Math 322

Changes today: 8.6

Feb 15th: Review + 9.1-2

Feb 17th: Exam 1

Feb 22nd: 9.3 (back on track)

Q's 8.5 #33

Example 5 R_4

S = Set of all strings

$a R_n b$ iff $a = b$
or both are at least
n characters long and
share 1st n charac.

ex: R_4 ex abcd R_4 abcdef

ab R_4 ab

33a) $[010]_{R_4} = \{ s \mid 010 \, R_4 \, s \}$

$= 010$

33b) $[1011]_{R_4} = \{ s \mid 1011 \, R_4 \, s \}$

$= \{1011, 10111, 10110, 101111, 101110,$

$101101, 101100, \ldots \}$

= 1011 followed by anything

Note: could you show R_4 is an equiv. relation?

need this

show: symmetric

$\forall s \forall t \, (s R_4 t \to t R_4 s)$

show: reflexive

$\forall s \, (s R_4 s)$

show: transitive

$\forall s \forall t \forall u \, (s R_4 t \land t R_4 u \to s R_4 u)$

8.6 Partial Orderings

R a relation on set S.

$\underbrace{\text{a relation}}$ Rule that relates a to b for aRb.

Need:
1. anti sym.
2. reflexive
3. transitive

Why the word "partial"?

ex: aRb when $a \le b$

(1) Reflexive? $\forall a\, aRa$

$\equiv \forall a (a \le a) \equiv T$ is reflexive

(2) antisym?

$\forall a \forall b (aRb \wedge bRa \rightarrow a = b)$

$\equiv \forall a \forall b (a \ge b \wedge b \ge a \rightarrow a = b) \equiv T$

is anti-sym.

(3) transitive?

$\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

$\equiv \forall a \forall b \forall c\,(a \geq b \wedge b \geq c \rightarrow a \geq c)$

$\equiv T$

is transitive.

b/c $a \geq b$ or $a \leq b$ are the "classic" orderings then...

① if R is a partial ordering write it as $\succcurlyeq$ — ref., antisym, trans

② Remember if R is an equiv. rel. write it as $\sim$ — ref., sym, trans.

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ex consider divisible as an operator.

aRb if $a \mid b$

ex $2 \mid 4$ so $2R4$

ex $5 \mid 20$ so $5R20$

is it a partial ordering?

① Reflexive?

$\forall a(aRa)$

$\equiv \forall a(a|a) \equiv T$

② anti sym?

$\forall a \forall b (aRb \land bRa \rightarrow a=b)$

$\equiv \forall a \forall b (a|b \land b|a \rightarrow a=b)$

$\equiv \forall a \forall b (b=a\cdot k \land a=\boxed{b\cdot l} \rightarrow a=b)$

$b=(b\cdot l)\cdot k$

$b=b\cdot l\cdot k \rightarrow l=k=1$

$b=a$ True

③ trans? $\boxed{\text{yes.}}$

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..

R is $\leq$ every number can compare.

R is | $\underbrace{2R8}$ but $2 \not R 5$

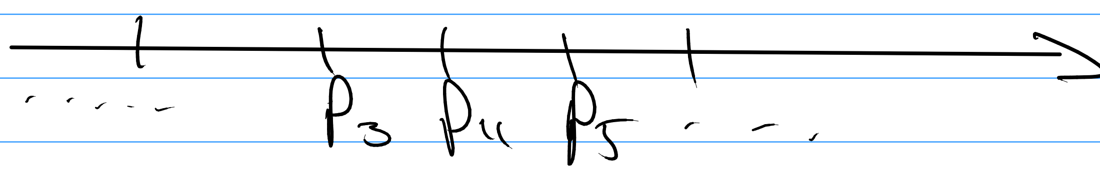
Def: If $a \preceq b$ or $b \preceq a$ then a, b are <u>comparable</u>.

If <u>not</u> a, b are <u>incomparable</u>

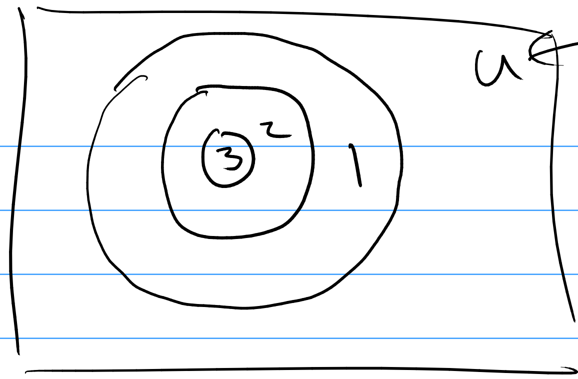
In a partial ordering not everyone can be compared.

(ex) If everyone compares to everyone then $\preceq$ is a <u>total ordering</u>. $\leftarrow$ partial order (+) everyone compares

(ex) R was $\leq$ is a total order. on the integers.



(E) If $\preceq$ is a total order and every non-empty subset of S has a least element $\rightarrow$ $\boxed{\text{well-ordered}}$



← all relations

$1 \equiv$ partial order

$2 \equiv$ total order

$3 \equiv$ well order

Notation: if S is ordered by $\preccurlyeq$

$(S, \preccurlyeq) \leftarrow$ poset

Ex's $(\mathbb{Z}, |)$ is a poset

$(\mathbb{Z}, \leq)$ is a poset (and a total order)

$(\mathbb{Z}^+, \leq)$ is a poset (and a well-order)

Apps

① lexicographic ordering. (Dictionary Sort) if mark vs markeous

ma∅ΔK vs ab∅∅☺ v

$s_1 = (a_1, a_2, a_3, \dots, a_n)$ vs $s_2 = (b_1, b_2, \dots, b_m)$

$S_1 \prec S_2$

if $a_1 \prec b_1$

or $a_1 = b_1 \wedge a_2 \prec b_2$ element partial order

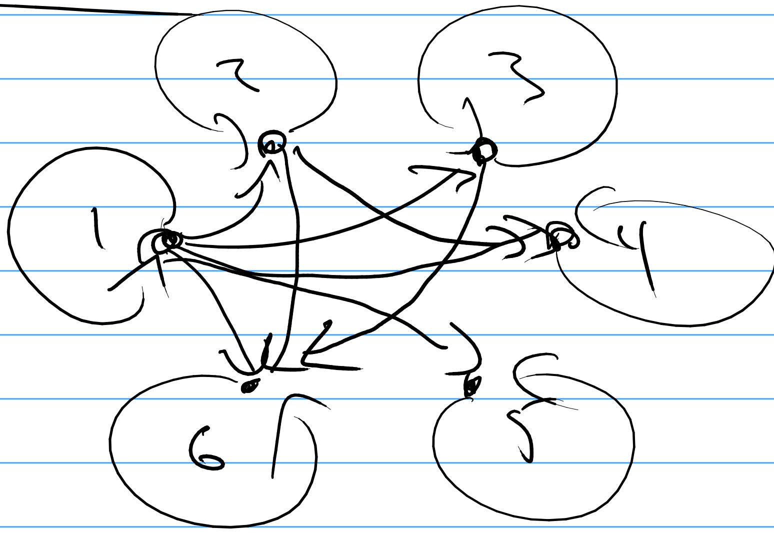
or $a_2 = b_2 \wedge a_3 \prec b_3$

⋮

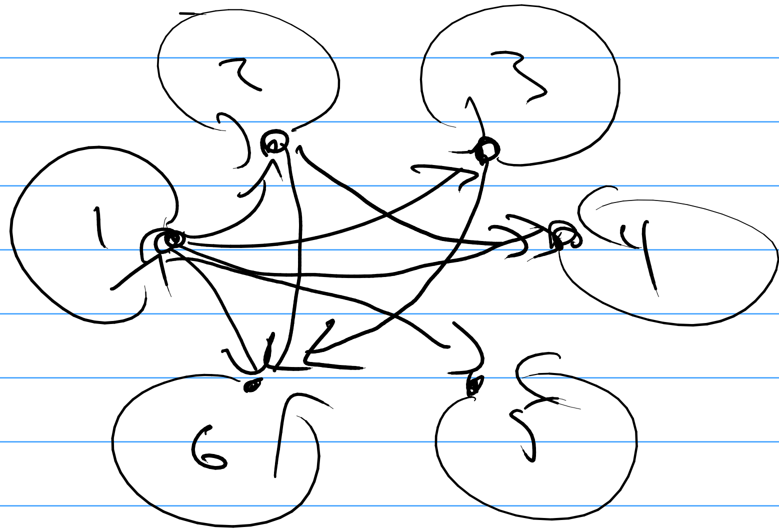
S_1 is shorter than S_2 and all symbols in S_1 are in S_2.

② Visual Version (Hasse Diagram)

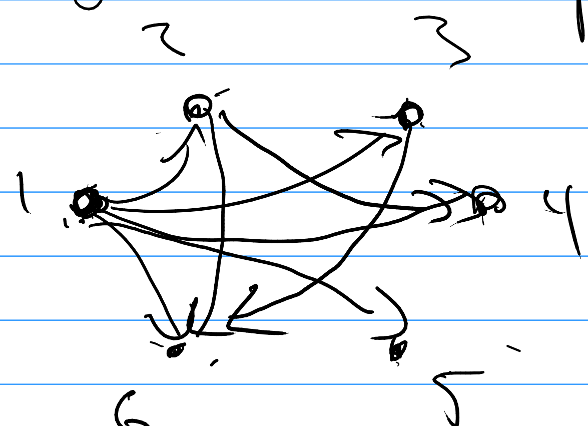
digraph $(\{1,2,3,4,5,6\}, \mid)$



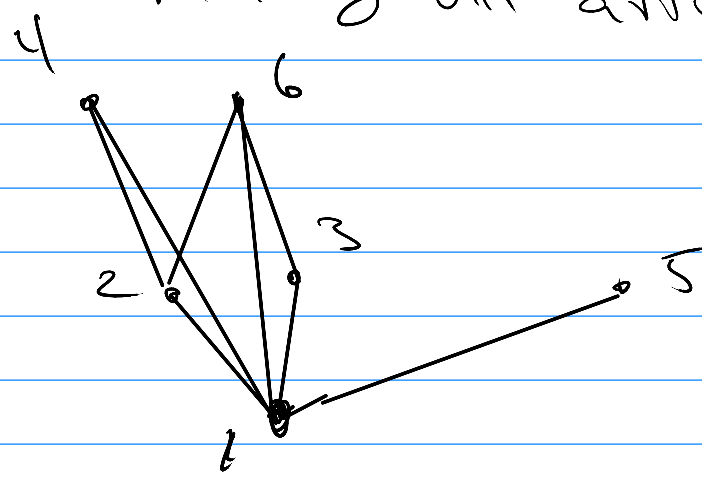
Hasse Diagram simplifies the digraph of a poset.



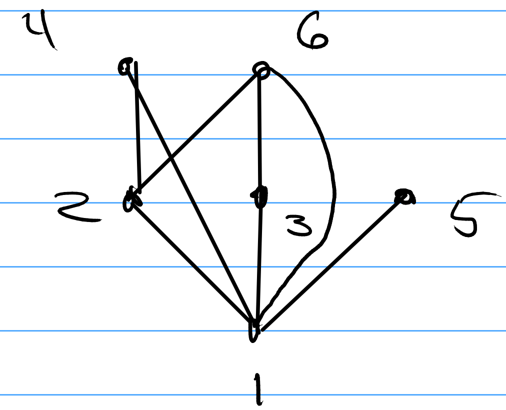
① don't draw loops (we know they are there)



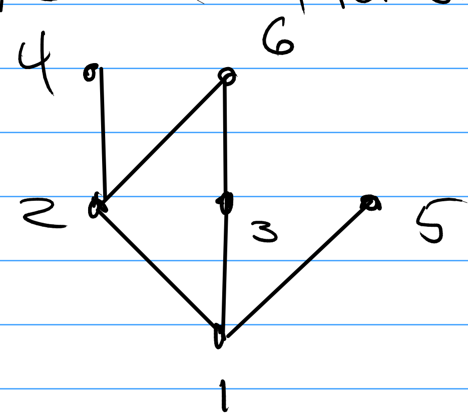
③ get rid of arrows by making all arrows point up.



(4) if one edge move one level.



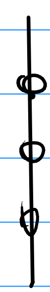
(5) remove transitive edges



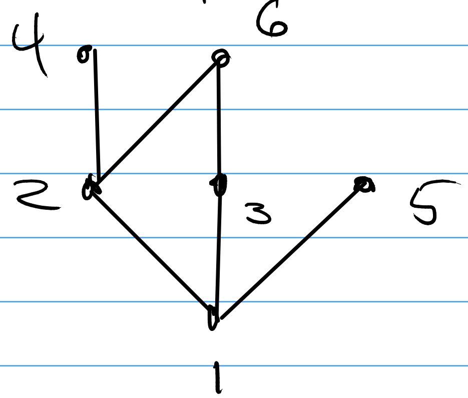
Hasse Diagram

Why this visual?

a) total orderings



b) Maximal / minimal elements
(tops & bottoms)



Maximals: 4, 6, 5

Minimals: 1

b) greatest / least elements.

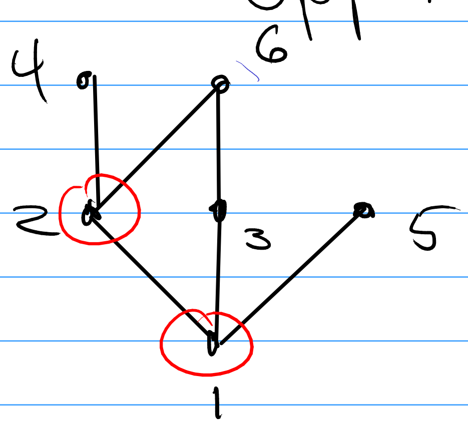
above <u>everyone</u> below <u>everyone</u>.

<u>ex:</u> greatest: none

least: 1

c) given a subset does it have bounds?

upperbounds / lower bounds



subset = {2, 1}

upperbound: @ or above the set

{2, 4, 6}

lowerbound: @ or below the set

$\{1\}$