Math 322

Relations

if R is a subset of $A \times B$ it is a **binary relation** or just **relation**

Notation: $(a,b) \in R$ $\quad aRb$

$(a,b) \notin R$ $\quad a\not{R}b$

Note: if R is a subset of $A \times B$...

How many possible relations are there?

$\to$ How many subsets of $A \times B$?

Remember $P(S)$ = set of **all** subsets

$\to$ Find $|P(S)| = 2 \cdot 2 \cdot \cdots \cdot 2 = 2^{|S|}$

$S = \{s_1, s_2, \ldots, s_n\}$

1 0

Ans: $2^{|A \times B|} = 2^{|A| \cdot |B|}$

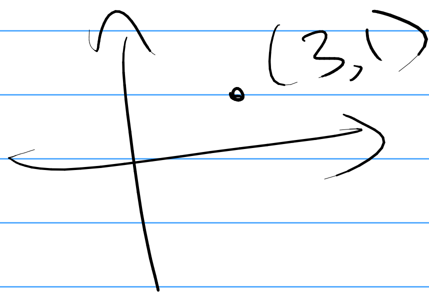
ex: how many relations from A to B?

$|A| = 20 \quad |B| = 10$

ex $R_1 = \{(c,9), (c,7), (d,0)\}$

total: $2^{20 \cdot 10} = 2^{200}$

If R is a subset of $\boxed{A \times A}$.



$R \times R$

Say R is a relation on A.

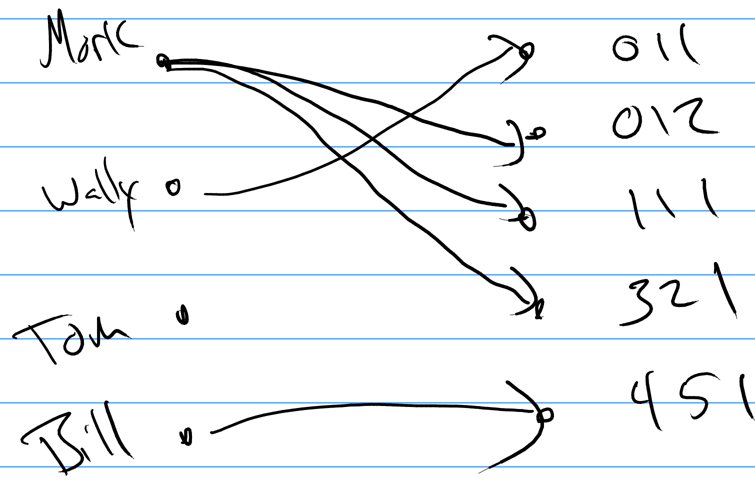
Visualization

① List the pairs (tuples)

$R = \{(\text{Mark}, \text{Math}321), (\text{Wally}, \text{Math}011)\}$

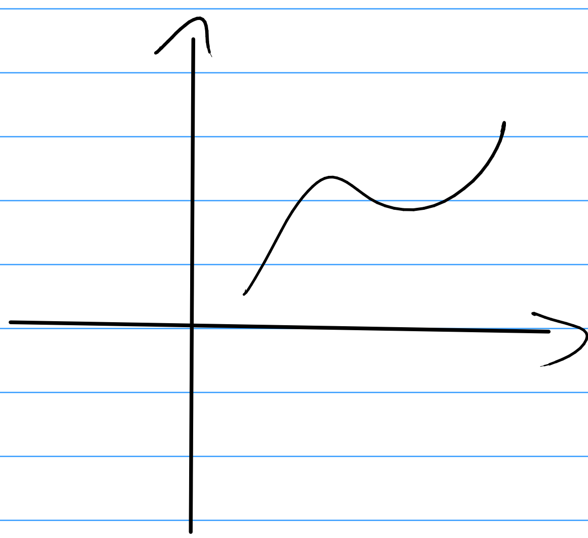
② Set builder notation $\{(a,b) \mid P(a,b)\}$

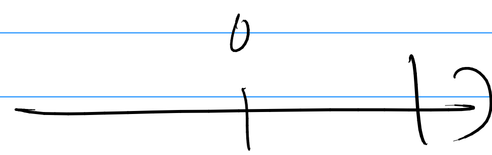
3 Arrow diagram



4 Table

	011	012	321	322
Mark			X	X
Wally	X	X		
Tom				





Functions:

$F : A \to B$

domain (A) — codomain (B)

obviously $F \in$ Relations.

Types

R a relation on set A $(|A| = n)$

there are $2^{|A|^2} = 2^{n^2}$

notation: $\boxed{(a,b) \in R \;/\; aRb}$

① if $\forall a \, (aRa)$

R is reflexive.

ex: A is set of all web pages.

$R_1 = \{ (a,b) \mid$ everyone who has visited a has visited $b \}$

? Reflexive: $\boxed{\forall a \, (aR_1a)} \equiv T$?

ex for all webpages a, everyone who has visited a has visited a. $\boxed{\text{True}}$

$R_2 = \{ (a,b) \mid a$ and b have a common link $\}$

$\forall a \, (a R_2 a)$

for all webpages a, a and a have a common link.

<html>
</html> ← counter example

so R_2 is <u>not</u> reflexive.

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2 $\forall a \, (a \not R a)$ irreflexive.

R_1 <u>not irreflexive.</u>

$A = \{a, b\}$

$R_3 = \{(a, a)\}$

(b, b) so <u>not</u> reflexive.

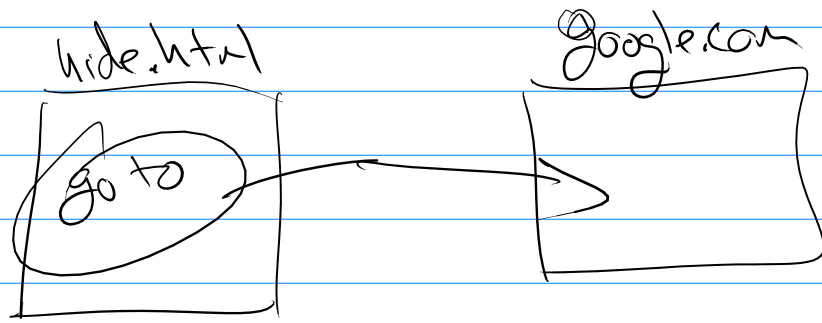
b/c (a, a) then <u>not</u> irreflexive.

③ Symmetric

$$\forall a \forall b \, (aRb \rightarrow bRa)$$

$R_1 = \{ (a,b) \mid$ everyone who has visited a has visited b $\}$

ex



(hide, google)

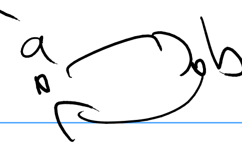
Counterexample. | Not. Symmetric

Note: $(F \rightarrow p) \equiv T$

ex $R = \{ \}$ on $A = \{a, b\}$

not ref.
is irref.
is sym.

Sym $\forall a \forall b (\boxed{aRb} \rightarrow bRa)$

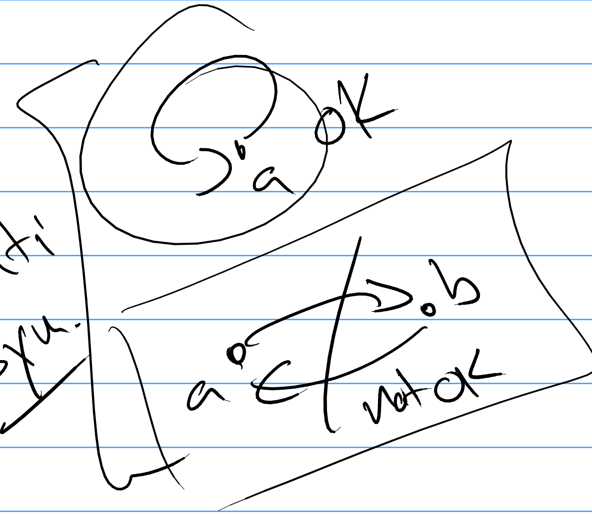
Sym 

④ Antisymmetric

$$\forall a \forall b (aRb \wedge bRa \rightarrow a = b)$$

$$\equiv \forall a \forall b (a \neq b \rightarrow [a \not R b \vee b \not R a])$$

$$\equiv \forall a \forall b (a \neq b \rightarrow \neg(aRb \wedge bRa))$$ Anti ~~sym.~~



$$(p \rightarrow q) \equiv (\neg p \vee q)$$

$$\equiv \neg(p \wedge \neg q)$$

$$\equiv \forall a \forall b \neg(aRb \wedge bRa \wedge a \neq b)$$

$$\equiv \neg \exists a \exists b (aRb \wedge bRa \wedge a \neq b)$$

⑤ $\boxed{\text{asymmetric}}$

$$\forall a \forall b (aRb \rightarrow b \not R a)$$

irreflexive and antisym.

$R = \{\}$

1. Not reflexive
2. IS irreflexive
3. IS sym.
4. IS antisym
5. IS asym

Vacuous.

6. Transitive

$$\forall a \forall b \forall c (aRb \land bRc \to aRc)$$

ex $R = \{\}$ is transitive!

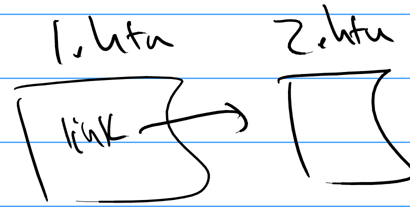
ex $R = \{(a,b) \mid \text{there is a page that has links to } a \text{ and } b\}$

Reflexive? $\forall a (aRa)$

No. Counterexample: a page no one links to.

Irreflexive? $\forall a (a \not R a)$

No. Counterexample



$(2,2)$

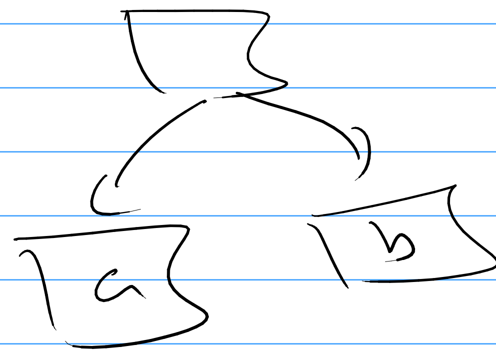
Sym? $\forall a \forall b (aRb \to bRa)$

$f(a,b)$ $\{a\}$ $\{b\}$ $\boxed{\text{Yes}}$

antisym?: $\forall a \forall b \, (aRb \wedge bRa \to a=b)$

$\equiv \forall a \forall b \, (a \neq b \to \neg(aRb \wedge bRa))$

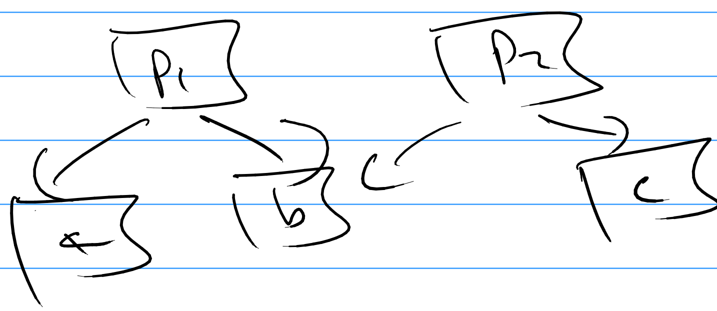
No. Counter example.



asym?: No (not irr, not antisym)

trans?: $\forall a \forall b \forall c \, (aRb \wedge bRc \to aRc)$

No



Counter example.

Note: R is a set.

all set ops. apply!

$\cup$, $\cap$, $-$, $\oplus$

plus composition!