CIS 770: Formal Language Theory

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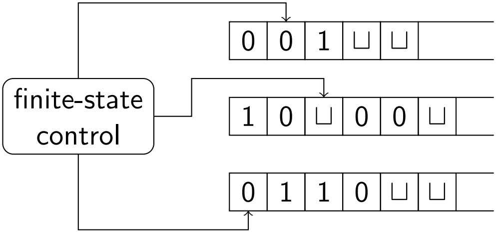
Spring 2016

Multi-Tape Turing Machine



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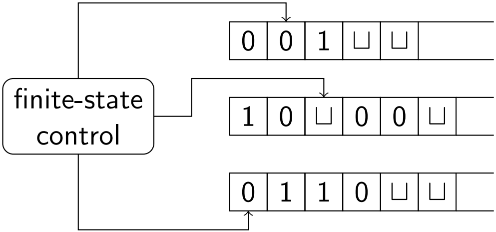

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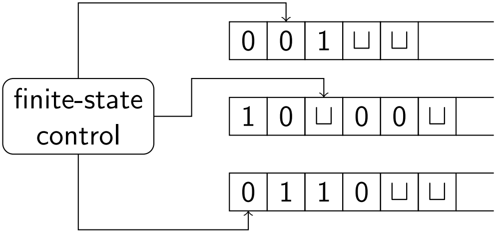

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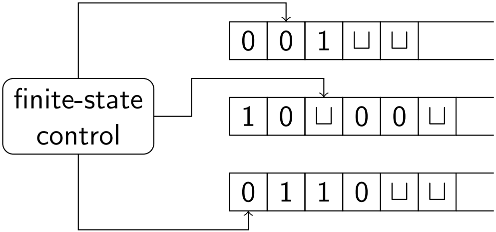

- Input on Tape 1
- Initially all heads scanning cell 1, and tapes 2 to k blank
- In one step: Read symbols under each of the k-heads, and depending on the current control state, write new symbols on the tapes, move the each tape head (possibly in different directions), and change state.

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Formal Definition

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- $\delta : (Q \setminus \{q_{\mathrm{acc}}, q_{\mathrm{rej}}\}) \times \Gamma^k \to Q \times (\Gamma \times \{\mathrm{L}, \mathrm{R}\})^k$ is the transition function.

Computation, Acceptance and Language

- A configuration of a multi-tape TM must describe the state, contents of all k-tapes, and positions of all k-heads. Thus, $\mathrm{C} \in Q \times (\Gamma^*\{*\}\Gamma\Gamma^*)^k$, where $*$ denotes the head position.

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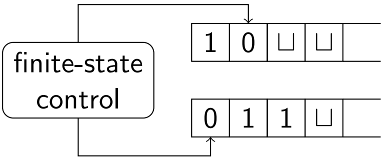
- How do we simulate the movement of k independent heads?

Storing Multiple Tapes



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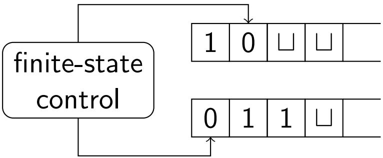

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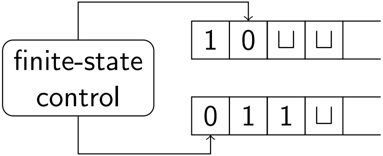

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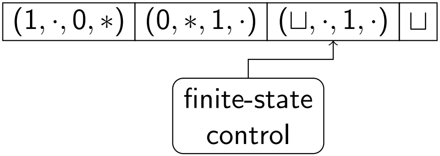

1-tape equivalent single(M)

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- Once again, scan the tape, change all relevant contents, "move" heads (i.e., move $*$s), and change state.

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Formal construction in notes.

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- Idea 2: $\mathrm{det}(M)$ will simulate M on each possible sequence of computation steps that M may try in each step.

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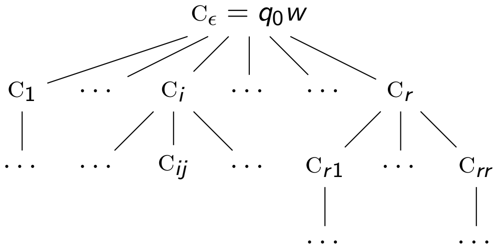

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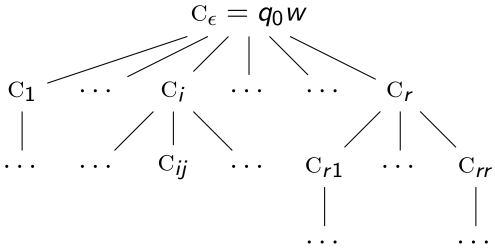

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- $\mathrm{C}_{i_1 i_2 \cdots i_n}$ is the configuration of M after n-steps, where choice i_1 is taken in step 1, i_2 in step 2, and so on.
- Input w is accepted iff $\exists$ accepting configuration in tree.

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Observe that $\det(M)$ may not terminate if w is not accepted.

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- Tape 3, called choice tape, will store the current sequence of nondeterministic choices

Execution of det(M)

1. Initially: Input tape contains w, simulation tape and choice tape are blank
2. Copy contents of input tape onto simulation tape
3. Simulate M using simulation tape as its (only) tape
   1. At the next step of M, if state is q, simulation tape head reads X, and choice tape head reads i, then simulate the ith possibility in $\delta(q, X)$; if i is not a valid choice, then goto step 4
   2. After changing state, simulation tape contents, and head position on simulation tape, move choice tape's head to the right. If Tape 3 is now scanning $\sqcup$, then goto step 4
   3. If M accepts then accept and halt, else goto step 3(1) to simulate the next step of M.
4. Write the lexicographically next choice sequence on choice tape, erase everything on simulation tape and goto step 2.

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- If M accepts w then there is a sequence of choices that will lead to acceptance. det(M) will eventually have this sequence on choice tape, and then its simulation M will accept.
- If M does not accept w then no sequence of choices leads to acceptance. det(M) will therefore never halt!

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- Initially, the program instructions are stored in a contiguous block of memory locations starting at location 1. All registers and memory locations, other than those storing the program, are set to 0.

Instruction Set

- `add X, Y`: Add the contents of registers X and Y and store the result in X.
- `loadc X, I`: Place the constant I in register X.
- `load X, M`: Load the contents of memory location M into register X.
- `loadI X, M`: Load the contents of the location "pointed to" by the contents of M into register X.
- `store X, M`: store the contents of register X in memory location M.
- `jmp M`: The next instruction to be executed is in location M.
- `jmz X, M`: If register X is 0, then jump to instruction M.
- `halt`: Halt execution.

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Proof sketch in the notes. □

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- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
- Restricted Turing Machine models: queue machines, 2-stack machines, 2-counter machines, ...

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- Strong evidence based on the fact that many attempts to define computation yield the same expressive power

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Consequences

- In the course, we will use an informal pseudo-code to argue that a problem/language can be solved on Turing machines
- To show that something can be solved on Turing machines, you can use any programming language of choice, *unless the problem specifically asks you to design a Turing Machine*