CIS 770: Formal Language Theory

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Parenthesis Matching

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  - The concatenation of two correctly matched expressions, or
  - It must begin with (and end with) and moreover, once the first and last symbols are removed, the resulting string must correspond to a valid expression.

Taking E to be the set of correct expressions, the inductive definition can be succinctly written as

$$
\begin{aligned}
E &\rightarrow \epsilon \\
E &\rightarrow EE \\
E &\rightarrow (E)
\end{aligned}
$$

English Sentences

English sentences can be described as

$$
\begin{aligned}
\langle S \rangle &\rightarrow \langle NP \rangle \langle VP \rangle \\
\langle NP \rangle &\rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\
\langle VP \rangle &\rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\
\langle PP \rangle &\rightarrow \langle P \rangle \langle CN \rangle \\
\langle CN \rangle &\rightarrow \langle A \rangle \langle N \rangle \\
\langle CV \rangle &\rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\
\langle A \rangle &\rightarrow \text{a} \mid \text{the} \\
\langle N \rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
\langle V \rangle &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees} \\
\langle P \rangle &\rightarrow \text{with}
\end{aligned}
$$

English Sentences
Examples

$$\overbrace{\underbrace{\text{a}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}}^{\text{verb-phrs}}$$

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Applications

Such rules (or grammars) play a key role in

- Parsing programming languages and natural languages
- Markup Languages like HTML and XML.
- Modelling software

Grammars

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- R is a finite set of rules or productions. Each production is of the form $\alpha \to \beta$ where $\alpha, \beta \in (V \cup \Sigma)^*$
- $S \in V$ is the start symbol; it is the variable that represents the language being defined. Other variables represent auxiliary languages that are used to define the language of the start symbol.

Example of a CFG

Example

Let $G_{\mathrm{par}} = (V, \Sigma, R, S)$ be

- $V = \{E\}$
- $\Sigma = \{(,)\}$
- $R = \{E \to \epsilon, E \to EE, E \to (E)\}$
- $S = E$

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$G_{\mathrm{pal}} = (\{S\}, \{0,1\}, R, S)$ defines palindromes over $\{0,1\}$, where R is

$$S \to \epsilon$$
$$S \to 0$$
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Or more briefly, $R = \{S \to \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$

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Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$
$G_{\mathrm{exp}} = (\{E, I, N\}, \{a, b, 0, 1, (,), +, *, -\}, R, E)$ where R is

$$\begin{array}{l} E \to I \mid N \mid E + E \mid E * E \mid (E) \\ I \to a \mid b \mid Ia \mid Ib \\ N \to 0 \mid 1 \mid N0 \mid N1 \mid -N \mid +N \end{array}$$

More Examples

Example

Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

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\begin{array}{lll}
S \rightarrow aSBC \mid aBC & CB \rightarrow HB & HB \rightarrow HC \\
HC \rightarrow BC & aB \rightarrow ab & bB \rightarrow bb \\
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\end{array}
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Derivation

Expand the start symbol using one of its rules. Then expand the resulting string by replacing one of its substrings that matches the LHS of a rule by the RHS. Repeat until you get a string of terminals.

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we have

$$\begin{array}{rcl} E & \Rightarrow & E * E \Rightarrow E * N \Rightarrow E * -N \Rightarrow E * -N \Rightarrow E * -1 \\ & \Rightarrow & (E) * -1 \Rightarrow (E + E) * -1 \Rightarrow (E + I) * -1 \\ & \Rightarrow & (E + a) * -1 \Rightarrow (I + a) * -1 \Rightarrow (a + a) * -1 \end{array}$$

Formal Definition

Definition

Let $G = (V, \Sigma, R, S)$ be a grammar. We say $\gamma_1\alpha\gamma_2 \Rightarrow_G \gamma_1\beta\gamma_2$, where $\gamma_1, \gamma_2, \alpha, \beta \in (V \cup \Sigma)^*$ if $\alpha \to \beta$ is a rule of G.

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Notation

When G is clear from the context, we will write $\Rightarrow$ and $\overset{*}{\Rightarrow}$ instead of $\Rightarrow_G$ and $\overset{*}{\Rightarrow}_G$.

Language of a Grammar

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The language of a grammar $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from S using the rules in R.

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The language of a grammar $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from S using the rules in R. In other words,

$$L(G) = \{w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow} w\}$$

Example I

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Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

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&\Rightarrow aaaBBCCBC \Rightarrow aaaBBCHBC \Rightarrow aaaBBCHCC \Rightarrow aaaBBCBCC \\
&\Rightarrow aaaBBHBCC \Rightarrow aaaBBHCCC \Rightarrow aaaBBBCCC \Rightarrow aaabBBCCC \\
&\Rightarrow aaabbBCCC \Rightarrow aaabbbCCC \Rightarrow aaabbbcCC \Rightarrow aaabbbccC \Rightarrow aaabbbccc
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$$L(G) = \{a^n b^n c^n \mid n \geq 0\}$$

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$L(G) = \{a^i \mid i \text{ is a power of } 2\}$

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$G_{\mathrm{pal}} = (\{S\}, \{0,1\}, \{S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}, S)$ we have

$$S \Rightarrow 0S0 \Rightarrow 00S00 \Rightarrow 001S100 \Rightarrow 0010100$$

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Notation

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Context-Free Language

Definition

The language of CFG $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from S using the rules in R. In other words,

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Definition

A language L is said to be context-free if there is a CFG G such that $L = L(G)$.

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Proposition

$L(G_{\mathrm{pal}}) = L_{\mathrm{pal}}$

Proof.

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- Base Cases: If $|w| = 0$ or $|w| = 1$ then $w = \epsilon$ or 0 or 1. And $S \to \epsilon \mid 0 \mid 1$.

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- Base Cases: If $|w| = 0$ or $|w| = 1$ then $w = \epsilon$ or 0 or 1. And $S \rightarrow \epsilon \mid 0 \mid 1$.
- Induction Step: If $|w| \geq 2$ and $w = w^R$ then it must begin and with the same symbol.

Proving Correctness of CFG

$L_{\mathrm{pal}} \subseteq L(G_{\mathrm{pal}})$

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- Induction Step: If $|w| \geq 2$ and $w = w^R$ then it must begin and with the same symbol. Let $w = 0x0$. Now, $w^R = 0x^R0 = w = 0x0$; thus, $x^R = x$. By induction hypothesis, $S \stackrel{*}{\Rightarrow} x$. Hence $S \Rightarrow 0S0 \stackrel{*}{\Rightarrow} 0x0$. If $w = 1x1$ the argument is similar.

Proof (contd).

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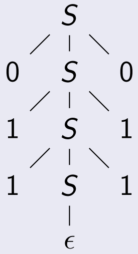

Example Parse Tree with yield 011110

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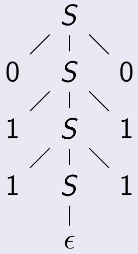

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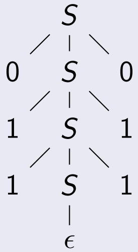

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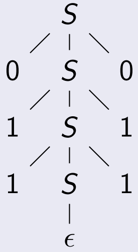

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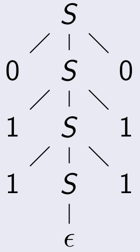

Example Parse Tree with yield 011110

Yield of a parse tree is the concatenation of leaf labels (left–right)

Parse Trees and Derivations

Proposition

Let $G = (V, \Sigma, R, S)$ be a CFG. For any $A \in V$ and $\alpha \in (V \cup \Sigma)^*$, $A \stackrel{*}{\Rightarrow} \alpha$ iff there is a parse tree with root labeled A and whose yield is α.

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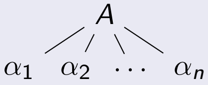
Proof.

($\Rightarrow$): Proof by induction on the number of steps in the derivation.

- Base Case: If $A \Rightarrow \alpha$ then $A \to \alpha$ is a rule in G. There is a tree of height 1, with root A and leaves the symbols in α.



Parse Tree for Base Case

Parse Trees for Derivations

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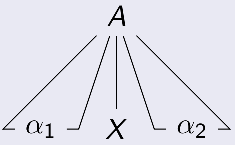

Parse Tree for Induction Step

Parse Trees for Derivations

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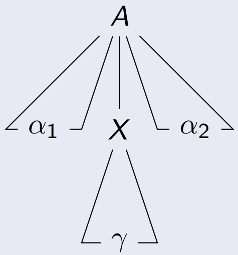
- Add leaves $X_1, \ldots X_n$ and make them children of X. New tree is a parse tree with desired yield. $\cdots\!\!\longrightarrow$



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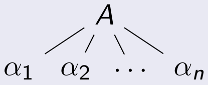

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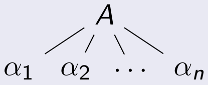

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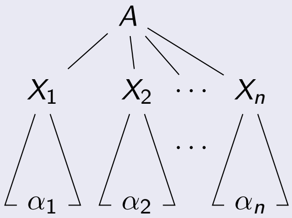

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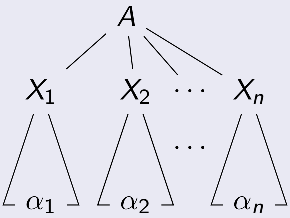

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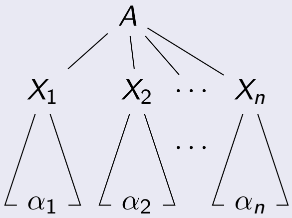

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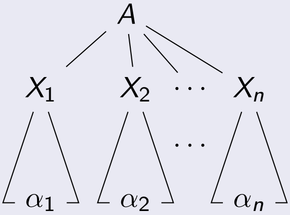

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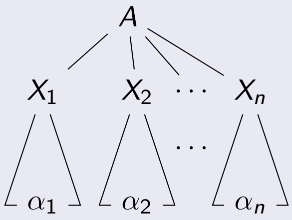

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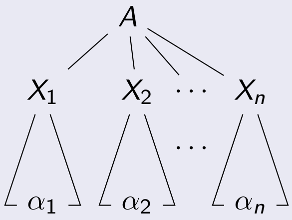

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For a CFG G with variable A the following are equivalent

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Context-free-ness

CFGs have the property that if $X \stackrel{*}{\Rightarrow} \gamma$ then $\alpha X \beta \stackrel{*}{\Rightarrow} \alpha\gamma\beta$

Example: English Sentences

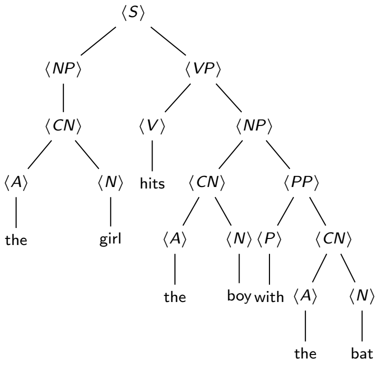
English sentences can be described as

$$
\begin{aligned}
\langle S \rangle &\to \langle NP \rangle \langle VP \rangle \\
\langle NP \rangle &\to \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\
\langle VP \rangle &\to \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\
\langle PP \rangle &\to \langle P \rangle \langle CN \rangle \\
\langle CN \rangle &\to \langle A \rangle \langle N \rangle \\
\langle CV \rangle &\to \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\
\langle A \rangle &\to \text{a} \mid \text{the} \\
\langle N \rangle &\to \text{boy} \mid \text{girl} \mid \text{bat} \\
\langle V \rangle &\to \text{hits} \mid \text{likes} \mid \text{sees} \\
\langle P \rangle &\to \text{with}
\end{aligned}
$$

Multiple Parse Trees

Example 1

The sentence "the girl hits the boy with the bat" has the following parse tree



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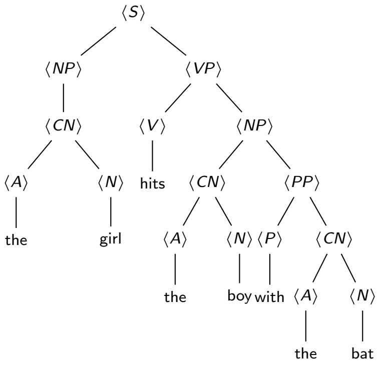

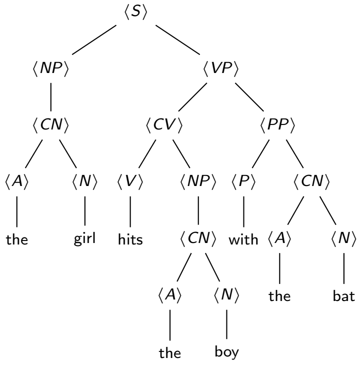

Example: Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$

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$$\begin{aligned} E &\rightarrow I \mid N \mid -N \mid E + E \mid E * E \mid (E) \\ I &\rightarrow a \mid b \mid Ia \mid Ib \\ N &\rightarrow 0 \mid 1 \mid N0 \mid N1 \end{aligned}$$

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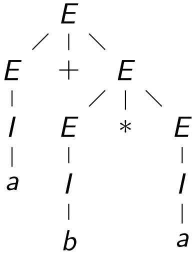

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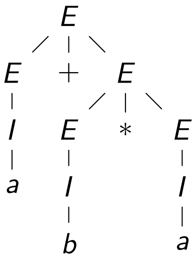

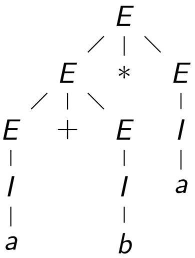

Ambiguity

Definition

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Warning!

Existence of two derivations for a string does not mean the grammar is ambiguous!

Removing Ambiguity

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An Example

Recall, G_{exp} has the following rules

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New CFG G'_{exp} has the rules

$$
\begin{aligned}
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&N \rightarrow 0 \mid 1 \mid N0 \mid N1\\
&F \rightarrow I \mid N \mid -N \mid (E)\\
&T \rightarrow F \mid T * F\\
&E \rightarrow T \mid E + T
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Ambiguity: Computational Problems

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Problem: Given CFG G, find CFG G' such that $L(G) = L(G')$ and G' is unambiguous.

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Problem: Given CFG G, determine if G is ambiguous.
The problem is undecidable.

Problem: Is it the case that for every CFG G, there is a grammar G' such that $L(G) = L(G')$ and G' is unambiguous, *even if* G' *cannot be constructed algorithmically*?

Inherently Ambiguous Languages

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Consider

$$L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

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