Math 243

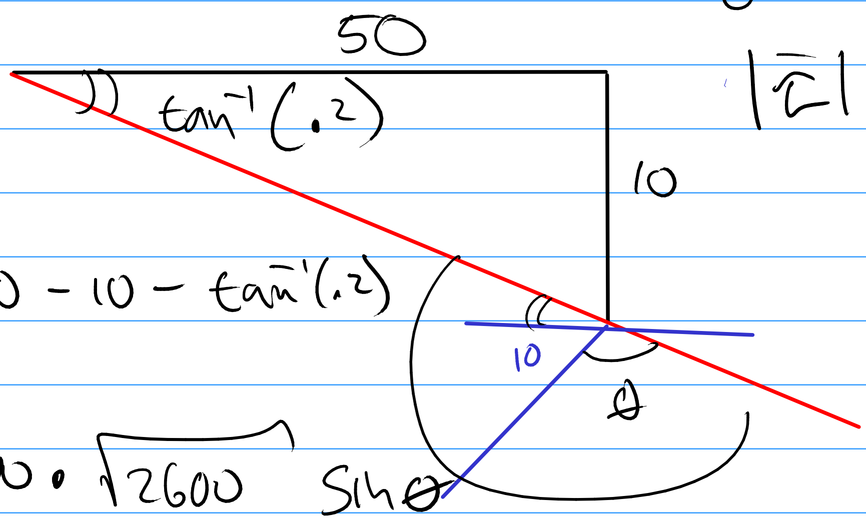
Midterm

① $\vec{a} + 4\vec{b} = \langle 2, 7, -1 \rangle$

$|\vec{a} - \vec{b}| = \sqrt{29}$

② $\vec{a} \cdot \vec{b} \neq 0$ not orthogonal

③



$|\vec{\tau}| = |\vec{F}||\vec{r}|\sin\theta$

$\theta = 180 - 10 - \tan^{-1}(.2)$

$|\vec{\tau}| = 200 \cdot \sqrt{2600}\ \sin\theta$

$= 200\sqrt{2600}\ \sin(170 - \tan^{-1}(.2))$

④ $(2, 4, -3)$ $(3, -1, 1)$ pts.

$\vec{v} = \langle -1, 5, -4 \rangle$

$\langle x, y, z \rangle = \langle 3, -1, 1 \rangle + t\langle -1, 5, -4 \rangle$

⑤ $(1, 3, 2)$ $(3, -1, 6)$ $(5, 2, 0)$

$\vec{v}_1 = \langle 2, -4, 4 \rangle$

$\vec{v}_2 = \langle 4, -1, -2 \rangle$ $\vec{n} = \vec{v}_1 \times \vec{v}_2$

do the work | $\vec{n} = \langle a, b, c \rangle$ + a point $(5, 2, 0)$

$$a(x-5) + b(y-2) + c(z-0) = 0$$

$$ax + by + cz = d$$

$$\int_1^2 x^2 \ln x \, dx = \frac{1}{3}x^2 \ln x \Big|_1^2 - \int_1^2 \frac{1}{3}x^2 \, dx$$

$u = \ln x \rightarrow du = \frac{1}{x}\,dx$

$dv = x^2\,dx \rightarrow v = \frac{1}{3}x^3$

= Finish

7) $$\int \frac{x}{\sqrt{3-x^2}}\,dx$$ let $x = \sqrt{3}\sin\theta$

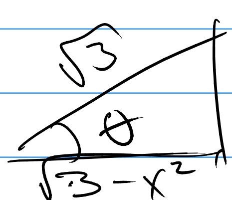
$dx = \sqrt{3}\cos\theta\,d\theta$

$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3 - 3\sin^2\theta}} \sqrt{3}\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{3}\sin\theta}{\cancel{\sqrt{3}\cos\theta}} \cancel{\sqrt{3}\cos\theta}\,d\theta$$

$$= \sqrt{3}\int \sin\theta\,d\theta = -\sqrt{3}\cos\theta + C$$

$\sin\theta = \frac{x}{\sqrt{3}}$



$$= -\cancel{\sqrt{3}}\,\frac{\sqrt{3-x^2}}{\cancel{\sqrt{3}}} + C$$

$= -\sqrt{3-x^2} + C$

or

$$\int \frac{x}{\sqrt{3-x^2}}\,dx = -\frac{1}{2}\int u^{-1/2}\,du$$

let $u = 3-x^2$ $\qquad = -\sqrt{u} + C$

$du = -2x\,dx$

$$= \boxed{-\sqrt{3-x^2} + C}$$

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8 $$\int \frac{x^3+2x^2-x-2+1}{x^3+2x^2-x-2}\,dx$$

$$\int 1 + \frac{1}{\underbrace{x^3+2x^2-x-2}}\,dx$$

$x^2(x+2) - 1(x+2)$

$\underset{A}{(x+2)}\underset{B}{(x+1)}\underset{C}{(x-1)}$

$1 = A(x+1)(x-1) + B(x+2)(x-1) + C(x+2)(x+1)$

$C = 1/6 \qquad B = -1/2 \qquad A = 1/3$

$= x + \frac{1}{3}\ln|x+2| - \frac{1}{2}\ln|x+1| + \frac{1}{6}\ln|x-1| + C$

$$\int \frac{1}{\sqrt{e^{2x}+3}}\,dx$$ let $u = e^x$

$du = e^x\,dx$

$$\int \frac{1}{u\sqrt{u^2+3}}\,du$$ use #51

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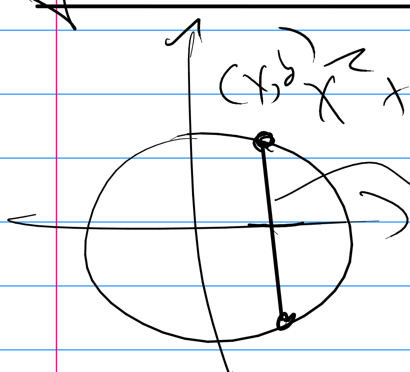
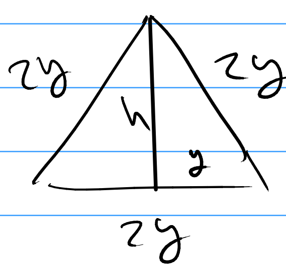
(11) $$\int_1^\infty \frac{1}{(x-1)^2}\,dx = \int_1^2 \frac{1}{(x-1)^2}\,dx + \int_2^\infty \frac{1}{(x-1)^2}\,dx$$

$$\lim_{t\to 1^+} \int_t^2 \frac{1}{(x-1)^2}\,dx + \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\,dx$$

Show div. conv.



line $= 2y = 2\boxed{\sqrt{1-x^2}} = y$



$h = \sqrt{4y^2 - y^2} = \sqrt{3}\,y$

triangle: $\frac{1}{2} b\cdot h = \frac{1}{2}(2y)(\sqrt{3}\,y) = \sqrt{3}\,y^2$

$$\int_{-1}^{1} \sqrt{3}(1-x^2)\,dx = \text{finish.}$$

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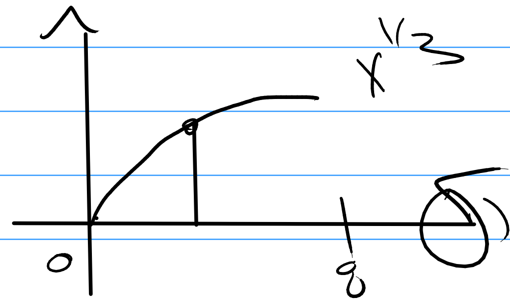

$$\int_0^8 \pi r^2 \, dx$$

$$\pi \int_0^8 x^{2/3} \, dx$$

$= \underline{\underline{\text{finish.}}}$

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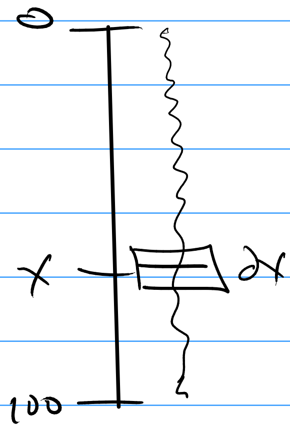

100ft @ 200 lb

2 lb/ft

$$\int_0^{100} 2x \, dx$$

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$$\frac{dy}{dx} = \frac{y \cos x}{1+y^2} \qquad \boxed{y(0)=1}$$

$$\int \boxed{\frac{1+y^2}{y}} \, dy = \int \cos x \, dx$$

$$\int \left(\frac{1}{y} + y\right) dy = \int \cos x \, dx$$

$$\ln|y| + \tfrac{1}{2}y^2 = \sin x + C$$

$$\ln|y| + \tfrac{1}{2}y^2 = \sin x + \tfrac{1}{2}$$

$$\boxed{\text{Convergence of } \sum_{n=1}^{\infty} a_n}$$

Why?

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

$$S = a + ar + ar^2 + \cdots$$

$$rS = ar + ar^2 + ar^3 + \cdots$$

$$S - rS = a$$

$$S = \frac{a}{1-r}$$

for all $|r|<1$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r|<1$$

$$f(x) = \sum_{n=1}^{\infty} ax^{n-1} = \frac{a}{1-x} \qquad |x|<1$$

if $|x|<1$

$$a + ax + ax^2 + ax^3 + \boxed{ax^n} + = \boxed{\frac{a}{1-x}}$$

Comparison test

if a smaller series diverges so will a larger.

if a larger series converges so will a smaller

Divergence test

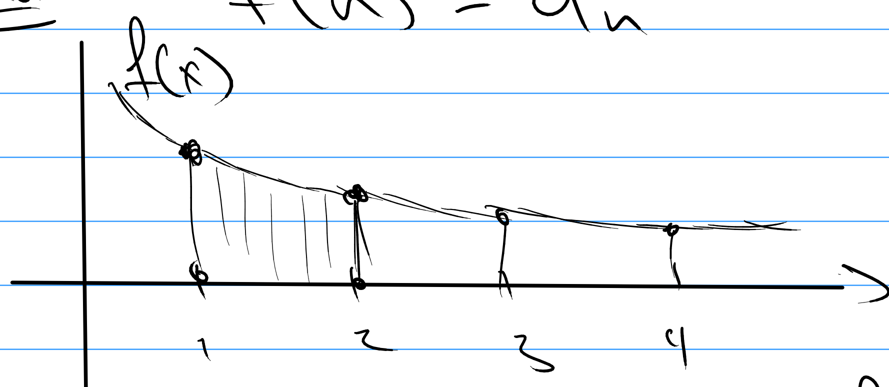
$\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$

if $a_n \not\to 0$ as $n \to \infty$ $\to$ $\sum a_n$ <u>diverges</u>.

Integral Test.

f is cont., positive, decreasing on $[1, \infty)$

<u>and</u> $f(n) = a_n$



Th^m: $\sum_{n=1}^{\infty} a_n$ converges <u>iff</u> $\int_1^{\infty} f(x)\,dx$ converges.

(a) both converge

or

(b) both diverge.

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ex $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ $\quad f(x) = \frac{1}{x^{1/4}}$

$f'(x) = -\frac{1}{4} \frac{1}{x^{3/4}} < 0$

$\rightarrow$ f is dec.

$$\int_1^{\infty} \frac{1}{x^{1/4}}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/4}\,dx$$

$$= \lim_{t\to\infty} \frac{4}{3} x^{3/4} \Big|_1^t = \lim_{t\to\infty} \frac{4}{3} \underbrace{t^{3/4}}_{\to\infty} - \frac{4}{3}$$

$\int_1^{\infty} \frac{1}{x^{1/4}}\,dx$ diverges

$\rightarrow$ $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \text{ diverges}}$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad f(x) = \frac{1}{x^p} = x^{-p}$

$f'(x) = -p \, \underbrace{x^{-p-1}}_{\text{always pos}} < 0$

Need dec.

For $f(x) = \frac{1}{x^p}$ to be dec.

$p > 0$

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$$

$$\lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{1}{1-p} x^{1-p} \Big|_1^t$$

∞

$$= \frac{1}{1-p} \lim_{t\to\infty} \left[\cancel{t^{1-p}} - 1 \right]$$

given $p > 0$

$p = 1/2 \to \infty$

$t^{1-p} \to \infty$ if $0 < p < 1$ $\boxed{\text{div.}}$

$t^{1-p} \to 0$ if $p > 1$ $\boxed{\text{conv}}$

$(p = 1?)$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t = \lim_{t\to\infty} \ln t = \infty$$

$\boxed{\text{div.}}$

$\boxed{\text{p-series test}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$

diverges otherwise

$\boxed{\text{Comparison test}}$

$0 < a_n \leq b_n$

a) if b_n conv. $\rightarrow$ a_n conv.

b) if a_n div. $\rightarrow$ b_n div.

ex's $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

Integral test $\displaystyle\int_1^{\infty} \frac{1}{x^2+x+1}\,dx = ?$

$\boxed{\text{try}}$ Comparison... guess $\boxed{\text{conv.}}$

$\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ b/c $n > 0$

$\dfrac{1}{n^2} > \dfrac{1}{n^2+n+1}$

by p-series test $\frac{1}{n^2}$ conv.

by comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ conv.

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ex $\sum_{n=1}^{\infty} n e^{-n} = \sum_{n=1}^{\infty} \frac{n}{e^n}$

try div. test $\lim_{n \to \infty} \frac{n}{e^n} = 0$ so div. test fails.

$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

try integral test. $f(x) = \frac{x}{e^x}$

test decreasing. $f'(x) = \frac{e^x - x e^x}{e^{2x}} = \frac{1-x}{e^x}$

$f'(x) < 0 \quad x > 1$

so it is dec on $[1, \infty)$

$\int_1^{\infty} x e^{-x} dx = \lim_{t \to \infty} \left[\int_1^t x e^{-x} dx \right]$

let $u = x \qquad du = dx$

$dv = e^{-x} dx \qquad v = -e^{-x}$

$\lim_{t \to \infty} \left[-x e^{-x} \Big|_1^t + \int_1^t e^{-x} dx \right]$

$$= \lim_{t\to\infty} \left[-xe^{-x} \Big|_1^t - e^{-x} \Big|_1^t \right]$$

$$= \lim_{t\to\infty} \left[-\cancel{te^{-t}}^{\to 0} + e^{-1} - \cancel{e^{-t}}^{\to 0} + e^{-1} \right]$$

$$= 2e^{-1} \quad \text{converges}$$

by integral test

$$\sum_{n=1}^{\infty} ne^{-n} \quad \text{converges.}$$

$\boxed{\text{Limit Comparison test}}$ $a_n, b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

$\Rightarrow$ both conv. <u>or</u> both divergent

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-n-1}$ <u>guess conv.</u>

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-n-1}} = \lim_{n\to\infty} \frac{n^2-n-1}{n^2} = 1 > 0$$

by limit comparison b/c $\sum \frac{1}{n^2}$ conv. $\sum \frac{1}{n^2-n-1}$ conv.

ex $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{n^4}+\frac{4}{n^{3/2}}\right)$

if $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n^4}$ ① conv. <u>and</u> $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^{3/2}}$ ② conv.

<u>then</u> $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{n^4}+\frac{4}{n^{3/2}}\right)$ conv.

① $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n^4} = 5\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4}}$ by p-series this conv.

so $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n^4}$ <u>conv.</u>

② same as for $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^{3/2}}$

being conv.

$\rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{n^4}+\frac{4}{n\sqrt{n}}\right)$ $\boxed{\text{conv.}}$

ex $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+\sin n}{10^n}\right)$ <u>guess conv.</u>

<u>comparison</u>

$$\frac{1+\sin n}{10^n} \leq \frac{1+1}{10^n} = \frac{2}{10^n}$$ <u>conv?</u>

$$\sum \frac{2}{10^n}$$

$$\int_0^\infty \frac{2}{10^x}\,dx = \lim_{t\to\infty} 2\int_0^t 10^{-x}\,dx$$

$$= \lim_{t\to\infty} \frac{-2}{\ln 10} 10^{-x} \Big|_0^t$$

$$= \lim_{t\to\infty} \cancel{\frac{-2}{\ln 10} 10^{-t}}^{\,0} + \frac{2}{\ln 10} \quad \boxed{\text{conv.}}$$

Integral test $\Rightarrow$ $\sum \frac{2}{10^n}$ conv.

$\Rightarrow$ by comparison test

$$\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n} \quad \text{conv.}$$

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

try Integral test: let $f(x) = \dfrac{\ln x}{x^p}$

$$f'(x) = \frac{x^{p-1} - \ln x \; p\, x^{p-1}}{x^{2p}} = \boxed{\frac{1 - p\ln x}{x^{2p} \cdot x^{1-p}}}$$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^p}\,dx$$

Have: $\lim_{t\to\infty} \cancel{\frac{\ln t}{t^{p-1}}} \to 0$ $\qquad$ $\lim_{t\to\infty} \cancel{\frac{1}{t^{p-1}}} \to 0$

$p > 1$ $\qquad$ $p > 1$

Conv. $\quad p > 1$

$$\frac{1}{n^p} \quad \text{vs} \quad \frac{\ln n}{n^p}$$

Conv. $\quad p > 1$

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{1}{(n^3+1)^{1/2}}} = \lim_{n\to\infty} \left(\frac{n^3+1}{n^3}\right)^{1/2}$$

$$= \lim_{n\to\infty} \left(1 + \cancel{\frac{1}{n^3}}^{\,0}\right)^{1/2} = 1 > 0$$

by limit comparison test

b/c $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv. (p-series)

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ conv.
