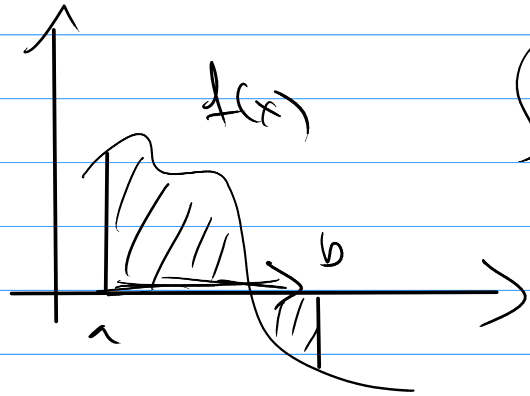
Math 293

Chapter 7 Applications of Integration.

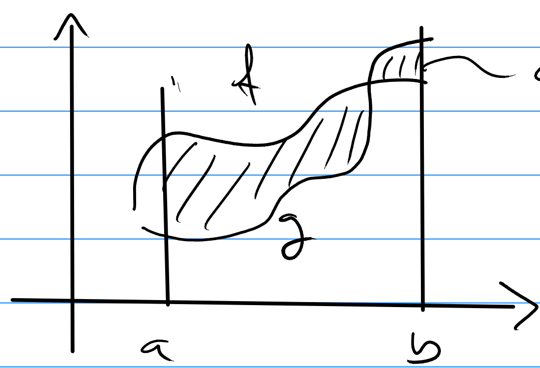
(1) Area (2) Volume



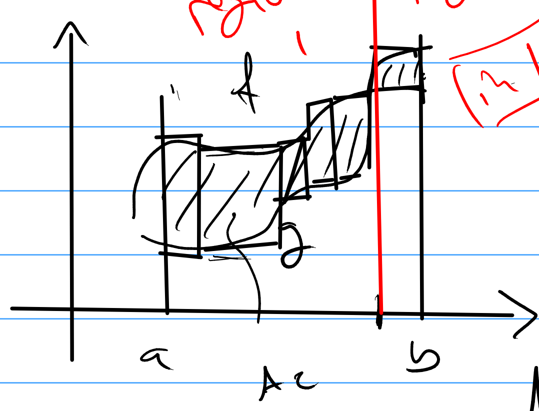
$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{pos or neg.}}\underbrace{\Delta x_i}_{\text{pos}}$$

net signed area

Area ← always positive



area between the curves?



$$A_i = [\underbrace{f(x_i^*)}_{\text{top}} - \underbrace{g(x_i^*)}_{\text{bottom}}]\,\Delta x_i$$

$$A \approx \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x_i$$

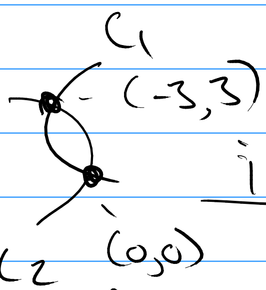
$$A = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \underbrace{\left(f(x_i^*) - g(x_i^*)\right)}_{\text{function}} \Delta x_i$$

Area between $f(x) \geq g(x)$ $= \boxed{\int_a^b (f(x) - g(x))\,dx}$

$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

ex area between $x = y^2 - 4y$: c_1

$x = 2y - y^2$: c_2

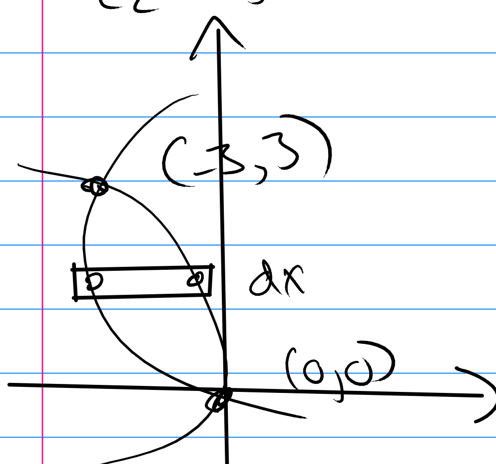


intercepts.

$y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$2y(y-3) = 0$



$y = 0 \qquad y = 3$

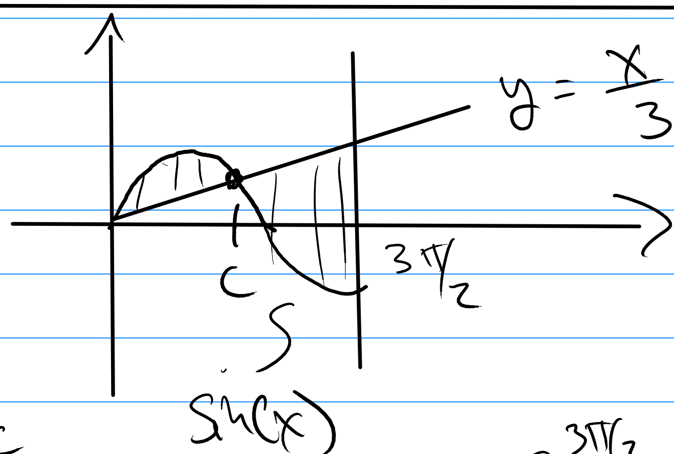
$x = 0 \qquad x = -3$

$$\int_0^3 \left((2y - y^2) - (y^2 - 4y)\right) dy$$

$$= \int_0^3 (-2y^2 + 6y)\,dy = -\frac{2}{3}y^3 + 3y^2 \Big|_0^3$$

$= -18 + 27 = \boxed{9}$

Ex



$\sin x = \frac{x}{3}$

$\sin x - \frac{x}{3} = 0$

$$\int_0^c \left(\sin x - \frac{x}{3}\right)dx + \int_c^{3\pi/2} \left(\frac{x}{3} - \sin x\right) dx$$

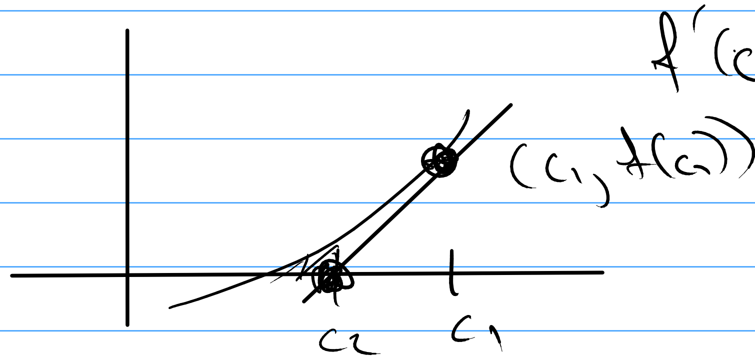
$$\left(-\cos x - \frac{x^2}{6}\right)\Big|_0^c + \left(\frac{x^2}{6} + \cos x\right)\Big|_c^{3\pi/2}$$

$$\left[\left(-\cos c - \frac{c^2}{6}\right) - (-1)\right] + \left[\left(\frac{9\pi^2}{24}\right) - \left(\frac{c^2}{6} + \cos c\right)\right]$$

$$\boxed{-2\cos c - \frac{c^2}{3} + 1 + \frac{9\pi^2}{24}}$$

c is? $f(c) = \sin c - \frac{c}{3} = 0$

$f'(c_1) = \cos c_1 - \frac{1}{3}$



$$y - f(c_1) = f'(c_1)(c - c_1)$$

to find c_1 let $y = 0$

$$-f(c_1) = f'(c_1)(c_2 - c_1)$$

$$-\frac{f(c_1)}{f'(c_1)} = c_2 - c_1$$

See p. 237

Newton's method $\to$ $c_2 = c_1 - \dfrac{f(c_1)}{f'(c_1)}$

$\vdots$

$c_n \to \boxed{c^*}$ where it crosses.

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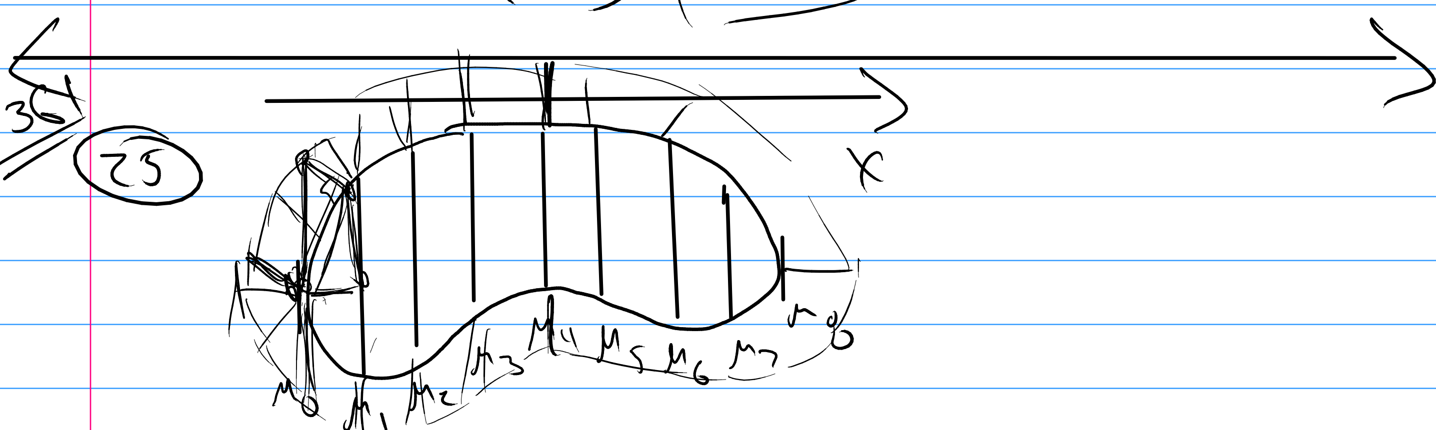
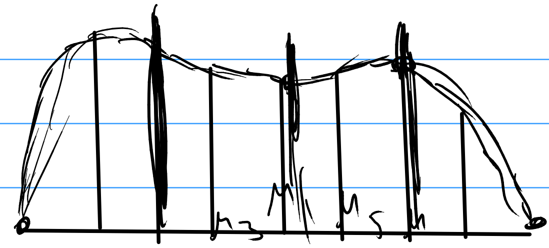


$M_0 = 0$, $M_1 = 6.2$, $M_2 = 7.2$, $M_3 = 6.8$, $M_4 = 5.6$

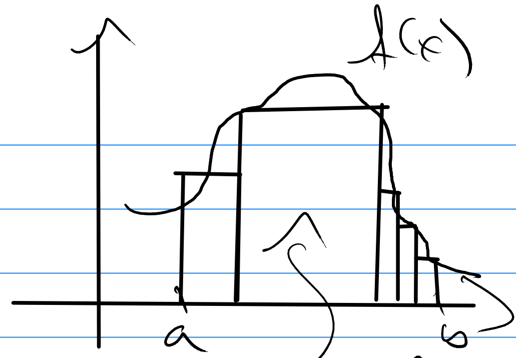
$M_5 = 5.0$ $\quad M_6 = 4.8$ $\quad M_7 = 4.8$ $\quad M_8 = 0$

$\Delta x = 2$

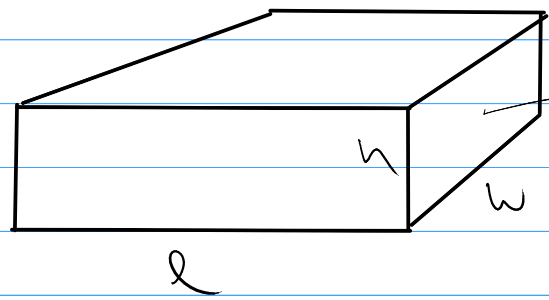


$$\text{Area} \approx \frac{2}{3}\big(0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0\big)$$

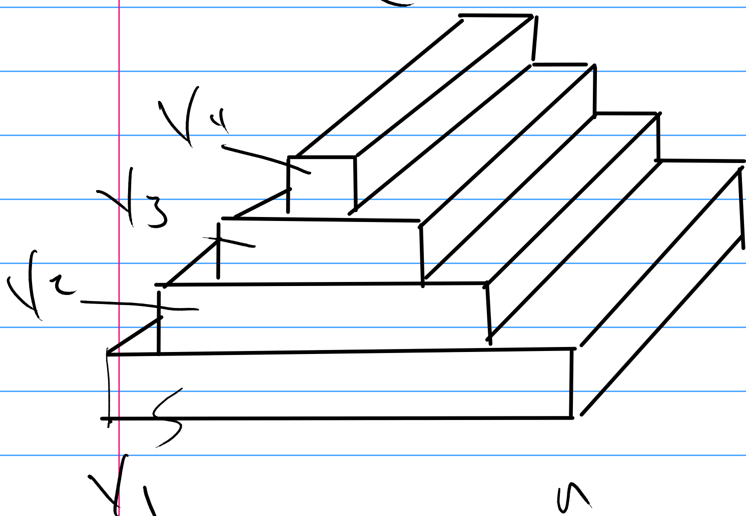
Volume



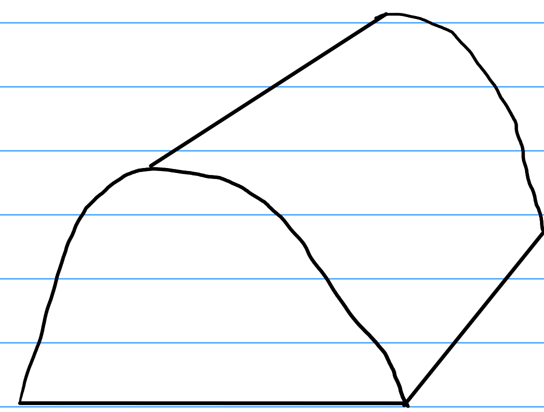
sum of rectangles $\approx$ Area



$V = l \cdot w \cdot h$

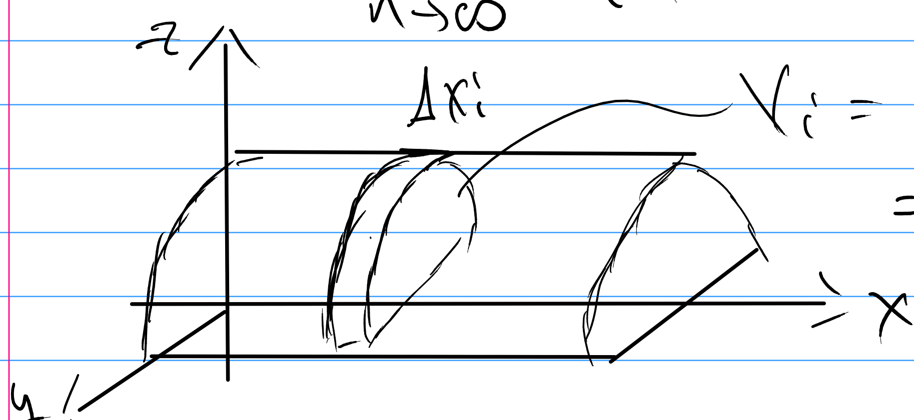


$\approx$



$$V \approx \sum_{i=1}^{n} V_i$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} V_i$$



V_i = Slices Volume

= (Area of cross section) Δx_i

Area of cross section = $A(x_i^*)$

Volume of cross section $A(x_i^*)\,\Delta x_i$

$$\text{Volume} = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\,\Delta x_i$$

$$\text{Volume} = \int_a^b A(x)\,dx$$

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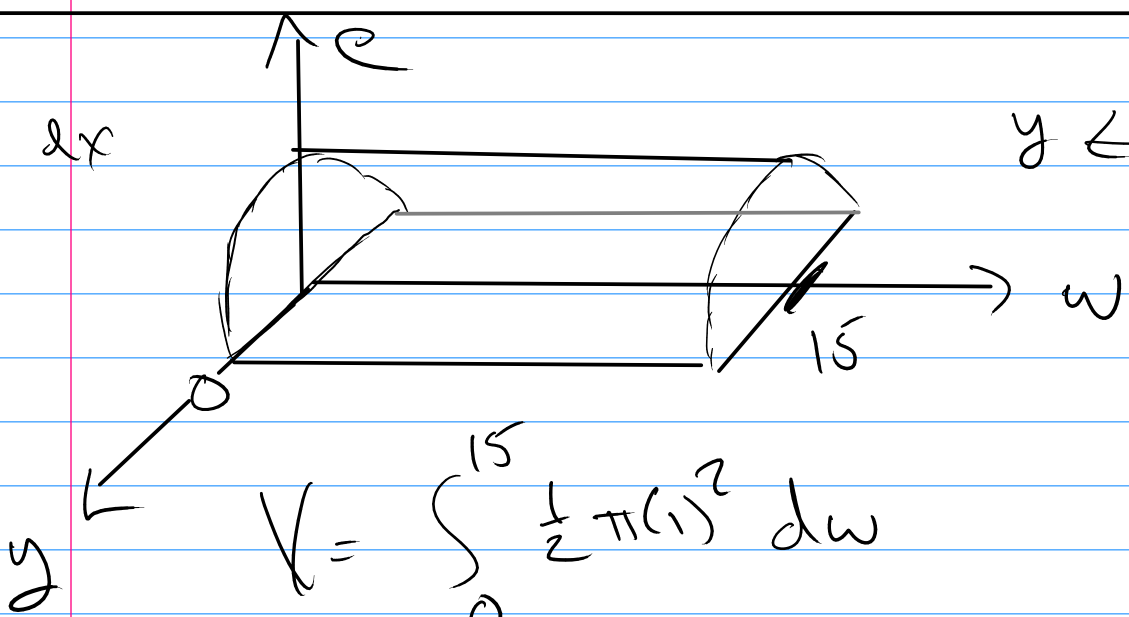
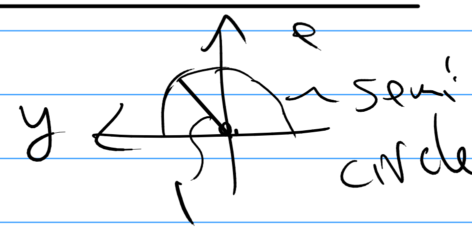
dx

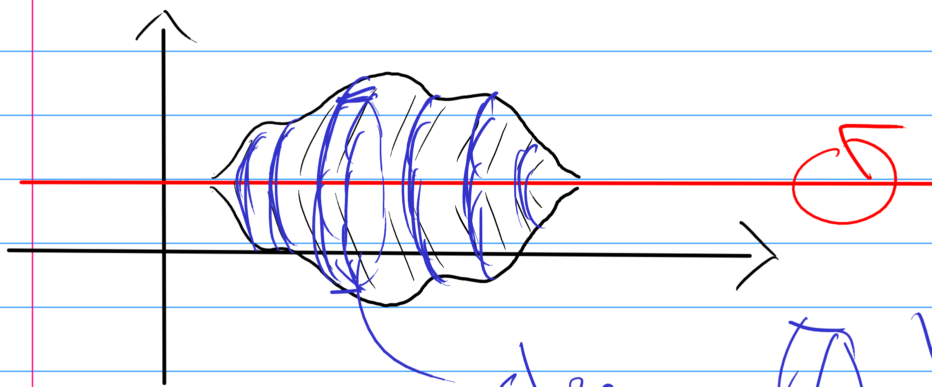




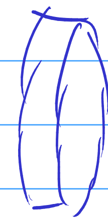
$$V = \int_0^{15} \tfrac{1}{2}\pi(1)^2\,dw$$

$$V = \tfrac{1}{2}\pi w \Big|_0^{15} = \boxed{\tfrac{15}{2}\pi \text{ units}^3}$$

Solids of Revolution

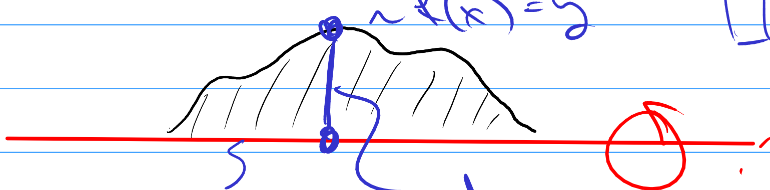


slice



$V_i = \pi r^2 \cdot dx$

disk



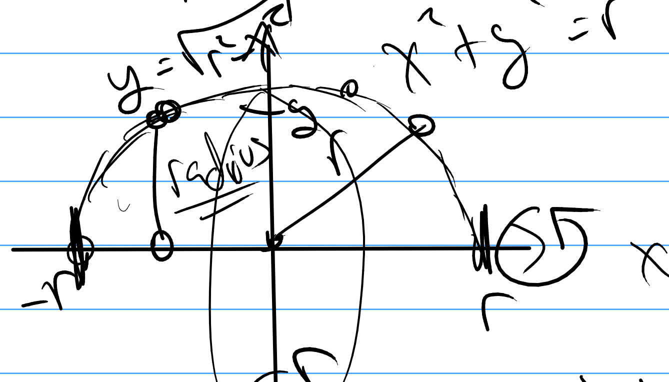
$y=c$

$\underline{\text{radius}} = f(x)-c$

Volume by disks

$$V=\int_a^b \pi \underbrace{r^2}_{\text{radius due to rotation}} dx$$

ex **Sphere**



$$V=\int_{-r}^{r}\pi(\text{radius})^2\,dx$$

$$=\int_{-r}^{r}\pi y^2\,dx = \int_{-r}^{r}\pi(r^2-x^2)\,dx$$

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = 2\pi \int_{0}^{r} (r^2 - x^2)\,dx$$

$$V = 2\pi \left(r^2 x - \tfrac{1}{3} x^3 \right)\Big|_0^r$$

$$= 2\pi \left[\left(r^3 - \tfrac{1}{3} r^3 \right) - (0) \right]$$

$$= \boxed{\tfrac{4}{3}\pi r^3}$$

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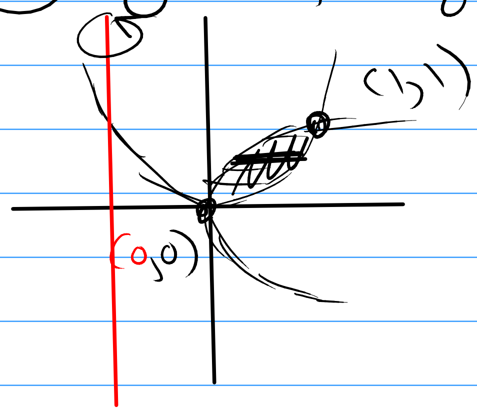
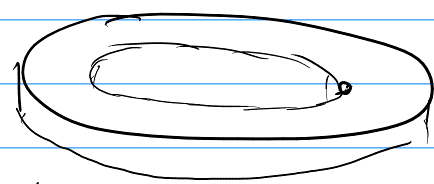
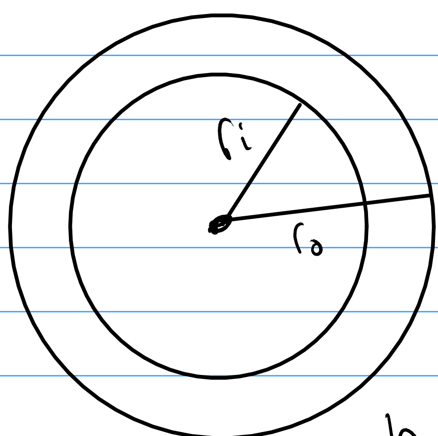
ex $y = x^2,\ x = y^2$ about $x = -1$





$$\int_a^b \boxed{A(y)}\,dy$$

cross sectional Area



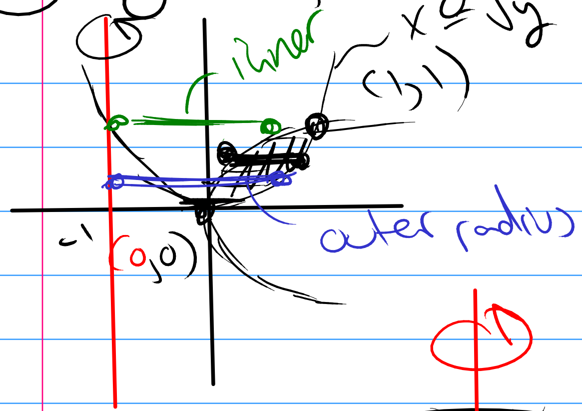
$$A = \pi r_o^2 - \pi r_i^2$$

$$A = \pi (r_o^2 - r_i^2)$$

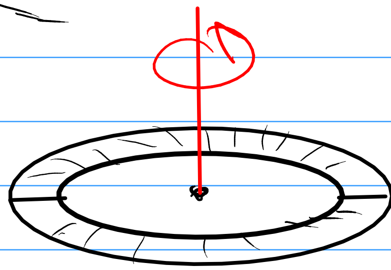
<u>Washers</u>: $V = \int_a^b \pi (r_o^2 - r_i^2)\,dy$

outer inner

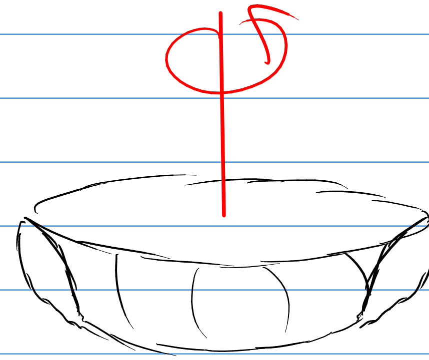
ex $y = x^2$, $x = y^2$ about $x = -1$



$$V = \int_0^1 \pi\left((\sqrt{y}+1)^2 - (y^2+1)^2\right)dy$$



one slice



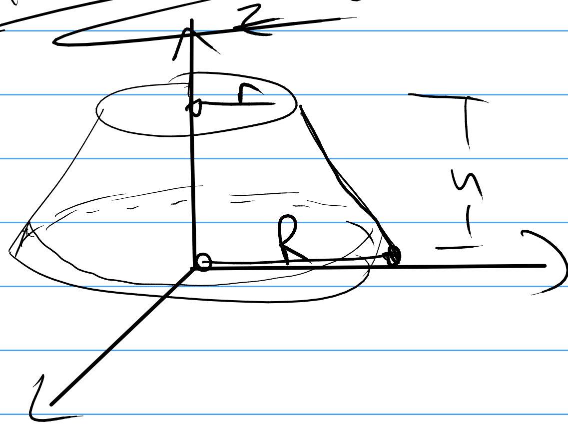
$$V = \int_0^1 \pi\left((\sqrt{y}+1)^2 - (y^2+1)^2\right)dy$$

$$V = \pi\int_0^1 \left(y + 2\sqrt{y} \cancel{+1} - y^4 - 2y^2 \cancel{-1}\right)dy$$

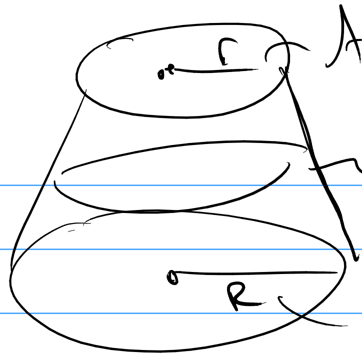
$$= \pi\int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2\right)dy$$

$$= \pi\left(\quad \text{Finish!} \right)$$

Harder Slicing



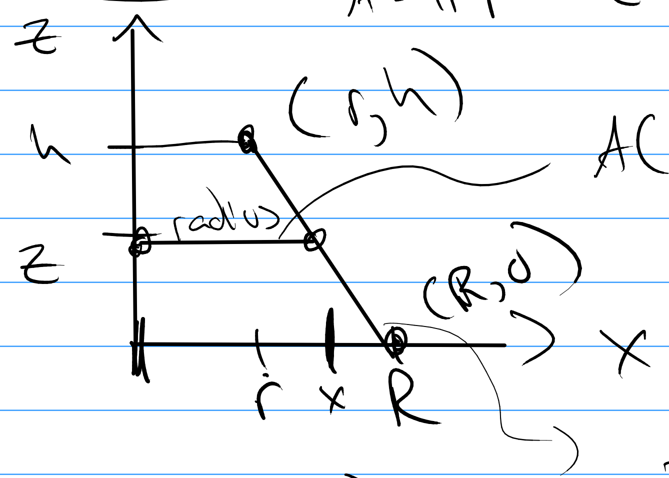
$$V = \int_0^h \underbrace{A(z)}_{\text{cross sections}}\,dz$$



$A = \pi r^2 \quad z = h$

$A(z) = ?$

$A = \pi R^2 \quad z = 0$



$A(z) = \pi\,(\text{radius})^2$

$$z - 0 = \frac{-h}{R-r}(x - R)$$

$$y - y_0 = m(x - x_0)$$

$$z = \frac{-h}{R-r}(x-R)$$

$$x = \boxed{R + \frac{r-R}{h} z} = \text{radius}$$

$$A(z) = \pi\left(R + \frac{r-R}{h} z\right)^2$$

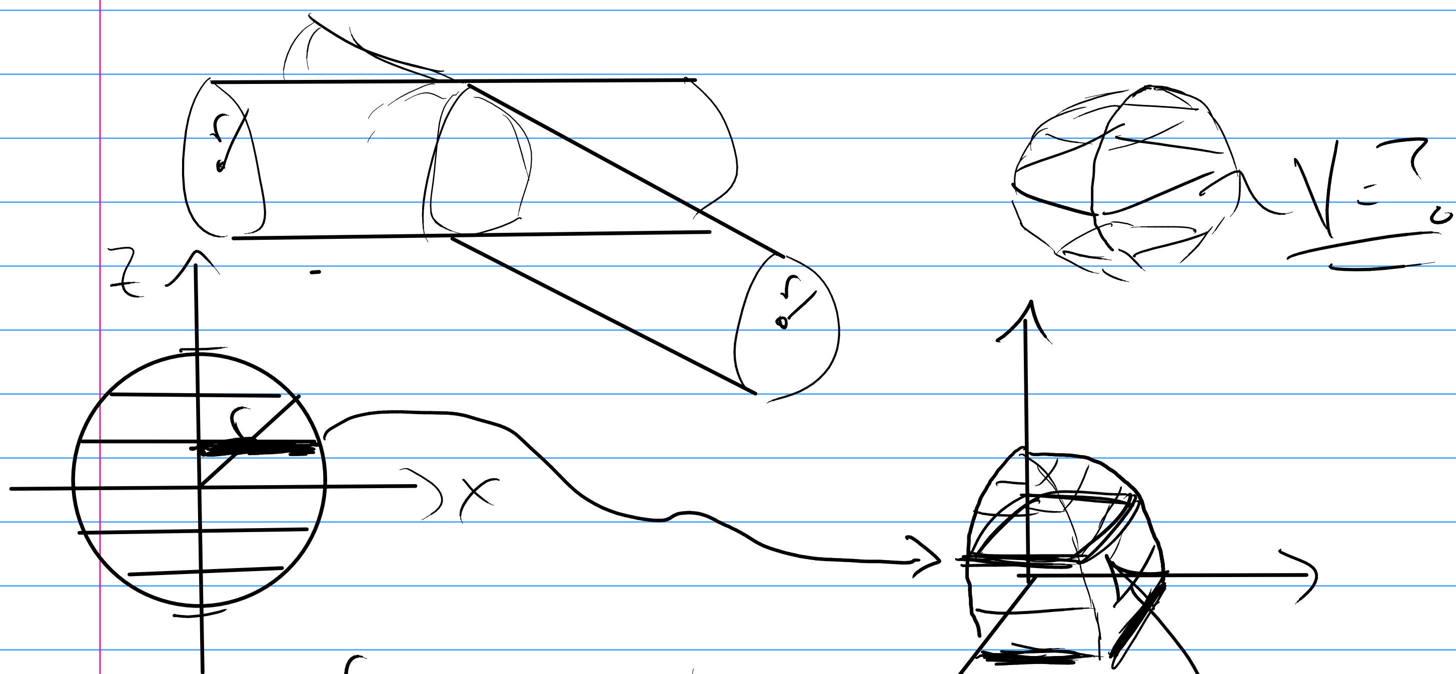
$$V = \int_0^h \pi\left(R + \frac{r-R}{h} z\right)^2 dz$$

$$V = \frac{\pi h}{3(r-R)}\left(R + \frac{r-R}{h} z\right)^3 \Bigg|_0^h$$

$$V = \frac{\pi}{3}\,\frac{h}{r-R}\left[\left(\cancel{R} + r - \cancel{R}\right)^3 - R^3\right]$$

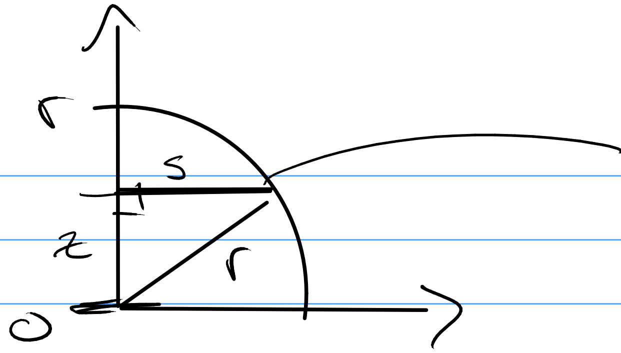
$$V = \frac{h\pi}{3(r-R)}\left[r^3 - R^3\right]$$

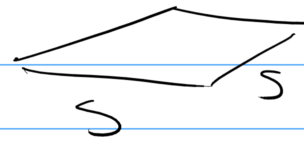
$$V = \boxed{\frac{h\pi}{3}\left[r^2 + rR + R^2\right]}$$



$$V = \int_{-r}^{r} \underbrace{A(z)}\,dz$$

$\boxed{\text{cross sectional Area}} = 4 * \text{Area in } 1^{st} \text{ octant}$





$A_1 = s^2$

$s^2 + z^2 = r^2$

$$A_1(z) = r^2 - z^2$$

$$V = \int_{-r}^{r} 4(r^2 - z^2)\,dz$$

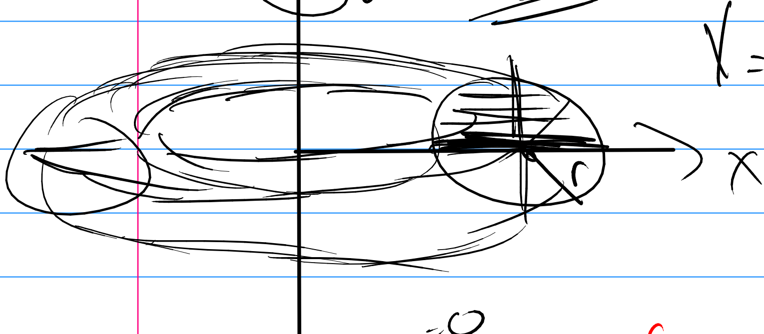
$$= 8\int_0^r (r^2 - z^2)\,dz$$

$$= 8\ r^2 z - \tfrac{1}{3}z^3 \Big|_0^r$$

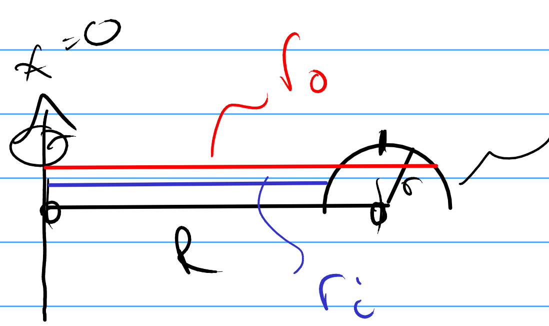
$$= 8\left(r^3 - \tfrac{1}{3}r^3\right)$$

$$= \boxed{\tfrac{16}{3}r^3}$$

Avg washers.



$$V = 2\int_0^r \pi\left[r_o^2 - r_i^2\right]dy$$



$(x-R)^2 + y^2 = r^2$

$(x-R)^2 = r^2 - y^2$

$$x - R = \pm\sqrt{r^2 - y^2}$$

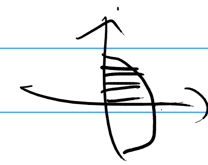
r_o: $x = R + \sqrt{r^2 - y^2}$ is)

r_i: $x = R - \sqrt{r^2 - y^2}$ is (

$$V = 2\int_0^r \pi\left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2\right] dy$$

$$V = 2\pi\int_0^r \left(\cancel{R^2} + 2R\sqrt{r^2 - y^2} + \cancel{(r^2 - y^2)}\right) - \left(\cancel{R^2} - 2R\sqrt{r^2 - y^2} + \cancel{(r^2 - y^2)}\right) dy$$

$$V = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$$



$\frac{1}{2}$ circle from y from 0 to r
is $\frac{1}{4}$ area circle.

$$V = 8\pi R \tfrac{1}{4}\pi r^2 = \boxed{2\pi^2 R r^2}$$