CIS 770: Formal Language Theory

Pavithra Prabhakar

Kansas State University

Spring 2015

Non-Context Free Languages

Question

Are there languages that are not context-free?

Non-Context Free Languages

Question

Are there languages that are not context-free? What about $L = \{a^n b^n c^n \mid n \geq 0\}$?

Non-Context Free Languages

Question

Are there languages that are not context-free? What about $L = \{a^n b^n c^n \mid n \geq 0\}$?

Answer

L is not context-free, because

- Recognizing if $w \in L$ requires remembering the number of as seen, bs seen and cs seen
- We can remember one of them on the stack (say as) , and compare them to another (say bs) by popping, but not to both bs and cs

Non-Context Free Languages

Question

Are there languages that are not context-free? What about $L = \{a^n b^n c^n \mid n \geq 0\}$?

Answer

L is not context-free, because

- Recognizing if $w \in L$ requires remembering the number of as seen, bs seen and cs seen
- We can remember one of them on the stack (say as) , and compare them to another (say bs) by popping, but not to both bs and cs

The precise way to capture this intuition is through the pumping lemma

Pumping Lemma for CFLs

Informal Statement

For all sufficiently long strings z in a context free language L, it is possible to find two substrings, not too far apart, that can be simultaneously pumped to obtain more words in L.

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

Pumping Lemma for CFLs
Formal Statement

Lemma

If L *is a CFL, then* $\exists p$ *(pumping length) such that* $\forall z \in L$*, if* $|z| \geq p$ *then* $\exists u, v, w, x, y$ *such that* $z = uvwxy$

1. $|vwx| \leq p$

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$

Pumping Lemma for CFLs

Formal Statement

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^i wx^i y \in L$

Two Pumping Lemmas side-by-side

Context-Free Languages

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^iwx^iy \in L$

Two Pumping Lemmas side-by-side

Context-Free Languages

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^iwx^iy \in L$

Regular Languages

If L is a regular language, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w$ such that $z = uvw$

1. $|uv| \leq p$
2. $|v| > 0$
3. $\forall i \geq 0.\ uv^iw \in L$

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender	**Challenger**

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender	**Challenger**
Pick pumping length p	

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
		Pick $z \in L$ s.t. $\lvert z \rvert \geq p$

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z \rvert \geq p$

Pumping Lemma for CFLs

Game View

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z\rvert \geq p$
Divide z into u, v, w, x, y s.t. $\lvert vwx\rvert \leq p$, and $\lvert vx\rvert > 0$		

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z \rvert \geq p$
Divide z into u, v, w, x, y s.t. $\lvert vwx \rvert \leq p$, and $\lvert vx \rvert > 0$	$\xrightarrow{u,v,w,x,y}$	

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $|z| \geq p$
Divide z into u, v, w, x, y s.t. $|vwx| \leq p$, and $|vx| > 0$	$\xrightarrow{u,v,w,x,y}$	
		Pick i, s.t. $uv^iwx^iy \notin L$

Pumping Lemma for CFLs

Game View

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $|z| \geq p$
Divide z into u, v, w, x, y s.t. $|vwx| \leq p$, and $|vx| > 0$	$\xrightarrow{u,v,w,x,y}$	
	$\xleftarrow{i}$	Pick i, s.t. $uv^iwx^iy \notin L$

Pumping Lemma for CFLs

Game View

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		**Challenger**
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z \rvert \geq p$
Divide z into u, v, w, x, y s.t. $\lvert vwx \rvert \leq p$, and $\lvert vx \rvert > 0$	$\xrightarrow{u,v,w,x,y}$	
	$\xleftarrow{i}$	Pick i, s.t. $uv^iwx^iy \notin L$

Pumping Lemma: If L is CFL, then there is always a winning strategy for the defender (i.e., challenger will get stuck).

Pumping Lemma for CFLs

Game View

Game between Defender, who claims L satisfies the pumping condition, and Challenger, who claims L does not.

Defender		Challenger
Pick pumping length p	$\xrightarrow{p}$	
	$\xleftarrow{z}$	Pick $z \in L$ s.t. $\lvert z \rvert \geq p$
Divide z into u, v, w, x, y s.t. $\lvert vwx \rvert \leq p$, and $\lvert vx \rvert > 0$	$\xrightarrow{u,v,w,x,y}$	
	$\xleftarrow{i}$	Pick i, s.t. $uv^iwx^iy \notin L$

Pumping Lemma: If L is CFL, then there is always a winning strategy for the defender (i.e., challenger will get stuck).

Pumping Lemma (in contrapositive): If there is a winning strategy for the challenger, then L is not CFL.

Consequences of Pumping Lemma

- If L is context-free then L satisfies the pumping lemma.

Consequences of Pumping Lemma

- If L is context-free then L satisfies the pumping lemma.
- If L satisfies the pumping lemma that does not mean L is context-free

Consequences of Pumping Lemma

- If L is context-free then L satisfies the pumping lemma.
- If L satisfies the pumping lemma that does not mean L is context-free
- If L does not satisfy the pumping lemma (i.e., challenger can win the game, *no matter* what the defender does) then L is not context-free.

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p \in L_{anbncn}$.

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p \in L_{anbncn}$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p \in L_{anbncn}$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.
- Since $|vwx| \leq p$, vwx cannot contain all three of the symbols a, b, c, because there are p bs. So vwx either does not have any as or does not have any bs or does not have any cs.

$\square$

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *is not a CFL.*

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p \in L_{anbncn}$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.
- Since $|vwx| \leq p$, vwx cannot contain all three of the symbols a, b, c, because there are p bs. So vwx either does not have any as or does not have any bs or does not have any cs. Suppose, (wlog) vwx does not have any as.

□

Example I

Proposition

$L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ is not a CFL.

Proof.

Suppose L_{anbncn} is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p \in L_{anbncn}$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.
- Since $|vwx| \leq p$, vwx cannot contain all three of the symbols a, b, c, because there are p bs. So vwx either does not have any as or does not have any bs or does not have any cs. Suppose, (wlog) vwx does not have any as. Then $uv^0 wx^0 y = uwy$ contains more as than either bs or cs. Hence $uwy \notin L$. □

Example II

Proposition

$L_{a=c \wedge b=d} = \{a^i b^j c^i d^j \mid i, j \geq 0\}$ *is not a CFL.*

Example II

Proposition

$L_{a=c \wedge b=d} = \{a^i b^j c^i d^j \mid i, j \geq 0\}$ *is not a CFL.*

Proof.

Suppose $L_{a=c \wedge b=d}$ is context-free. Let p be the pumping length.

Example II

Proposition

$L_{a=c\wedge b=d} = \{a^i b^j c^i d^j \mid i,j \geq 0\}$ *is not a CFL.*

Proof.

Suppose $L_{a=c\wedge b=d}$ is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p d^p \in L$.

Example II

Proposition

$L_{a=c\wedge b=d} = \{a^i b^j c^i d^j \mid i,j \geq 0\}$ *is not a CFL.*

Proof.

Suppose $L_{a=c\wedge b=d}$ is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p d^p \in L$.
- Since $|z| > p$, there are u,v,w,x,y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.

Example II

Proposition

$L_{a=c \wedge b=d} = \{a^i b^j c^i d^j \mid i, j \geq 0\}$ *is not a CFL.*

Proof.

Suppose $L_{a=c \wedge b=d}$ is context-free. Let p be the pumping length.

- Consider $z = a^p b^p c^p d^p \in L$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.
- Since $|vwx| \leq p$, v, x cannot contain both as and cs, nor can it contain both bs and ds. Further $|vx| > 0$. Now $uv^0 wx^0 y = uwy \notin L$, because it either contains fewer as than cs, or fewer cs than as, or fewer bs than ds, or fewer ds than bs. □

Example III

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Example III

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

$\square$

Example III
Wrong Proof

Proposition
$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.
Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.

$\square$

Example III

Wrong Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.
- z can be pumped if we make the following division.

$$\overbrace{\underbrace{00\cdots00}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{w}}^{0^p1}\ \overbrace{\underbrace{0}_{x}\ \underbrace{00\cdots001}_{y}}^{0^p1}$$

$\square$

Example III

Wrong Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.
- z can be pumped if we make the following division.

$$\overbrace{\underbrace{00\cdots00}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{w}}^{0^p1}\ \overbrace{\underbrace{0}_{x}\ \underbrace{00\cdots001}_{y}}^{0^p1}$$

- So is E CFL?

□

Example III
Wrong Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.
- z can be pumped if we make the following division.

$$\overbrace{\underbrace{00\cdots00}_{u}\;\underbrace{0}_{v}\;\underbrace{1}_{w}}^{0^p1}\;\overbrace{\underbrace{0}_{x}\;\underbrace{00\cdots001}_{y}}^{0^p1}$$

- So is E CFL? No!

□

Example III

Wrong Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.
- z can be pumped if we make the following division.

$$\overbrace{\underbrace{00\cdots00}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{w}}^{0^p1}\ \overbrace{\underbrace{0}_{x}\ \underbrace{00\cdots001}_{y}}^{0^p1}$$

- So is E CFL? No! Does E satisfy the pumping lemma?

$\square$

Example III
Wrong Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p10^p1 \in L$.
- z can be pumped if we make the following division.

$$\overbrace{\underbrace{00\cdots00}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{w}}^{0^p1}\ \overbrace{\underbrace{0}_{x}\ \underbrace{00\cdots001}_{y}}^{0^p1}$$

- So is E CFL? No! Does E satisfy the pumping lemma? No!

□

Example III
Corrected Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ *is not a CFL.*

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p 1^p 0^p 1^p \in L$.

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p1^p0^p1^p \in L$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^iwx^iy \in L$ for all $i \geq 0$.

Example III

Corrected Proof

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p 1^p 0^p 1^p \in L$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^i wx^i y \in L$ for all $i \geq 0$.
- vwx must straddle the midpoint of z.

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p1^p0^p1^p \in L$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^iwx^iy \in L$ for all $i \geq 0$.
- vwx must straddle the midpoint of z.
    - Suppose vwx is only in the first half. Then uv^0wx^0y is of the form $0^i1^j0^p1^p$, where $0 \leq i, j \leq p$ and either i or j is strictly less than p.

Proposition

$E = \{ww \mid w \in \{0,1\}^*\}$ is not a CFL.

Proof.

Suppose E is context-free. Let p be the pumping length.

- Consider $z = 0^p1^p0^p1^p \in L$.
- Since $|z| > p$, there are u, v, w, x, y such that $z = uvwxy$, $|vwx| \leq p$, $|vx| > 0$ and $uv^iwx^iy \in L$ for all $i \geq 0$.
- vwx must straddle the midpoint of z.
  - Suppose vwx is only in the first half. Then uv^0wx^0y is of the form $0^i1^j0^p1^p$, where $0 \leq i, j \leq p$ and either i or j is strictly less than p.
  - The argument is similar if vwx is only in the second half. $\cdots\longrightarrow$

Proof (contd).

Example III

Corrected Proof

Proof (contd).

- Suppose vwx straddles the middle.

Proof (contd).

- Suppose vwx straddles the middle. Then uv^0wx^0y must be of the form $0^p1^i0^j1^p$, where either i or j is not p.

Proof (contd).

- Suppose vwx straddles the middle. Then uv^0wx^0y must be of the form $0^p1^i0^j1^p$, where either i or j is not p. Thus, $uv^0wx^0y \notin E$. $\square$

Proof of Pumping Lemma

Recall . . .

Lemma

If L is a CFL, then $\exists p$ (pumping length) such that $\forall z \in L$, if $|z| \geq p$ then $\exists u, v, w, x, y$ such that $z = uvwxy$

1. $|vwx| \leq p$
2. $|vx| > 0$
3. $\forall i \geq 0.\ uv^i wx^i y \in L$

Chomsky Normal Form

CNF

Productions are of the form $A \rightarrow BC$ or $A \rightarrow a$

Reduction to Normal Form

For every grammar G such that $\epsilon \notin L(G)$, there is an equivalent grammar G' in CNF such that $L(G) = L(G')$.

Chomsky Normal Form

CNF

Productions are of the form $A \rightarrow BC$ or $A \rightarrow a$

Reduction to Normal Form

For every grammar G such that $\epsilon \notin L(G)$, there is an equivalent grammar G' in CNF such that $L(G) = L(G')$.

Unit Productions and ϵ-productions

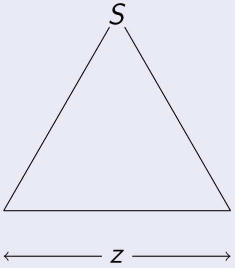
- Unit Productions: Rules of the form $A \rightarrow B$
- ϵ Productions: Rules of the form $A \rightarrow \epsilon$

Proof Idea

Let G be a CFG in Chomsky Normal Form such that $L(G) = L$.
Let z be a "very long" string in L ("very long" made precise later).

Proof Idea

Let G be a CFG in Chomsky Normal Form such that $L(G) = L$.
Let z be a "very long" string in L ("very long" made precise later).

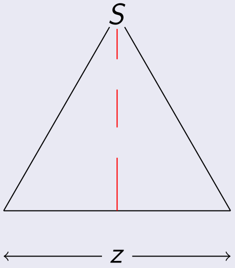


Parse Tree for z

- Since $z \in L$ there is a parse tree for z

Proof Idea

Let G be a CFG in Chomsky Normal Form such that $L(G) = L$.
Let z be a "very long" string in L ("very long" made precise later).

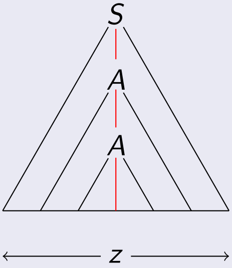


Parse Tree for z

- Since $z \in L$ there is a parse tree for z
- Since z is very long, the parse tree (which is a binary tree) must be "very tall"

Proof Idea

Let G be a CFG in Chomsky Normal Form such that $L(G) = L$. Let z be a "very long" string in L ("very long" made precise later).

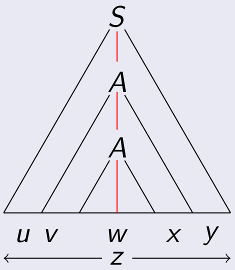


Parse Tree for z

- Since $z \in L$ there is a parse tree for z
- Since z is very long, the parse tree (which is a binary tree) must be "very tall"
- The longest path in the tree, by pigeon hole principle, must have some variable (say) A repeat.

Proof Idea

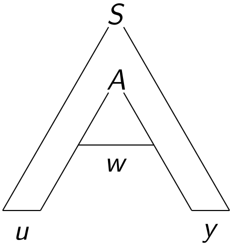
Let G be a CFG in Chomsky Normal Form such that $L(G) = L$. Let z be a "very long" string in L ("very long" made precise later).



Parse Tree for z

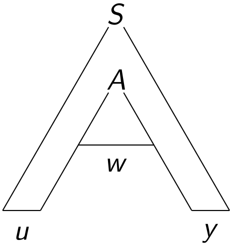
- Since $z \in L$ there is a parse tree for z
- Since z is very long, the parse tree (which is a binary tree) must be "very tall"
- The longest path in the tree, by pigeon hole principle, must have some variable (say) A repeat. Let u, v, w, x, y be as shown.

Pumping down

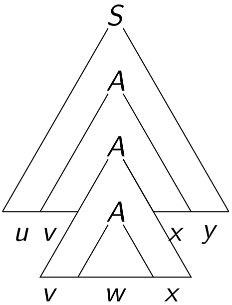


Pumping zero times

Pumping down and up

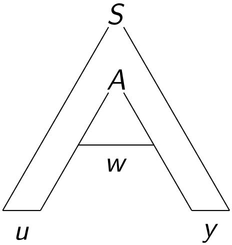


Pumping zero times

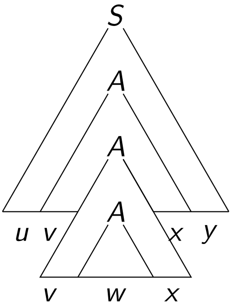


Pumping two times

Pumping down and up



Pumping zero times



Pumping two times

- Thus, uv^iwx^iy has a parse tree, for any i.

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

- Consider a parse tree for z. Height of this tree is at least $k + 1$

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

- Consider a parse tree for z. Height of this tree is at least $k + 1$
    - Parse trees of G are binary trees

Proof of Pumping Lemma

Existence of tall parse trees

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

- Consider a parse tree for z. Height of this tree is at least $k + 1$
  - Parse trees of G are binary trees
  - Fact: A binary tree of height h has at most 2^{h-1} leaves

Proof.

Let G be a grammar in Chomsky Normal Form with k variables such that $L(G) = L$. Take $p = 2^k$. Consider $z \in L$ such that $|z| \geq p = 2^k$.

- Consider a parse tree for z. Height of this tree is at least $k + 1$
  - Parse trees of G are binary trees
  - Fact: A binary tree of height h has at most 2^{h-1} leaves
  - $|z|$ =Number of leaves in parse tree of $z = 2^k \leq 2^{h-1}$. Thus, $h \geq k + 1$.

Proof (contd).

Proof (contd).

- A parse tree for z has a path of length $k+1$

Proof (contd).

- A parse tree for z has a path of length $k+1$
- A path of length $k+1$ has $k+2$ vertices, out of which the last one is leaf that is labelled by a terminal

Proof (contd).

- A parse tree for z has a path of length $k+1$
- A path of length $k+1$ has $k+2$ vertices, out of which the last one is leaf that is labelled by a terminal; thus, there are at least $k+1$ internal vertices on path.

Proof of Pumping Lemma

Repeated Variables

Proof (contd).

- A parse tree for z has a path of length $k+1$
- A path of length $k+1$ has $k+2$ vertices, out of which the last one is leaf that is labelled by a terminal; thus, there are at least $k+1$ internal vertices on path.
- Thus, there must be two vertices n_1 and n_2 on this path such that n_1 and n_2 have the same label (say A) and n_1 is an ancestor of n_2.

Proof (contd).

- A parse tree for z has a path of length $k+1$
- A path of length $k+1$ has $k+2$ vertices, out of which the last one is leaf that is labelled by a terminal; thus, there are at least $k+1$ internal vertices on path.
- Thus, there must be two vertices n_1 and n_2 on this path such that n_1 and n_2 have the same label (say A) and n_1 is an ancestor of n_2.
- Let the yield of tree rooted at n_2 be w, and yield of n_1 be vwx.

Proof (contd).

- A parse tree for z has a path of length $k + 1$
- A path of length $k + 1$ has $k + 2$ vertices, out of which the last one is leaf that is labelled by a terminal; thus, there are at least $k + 1$ internal vertices on path.
- Thus, there must be two vertices n_1 and n_2 on this path such that n_1 and n_2 have the same label (say A) and n_1 is an ancestor of n_2.
- Let the yield of tree rooted at n_2 be w, and yield of n_1 be vwx. Yield of the root $= z$ is say $uvwxy$.

Proof (contd).

Proof (contd).

- Height of n_1 can be assumed to be at most $k + 1$

Proof (contd).

- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.

Proof (contd).

- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.
- $n_1 \neq n_2$.

Proof of Pumping Lemma

Properties of u, v, w, x, y

Proof (contd).

- Height of n_1 can be assumed to be at most $k+1$; thus, the yield of n_1 (vwx) is at most $2^k = p$.
- $n_1 \neq n_2$. Since the grammar has no ϵ-productions and no unit-productions, $vwx \neq w$. i.e., $|vx| > 0$.

Proof of Pumping Lemma

Pumping the strings

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

Proof of Pumping Lemma

Pumping the strings

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \stackrel{*}{\Rightarrow} vAx$.

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \stackrel{*}{\Rightarrow} vAx$.
- There is a parse tree with yield w and root A; this is the tree rooted at n_2. Thus, $A \stackrel{*}{\Rightarrow} w$.

Proof of Pumping Lemma

Pumping the strings

Proof (contd).

Based on the parse tree for z, and definitions of u, v, w, x, y, we have

- There is a parse tree with yield uAy and root S, obtained by not expanding n_1. Thus, $S \stackrel{*}{\Rightarrow} uAy$.
- There is a parse tree with yield vAx and root A, obtained from n_1 and not expanding n_2. Thus, $A \stackrel{*}{\Rightarrow} vAx$.
- There is a parse tree with yield w and root A; this is the tree rooted at n_2. Thus, $A \stackrel{*}{\Rightarrow} w$.

Putting it together, we have

$$S \stackrel{*}{\Rightarrow} uAy \stackrel{*}{\Rightarrow} uvAxy \stackrel{*}{\Rightarrow} uvvAxxy \stackrel{*}{\Rightarrow} \cdots \stackrel{*}{\Rightarrow} uv^iAx^iy \stackrel{*}{\Rightarrow} uv^iwx^iy \quad \square$$