Kansas State University | Department of Mathematics

Applied Matrix Theory - Math 551

Notes on the Singular Value Decomposition

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Any $m \times n$ matrix A be factored or decomposed as

$$A = USV^T$$

where the matrices U, S, and V satisfy the following properties:

(i) U is an $m \times m$ orthogonal matrix,

(ii) V is an $n \times n$ orthogonal matrix, and

(iii) S is an $m \times n$ matrix whose off-diagonal entries are all 0's and whose diagonal entries are nonnegative numbers arranged in decreasing order.

The factorization of A given above is called a **singular value decomposition** (SVD) of A and the numbers along the diagonal of S are called the **singular values** of A.

Example 1. The matrix $A = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 2 & -2 & 1 \end{bmatrix}$ has singular value decomposition

$$\begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

Then the singular values of A are 30, 15, and 3.

Example 2. The matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ has singular value decomposition

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Then the singular values of A are 2 and 0.

Example 3. The matrix $A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix}$ has singular value decomposition

$$\begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}.$$

Then the singular values of A are 12, 6 and 0.

Singular values and rank.

The number of nonzero singular values of a matrix A (counting repetition) coincides with the rank of A. In this way, the rank of the matrix A in example 1 is 3, the rank of the matrix A in example 2 is 1, and the rank of the matrix A in example 3 is 2.

Singular Value Decomposition and approximation.

Check out the following computations.

From example 1:

$$30 \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \end{bmatrix} + 15 \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & -2/3 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$=30 \begin{bmatrix} 1/5 & 2/5 & 2/5 \\ 4/15 & 8/15 & 8/15 \\ 0 & 0 & 0 \end{bmatrix} + 15 \begin{bmatrix} -8/15 & -4/15 & 8/15 \\ 2/5 & 1/5 & -2/5 \\ 0 & 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -8 & -4 & 8 \\ 6 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 2 & -2 & 1 \end{bmatrix} = A$$

From example 2:

$$2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = A$$

From example 3:

$$12\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\begin{bmatrix}2/3 & 2/3 & 1/3\end{bmatrix}+6\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}\begin{bmatrix}-2/3 & 1/3 & 2/3\end{bmatrix}$$
$$=\begin{bmatrix}4&4&2\\4&4&2\\4&4&2\\4&4&2\end{bmatrix}+\begin{bmatrix}-2&1&2\\2&-1&-2\\2&-1&-2\\-2&1&2\end{bmatrix}=\begin{bmatrix}2&5&4\\6&3&0\\6&3&0\\2&5&4\end{bmatrix}=A$$

In general, if A is an $m \times n$ matrix that has exactly r nonzero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, and denoting by $\vec{u}_1, \cdots, \vec{u}_m$ the columns of the matrix U, and by $\vec{v}_1, \cdots, \vec{v}_n$ the columns of the matrix V, then:

$$\begin{aligned}
\underset{m\times n}{A} &= \underset{m\times m}{U}\ \underset{m\times n}{S}\ \underset{n\times n}{V}^{T}\\
&= \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m\\ \vdots & \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 & & & \cdots & & 0\\ 0 & \sigma_2 & 0 & & \cdots & & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots\\ & & & \sigma_r & 0 & \cdots & 0\\ 0 & 0 & 0 & & & & \end{pmatrix}\begin{pmatrix} \cdots & \vec{v}_1^T & \cdots\\ \cdots & \vec{v}_2^T & \cdots\\ \cdots & \vdots & \cdots\\ & \vec{v}_n^T & \end{pmatrix}\\
&= \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & \cdots & \sigma_r\vec{u}_r & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}\begin{pmatrix} \cdots & \vec{v}_1^T & \cdots\\ \cdots & \vec{v}_2^T & \cdots\\ \cdots & \vdots & \cdots\\ & \vec{v}_n^T & \end{pmatrix}\\
&= \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \cdots + \sigma_r\vec{u}_r\vec{v}_r^T
\end{aligned}$$

Since A can be written as a sum, it is sometimes possible to approximate A by throwing away the smaller singular values (remember that the singular values are arranged in decreasing order):

$$A = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \cdots + \sigma_r\vec{u}_r\vec{v}_r^T \approx \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T$$

where $k < r$.

The total storage for the matrix A is in general mn units of memory (bytes for example) since A has mn entries. However the total storage for the matrix

$$A_k = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T$$

can be arranged to be $k(m+n+1)$. Why?

The SVD and Matlab

The basic Matlab command to produce the singular value decomposition of any matrix A is the command *svd*. Here are some examples

Example 4. Consider the matrix

```
>> A=[1 2 3 4 5; 6 7 8 9 0; 7 6 5 4 3; 0 -1 -1 -1 -1]

A =

     1     2     3     4     5
     6     7     8     9     0
     7     6     5     4     3
     0    -1    -1    -1    -1
```

Typing

```
>> [U S V]=svd(A)
```

will produce the matrices

```
U =

   -0.3004   -0.8604   -0.3505   -0.2159
   -0.7592    0.4628   -0.4550   -0.0491
   -0.5709   -0.1382    0.8086    0.0323
    0.0859    0.1627    0.1274   -0.9746


S =

   19.6135         0         0         0         0
         0    5.0056         0         0         0
         0         0    3.7377         0         0
         0         0         0    0.5340         0


V =

   -0.4513    0.1896    0.6903   -0.5327    0.0000
   -0.4806    0.1053    0.2244    0.7355    0.4082
   -0.5055    0.0534   -0.2075    0.1785   -0.8165
   -0.5305    0.0016   -0.6393   -0.3785    0.4082
   -0.1683   -0.9747    0.1461   -0.0150   -0.0000
```

such that

```
>> U*S*V'

ans =

    1.0000    2.0000    3.0000    4.0000    5.0000
    6.0000    7.0000    8.0000    9.0000    0.0000
    7.0000    6.0000    5.0000    4.0000    3.0000
   -0.0000   -1.0000   -1.0000   -1.0000   -1.0000
```

and

```
>> U*U'

ans =

    1.0000   -0.0000    0.0000   -0.0000
   -0.0000    1.0000    0.0000    0.0000
    0.0000    0.0000    1.0000   -0.0000
   -0.0000    0.0000   -0.0000    1.0000

>> V*V'

ans =

    1.0000   -0.0000    0.0000    0.0000   -0.0000
   -0.0000    1.0000    0.0000    0.0000    0.0000
    0.0000    0.0000    1.0000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    1.0000    0.0000
   -0.0000    0.0000    0.0000    0.0000    1.0000
```

That is, U and V are orthogonal matrices.

Example 5. In this example let's keep working on the matrix A from Example 4 and use the Matlab command *svds* to produce truncated singular value decompositions. For instance, if we only want to keep the first 3 singular values of A, we do

```
>> [U3 S3 V3]=svds(A,3)

U3 =

   -0.3004    0.8604    0.3505
```

```
   -0.7592   -0.4628    0.4550
   -0.5709    0.1382   -0.8086
    0.0859   -0.1627   -0.1274


S3 =

   19.6135         0         0
         0    5.0056         0
         0         0    3.7377


V3 =

   -0.4513   -0.1896   -0.6903
   -0.4806   -0.1053   -0.2244
   -0.5055   -0.0534    0.2075
   -0.5305   -0.0016    0.6393
   -0.1683    0.9747   -0.146
```

Notice that in this case the product $U3*S3*V3^T$ will only approximate (and not equal) A. Indeed, the difference equals

```
>> U3*S3*V3'-A

ans =

   -0.0614    0.0848    0.0206   -0.0436   -0.0017
   -0.0140    0.0193    0.0047   -0.0099   -0.0004
    0.0092   -0.0127   -0.0031    0.0065    0.0003
   -0.2773    0.3828    0.0929   -0.1970   -0.0078
```

which is not the zero matrix.

Example 6. If we are only interested in the first two singular values of A from Example 4, we do

```
>> [U2 S2 V2]=svds(A,2)

U2 =

   -0.3004   -0.8604
   -0.7592    0.4628
   -0.5709   -0.1382
```

```
    0.0859    0.1627


S2 =

   19.6135         0
         0    5.0056


V2 =

   -0.4513    0.1896
   -0.4806    0.1053
   -0.5055    0.0534
   -0.5305    0.0016
   -0.1683   -0.9747
```

In this case the difference equals

```
>> U2*S2*V2'-A

ans =

    0.8430    0.3788   -0.2512   -0.8812    0.1896
    1.1599    0.4009   -0.3481   -1.0971    0.2480
   -2.0773   -0.6909    0.6239    1.9388   -0.4412
   -0.6061    0.2759    0.1917    0.1075   -0.0774
```

Notice that the entries in this difference are larger than the difference computed in Example 5 using 3 singular values.

Decomposing with respect to singular values vs. decomposing with respect to eigenvalues

In what follows we discuss $A = USV^T$ **versus** $A = PDP^{-1}$.

Remember that an $n \times n$ matrix A is called diagonalizable if there exist $n \times n$ matrices P invertible and D diagonal such that

$$A = PDP^{-1}. \tag{1}$$

In such case, the diagonal entries in D are the eigenvalues of A and the columns of P are eigenvectors of A. The diagonal factorization (1) has certain limitations in terms of its implementation. For instance, it restricts the size of A to be a square matrix and; even worse, there exist square matrices that do not admit a decomposition of the form (1) (i.e., there are square matrices which are not diagonalizable).

The powerful singular value decomposition

$$A = USV^T \tag{2}$$

has a number of advantages over the diagonal decomposition (1). Namely, the singular value decomposition can **always** be performed on any matrix, square or not. The price to pay is that, as opposed to the diagonal decomposition (1) where we have P and P^{-1}, in the singular value decomposition we have the matrices U and V^T where one is not necessarily the inverse of the other. However, since U and V are orthogonal matrices, that is,

$$U^T = U^{-1} \quad \text{and} \quad V^T = V^{-1},$$

the singular value decomposition (2) provides a flexible, useful, always implementable alternative to the (also useful but not always implementable) diagonal decomposition (1).

The singular values of A and the eigenvalues of A^TA and AA^T

Let's consider the following example:

```
A =

     1    -2     1
     0     1     1
     2     2     2
     1     1     1
```

A is a 4×3 matrix and its singular value decomposition is given by

```
>> [U S V]=svd(A)

U =

    0.0287    0.9851    0.1696         0
   -0.2983   -0.1535    0.9420         0
   -0.8533    0.0694   -0.2589   -0.4472
   -0.4267    0.0347   -0.1295    0.8944

S =

    4.0495         0         0
         0    2.4811         0
         0         0    0.6677
         0         0         0
```

```
V =

   -0.5197    0.4670   -0.7154
   -0.6146   -0.7860   -0.0666
   -0.5934    0.4051    0.6955
```

That is, the matrix S stores the three singular values of A. Let's now consider the 3×3 symmetric matrix $A^T A$. As with any symmetric matrix, all its eigenvalues will be real numbers and there will be a basis of $\mathbf{R}^3$ made of its eigenvectors. Moreover, the eigenvalues of $A^T A$ will be non-negative. This is due to the fact that if λ is an eigenvalue of $A^T A$, then we will have

$$A^T A v = \lambda v,$$

for some non-zero vector v. Now, computing dot products we write

$$\lambda \|v\|^2 = \lambda(v \cdot v) = (\lambda v \cdot v) = (A^T A v \cdot v) = (Av \cdot Av) = \|Av\|^2,$$

where we have used that we can always move any matrix M from one factor of the dot product to the other by transposition, that is, we always have

$$(Mu \cdot w) = (u \cdot M^T w) \quad \forall u, w.$$

Thus, the computation above yields $\lambda\|v\|^2 = \|Av\|^2$ and, since norms are always non-negative and $\|v\| > 0$ (because v is a non-zero vector), we deduce that $\lambda \geq 0$ for every eigenvalue λ of $A^T A$.

Let's look at the eigenvalues of $A^T A$.

```
>> [P D]=eig(A'*A)

P =

    0.7154    0.4670    0.5197
    0.0666   -0.7860    0.6146
   -0.6955    0.4051    0.5934


D =

    0.4458         0         0
         0    6.1558         0
         0         0   16.3984
```

As expected, all the eigenvalues of $A^T A$, along the diagonal of D, are non-negative numbers. Then we can take square roots of them by doing

```
>> sqrt(D)

ans =

    0.6677         0         0
         0    2.4811         0
         0         0    4.0495
```

And we find that if A is an $m \times n$ matrix with $m \geq n$, "*the square roots of the eigenvalues of* $A^T A$ *are precisely the singular values of* A".

Let's do another example. Now for an $m \times n$ matrix C with $m < n$ (as opposed to the previous example where we had $n > m$).

```
C =

     3     1     1     3     1
     2     0     3     3     4
     2     2     1     2     1
```

Let's look at the singular values of C.

```
>> [U S V]=svd(C)
U =
   -0.5341    0.5365   -0.6534
   -0.7328   -0.6792    0.0413
   -0.4216    0.5009    0.7559

S =
    7.9885         0         0         0         0
         0    2.8435         0         0         0
         0         0    1.0483         0         0

V =
   -0.4896    0.4406   -0.3489   -0.6663    0.0218
   -0.1724    0.5410    0.8189    0.0576    0.0611
   -0.3948   -0.3518    0.2160   -0.0823   -0.8167
   -0.5813    0.2017   -0.3095    0.7239    0.0394
   -0.4866   -0.5906    0.2554   -0.1481    0.5721
```

The 3 singular values of C are stored in S. Let's now look at the eigenvalues of the 3×3 symmetric matrix CC^T. Notice that we're doing CC^T (as opposed to $C^T C$) to obtain a 3×3 matrix.

```
>> [P D]=eig(C*C')

P =
    0.6534    0.5365    0.5341
   -0.0413   -0.6792    0.7328
   -0.7559    0.5009    0.4216

D =
    1.0989         0         0
         0    8.0857         0
         0         0   63.8155
```

A similar reasoning as above shows that all the eigenvalues of CC^T are non-negative, as illustrated by the diagonal entries of D above. Taking square roots of D yields

```
>> sqrt(D)

ans =
    1.0483         0         0
         0    2.8435         0
         0         0    7.9885
```

And we find that if C is an $m \times n$ matrix with $m \leq n$, "*the square roots of the eigenvalues of CC^T are precisely the singular values of C*".