Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

How to find bases of subspaces

Created by Prof. Diego Maldonado and Prof. Virginia Naibo

At this point we know that given any $m \times n$ matrix A there are three subspaces associated to A. Namely, the null space of A (which is a subspace of $\mathbf{R}^n$), denoted by $null(A)$; the column space of A (which is a subspace of $\mathbf{R}^m$), denoted by $col(A)$; and the row space of A (which is a subspace of $\mathbf{R}^n$), denoted by $row(A)$.

These notes will help you to find bases for those subspaces. First, notice that what seems to be three tasks actually reduces to two. Indeed, since the rows of A are the same as the columns of A^T (the transpose of A), once you know how to find bases for column spaces, then (by transposing the matrix under consideration) you will be able to find bases for row spaces. Hence, we only need to focus on finding bases for $null(A)$ and $col(A)$, where A is an arbitrary matrix.

Let's do some problems.

Two ways of finding bases for $null(A)$

(1) Consider the 2×4 matrix

```
A =

    -1    -2    -1     1
     2     0    -1     2
```

The first way to find a basis for $null(A)$ is immediate and it involves the Matlab command *null*. In this case we do

```
>> null(A)

ans =

    0.1737   -0.5555
   -0.3902    0.5168
    0.8662    0.1547
    0.2594    0.6328
```

which means that the vectors $v_1 = \begin{bmatrix} 0.1737 \\ -0.3902 \\ 0.8662 \\ 0.2594 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -0.5555 \\ 0.5168 \\ 0.1547 \\ 0.6328 \end{bmatrix}$ form a basis of $null(A)$. In symbols, the set $\mathcal{B} := \{v_1, v_2\}$ is a basis for $null(A)$. Simple! But not very insightful... Let's see another way.

By definition, we have that $null(A)$ is the collection of all the solutions to the homogeneous system $Ax = 0$ (here the 0 is the zero vector in $\mathbf{R}^m$). That is,

$$null(A) = \{x \in \mathbf{R}^n : Ax = 0\}.$$

Since we're now talking about systems, we have to do the $rref$. Here we go

```
>> b=[0 0]'

b =

     0
     0

>> rref([A b])

ans =

    1.0000         0   -0.5000    1.0000         0
         0    1.0000    0.7500   -1.0000         0
```

and we interpret that $x_1 = .5x_3 - x_4$, $x_2 = -.75x_3 + x_4$ with x_3 and x_4 free variables. That is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} .5x_3 - x_4 \\ -.75x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} .5 \\ -.75 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

That is, every vector x in $null(A)$ can be expressed as a linear combination of the vectors $\begin{bmatrix} .5 \\ -.75 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Moreover, these two vectors are linearly independent (check it!). Therefore, the set

$$\mathcal{C} = \left\{ \begin{bmatrix} .5 \\ -.75 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is also a basis for $null(A)$ (since the vectors in $\mathcal{C}$ generate everything in $null(A)$ and they do it without redundancies, meaning that the vectors in $\mathcal{C}$ are linearly independent).

Notice that the basis $\mathcal{B}$ above has different vectors from the ones in the basis $\mathcal{C}$. So what? Bases are not unique, there are infinitely many bases for each subspace. We just found two for the subspace $null(A)$.

Just for the sake of playing around, let's check that the bases $\mathcal{B}$ and $\mathcal{C}$ generate (by linear combinations) the same subspace.

Let's put $u_1 = \begin{bmatrix} .5 \\ -.75 \\ 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, so that $\mathcal{C} = \{u_1, u_2\}$. The sets $\mathcal{B}$ and $\mathcal{C}$ will generate the same subspace if all the elements of $\mathcal{B}$ generate all the elements in $\mathcal{C}$ and vice versa. Let's see that we can generate v_1 as a linear combination of u_1 and u_2 (the other cases will be similar). We do,

```
>> v1

v1 =

    0.1737
   -0.3902
    0.8662
    0.2594

>> v2

v2 =

   -0.5555
    0.5168
    0.1547
    0.6328
>> u1

u1 =

    0.5000
   -0.7500
    1.0000
         0

>> u2
```

```
u2 =

    -1
     1
     0
     1
>> rref([u1 u2 v1])

ans =

    1.0000         0    0.8662
         0    1.0000    0.2594
         0         0         0
         0         0         0
```

And we interpret, to get that v_1 can be expressed as a linear combination of u_1 and u_2. Moreover,

$$v_1 = 0.8662 u_1 + 0.2594 u_2.$$

In all this discussion, the crucial point is to use the "key to success", which is the ability to smoothly go from a system and its matrix form to its vector form and vice versa.

Along the same lines you can prove that the vectors of $\mathcal{B}$ generate all the vectors in $\mathcal{C}$ and vice versa, as we wanted.

Geometric interpretation. For the matrix A as above, the subspace $null(A)$ is a (two-dimensional) plane in $\mathbf{R}^4$ generated by the vectors u_1 and u_2 (or, alternatively, by the vectors v_1 and v_2, since we just saw that they generate the same subspace of $\mathbf{R}^4$).

Next, another problem

(2) Find a basis for the subspace

$$\nu = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 / 2x_1 + x_2 - x_3 = 0, \ \ -x_1 + 2x_2 + x_3 = 0\}.$$

The first thing we notice is that ν is precisely the set of all the solutions to the system $Ax = 0$, where

```
>> A=[2 1 -1; -1 2 1]

A =

     2     1    -1
    -1     2     1
```

The we do

```
>> null(A)

ans =

    0.5071
   -0.1690
    0.8452
```

to conclude that the set $\mathcal{B} = \{v\}$, where $v = \begin{bmatrix} 0.5071 \\ -0.1690 \\ 0.8452 \end{bmatrix}$, is a basis for $null(A)$. By using the second method, we get

```
>> b=[0 0]'

b =

     0
     0

>> rref([A b])

ans =

    1.0000         0   -0.6000         0
         0    1.0000    0.2000         0
```

After interpreting, we get that $x_1 = .6x_3$, $x_2 = -.2x_3$ and x_3 is free. Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .6x_3 \\ -.2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} .6 \\ -.2 \\ 1 \end{bmatrix}.$$

Hence, every vector $x \in \nu$ can be expressed as a multiple of the vector $u = \begin{bmatrix} .6 \\ -.2 \\ 1 \end{bmatrix}$. In other words, the set $\{u\}$ is also a basis for $\nu = null(A)$. What is the relation between u and v? Clearly, since these are two vectors which generate the same subspace (that is, all their multiples), then each one of these vectors is a multiple of other. Indeed, we can check that

```
>> u

u =
```

```
    0.6000
   -0.2000
    1.0000

>> v

v =

    0.5071
   -0.1690
    0.8452

>> .8462*u

ans =

    0.5077
   -0.1692
    0.8462
```

That is, $v = .8462u$.

Geometric interpretation. For the matrix A as above, the subspace $null(A)$ is the (one-dimensional) line in $\mathbf{R}^3$ that passes through the origin and is parallel to the vector u (or, alternatively, the vector v, as they generate the same line).

Finding bases for $col(A)$. Now we move on to finding bases for $col(A)$

(3) Let P be the matrix

```
P =

     2     3    -1     0    -5
     3    -2    -2     3     9
     1     1    -8     3     2
     2     3     0    -7     7
```

And let's label the columns of P as $u_1, \ldots, u_5$, that is,

```
>> u1=P(:,1);
>> u2=P(:,2);
>> u3=P(:,3);
>> u4=P(:,4);
>> u5=P(:,5);
```

By definition, $col(P)$ is the subspace formed by all the possible linear combinations of the columns $u_1, \ldots, u_5$. In symbols,

$$col(P) = span\{u_1, u_2, u_3, u_4, u_5\}$$

where the word "span" in this case stands for "all the possible linear combinations of".

In order to obtain a basis for $col(P)$ we must eliminate any possible redundancy (that is, linear dependence) that there might be in the $u_1, \ldots, u_5$. Let's do this in an algorithmic fashion (as a computer would do it). We first take u_1 which is a non-zero column vector and we keep it. Then we consider $\{u_1, u_2\}$ and ask the question: is this set linearly independent? We can answer this question by simply doing

```
>> b=[0 0 0 0]'

b =

     0
     0
     0
     0

>> rref([u1 u2 b])

ans =

     1     0     0
     0     1     0
     0     0     0
     0     0     0
```

And we interpret that the only solution $x = [x_1, x_2]'$ to the system (in vector form) $x_1u_1 + x_2u_2 = b$ is the zero solution $x = [0, 0]'$. By definition of linear independence, this tells us that the set of vectors $\{u_1, u_2\}$ is indeed a l.i. (linearly independent) set. So we keep both, u_1 and u_2, and then add u_3 to the set, and ask the question: is the set of vectors $\{u_1, u_2, u_3\}$ l.i.? To answer the question we do

```
>> rref([u1 u2 u3 b])

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     0
```

And, again, we interpret and conclude that the set $\{u_1, u_2, u_3\}$ is indeed l.i., we keep that set and add u_4, and ask the question: is the set $\{u_1, u_2, u_3, u_4\}$ l.i.? In order to answer we do

```
>> rref([u1 u2 u3 u4 b])

ans =

     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
```

And interpret and conclude that the set $\{u_1, u_2, u_3, u_4\}$ is indeed l.i., we keep u_1, u_2, u_3, u_4, and add u_5, and ask the question: is the set $\{u_1, u_2, u_3, u_4, u_5\}$ l.i.? In order to answer that we do

```
>> rref([u1 u2 u3 u4 u5 b])

ans =

    1.0000         0         0         0    1.9114         0
         0    1.0000         0         0   -3.3173         0
         0         0    1.0000         0   -1.1291         0
         0         0         0    1.0000   -1.8756         0
```

We interpret and realize that the homogeneous system (in vector form) $x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 + x_5u_5 = b$ is a consistent system with infinitely many solutions (one free variable). Thus, by definition, the set $\{u_1, u_2, u_3, u_4, u_5\}$ is not linearly independent.

So, at this point we have that the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent, but the set $\{u_1, u_2, u_3, u_4, u_5\}$ is not. This tells us that the last vector we added, u_5, can be written as a linear combination of the vectors we had. Indeed, if we do

```
>> rref([u1 u2 u3 u4 u5])

ans =

    1.0000         0         0         0    1.9114
         0    1.0000         0         0   -3.3173
         0         0    1.0000         0   -1.1291
         0         0         0    1.0000   -1.8756
```

We see that u_5 can be written as a linear combination of u_1, u_2, u_3, and u_4 as follows,

$$u_5 = 1.9114u_1 - 3.3173u_2 - 1.1291u_3 - 1.8756u_4.$$

That is, the vector u_5 is redundant (we can generate it by using u_1, u_2, u_3, and u_4) and we drop it.

Finally, we conclude that a basis for $col(P)$ is given by the set $\{u_1, u_2, u_3, u_4\}$. That is, the set $\{u_1, u_2, u_3, u_4\}$ is a linearly independent set that spans all of $col(P)$, in symbols,

$$col(P) = span\{u_1, u_2, u_3, u_4\}.$$

Geometric interpretation. For the matrix P as above, we have that $col(P)$ has dimension 4 (since we found a basis for $col(P)$ consisting of 4 vectors). But, since the vectors u_1, u_2, u_3, u_4 are vectors in $\mathbf{R}^4$ (which has dimension 4) and they generate a subspace of dimension 4, then they must span everything in $\mathbf{R}^4$! (Just as 2 linearly independent vectors in $\mathbf{R}^2$ will generate/span all of $\mathbf{R}^2$, 3 linearly independent vectors in $\mathbf{R}^3$ will generate all of $\mathbf{R}^3$, and so on). Hence, in this case we have that $col(P)$ is all of $\mathbf{R}^4$, that is, $col(P) = \mathbf{R}^4$.

Let's do another one

(4) Find a basis for $col(Q)$ where

```
Q =

     1     2    -3     4     5    -1
     2    -2     1     2     0     1
     3     4    -2    -7     2     2
```

The first step is to label the columns. We do

```
>> u1=Q(:,1);
>> u2=Q(:,2);
>> u3=Q(:,3);
>> u4=Q(:,4);
>> u5=Q(:,5);
>> u6=Q(:,6);
```

And start running the machine. Since u_1 is a non-zero vector, we keep it and add u_2. Is the set $\{u_1, u_2\}$? In order to answer this, we do

```
>> b=[0 0 0]'

b =
```

```
     0
     0
     0

>> rref([u1 u2 b])

ans =

     1     0     0
     0     1     0
     0     0     0
```

And interpret that the set $\{u_1, u_2\}$ is indeed l.i. (no redundancies there). Now we add u_3 and do

```
>> rref([u1 u2 u3 b])

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
```

And interpret that the set $\{u_1, u_2, u_3\}$ is indeed l.i. Now we add u_4 and do

```
>> rref([u1 u2 u3 u4 b])

ans =

    1.0000         0         0   -0.4286         0
         0    1.0000         0   -3.2500         0
         0         0    1.0000   -3.6429         0
```

And interpret that u_4 is a linear combination of u_1, u_2, and u_3, so it's redundant and we drop it. Let's now grab u_5 and ask the question: is the set $\{u_1, u_2, u_3, u_5\}$ l.i.? To answer this we do

```
>> rref([u1 u2 u3 u5 b])

ans =

    1.0000         0         0    0.2857         0
         0    1.0000         0   -0.7500         0
         0         0    1.0000   -2.0714         0
```

And interpret that u_5 is a linear combination of u_1, u_2, and u_3, so it's redundant and we drop it. Let's finally grab u_6 and ask the question: is the set $\{u_1, u_2, u_3, u_6\}$ l.i.? To answer this we do

```
>> rref([u1 u2 u3 u6 b])

ans =

    1.0000         0         0    0.5714         0
         0    1.0000         0    0.5000         0
         0         0    1.0000    0.8571         0
```

And interpret that u_6 is a linear combination of u_1, u_2, and u_3, so it's redundant and we drop it.

Therefore, the set $\{u_1, u_2, u_3\}$ is a basis for $col(Q)$. That set generates (by linear combinations) every single column in Q and it does it without redundancies (that is, it's a l.i. set).

Geometric interpretation. For the matrix Q as above we have that $col(Q) = \mathbf{R}^3$. Since three linearly independent vectors in $\mathbf{R}^3$ will span all of $\mathbf{R}^3$.

Now, let's do one for $row(A)$

(5) Find a basis for $row(A)$ where

```
>> A=[2 3 4 -1; 2 3 5 2; 4 2 -1 1]

A =

     2     3     4    -1
     2     3     5     2
     4     2    -1     1
```

As we said, $row(A) = col(A^T)$, where A^T is the transpose of A, let's call it P. That is,

```
>> P=A'

P =

     2     2     4
     3     3     2
     4     5    -1
    -1     2     1
```

That is, $row(A) = col(P)$. But we already know how to find a basis for $col(P)$! And that basis will be a basis for $row(A)$ as well.