Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

How to find eigenvalues and eigenvectors

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We say that a (real or complex) number λ is an *eigenvalue* of an $n \times n$ matrix A if and only if there exists a non-zero vector $v \in \mathbf{R}^n$ such that

$$Av = \lambda v. \tag{1}$$

In such case, we say that v is an *eigenvector of A associated to the eigenvalue* λ. Notice that (1) holds true if and only if $(A - \lambda I)v = 0$ if and only if $v \in null(A - \lambda I)$. That is, the collection of all the eigenvectors associated to the eigenvalue λ is precisely the linear subspace $null(A - \lambda I)$.

Now we put together some results to deduce a way to find eigenvalues (remember that for every $n \times n$ matrix M we always have $n = rank(M) + nullity(M)$): we have that λ is an eigenvalue of A if and only if $null(A - \lambda I) \neq \{0\}$ if and only if $nullity(A - \lambda I) := \dim(\mathrm{null}(A - \lambda I)) \geq 1$ if and only if $rank(A - \lambda I) \leq n - 1$ if and only if $\det(A - \lambda I) = 0$.

That is, λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. Let's see an example.

Example 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Then we have

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$$

so that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 6.$$

That is, for every (real or complex) number λ we find that $\det(A - \lambda I)$ coincides with the polynomial $p(\lambda) = \lambda^2 - 5\lambda + 6$. It follows that, by definition, a number λ is an eigenvalue of A if and only if λ is a root of the polynomial p, that is, if and only if $p(\lambda) = 0$. By solving for λ in $p(\lambda) = \lambda^2 - 5\lambda + 6 = 0$, we easily find that the eigenvalues of A are $\lambda = 2$ and $\lambda = 3$.

The polynomial $p(\lambda)$ is called the *characteristic polynomial* of the matrix A. In general, for an $n \times n$ matrix A, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ will be a polynomial of degree n whose roots are precisely the eigenvalues of A. Notice that,

$$p(0) = \det(A). \tag{2}$$

Next, let's find the eigenvectors associated to each eigenvalue. By definition, a vector v is an eigenvector associated to the eigenvalue $\lambda = 2$ if and only if $v \in \text{null}(A - 2I)$. That is, v solves the system $(A - 2I)v = 0$. In order to solve we do

```
>> A

A =

     1     1
    -2     4

>> I=eye(2)

I =

     1     0
     0     1

>> b=[0 0]'

b =

     0
     0

>> rref([A - 2*I, b])

ans =

     1    -1     0
     0     0     0
```

That is, $\text{null}(A-2I) = span\{[1,\ 1]'\}$. In other words, the eigenvectors of A associated to the eigenvalue $\lambda = 2$ are all the multiples of the vector $[1,\ 1]'$. Next, let's find the eigenvectors associated to $\lambda = 3$

```
>> rref([A - 3*I, b])

ans =

    1.0000   -0.5000         0
         0         0         0
```

That is, $\text{null}(A - 3I) = span\{[1/2,\ 1]'\}$. In other words, the eigenvectors of A associated to the eigenvalue $\lambda = 3$ are all the multiples of the vector $[1/2,\ 1]'$.

A quick way to find eigenvalues and eigenvectors using Matlab is

```
>> [P D]=eig(A)

P =

   -0.7071   -0.4472
   -0.7071   -0.8944


D =

     2     0
     0     3
```

where Matlab stores the eigenvalues along the diagonal of the diagonal matrix D and the corresponding eigenvectors must be read column-wise from the matrix P. Also, the columns of P are normalized to have norm 1.

Notice that

```
>> A*P

ans =

   -1.4142   -1.3416
   -1.4142   -2.6833

>> P*D

ans =

   -1.4142   -1.3416
   -1.4142   -2.6833
```

That is,

$$AP = PD. \tag{3}$$

Let's do another one. Consider the matrix

$$B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

and do

```
>> B=[3 2 -1; 2 2 4; 1 1 2]

B =

     3     2    -1
     2     2     4
     1     1     2

>> [L R]=eig(B)

L =

   -0.7174    0.5034    0.5774
    0.6231    0.7728   -0.8083
    0.3115    0.3864    0.1155


R =

    1.6972         0         0
         0    5.3028         0
         0         0         0
```

Notice that $\lambda = 0$ is an eigenvalue. That is, $p(0) = 0$ (here p is the characteristic polynomial of B). From equation (2) we infer that $\det(B) = 0$ and B that is not invertible. Notice, again, that $BL = LR$, just like in equation (3),

```
>> B*L

ans =

   -1.2176    2.6695   -0.0000
    1.0575    4.0981   -0.0000
    0.5288    2.0491   -0.0000
```

```
>> L*R

ans =

   -1.2176    2.6695         0
    1.0575    4.0981         0
    0.5288    2.0491         0
```

We say that an $n \times n$ matrix A is *diagonalizable* if there is a basis of $\mathbf{R}^n$ made out of eigenvectors of A. For instance, in our first example we have

```
>> [P D]=eig(A)

P =

   -0.7071   -0.4472
   -0.7071   -0.8944


D =

     2     0
     0     3
```

The columns of P are eigenvectors of A: do they form a basis of $\mathbf{R}^2$? Yes, as it's clearly seen from doing

```
>> rank(P)

ans =

     2
```

Hence, we're able to compute P^{-1}, and from (3) we can now write

$$A = PDP^{-1}$$

where P contains the eigenvectors of A and D is diagonal.

Is the matrix B above diagonalizable? Let's do

```
>> rank(L)
```

```
ans =

     3
```

Consequently, B is diagonalizable since it's possible to form a basis of $\mathbf{R}^3$ by using some of its eigenvectors (the columns of L). By the way, notice that the concepts of "invertible" and "diagonalizable" are not related. Matrices can be invertible and diagonalizable, or one but not the other, or neither one.

Along the lines of (3), we can write

$$B = LRL^{-1}$$

where the columns of L are eigenvectors of B and the diagonal elements of the diagonal matrix R are the eigenvalues of B.

Let's do some more examples.

Example 2. Consider the matrix

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Is M diagonalizable? Let's do

```
>> M=[1 -1; 0 1]

M =

     1    -1
     0     1

>> [E F]=eig(M)

E =

    1.0000    1.0000
         0    0.0000


F =

     1     0
```

```
     0     1

>> rank(E)

ans =

     1
```

From the information above we deduce M is not diagonalizable since $\text{rank}(E) = 1 < 2$.

Suppose that λ is an eigenvalue of an $n \times n$ matrix A, that is,

$$Av = \lambda v,$$

for some non-zero vector $v \in \mathbf{R}^n$. If we left-multiply the equation above by A we obtain

$$A^2 v = A(\lambda v) = \lambda A v = \lambda^2 v.$$

That is, $A^2 v = \lambda^2 v$. Then, we obtain the following result: if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2, and so on... And everything with the same eigenvectors!

Now suppose we are given the polynomial

$$q(\lambda) = (3 - \lambda)(1 - \lambda)(-4 - \lambda) \tag{4}$$

and we are told that $q(\lambda)$ is the characteristic polynomial of some matrix C, what can we say about the matrix C? The very first thing would be to deduce that C is a 3×3 matrix (because q has degree 3). We can also deduce that the eigenvalues of C are $\lambda = 3$, $\lambda = 1$, and $\lambda = -4$ (because those are the roots of q). Also, we can say that C is invertible. Indeed, since $q(0) = \det(C)$ and $q(0) = (3)(1)(-4) = -12 \neq 0$, we obtain that C is invertible (i.e., C^{-1} exists). We can also infer that the eigenvalues of C^2 are 9, 1, and 16. We can say all this about C without even looking at it! Just by knowing its characteristic polynomial. The next question is: Is C diagonalizable? In principle we would need to take a look at the eigenvectors of C, but here is a nice criterion:

A sufficient (but not necessary) condition for diagonalization: If all the eigenvalues of an $n \times n$ matrix are distinct, in the sense that they're all different from one another (no repetitions), then the matrix is diagonalizable.

It's important to notice that this implication goes in only one direction. There are matrices with repeated eigenvalues which are diagonalizable. For instance, the identity matrix I has all its eigenvalues equal to 1, and it's diagonalizable (check it!).

Hence, from the criterion above, the matrix C with characteristic polynomial (4) is diagonalizable.

Can C be an orthogonal matrix? Well, the answer is no. Indeed, *every orthogonal matrix has determinant equal to 1 or* -1. This follows from the fact that whenever

$$I = QQ^t, \tag{5}$$

then

$$1 = \det(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)\det(Q) = \det(Q)^2,$$

(where we have used that the determinant of the product of two matrices is the product of their determinants and the fact that the determinant of a matrix coincides with the determinant of its transpose). That is, whenever (5) holds (i.e., Q is orthogonal), then $\det(Q)^2 = 1$. Therefore, $\det(Q) = 1$ or $\det(Q) = -1$. Since $\det(C) = -12$, we obtain that C is not orthogonal.

In all the cases thus far, the eigenvalues have been real numbers. Will that always be the case? No. For instance, take the matrix

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and do

```
>> U=[1 -1; 1 1]

U =
     1    -1
     1     1

>> [H J]=eig(U)

H =
   0.7071             0.7071
        0 - 0.7071i        0 + 0.7071i

J =
   1.0000 + 1.0000i        0
        0             1.0000 - 1.0000i
```

That is, in this example the matrix U has eigenvalues $1 + i$ and $1 - i$ (here i is the square root of -1, the so-called imaginary unit). Is U diagonalizable? Let's see

```
>> rank(H)

ans =

     2
```

So, yes, U is diagonalizable in the field of the complex numbers (not in the field of the real numbers).

What other criteria for diagonalizable matrices are there? There is, for instance, the next one

Diagonalization criterion for symmetric matrices: If an $n \times n$ matrix is symmetric (that is, if it coincides with its transpose), then the matrix is diagonalizable; moreover, there exists an orthonormal basis made out of its eigenvectors. Also, a symmetric matrix will always have real eigenvalues. Let's look at an example.

Example 3. Consider the symmetric matrix

```
S =

    14     6     3
     6    24    12
     3    12     6
```

and do

```
>> [P D]=eig(S)

P =

   -0.0000   -0.9398   -0.3419
   -0.4472    0.3058   -0.8405
    0.8944    0.1529   -0.4203


D =

    0.0000         0         0
         0   11.5597         0
         0         0   32.4403
```

Then we have that P is an orthogonal matrix, indeed,

```
>> P*P'

ans =
```

```
    1.0000         0   -0.0000
         0    1.0000         0
   -0.0000         0    1.0000
```

Thus, not only do the columns of P form a basis for $\mathbf{R}^3$, they form an orthonormal basis! That's because S is symmetric.