Math 322

8.3 R is a relation on set A,
R is a subset of $A \times A$

Representing Relations

① Matrix (zero-one matrix) Review 3.8

given R, a relation on A.

$$M_R = [m_{ij}]$$

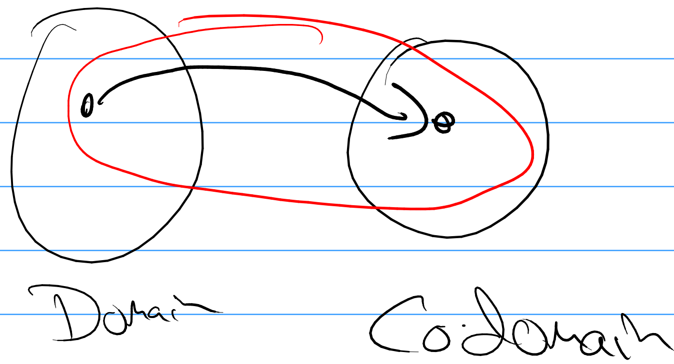
$$m_{ij} = \begin{cases} 1 & a_i \, R \, b_j \\ 0 & a_i \not R \, b_j \end{cases}$$

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R = \{(a,a), (a,d), (b,b), (b,c),$
$(c,a), (c,b), (d,a)\}$

② Digraph (directed graph)

Idea:

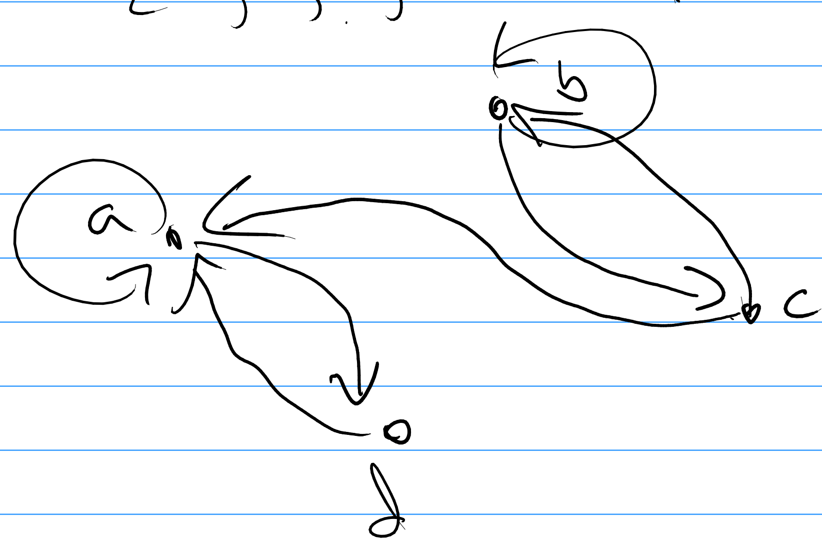


is a set, V, of vertices (V is A) and a set, E, of edges which represent the ordered pairs. (E is R)

ex

$R = \{(a,a), (a,d), (b,b), (b,c), (c,a), (c,b), (d,a)\} = E$

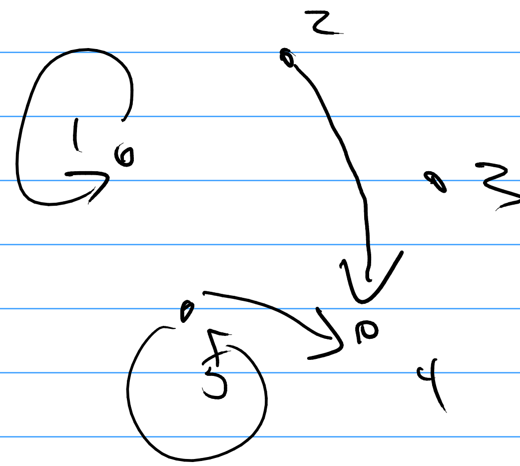
$A = \{a,b,c,d\} = V$



$A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (2,4), (5,4), (5,5)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



Types

Reflexive: $\forall a\,(aRa)$

Matrix: $M_R \leftarrow$ if Matrix has main diagonal of 1's

Digraph: every point has a loop.

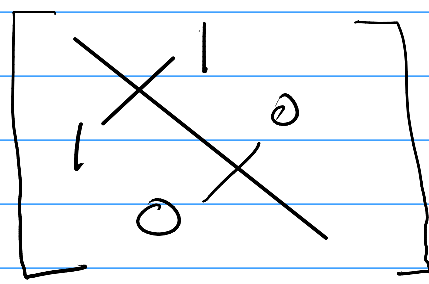
Irreflexive: $\forall a\,(a \not R a)$

Matrix: M_R all zeros on main diag.

Digraph: no loops

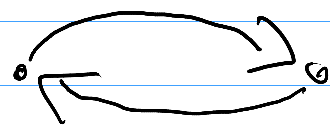
Symmetric $\forall a \forall b\,(aRb \to bRa)$

Matrix: M_R



Mirror on main diagonal

Digraph:

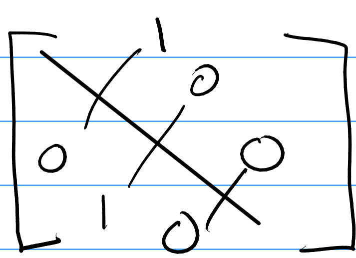


edges are paired

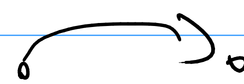
Anti Symmetric $\forall a \forall b\,(aRb \land bRa \to a = b)$

$\equiv \forall a \forall b\,(a \neq b \to \neg(aRb \land bRa))$

M_R Matrix:



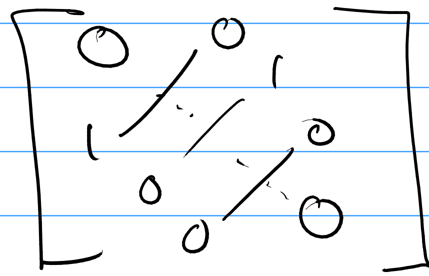
digraph:



no paired edges

asymmetric $\forall a \forall b\,(aRb \rightarrow b \not R a)$

Matrix M_R



digraph: no loops and no paired edges

Transitive $\forall a \forall b \forall c\,(aRb \wedge bRc \rightarrow aRc)$

Matrix — not to helpful (yet)

digraph —

ex

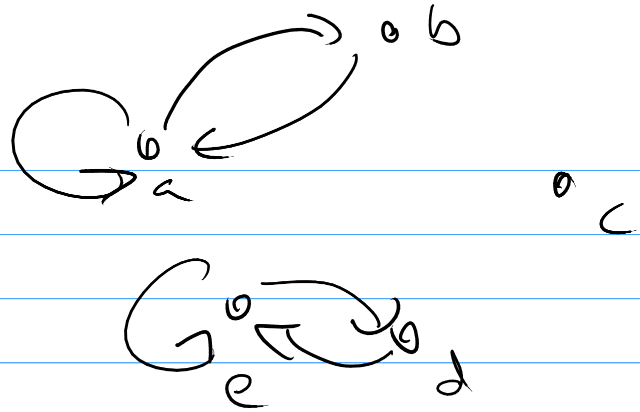
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & \boxed{0} & 1 & 1 \\ c & 1 & 1 & 0 \end{array}$$

Properties

Not	IS
ref.	
irr.	
sym	
antisym	
asym	
trans.	

$(b,c)\ (c,a) \rightarrow (b,a)$

ex



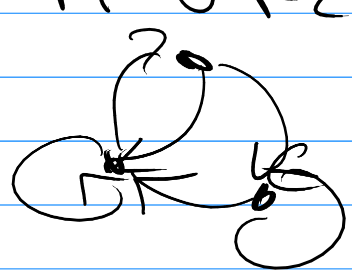
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

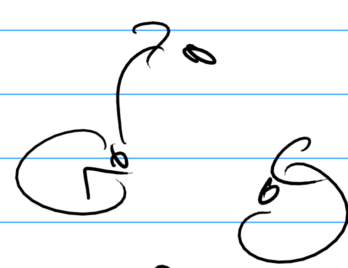
Properties.

Is	Not
Sym	ref. $b \not R b$
	irr. aRa
	antisym $aRb \wedge bRa$
	asym aRa
	Tran $bRa \wedge aRb \rightarrow b \not R b$

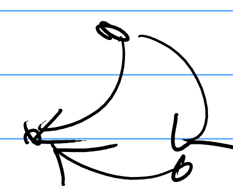
Operations Note: Relations are sets

$R_1 \cup R_2$





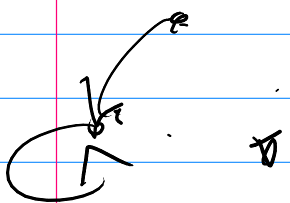
R_1

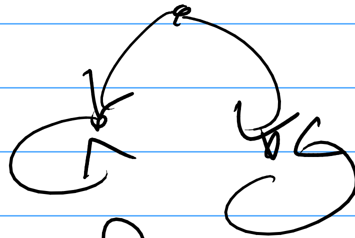


R_2

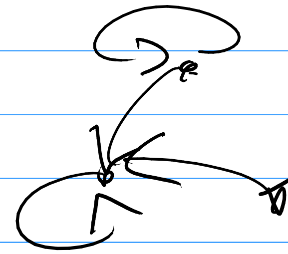
$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \quad \text{(bitwise or)}$$

$R_1 \cap R_2$





R_1



R_2

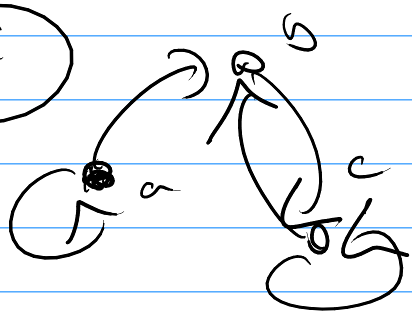
$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} \quad \text{(bitwise and)}$$

S and R are on A.

$$M_{S \circ R} = M_R \odot M_S$$

$$M_{R^n} = M_R^{[n]}$$

ex



$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Closure

if Relation R on set A does not have property P. ← ref? irr? sym? etc.

only: reflexive, symmetric, and transitive can be created by adding edges.

Def: R is a relation on set A. P is a relational property. If there is a relation S with property P such that

① $R \subseteq S$ (adding edges to R)

② S is a subset of all relations with property P that also contain R. (S is the "smallest")

① Reflexive: $\Delta = \{(a,a) \mid aRa\}$

reflexive closer $R \cup \Delta$.

ex $M_R = \begin{array}{c|ccc} & a & b & c \\ a & 1 & 0 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 1 & 1 \end{array}$

~~closure is (b,b)~~

ref. closure of R

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\boxed{\text{Symmetric}}$

$R^{-1} = \{(b,a) \mid aRb\}$

$M_{R^{-1}} = M_R^T$

Sym closure $M_{R \cup R^{-1}} = M_R \vee M_{R^{-1}}$

ex

from above

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & \cancel{0} & 1 \\ 1 & 1 & 1 \end{bmatrix}$$