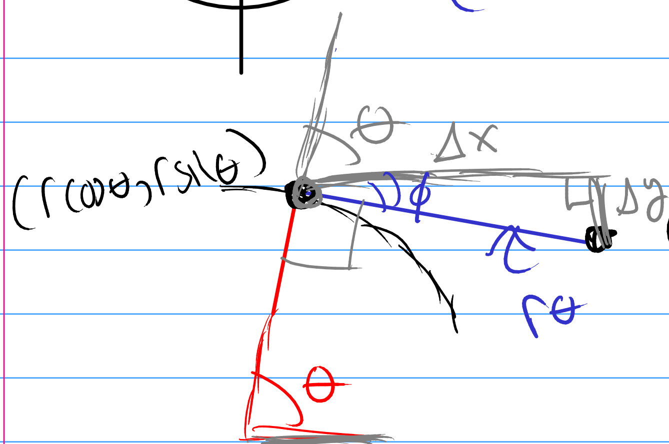
Arclength $L = \int_{\alpha}^{\beta} \sqrt{(y'(t))^2 + (x'(t))^2}\,dt$



~ Involute

$(x,y) = (x(\theta), y(\theta))$

~ $(r\cos\theta, r\sin\theta)$



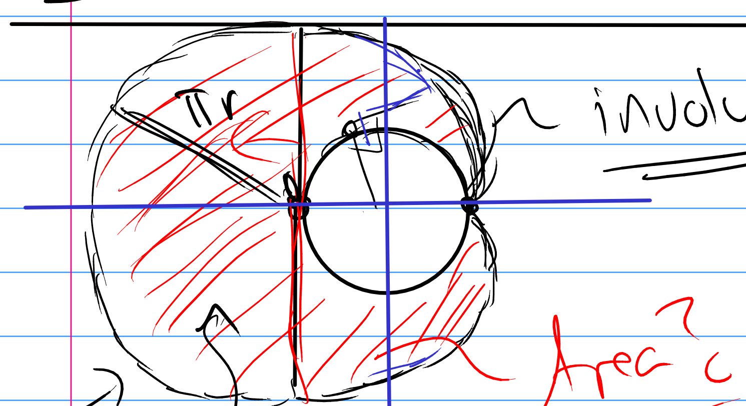
$\phi = \frac{\pi}{2} - \theta$

$\Delta x = r\theta \cos\phi$

$\Delta y = r\theta \sin\phi$

$$x = r\cos\theta + r\theta\cos\left(\tfrac{\pi}{2} - \theta\right)$$
$$y = r\sin\theta - r\theta\sin\left(\tfrac{\pi}{2} - \theta\right)$$

$$\boxed{\begin{aligned} x(\theta) &= r(\cos\theta + \theta\sin\theta) \\ y(\theta) &= r(\sin\theta - \theta\cos\theta) \end{aligned}}$$ Involute.



Area? $= \frac{1}{2}$ circle (left side)

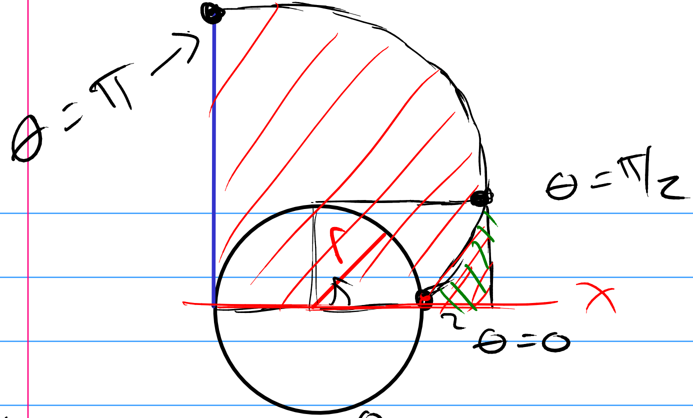
$+$ involute on right

$-$ small circle

$\frac{1}{2}$ circle

$\frac{1}{2}\pi(\pi r)^2$

$\frac{1}{2}\pi^3 r^2$



$$\int_a^b y\,dx = \int_\alpha^\beta y(t)\,x'(t)\,dt$$

$$\int_\pi^{\pi/2} y\,dx - \int_0^{\pi/2} y\,dx$$

$$\int_\pi^{\pi/2} y\,dx + \int_{\pi/2}^{0} y\,dx = \int_\pi^0 y\,\underbrace{dx}_{x'\,d\theta} \leftarrow \left(\text{area of upper Involute}\right)$$

$$y = r\sin\theta - r\theta\cos\theta$$

$$x = r\cos\theta + r\theta\sin\theta$$

$$\int_\pi^0 (r\sin\theta - r\theta\cos\theta)\cdot\overbrace{(\cancel{-r\sin\theta} + \cancel{r\sin\theta} + r\theta\cos\theta)}^{x'}\,d\theta$$

$$= \int_0^\pi (r\sin\theta - r\theta\cos\theta)(r\theta\cos\theta)\,d\theta$$

$$= \frac{(\pi^3 + 3\pi)\,r^2}{6} \quad \text{(by Maxima)}$$

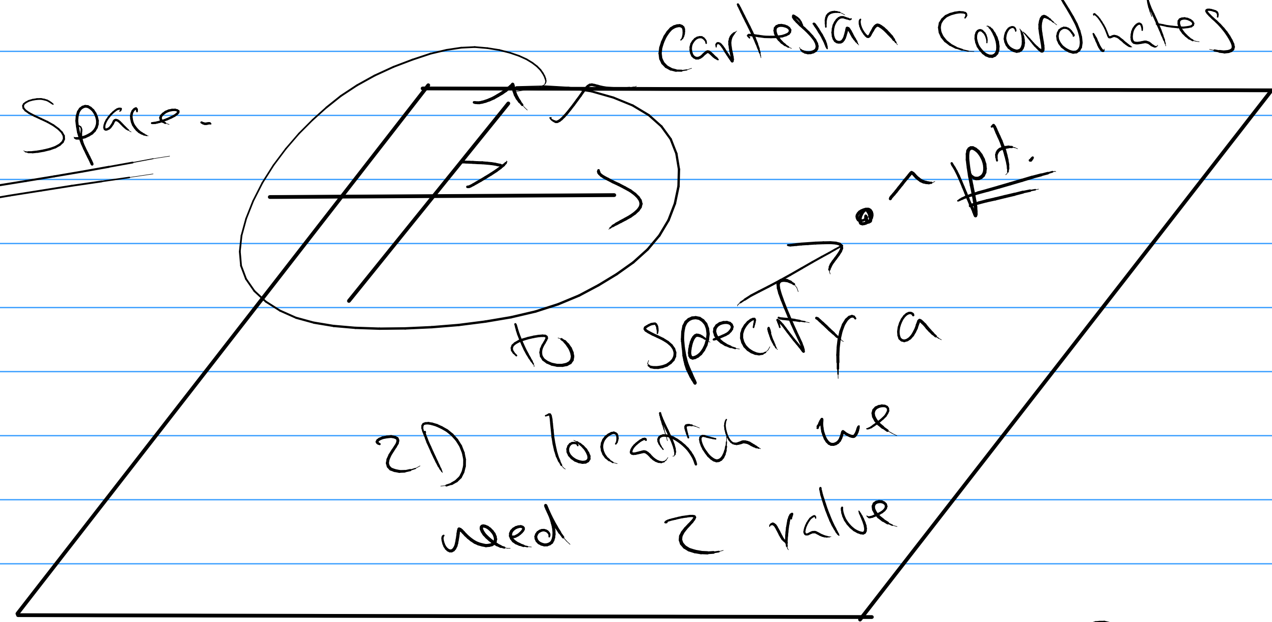
$$\text{Ans} = \underbrace{\tfrac{1}{2}\pi^3 r^2}_{\text{left } \frac{1}{2} \text{ circ.}} + \underbrace{2\,\frac{(\pi^3 + 3\pi)}{6}\,r^2}_{2 \text{ involutes}} - \underbrace{\pi r^2}_{\text{small circle}}$$

$$= \tfrac{1}{2}\pi^3 r^2 + \tfrac{1}{3}\pi^3 r^2 + \cancel{\pi r^2} - \cancel{\pi r^2}$$

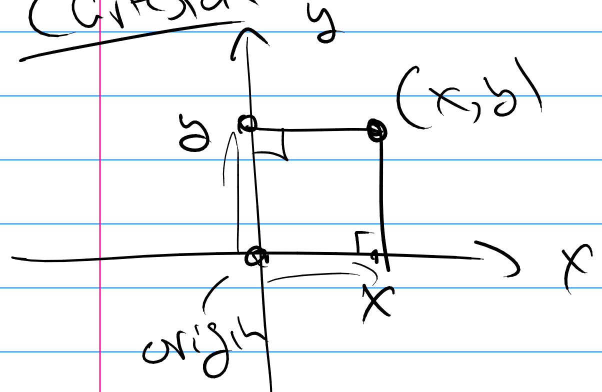
$$= \boxed{\frac{5\pi^3 r^2}{6}}$$

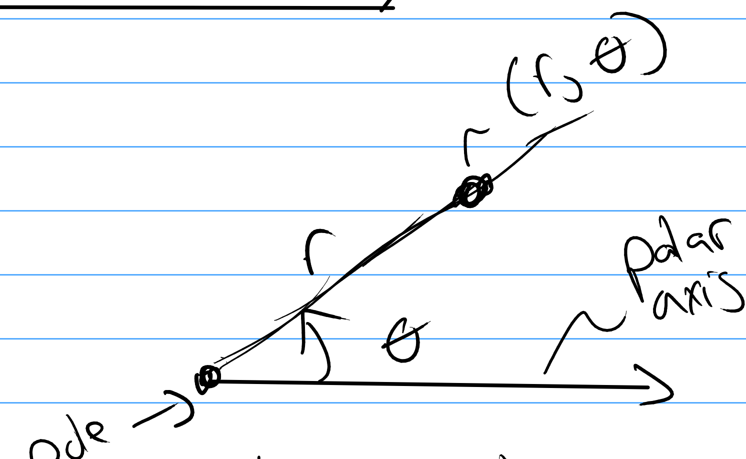
{9.3} Polar Coordinates

2D Space.

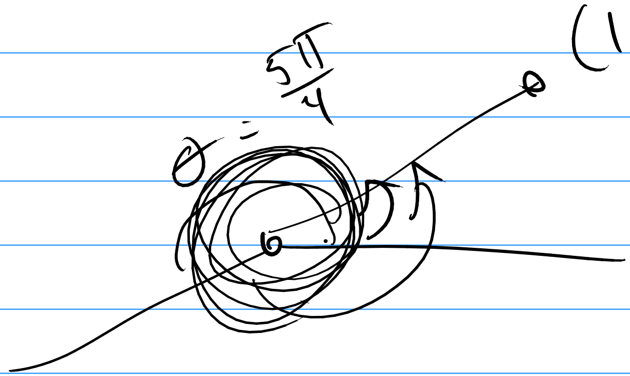


Cartesian





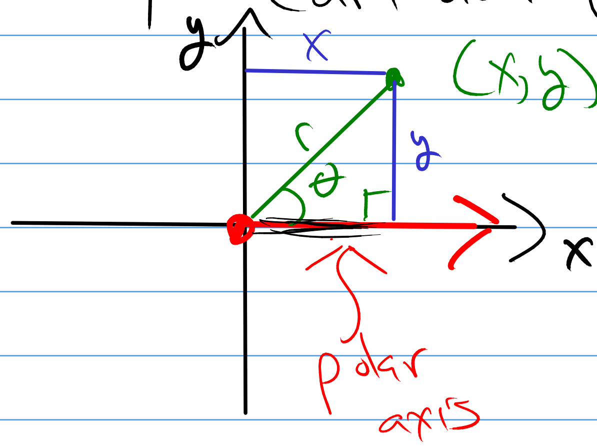
polar coordinates



$$(1, \pi/4) = (-1, 5\pi/4)$$

$$= (30\pi + \pi/4, 1)$$

Overlay Cartesian (+) Polar



$$x = r\cos\theta$$
$$y = r\sin\theta$$

or

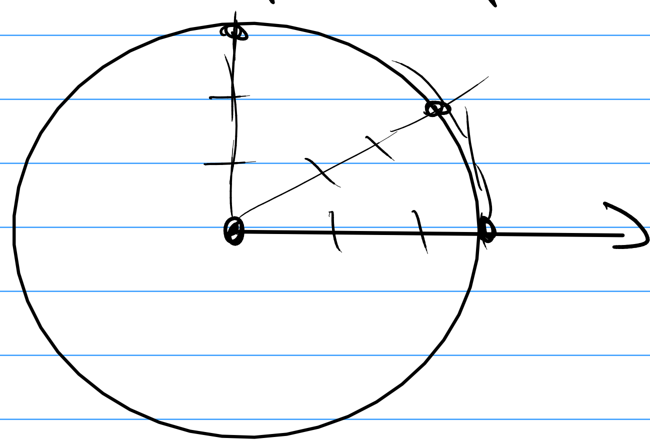
$$\tan\theta = \frac{y}{x}$$
$$r^2 = x^2 + y^2$$

Polar Curves

$r = f(\theta)$ explicit polar curve

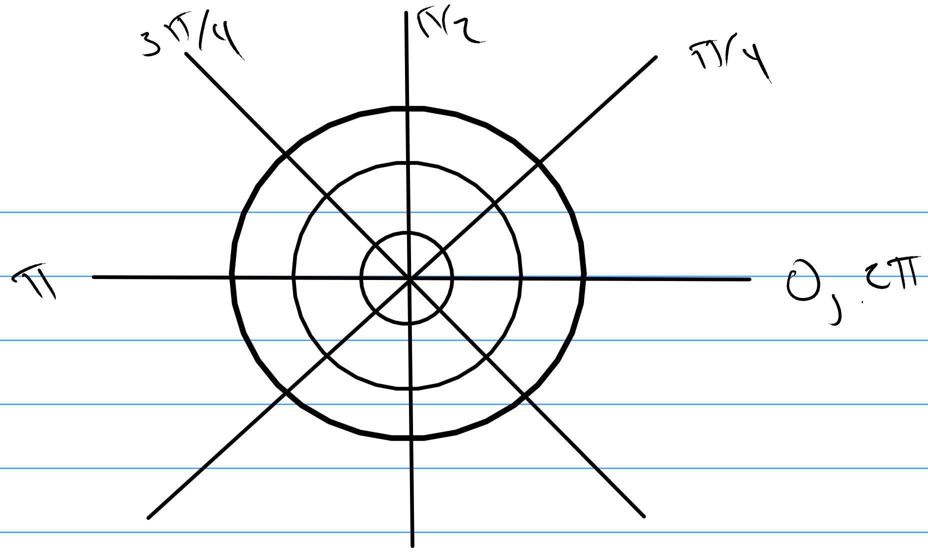
$F(r,\theta) = 0$ implicit polar curve

ex $r = 3$

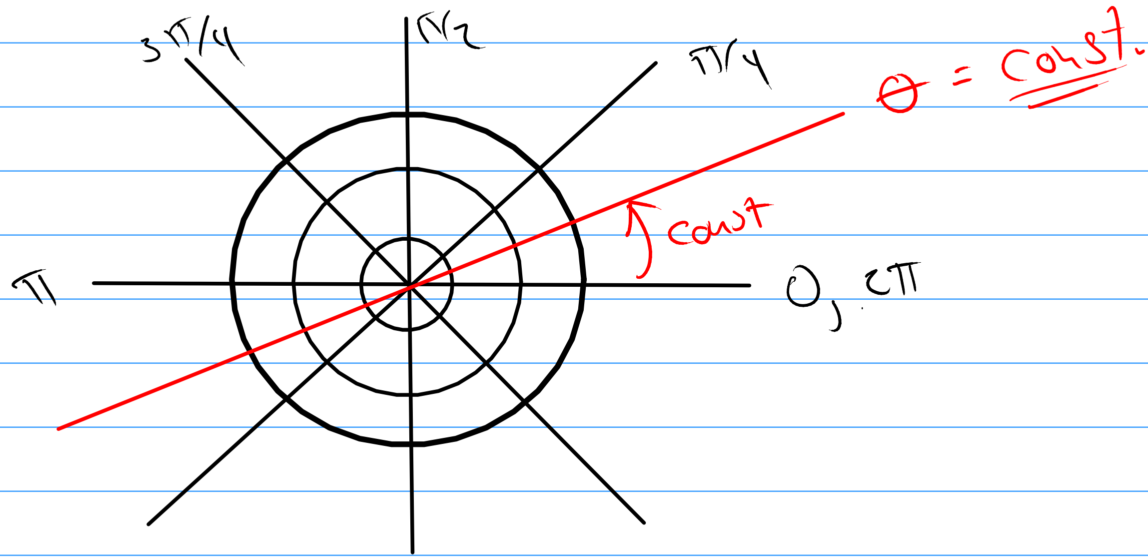


$r = \text{const}$ ← circle centered @ pole of radius = const.

Polar graph paper



$\theta = \text{const}$



ex

$$r = 2\cos\theta$$

$$r \cdot r = r \cdot 2 \cdot \cos\theta$$

$$r^2 = 2 \cdot r\cos\theta$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

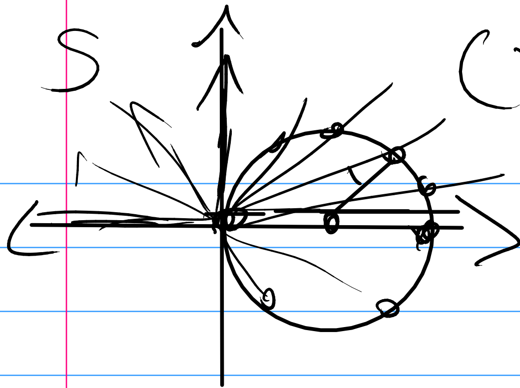
$$(x^2 - 2x + 1) + y^2 = 1 \rightarrow \boxed{(x-1)^2 + y^2 = 1}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$



$(x-1)^2 + y^2 = 1$

$r = 2\cos\theta$

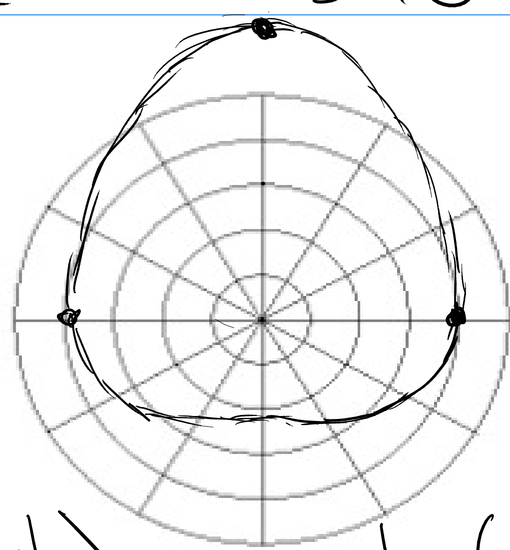
$r = n\cos\theta$

circle centered on x-axis of diameter n passing through origin

$r = n\sin\theta$

circle centered on y-axis of diameter n passing through origin

ex $r = 2 + \sin\theta$



Cardioids

$r = a + b\sin\theta$

$r = a + b\cos\theta$

$a > b$ (dimple) $a = b$ (heart) $b > a$ (loop)

rose curves

$r = \sin(n\theta)$

$r = \cos(n\theta)$

$n \equiv$ odd $\rightarrow$ n petals

$n \equiv$ even $\rightarrow$ $2n$ petals

$n \equiv$ not an integer $\rightarrow$??, **pretty**!

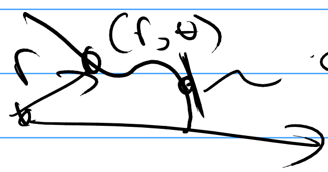
$\boxed{\text{Slope}}$ $\quad x(\theta),\ y(\theta)$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$$

Polar:

$$\left.\begin{aligned} x(\theta) &= r \cdot \cos\theta \\ y(\theta) &= r \cdot \sin\theta \end{aligned}\right\} \quad r = f(\theta)$$

$$x(\theta) = f(\theta)\cos\theta \qquad y(\theta) = f(\theta)\sin\theta$$

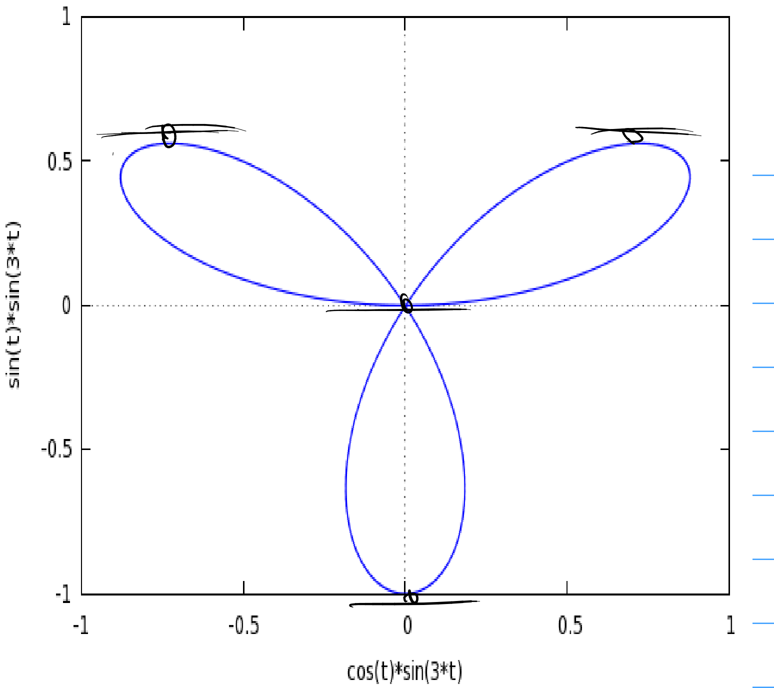


$\frac{dy}{dx}$ = slope

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[y(\theta)]}{\frac{d}{d\theta}[x(\theta)]} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$\boxed{ex}$ $r = \sin(3\theta)$

$\frac{dy}{dx} \stackrel{?}{=} 0$



$x(\theta) = \sin(3\theta)\cos\theta$

$y(\theta) = \sin(3\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{3\cos(3\theta)\sin\theta + \sin 3\theta\cos\theta}{3\cos(3\theta)\cos\theta - \sin 3\theta\sin\theta}$$

$\frac{dy}{dx} = 0$ means $3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta = 0$

$$3\cos 3\theta\sin\theta = -\sin 3\theta\cos\theta$$

$$3\cot 3\theta = -\cot\theta$$

$$3\cot(2\theta+\theta) + \cot(\theta) = 0$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$\boxed{\cos(3\theta)} = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta$ etc.

$r = e^{\theta}$ Find vert. / horz. tangents.

$x(\theta) = e^{\theta}\cos\theta$
$y(\theta) = e^{\theta}\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[y(\theta)]}{\frac{d}{d\theta}[x(\theta)]} = \frac{e^{\theta}\sin\theta + e^{\theta}\cos\theta}{e^{\theta}\cos\theta - e^{\theta}\sin\theta}$$

① horz: $e^{\theta}\sin\theta + e^{\theta}\cos\theta = 0$

② vert: $e^{\theta}\cos\theta - e^{\theta}\sin\theta = 0$

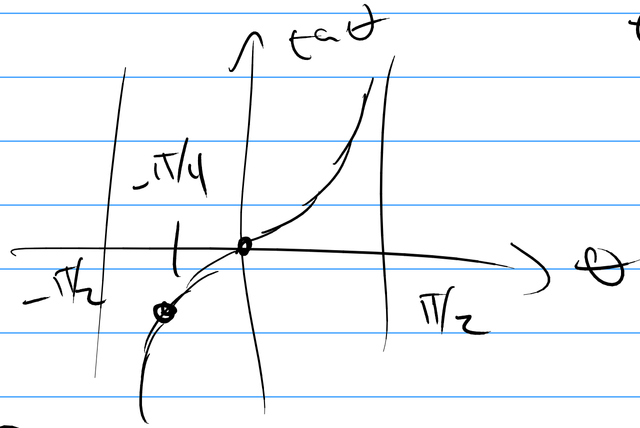
① $e^{\theta}\sin\theta + e^{\theta}\cos\theta = 0$

$e^{\theta}(\sin\theta + \cos\theta) = 0$

$e^{\theta} = 0$ $\sin\theta + \cos\theta = 0$

Never $\sin\theta = -\cos\theta$

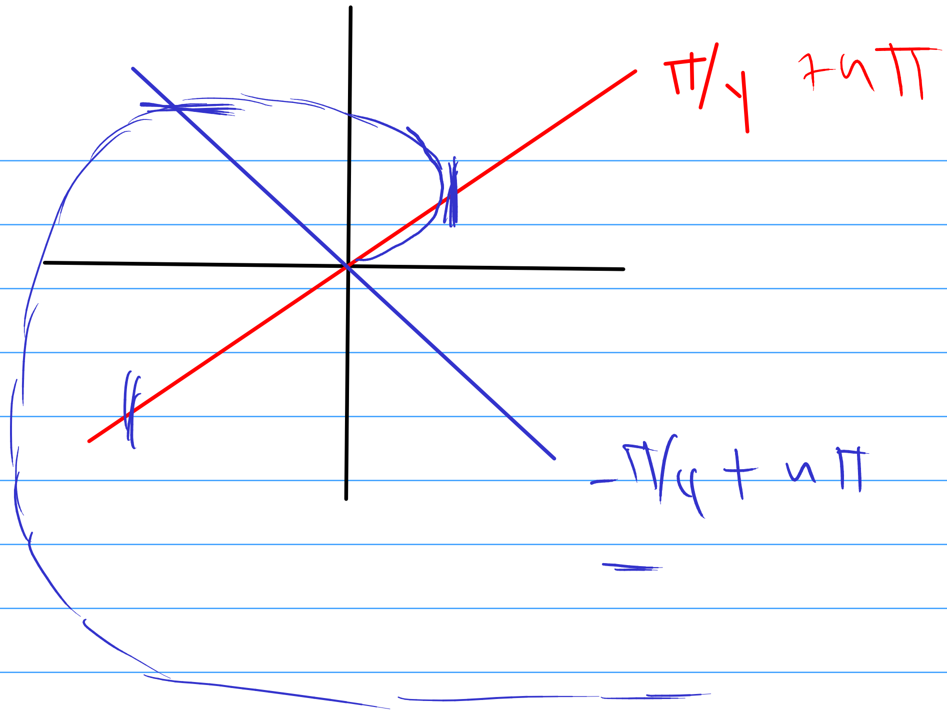
$\tan\theta = -1$



$\theta = -\pi/4 + n\pi$

② $e^{\theta}\cos\theta - e^{\theta}\sin\theta = 0$ $\tan\theta = 1$

$e^{\theta}(\cos\theta - \sin\theta) = 0$ $\theta = \pi/4 + n\pi$



π/4 + nπ
-π/4 + nπ