Kansas State University | Department of Mathematics

**Applied Matrix Theory - Math 551**
Solutions to the exercises in the study guide for Test 1
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**1.** In each of the problems (i)-(v) below, the matrix rref($[\mathbf{A} \mid \mathbf{b}]$) is given. Here we use the notation $[\mathbf{A} \mid \mathbf{b}]$ to indicate the augmented matrix of a linear system of equations with matrix of coefficients $\mathbf{A}$ and right-hand side vector $\mathbf{b}$. For each case (i)-(v) below determine the following:

(a) Number of equations of the system, number of unknowns, size of $\mathbf{A}$ and size of $[\mathbf{A} \mid \mathbf{b}]$.

(b) rref($\mathbf{A}$)

(c) Use the rank-based solvability criterion to determine whether the system is consistent or inconsistent. In case of consistency determine the number of solutions and free variables (if any).

(d) Find the solution(s) of the consistent systems.

(i) $\operatorname{rref}([\mathbf{A} \mid \mathbf{b}]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$

(ii) $\operatorname{rref}([\mathbf{A} \mid \mathbf{b}]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(iii) $\operatorname{rref}([\mathbf{A} \mid \mathbf{b}]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(iv) $\operatorname{rref}([\mathbf{A} \mid \mathbf{b}]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

(v) $\operatorname{rref}([\mathbf{A} \mid \mathbf{b}]) = \begin{bmatrix} 1 & 2 & 0 & 0 & -7 & -3 \\ 0 & 0 & 1 & 0 & 8 & 5 \\ 0 & 0 & 0 & 1 & 0 & -10 \end{bmatrix}$

**Solution**.

(i) (a) 3 equations, 4 unknowns, size($\mathbf{A}$) $= 3 \times 4$, size($[\mathbf{A} \mid \mathbf{b}]$) $= 3 \times 5$.

(b) Delete the last column of rref($[\mathbf{A} \mid \mathbf{b}]$) to obtain

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) By definition of rank of a matrix (that is, the number of non-zero rows in its rref) we find that rank$(\mathbf{A})$ = rank$([\mathbf{A} \mid \mathbf{b}])$ = 3. Hence, the system is consistent. There are 4-3=1 free variables ($x_3$).

(d) All solutions take the form $[7, -x_3, x_3, -4]'$, where $x_3$ is a real number (the free variable!).

(ii) (a) 5 equations, 3 unknowns, size$(\mathbf{A}) = 5 \times 3$, size$([\mathbf{A} \mid \mathbf{b}]) = 5 \times 4$.

(b) Delete the last column of rref$([\mathbf{A} \mid \mathbf{b}])$ to obtain

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) By definition of rank of a matrix (that is, the number of non-zero rows in its rref) we find that rank$(\mathbf{A})$ = rank$([\mathbf{A} \mid \mathbf{b}])$ = 3. Hence, the system is consistent. There are 3-3=0 free variables.

(d) $(0, 0, 0)$ (unique solution).

(iii) (a) 5 equations, 4 unknowns, size$(\mathbf{A}) = 5 \times 4$, size$([\mathbf{A} \mid \mathbf{b}]) = 5 \times 5$.

(b) Delete last column of rref$([\mathbf{A} \mid \mathbf{b}])$ to obtain

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) By using the definition of rank we see that rank$(\mathbf{A}) <$ rank$([\mathbf{A} \mid \mathbf{b}])$. Consequently, the system is inconsistent. The fourth row also shows inconsistency.

(d) There are no solutions.

(iv) (a) 3 equations, 4 unknowns, size$(\mathbf{A}) = 3 \times 4$, size$([\mathbf{A} \mid \mathbf{b}]) = 3 \times 5$.

(b) Delete last column of rref$([\mathbf{A} \mid \mathbf{b}])$ to obtain

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(c) By using the definition of rank we get rank$(\mathbf{A})$ = rank$([\mathbf{A} \mid \mathbf{b}])$ = 3. Hence, the system is consistent. There are 4-3=1 free variables (say, $x_4$).

(d) All vectors of the form $[7, 4, 1 - x_4, x_4]'$, where $x_4$ is any real number, are the solutions to the system.

(v) (a) 3 equations, 5 unknowns, size$(\mathbf{A}) = 3 \times 5$, size$([\mathbf{A} \mid \mathbf{b}]) = 3 \times 6$.

(b) Delete last column of rref$([\mathbf{A} \mid \mathbf{b}])$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) We use the definition of rank, to obtain $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \mid \mathbf{b}]) = 3$. Therefore, the system is consistent. There are 5-3=2 free variables ($x_2$ and $x_5$).

(d) All vectors of the form $[-3 - 2x_2 + 7x_5, x_2, 5 - 8x_5, -10, x_5]'$, where $x_2$ and $x_5$ are any real numbers, are the solutions to the system.

**2.** Are there real numbers $x_1$, $x_2$, and $x_3$ such that

$$x_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}?$$

**Solution**. The first thing we should immediately notice is that the equation above is nothing but the vector form of a system $Ax = b$ where the matrix of coefficients is given by

$$A = \begin{bmatrix} 0 & 1 & -4 \\ 1 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

and the right hand side vector is given by

$$b = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}.$$

Hence, by doing

```
>> rref([A b])

ans =

     1     0     0   100
     0     1     0   -30
     0     0     1    -9
```

we realize that the answer is yes and obtain

$$100 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 30 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 9 \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$$

**3.** The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is the adjacency matrix of a graph $G$. Is $G$ a directed or undirected graph? Draw the graph $G$. How many walks of length 4 are there from vertex 1 to vertex 3? How many walks of length less than or equal to 4 are there from vertex 1 to vertex 3? Explain using powers of $A$.

**Solution**. $G$ is a directed graph since the adjacency matrix $A$ is not symmetric. For example, the entry in row 4 and column 1 is different from the entry in row 1 and column 4. Don't forget to draw the graph.

The number of walks of length 4 from vertex 1 to vertex 3 is given by the entry in row 1 and column 3 of the matrix $A^4$. We have

$$A^4 = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 5 & 3 & 7 & 7 \\ 3 & 1 & 4 & 3 \end{bmatrix}.$$

Then, there are 3 walks of length 4 from vertex 1 to vertex 3.

The number of walks of length less than or equal to 4 from vertex 1 to vertex 3 is given by the entry in row 1 and column 3 of the matrix $A + A^2 + A^3 + A^4$. We have

$$A + A^2 + A^3 + A^4 = \begin{bmatrix} 4 & 3 & 5 & 5 \\ 3 & 2 & 2 & 3 \\ 10 & 5 & 14 & 13 \\ 5 & 2 & 8 & 6 \end{bmatrix}.$$

Therefore there are 5 walks of length less than or equal to 4 from vertex 1 to vertex 3.

**4.** $A$ is a $4 \times 3$ matrix such that

$$A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 9 \end{bmatrix} \qquad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}.$$

Find the matrix $A$.

**Solution**. Denote by $a_{ij}$ the entry in row $i$ and column $j$ of the matrix $A$. Note that

$$A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_{11} \\ 2a_{21} \\ 2a_{31} \\ 2a_{41} \end{bmatrix}.$$

Then the first column of $A$ is given by $a_{11} = 1/2$, $a_{21} = 1$, $a_{31} = 3/2$, $a_{41} = 9/2$. Proceed similarly to obtain the second and third columns of $A$.

**5.** (a) In this problem $A$, $B$ and $C$ are matrices. Recall that if a matrix $A$ has size $m \times n$, then its transpose $A^T$ has size $n \times m$. Suppose that the size of $A^T B$ is $4 \times 3$ and the size of $BC$ is $6 \times 5$.

(i) What are the sizes of $A$, $B$ and $C$?

(ii) Determine whether the following operations are well defined and, if so, give the size of the resulting matrix: $C^T B^T$, $A + C$, $2A$.

(b) How should the sizes of the matrices $A$ and $B$ be related for the sum $A + B^T$ to make sense?

(c) Give an example of a symmetric matrix $S$ (that is, $S = S^T$). Give an example of a *skew symmetric matrix* $R$, that is, $R = -R^T$.

**Solution**. Use that any matrix product $PQ$ is defined only when the number of columns of $P$ equals the number of rows of $Q$. That is, if $P$ is $m \times n$ and $Q$ is $r \times s$, then the product $PQ$ makes sense if and only if $n = r$. In such case, the size of $PQ$ is $m \times s$.

(a) (i) size$(A) = 6 \times 4$, size$(B) = 6 \times 3$, size$(C) = 3 \times 5$.

(ii) $C^T B^T$ is defined and is of size $5 \times 6$, $A + C$ is not defined, $2A$ is defined and is of size $6 \times 4$.

Don't forget to do (b) and (c).

**6.** Consider an economy which has a steel plant, a coal mine and a transportation industry. To produce \$1 value of steel requires inputs of 50 cents from the steel plant, 30 cents from the coal mine, and 10 cents from transportation. To produce \$1 value of coal requires inputs of 10 cents from the steel plant, 20 cents from the coal mine, and 30 cents from transportation. Each dollar's worth of transportation output requires inputs of 10 cents from the steel plant, 40 cents from the coal mine, and 5 cents from transportation. Assume that the outside demand for the current production period is 2 million dollars for steel, 1.5 million dollars for coal, and \$500,000 for transportation. How much should each industry produce to satisfy the demands?

**Solution**. Let $x_1$, $x_2$, and $x_3$ be the dollar value (in millions) of the outputs of the steel plant, the coal mine, and the transportation industry, respectively. Then by asking (and answering) the key questions: "How much steel is consumed?" "How much coal is consumed?" "How much transportation is consumed?", and equating those consumptions to the corresponding productions, we obtain the following system (do it!)

$$(S)\left\{\begin{array}{rcrcrcrl} 0.5x_1 & + & 0.1x_2 & + & 0.1x_3 & + & 2 = & x_1 \\ 0.3x_1 & + & 0.2x_2 & + & 0.4x_3 & + & 1.5 = & x_2 \\ 0.1x_1 & + & 0.3x_2 & + & 0.05x_3 & + & 0.5 = & x_3 \end{array}\right.$$

From which we extract the **consumption matrix** $C$ given by

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.05 \end{bmatrix}$$

and the **external demand vector** $d$,

$$d = \begin{bmatrix} 2 \\ 1.5 \\ 0.5 \end{bmatrix}$$

We can solve the original system by moving all the variables to one side and the constant terms to the other. Namely,

$$(S') \begin{cases} -0.5x_1 + 0.1x_2 + 0.1x_3 = -2 \\ 0.3x_1 - 0.8x_2 + 0.4x_3 = -1.5 \\ 0.1x_1 + 0.3x_2 - 0.95x_3 = -0.5 \end{cases}$$

Using a calculator, the rref of the augmented matrix of this system equals

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5.6470 \\ 0 & 1 & 0 & 5.4067 \\ 0 & 0 & 1 & 2.8281 \end{array}\right]$$

Therefore, the steel plant should produce output for $5.6470$ million dollars, the coal mine should produce output for $5.4067$ million dollars, and the transportation industry should produce output for $2.8281$ million dollars.

Notice that the original system $(S)$ can be expressed as

$$Cx + d = x,$$

where $x = (x_1, x_2, x_3)'$. Equivalently,

$$(C - I)x = -d,$$

where $I$ is the $3 \times 3$ identity matrix. Hence, we could have also solved for $x$ by doing

$$>> rref([C - I, -d]),$$

and notice that the matrix $[C - I, -d]$ is exactly the augmented matrix for the system $(S')$. Therefore, both ways to solve for $x$ will yield the same answer (naturally!).

**7.** My aunt Lila's house is infested with ants. My friend Rainman and I visit my aunt daily because she makes a really good stew. The stew usually has ants in it, but it's no big deal. Anyway, about a month ago, right after lunch time, good old Rainman took a glance at the house and, almost immediately, he said to me: "There are exactly 6,000 ants in this house". "And how many anthills?", I asked. A couple of hours later, he replied: "Two". "Where are they?" I inquired. "There is one in the basement and one in the master bedroom. The one in the basement has 2,000 ants", he said. Right away, Rainman and I took up a meticulous study of the circulation of the ants in the house. After a month of research here is what we got: each day, 40% of the ants in the anthill located in the basement moved to the anthill in the master bedroom. In turn, 30% of the ants in the master bedroom would move to the anthill in the basement. Coincidentally enough, 10% of the ants in the master bedroom would patiently find their way into the stew pot. A few moments ago I asked Rainman: "How many ants do you think we ate today?". He couldn't answer because he suddenly felt very sick. Was he overreacting?

**Solution**.

Let $a_k$ be the number of ants in the basement by day $k$, $b_k$ the number of ants in the master bedroom by day $k$, and $c_k$ be the number of ants in the stew pot by day $k$.

When Rainman took the first glance, at day $k = 0$ say, the situation was $a_0 = 2000, b_0 = 4000$ and $c_0 = 0$. $c_0 = 0$ because Rainman took a first glance right after lunch, when the number of ants in the pot was zero (because they were eaten!). Thus, the vector that describes the initial situation is

$$u_0 = [2000\ 4000\ 0]'.$$

From the study of the circulation of the ants we get

$$\begin{aligned} a_{k+1} &= 0.6\,a_k + 0.3\,b_k + 0\,c_k \\ b_{k+1} &= 0.4\,a_k + 0.6\,b_k + 0\,c_k \\ c_{k+1} &= 0\,a_k + 0.1\,b_k + 0\,c_k. \end{aligned}$$

Therefore, the step matrix is

$$A = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}.$$

Thus, the situation one month (30 days) after the first ant counting is encoded in the vector $u_{30}$, given by

$$u_{30} = A^{30}u_0.$$

Using our calculator we obtain that

$$u_{30} = [523.4445\ 604.4216\ 63.8647]'$$

That is, the number of ants that were eaten during the $30th$ day is 63. Which is not bad, or good either. However, if you take a look at the situation at $k = 1$, that is, on the day following Rainman's first glance we get

$$u_1 = Au_0,$$

using a calculator (or just by hand) it turns out that

$$u_1 = [2400\ 3200\ 400]'.$$

That is, the number of ants eaten on day $k = 1$ was 400 (much more that 63). Clearly, Rainman was overreacting...

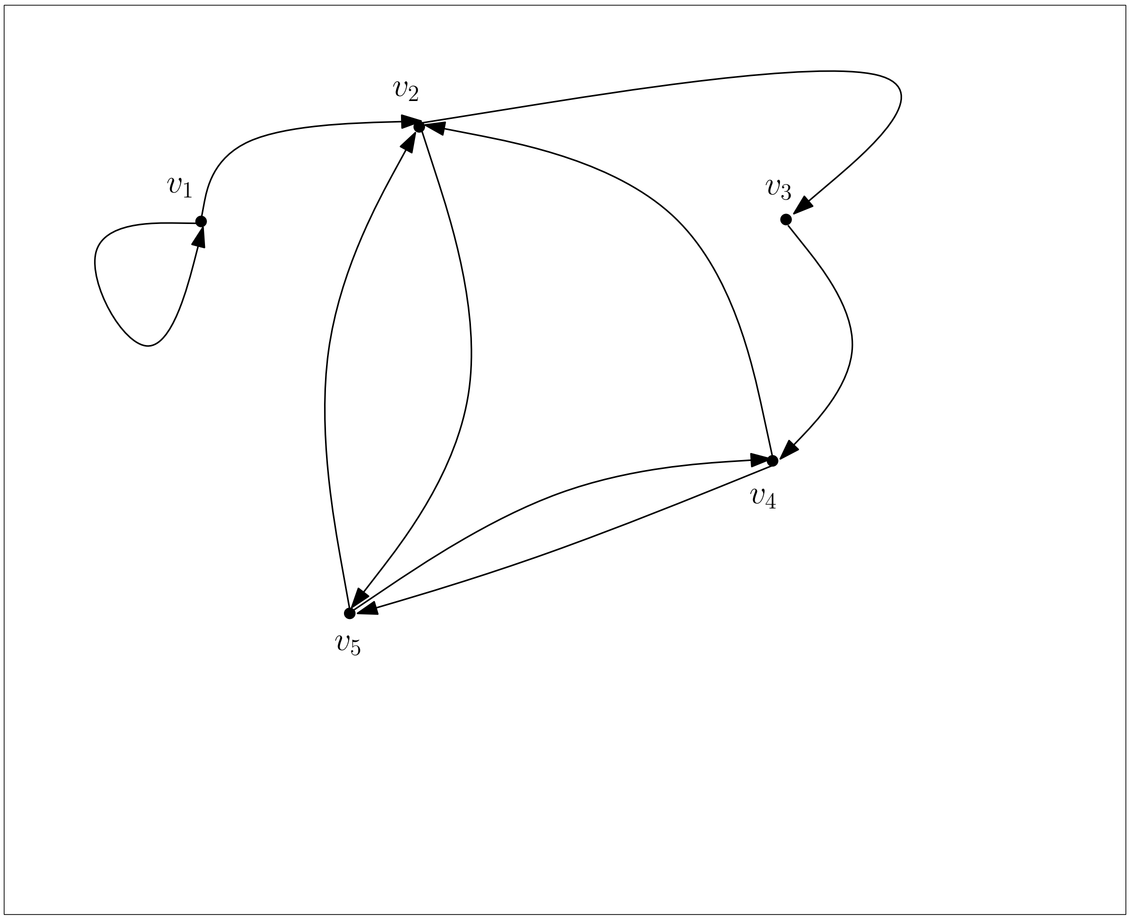


Figure 1: Directed graph representing a 5-page web

**8.** Graphs and websites. Suppose that a 5-page web is represented by the graph in Figure 1 in such a way that an edge going from vertex $v_i$ to vertex $v_j$ means that there is a link from page $v_i$ to page $v_j$. By departing from page 4 and clicking exactly 8 times, in how many ways can we end up on page 5? Explain using powers of the adjacency matrix.

**Solution**. The graph in Figure 1 is represented by the following adjacency matrix.

```
A =
        1       1       0       0       0
        0       0       1       0       1
        0       0       0       1       0
        0       1       0       0       1
        0       1       0       1       0
```

Since we are interested in walks of length 8 (8 clicks), we compute

```
>> A^8

ans =

        1      44      24      36      44
        0      37      20      32      36
        0      24      13      20      24
        0      44      24      37      44
        0      44      24      36      45
```

and conclude that there are 44 ways to get from page 4 to page 5 in 8 clicks.

**9.** (a) Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ be points in the plane. Let $y = ax^3 + bx^2 + cx + d$ be a cubic curve passing through all points (assuming there is one). What system of linear equations must be solved in order to find the coefficients $a$, $b$, $c$, and $d$? Identify the corresponding augmented matrix.

(b) Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ be points in the plane. Identify the cases in which there is only one cubic curve going through all points, infinitely many cubic curves going through all points, and no cubic curve going through all points.

**Solution**.

(a) Consider a cubic curve with equation $y = ax^3 + bx^2 + cx + d$. We want to find (if possible) coefficients $a$, $b$, $c$ and $d$ so that the cubic curve passes through the given points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$. This means that $a$, $b$, $c$ and $d$ must satisfy the following linear system of equations:

$$\begin{array}{ccccccccccc}
(x_1)^3 & a & + & (x_1)^2 & b & + & x_1 & c & + & d & = & y_1 \\
(x_2)^3 & a & + & (x_2)^2 & b & + & x_2 & c & + & d & = & y_2 \\
(x_3)^3 & a & + & (x_3)^2 & b & + & x_3 & c & + & d & = & y_3 \\
(x_4)^3 & a & + & (x_4)^2 & b & + & x_4 & c & + & d & = & y_4
\end{array}$$

The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} (x_1)^3 & (x_1)^2 & x_1 & 1 & y_1 \\ (x_2)^3 & (x_2)^2 & x_2 & 1 & y_2 \\ (x_3)^3 & (x_3)^2 & x_3 & 1 & y_3 \\ (x_4)^3 & (x_4)^2 & x_4 & 1 & y_4 \end{array}\right]$$

(b) **Case 1**: No cubic curve passing through all given points exists if at least two of the points have the same $x$-coordinate and different corresponding $y$-coordinates. For example, there is no cubic curve passing through the points $(2, 6)$, $(2, 1)$, $(0, 1)$, $(-9, 4)$.

**Case 2**: There is exactly one cubic curve passing through the given points if all the $x$-coordinates are different. For instance, there is exactly one cubic curve passing through the points $(-3, 2)$, $(2, 1)$, $(0, 1)$, $(-9, 4)$.

**Case 3:** There are infinitely many cubic curves passing through all given points if at least two of the points coincide and case 1 does not happen. For example, there are infinitely many cubic curves passing through the points $(2, 1)$, $(2, 1)$, $(0, 1)$, $(-9, 4)$ (which are actually 3 points). In general there are infinitely many cubic curves passing through one/two/three given points as long as case 1 does not happen.

**10.** Given a monochromatic digital image $A$ of size $512 \times 512$, write one line of MATLAB code that captures the upper left subimage of $A$ of size $256 \times 256$. Call that submatrix $S$.

**Solution**. $S = A(1 : 256, 1 : 256);$

**11.** Find real numbers $x$, $y$, and $z$ such that

$$\begin{bmatrix} x & y & -1 \\ -x & 4 & z \\ 2x & y/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -y \\ z \end{bmatrix}. \tag{1}$$

**Solution**. From the definition of matrix multiplication, we obtain the following equalities

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & - & 4 & = & 8 \\ -x & + & 8 & + & 4z & = & -y \\ 2x & + & y & + & 4 & = & z \end{array}\right.$$

which, by a simple rearrangement, yield the system

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 0z & = & 12 \\ -x & + & y & + & 4z & = & -8 \\ 2x & + & y & - & z & = & -4 \end{array}\right.$$

Now, we simply do

```
>> rref([1 2 0 12; -1 1 4 -8; 2 1 -1 -4])
```

```
ans =

     1     0     0   -12
     0     1     0    12
     0     0     1    -8
```

to find that $x = -12$, $y = 12$ and $z = -8$. At this point is always a good idea to take these values of $x$, $y$, and $z$, and check that they indeed satisfy the original matrix equality (1). Do it!

**12.** If we type

```
>> diag([3 4 -2])*diag([-2 1 8])
```

in Matlab and hit Enter, what is the answer?

**Solution**.

```
ans =

    -6     0     0
     0     4     0
     0     0   -16
```

**13.** A system $(S)$ has the following augmented matrix

```
M =

     2     1     3     1     2    -1
     3     4     5     1     1     2
     4     5     6     7     0     4
```

What is the size of $(S)$? What is the matrix of coefficients of $(S)$? Is $(S)$ consistent or inconsistent? If consistent, find its solution(s). If there are free variables, express the solution in terms of the free variables.

**Solution**. $(S)$ is a $3 \times 5$ system (not a $3 \times 6$ system), its matrix of coefficients is

```
A =

     2     1     3     1     2
     3     4     5     1     1
     4     5     6     7     0
```

In order to determine consistency, we do

```
>> rref(M)

ans =

    1.0000         0         0   13.6667   -2.3333    3.0000
         0    1.0000         0    1.6667   -1.3333    2.0000
         0         0    1.0000   -9.3333    2.6667   -3.0000
```

and interpret. The solutions are given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 - 13.6667x_4 + 2.3333x_5 \\ 2 - 1.6667x_4 + 1.3333x_5 \\ -3 + 9.3333x_4 - 2.6667x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13.6667 \\ -1.6667 \\ 9.3333 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2.3333 \\ 1.3333 \\ -2.6667 \\ 0 \\ 1 \end{bmatrix},$$

where $x_4$ and $x_5$ are the free variables.

**14.** Geometric interpretation of linear systems. In Figure 2 we see the graphs of three planes in three-dimensional space whose equations form a linear system.

Which one of the following systems is compatible with Figure 2?

$$(S_1) \begin{cases} -3x + y + \frac{7}{10}z = 1 \\ 9x - 3y - \frac{21}{10}z = -3 \\ x - y + \frac{1}{4}z = 0 \end{cases}$$

$$(S_2) \begin{cases} -3x + y + \frac{7}{10}z = 1 \\ -5x - 8y + z = -3 \\ 10x + 16y - 2z = 6 \end{cases}$$

$$(S_3) \begin{cases} -30x + 10y + 7z = 10 \\ 5x + y - z = 0 \\ x - y + \frac{1}{4}z = 6 \end{cases}$$

**Solution**. The answer is $(S_3)$. By using our rref skills, we see that systems $(S_1)$ and $(S_2)$ each yields a free variable. Hence, they have infinitely many solutions. The geometrical interpretation of this phenomenon is that the corresponding planes intersect in a line, which is against the configuration in Figure 2.

On the other hand, by solving the system $(S_3)$ we see that there is exactly one solution. That is, the corresponding planes in $(S_3)$ intersect in a single point (whose coordinates are given by our solution). This is exactly the situation depicted in Figure 2.

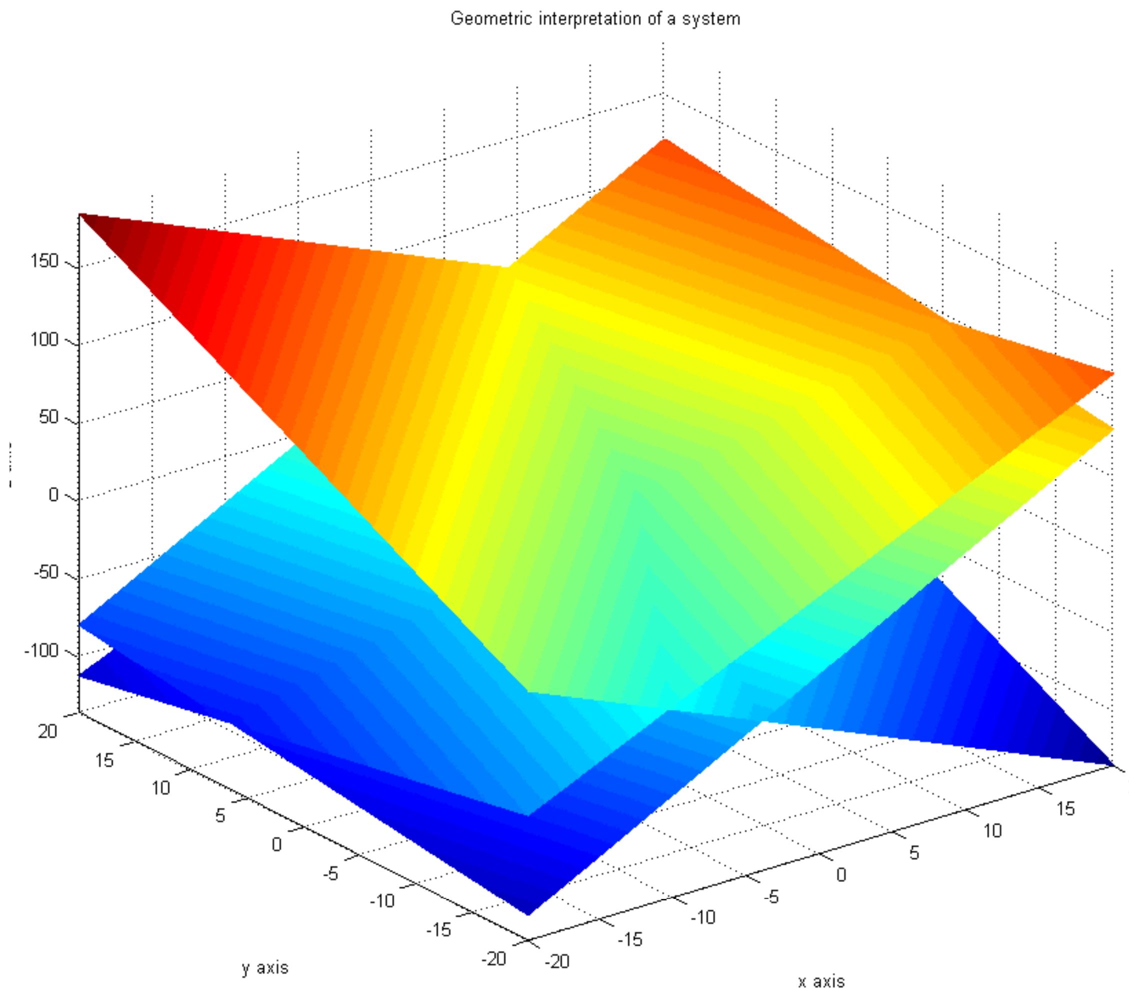


Figure 2: Three planes in $\mathbf{R}^3$