Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

On the notion of *span*

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We know that given (column) vectors u, u_1, $u_2, \ldots$, u_n in $\mathbf{R}^m$, the expression "*u is a linear combination of* u_1, $u_2, \ldots$, u_n" means that there exist real numbers x_1, x_2, $\ldots$, x_n (some or all of which could be zero or not) such that

$$u = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n. \tag{1}$$

Another way to signify exactly the same thing is to say "*u belongs to the sub-space spanned by* u_1, $u_2, \ldots$, u_n", in symbols, "$u \in \mathit{span}(u_1, u_2, \ldots, u_n)$". That is, $\mathit{span}(u_1, u_2, \ldots, u_n)$ denotes the set of all possible linear combinations of the vectors u_1, $u_2, \ldots$, u_n.

Yet another way to signify that "u is a linear combination of the vectors u_1, $u_2, \ldots$, u_n" is to say that "u is spanned by the vectors u_1, $u_2, \ldots$, u_n."

Therefore, the questions "Is u a linear combination of u_1, $u_2, \ldots$, u_n?", "Does u belong to $\mathit{span}(u_1, u_2, \ldots, u_n)$?", and "Is u spanned by $u_1, u_2, \ldots, u_n$?" are all equivalent.

Example 1. Given the vectors

$$u = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \text{ and } \quad u_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

Does u belong to $\mathit{span}(u_1, u_2, u_3)$?

Well, by definition of *span*, the question is equivalent to "Is u a linear combination of u_1, u_2, and u_3? And, by definition of linear combination, the question is equivalent to "Are there real numbers x_1, x_2, and x_3 such that

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = u?'' \tag{2}$$

And this last question can be immediately addressed by realizing that (2) is the vector form of a system whose matrix of coefficients has the vectors u_1, u_2, and u_3 as its columns and whose right-hand side vector is the vector u. Hence, the question is equivalent to "Are there solutions x_1, x_2, and x_3 to such system?", which we answer by doing, for instance,

```
>> u=[2 1 1]'

u =

     2
     1
     1

>> u1=[0 1 1]'

u1 =

     0
     1
     1

>> u2=[1 0 -4]'

u2 =

     1
     0
    -4

>> u3=[4 1 3]'

u3 =

     4
     1
     3

>> rref([u1 u2 u3 u])

ans =

    1.0000         0         0    0.5556
         0    1.0000         0    0.2222
         0         0    1.0000    0.4444
```

And we interpret that the system has a solution; namely, $x_1 = 0.5556$, $x_2 = 0.2222$ and $x_3 = 0.4444$. Consequently, the vector u does belong to $span(u_1, u_2, u_3)$, and the scalars that make (2) true are given by $x_1 = 0.5556$, $x_2 = 0.2222$ and $x_3 = 0.4444$. That is,

$$u = 0.5556u_1 + 0.2222u_2 + 0.4444u_3.$$

Notice how the steps *translation* and *interpretation* were successfully taken to answer the initial question. We translated the original question into equivalent questions until we rephrased the question as a question involving systems, and systems we can handle.

Example 2. Given the vectors

$$w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \quad w_2 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

Is w spanned by w_1 and w_2? In other words, is w a linear combination of w_1 and w_2?

In order to answer this question we do

```
>> w=[1 1]'

w =

     1
     1

>> w1=[2 1]'

w1 =

     2
     1

>> w2=[1 1/2]'

w2 =

    1.0000
    0.5000

>> rref([w1 w2 w])

ans =
```

```
    1.0000    0.5000         0
         0         0    1.0000
```

And we interpret that the associated system is inconsistent. Hence, there are no values of x_1 and x_2 that will make $x_1w_1 + x_2w_2$ equal to w. In this case the answer is no, the vector w is not spanned by the vectors w_1 and w_2. Equivalent ways to indicate this would be to write $w \notin span(w_1, w_2)$ or to say "w is not a linear combination of the vectors w_1 and w_2".

Example 3. Given the matrix

```
P =

     1     3    -1     4
    -1     3     4     2
     3     3     3     3
     4     6     2     7
```

Determine which ones of its columns are spanned by the rest (of its columns).

In order to do this we label the columns of P, for instance, as u_1, u_2, u_3, and u_4. That is,

```
>> u1=P(:,1)

u1 =

     1
    -1
     3
     4

>> u2=P(:,2)

u2 =

     3
     3
     3
     6

>> u3=P(:,3)
```

```
u3 =

    -1
     4
     3
     2

>> u4=P(:,4)

u4 =

     4
     2
     3
     7
```

Now, is u_1 spanned by the other columns? In other words, is u_1 a linear combination of u_2, u_3, and u_4? To answer this we only have to do

```
>> rref([u2 u3 u4 u1])

ans =

     1     0     0    -7
     0     1     0     2
     0     0     1     6
     0     0     0     0
```

To obtain that the answer is Yes. Moreover, we have $u_1 = -7u_2 + 2u_3 + 6u_4$. Now, is u_2 a linear combination of the other vectors? By doing

```
>> rref([u1 u3 u4 u2])

ans =

    1.0000         0         0   -0.1429
         0    1.0000         0    0.2857
         0         0    1.0000    0.8571
         0         0         0         0
```

So, yes, it is a linear combination and we have $u_2 = -0.1429u_1 + 0.2857u_3 + 0.8571u_4$.

The same questions but now with u_3 and u_4 lead to

```
>> rref([u1 u2 u4 u3])

ans =

    1.0000         0         0    0.5000
         0    1.0000         0    3.5000
         0         0    1.0000   -3.0000
         0         0         0         0

>> rref([u1 u2 u3 u4])

ans =

    1.0000         0         0    0.1667
         0    1.0000         0    1.1667
         0         0    1.0000   -0.3333
         0         0         0         0
```

And we conclude that each column of P can be expressed as a linear combination of the remaining ones.

Let's do another one.

Example 4. Consider the matrix

```
>> R=[1 2  -1 0; 0 2 1 0; 0 0 0 3]

R =

     1     2    -1     0
     0     2     1     0
     0     0     0     3
```

Determine which ones of its columns can be spanned by the remaining ones. Again, let's put a handle on the columns of R. For instance,

```
>> v1=R(:,1)

v1 =

     1
```

```
     0
     0

>> v2=R(:,2)

v2 =

     2
     2
     0

>> v3=R(:,3)

v3 =

    -1
     1
     0

>> v4=R(:,4)

v4 =

     0
     0
     3
```

Is v_1 a linear combination of v_2, v_3, and v_4? Let's see...

```
>> rref([v2 v3 v4 v1])

ans =

    1.0000         0         0    0.2500
         0    1.0000         0   -0.5000
         0         0    1.0000         0
```

Yes! Moreover, we can write

$$v_1 = 0.2500v_2 - 0.5000v_3 + 0v_4 = 0.2500v_2 - 0.5000v_3.$$

How about v_2? Is it a linear combination of the other ones? Let's see

```
>> rref([v1 v3 v4 v2])

ans =

     1     0     0     4
     0     1     0     2
     0     0     1     0
```

Indeed it is. We have

$$v_2 = 4v_1 + 2v_3 + 0v_4 = 4v_1 + 2v_3.$$

How about v_3? Is it a linear combination of the other ones? Let's see

```
>> rref([v1 v2 v4 v3])

ans =

    1.0000         0         0   -2.0000
         0    1.0000         0    0.5000
         0         0    1.0000         0
```

Yes again. We have

$$v_3 = -2v_1 + 0.5v_2 + 0v_4 = -2v_1 + 0.5v_2.$$

How about v_4? Let's see

```
>> rref([v1 v2 v3 v4])

ans =

    1.0000         0   -2.0000         0
         0    1.0000    0.5000         0
         0         0         0    1.0000
```

No solutions! Therefore, v_4 cannot be expressed as a linear combination of v_1, v_2, and v_3. In other words, v_4 cannot be spanned by v_1, v_2, and v_3. In symbols, $v_4 \notin span(v_1, v_2, v_3)$.

Hence, only the first three columns of R can be written as linear combinations of the remaining ones.