Math 322

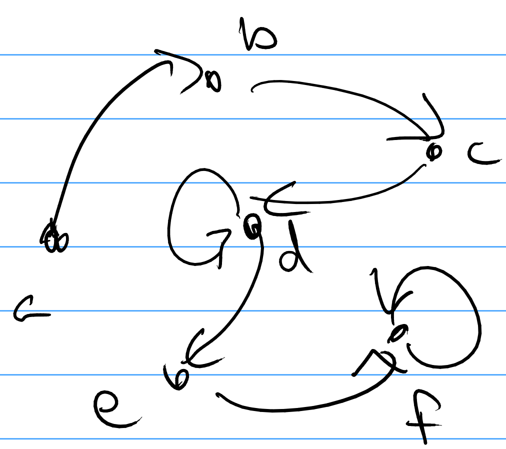
Note: Wednesday's homework now due Thursday 11pm.

☆ - Chat <u>hi</u> if you get connected.

- type in questions.

R is a relation on set A

<u>digraph</u>



aRb

the least num. of edges

<u>8.4</u> $\boxed{\text{Closures}}$ ← add edges to R

to make a relation that

is <u>reflexive</u>, <u>symmetric</u>, <u>or</u> <u>transitive</u>

① Reflexive Closure

$R \cup \Delta$ where $\Delta = \{(a,a) \mid a \in A\}$

② Symmetric Closure

$R \cup R^{-1}$ where $R^{-1} = \{(b,a) \mid aRb\}$

ex) $R = \{(a,b) \mid a \neq b\}$ on $A = \mathbb{Z}$

integers

reflexive closure.

$\Delta = \{(a,a) \mid a \in \mathbb{Z}\}$

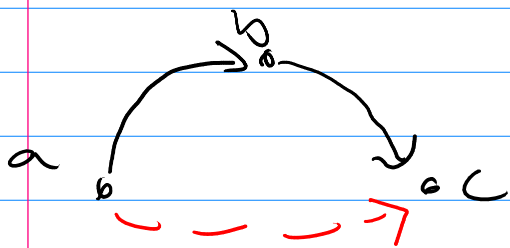
$= \{\ldots, (-2,-2), (-1,-1), (0,0), (1,1) \ldots\}$

$= \{(a,b) \mid a = b\}$

$R \cup \Delta = \{(a,b) \mid a \neq b \vee a = b\}$

$= \mathbb{Z} \times \mathbb{Z}$

Transitive Closure?



$aRb \wedge bRc$

but $a \not R c$

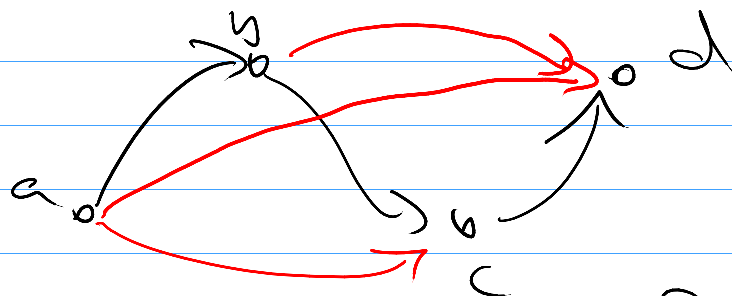
counter example

$\rightarrow$ not transitive.

Idea

① check for all 2-step edges

② join (union) the transitive edges.



How to do this for any R?

Remember:

thm R is trans iff $\forall n\ R^n \subseteq R$.

Background work.

A) Paths

a path from a to z in a digraph is a sequence of edges ...

$$e_1 = (\overset{x_0}{\overset{\shortparallel}{a}}, x_1)$$
$$e_2 = (x_1, x_2)$$
$$\vdots$$
$$e_n = (x_{n-1}, \underset{x_n}{\underset{\shortparallel}{z}})$$

$n \equiv$ length of the path

If there is no confusion you can list the vertices

path is $a = \underbrace{x_0, x_1}_{\text{edge}}, x_2, \dots, x_{n-1}, x_n = z$

Note: n is the path length and the number of commas

$(n+1)$ vertices

B) tie paths to R^n

we saw this last time with M_{R^2}

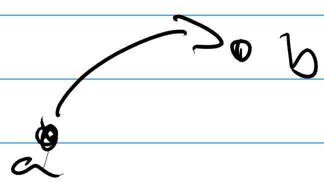
vs $M_{R'}$

$\boxed{\text{Th}^{m}}$ R is a relation on set A.

there is a path of length $n \in \mathbb{Z}^+$ from a to b iff $\underbrace{(a,b) \in R^n}_{a R^n b}$

$\boxed{\text{Pf:}}$ $P(k)$: "there is a path of length k from a to b iff $(a,b) \in R^k$"

Basis: $P(1)$: "there is a path of length 1 iff $(a,b) \in R$"



$\boxed{\text{true}}$

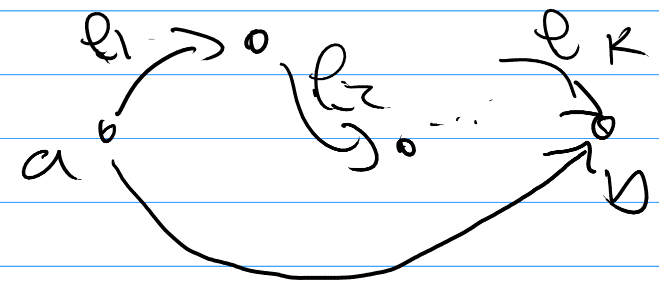
Inductive: show $P(k) \to P(k+1)$

assume path length k iff $(a,b) \in R^k$

show: path length $k+1$ iff $(a,b) \in R^{k+1}$

path length k iff $(a,b) \in R^k$

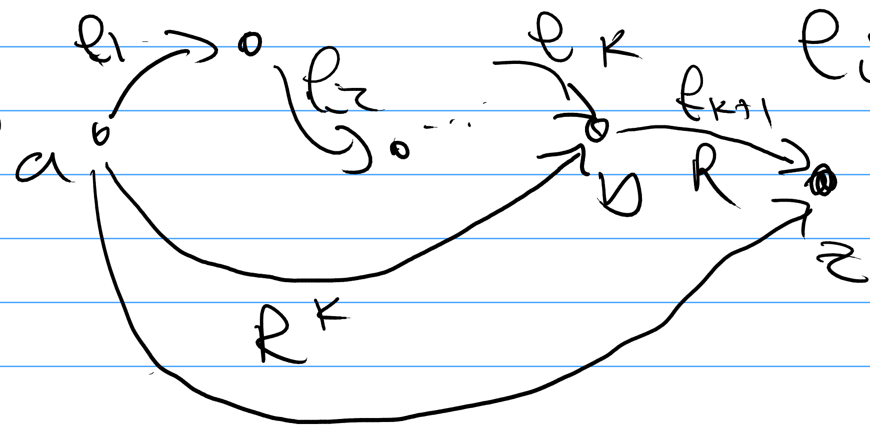
$P(k)$:



$e_i \in R$

$$R^k = \underbrace{R \circ R \circ \cdots \circ R}_{k \text{ times}}$$

$P(k+1)$:



$e_i \in R$

$$R^{k+1} = R^k \circ R$$

R^n is same as paths of length n.

c) **Def:** $R^1 \leftarrow$ all (a,b) of path length 1

$R^2 \leftarrow$ all (a,b) of path length 2

etc.

$\boxed{R^*} = R^1 \cup R^2 \cup R^3 \cup \cdots$

is the connetivity relation, all (a,b) such that there is a path of any length $n \geq 1$ from a to b.

So $R^* = \bigcup_{n=1}^{\infty} R^n$

thm: R^* is the transitive closure.

(proof, see text)

D) $R^* = R^1 \cup R^2 \cup \cdots$

Means: $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots$ forever!

How to do this?

Pigeonhole Principle (+) paths.

$\rightarrow x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$

the path from a to b is length n.

If $|A| = m$ and you have fewer ($m < n$) number of vertices then the length

then the list...

$x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$

Must have at least one vertex listed two or more times. (P.P.)

Say it was $\hat{x}$ is a loop.

$x_0, x_1, \ldots, \overbrace{\hat{x}, \ldots, \hat{x}}, \ldots, x_n$

So you can cut out the loop and shorten the path.

Lemma if $|A| = n$ and you have any path from a to b then you must have a path of length $\leq n$ from a to b.

$$M_{R^*} = \underbrace{M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}} \vee M_R^{[n+1]} \vee \cdots$$

any

connection here, has a path of lower length here

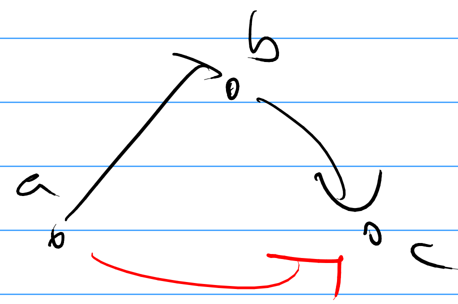
E) Transitive Closure

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}$$

Now that we have an algorithm to find trans. closure...

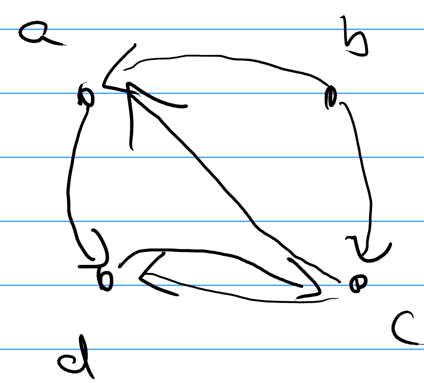
Let's make an easier one!

$\boxed{\text{Warshall's Alg.}}$



ex

$$W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$



(b,a)
(a,d)
$\downarrow$
(b,d)

(c,a)
(a,d)
$\downarrow$
(c,d)

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} \cancel{0} & 0 & \cancel{0} & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \cancel{0} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad M_{R^*} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

trans. closure!

Note: WSU closed Wed!