"ለአሸምጠው እንደ ነበረ
ህግ ያስገድዳል።"

⇒ <u>ለማ አበነቀቁያ</u>

« ከ አቴሌትን ወደ ገንዘቤ ካምፓኝ የተመላላገቾ (2^{nd} year)
ሌት ተማሪዎች የሠራን እንም <u>ገለማማፊት</u>
እንዲያነታገበሩኝ ካለ በንፃህትና መቶ ቆሎው »

AHAND BOOK OF APPLIED MATHEMATICS-III

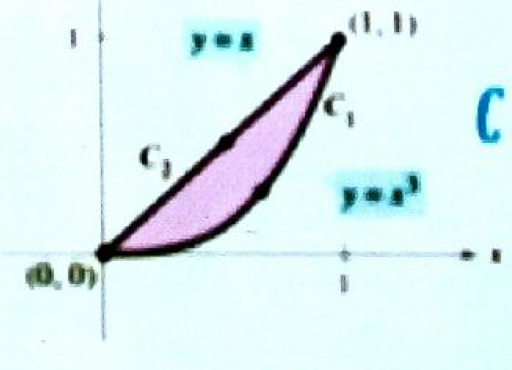
PROBLEM SOLVING APPROACH

For Engineering, Science and Technology Students

Method of Variation of Parameters:

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

Additivity of Line Integrals



$$C = C_1 \cup C_2$$

$$\int_C r(t)dt = \int_{C_1} r(t)dt + \int_{C_2} r(t)dt$$

Author: Begashaw Moltot

Qualifications: MED+MSC

2009

A HAND BOOK OF APPLIED MATHEMATICS-III

Revised Edition

By Begashaw Moltot Zemedhun

2009 E.C.

Contents

CHAPTER-1

ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER-2

Laplace Transform and its Applications

CHAPTER-3: CALCULUS OF VECTOR VALUED FUNCTIONS

CHAPTER-4

FOURIER-ANALYSIS

CHAPTER-5

CALCULUS OF COMPLEX FUNCTIONS

CHAPTER-1

Ordinary Differential Equations (ODE)

1.1 Definition and Classifications of Differential Equations

An equation containing (involving) derivative of the dependent variable with respect to independent variables is known as *differential equation* (DE).

There are two basic types of differential equations.

a) **Ordinary Differential Equation (ODE):** A differential equation involving derivatives of the dependent variable with respect to *one* independent variable.

b) **Partial Differential Equation (PDE):** A differential equation containing the derivatives of the dependent variable with respect to *more than one* independent variables.

Notations: $\frac{dy}{dx}=y', \frac{d^2y}{dx^2}=y'', \frac{d^3y}{dx^3}=y''',..., \frac{d^ny}{dx^n}=y^{(n)}$ in ODE.

Examples:

a) Examples of ODE: i) $\frac{dy}{dx}=2x$ ii) $y''+4y=xe^{3x}$ iii) $\frac{d^3y}{dx^3}-6\frac{dy}{dx}=0$

b) Examples of PDE: i) $\frac{\partial z}{\partial x}=4xy$ ii) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}=x^2+y^2$ iii) $z_{xx}=2y$

Order and Degree of Differential Equations:

a) **Order of a Differential Equation:** It refers the order of the term with the highest derivative in the Differential Equation.

In a differential equation, y' represents first order, y'' represents second order, y''' represents third order and in general, $y^{(n)}$ represents n^{th} order derivative. So, if we get derivatives of different orders in a single differential equation, we take the highest order as the order of that differential equation.

b) **Degree of a Differential Equation:** It refers the highest power or exponent (positive integer only) of the term with the highest order derivative.

Remarks: Main points about Order and Degree

1) The degree of a DE is identified after the exponents of all the terms with any order derivatives are made to have positive integer exponents. This may need squaring or cubing both sides of the equation.

Besides, if the term with the highest order is part of an operation like multiplication, you have to simplify the operation before reading the degree.

2) Degree of a differential equation is defined only if the DE is a polynomial in all the terms with the derivatives $y', y'', y''', \dots, y^{n}$. In general, the degree of a differential equation that contains any of the following exceptions need not be defined. Because of this, every DE has order but may not have degree.

Exceptions: $e^{\frac{dy}{dx}}$ or $e^{y'}$, $\ln\frac{dy}{dx}$ or $\ln y'$, $\sin(y')$ or $\sin(\frac{dy}{dx})$, $\cos(y')$, $\sin^{-1}y'$, $\cos^{-1}y'$.

Examples: Identify the order and degree of the following DEs.

a) $y'''^5+4y''^7-2y^8=x^9$. For this DE, order $n=3$ and degree $d=5$.

Why the degree is 5? Because the term with the highest order derivative is y''' and its exponent is 5.

b) $\left(\frac{dy}{dx}\right)^3-4x\left(\frac{d^3y}{dx^3}\right)^2+\frac{d^4y}{dx^4}=0$. Here, order $n=4$ and degree $d=1$.

Why the degree is 1? This is because the term with the highest order derivative in the given DE is $\frac{d^4y}{dx^4}$ and its power is 1.

c) $y''^4-y'^{\frac{3}{2}}=0$. Here, order $n=2$ and degree $d=8$. How? In this DE, we cannot read the degree directly because the exponent of y' is not integer.

So, first square both sides to make it integer.

That is $y''^4-y'^{\frac{3}{2}}=0 \Rightarrow y''^4=y'^{\frac{3}{2}} \Rightarrow (y''^4)^2=(y'^{\frac{3}{2}})^2 \Rightarrow y''^8-y'^3=0 \Rightarrow d=8$.

d) $y''^3+\sin 4y'=x$. Here, order $n=2$ but its degree is not defined because it is not a polynomial with respect to y'. (This is because of the term $\sin 4y'$).

e) $y''^3+\sin 4y=x$. Here, order $n=2$ and its degree is also defined which is $d=3$ even though it is not polynomial in y.

f*) $y''^4(y''^3-y')^2=x$. Here, order $n=2$ and degree $d=10$. (How?)

1.2 Linear and Non-linear Differential Equations

Definition: A differential equation is said to be linear if and only if the following two conditions are satisfied:

i) The dependent variable and its derivatives in all terms have first degree.

ii) There is no term involving product of the dependent variable and any order of its derivatives.

A differential equation which violates either of these two conditions is said to be *non-linear* differential equation. In general, any differential equation that contains at least one of the following terms is automatically non-linear.

$$\sin(y'), \cos(y'), \sec(y'), \ln y', e^{y'}, \sin^{-1} y', \cos^{-1} y' .$$

Examples:

a) The DE $\frac{d^2y}{dx^2} = 5x\frac{dy}{dx}$ has order $n = 2$ and degree $n = 1$. Thus, it is linear.

b) The DE $y''^4 + 3y' - y = 0$ has order $n = 2$ and degree $n = 4$. It violates the first condition of the definition. Thus, it is non-linear.

c) The DE $y''' + yy' = 1$ has order $n = 2$ and degree $n = 1$ but it contains the term yy' which is the product of the dependent variable and its derivative. It violates the second condition of the definition and thus it is non-linear.

d) The DE $y'' = \sqrt{1+x^2}$ has order $n = 2$ and degree $n = 1$. Thus, it is linear.

e) The DE $y'' = \sqrt{1+y'^2}$ has order $n = 2$ and degree $n = 2$. It violates the first condition and thus it is non-linear.

f) Consider the DEs: $\sqrt{y''+x} = y$ and $\sqrt{y''+y} = x$. Which equation is linear and which equation is non-linear. Why?

1.3 Solutions of Differential Equations

Definition: Any function (involving the independent and dependent variables) which satisfies the given DE whenever substituted is called solution of the DE.

Types of solutions: There are two forms of solutions for a given DE (if it has).

a) **General solution:** The solution of a DE which contains arbitrary constants in its expression is called *general solution* (*primitive*).

b) **Particular solution:** The solution of a DE free from arbitrary constants (that does not contain arbitrary constants) is called a *particular solution*. Usually, particular solutions are solutions obtained from the general solution by assigning particular values to the arbitrary constants.

Examples:

a) $y = 3e^{2x} - 4x$ is the solution of the DE $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 8$.

Here, $y = 3e^{2x} - 4x, \dfrac{dy}{dx} = 6e^{2x} - 4, \dfrac{d^2y}{dx^2} = 12e^{2x}$.

Then, $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 12e^{2x} - 2(6e^{2x} - 4) = 12e^{2x} - 12e^{2x} + 8 = 8$

This means $y = 3e^{2x} - 4x$ satisfies the given DE when substituted.

b) Consider the DE: $y''' = 8y$. Can both $y = 3e^{2x}$ and $y = ce^{2x}$ be solutions?

Here, $y = 3e^{2x}, y' = 6e^{2x}, y'' = 12e^{2x}, y''' = 24e^{2x}$. So, $y''' = 24e^{2x} = 8(3e^{2x}) = 8y$.

Again, $y = ce^{2x}, y' = 2ce^{2x}, y'' = 4ce^{2x}, y''' = 8ce^{2x}$. So, $y''' = 8ce^{2x} = 8(ce^{2x}) = 8y$.

Therefore, both $y = 3e^{2x}$ and $y = ce^{2x}$ are solutions.

Besides, $y = ce^{2x}$ is general solution because it contains arbitrary constant c in its expression. But $y = 3e^{2x}$ is particular solution obtained by assigning $c = 3$.

c) For arbitrary constants c_1 and c_2, $y = c_1 + c_2e^{-x} + x^3$ is the solution of the DE $y'' + y' - 6x = 3x^2$. Therefore, it is a general solution. But if we give $c_1 = 0$ and $c_2 = 2$, we obtain $y = 2e^{-x} + x^3$ which is a particular solution.

d) $y = x^3$ is a solution of $y''' - 2xy' + 6y = 6$ but $y = x^2$ is not.

e) If $y = e^{2x}$ is the solution of the DE $\frac{d^2y}{dx^2} + 3y' - ky = 0$, then find k.

Solution: Since is $y = e^{2x}$ is given to be a solution, it must satisfy the DE.

That is $y = e^{2x} \Rightarrow y' = 2e^{2x}, y'' = 4e^{2x} \Rightarrow \frac{d^2y}{dx^2} + 3y' - ky = 0$

$$\Rightarrow 4e^{2x} + 6e^{2x} - ke^{2x} = 0 \Rightarrow k - 10 = 0 \Rightarrow k = 10$$

d) For what value of a does $y = x^2$ is the solution of the differential equation given by $4x\frac{d^2y}{dx^2} + axy' - 6y = 2x^2 + 8x$?

Solution: Since $y = x^2$ is a solution, it must satisfy the given DE.

That is $y = x^2 \Rightarrow y' = 2x, y'' = 2 \Rightarrow 4x\frac{d^2y}{dx^2} + axy' - 6y = 2x^2 + 8x$

$$\Rightarrow 8x + 2ax^2 - 6x^2 = 2x^2 + 8x \Rightarrow 2a - 6 = 2 \Rightarrow a = 4$$

1.4 Initial Value Problems (IVP)

The problem of finding a function solution satisfying a differential equation and an initial condition is called *an initial value problem* (IVP).

Consider a differential equation $\frac{dy}{dx} = f(x,y)$. Then the initial value problem for such first order DEs is of the form $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$. Then, the problem of finding a function which satisfies this differential equation with the given additional condition is said to be *Initial Value Problem.*

The condition $y(x_0) = y_0$ is called initial condition.

Examples: *a)* $xy' + 2y = 4x^2,\ y(1) = 3$ *b)* $y' = 2y + 6,\ y(0) = 2$

Note: To solve IVPs, first find the general solution of the given DE using any method and finally find the constants using the given initial conditions.

1.5 Solving First Order Differential Equations

Solving a differential equation means finding the unknown function which satisfies the given differential equation.

Forms of First Order Differential Equation:

Any DE of the form $M(x,y)dx + N(x,y)dy = 0$ or $\frac{dy}{dx} = f(x,y)$ is said to be first order differential equation. Here, under we will discuss how to find the general solution for such form of DE.

Since there is no general method to solve all forms of DEs, we will see different methods for different forms of DEs.

Methods of Solutions for First Order Differential Equations:

1) Separable Differential Equations: Method of Separation
2) Homogeneous Differential Equations: Change of Variable
3) Exact Differential Equations: Method of Exactness
4) Non-Exact Differential Equations: Integrating Factor Method
5) Bernoulli's Differential Equations: Transformation (Reduction)

1.5.1 Separable Differential Equations: Method of Separation

Separable Differential Equation:

Any first order differential equation $M(x,y)dx + N(x,y)dy = 0$ which can be expressible in the form $g(y)dy = f(x)dx$ is said to be ***separable differential equation***. Here, the form $g(y)dy = f(x)dx$ is said to be the separable form.

Method of solution: The *general solution* of such separable DE is obtained by integrating the separable form $g(y)dy = f(x)dx$ both sides.

That is $\int g(y)dy = \int f(x)dx$.

Examples:

1. Verify that the following DEs are separable and solve them.

a) $(xy^4 - y^4)dx - (x^3y^2 - 3x^3)dy = 0$ b) $(x^2+1)\frac{dy}{dx} - \frac{x}{2y} = 0$

c) $24dy - (x^2y^3 - 4x^2y + y^3 - 4y)dx = 0$ d) $e^{x+2y}dx - e^{2x-y}dy = 0$

e) $\tan x\sin^2 ydx + \cos^2 x\cot ydy = 0$ f) $e^y \sin xdx - \cos^2 xdy = 0$

g) $(e^y + 3)\cos xdx - e^y \sin xdy = 0$ h) $\frac{dy}{dx} = e^{x-y} + 2xe^{-y}$

i) $(y^2 + xy^2)dx + (x^2 - x^2y)dy = 0$ j) $3(1+x^2)dy = 2xy(y^3 - 1)dx$

Solution:

a) $(x^2+1)\frac{dy}{dx} - \frac{x}{2y} = 0 \Rightarrow (x^2+1)dy = \frac{x}{2y}dx \Rightarrow 2ydy = \frac{x}{x^2+1}dx.$

Hence, the DE is separable.

Thus, $2ydy = \frac{x}{x^2+1}dx \Rightarrow \int 2ydy = \int \frac{x}{x^2+1}dx \Rightarrow y^2 = \frac{1}{2}\ln(x^2+1) + c$

b) $(xy^4 - y^4)dx - (x^3y^2 - 3x^3)dy = 0 \Rightarrow y^4(x-1)dx - x^3(y^2-3)dy = 0$

$$\Rightarrow \left(\frac{y^2-3}{y^4}\right)dy = \left(\frac{x-1}{x^3}\right)dx \Rightarrow \left(\frac{1}{y^2} - \frac{3}{y^4}\right)dy = \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$$

c) $24dy - (x^2y^3 - 4x^2y + y^3 - 4y)dx = 0 \Rightarrow 24dy = [x^2(y^3-4y) + y^3 - 4y]dx$

$$\Rightarrow 24dy = (y^3-4y)(x^2+1)dx \Rightarrow \frac{24}{y^3-4y}dy = (x^2+1)dx$$

Hence, the DE is separable and by PFD, $\frac{24}{y^3-4y} = \frac{-6}{y} + \frac{3}{y-2} + \frac{3}{y+2}$.

d) $e^x dx - \frac{e^{2x}}{e^{3y}}dy = 0 \Rightarrow e^{-x}dx = e^{-3y}dy \Rightarrow \int e^{-x}dx = \int e^{-3y}dy \Rightarrow -e^{-x} = \frac{-1}{3}e^{-3y} + c$

So, $\int \frac{24}{y^3-4y}dy = \int\left(\frac{-6}{y} + \frac{3}{y-2} + \frac{3}{y+2}\right)dy = \int (x^2+1)dx \Rightarrow 3\ln\left|\frac{y^2-4}{y^2}\right| = \frac{x^3}{3} + x.$

e) $\frac{\cot y}{\sin^2 y}dy = \frac{-\tan x}{\cos^2 x}dx \Rightarrow \cos y\sin^{-3} y = -\sin x\cos^{-3} x$. Hence, it is separable.

So, $\int \cos y\sin^{-3} ydy = -\int \sin x\cos^{-3} xdx \Rightarrow \frac{1}{2\sin^2 y} = \frac{1}{2\cos^2 x} + c$

f) $e^{-y}dy = \frac{\sin x}{\cos^2 x}dx \Rightarrow \int e^{-y}dy = \int \frac{\sin x}{\cos^2 x}dx \Rightarrow -e^{-y} = \sec x + c$

g) $\frac{e^y}{e^y+3}dy = \frac{\sin x}{\cos x}dx \Rightarrow \int \frac{e^y}{e^y+3}dy = \int \frac{\sin x}{\cos x}dx \Rightarrow (e^y+3)\cos x = c$

h) $\frac{dy}{dx} = e^{x-y} + 2xe^{-y} \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{2x}{e^y} \Rightarrow e^y dy = (e^x + 2x)dx$

Hence, $e^y dy = (e^x+2x)dx \Rightarrow \int e^y dy = \int (e^x + 2x)dx \Rightarrow e^y = e^x + x^2 + c$

i) $(y^2 + xy^2)dx + (x^2 - x^2y)dy = 0 \Rightarrow y^2(1+x)dx + x^2(1-y)dy = 0$

So, $\frac{1-y}{y^2}dy = -\frac{1+x}{x^2}dx \Rightarrow \int \frac{1-y}{y^2}dy = -\int \frac{1+x}{x^2}dx \Rightarrow -\frac{1}{y} - \ln|y| = \frac{1}{x} + \ln|x| + cx$

j) $\frac{3}{y(y^3-1)}dy = \frac{2x}{1+x^2}dx \Rightarrow \frac{3}{y(y-1)(y^2+y+1)}dy = \frac{2x}{1+x^2}dx$

2. Solve the following IVPs using separable method

a) $2xe^{x^2}dx + (y^3-1)dy = 0, y(0) = 2$

b) $\frac{dy}{dx} = 6xe^{y-x^2}, y(0) = 0$

c) $y' + y^2\cos 2x = 0, y(0) = 1$

d) $x\frac{dy}{dx} + \cot y = 0,\ y(\sqrt{2}) = \frac{\pi}{4}$

Solution: Very Important: To solve IVPs with $y(x_0) = y_0$,

First: Find the general solution using appropriate method

Second: Find the constant by putting $x = x_0$ and $y = y_0$ in the general solution.

a) $2xe^{x^2}dx + (y^3-1)dy = 0 \Rightarrow (y^3-1)dy = -2xe^{x^2}dx$

$$\Rightarrow \int (y^3-1)dy = \int -2xe^{x^2}dx \Rightarrow \frac{y^4}{4} - y = -e^{x^2} + c$$

Now, find the arbitrary constant by using $x = 0,\ y = 2$.

That is $\frac{y^4}{4} - y = -e^{x^2} + c \Rightarrow \frac{(2)^4}{4} - 2 = -e^0 + c \Rightarrow -1 + c = 2 \Rightarrow c = 3$

Therefore, the solution that satisfies the given IVP is $\frac{y^4}{4} - y = -e^{x^2} + 3$.

b) $\frac{dy}{dx} = 6xe^{y-x^2} \Rightarrow \frac{dy}{e^y} = 6xe^{-x^2}dx \Rightarrow \int e^{-y}dy = \int 6xe^{-x^2}dx \Rightarrow -e^{-y} = -3e^{-x^2} + c$

$y(0) = 0 \Rightarrow -1 = -3 + c \Rightarrow c = 2 \Rightarrow -e^{-y} = -3e^{-x^2} + 2 \Rightarrow e^{-y} = 3e^{-x^2} - 2$

Equations Reducible to Separable: Any DE of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to separable form using the substitution $t = ax+by+c$.

Then, $\frac{dt}{dx} = a + bf(t) \Rightarrow dx = \frac{dt}{a+bf(t)} \Rightarrow \int dx = \int \frac{dt}{a+bf(t)} + c$

Examples: By reducing into separable form, solve the following DEs:

a) $\frac{dy}{dx} = (9x+y+5)^2$ *b)* $(x+y+1)^2 \frac{dy}{dx} = 1$ *c)* $\frac{dy}{dx} = \frac{2x+2y+3}{x+y+1}$

d) $\frac{dy}{dx} = 2x+y$ *e)* $\frac{dy}{dx} = \frac{2x+3y+5}{4x+6y-3}$ *f)* $\frac{dy}{dx} = \sec(x+5y)$

Solution: We use the above substitution rule

a) Let $t = 9x+y+5 \Rightarrow \frac{dt}{dx} = 9 + \frac{dy}{dx}$. But from the given $\frac{dy}{dx} = t^2$.

So, $\frac{dt}{dx} = 9+t^2 \Rightarrow \frac{dt}{9+t^2} = dx \Rightarrow \int \frac{dt}{9+t^2} = \int dx \Rightarrow \frac{1}{3}\tan^{-1}\frac{t}{3} = x+c$

$$\Rightarrow \frac{1}{3}\tan^{-1}\left(\frac{9x+y+5}{3}\right) = x+c \Rightarrow y = 3\tan(3x+3c) - 9x - 5$$

b) Let $t = x+y+1 \Rightarrow \frac{dt}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

Then, $(x+y+1)^2 \frac{dy}{dx} = 1 \Rightarrow t^2\left(\frac{dt}{dx} - 1\right) = 1 \Rightarrow \frac{dt}{dx} = 1 + \frac{1}{t^2} \Rightarrow \frac{t^2 dt}{t^2+1} = dx$

$$\Rightarrow \int \frac{t^2 dt}{t^2+1} = \int dx \Rightarrow \int\left(1 - \frac{1}{t^2+1}\right)dt = \int dx = t - \tan^{-1} t = x+c$$

$$\Rightarrow x+y+1-\tan^{-1}(x+y+1) = x+c \Rightarrow y = \tan^{-1}(x+y+1) - 1 + c$$

c) Let $t = x+y+1 \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$.

Then, $\frac{dy}{dx} = \frac{2x+2y+3}{x+y+1} = \frac{2(x+y)+3}{x+y+1} \Rightarrow \frac{dt}{dx} - 1 = \frac{2(t-1)+3}{t}$

$$\Rightarrow \frac{dt}{dx} = 3 + \frac{1}{t} \Rightarrow \int\left(3+\frac{1}{t}\right)dt = \int dx \Rightarrow 3t + \ln|t| = x+c$$

$$\Rightarrow 3(x+y+1) + \ln|x+y+1| = x+c \Rightarrow 2x+3y+\ln|x+y+1| = c$$

1.5.2 Homogeneous Differential Equations

A function $f(x,y)$ is said to be *homogenous* of degree 1 if for any parameter $t \neq 0$, $f(tx,ty) = f(x,y)$. In general, f is *homogenous* function of degree n if and only if for any parameter $t \neq 0$, $f(tx,ty) = t^n f(x,y)$.

Homogenous Differential Equation:

A differential equation of the form $\frac{dy}{dx} = f(x,y)$ is said to be *homogenous* if $f(x,y)$ is homogeneous function. Otherwise, it is said to be non-homogeneous.

Test of Homogeneity:

A differential equation $\frac{dy}{dx} = f(x,y)$ is *homogenous* of degree 1 if and only if for any parameter $t \neq 0$, $f(tx,ty) = f(x,y)$.

Method of Solutions: By Reducing into Separable DEs

First: Use Test of Homogeneity to check whether it is homogeneous or not.

Second: Change the homogeneous DE into separable DE.

To change into separable DE, use the substitutions: $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$.

This will change the given DE into separable DE in the variables x and v. Then, integrating the separable form gives the general solution in terms of x and v. Finally, by solving $y = vx$ for v, express the answer in terms of x and y.

Examples:

1. Check whether the differential equations are *homogenous* or not. For these which are homogeneous, solve using homogeneous method.

a) $\frac{dy}{dx} = \frac{x+y}{x}$ *b)* $2xy\frac{dy}{dx} = x^2 + y^2$

c) $xdy = (y + x\csc\frac{y}{x})dx$ *d)* $xdy = y(1 + \ln\frac{y}{x})dx$

e) $x^2dy = (y^2 - xy)dx$ *f)* $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$

g) $(x-y)dx = (x+y)dy$ *h)* $(x^2 - y^2)dx = 2xydy$

i) $(y^2 - 2xy)dx = (x^2 - 2xy)dy$ *j)* $xy' = y(\ln y - \ln x)$

Solution:

First: Use Test of Homogeneity: To use this test for DE, first express the given DE in the form $\frac{dy}{dx}=f(x,y)$.Then, apply the test on f.

a) Then the DE is the form $\frac{dy}{dx}=f(x,y)$ where $f(x,y)=\frac{x+y}{x}$.

Here, for any $t\neq 0$, $f(tx,ty)=\frac{tx+ty}{tx}=\frac{x+y}{x}=f(x,y)$.

Thus, f is homogeneous which implies the DE itself is *homogenous.*

$$\frac{dy}{dx}=\frac{x+y}{x}=1+\frac{y}{x}\Rightarrow v+x\frac{dv}{dx}=1+v\Rightarrow x\frac{dv}{dx}=1\Rightarrow dv=\frac{dx}{x}$$

$$\Rightarrow\int dv=\int\frac{1}{x}dx\Rightarrow v=\ln|x|+c\Rightarrow\frac{y}{x}=\ln|x|+c\Rightarrow y=x\ln|x|+cx$$

b) $2xy\frac{dy}{dx}=x^2+y^2\Rightarrow\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$. Then the DE is expressible in the form $\frac{dy}{dx}=f(x,y)$ where the function is $f(x,y)=\frac{x^2+y^2}{2xy}$.

Here, for any $t\neq 0$, $f(tx,ty)=\frac{t^2x^2+t^2y^2}{2t^2xy}=\frac{t^2(x^2+y^2)}{2t^2xy}=\frac{x^2+y^2}{2xy}=f(x,y)$

Thus, f is homogeneous which implies the DE itself is *homogenous.*

Second: Use the substitution $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$ to change into separable.

$$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{x^2+x^2v^2}{2vx^2}\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^2}{2v}\Rightarrow\frac{2v}{v^2-1}dv=-\frac{dx}{x}$$

$$\Rightarrow\int\frac{2v}{v^2-1}dv=-\int\frac{1}{x}dx\Rightarrow\ln|v^2-1|=-\ln|x|+c$$

$$\Rightarrow\ln|v^2-1|+\ln|x|=c\Rightarrow\ln|(v^2-1)x|=c\Rightarrow(v^2-1)x=c'$$

$$\Rightarrow\left(\frac{y^2}{x^2}-1\right)x=c'\Rightarrow\left(\frac{y^2-x^2}{x^2}\right)x=c'\Rightarrow y^2-x^2=c'x$$

c) Here, $xdy = (y + x\csc\frac{y}{x})dx \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \csc\frac{y}{x}$.Here, for any $t \neq 0$,

$$f(x,y) = \frac{y}{x} + \csc\frac{y}{x} \Rightarrow f(tx,ty) = \frac{ty}{tx} + \csc\frac{ty}{tx} = \frac{y}{x} + \csc\frac{y}{x} = f(x,y).$$

Thus, f is homogeneous which implies the DE itself is *homogenous.*

Second: Use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ to change into separable.

$$\frac{dy}{dx} = \frac{y}{x} + \csc\frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v + \csc v \Rightarrow x\frac{dv}{dx} = \csc v \Rightarrow \frac{1}{\csc v}dv = \frac{1}{x}dx$$

$$\Rightarrow \int \sin v dv = \int \frac{1}{x}dx \Rightarrow -\cos v = \ln|x| + c \Rightarrow -\cos\frac{y}{x} = \ln|x| + c$$

d) $xdy = y(1+\ln\frac{y}{x})dx \Rightarrow \frac{dy}{dx} = \frac{y}{x}(1+\ln\frac{y}{x}) = \frac{y}{x} + \frac{y}{x}\ln\frac{y}{x}$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y}{x}.\ln\frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v + v\ln v \Rightarrow x\frac{dv}{dx} = v\ln v$$

$$\Rightarrow \frac{1}{v\ln v} = \frac{1}{x}dx \Rightarrow \int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx \Rightarrow \ln|\ln v| = \ln|x| + c$$

$$\Rightarrow \ln\left|\ln\frac{y}{x}\right| = \ln|x| + c \text{ or } \ln\frac{y}{x} = cx \text{ or } y = xe^{cx}$$

e) $\frac{dy}{dx} = \frac{y^2 - xy}{x^2} \Rightarrow v + x\frac{dv}{dx} = v^2 - v \Rightarrow x\frac{dv}{dx} = v^2 - 2v \Rightarrow \frac{1}{v^2 - 2v}dv = \frac{1}{x}dx$

$$\Rightarrow (\frac{1/2}{v-2} - \frac{1/2}{v})dv = \frac{1}{x}dx \Rightarrow \int(\frac{1/2}{v-2} - \frac{1/2}{v})dv = \int\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}\ln|v-2| - \frac{1}{2}\ln|v| = \ln|x| + c \Rightarrow \frac{1}{2}\ln\left|\frac{y}{x} - 2\right| - \frac{1}{2}\ln\left|\frac{y}{x}\right| = \ln|x| + c$$

f) $v + x\frac{dv}{dx} = v + \sin v \Rightarrow x\frac{dv}{dx} = \sin v \Rightarrow \frac{dv}{\sin v} = \frac{dx}{x} \Rightarrow \int\frac{1}{\sin v}dv = \int\frac{1}{x}dx$

$$\Rightarrow \int \csc v dv = \int \frac{1}{x}dx \Rightarrow |\csc v - \cot v| = |x| + c \Rightarrow 1 - \cos\frac{y}{x} = cx\sin\frac{y}{x}$$

g) $\frac{dy}{dx}=\frac{x-y}{x+y}\Rightarrow v+x\frac{dv}{dx}=\frac{1-v}{1+v}\Rightarrow x\frac{dv}{dx}=\frac{1-2v-v^2}{1+v}\Rightarrow \frac{v+1}{v^2+2v-1}dv=-\frac{1}{x}dx$

But by substitution, $u=v^2+2v-1\Rightarrow du=2(v+1)dv$,

$$\int(\frac{v+1}{v^2+2v-1})dv=-\int\frac{1}{x}dx\Rightarrow\int\frac{1}{2u}du=-\ln|x|+c\Rightarrow\frac{1}{2}\ln|u|=-\ln|x|+c$$

$$\Rightarrow \ln\left|v^2+2v-1\right|+2\ln|x|=c\Rightarrow\ln\left|(v^2+2v-1)x^2\right|=c$$

$$\Rightarrow\left(\frac{y^2}{x^2}+\frac{2y}{x}-1\right)x^2=c\Rightarrow y^2+2xy-x^2=c\Rightarrow x^2-y^2=2xy+c$$

h) $\frac{dy}{dx}=\frac{x^2-y^2}{2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{x^2-v^2x^2}{2x^2v}=\frac{1}{2v}-\frac{v}{2}\Rightarrow x\frac{dv}{dx}=\frac{1}{2v}-\frac{v}{2}-v$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{2v}\Rightarrow\frac{2vdv}{1-3v^2}=\frac{dx}{x}\Rightarrow\int\frac{2vdv}{1-3v^2}=\int\frac{dx}{x}$$

$$\Rightarrow-\frac{1}{3}\ln\left|1-3v^2\right|=\ln|x|+c\Rightarrow x^3(1-3v^2)=c$$

$$\Rightarrow x^3(\frac{x^2-3y^2}{x^2})=c\Rightarrow x(x^2-3y^2)=c$$

i) $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2x^2-2vx^2}{x^2-2vx^2}\Rightarrow x\frac{dv}{dx}=\frac{3v^2-3v}{1-2v}$

$$\Rightarrow\frac{1-2v}{3v^2-3v}dv=\frac{1}{x}dx\Rightarrow\int\frac{1-2vdv}{3v^2-3v}=\int\frac{1}{x}dx$$

$$\Rightarrow\int\frac{1dv}{3v^2-3v}-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx\Rightarrow\frac{1}{3}\int\frac{1}{v(v-1)}dv-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx$$

$$\Rightarrow\frac{1}{3}\int\left(\frac{1}{v-1}-\frac{1}{v}\right)dv-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx\Rightarrow\frac{1}{3}\ln|v-1|-\frac{1}{3}\ln|v|-\frac{2}{3}\ln|v-1|=\ln|x|+c$$

$$\Rightarrow-\frac{1}{3}\ln|v-1|-\frac{1}{3}\ln|v|=\ln|x|+c\Rightarrow\ln|v-1|+\ln|v|=-3\ln|x|+c\Rightarrow xy(y-x)=c$$

j) $xy'=y(\ln y-\ln x)\Rightarrow x\frac{dy}{dx}=y(\ln\frac{y}{x})\Rightarrow\frac{dy}{dx}=\frac{y}{x}(\ln\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v\ln v$

$$\Rightarrow\int\frac{1}{v(\ln v-1)}dv=\int\frac{1}{x}dx\Rightarrow\ln(\ln v-1)=\ln|x|+c$$

$$\Rightarrow\ln v-1=Cx\Rightarrow\ln\frac{y}{x}=Cx+1\Rightarrow\frac{y}{x}=e^{Cx+1}\Rightarrow y=xe^{Cx+1}$$

2. Solve the following homogenous DEs

$a^*)\ xdy - ydx = \sqrt{x^2+y^2}\,dx$ $\qquad$ $b)\ y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

$c)\ x\sin\dfrac{y}{x}dy = (x + y\sin\dfrac{y}{x})dx$ $\qquad$ $d^*)\ xdy = (y + y(\ln\frac{y}{x})^{-5})dx$

$e)\ (x^3 - y^3)dx + xy^2dy = 0$ $\qquad$ $f)\ (2y-3x)dx + xdy = 0$

$g)\ \dfrac{dy}{dx} = \dfrac{y}{x} + \tan(\dfrac{y}{x})$ $\qquad$ $h)\ (x^2+y^2)\dfrac{dy}{dx} = xy$

$i)\ y^2dx + (xy + x^2)dy = 0$ $\qquad$ $j)\ x(x-y)dy + y^2dx = 0$

Solution:

$$a)\ xdy - ydx = \sqrt{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow \frac{1}{\sqrt{1+v^2}}dv = \frac{1}{x}dx$$

$$\Rightarrow \int\frac{1}{\sqrt{1+v^2}}dv = \int\frac{1}{x}dx \Rightarrow \int\frac{1}{\sqrt{1+v^2}}dv = \ln|x| + c$$

Now, using trig substitution, $v = \tan\theta \Rightarrow dv = \sec^2\theta d\theta$.

$$\int\frac{1}{\sqrt{1+v^2}}dv = \int\frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}}d\theta = \int\sec\theta d\theta = \ln|\sec\theta + \tan\theta| = \ln\left|\sqrt{1+v^2} + v\right|$$

So, $\displaystyle\int\frac{1}{\sqrt{1+v^2}}dv = \ln|x| + c \Rightarrow \ln\left|\sqrt{1+v^2} + v\right| = \ln|x| + c \Rightarrow \ln\left|\sqrt{1+\tfrac{y^2}{x^2}} + \frac{y}{x}\right| = \ln|x| + c$

$$\Rightarrow \ln\left|\sqrt{\frac{x^2+y^2}{x^2}} + \frac{y}{x}\right| = \ln|x| + c \Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}}{x} + \frac{y}{x}\right| = \ln|x| + c$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}+y}{x}\right| - \ln|x| = c \Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}+y}{x^2}\right| = c$$

$$\Rightarrow \frac{\sqrt{x^2+y^2}+y}{x^2} = e^c \Rightarrow \sqrt{x^2+y^2} + y = Cx$$

b) $\frac{dy}{dx}=\frac{y^2}{xy-x^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2}{v-1}\Rightarrow x\frac{dv}{dx}=\frac{v}{v-1}\Rightarrow\frac{v-1}{v}dv=\frac{1}{x}dx$

$$\Rightarrow(\frac{v-1}{v})dv=\frac{1}{x}dx\Rightarrow\int(1-\frac{1}{v})dv=\int\frac{1}{x}dx\Rightarrow v-\ln|v|=\ln|x|+c$$

$$\Rightarrow v=\ln|v|+\ln|x|+c\Rightarrow v=\ln|cvx|\Rightarrow\frac{y}{x}=\ln|cy|\Rightarrow y=x\ln|cy|$$

c) $x\sin\frac{y}{x}dy=(x+y\sin\frac{y}{x})dx\Rightarrow\sin\frac{y}{x}.\frac{dy}{dx}=1+\frac{y}{x}\sin(\frac{y}{x}).$

$$\Rightarrow\sin v.(v+x\frac{dv}{dx})=1+v\sin v\Rightarrow v\sin v+x\sin v\frac{dv}{dx}=1+v\sin v$$

$$\Rightarrow x\sin v\frac{dv}{dx}=1\Rightarrow\sin v dv=\frac{1}{x}dx\Rightarrow\int\sin v dv=\int\frac{1}{x}dx$$

$$\Rightarrow-\cos v=\ln|x|+c\Rightarrow-\cos\frac{y}{x}=\ln|x|+c$$

d) $xdy=(y+y(\ln\frac{y}{x})^{-5})dx\Rightarrow\frac{dy}{dx}=\frac{y}{x}+\frac{y}{x}(\ln\frac{y}{x})^{-5}$

$$\Rightarrow v+x\frac{dv}{dx}=v+24v(\ln v)^{-5}\Rightarrow x\frac{dv}{dx}=v(\ln v)^{-5}$$

$$\Rightarrow\frac{1}{v(\ln v)^{-5}}dv=\frac{1}{x}dx\Rightarrow\frac{(\ln v)^5}{v}dv=\frac{1}{x}$$

$$\Rightarrow\int\frac{(\ln v)^5}{v}dv=\int\frac{1}{x}dx\Rightarrow\int t^5dt=\int\frac{1}{x}dx\qquad(\text{Using }\ln v=t\Rightarrow\frac{1}{v}dv=dt)$$

$$\Rightarrow\frac{t^6}{6}=\ln|x|+c\Rightarrow\frac{(\ln v)^6}{6}=\ln|x|+c\Rightarrow\frac{1}{6}(\ln\frac{y}{x})^6=\ln|x|+c$$

e) $\frac{dy}{dx}=\frac{y^3-x^3}{xy^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v^3-1}{v^2}\Rightarrow x\frac{dv}{dx}=\frac{-1}{v^2}\Rightarrow-v^2dv=\frac{1}{x}dx$

$$\Rightarrow\int-v^2dv=\int\frac{1}{x}dx\Rightarrow\frac{-v^3}{3}=\ln|x|+c\Rightarrow-\frac{y^3}{3x^3}=\ln|x|+c$$

$f)\ \frac{dy}{dx}=\frac{3x-2y}{x}\Rightarrow v+x\frac{dv}{dx}=3-2v\Rightarrow x\frac{dv}{dx}=3-3v$

$$\Rightarrow \frac{1}{3-3v}dv=\frac{1}{x}dx\Rightarrow \int\frac{1}{3-3v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{3}\ln|1-v|=\ln|x|+c\Rightarrow \ln\left|1-\frac{y}{x}\right|=-3\ln|x|+c$$

$g)\ \frac{dy}{dx}=\frac{y}{x}+\tan(\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v+\tan v\Rightarrow x\frac{dv}{dx}=\tan v\Rightarrow \frac{dv}{\tan v}=\frac{dx}{x}$

$$\Rightarrow \int\frac{\cos v}{\sin v}dv=\int\frac{1}{x}dx\Rightarrow \ln|\sin v|=\ln|x|+c\Rightarrow \sin v=cx\Rightarrow \sin(\frac{y}{x})=cx$$

$h)\ \frac{dy}{dx}=\frac{xy}{x^2+y^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v}{1+v^2}\Rightarrow x\frac{dv}{dx}=-\frac{v^3}{1+v^2}\Rightarrow \frac{1+v^2}{v^3}dv=-\frac{1}{x}dx$

$$\Rightarrow (\frac{1}{v}+\frac{1}{v^3})dv=-\frac{1}{x}dx\Rightarrow \int(\frac{1}{v}+\frac{1}{v^3})dv=\int-\frac{1}{x}dx$$

$$\Rightarrow \ln|v|-\frac{1}{2v^2}=-\ln|x|+c\Rightarrow \ln\left|\frac{y}{x}\right|+\ln|x|-\frac{x^2}{2y^2}=c\Rightarrow \ln|y|-\frac{x^2}{2y^2}=c$$

$i)\ \frac{dy}{dx}=\frac{-y^2}{x^2+xy}\Rightarrow v+x\frac{dv}{dx}=\frac{-v^2}{1+v}\Rightarrow x\frac{dv}{dx}=\frac{-2v^2-v}{1+v}\Rightarrow \frac{v+1}{2v^2+v}dv=\frac{-1}{x}dx$

But by Partial Fraction decomposition, $\frac{v+1}{2v^2+v}=\frac{v+1}{v(2v+1)}=\frac{1}{v}-\frac{1}{2v+1}$.

Thus, $\int(\frac{v+1}{2v^2+v})dv=-\int\frac{1}{x}dx\Rightarrow \int(\frac{1}{v}-\frac{1}{2v+1})dv=-\ln|x|+c$

$$\Rightarrow \ln|v|-\frac{1}{2}\ln|2v+1|=-\ln|x|+c\Rightarrow \ln|v|+\ln|x|-\frac{1}{2}\ln|2v+1|=c$$

$$\Rightarrow \ln|vx|+\ln\sqrt{\frac{1}{2v+1}}=c\Rightarrow \ln|y|+\ln\sqrt{\frac{x}{2y+x}}=c$$

$j)\ \frac{dy}{dx}=\frac{y^2}{xy-x^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2}{v-1}\Rightarrow x\frac{dv}{dx}=\frac{v}{v-1}\Rightarrow \frac{v-1}{v}dv=\frac{1}{x}dx$

$$\Rightarrow (1-\frac{1}{v})dv=\frac{1}{x}dx\Rightarrow \int(1-\frac{1}{v})dv=\int\frac{1}{x}dx$$

$$\Rightarrow v-\ln|v|=\ln|x|+c\Rightarrow \frac{y}{x}=\ln|y|+c$$

3. Solve the following problems using the Methods of homogenous DEs.

$a)\ (x\sec\frac{y}{x}+y)dx-xdy=0, y(1)=0$ $\qquad b)\ xdy-(2xe^{\frac{-y}{x}}+y)dx=0, y(1)=0$

$c)\ 6y^2dy-\frac{6y^3}{x}dx=x^2e^{-\frac{y^3}{x^3}}dx$ $\qquad d)\ xdy=(y+x\tan\frac{y}{x})dx$

$e)\ (x^2+xy)dy=(x^2+y^2)dx$ $\qquad f)\ xdy=(y+x\cos\frac{y}{x})dx$

Solution: First, find the general solution and then use the initial conditions.

$a)\ \frac{dy}{dx}=\frac{y}{x}+\sec\frac{y}{x}\Rightarrow v+x\frac{dv}{dx}=v+\sec v\Rightarrow x\frac{dv}{dx}=\sec v\Rightarrow \cos v dv=\frac{1}{x}dx$

$$\Rightarrow\int\cos v dv=\int\frac{1}{x}dx\Rightarrow \sin v=\ln|x|+c\Rightarrow \sin\frac{y}{x}=\ln|x|+c$$

Besides, $y(1)=0\Rightarrow \sin 0=\ln 1+c\Rightarrow c=0\Rightarrow \sin\frac{y}{x}=\ln|x|\Rightarrow x=e^{\sin\frac{y}{x}}$

$b)\frac{dy}{dx}=2e^{\frac{-y}{x}}+\frac{y}{x}\Rightarrow x\frac{dv}{dx}=2e^{-v}\Rightarrow e^{v}dv=\frac{2}{x}dx\Rightarrow\int e^{v}dv=\int\frac{2}{x}dx\Rightarrow e^{\frac{y}{x}}=\ln x^2+c$

Besides, $y(1)=0\Rightarrow e^{0}=\ln 1+c\Rightarrow c=1\Rightarrow e^{\frac{y}{x}}=\ln x^2+1$

$c)\ 6y^2dy=(\frac{6y^3}{x}+x^2e^{-\frac{y^3}{x^3}})dx\Rightarrow 6\frac{dy}{dx}=\frac{6y}{x}+\frac{x^2}{y^2}e^{-\frac{y^3}{x^3}}\Rightarrow 6(v+x\frac{dv}{dx})=6v+\frac{1}{v^2}e^{-v^3}$

$$\Rightarrow 6v^2e^{v^3}dv=\frac{1}{x}dx\Rightarrow\int 6v^2e^{v^3}dv=\int\frac{1}{x}dx\Rightarrow 2e^{v^3}=\ln|x|+c\Rightarrow 2e^{\frac{y^3}{x^3}}=\ln|x|+c$$

$d)\ \frac{dy}{dx}=\frac{y}{x}+\tan(\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v+\tan v\Rightarrow x\frac{dv}{dx}=\tan v\Rightarrow\frac{1}{\tan v}dv=\frac{1}{x}dx$

$$\Rightarrow\cot v dv=\frac{1}{x}dx\Rightarrow\int\cot v dv=\int\frac{1}{x}dx\Rightarrow\int\frac{\cos v}{\sin v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow\ln|\sin v|=\ln|x|+c\Rightarrow\ln\left|\sin\frac{y}{x}\right|=\ln|x|+c \text{ or } \sin\frac{y}{x}=cx$$

$e)\ \frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}\Rightarrow\int(1+\frac{2}{v-1})dv=-\int\frac{1}{x}dx\Rightarrow\frac{y}{x}+\ln\left|\frac{(y-x)^2}{x}\right|=c$

$f)\ \int\sec v dv=\int\frac{1}{x}dx\Rightarrow\ln|\sec v+\tan v|=\ln|x|+c\Rightarrow\ln\left|\sec\frac{y}{x}+\tan\frac{y}{x}\right|=\ln|x|+c$

1.5.3 Exact Differential Equations: Method of Exactness

Definition: A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact differential equation if there exists a function $u(x,y)$ such that $u_x(x,y) = M(x,y)$ & $u_y(x,y) = N(x,y)$.

The function $u(x,y)$ with such properties is called *potential function.*

But determining exactness by finding the potential function $u(x,y)$ is a difficult task. So, we have the following test for exactness.

Test for exactness: Any DE of the form $M(x,y)dx + N(x,y)dy = 0$ is exact if and only if $M_y = N_x$ or $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial X}$.

Solving exact DE: Method of Exactness

Suppose $M(x,y)dx + N(x,y)dy = 0$ is exact. If $u(x,y)$ is its potential function, then, the general solution is $u(x,y) = c$ where c is arbitrary constant.

This means that once the DE is exact its general solution is obtained from the knowledge of the potential function $u(x,y)$ that satisfies the definition.

So, the basic task is how to determine the potential function.

Procedures to find the general solution of exact DEs:

Step-1: Integrate $u_x(x,y) = M(x,y)$ w.r.t x to get $u(x,y) = \int M(x,y)dx + g(y)$

Here, assume y as constant and g(y) as constant of integration.

Step-2: To find g(y), equate $u_y(x,y) = N(x,y)$ from step-1.

That is $u_y(x,y) = \dfrac{\partial}{\partial y}\int M(x,y)dx + \dfrac{d}{dy}[g(y)] = N(x,y)$

Step-3: Integrate the result in step-2 with respect to y to obtain $g(y)$.

Therefore, the general solution is $u(x,y) = \int M(x,y)dx + g(y) = c$ where $g(y)$ is to be substituted by the result in step-3.

Short-Cut Formula:

By collecting and rearranging the results from **step-1** to **step-3** in the above procedures, we get the following short-cut formula for the potential function.

Potential function: $u(x,y) = \int M dx + \int [N - \int M_y dx] dy$.

1. Verify that the following DEs are exact and solve them.

a) $(y^3e^x + 2\cos x)dx + (3y^2e^x - 4y)dy = 0$

b) $3x^2ydx + (6y + x^3)dy = 0$

c) $(5x + 4y)dx + (4x - 8y^3)dy = 0$

d) $(2y^2x - 3)dx + (2yx^2 + 4)dy = 0$

e) $-2xy\sin(x^2)dx + \cos(x^2)dy = 0$

f) $ye^{xy}dx + (2y + xe^{xy})dy = 0$

g) $(e^x \sin y + 3x^2)dx + e^x \cos ydy = 0$

h) $2x\sin ydx + x^2 \cos ydy = 0$

Solution: For your understanding, first let's follow the procedures and then use the short-cut formula. Please compare the results in each problem.

a) $M = y^3e^x + 2\cos x, N = 3y^2e^x - 4y \Rightarrow M_y = 3y^2e^x, N_x = 3y^2e^x$.

Since $M_y = N_x = 3y^2e^x$, the differential equation is exact.

To clarify the above steps, let's solve step by step only this part.

Step-1: $u_x(x,y) = y^3e^x + 2\cos x \Rightarrow u(x,y) = y^3e^x + 2\sin x + g(y)$

Step-2: Equate $u_y(x,y) = N(x,y)$.

From step-1, $u(x,y) = y^3e^x + 2\sin x + g(y) \Rightarrow u_y(x,y) = 3y^2e^x + g'(y)$.

Then, $u_y(x,y) = N(x,y) \Rightarrow 3y^2e^x + g'(y) = 3y^2e^x - 4y \Rightarrow g'(y) = -4y$.

Step-3: Integrate the result in step-2 with respect toy.

That is $g'(y) = -4y \Rightarrow \int g'(y)dy = \int -4ydy \Rightarrow g(y) = -2y^2$.

So, using $g(y) = -2y^2$, the potential function is $u(x,y) = y^3e^x + 2\sin x - 2y^2$.

Using the short-cut formula: Here, $\mathbf{M} = e^xy^3 + 2\cos x$, $\mathbf{N} = 3y^2e^x - 4y$.

$$u(x,y) = \int \mathbf{M}dx + \int[\mathbf{N} - \int \mathbf{M}_y dx]dy$$

$$= \int(e^xy^3 + 2\cos x)dx + \int[3y^2e^x - 4y - \int 3y^2e^x dx]dy$$

$$= e^xy^3 + 2\sin x + \int(3y^2e^x - 4y - 3y^2e^x)dy$$

$$= e^xy^3 + 2\sin x + \int -4ydy = e^xy^3 + 2\sin x - 2y^2$$

Therefore, the general solution is $e^xy^3 + 2\sin x - 2y^2 = c$.

b) Here, $M(x,y) = 3x^2y,\ N = 6y + x^3 \Rightarrow M_y = 3x^2 = N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=3x^2y, \mathbf{N}=6y+x^3$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int 3x^2y dx+\int[6y+x^3-\int 3x^2dx]dy=x^3y+3y^2$$

Therefore, the general solution is $u(x,y)=x^3y+3y^2=c$.

c) $M=5x+4y, M_y=4,\ N=4x-8y^3,\ N_x=4 \Rightarrow M_y=N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=5x+4y, \mathbf{N}=4x-8y^3$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int(5x+4y)dx+\int[4x-8y^3-\int 4dx]dy=x^3y+3y^2$$

$$=\frac{5}{2}x^2+4xy+\int -8y^3dy=\frac{5}{2}x^2+4xy-2y^4$$

Therefore, the general solution is $\frac{5x^2}{2}+4xy-2y^4=c$.

d) Here, $M=2y^2x-3, N=2yx^2+4 \Rightarrow M_y=4xy=N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=2y^2x-3, \mathbf{N}=2yx^2+4$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int(2y^2x-3)dx+\int[2yx^2+4-\int 4xydx]dy=x^2y^2-3x+4y$$

Hence, $u(x,y)=y^2x^2-3x+4y=c$ is the general solution.

e) $M=-2xy\sin(x^2), M_y=-2x\sin(x^2),\ N=\cos(x^2), N_x=-2x\sin(x^2)$

Using the short-cut formula: Here, $\mathbf{M}=-2xy\sin(x^2), \mathbf{N}=\cos(x^2)$. Then

$$u(x,y)=\int -2xy\sin(x^2)dx+\int[\cos(x^2)-\int -2x\sin(x^2)dx]dy=y\cos(x^2)$$

Hence, $u(x,y)=y\cos(x^2)=c$ is the general solution.

f) $M_y=e^{xy}+xye^{xy},\ N_x=e^{xy}+xye^{xy} \Rightarrow M_y=N_x$

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathrm{M} = ye^{xy}, \mathrm{N} = 2y + xe^{xy}$ .Then

$$u(x,y) = \int \mathrm{M}dx + \int [\mathrm{N} - \int \mathrm{M}_y dx]dy$$

$$= \int ye^{xy}dx + \int [2y + xe^{xy} - \int (e^{xy} + xye^{xy})dx]dy$$

$$= x + e^{xy} + \int 2ydy = x + e^{xy} + y^2$$

Hence, the general solution is $u(x,y) = x + y^2 + e^{xy} = c$.

g) $M_y = e^x \cos y,\ N_x = e^x \cos y \Rightarrow M_y = N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathrm{M} = e^x \sin y + 3x^2, \mathrm{N} = e^x \cos y$ .Then

$$u(x,y) = \int \mathrm{M}dx + \int [\mathrm{N} - \int \mathrm{M}_y dx]dy$$

$$= \int (e^x \sin y + 3x^2)dx + \int [e^x \cos y - \int e^x \cos y dx]dy = e^x \sin y + x^3$$

Thus, the solution is $e^x \sin y + x^3 = c$.

h) $M = 2x\sin y,\ N = x^2 \cos y \Rightarrow M_y = 2x\cos y,\ N_x = 2x\cos y \Rightarrow M_y = N_x$.

Using the short-cut formula: Here, $\mathrm{M} = 2x\sin y, \mathrm{N} = x^2 \cos y$ .Then

$$u(x,y) = \int \mathrm{M}dx + \int [\mathrm{N} - \int \mathrm{M}_y dx]dy$$

$$= \int 2x\sin ydx + \int [x^2 \cos y - \int 2x\cos ydxdy = x^2 \sin y$$

Therefore, the general solution is $U(x,y) = c \Rightarrow x^2 \sin y = c$.

2. Check the exactness and solve the following DEs

a) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0, y(0) = 3$

b) $(e^y - ye^x)dx + (xe^y - e^x)dy = 0, y(3) = 0$

c) $(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0, y(2) = 0$

d) $(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2y\cos x)dy = 0,\ y(0) = \pi$

e) $2xydx + (x^2 + \cos y)dy = 0$

f) $(2xy + 3x^2)dx + x^2dy = 0$

Solution:

a) $M = x^2 - 4xy - 2y^2, N = y^2 - 4xy - 2x^2 \Rightarrow M_y = -4x - 4y = N_x$

The DE is exact.

Using the short-cut formula:

$$u(x,y)=\int \mathrm{M}dx+\int[\mathrm{N}-\int \mathrm{M}_y dx]dy$$

$$=\int(x^2-4xy-2y^2)dx+\int[y^2-4xy-2x^2-\int(-4x-4y)dx]dy$$

$$=\frac{x^3}{3}-2x^2y-2xy^2+\int y^2dy=\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}$$

The general solution is $\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}=c\Rightarrow x^3-6x^2y-6xy^2+y^3=c$

But $y(0)=3\Rightarrow c=27\Rightarrow x^3-6x^2y-6xy^2+y^3=27$

b) $M=e^y-ye^x,\ N=xe^y-e^x\Rightarrow M_y=e^y-e^x=N_x$

The DE is exact. So, by themethod of exactness,

$$u(x,y)=\int(e^y-ye^x)dx+\int[xe^y-e^x-\int(e^y-e^x)dx]dy=xe^y-ye^x$$

The general solution is $xe^y-ye^x=c$. But $y(3)=0\Rightarrow c=3\Rightarrow xe^y-ye^x=3$

c) $M_y=\cos x,\ N_x=\cos x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Thus, the general solution is $x^3-y^4+y\sin x=c$.

Besides, $y(2)=0\Rightarrow c=8\Rightarrow x^3-y^4+y\sin x=8$.

d) $M_y=e^x\cos y-2\cos x,\ N_x=e^x\cos y-2\cos x\Rightarrow M_y=N_x$

Thus, the general solution is $u(x,y)=e^x\sin y+2y\cos x=c$

Besides, $y(0)=\pi\Rightarrow c=2\pi\Rightarrow e^x\sin y+2y\cos x=2\pi$.

f) $M_y=2x,\ N_x=2x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Using the short-cut formula: Here, $\mathrm{M}=2xy, \mathrm{N}=x^2+\cos y$. Then

$$u(x,y)=\int 2xydx+\int[x^2+\cos y-\int 2xdx]dy=x^2y+\sin y$$

Hence, the general solution is $x^2y+\sin y=c$.

g) $M_y=2x,\ N_x=2x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Using the short-cut formula: Here, $\mathrm{M}=2xy+3x^2, \mathrm{N}=x^2$.

Then $u(x,y)=\int(2xy+3x^2)dx+\int[x^2-\int 2xdx]dy=x^2y+x^3$

Hence, the general solution is $x^2y+x^3=c$.

1.6 Non-Exact ODEs: Method of Integrating Factors

So far, we discussed how to solve DEs when they are separable, homogeneous or exact. However, there are many situations that do not fit to either of such cases. So, our next discussion focuses on how to solve DEs that are neither of the above forms. Consider the non-exact DE $P(x,y)dx+Q(x,y)dy=0$. Now, multiply this equation by a nonzero function, say μ (it will be a function of x, y, or both) such that the resulting equation $\mu Pdx+\mu Qdy=0$ is exact.

The function μ which is used to change the non-exact differential equation $P(x,y)dx+Q(x,y)dy=0$ into an equivalent exact DE of $\mu Pdx+\mu Qdy=0$ is known as *Integrating Factor.* For example, the equation $2ydx+xdy=0$ is not exact but if we multiply it by $\mu(x)=-x$, it becomes $-2xydx-x^2dy=0$ such that $M(x,y)=-2xy,\ N(x,y)=-x^2 \Rightarrow M_y=-2x=N_x$ which is exact and its solution is obtained easily. But, here the main problem is how to choose or select the function μ which is used as multiplier to change the non-exact DEs into exact DE. Even though there is no hard and fast rule on how to find integrating factor, any way let's see the general procedure to find such function. From the condition of exactness, the DE $\mu Pdx+\mu Qdy=0$ will be exact if and only if $\frac{\partial}{\partial y}(\mu P)=\frac{\partial}{\partial x}(\mu Q)$. Then, by product rule, $\mu_y P+\mu P_y=\mu_x Q+\mu Q_x$.

Now, solving this equation for μ is too complicated. So, to simplify the complication let's consider different cases for μ.

Case-1: Suppose μ is a function of x only. Then, using the relation between partial and ordinary derivatives, we have

$$\mu_y P+\mu P_y=\mu_x Q+\mu Q_x \Rightarrow \mu P_y=\frac{d\mu}{dx}Q+\mu Q_x,\quad (\because \mu_y=0,\ \mu_x=\frac{d\mu}{dx})$$

Again, dividing this result by μQ gives

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{P_y-Q_x}{Q}\Rightarrow\frac{\mu'}{\mu}=\frac{P_y-Q_x}{Q}\quad (\because \frac{d\mu}{dx}=\mu')$$

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The function μ which is used to change the non-exact differential equation $P(x,y)dx+Q(x,y)dy=0$ into an equivalent exact DE of $\mu Pdx+\mu Qdy=0$ is known as *Integrating Factor*. For example, the equation $2ydx+xdy=0$ is not exact but if we multiply it by $\mu(x)=-x$, it becomes $-2xydx-x^2dy=0$ such that $M(x,y)=-2xy,\ N(x,y)=-x^2 \Rightarrow M_y=-2x=N_x$ which is exact and its solution is obtained easily. But, here the main problem is how to choose or select the function μ which is used as multiplier to change the non-exact DEs into exact DE. Even though there is no hard and fast rule on how to find integrating factor, any way let's see the general procedure to find such function. From the condition of exactness, the DE $\mu Pdx+\mu Qdy=0$ will be exact if and only if $\frac{\partial}{\partial y}(\mu P)=\frac{\partial}{\partial x}(\mu Q)$. Then, by product rule, $\mu_y P+\mu P_y=\mu_x Q+\mu Q_x$.

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Again, dividing this result by μQ gives

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{P_y-Q_x}{Q} \Rightarrow \frac{\mu'}{\mu}=\frac{P_y-Q_x}{Q} \quad (\because \frac{d\mu}{dx}=\mu')$$

Now, integrating this equation with respect to x gives

$$\int \frac{\mu'}{\mu} dx = \int \frac{P_y - Q_x}{Q} dx \Rightarrow \mu(x) = e^{\int f(x)dx}, \text{ where } f(x) = \frac{P_y - Q_x}{Q}$$

From this explanation, we can state the following theorem

Theorem: [Integrating Factor of the form $\mu(x)$]:

If $P(x,y)dx + Q(x,y)dy = 0$ is non-exact such that $\mu Pdx + \mu Qdy = 0$ is exact and $\frac{P_y - Q_x}{Q}$ depends only on x, then the integrating factor is $\mu(x) = e^{\int \frac{P_y - Q_x}{Q}dx}$.

Case-2: Suppose μ is a function of y only.

Theorem: [Integrating Factor of the form $\mu(y)$]:

If $P(x,y)dx + Q(x,y)dy = 0$ is non-exact such that $\mu Pdx + \mu Qdy = 0$ is exact and $\frac{Q_x - P_y}{P}$ depends only on y, then the integrating factor is $\mu(y) = e^{\int \frac{Q_x - P_y}{P}dy}$.

Examples:

1. Verify that the following DEs are not exact and solve by finding the appropriate Integrating Factor:

$a)\ (8x^5 + 3y^4)dx + 4xy^3dy = 0$ $\qquad b)\ 6xydx + (4y + 9x^2)dy = 0$

$c)\ (x^4 + xy)dx + (x^2 + xy)dy = 0$ $\qquad d)\ (x^2y^2 + y)dx + (y^2 - x)dy = 0$

$e)\ (x^4 + y^2)dx - xydy = 0$ $\qquad f)\ ydx + (3x - y + 3)dy = 0$

$g)\ (4xy + 2y^2)dx + (2xy + 5y^4)dy = 0$ $\qquad h)(2x^2 + y)dx + (x^2y - x)dy = 0$

Solution:

a) Here, $P = 8x^5 + 3y^4, Q = 4xy^3 \Rightarrow P_y = 12y^3, Q_x = 4y^3$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

So, to solve this DE, first find the integrating factor that changes it into exact.

But, $\frac{P_y - Q_x}{Q} = \frac{12y^3 - 4y^3}{4xy^3} = \frac{8y^3}{4xy^3} = \frac{2}{x} = f(x)$ which depends only on x.

Hence, the integrating factor is $\mu(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Now, multiply by $\mu(x) = x^2$ to get the new exact DE.

Hence, the new exact DE is $(8x^7 + 3x^2y^4)dx + 4x^3y^3dy = 0$.

Next, solve by method of exactness. Let $M = 8x^7 + 3x^2y^4$, $N = 4x^3y^3$.

Then integrating $u_x(x, y) = M(x, y)$ and integrate with respect to x.

That is $\int u_x(x, y)dx = \int (8x^7 + 3x^2y^4)dx \Rightarrow u(x, y) = x^8 + x^3y^4 + g(y)$.

Differentiating $u(x,y) = x^8 + x^3y^4 + g(y)$ with respect to y and equating with N gives $u_y(x,y) = 4x^3y^3 + g'(y) = 4x^3y^3 \Rightarrow g'(y) = 0 \Rightarrow g(y) = k$.

Therefore, the solution is $u(x,y) = c \Rightarrow x^8 + x^3y^4 = c$.

b) $P = 6xy, Q = 4y + 9x^2 \Rightarrow P_y = 6x \neq 18x = Q_x$. This means it is not exact.

Besides, $\dfrac{Q_x - P_y}{P} = \dfrac{18x - 6x}{6xy} = \dfrac{2}{y} = f(y) \Rightarrow \mu(y) = e^{\int \frac{2}{y}dy} = y^2$

Hence, the exact DE is $6xy^3dx + (4y^3 + 9x^2y^2)dy = 0$.

Next, solve by method of exactness. Let $M = 6xy^3$, $N = 4y^3 + 9x^2x^2$.

Then integrating $u_x(x, y) = M(x, y)$ and integrate with respect to x.

That is $\int u_x(x, y)dx = \int 6xy^3dx \Rightarrow u(x, y) = 3x^2y^3 + g(y)$.

Differentiating $u(x,y) = 3x^2y^3 + g(y)$ with respect to y and equating with N gives $u_y(x,y) = 9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$.

Integrating with respect to y gives $\int g'(y)dy = \int 4y^3dy \Rightarrow g(y) = y^4$.

Therefore, the general solution is $3x^2y^3 + y^4 = c.$

c) $M_y = x, N_x = 2x + y \Rightarrow M_y \neq N_x$. Hence, the equation is not exact.

Besides, $\dfrac{M_y - N_x}{N} = \dfrac{x - 2x - y}{x^2 + xy} = \dfrac{-(x+y)}{x(x+y)} = \dfrac{-1}{x}$. Hence, the integrating factor is $\mu(x) = e^{\int \frac{-1}{x}dx} = \dfrac{1}{x}$ and the new exact DE is $(x^3 + y)dx + (x + y)dy = 0$.

Next, solve by method of exactness. Let $M = x^3 + y$, $N = x + y$.

Then integrating $u_x(x, y) = M(x, y)$ and integrate with respect to x.

That is $\int u_x(x, y)dx = \int (x^3 + y)dx \Rightarrow u(x, y) = \dfrac{x^4}{4} + xy + g(y)$.

Differentiating $u(x,y)=\frac{x^4}{4}+xy+g(y)$ with respect to y and equating with N gives $u_y(x,y)=x+g'(y)=x+y \Rightarrow g'(y)=y$.

Integrating $g'(y)=y$ with respect to y gives us $g(y)=\frac{y^2}{2}$.

Therefore, the general solution is $\frac{x^4}{4}+xy+\frac{y^2}{2}=c$.

d) Here, $P=x^2y^2+y, Q=y^2-x \Rightarrow P_y=2xy^2+1, Q_x=-1$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

To solve this DE, first change it into exact using integrating factor.

But, $\frac{Q_x-P_y}{P}=\frac{-1-(2xy^2+1)}{x^2y^2+y}=\frac{-2(1+xy^2)}{y(x^2y+1)}=-\frac{2}{y}=f(y)$ depends on y.

Hence, $\mu(y)=e^{\int f(y)dy}=e^{\int -\frac{2}{y}dy}=e^{-2\ln y}=e^{\ln(\frac{1}{y^2})}=\frac{1}{y^2}$.

Now, multiply by $\mu(y)=\frac{1}{y^2}$ to get the new exact DE.

Hence, the new exact DE is $(x^2+\frac{1}{y})dx+(1-\frac{x}{y^2})dy=0$.

Next, solve by method of exactness. Let $M=x^2+\frac{1}{y}, N=1-\frac{x}{y^2}$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(x^2+\frac{1}{y})dx \Rightarrow u(x,y)=\frac{x^3}{3}+\frac{x}{y}+g(y)$.

Differentiating $u(x,y)=\frac{x^3}{3}+\frac{x}{y}+g(y)$ with respect to y and equating with N gives $u_y(x,y)=-\frac{x}{y^2}+g'(y)=1-\frac{x}{y^2} \Rightarrow g'(y)=1$.

Integrating $g'(y)=1$ with respect to y gives us $g(y)=y$.

Therefore, the general solution is $u(x,y)=c \Rightarrow \frac{x^3}{3}+\frac{x}{y}+y=c$.

e) $P = x^4 + y^2, Q = -xy \Rightarrow P_y = 2y \neq -y = Q_x$. This means it is not exact.

Besides, $\frac{P_y - Q_x}{Q} = -\frac{3}{x} = f(x) \Rightarrow \mu(x) = e^{\int \frac{-3}{x} dx} = \frac{1}{x^3}$

Hence, the exact DE is $(x + \frac{y^2}{x^3})dx - \frac{y}{x^2}dy = 0$ and thus using the method of exactness, we get the solution to be $x^2 - \frac{y^2}{x^2} = c$

f) $P = y, Q = 3x - y + 3 \Rightarrow P_y = 1 \neq 3 = Q_x$. This means it is not exact.

Besides, $\frac{Q_x - P_y}{P} = \frac{2}{y} = f(y) \Rightarrow \mu(y) = e^{\int \frac{2}{y} dy} = y^2$

Hence, the exact DE is $y^3 dx + (3xy^2 - y^3 + 3y^2)dy = 0$.

Next, solve by method of exactness. Let $M = y^3$, $N = 3xy^2 - y^3 + 3y^2$.

Then integrating $u_x(x, y) = M(x, y)$ and integrate with respect to x.

That is $\int u_x(x, y)dx = \int y^3 dx \Rightarrow u(x, y) = xy^3 + g(y)$.

Differentiating $u(x, y) = xy^3 + g(y)$ with respect to y and equating with N gives $u_y(x, y) = 3xy^2 + g'(y) = 3xy^2 - y^3 + 3y^2 \Rightarrow g'(y) = -y^3 + 3y^2$.

Integrate $g'(y) = -2y^3 + 3y^2$ with respect to y

That is $\int g'(y)dy = \int(-y^3 + 3y^2)dy \Rightarrow g(y) = -\frac{y^4}{4} + y^3$.

Therefore, the general solution is $xy^3 - \frac{y^4}{4} + y^3 = c$.

g) Here, $P = 4xy + 2y^2, Q = 2xy + 5y^4 \Rightarrow P_y = 4x + 4y, Q_x = 2y$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

But, $\frac{Q_x - P_y}{P} = \frac{2y - (4x + 4y)}{4xy + 2y^2} = \frac{-(4x + 2y)}{y(4x + 2y)} = -\frac{1}{y} = f(y)$ depends on y.

Hence, the integrating factor is $\mu(y) = e^{\int f(y)dy} = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = e^{\ln(\frac{1}{y})} = \frac{1}{y}$.

Hence, the new exact DE is $(4x + 2y)dx + (2x + 5y^3)dy = 0$.

Next, solve by method of exactness. Let $M=4x+2y$, $N=2x+5y^3$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(4x+2y)dx \Rightarrow u(x,y)=2x^2+2xy+g(y)$.

Differentiating $u(x,y)=2x^2+2xy+g(y)$ with respect to y and equating with N gives $u_y(x,y)=2x+g'(y)=2x+5y^3 \Rightarrow g'(y)=5y^3$.

Integrating $g'(y)=5y^3$ with respect to y gives us $g(y)=\frac{5y^4}{4}$.

Therefore, the general solution is $u(x,y)=c \Rightarrow 2x^2+2xy+\frac{5y^4}{4}=c$.

h) $P=2x^2+y, Q=x^2y-x \Rightarrow P_y=1 \neq 2xy-1=Q_x$. This means it is not exact.

Besides, $\frac{P_y-Q_x}{Q}=\frac{1-(2xy-1)}{x^2y-x}=\frac{-2(xy-1)}{x(xy-1)}=-\frac{2}{x}=f(x) \Rightarrow \mu(x)=e^{\int -\frac{2}{x}dy}=\frac{1}{x^2}$

Hence, the new exact DE is $(2+\frac{y}{x^2})dx+(y-\frac{1}{x})dy=0$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(2+\frac{y}{x^2})dx \Rightarrow u(x,y)=2x-\frac{y}{x}+g(y)$ and

$u_y(x,y)=-\frac{1}{x}+g'(y)=y-\frac{1}{x} \Rightarrow g'(y)=y \Rightarrow g(y)=\frac{y^2}{2}$.

Therefore, the general solution is $2x-\frac{y}{x}+\frac{y^2}{2}=c$.

2. Verify whether the following DEs are not exact and solve by finding the appropriate Integrating Factor:

a) $(y^2+2xy)dx+(4x^2+5xy+6)dy=0$ b) $ydx+(2xy-e^{-2y})dy=0$

c) $3x^2y^2dx+(2x^3y+x^3y^4)dy=0$ d) $2xydx+y^2dy=0$

e) $(y-e^{x+y})dx-(1+xe^{x+y})dy=0$ f) $(3x^2y-x^2)dx+dy=0$

g) $(y+x^4)dx-xdy=0$ h) $(1+y)dx+(1-x)dy=0$

Solution:

a) $P_y=2y+2x, Q_x=8x+5y \Rightarrow P_y \neq Q_x$. Hence, the equation is not exact.

Besides, $\dfrac{Q_x - P_y}{P} = \dfrac{8x+5y-(2y+2x)}{y^2+2xy} = \dfrac{3(2x+y)}{y(y+2x)} = \dfrac{3}{y} = f(y)$.

Hence, the integrating factor is $\mu(y) = e^{\int \frac{3}{y}dy} = e^{3\ln y} = e^{\ln y^3} = y^3$.

Then, the new exact DE is $(y^5 + 2xy^4)dx + (4x^2y^3 + 5xy^4 + 6y^3)dy = 0$.

Here, $M_y = 5y^4 + 8xy^3$, $N_x = 8xy^3 + 5y^4 \Rightarrow M_y = N_x$. So, it is exact.

Thus, the general solution is obtained as follow:

Using the short-cut formula:

$$
\begin{aligned}
u(x,y) &= \int \mathrm{M}dx + \int [\mathrm{N} - \int \mathrm{M}_y dx]dy \\
&= \int (y^5 + 2xy^4)dx + \int [4x^2y^3 + 5xy^4 + 6y^3 - \int (5y^4 + 8xy^3)dx]dy \\
&= xy^5 + x^2y^4 + \int 6y^3 dy = xy^5 + x^2y^4 + \frac{3}{2}y^4
\end{aligned}
$$

Therefore, the general solution is $u(x,y) = xy^5 + x^2y^4 + \frac{3}{2}y^4 = c$.

b) $M_y = 1$, $N_x = 2y \Rightarrow M_y \neq N_x$. Hence, the equation is not exact. Besides, $\dfrac{N_x - M_y}{M} = \dfrac{2y-1}{y} = 2 - \dfrac{1}{y} = f(y)$. Hence, the integrating factor is

$\mu(y) = e^{\int\left(2-\frac{1}{y}\right)dy} = e^{2y - \ln|y|} = e^{2y}.e^{\ln\left|\frac{1}{y}\right|} = \dfrac{e^{2y}}{y}$ and the new exact DE is

$e^{2y}dx + (2xe^{2y} - \frac{1}{y})dy = 0$. Thus, the solution is

$u_x(x,y) = M(x,y) = e^{2y} \Rightarrow u(x,y) = xe^{2y} + g(y)$

$u_y(x,y) = N(x,y) \Rightarrow 2xe^{2y} + g'(y) = 2xe^{2y} - \dfrac{1}{y} \Rightarrow g(y) = -\ln|y|$

Thus, the solution is $xe^{2y} - \ln|y| = c$.

c) $M_y = 6x^2y$, $N_x = 6x^2y + 3x^2y^4 \Rightarrow M_y \neq N_x$. Hence, the equation is not exact. Besides, $\dfrac{N_x - M_y}{M} = \dfrac{6x^2y + 3x^2y^4 - 6x^2y}{3x^2y^2} = y^2$.

Hence, the integrating factor is $\mu(y)=e^{\int y^2dy}=e^{\frac{y^3}{3}}$ and the new exact DE is

$(3x^2y^2)e^{\frac{y^3}{3}}dx+(2x^3y+x^3y^4)e^{\frac{y^3}{3}}dy=0$.

$u_x(x,y)=3x^2y^2e^{\frac{y^3}{3}}\Rightarrow u(x,y)=x^3y^2e^{\frac{y^3}{3}}+g(y)$,

$u_y(x,y)=(2x^3y+x^3y^4)e^{\frac{y^3}{3}}+g'(y)=(2x^3y+x^3y^4)e^{\frac{y^3}{3}}\Rightarrow g(y)=k$

d) $M_y=2x,\ N_x=0\Rightarrow M_y\neq N_x$. Hence, the equation is not exact. Besides,

$\frac{N_x-M_y}{M}=\frac{0-2x}{2xy}=-\frac{1}{y}=f(y)$. Hence, the integrating factor is

$\mu(y)=e^{\int\left(-\frac{1}{y}\right)dy}=e^{-\ln|y|}=\frac{1}{y}$ and the new exact DE is $2xdx+ydy=0$.

Thus, the solution is $x^2+y^2/2=c$.

e) Since $P_y=1-e^{x+y}\neq-(1+x)e^{x+y}=Q_x$, the DE is not exact. But,

$\frac{P_y-Q_x}{Q}=\frac{1-e^{x+y}+(1+x)e^{x+y}}{-(1+xe^{x+y})}=-1$. Hence, $\mu(x)=e^{\int f(x)dx}=e^{-x}$ and the new exact DE is $(ye^{-x}-e^y)dx-(xe^y+e^{-x})dy=0$.

Here, let $M=ye^{-x}-e^y,\ N=-(xe^y+e^{-x})$. Therefore, integrating $u_x(x,y)=M(x,y)$ with respect to x gives $u(x,y)=-xe^y-ye^{-x}+h(y)$. Then, differentiating $u(x,y)=-xe^y-ye^{-x}+h(y)$ w.r.t y and equating with N gives

$u_y(x,y)=-xe^y-e^{-x}+h'(y)=-(xe^y+e^{-x})\Rightarrow h'(y)=0\Rightarrow h(y)=c$

Therefore, the solution is $u(x,y)=c\Rightarrow xe^y+ye^{-x}=c$

f) Since $P_y=3x^2\neq 0=Q_x$, the DE is not exact.

But, $\frac{P_y-Q_x}{Q}=3x^2=f(x)$. Hence, the integrating factor is $\mu(x)=e^{\int f(x)dx}=e^{x^3}$

and the new exact DE is $(3x^2y-x^2)e^{x^3}dx+e^{x^3}dy=0$.

Now, let $M(x,y)=(3x^2y-x^2)e^{x^3},\ N(x,y)=e^{x^3}$.

Method of Integrating Factor for First Order Linear DEs:

Suppose $y'+p(x)y=f(x)$ is first order linear DE.

Then, $\frac{dy}{dx}+p(x)y=f(x)\Rightarrow[p(x)y-f(x)]dx+dy=0$.

Here, $P(x,y)=p(x)y-f(x),\quad Q(x,y)=1\Rightarrow P_y=p(x),\ Q_x=0\Rightarrow P_y\neq Q_x$

Thus, the DE is not exact. Besides, $\frac{P_y-Q_x}{Q}=p(x)$ is the function of x.

Hence, the integrating factor is $\mu(x)=e^{\int p(x)dx}$.

Furthermore, multiplying the DE with this integrating factor gives

$e^{\int p(x)dx}[p(x)y-f(x)]dx+e^{\int p(x)dx}dy=0$ which is exact.

Then, the general solution becomes $y(x)=e^{-\int p(x)dx}[\int e^{\int p(x)dx}f(x)dx+C]$.

Examples:

1. Find the general solution of the following first order linear DEs

a) $xy'-2y=x^3e^x$ b) $xy'+y=\frac{1}{x}$ c) $y'+y=\sin x$

d) $y'+6x^2y=x^2$ e) $(2y-3x)dx+xdy=0$ f) $xy'+y=x^2+1$

Solution: First change in the standard form $y'+p(x)y=f(x)$

a) $y'-\frac{2}{x}y=x^2e^x\Rightarrow\mu(x)=e^{\int\frac{-2}{x}dx}=\frac{1}{x^2}\Rightarrow y(x)=x^2\left(\int e^xdx+c\right)=x^2e^x+cx^2$

b) $xy'+y=\frac{1}{x}\Rightarrow y'+\frac{1}{x}y=\frac{1}{x^2}\Rightarrow\mu=e^{\int\frac{1}{x}dx}=x\Rightarrow y=\frac{1}{x}\int\frac{1}{x}dx+\frac{c}{x}=\frac{\ln|x|}{x}+\frac{c}{x}$

c) $\mu(x)=e^{\int xdx}=e^x\Rightarrow y=e^{-x}\int e^x\sin xdx+Ce^{-x}\Rightarrow y=\frac{1}{2}(\sin x-\cos x)+Ce^{-x}$

d) $\mu(x)=e^{\int 6x^2dx}=e^{2x^3}\Rightarrow y=e^{-2x^3}\int x^2e^{2x^3}dx+ce^{-2x^3}=ce^{-2x^3}+\frac{1}{6}$

e) $(2y-3x)dx+xdy=0\Rightarrow\frac{dy}{dx}+\frac{2}{x}y=3\Rightarrow\mu=e^{\int\frac{2}{x}dx}=x^2\Rightarrow y=x+\frac{c}{x^2}$

2. Solve the following IVPs.

a) $y'+5y=3e^{x}-1,\ \ y(0)=1$ b) $y'+y\tan x=\sin 2x,\ \ y(0)=1$

c) $xe^{x^2}dx+(y^3-1)dy=0,\ \ y(0)=0$ d) $xy'+2y=4x^2,\ \ y(1)=3$

Solution: Use integrating factor method for first order linear DEs

a) The integrating factor is $\mu(x)=e^{\int 5dx}=e^{5x}$.Then, the general solution is

$$y(x)=e^{-5x}[\int e^{5x}(3e^{x}-1)dx+c]=e^{-5x}\left(\frac{1}{2}e^{6x}-\frac{1}{5}e^{5x}+c\right)=\frac{1}{2}e^{x}+ce^{-5x}-\frac{1}{5}.$$

Now, find c using the initial condition. That is $y(0)=1\Rightarrow c=\frac{7}{10}$.

Hence, $y(x)=\frac{1}{2}e^{x}+\frac{7}{10}e^{-5x}-\frac{1}{5}$.

b) The integrating factor is $\mu(x)=e^{\int \tan x dx}=\frac{1}{\cos x}$.Then, the general solution is

$$y(x)=\cos x[\int\frac{1}{\cos x}\sin 2xdx+c]=\cos x(-2\cos x+c)=c\cos x-2\cos^2 x$$

Now, find c using the initial condition. That is $y(0)=1\Rightarrow c=3$.

Hence, $y(x)=3\cos x-2\cos^2 x$.

c) $xe^{x^2}dx+(y^3-1)dy=0\Rightarrow(y^3-1)dy=-xe^{x^2}dx$

$$\Rightarrow\int(y^3-1)dy=-\int xe^{x^2}dx\Rightarrow\frac{y^4}{4}-y=-\frac{e^{x^2}}{2}+c$$

Now, find c using the initial condition. That is $y(0)=0\Rightarrow c=\frac{1}{2}$.

Hence, $\frac{y^4}{4}-y=\frac{1}{2}-\frac{e^{x^2}}{2}$.

d) $xy'+2y=4x^2\Rightarrow y'+\frac{2}{x}y=4x$. Hence, $\mu(x)=e^{\int\frac{2}{x}dx}=x^2$.

Then, $y(x)=e^{-\int p(x)dx}[\int e^{\int p(x)dx}f(x)dx+C]=\frac{1}{x^2}\left(\int 4x^3dx+c\right)=x^2+\frac{c}{x^2}$.

Now, find c. That is $y(1)=3\Rightarrow c=2$. Hence, $y(x)=x^2+\frac{2}{x^2}$.

1.7 Bernoulli's Differential Equations

Definition: Differential equations of the form $y'+p(x)y = f(x)y^n$, $n \in R$ are known as **Bernoulli's Differential Equations.** If $n \neq 0,1$, the DE is nonlinear. Such non linear differential equations are transformed into linear as follow.

First, multiply both sides by $\frac{1}{y^n}$. That is $y^{-n}y'+p(x)y^{1-n} = f(x)$

Using the substitution $z = y^{1-n}$, $z'=\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx} \Rightarrow y'=\frac{z'}{(1-n)y^{-n}}$.

Putting this in the equation $y^{-n}y'+p(x)y^{1-n} = f(x)$, we have

$$y^{-n}y'+p(x)y^{1-n} = f(x) \Rightarrow y^{-n}.\frac{z'}{(1-n)y^{-n}}+p(x)y^{1-n} = f(x)$$

$$\Rightarrow \frac{z'}{1-n}+p(x)z = f(x) \Rightarrow z'+(1-n)p(x)z = (1-n)f(x)$$

Therefore, we get firs order linear DE $z'+(1-n)p(x)z=(1-n)f(x)$.

Simple steps to solve Bernoulli's Equations $y'+p(x)y = f(x)y^n$, $n \in R$

First: Identify $p(x), f(x), n$ from the given problem.

Then, use the following formula to get the general solution.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}$.

General solution: $y^{1-n}=\frac{1}{\mu(x)}\int(1-n)\mu(x)f(x)dx+\frac{c}{\mu(x)}$ where c is constant.

Examples:

1. Solve the following Bernoulli's Differential Equations.

a) $y'+\frac{y}{x}=x^2y^2; x>0$ *b)* $y'-y=xy^2, y(0)=-1$ *c)* $y'-\frac{3y}{x}=x^4y^{1/3}$

d) $2xy\frac{dy}{dx}-y^2=x^2$ *e)* $y'=y-\frac{1}{4}y^{3/2}, y(0)=\frac{1}{4}$ *f)* $y'-2y=6y^3$

g) $y'=\frac{y}{x}-y^2$ *h)* $2xy'=10x^3y^5+y$ *i)* $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$

Solution:

a) Here $p(x)=\dfrac{1}{x}$, $f(x)=x^2$, $n=2$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\dfrac{1}{x}$.

General solution:

$$y^{1-n}=\frac{1}{\mu(x)}\int(1-n)\mu(x)f(x)dx+\frac{c}{\mu(x)}$$

$$\Rightarrow y^{1-2}=x\int(1-2).\frac{1}{x}.x^2dx+cx$$

$$\Rightarrow y^{-1}=x\int-xdx+cx\Rightarrow\frac{1}{y}=-\frac{x^3}{2}+cx\Rightarrow y=\frac{2}{2cx-x^3}$$

b) $y'-y=xy^2$. Here $p(x)=-1$, $f(x)=x$, $n=2$. Then,

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int 1dx}=e^x$.

General solution:

$$y^{-1}=\frac{1}{e^x}\int-xe^xdx+\frac{c}{e^x}=-e^{-x}(xe^x-e^x)+ce^{-x}\Rightarrow y=\frac{1}{1-x+ce^{-x}}.$$

Besides, $y(0)=-1\Rightarrow\dfrac{1}{1+c}=-1\Rightarrow c=-2\Rightarrow y=\dfrac{1}{1-x-2e^{-x}}$.

c) Here $p(x)=-\dfrac{3}{x}$, $f(x)=x^4$, $n=\dfrac{1}{3}$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=\dfrac{1}{x^2}$.

General solution: $y^{2/3}=x^2\displaystyle\int\frac{2x^2}{3}dx+cx^2=\frac{2x^5}{9}+cx^2\Rightarrow y^2=(\frac{2x^5}{9}+cx^2)^3$.

d) First rearrange in Bernoulli's form to identify $p(x)$, $f(x)$, n.

That is $2xy\dfrac{dy}{dx}-y^2=x^2\Rightarrow\dfrac{dy}{dx}-\dfrac{1}{2x}y=\dfrac{x}{2}y^{-1}\Rightarrow p(x)=\dfrac{-1}{2x},f(x)=\dfrac{x}{2},n=-1$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int(2)(-\frac{1}{2x})dx}=e^{\int-\frac{1}{x}dx}=e^{-\ln x}=\dfrac{1}{x}$.

General solution: $y^2=x\displaystyle\int dx+cx=x^2+cx$.

e) $y' = y - \frac{1}{4}y^{3/2} \Rightarrow y' - y = -\frac{1}{4}y^{3/2}$. Here $p(x) = -1$, $f(x) = \frac{-1}{4}$, $n = \frac{3}{2}$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{1}{2}dx} = e^{\frac{x}{2}}$.

General solution:

$$y^{1-n} = \frac{1}{\mu(x)}\int (1-n)\mu(x)f(x)dx + \frac{c}{\mu(x)} \Rightarrow y^{-1/2} = e^{\frac{-x}{2}}\left(\int \frac{1}{8}e^{\frac{x}{2}}dx + c\right) = \frac{1}{4} + ce^{\frac{-x}{2}}.$$

f) Here $p(x) = -2$, $f(x) = 6$, $n = 3$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int 4dx} = e^{4x}$.

General solution:

$$y^{-2} = e^{-4x}\int -12e^{4x}dx + ce^{-4x} = ce^{-4x} - 3 \Rightarrow \frac{1}{y^2} = ce^{-4x} - 3 \Rightarrow y = \pm\frac{1}{\sqrt{ce^{-4x} - 3}}$$

g) Here, $y' = \frac{y}{x} - y^2 \Rightarrow y' - \frac{1}{x}y = -y^2 \Rightarrow p(x) = -\frac{1}{x}$, $f(x) = -1$, $n = 2$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{1}{x}dx} = x$.

General solution: $y^{-1} = \frac{1}{x}\int x + \frac{c}{x} \Rightarrow \frac{1}{y} = \frac{x}{2} + \frac{c}{x} = \frac{x^2 + c}{2x} \Rightarrow y = \frac{2x}{x^2 + c}$

h) Here, $2xy' = 10x^3y^5 + y \Rightarrow y' = 5x^2y^5 + \frac{1}{2x}y \Rightarrow y' - \frac{1}{2x}y = 5x^2y^5$.

Then, we have $p(x) = -\frac{1}{2x}$, $f(x) = 5x^2$, $n = 5$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{2}{x}dx} = x^2$.

General solution:

$$y^{-4} = \frac{1}{x^2}\int -20x^4dx + \frac{c}{x^2} = -4x^5 + \frac{c}{x^2} \Rightarrow \frac{1}{y^4} = -4x^5 + \frac{c}{x^2} \Rightarrow y^4 = \frac{x^2}{C - 4x^5}$$

1.8 Second Order Linear Differential Equations) with Constant Coefficients (SOLDE

Second–Order-Linear-Differential Equations with constant coefficients are equations of the form $ay''+by'+cy = f(x)$..(*)

Here, if $f(x) = 0$, then the differential equation is known as homogeneous and if $f(x) \neq 0$, it is known as non-homogeneous.

Examples:

$$\left.\begin{array}{l} \text{i) } y''-3y'-4y=0 \\ \text{ii) } 7y''+6y'+5y=0 \end{array}\right\} \text{are homogeneous differential equations.}$$

$$\left.\begin{array}{l} \text{iii) } y''+y'-2y=6x^2 \\ \text{iv) } y''-8y'-7y=2e^{3x} \end{array}\right\} \text{are non-homogeneous differential equations.}$$

1.8.1 Solutions and their Properties

Particular and Complementary solutions:

Consider second -order linear non homogeneous differential equation and the corresponding (reduced) homogeneous differential equation of the form

$$\begin{cases} ay''+by'+cy = f(x)\text{..(i)} \\ ay''+by'+cy = 0\text{..(ii)} \end{cases}$$

Then, any function free of arbitrary constants that satisfies the DE in **(i)** is said to be particular solution and denoted by y_p.

The general solution of the corresponding homogeneous DE given in **(ii)** is said to be complementary solution and denoted by y_c.

Fundamental Set of Solutions and Superposition Principle

Any set $F = \{y_1, y_2\}$ of *linearly* independent solutions of the homogenous differential equation $ay''+by'+cy = 0$ is said to be fundamental set of solutions.

Superposition Principle:

Let y_1 and y_2 be any two solutions of the equation $ay''+by'+cy=0$. Then, the linear combination $y=c_1y_1+c_2y_2y$ is also a solution.

In particular, if y_1 and y_2 are fundamental (linearly independent) solutions, then $y=c_1y_1+c_2y_2$ is its general solution and this is called complementary solution denoted by $y_c=c_1y_1+c_2y_2$.

So, once we have the fundamental solutions to $ay''+by'+cy=0$, we can easily determine its general solution from these fundamental solutions.

Here, to determine the general solution of $ay''+by'+cy=0$, it is sufficient to get any two we fundamental or linearly independent solutions y_1 and y_2.

Question:

How to check whether any two solutions are fundamental or not?

Wronskian Test and Fundamental Solutions:

Wronskian: Let f and g be differentiable functions. Then, the determinant defined by $W(f,g)=\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}=f.g'-f'.g$ is known as Wronskian of f and g.

Linearly Independent Functions: Any two functions are said to be linearly independent if and only if their Wronskian is non-zero.

Example: Verify that $f(x)=e^{-2x}$ and $g(x)=e^{5x}$ are linearly independent.

Solution: The Wronskian of f and g is $W(f,g)(x)=\begin{vmatrix} e^{-2x} & e^{5x} \\ -2e^{-2x} & 5e^{5x} \end{vmatrix}=7e^{3x}$.

Since $W(f,g)(x)=7e^{3x}\neq 0, \forall x\in R$, then f and g are linearly independent.

Wronskian Test (For Fundamental Solutions):

Any two solutions y_1 and y_2 of the equation $ay''+by'+cy=0$ are fundamental or linearly independent solutions if and only if $W(y_1,y_2)(x)\neq 0$ for all x.

1.8.2 Second Order Homogeneous Linear Differential Equations with constant coefficients (SOHLDE)

General Form: $ay''+by'+cy=0$ where a and b are constants.

Form of Solution: The solutions are of the form $y=e^{rx}$. (Oh! How?)

Question: What is the basic task to get the solution? As you see in $y=e^{rx}$, the constant r in the exponent is arbitrary. If we know the value of r, then the solution is known. Therefore, the basic task is to determine r.

Method to determine r: Assume $y=e^{rx}$ is the solution of $ay''+by'+cy=0$.

Since $y=e^{rx}$ is assumed to be the solution of $ay''+by'+cy=0$, it must satisfy this equation whenever substituted. Here, $y=e^{rx}, y'=re^{rx}, y''=r^2e^{rx}$.

Then, substitute these in the equation $ay''+by'+cy=0$.

$$ay''+by'+cy=0 \Rightarrow ar^2e^{rx}+bre^{rx}+ce^{rx}=0$$

$$\Rightarrow e^{rx}(ar^2+br+c)=0$$

$$\Rightarrow e^{rx}=0 \text{ or } ar^2+br+c=0$$

Since $e^{rx}\neq 0$, we must have $ar^2+br+c=0$.

Note:

i) The equation $ar^2+br+c=0$ is called **Auxiliary** (Characteristics) equation of the differential equation $ay''+by'+cy=0$.

ii) The function $y=e^{rx}$ is a solution of $ay''+by'+cy=0$ if and only if the constant r is the solution of the quadratic equation $ar^2+br+c=0$.

That is $ar^2+br+c=0 \Rightarrow r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Here, we may get two distinct real roots, single real root, or complex roots depending on the sign of the expression b^2-4ac under the radical sign. Since the type of the root determines the form of the solution of the DE, let's see the three different cases based on the type of the roots. The forms of the solutions based on the natures of the roots are summarized using table as follow.

Forms of Fundamental and General Solution for $ay''+by'+cy=0$

Cases	Roots of the Auxiliary equation $ar^2+br+c=0$	Fundamental Solutions y_1, y_2	Complementary Solution y_c
I) $b^2-4ac>0$	Two Real Roots $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$	$\begin{cases} y_1=e^{r_1x} \\ y_2=e^{r_2x} \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=c_1e^{r_1x}+c_2e^{r_2x}$
II) $b^2-4ac=0$	Single Root $r_1=-\dfrac{b}{2a}$	$\begin{cases} y_1=e^{r_1x} \\ y_2=xe^{r_1x} \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=(c_1+c_2x)e^{r_1x}$
III) $b^2-4ac<0$	Complex Roots $r_{1,2}=\alpha\pm\beta i$ $\alpha=\dfrac{-a}{2}$, $\beta=\dfrac{\sqrt{a^2-4b}}{2}$	$\begin{cases} y_1=e^{\alpha x}\cos\beta x \\ y_2=e^{\alpha x}\sin\beta x \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=c_1e^{\alpha x}\cos\beta x$ $+$ $c_2e^{\alpha x}\sin\beta x$

Notice about case-III: For the two complex roots $r_1=\alpha+\beta i, r_2=\alpha-\beta i$.

Here, $y_1=e^{(\alpha+\beta i)x}$ and $y_2=e^{(\alpha-\beta i)x}$ are the fundamental solutions. But these are complex solutions while our problem is real. So, we have to change these solutions in to their real forms. This is possible by using **Euler's formula.**

$$\begin{cases} y_1=e^{(\alpha+\beta i)x}=e^{\alpha x}e^{\beta ix}=e^{\alpha x}(\cos\beta x+i\sin\beta x) \\ y_2=e^{(\alpha-\beta i)x}=e^{\alpha x}e^{-\beta ix}=e^{\alpha x}(\cos\beta x-i\sin\beta x) \end{cases} \Rightarrow \begin{cases} y_1^*=\dfrac{1}{2}(y_1+y_2)=e^{\alpha x}\cos\beta x \\ y_2^*=\dfrac{1}{2i}(y_1-y_2)=e^{\alpha x}\sin\beta x \end{cases}$$

Hence, the corresponding real solutions are $y_1=e^{\alpha x}\cos\beta x$, $y_2=e^{\alpha x}\sin\beta x$

Examples:

1. Solve the differential equations using the above procedures.

$a)\ y''-5y'+6y=0$ $\quad b)\ y''+8y'+16y=0$ $\quad c)\ y''-4y'+13y=0$

$d)\ 2y''-7y'-4y=0$ $\quad e)\ y''+7y'=0$ $\quad f)\ y''+9y=0$

$g)\ 4y''-4y'+y=0$ $\quad h)\ y''+4y'+7y=0$ $\quad i)\ y''+2\sqrt{2}y'+2y=0$

Solution:

a) Here, the characteristics equation is $r^2-5r+6=0$.

Solving this gives us $r^2-5r+6=0 \Rightarrow (r-2)(r-3)=0 \Rightarrow r_1=2, r_2=3$.

Hence, the fundamental solutions are $y_1=e^{2x}, y_2=e^{3x}$.

Therefore, by *case-I*, the solution is $y=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{2x}+c_2e^{3x}$.

b) Here, the characteristics equation is $r^2+8r+16=0$.

Solving this gives us $r^2+8r+16=0 \Rightarrow (r+4)^2=0 \Rightarrow r_1=r_2=-4$.

Thus, the fundamental solutions are $y_1=e^{-4x}, y_2=xy_1=xe^{-4x}$.

Hence, by *case-II*, the general solution is $y=c_1e^{-4x}+c_2xe^{-4x}$.

c) Here, the characteristics equation is $r^2-4r+13=0$.

Solve this using quadratic formula give the following complex roots.

$$r^2-4r+13=0 \Rightarrow r=\frac{4\pm\sqrt{-36}}{2}=\frac{4\pm 6i}{2}=2\pm 3i.$$

From these complex roots, we have $\alpha=2$ and $\beta=3$. Thus, the fundamental solutions are $y_1=e^{\alpha x}\cos\beta x=e^{2x}\cos 3x,\ y_2=e^{\alpha x}\sin\beta x=e^{2x}\sin 3x$

Hence, by *case-III*, the general solution is $y=e^{2x}(c_1\cos 3x+c_2\sin 3x)$.

d) Here, the characteristics equation is $2r^2-7r-4=0$.

$$2r^2-7r-4=0 \Rightarrow r=\frac{7\pm\sqrt{81}}{4}=\frac{7\pm 9}{4} \Rightarrow r_1=4, r_2=-\frac{1}{2}.$$

Hence, the fundamental solutions are $y_1=e^{4x}, y_2=e^{-\frac{1}{2}x}$.

Therefore, by *case-I*, the general solution is $y=c_1e^{4x}+c_2e^{-\frac{1}{2}x}$.

e) Here, $r^2+7r=0 \Rightarrow r(r+7)=0 \Rightarrow r=0, r=-7$.

Hence, the fundamental solutions are $y_1=1, y_2=e^{-7x}$

Therefore, the general solution is $y=c_1y_1+c_2y_2=c_1+c_2e^{-7x}$.

f) Here, $r^2+9=0 \Rightarrow r^2=-9 \Rightarrow r=\pm\sqrt{-9} \Rightarrow r_1=3i,\ r_2=-3i$.

Hence, by *case-III*, $y=e^{\alpha}(c_1\cos\beta x+c_2\sin\beta xe^{r_1x})=c_1\cos 2x+c_2\sin 2x$.

2. Find the values of the constant k for which the DE $y''+ky'+ky=0$ has a general solution of the form $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$.

Solution: The characteristics equation is $r^2+kr+k=0$. Then the DE will have a general solution of the form $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ if and only if $r^2+kr+k=0$ has complex roots. But the roots will be complex if and only if

$$k^2-4k<0 \Leftrightarrow k(k-4)<0 \Leftrightarrow 0<k<4.$$

Obtaining HLDE from its General solutions:

Once we understand the forms of solution of second order homogeneous DEs, we can also determine the reduced DE from its general solution as follow.

First: Identify the fundamental solutions y_1 and y_2 from the given general solution and obtain the roots r_1 and r_2 from these fundamental solutions.

Second: Form the characteristics equation using the roots obtained.

That is $(r-r_1)(r-r_2)=0 \Rightarrow r^2-(r_1+r_2)r+r_1.r_2=0$.

Third: Deduce the DE. It is $y''-(r_1+r_2)y'+r_1.r_2y=0$.

Examples: Find second order LHDE whose general solution is given.

a) $y=c_1e^x+c_2e^{3x}$ *b)* $y=c_1\sin\sqrt{2}x+c_2\cos\sqrt{2}x$ *c)* $y=c_1e^{2x}+c_2xe^{2x}$

Solution: First identify the roots of the characteristics equation.

a) Using *case-I*, $y=c_1e^x+c_2e^{3x} \Rightarrow y_1=e^x, y_2=e^{3x} \Rightarrow r_1=1, r_2=3$.

So, the characteristics equation is determined as follow.

$(r-r_1)(r-r_2)=0 \Rightarrow (r-1)(r-3)=0 \Rightarrow r^2-4r+3=0$.

Therefore, the required homogeneous DE is $y''-4y'+3y=0$.

b) Here, using case-III, $y_1=\sin\sqrt{2}x, y_2=\cos\sqrt{2}x \Rightarrow r_1=\sqrt{2}\mathrm{i}, r_2=-\sqrt{2}\mathrm{i}$

So, $(r-r_1)(r-r_2)=0 \Rightarrow (r-\sqrt{2}\mathrm{i})(r+\sqrt{2}\mathrm{i})=0 \Rightarrow r^2+4=0$

Therefore, the required Homogeneous DE is $y''+4y=0$.

c) Using *case-II*, $y=c_1e^{2x}+c_2xe^{2x} \Rightarrow y_1=e^{2x}, y_2=xe^{2x} \Rightarrow r_1=r_2=2$.

So, $(r-r_1)(r-r_2)=0 \Rightarrow (r-2)(r-2)=0 \Rightarrow r^2-4r+4=0$.

Therefore, the required Homogeneous DE is $y''-4y'+4y=0$.

1.8.3 Initial and Boundary Value Problems (IVP and BVP)

A differential equation together with some specific conditions on the dependent variable and its derivatives which are given **at the same value** of the independent variable is known as Initial Value Problems (IVPs). The specific conditions are said to be initial conditions. On the other hand, if the specific conditions are given at **different values** of the independent variable, the problem is known as Boundary Value Problem (BVPs) and the specific conditions are said to be boundary conditions.

i) Forms of Initial Value Problems (IVPs)

$$\left.\begin{cases} ay''+by'+cy=f(x)-\text{DE} \\ y(x_0)=y_0,\ y'(x_0)=y_1-\text{ICs} \\ \qquad\qquad \downarrow \\ (\text{Both } y \text{ and } y' \text{at the same value } x=x_0) \end{cases}\right\rfloor \text{IVP}$$

ii) Forms of Boundary Value Problems (BVPs)

$$\left.\begin{cases} ay''+by'+cy=f(x)-\text{DE} \\ y(x_0)=y_0,\ y'(x_1)=y_1-\text{BCs} \\ \qquad\qquad \downarrow \\ (\text{Here } y \text{ and } y' \text{tare given at diiferent values } x=x_0, x=x_1) \end{cases}\right\rfloor \text{BVP}$$

Examples:

1. Solve the following IVPs

$c)\ y''+3y'-4y=0, y(0)=4, y'(0)=14$ **$b)\ y''-3y'=0, y(0)=7, y'(0)=-9$**

$a)\ y''+y=0, y(0)=2, y'(0)=0$ **$d)\ y''-4y'+4y=0, y(0)=1, y'(0)=4$**

$e)\ y''+4\pi^2 y=0,\ y(1)=3, y'(1)=4$ **$f)\ y''+2y=0, y(\pi)=1, y'(\pi)=0$**

Solution:

a) Here, the characteristics equation is $r^2+3r-4=0 \Rightarrow r=1,-4$.

Thus, the general solution is $y=c_1e^x+c_2e^{-4x}$.

Now, let's determine the constants c_1, c_2 using the initial conditions.

That is $y(0)=4, y'(0)=14 \Rightarrow \begin{cases} c_1+c_2=4 \\ c_1-4c_2=14 \end{cases} \Rightarrow c_1=6, c_2=-2$.

Therefore, the solution is $y=6e^x-2e^{-4x}$.

b) Here, the characteristics equation is $r^2 - 3r = 0 \Rightarrow r = 0,3$.

Thus, the general solution is $y = c_1 + c_2 e^{3x}$. Now, determine c_1, c_2.

That is $y(0) = 7, y'(0) = -9 \Rightarrow \begin{cases} c_1 + c_2 = 7 \\ 3c_2 = -9 \end{cases} \Rightarrow c_2 = -3, c_1 = 10$.

Therefore, the solution is $y = 10 - 3e^{3x}$.

c) Here, the characteristics equation is $r^2 + 1 = 0 \Rightarrow r = \pm i$. Thus, the general solution is $y = c_1 \cos x + c_2 \sin x$. Now, let's determine c_1, c_2.

That is $y(0) = 2 \Rightarrow c_1 = 2$, $y'(0) = 0 \Rightarrow c_2 = 0$. Therefore, $y = 2\cos x$.

d) Here, $r^2 - 4r + 4 = 0 \Rightarrow r = 2$. Thus, the general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$

and $y(0) = 1, y'(0) = 4 \Rightarrow \begin{cases} c_1 = 1 \\ 2c_1 + c_2 = 4 \end{cases} \Rightarrow c_1 = 1, c_2 = 2$. So, $y = (1 + 2x)e^{2x}$.

2. Solve the following BVPs

a) $y'' + 4y = 0, y(0) = 2, y'(\pi) = -6$ *b)* $y'' + 4y' + 4y = 0, y(0) = 6, y(3) = 0$

Solution:

a) Here, the characteristics equation is $r^2 + 4 = 0$.

Solving this gives us $r^2 + 4 = 0 \Rightarrow r_1 = 2i$, $r_2 = -2i$.

Hence, the general solution is $y = c_1 \cos 2x + c_2 \sin 2x$.

Now, let's determine the constants c_1, c_2 using the boundary conditions.

$$y(0) = 2, y'(\pi) = -6 \Rightarrow \begin{cases} c_1 \cos 0 + c_2 \sin 0 = 2 \\ -2c_1 \sin(2\pi) + 2c_2 \cos(2\pi) = -6 \end{cases} \Rightarrow c_1 = 2, c_2 = -3.$$

Hence, the solution of the BVP is $y = 2\cos 2x - 3\sin 2x$

b) Here, $r^2 + 4r + 4 = 0 \Rightarrow r = -2$. The solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

Now, let's determine the constants c_1, c_2 using the boundary conditions.

Therefore, the solution of the BVP is $y = 6e^{-2x} - 2xe^{-2x}$.

1.8.4 Solving Non-homogeneous Linear Differential Equations (SONHLDE) with Constant Coefficients

Form of SONHLDE with constant coefficients: $ay''+by'+cy=f(x)$.

Particular solution of $ay''+by'+cy=f(x)$:

Any function y_p free of arbitrary constants that satisfies this SOHLDE is said to be a particular solution.

Theorem (The General Solution Theorem, GST):

If y_p is any particular solution of the non-homogeneous DE $ay''+by'+cy=f(x)$ and y_c is the complementary solution of the homogeneous part $ay''+by'+cy=0$, then the general solution of $ay''+by'+cy=f(x)$ is given by $y=y_c+y_p$.

In short, this theorem says that the general solution of $ay''+by'+cy=f(x)$ is the sum of the general solution of the corresponding HLDE $ay''+by'+cy=0$ and any particular solution of $ay''+by'+cy=f(x)$. That is $y=y_c+y_p$.

So far, we have seen how to find y_c of $ay''+by'+cy=0$ but how to get y_p?

Procedures to solve $ay''+by'+cy=f(x)$.

First: Solve $ay''+by'+cy=0$ and obtained the solution $y_c=c_1y_1+c_2y_2$.

Second: Find any particular solution y_p of $ay''+by'+cy=f(x)$.

Third: Form the general solution $y=c_1y_c+c_2y_2+y_p$ using the GST.

There are different methods to find y_p.

1. Method of Undetermined Coefficients (MUCs)
2. Variation of Parameters (VPs)
3. The Operator Method (OM)
4. Diagonalization Method (DM)
5. Laplace Transform Method
6. Power Series Method

1.11.1 Method of Undetermined Coefficients (MUCs)

Suppose we want to solve $ay''+by'+cy=f(x)$ where the coefficients a,b and c are constants using the Method of Undetermined coefficient.

Main principles to notice about MUCs:

i) Assumption: The method of undetermined coefficients assumes that the solution to the DE equation is the same form as $f(x)$.

ii) Starts with trial form: Making educated guess

Once we assume y_p has the same form as $f(x)$, the method proceeds with an educated guess by expressing y_p using undetermined coefficients.

iii) Coefficient Determination: From the trial form obtain, y_p, y'_p and y''_p and substitute in $ay''+by'+cy=f(x)$ to determine the coefficients. With this it is possible to determine the undetermined coefficients to be determined.

For this reason, the method is named as Method of Undetermined Coefficients.

Conditions to use the method: When do we use the method?

The general method is limited to non-homogeneous linear DE of $ay''+by'+cy=f(x)$ with the assumption that the coefficients are constants, and $f(x)$ is only of the form $p_n(x)$, e^{ax}, $\sin\alpha x$, $\cos\beta x$, $p_n(x)e^{ax}$, and their combinations like $p_n(x)\sin\alpha x$, $p_n(x)\cos\beta x$, $p_n(x)e^{ax}\sin\alpha x + p_n(x)e^{ax}\cos\beta x$.

Exceptions: When does the method fails?

The Method of Undetermined Coefficient (MUCs) is not applicable if the function $f(x)$ is of the form $\frac{1}{x}, \ln x, \sqrt{x}, \tan x, \cos^{-1} x, \sec x, \cot x, \csc x$ or any other transcendental functions.

TABLE-1.2: The basic Trial Forms of Particular Solution	
The form of $f(x)$	Trial form of y_p
$f(x)=ae^{kx}$	$y_p=Ae^{kx}$
$f(x)=ax+b$	$y_p=Ax+B$
$f(x)=ax^2+bx+c$	$y_p=Ax^2+Bx+C$
$f(x)=\begin{cases} a\sin kx \text{ or } b\cos kx \\ a\sin kx+b\cos kx \end{cases}$	$y_p=A\sin kx+B\cos kx$
$f(x)=\begin{cases} a\sinh kx \text{ or} \\ b\cosh kx \end{cases}$	$y_p=A\sinh kx+B\cosh kx$
$f(x)=\begin{cases} a\sin kx+b\cos mx \\ a\sin kx+b\sin mx \end{cases}, k\neq m$	$y_p=A\sin kx+B\cos kx+C\sin mx+D\cos mx$
$f(x)=ae^{kx}+be^{mx}, k\neq m$	$y_p=Ae^{kx}+Be^{mx}$
For Product forms	
$f(x)=(ax+b)e^{kx}$	$y_p=(Ax+B)e^{kx}$
$f(x)=\begin{cases} (ax+b)\sin kx \text{ or} \\ (ax+b)\cos kx \end{cases}$	$y_p=(Ax+B)\sin kx+(Cx+D)\cos kx$
$f(x)=ae^{mx}\sin kx \text{ or } be^{mx}\cos kx$	$y_p=Ae^{mx}\sin kx+Be^{mx}\cos kx$
$f(x)=\begin{cases} (ax+b)e^{mx}\sin kx \\ (ax+b)e^{mx}\cos kx \end{cases}$	$y_p=(Ax+B)e^{mx}\sin kx+(Cx+D)e^{mx}\cos kx$
$f(x)=(ax^2+bx+c)e^{kx}$	$y_p=(Ax^2+Bx+C)e^{kx}$

Cautions:

The table will give you only **hints** on how to guess y_p based on $f(x)$.

Always, ASK yourself the following questions about y_p:

Does what we guess for y_p always work? **No!**

How do we know when it does not work? **From analysis of roots!**

How do we correct if it does not work? **Use Modification Rules!**

Examples: Solve the following DEs using MUCs

a) $y''-3y'-4y=12e^{2x}$ b) $y''-2y'+y=x^2-x+3$ c) $y''-3y'-4y=34\sin x$

d) $y''+y'-2y=2xe^{-x}$ e) $y''+2y=3x\sin x$ f) $y''+2y'+5y=e^{x}\cos 2x$

Solution:

a) **Step-1:** Find the complementary solution y_c of $y''-3y'-4y=0$.

Here, the characteristics equation is $r^2-3r-4=0$. Solving this gives us

$r^2-3r-4=0\Rightarrow(r+1)(r-4)=0\Rightarrow r_1=-1, r_2=4$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find the particular solution y_p having the same form as $f(x)=12e^{2x}$

From the table, it seems of the form $y_p=ae^{2x}$. Then, $y'_p=2ae^{2x}$, $y''_p=4ae^{2x}$.

Now, determine the constant a by substituting these values in the given DE.

That is $y''-3y'-4y=12e^{2x}\Rightarrow 4ae^{2x}-6ae^{2x}-4ae^{2x}=12e^{2x}$

$$\Rightarrow -6ae^{2x}=12e^{2x}\Rightarrow a=-2$$

Thus, $y_p=ae^{2x}=-2e^{2x}$. Therefore, $y=y_c+y_p=c_1e^{-x}+c_2e^{4x}-2e^{2x}$.

b) **Step-1:** Find the complementary solution y_c of $y''-2y'+y=0$.

Here, $r^2-2r+1=0\Rightarrow(r-1)^2=0\Rightarrow r=1$.

Hence, the complementary solution is $y_c=c_1e^{x}+c_2xe^{x}$.

Step-2: Find the particular solution y_p of the form $f(x)=x^2-x+3$.

That is y_p is of the form $y_p=ax^2+bx+c$.

Then, substitute $y_p=ax^2+bx+c$, $y'_p=2ax+b$, $y''_p=2a$ in the DE.

$$y''-2y'+y=x^2-x+3\Rightarrow 2a-2(2ax+b)+ax^2+bx+c=x^2-x+3$$
$$\Rightarrow ax^2+(b-4a)x+2a-2b+c=x^2-x+3$$
$$\Rightarrow a=1, b-4a=-1, 2a-2b+c=3\Rightarrow a=1, b=3, c=7$$

So, the particular solution is $y_p=x^2+3x+7$.

Hence, the general solution is $y=y_C+y_p=c_1e^{x}+c_2xe^{x}+x^2+3x+7$

c) **Step-1:** Find the complementary solution y_c of $y''-3y'-4y=0$.

It is the same as in part (a). That is $y_c=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find the particular solution y_p of the form $f(x)=34\sin x$

It seems of the form $y_p=a\sin x+b\cos x$.

Then, $y'_p=a\cos x-b\sin x$, $y''_p=-a\sin x-b\cos x$.

Now, determine a,b by substituting these values in the given DE.

$$y''-3y'-4y=34\sin x$$
$$\Rightarrow -a\sin x-b\cos x-3(a\cos x-b\sin x)-4(a\sin x+b\cos x)=34\sin x$$
$$\Rightarrow (-5a+3b)\sin x+(-3a-5b)\cos x=34\sin x$$

Equation the coefficients of $\sin x, \cos x$ on both sides, we have

$$\begin{cases} -3a-5b=0\Rightarrow b=\dfrac{-3}{5}a, \\ -5a+3b=34\Rightarrow \dfrac{-34}{5}a=34\Rightarrow a=-5, b=\dfrac{-3}{5}a\Rightarrow b=3 \end{cases}$$

Thus, $y_p=a\sin x+b\cos x=-5\sin x+3\cos x$.

Therefore, $y=y_c+y_p=c_1e^{-x}+c_2e^{4x}-5\sin x+3\cos x$.

d) First, find the complementary solution y_c of $y''+y'-2y=0$.

Here, $r^2+r-2=0\Rightarrow (r-1)(r+2)=0\Rightarrow r_1=1, r_2=-2$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{x}+c_2e^{-2x}$.

Next, find the particular solution y_p of the form $f(x)=2xe^{-x}$.

It seems of the form $y_p=(ax+b)e^{-x}$.

Then, $y'_p=ae^{-x}-(ax+b)e^{-x}$, $y''_p=-2ae^{-x}+(ax+b)e^{-x}$.

Thus, substitute these values to determine the coefficients.

$$y''+y'-2y=2xe^{-x}$$
$$\Rightarrow -2ae^{-x}+(ax+b)e^{-x}+ae^{-x}-(ax+b)e^{-x}-2(ax+b)e^{-x}=2xe^{-x}$$
$$\Rightarrow -2axe^{-x}-(a+2b)e^{-x}=2xe^{-x}$$
$$\Rightarrow -2a=2, -a-2b=0\Rightarrow a=-1, b=\frac{1}{2}\Rightarrow y_p=(\frac{1}{2}-x)e^{-x}$$

Therefore, the general solution is $y=y_c+y_p=c_1e^{x}+c_2e^{-2x}+(\frac{1}{2}-x)e^{-x}$.

e) $y''+2y=0 \Rightarrow r^2+2=0 \Rightarrow r=\pm\sqrt{2}i \Rightarrow y_c = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$

Now, lets' find the particular solution y_p of the form $f(x)=3x\sin x$.

Here, y_p is of the form $y_p=(ax+b)\cos x+(cx+d)\sin x$. Then, we have

$y'_p = a\cos x-(ax+b)\sin x+c\sin x+(cx+d)\cos x$

$y''_p = -2a\sin x-(ax+b)\cos x+2c\cos x-(cx+d)\sin x$

So, $y''+2y=3x\sin x$

$\Rightarrow (2c+b)\cos x+(d-2a)\sin x+ax\cos x+cx\sin x=3x\sin x$

$\Rightarrow 2c+b=0, d-2a=0, a=0, c=3 \Rightarrow a=0, b=-6, d=0$

$\Rightarrow y_p = 3x\sin x-6\cos x$

Hence, the general solution is $y=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x+3x\sin x-6\cos x$

f) $r^2+2r+5=0 \Rightarrow r=-1+2i,-1-2i$. Hence, the fundamental solutions are $y_1=e^{-x}\cos 2x,\ y_2=e^{-x}\sin 2x$. Now, find y_p of the form $f(x)=e^x\cos 2x$. It is of the form $y_p=ae^x\sin 2x+be^x\cos 2x$.

Then, $\begin{cases} y'_p=(a-2b)e^x\sin 2x+(2a+b)e^x\cos 2x, \\ y''_p=-(3a+4b)e^x\sin 2x+(4a-3b)e^x\cos 2x \end{cases}$

$y''+2y'+5y=e^x\cos 2x$

$$\Rightarrow (4a-8b)e^x\sin 2x+(8a+4b)e^x\cos 2x=e^x\cos 2x$$

$$\Rightarrow 4a-8b=0,\ 8a+4b=1 \Rightarrow a=2b,\ 8a+4b=1$$

$$\Rightarrow 16b+4b=1 \Rightarrow b=\frac{1}{20},\ a=\frac{1}{10}$$

$$\Rightarrow y_p=\frac{1}{10}e^x\sin 2x+\frac{1}{20}e^x\cos 2x=\frac{e^x}{10}(2\sin 2x+\cos 2x)$$

Remark (Sum Rule for Trial forms):

In the DE $ay''+by'+cy=f(x)$, the function $f(x)$ may be the sum (difference) of functions like $f(x)=g(x)+h(x)$,in such case, guess the trial forms y_{p1} for $g(x)$ and y_{p2} for $h(x)$ separately as the particular solutions.

Then, their sum $y_p=y_{p1}+y_{p2}$ is a trail form for $f(x)=g(x)+h(x)$.

Examples: Solve the following DEs using MUC.

$a)\ y''+4y'+5y=30e^x+10x-7$ $\qquad$ $b)\ y''+4y=8x+9\sin x$

$c)\ y''-y'-2y=2x^2+e^x$ $\qquad$ $d)\ y''+5y'+6y=4e^{-x}+5\sin x$

Solution:

Step-1: Find the complementary solution y_c of $y''+4y'+5y=0$.

Here, $r^2+4r+5=0 \Rightarrow r=-2\pm i$. Then, $y_c=c_1e^{-2x}\cos x+c_2e^{-2x}\sin x$

Step-2: Find y_p which is the same form as $f(x)=30e^x+10x-7$.

Here, $f(x)=g(x)+h(x)$ where $g(x)=30e^x$, $h(x)=10x-7$.

So, guess $y_{p1}=Ae^x$ for $g(x)=30e^x$ and $y_{p2}=Bx+C$ for $h(x)=9\sin x$.

Then, y_p for $f(x)=30e^x+10x-7$ becomes $y_p=y_{p1}+y_{p2}=Ae^x+Bx+C$.

That is $y_p=ae^x+bx+c \Rightarrow y'_p=ae^x+b,\ y''_p=ae^x$.

So, $y''+4y'+5y=30e^x+10x-7$

$$\Rightarrow Ae^x+4(Ae^x+B)+5(Ae^x+Bx+C)=30e^x+10x-7$$

$$\Rightarrow Ae^x+4Ae^x+4B+5Ae^x+5Bx+5C=30e^x+10x-7$$

$$\Rightarrow 10Ae^x+5Bx+4B+5C=30e^x+10x-7$$

$$\Rightarrow \begin{cases} 10A=30 \Rightarrow A=3 \\ 5B=10 \Rightarrow B=2 \\ 4B+5C=-7 \Rightarrow C=-3 \end{cases} \Rightarrow y_p=3e^x+2x-3$$

Therefore, $y=y_c+y_p=c_1e^{-2x}\cos x+c_2e^{-2x}\sin x+3e^x+2x-3$.

b) **Step-1:** Find the complementary solution y_c of $y''+4y=0$.

Here, the characteristics equation $r^2+4=0 \Rightarrow r=\pm 2i$.

So, the complementary solution is $y_c=c_1\cos 2x+c_2\sin 2x$.

Step-2: Find the particular solution y_p of the form $f(x)=8x+9\sin x$

Guess $y_{p1}=Ax+B$ for $g(x)=8x$ and $y_{p2}=C\cos x+D\sin x$ for $h(x)=9\sin x$.

Then, $y_p=y_{p1}+y_{p2}=Ax+B+C\cos x+D\sin x$ for $f(x)=8x+9\sin x$.

Hence, $y'_p=A-C\sin x+D\cos x$, $y''_p=-C\cos x-D\sin x$.

Now, determine the constants A,B,C,D.

$y''+4y=8x+9\sin x$

$$\Rightarrow -C\cos x-D\sin x+4[Ax+B+C\cos x+D\sin x]=8x+9\sin x$$

$$\Rightarrow 4A=8, 4B=0, 3C=0, 3D=9 \Rightarrow A=2, B=0, C=0, D=3$$

Hence, the particular solution is $y_p=2x+3\sin x$.

Therefore, $y=c_1\cos 2x+c_2\sin 2x+2x+3\sin x$.

c) **Step-1:** Find the complementary solution y_c of $y''-y'-2y=0$.

Here, the characteristics equation is $r^2-r-2=0$. Solving this gives us $r^2-r-2=0 \Rightarrow (r+1)(r-2)=0 \Rightarrow r_1=-1, r_2=2$. So, $y_c=c_1e^{-x}+c_2e^{2x}$.

Step-2: Find the particular solution y_p having the same form as $f(x)=4x^2+8e^x$. Here, we guess $y_{p1}=ax^2+bx+c$ for $g(x)=4x^2$ and $y_{p2}=de^x$ for $h(x)=8e^x$. Then, $y_p=y_{p1}+y_{p2}=ax^2+bx+c+de^x$ for $f(x)=4x^2+8e^x$. Hence, $y'_p=2ax+b+de^x$, $y''_p=2a+de^x$.

Now, determine the constants a,b,c,d.

$y''-y'-2y=4x^2+8e^x$

$$\Rightarrow 2a+de^x-[2ax+b+de^x]-2(ax^2+bx+c+de^x)=4x^2+8e^x$$

$$\Rightarrow -2ax^2-(2a+2b)x+2a-b-2c-2de^x=4x^2+8e^x$$

$$\Rightarrow -2a=4, -2a-2b=0, 2a-b-2c=0, -2d=8$$

$$\Rightarrow a=-2, b=2, c=-3, d=-4$$

Hence, the particular solution is $y_p=-2x^2+2x-3-4e^x$.

Therefore, $y=c_1e^{-x}+c_2e^{2x}-2x^2+2x-3-4e^x$.

Modification Rules for Trial Forms: Generalization on MUCs

Now, let's analyze the three questions about y_p that we have posed earlier.

- ✓ Does what we guess for y_p always work? **No!**
- ✓ How do we know when it does not work? **From analysis of roots!**
- ✓ How do we correct if it does not work? **Use Modification Rules!**

As we have discussed for the DE $ay''+by'+cy=f(x)$, we used an educated guess of the particular solution y_p based on the form of $f(x)$. But this is not always true. There are cases where the form of the particular solution y_p that we guessed may not work. In what follow, let's discuss the cases where the form of y_p is determined based on the relation between the roots r_1 and r_2 of the characteristics equation and some part of $f(x)$.

Modification Rule-1: For the form $f(x)=(A_nx^n+...+A_1x+A_0)e^{kx}$.

In such case, the form of the particular solution y_p of $ay''+by'+cy=0$ depends on r_1, r_2 and the exponent k.

i) If $r_1 \neq k, r_2 \neq k$, then $y_p=(a_nx^n+...+a_1x+a_0)e^{kx}$

ii) If $r_1=k$ or $r_2=k$ but $r_1 \neq r_2$, then $y_p=x(a_nx^n+...+a_1x+a_0)e^{kx}$

iii) If $r_1=r_2=k$, then $y_p=x^2(a_nx^n+...+a_1x+a_0)e^{kx}$

Examples: Solve the following DEs using undetermined coefficients.

a) $y''-3y'-4y=10e^{-x}$ *b)* $y''-3y'-4y=xe^{-x}$ *c)* $y''-2y'+y=6e^{x}$

d) $y''+6y'+9y=xe^{-3x}$ *e)* $y''+y'-2y=e^{x}+e^{-x}$ *f)* $y''-4y'+4y=xe^{2x}$

Solution:

a) Step-1: Find the complementary solution y_c of $y''-3y'-4y=0$

Here, the characteristics equation of is $r^2-3r-4=0$. Solving this gives us $r^2-3r-4=0 \Rightarrow (r+1)(r-4)=0 \Rightarrow r_1=-1, r_2=4$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find y_p .Here, $f(x)=10e^{kx}$, with $k=-1$, $r_1 \neq r_2$ and $r_1=-1, r_2=4$ but here $r_1=k$ .Thus, y_p is of the form $y_p=axe^{-x}$.

Then, $y'_p = ae^{-x} - axe^{-x}$, $y''_p = -2ae^{-x} + axe^{-x}$. So,

$$y''-3y'-4y = 2e^{-x} \Rightarrow -2ae^{-x} + axe^{-x} - 3(ae^{-x} - axe^{-x}) - 4ae^{-x} = 10e^{-x}$$
$$\Rightarrow -5ae^{-x} = 10e^{-x} \Rightarrow -5a = 10 \Rightarrow a = -2$$

Thus, $y_p = -2xe^{-x}$. Therefore, $y = y_c + y_p = c_1e^{-x} + c_2e^{4x} - 2xe^{-x}$.

b) Here, $r^2 - 3r - 4 = 0 \Rightarrow r_1 = -1, r_2 = 4$ and $f(x) = xe^{-x}$, with $k = -1$.

So, by the second part of rule-1, $y_p = x(ax+b)e^{-x} = (ax^2+bx)e^{-x}$.

Then, $y'_p = (b+2ax-bx-ax^2)e^{-x}$, $y''_p = (ax^2-4ax+bx+2a-2b)e^{-x}$.

So, $y''-3y'-4y = xe^{-x}$

$$\Rightarrow (ax^2-4ax+bx+2a-2b)e^{-x} - 3(b+2ax-bx-ax^2)e^{-x} - 4(ax^2+bx)e^{-x} = xe^{-x}$$

$$\Rightarrow (-10ax+2a-5b)e^{-x} = xe^{-x} \Rightarrow -10a = 1,\ 2a-5b = 0 \Rightarrow a = -\frac{1}{10}, b = -\frac{1}{25}$$

Thus, $y_p = (-\frac{x^2}{10} - \frac{x}{25})e^{-x}$.

c) **Step-1:** Find the complementary solution y_c of $y''-2y'+y=0$

Here, the solution of the corresponding homogenous equation is
$r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0 \Rightarrow r_1 = r_2 = 1$.

Hence, the complementary solution is $y_c = c_1e^x + c_2xe^x$.

Step-2: Find the particular solution y_p.

Here, $f(x) = 6e^{kx}$, with $k = 1, r_1 = r_2 = k = 1$. Thus, $y_p = ax^2e^x$.

Then, $y'_p = (2ax+ax^2)e^x$, $y''_p = (2a+4ax+ax^2)e^x$. So,

$$y''-2y'+y = 6e^x \Rightarrow (2a+4ax+ax^2)e^x - 2(2ax+ax^2)e^x + ax^2e^x = 6e^x$$
$$\Rightarrow 2ae^x = 6e^x \Rightarrow 2a = 6 \Rightarrow a = 3$$

Thus, $y_p = 3x^2e^x$. Therefore, $y = y_c + y_p = c_1e^x + c_2xe^x + 3x^2e^x$.

d) Here, $r^2 + 6r + 9 = 0 \Rightarrow r_1 = r_2 = -3$ and $f(x) = xe^{-3x}$, with $k = -3$.

So, by third part of rule-1, $y_p = x^2(ax+b)e^{-3x}$. (Complete it !)

e) **Step-1:** Find the complementary solution y_c of $y''+y'-2y = 0$.

Here, the characteristics equation is $r^2+r-2=0$. Solving this gives us

$r^2+r-2=0 \Rightarrow (r-1)(r+2)=0 \Rightarrow r_1=1, r_2=-2$. So, $y_c=c_1e^x+c_2e^{-2x}$.

Step-2: Find y_p having the same form as $f(x)=e^x+e^{-x}$. Here, we guess $y_{p1}=axe^x$ for $g(x)=e^x$ and $y_{p2}=be^{-x}$ for $h(x)=e^{-x}$. Then,

$y_p=y_{p1}+y_{p2}=axe^x+be^{-x}$ for $f(x)=e^x+e^{-x}$. Now, determine a and b.

$$y''+y'-2y=e^x+e^{-x}$$

$$\Rightarrow 2ae^x+axe^x+be^{-x}+ae^x+axe^x-be^{-x}-2[axe^x+be^{-x}]=e^x+e^{-x}$$

$$\Rightarrow 3ae^x-2be^{-x}=e^x+e^{-x} \Rightarrow a=\frac{1}{3}, b=-\frac{1}{2} \Rightarrow y_p=\frac{1}{3}xe^x-\frac{1}{2}e^{-x}$$

Therefore, $y=c_1e^x+c_2e^{-2x}+\frac{1}{3}xe^x-\frac{1}{2}e^{-x}$.

Modification Rule-2: For the form $f(x)=A_nx^n+\ldots+A_1x+A_0$.

In such case, the form of the particular solution y_p of $ay''+by'+cy=f(x)$ depends on the coefficients a, b.

i) If $b\neq 0$, then $y_p=a_nx^n+\ldots+a_1x+a_0$

ii) If $\mathrm{b}=0$, but $a\neq 0$, then $y_p=x(a_nx^n+\ldots+a_1x+a_0)$

iii) If $a=b=0$, then $y_p=x^2(a_nx^n+\ldots+a_1x+a_0)$

Examples: Find the particular solution of the following DEs.

a) $y''-3y'=18x^2+2$ b) $y''=24x$ c) $y''=9x^2+2x-6$

Solution:

a) Here, the characteristics equation is $r^2-3r=0 \Rightarrow a=-3, b=0$ and $f(x)=18x^2+2$. So, y_p is of the form $y_p=x(ax^2+bx+c)=ax^3+bx^2+cx$.

Then, $y'_p=3ax^2+2bx+c$, $y''_p=6ax+2b$. So,

$$y''-3y'-4y=2e^{-x} \Rightarrow 6ax+2b-3(3ax^2+2bx+c)=18x^2+2$$

$$\Rightarrow -9ax^2+(6a-6b)x+2b-3c=18x^2+2$$

$$\Rightarrow -9a=18, 6a-6b=0, 2b-3c=2 \Rightarrow a=-2, b=-2, c=-2$$

Thus, $y_p=x(-2x^2-2x-2)=-2x^3-2x^2-2x$.

b) Here, the characteristics equation is $r^2=0$. So, the coefficients of the characteristics equations are $a=b=0$ and $f(x)=24x$. So, y_p is of the form $y_p=x^2(ax+b)=ax^3+bx^2$. Then, $y'_p=3ax^2+2bx$, $y''_p=6ax+2b$.

But $y''=24x \Rightarrow 6ax+2b=24x \Rightarrow 6a=24, 2b=0 \Rightarrow a=4, b=0$

Thus, $y_p=4x^3$ and the general solution is $y=y_c+y_p=c_1+c_2x+4x^3$.

c) Here, the characteristics equation $r^2=0 \Rightarrow r_1=r_2=0$ which is a single root.

Then, the fundamental solutions are $y_1=e^0=1, y_2=xy_1=x$.

Next, let's determine the particular solution y_p. The direct trial form is

$y_p=ax^2+bx+c$ but it does not work because the coefficients of the characteristics equations $r^2=0$ are $a=b=0$. Besides, $f(x)=9x^2+2x-6$. Thus, by the above modification rule, the direct trial form of y_p must be modified as $y_p=x^2(ax^2+bx+c)=ax^4+bx^3+cx^2$.

Then, $y'_p=4ax^3+3bx^2+2cx$, $y''_p=12ax^2+6bx+2c$.

So, $y''=9x^2+2x-6 \Rightarrow 12ax^2+6bx+2c=9x^2+2x-6$

$$\Rightarrow 12a=9, 6b=2, 2c=-6 \Rightarrow a=\frac{3}{4}, b=\frac{1}{3}, c=-3$$

Thus, $y_p=\frac{3}{4}x^4+\frac{1}{3}x^3-3x^2$. Therefore, $y=c_1+c_2x+\frac{3}{4}x^4+\frac{1}{3}x^3-3x^2$.

Modification Rule-3: For the form $f(x)=(a\cos\beta x+b\sin\beta x)e^{\alpha x}$.

In such cases, the form of the particular solution y_p of $ay''+by'+cy=f(x)$ depends on the relation between the characteristics roots r_1, r_2 and α, β.

i) If $r_1 \neq \alpha+\beta i, r_2 \neq \alpha-\beta i$, then $y_p=(A\cos\beta x+B\sin\beta x)e^{\alpha x}$

ii) If $r_1=\alpha+\beta i, r_2=\alpha-\beta i$, then $y_p=x(A\cos\beta x+B\sin\beta x)e^{\alpha x}$

Examples: Find the particular solution of the following DEs.

a) $y''+y=\sin x$ *b)* $y''+2y'+5y=e^{-x}\sin 2x$ *c)* $y''+4y=x\cos 2x$

d) $y''+4y=8\cos 2x$ *e)* $y''-2y'+y=e^{x}\sin x$ *f)* $y''+16y=4\cos 4x$

Solution:

a) Here, $f(x)=\sin x$, with $\beta=1$, $\alpha=0$.

But $r^2+1=0 \Rightarrow r=\pm i \Rightarrow r_1=i=\alpha+\beta i,\ r_2=-i=\alpha-\beta i$

So, by part (ii) of modification rule-3, $y_p=x(a\cos x+b\sin x)$.

$y'_p=a\cos x+b\sin x+x(b\cos x-a\sin x), y''_p=(-2a-bx)\sin x+(2b-ax)\cos x$

So, $y''+y=\sin x \Rightarrow (-2a-bx)\sin x+(2b-ax)\cos x+x(a\cos x+b\sin x)=\sin x$

$$\Rightarrow -2a\sin x+2b\cos x=\sin x \Rightarrow -2a=1, 2b=0 \Rightarrow a=-\frac{1}{2}, b=0$$

Thus, $y_p=x(a\cos x+b\sin x)=-\frac{1}{2}x\cos x$.

b) $y''+2y'+5y=0 \Rightarrow r^2+2r+5=0 \Rightarrow r_1=-1+2i, r_2=-1-2i$

Now, lets' find y_p which is the same form as $f(x)=e^{-x}\sin 2x$.

Here, in $f(x)=e^{-x}\sin 2x$, we have $\alpha=-1$, $\beta=2$. But we have

$r_1=-1+2i, r_2=-1-2i \Rightarrow r_1=\alpha+\beta i,\ r_2=\alpha-\beta i$. Thus, y_p is of the form

$y_p=x(A\cos 2x+B\sin 2x)e^{-x}$. (Complete the solution!)

c) $y''+4y=0 \Rightarrow r_1=2i, r_2=-2i$. Now, lets' find y_p which is the same form as $f(x)=x\cos 2x$. Here, in $f(x)=x\cos 2x$, we have $\alpha=0$, $\beta=2$. But we have $r_1=2i, r_2=-2i \Rightarrow r_1=\alpha+\beta i,\ r_2=\alpha-\beta i$. Thus, y_p is of the form

$y_p=x[(Ax+B)\cos 2x+(Cx+D)\sin 2x]$. (Complete the solution!).

Miscellaneous Examples on MUCs:

The following problems will help you to test yourself whether you understood all the main concepts about MUCs. First try by yourself and then see the hints.

1. Solve the following DEs using MUCs.

a) $y''-y'-2y=4x^2$ b) $y''-3y'+2y=x^2+e^x$

c) $y''-5y'+6y=xe^{2x}$ d) $y''-6y'+9y=x+e^x$

e) $y''-2y'-3y=12xe^{2x}+6x-11$ f) $y''-3y'+2y=8x^2+1$

Solution:

a) **Step-1:** Find the complementary solution y_c of $y''-y'-2y=0$.

Here, the characteristics equation is $r^2-r-2=0$. Solving this gives us $r^2-r-2=0 \Rightarrow (r+1)(r-2)=0 \Rightarrow r_1=-1, r_2=2$. So, $y_c=c_1e^{-x}+c_2e^{2x}$.

Step-2: Find the particular solution y_p of the form $f(x)=4x^2$.

Here, we guess $y_p=ax^2+bx+c$. Hence, $y'_p=2ax+b$, $y''_p=2a$.

Now, determine the constants a,b,c.

$$
\begin{aligned}
y''-y'-2y=4x^2 &\Rightarrow 2a-[2ax+b]-2(ax^2+bx+c)=4x^2\\
&\Rightarrow -2ax^2-(2a+2b)x+2a-b-2c=4x^2\\
&\Rightarrow -2a=4, -2a-2b=0, 2a-b-2c=0\\
&\Rightarrow a=-2, b=2, c=-3
\end{aligned}
$$

Hence, the particular solution is $y_p=-2x^2+2x-3$.

Therefore, $y=c_1e^{-x}+c_2e^{2x}-2x^2+2x-3$.

b) Here, $r^2-3r+2=0 \Rightarrow (r-1)(r-2)=0 \Rightarrow r_1=1, r_2=2 \Rightarrow y_c=c_1e^x+c_2e^{2x}$.

Here, y_p seems of the form $y_p=y_{p1}+y_{p2}$ where $y_{p1}=ax^2+bx+c$, $y_{p2}=de^x$ but not because e^x is already in the solution $y_c=c_1e^x+c_2e^{2x}$.

Thus $y_{p2}=dxe^x$ such that $y_p=ax^2+bx+c+dxe^x$.

Then, $y'_p=2ax+b+(d+dx)e^x$, $y''_p=2a+(2d+dx)e^x$.

So, $y''-3y'+2y=x^2+e^x$

$$
\begin{aligned}
&\Rightarrow 2a+(2d+dx)e^x-3[2ax+b+(d+dx)e^x]+2[ax^2+bx+c+dxe^x]=x^2+e^x\\
&\Rightarrow 2ax^2+(-d-2)xe^x+(2b-6a)x-de^x+2a-3b+2c=x^2+e^x\\
&\Rightarrow 2a=1,,2b-6a=0,-d=1,2a-3b+2c=0 \Rightarrow a=\frac{1}{2}, b=\frac{3}{2}, c=\frac{7}{4}, d=-1
\end{aligned}
$$

Hence, $y_p=\frac{1}{2}x^2+\frac{3}{2}x+\frac{7}{4}-xe^x \Rightarrow y=c_1e^x+c_2e^{2x}+\frac{1}{2}x^2+\frac{3}{2}x+\frac{7}{4}-xe^x$.

c) Here, the characteristics equation is

$r^2-5r+6=0 \Rightarrow (r-2)(r-3)=0 \Rightarrow r_1=2, r_2=3$ and $f(x)=xe^{2x}$, with $k=2$.

So y_p is of the form $y_p=x(ax+b)e^{2x}=(ax^2+bx)e^{2x}$.

Then, $y'_p=(2ax^2+2ax+2bx+b)e^{2x}$, $y''_p=(4ax^2+8ax+4bx+2a+4b)e^{2x}$.

So, $y''-5y'+6y = xe^{2x} \Rightarrow (-2ax+2a-b)e^{2x} = xe^{2x}$

$\Rightarrow -2a = 1,\ 2a-b = 0 \Rightarrow a = -\frac{1}{2}, b = -1 \Rightarrow y_p = (-\frac{x^2}{2} - x)e^{2x}$.

Thus, $y = y_c + y_p = c_1e^{2x} + c_2e^{3x} - (\frac{x^2}{2} + x)e^{2x}$.

d) Here, $r^2-6r+9=0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$.Then, $y_c = c_1e^{3x} + c_2xe^{3x}$.

Now, lets' find y_p which is the same form as $f(x) = x+e^x$.

Here, y_p seems of the form $y_p = y_{p1} + y_{p2}$ where $y_{p1} = ax+b, y_{p2} = ce^x$.

Then, $y'_p = a+ce^x, y''_p = ce^x$.

So, $y''-6y'+9y = x+e^x \Rightarrow 4ce^x + 9ax - 6a + 9b = x + e^x$

$\Rightarrow 4c = 1, 9a = 1, 9b-6a = 0 \Rightarrow c = \frac{1}{4}, a = \frac{1}{9}, b = \frac{2}{27} \Rightarrow y_p = \frac{x}{9} + \frac{e^x}{4} + \frac{2}{27}$

e) **Step-1:** Find the complementary solution y_c of $y''-2y'-3y=0$.

Here, the characteristics equation is $r^2-2r-3=0$. Solving this gives us $r^2-2r-3=0 \Rightarrow (r+1)(r-3)=0 \Rightarrow r_1 = -1, r_2 = 3$. So, $y_c = c_1e^{-x} + c_2e^{3x}$.

Step-2: Guess y_p having the same form as $f(x) = 12xe^{2x} + 6x - 11$.

Here, $y_{p1} = (ax+b)e^{2x}$ for $h(x) = 12xe^{2x}$ and $y_{p2} = cx+d$ for $g(x) = 6x-11$.

Then, $y_p = y_{p1} + y_{p2} = (ax+b)e^{2x} + cx + d$ for $f(x) = 12xe^{2x} + 6x - 11$.

Hence, $y'_p = (2ax+2b+a)e^{2x} + c,\ y''_p = (4ax+4a+4b)e^{2x}$.

$$y''-2y'-3y = 12xe^{2x} + 6x - 11$$
$$\Rightarrow (-3ax+2a-3b)e^{2x} - 3cx - 2c - 3d = 12xe^{2x} + 6x - 11$$
$$\Rightarrow -3a = 12, 2a-3b = 0, -3c = 6,\ -2c-3d = -11$$
$$\Rightarrow a = -4, b = -\frac{8}{3}, c = -2 = 2, d = 5 \Rightarrow y_p = (-4x - \frac{8}{3})e^{2x} - 2x + 5$$

f) **Step-1:** Find the complementary solution y_c of $y''+3y'+4y=0$

Here, $r^2+3r+4=0 \Rightarrow r = \frac{-3 \pm \sqrt{7}i}{2} \Rightarrow \alpha = \frac{-3}{2}, \beta = \frac{\sqrt{7}}{2}$.

2. Using *Method of Undetermined Coefficients*, find the general solution.

a) $y''+y'-6y=x+e^{-3x}$ b) $y''-y=\cos 2x$ c) $y''-y'=x+e^{x}$

d) $y''+3y'=4x+6e^{-3x}$ e) $y''-2y'=e^{x}\sin x$ f) $y''+y'-2y=2xe^{-x}$

g) $y''+3y'+2y=4x+e^{3x}$ h) $y''+y'=3x+4e^{x}$ i) $y''-3y'+2y=2x^{2}+e^{x}$

Solution:

a) Here, $k^2+k-6=0 \Rightarrow (k+3)(k-2)=0 \Rightarrow k=-3,2$.

Hence, $y_c=c_1e^{-3x}+c_2e^{2x}$. Now, $y_p=ax+b+ce^{-3x}$. But the term ce^{-3x} already exists in y_c. So, it must be multiplied by x. That is $y_p=ax+b+cxe^{-3x}$.

Then, $y'_p=a+ce^{-3x}-3cxe^{-3x}, y''_p=-3ce^{-3x}-3ce^{-3x}+9cxe^{-3x}$

$$-3ce^{-3x}-3ce^{-3x}+9cxe^{-3x}+a+ce^{-3x}-3cxe^{-3x}-6(ax+b+cxe^{-3x})=x+e^{-3x}$$

$$\Rightarrow -5ce^{-3x}-6ax+a-6b=x+e^{-3x} \Rightarrow a=\frac{-1}{6}, b=\frac{-1}{36}, c=\frac{-1}{5}$$

Therefore, $y=y_c+y_p=c_1e^{-3x}+c_2e^{2x}-\frac{1}{6}x-\frac{1}{5}xe^{-3x}-\frac{1}{36}$.

b) Here, $r^2-1=0 \Rightarrow r_1=1, r_2=-1$ and $y_p=a\cos 2x+b\sin 2x$.

$y''-y=\cos 2x \Rightarrow -5a\cos 2x-5b\sin 2x=\cos 2x \Rightarrow -5a=1, -5b=0$

$$\Rightarrow a=-1/5, b=0 \Rightarrow y_p=-1/5\cos 2x \Rightarrow y=c_1e^{x}+c_2e^{-x}-1/5\cos 2x$$

c) Here, the characteristics equation $r(r-1)=0 \Rightarrow r_1=0, r_2=1$.

So, $y_c=c_1+c_2e^{x}$. Now, let's find y_p having the same form as $f(x)=x+e^{x}$. Since $b=0$(as in rule-2) and $r_2=1$ as in rule-1, we guess $y_{p1}=x(ax+b)$ for $g(x)=x$ and $y_{p2}=cxe^{x}$ for $h(x)=e^{x}$. Then,

$y_p=y_{p1}+y_{p2}=ax^2+bx+cxe^{x}$ for $f(x)=x+e^{x}$. Hence,

$y'_p=2ax+b+ce^{x}+cxe^{x}$, $y''_p=2a+2ce^{x}+cxe^{x}$ .Thus,

$y''-y'=x+e^{x} \Rightarrow 2a+2ce^{x}+cxe^{x}-[2ax+b+ce^{x}+cxe^{x}]=x+e^{x}$

$$\Rightarrow -2ax+ce^{x}+2a-b=x+e^{x} \Rightarrow -2a=1, c=1, 2a-b=0$$

$$\Rightarrow a=-\frac{1}{2}, b=-1, c=1 \Rightarrow y_p=-\frac{x^2}{2}-x+xe^{x}$$

d) Here, $r^2+3r=0 \Rightarrow r=0, r=-3$. So, $y_c=c_1+c_2e^{-3x}$. Now, let's find y_p having the same form as $f(x)=6e^{-3x}+4x$. Since $b=0$(as in form 2) and $r_2=-3$ as in form 1, we guess $y_{p1}=x(ax+b)$ for $g(x)=4x$ and $y_{p2}=cxe^{-3x}$ for $h(x)=6e^{-3x}$.

Then, $y_p=y_{p1}+y_{p2}=ax^2+bx+cxe^{-3x}$ for $f(x)=4x+6e^{-3x}$. Hence,

$y'_p=2ax+b+ce^{-3x}-3cxe^{-3x}$, $y''_p=2a-6ce^{-3x}+9cxe^{-3x}$. Thus,

$$y''+3y'=4x+6e^{-3x} \Rightarrow 2a-6ce^{-3x}+9cxe^{-3x}+3(2ax+b+ce^{-3x}-3cxe^{-3x})=4x+6e^{-3x}$$

$$\Rightarrow 2a-3ce^{-3x}+6ax+3b=4x+6e^{-3x}$$

$$\Rightarrow 6a=4, -3c=6, 2a+3b=0 \Rightarrow a=\frac{2}{3}, c=-2, b=-\frac{4}{9}$$

$$\Rightarrow y_p=\frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$$

Therefore, $y=c_1+c_2e^{-3x}+\frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$.

e) $y''-2y'=0 \Rightarrow r^2-2r=0 \Rightarrow r(r-2)=0 \Rightarrow r_1=0, r_2=2 \Rightarrow y_c=c_1+c_2e^{2x}$

Now, lets' find the particular solution y_p which is the same form as

$f(x)=e^x\sin x$. Here, y_p is of the form $y_p=(a\cos x+b\sin x)e^x$ and thus

$y'_p=(b\cos x-a\sin x+a\cos x+b\sin x)e^x, y''_p=(-2a\sin x+2b\cos x)e^x$

So, $y''-2y'=e^x\sin x$

$$\Rightarrow (-2a\sin x+2b\cos x)e^x-2(b\cos x-a\sin x+a\cos x+b\sin x)e^x=e^x\sin x$$

$$\Rightarrow -2ae^x\cos x-be^x\sin x=e^x\sin x \Rightarrow -2a=0, -2b=1 \Rightarrow a=0, b=-\frac{1}{2}$$

Hence, $y_p=-\frac{1}{2}e^x\sin x \Rightarrow y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$.

g) Here, $f(x)=4x+e^{3x}$ indicates $y_p=ax+b+ce^{3x}$.

Then, using $y'_p=a+3ce^{3x}$, $y''_p=9ce^{3x}$, we have

d) Here, $r^2+3r=0 \Rightarrow r=0, r=-3$. So, $y_c=c_1+c_2e^{-3x}$. Now, let's find y_p having the same form as $f(x)=6e^{-3x}+4x$. Since $b=0$(as in form 2) and $r_2=-3$ as in form 1, we guess $y_{p1}=x(ax+b)$ for $g(x)=4x$ and $y_{p2}=cxe^{-3x}$ for $h(x)=6e^{-3x}$.

Then, $y_p=y_{p1}+y_{p2}=ax^2+bx+cxe^{-3x}$ for $f(x)=4x+6e^{-3x}$. Hence,

$y'_p=2ax+b+ce^{-3x}-3cxe^{-3x}$, $y''_p=2a-6ce^{-3x}+9cxe^{-3x}$.Thus,

$$y''+3y'=4x+6e^{-3x} \Rightarrow 2a-6ce^{-3x}+9cxe^{-3x}+3(2ax+b+ce^{-3x}-3cxe^{-3x})=4x+6e^{-3x}$$

$$\Rightarrow 2a-3ce^{-3x}+6ax+3b=4x+6e^{-3x}$$

$$\Rightarrow 6a=4,-3c=6,2a+3b=0 \Rightarrow a=\frac{2}{3},c=-2,b=-\frac{4}{9}$$

$$\Rightarrow y_p=\frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$$

Therefore, $y=c_1+c_2e^{-3x}+\frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$.

e) $y''-2y'=0 \Rightarrow r^2-2r=0 \Rightarrow r(r-2)=0 \Rightarrow r_1=0, r_2=2 \Rightarrow y_c=c_1+c_2e^{2x}$

Now, lets' find the particular solution y_p which is the same form as $f(x)=e^x\sin x$. Here, y_p is of the form $y_p=(a\cos x+b\sin x)e^x$ and thus

$y'_p=(b\cos x-a\sin x+a\cos x+b\sin x)e^x, y''_p=(-2a\sin x+2b\cos x)e^x$

So, $y''-2y'=e^x\sin x$

$$\Rightarrow(-2a\sin x+2b\cos x)e^x-2(b\cos x-a\sin x+a\cos x+b\sin x)e^x=e^x\sin x$$

$$\Rightarrow -2ae^x\cos x-be^x\sin x=e^x\sin x \Rightarrow -2a=0,-2b=1 \Rightarrow a=0,b=-\frac{1}{2}$$

Hence, $y_p=-\frac{1}{2}e^x\sin x \Rightarrow y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$.

g) Here, $f(x)=4x+e^{3x}$ indicates $y_p=ax+b+ce^{3x}$.

Then, using $y'_p=a+3ce^{3x}, y''_p=9ce^{3x}$, we have

$$y''+3y'+2y=4x+e^{3x}\Rightarrow 9ce^{3x}+3(a+3ce^{3x})+2(ax+b+ce^{3x})=4x+e^{3x}$$

$$\Rightarrow 2ax+3a+2b+20ce^{3x}=4x+e^{3x}$$

$$\Rightarrow 2a=4, 3a+2b=0, 20c=1\Rightarrow a=2, b=-3, c=\frac{1}{20}$$

$$\Rightarrow y_p=2x+\frac{e^{3x}}{20}-3$$

h) Here, $r^2+r=0\Rightarrow r=0, r=-1$

$f(x)=4x+e^{3x}$ indicates $y_p=ax+b+ce^{3x}$.

Then, using $y'_p=a+3ce^{3x}$, $y''_p=9ce^{3x}$, we have

3. Using Method of Undetermined Coefficients, solve the IVPs and BVPs.

$a)\ y''-4y'=16x, y(0)=1, y'(0)=3 \quad b)\ y''-2y'+y=\sinh x, y(0)=\frac{1}{8}, y'(1)=e$

Solution:

a) Here, $r^2-4r=0\Rightarrow r=0,4$. Thus, $y_c=c_1+c_2e^{4x}$. Besides, as $f(x)=16x$, y_p is of the form $y_p=x(ax+b)$. Then, $y'_p=2ax+b$, $y''_p=2a$.

So, $y''-4y'=2x\Rightarrow 2a-4(2ax+b)=16x\Rightarrow a=-2, b=-1$

Hence, the general solution is $y=c_1+c_2e^{4x}-2x^2-x$

Here, $y(0)=1, y'(0)=3\Rightarrow \begin{cases} c_1+c_2=1 \\ 4c_2-1=3 \end{cases}\Rightarrow c_2=1, c_1=0$.

Therefore, the solution is $y=e^{4x}-2x^2-x$

4. Find a DE whose general solution is $y=c_1+c_2e^{-x}+\frac{x^2}{2}-x$

Solution: Here, from $y_c=c_1+c_2e^{-x}$, we can infer the roots $r_1=0, r_2=-1$.

So, the characteristics equation is $(r-0)(r+1)=0\Rightarrow r^2+r=0$.

Hence, the corresponding homogeneous DE is $y''+y'=0$.

Now, putting $y_p=\frac{x^2}{2}-x$, in $y''+y'=f(x)$ we get $y''+y'=x$.

1.8.4 Method of Variation of Parameters (VPs)

How the method is developed? Why it is so named?

The particular solution y_p is a *pseudo-linear* combination of the homogeneous equation. By a pseudo-linear combination we mean an expression that has the same form as a linear combination, but the constants in the linear combination are allowed to depend on x: $y_p(x)=u_1(x)y_1+u_2(x)y_2$.

In the combination $y_p(x)=u_1(x)y_1+u_2(x)y_2$, the parameters (the constants in the linear combination) u_1 and u_2 are assumed to be variables. That means the particular solution is the combination of the fundamental solutions with variable coefficients. That is why the method is so named.

Suppose y_1 and y_2 are fundamental solutions of $ay''+by'+cy=0$.

Then, we look for a pair of functions u_1 and u_2 that will make the combination given by $y_p(x)=u_1(x)y_1+u_2(x)y_2$ a solution of $ay''+by'+cy=f(x)$.

Putting the values $y_p(x), y'_p(x), y''_p(x)$ in $ay''+by'+cy=f(x)$ gives the linear system $\begin{cases} u'_1 y_1+u'_2 y_2=0 \\ u'_1 y'_1+u'_2 y'_2=f(x) \end{cases}$. In matrix form, $\begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}=\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Since y_1 and y_2 are linearly independent, $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}\neq 0$.

Hence, solving the above linear system by Cramer's rule, we have

$$u'_1(x)=\frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}}{W(y_1,y_2)}=\frac{-f(x)y_2}{W(y_1,y_2)},\quad u'_2(x)=\frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}}{W(y_1,y_2)}=\frac{f(x)y_1}{W(y_1,y_2)}.$$

Hence, by integration, $u_1(x)=-\int\frac{f(x)y_2}{W(y_1,y_2)}dx,\ u_2(x)=\int\frac{f(x)y_1}{W(y_1,y_2)}dx$.

Therefore, the particular solution $y_p(x)=u_1(x)y_1+u_2(x)y_2$ that we are looking for becomes $y_p(x)=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$.

Summary of Procedures to find y_p using VPs:

Objective: To solve the DE $ay''+by'+cy=f(x)$.

Step-1: Find y_1 and y_2 of part $ay''+by'+cy=0$ and compute their wronskian.

That is solve $ar^2+br+c=0$ and compute $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$.

Step-2: Find y_p using $y_p=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$.

Finally: The General solution is $y=y_C+y_p$ by superposition principle.

Examples:

1. Solve the following NHLDEs using Variation of Parameters

$a)\ y''+y=\sec x$ $\quad b)\ y''-2y+y=\frac{e^x}{x}$ $\quad c)\ y''+2y'+y=e^{-x}\ln x$

$d)\ y''+3y'+2y=e^{-2x}\cos x$ $\quad e)\ y''-5y'+6y=x$ $\quad f)\ y''-2y'+2y=e^x\tan x$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+y=0$

Here, $r^2+1=0\Rightarrow(r-i)(r+i)=0\Rightarrow r_1=-i, r_2=i$.

Hence, the fundamental solutions are $y_1=\cos x$, $y_2=\sin x$.

Then the Wronskian becomes $W(y_1,y_2)=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=1$.

Step-2: Find the particular solution y_p using the formula.

$$\begin{aligned} y_p &= -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx \\ &= -\cos x\int\sec x\sin x\,dx+\sin x\int\sec x\cos x\,dx \\ &= -\cos x\int\tan x\,dx+\sin x\int dx \\ &= \cos x\ln|\cos x|+x\sin x \end{aligned}$$

Therefore, the solution is $y=c_1\cos x+c_2\sin x+\cos x\ln|\cos x|+x\sin x$

b) **Step-1:** Here, $r^2-2r+1=0\Rightarrow r=1$. Hence, the fundamental solutions are $y_1=e^x$, $y_2=xe^x$ and their Wronskian becomes $W(y_1,y_2)=e^{2x}$.

Step-2: Find y_p.

$$y_p(x) = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^x\int\frac{e^x}{xe^{2x}}xe^x dx + xe^x\int\frac{e^x}{xe^{2x}}e^x dx$$

$$= -e^x\int dx + xe^x\int\frac{1}{x}dx = xe^x\ln|x| - xe^x$$

Therefore, the general solution is $y = c_1e^x + c_2xe^x + xe^x\ln|x| - xe^x$

c) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+2y'+y=0$

That is $r^2+2r+1=0 \Rightarrow (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1$.

Hence, the fundamental solutions are $y_1 = e^{-x}$, $y_2 = xe^{-x}$.

Then, the wronskian becomes $W(y_1,y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x}\end{vmatrix} = e^{-2x}$.

Step-2: Find the particular solution y_p

$$y_p = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^{-x}\int\frac{xe^{-x}(e^{-x}\ln x)}{e^{-2x}}dx + xe^{-x}\int\frac{e^{-x}(e^{-x}\ln x)}{e^{-2x}}dx$$

$$= -e^{-x}\int x\ln x dx + xe^{-x}\int \ln x dx$$

Now using by parts, $\int x\ln x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$ and $\int \ln x dx = x\ln x - x$

Hence, $y_p = -e^{-x}\int x\ln x dx + xe^{-x}\int \ln x dx = \frac{x^2}{2}e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$

Therefore, the solution is $y = c_1e^{-x} + c_2xe^{-x} + \frac{x^2}{2}e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$.

d) **Step-1:** Here, $r^2+3r+2=0 \Rightarrow r_1 = -2, r_2 = -1$.

Hence, the fundamental solutions are $y_1 = e^{-2x}$, $y_2 = e^{-x}$.

Then, their Wronskian becomes $W(y_1,y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x}\end{vmatrix} = e^{-3x}$.

Step-2: Find y_p.

$$y_p = -e^{-2x}\int \frac{e^{-2x}\cos x}{e^{-3x}} e^{-x}dx + e^{-x}\int \frac{e^{-2x}\cos x}{e^{-3x}} e^{-2x}dx$$

$$= -e^{-2x}\int \cos x dx + e^{-x}\int e^{-x}\cos x dx$$

$$= -e^{-2x}\sin x + \frac{e^{-x}}{2}(e^{-x}\sin x - e^{-x}\cos x) = \frac{-e^{-2x}}{2}(\sin x + \cos x)$$

Therefore, the general solution is $y = c_1e^{-2x} + c_2e^{-x} - \frac{e^{-2x}}{2}(\sin x + \cos x)$

e) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-5y'+6y=0$

As we did in part (a), $y_1 = e^{3x}$, $y_2 = e^{2x}$ and $W(y_1, y_2) = -e^{5x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx = -e^{3x}\int \frac{xe^{2x}}{-e^{5x}}dx + e^{2x}\int \frac{xe^{3x}}{-e^{5x}}dx$$

$$= e^{3x}\int xe^{-3x}dx - e^{2x}\int xe^{-2x}dx = e^{3x}\left(-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right) - e^{2x}\left(-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right)$$

$$= \frac{-x}{3} - \frac{1}{9} + \frac{x}{2} + \frac{1}{4} = \frac{x}{6} + \frac{5}{36}$$

Therefore, the general solution is $y = c_1e^{3x} + c_2e^{2x} + \frac{x}{6} + \frac{5}{36}$.

f) Here, $r^2 - 2r + 2 = 0 \Rightarrow r_1 = 1+i, r_2 = 1-i$. Hence, the fundamental solutions are $y_1 = e^x\cos x$, $y_2 = e^x\sin x$ and their Wronskian becomes

$$W(y_1, y_2) = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\sin x + e^x\cos x \end{vmatrix} = e^{2x}.$$

Now, find the particular solution y_p

$$y_p = -e^x\cos x\int \frac{\sin^2 x}{\cos x}dx + e^x\sin x\int \sin x dx$$

$$= -e^x\cos x\int (\sec x - \cos x)dx + e^x\sin x\int \sin x dx = -e^x\cos x\ln|\sec x + \tan x|$$

Therefore, $y = c_1e^x\cos x + c_2e^x\sin x - e^x\cos x\ln|\sec x + \tan x|$

Step-2: Find y_p.

$$y_p = -e^{-2x}\int \frac{e^{-2x}\cos x}{e^{-3x}} e^{-x}dx + e^{-x}\int \frac{e^{-2x}\cos x}{e^{-3x}} e^{-2x}dx$$

$$= -e^{-2x}\int \cos x dx + e^{-x}\int e^{-x}\cos x dx$$

$$= -e^{-2x}\sin x + \frac{e^{-x}}{2}(e^{-x}\sin x - e^{-x}\cos x) = \frac{-e^{-2x}}{2}(\sin x + \cos x)$$

Therefore, the general solution is $y = c_1e^{-2x} + c_2e^{-x} - \frac{e^{-2x}}{2}(\sin x + \cos x)$

e) **Step-1:** Find the fundamental solutions y_1, y_2 of $y'' - 5y' + 6y = 0$

As we did in part (a), $y_1 = e^{3x}$, $y_2 = e^{2x}$ and $W(y_1, y_2) = -e^{5x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx = -e^{3x}\int \frac{xe^{2x}}{-e^{5x}}dx + e^{2x}\int \frac{xe^{3x}}{-e^{5x}}dx$$

$$= e^{3x}\int xe^{-3x}dx - e^{2x}\int xe^{-2x}dx = e^{3x}\left(-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right) - e^{2x}\left(-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right)$$

$$= \frac{-x}{3} - \frac{1}{9} + \frac{x}{2} + \frac{1}{4} = \frac{x}{6} + \frac{5}{36}$$

Therefore, the general solution is $y = c_1e^{3x} + c_2e^{2x} + \frac{x}{6} + \frac{5}{36}$.

f) Here, $r^2 - 2r + 2 = 0 \Rightarrow r_1 = 1 + i, r_2 = 1 - i$. Hence, the fundamental solutions are $y_1 = e^x\cos x$, $y_2 = e^x\sin x$ and their Wronskian becomes

$$W(y_1, y_2) = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\sin x + e^x\cos x \end{vmatrix} = e^{2x}.$$

Now, find the particular solution y_p

$$y_p = -e^x\cos x\int \frac{\sin^2 x}{\cos x}dx + e^x\sin x\int \sin x dx$$

$$= -e^x\cos x\int (\sec x - \cos x)dx + e^x\sin x\int \sin x dx = -e^x\cos x\ln|\sec x + \tan x|$$

Therefore, $y = c_1e^x\cos x + c_2e^x\sin x - e^x\cos x\ln|\sec x + \tan x|$

2. Solve the following DEs using Variation of Parameters

$a)\ y''+9y=\csc 3x$ $\quad b)\ y''-2y'+2y=8e^{x}\sin x$ $\quad c)\ y''-9y=\frac{9x}{e^{3x}}$

$d)\ y''-y=\sinh 2x$ $\quad e)\ y''+y=\tan x$ $\quad f)\ y''+y=\cot x$

$g)\ y''+4y'+4y=e^{-x}$ $\quad h)\ y''-4y'+4y=x^{2}e^{x}$ $\quad i)\ y''+y=x\sin x$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+9y=0$

Here, the characteristics equation is $r^2+9=0$. Solving this gives us $r^2+9=0 \Rightarrow r_1=-3i, r_2=3i$.

Hence, the fundamental solutions are $y_1=\cos 3x,\ y_2=\sin 3x$ and their Wronskian becomes $W(y_1,y_2)=\begin{vmatrix}\cos 3x & \sin 3x\\ -3\sin 3x & 3\cos 3x\end{vmatrix}=3$.

Step-2: Find the particular solution y_p

$$y_p(x)=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$=-\cos 3x\int\frac{1}{3}\csc 3x\sin 3x\,dx+\sin 3x\int\frac{1}{3}\csc 3x\cos 3x\,dx$$

$$=-\frac{1}{3}\cos 3x\int dx+\frac{1}{3}\sin 3x\int\cot 3x\,dx=-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x|$$

Therefore, $y=c_1\cos 3x+c_2\sin 3x-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x|$

b) Here, the characteristics equation is $r^2-2r+2=0$. Solving this gives us $r^2-2r+2=0\Rightarrow r_1=1+i, r_2=1-i$. Hence, the fundamental solutions are $y_1=e^{x}\cos x,\ y_2=e^{x}\sin x$ and their Wronskian becomes

$$W(y_1,y_2)=\begin{vmatrix}e^{x}\cos x & e^{x}\sin x\\ e^{x}\cos x-e^{x}\sin x & e^{x}\sin x+e^{x}\cos x\end{vmatrix}=e^{2x}.$$

Now, find the particular solution y_p

$$y_p = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^x\cos x\int 8\sin^2 x dx + e^x\sin x\int 8\sin x\cos x dx$$

$$= -e^x\cos x\int 4(1-\cos 2x)dx + 4e^x\sin x\int\sin 2x dx$$

$$= -e^x\cos x(4x - 2\sin 2x) - 2e^x\sin x\cos 2x$$

Therefore, $y = c_1e^x\cos x + c_2e^x\sin x - e^x\cos x(4x-2\sin 2x) - 2e^x\sin x\cos 2x$

c) Here, $r^2-9=0\Rightarrow r_1=3, r_2=-3$. Hence, the fundamental solutions are $y_1=e^{3x}$, $y_2=e^{-3x}$ and $W(y_1,y_2)=-6$. Now, find y_p.

$$y_p = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^{3x}\int -\frac{9x}{6e^{3x}}(e^{-3x})dx + e^{-3x}\int -\frac{9x}{6e^{3x}}(e^{3x})dx$$

$$= \frac{3e^{3x}}{2}\int xe^{-6x}dx - \frac{3e^{-3x}}{2}\int xdx = \frac{-e^{-3x}}{24} - \frac{xe^{-3x}}{4} - \frac{3x^2e^{-3x}}{4}$$

d) **Step-1:** Here, $r^2-1=0\Rightarrow r_1=1, r_2=-1$.

Hence, the fundamental solutions are $y_1=e^x$, $y_2=e^{-x}$ and their Wronskian becomes $W(y_1,y_2)=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x}\end{vmatrix}=-2$.

Step-2: Find y_p. Observe that $f(x)=\sinh 2x=\dfrac{e^{2x}-e^{-2x}}{2}$

$$y_p(x) = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= e^x\int\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)e^{-x}dx - e^{-x}\int\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)e^x dx$$

$$= \frac{e^x}{4}\int(e^x - e^{-3x})dx - \frac{e^{-x}}{4}\int(e^{3x}-e^{-x})dx$$

$$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{12} - \frac{e^{2x}}{12} - \frac{e^{-2x}}{4} = \frac{1}{6}(e^{2x}-e^{-2x}) = \frac{1}{3}\sinh 2x$$

Therefore, the general solution is $y = c_1 e^x + c_2 e^{-x} + \frac{1}{3}\sinh 2x$

e) **Step-1:** Find the fundamental solutions y_1, y_2 of $y'' + y = 0$

Here, the characteristics equation is $r^2 + r = 0$. Solving this gives us
$r^2 + r = 0 \Rightarrow (r - i)(r + i) = 0 \Rightarrow r_1 = -i, r_2 = i$.

Hence, $y_1 = \cos x$, $y_2 = \sin x$ and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$.

Step-2: Find the particular solution y_p

$$y_p(x) = -y_1 \int \frac{f(x) y_2}{W(y_1, y_2)} dx + y_2 \int \frac{f(x) y_1}{W(y_1, y_2)} dx$$

$$= -\cos x \int \tan x \sin x dx + \sin x \int \tan x \cos x dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x dx = -\cos x \int \frac{1 - \cos^2 x}{\cos x} dx - \sin x \cos x$$

$$= \cos x \int (\cos x - \sec x) dx - \sin x \cos x$$

$$= \cos x \left(\int \cos x dx - \int \sec x dx \right) - \sin x \cos x = -\cos x \ln|\sec x + \tan x|$$

Therefore, the general solution is $y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

f) Here, $r^2 + 1 = 0 \Rightarrow r_1 = -i, r_2 = i$. Hence, the fundamental solutions are

$y_1 = \cos x$, $y_2 = \sin x$ and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$. Then,

$$y_p = -\cos x \int \cot x \sin x dx + \sin x \int \cot x \cos x dx$$

$$= -\cos x \int \cos x dx + \sin x \int \frac{\cos^2 x}{\sin x} dx$$

$$= -\cos x \sin x + \sin x \int (\csc x - \sin x) dx, \quad \cos^2 x = 1 - \sin^2 x$$

$$= -\cos x \sin x - \sin x \ln|\csc x + \cot x| + \sin x \cos x = -\sin x \ln|\csc x + \cot x|$$

Therefore, $y = c_1 \cos x + c_2 \sin x - \sin x \ln|\csc x + \cot x|$

g) **Step-1:** Here, $r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r_1 = r_2 = -2$.

Hence, the fundamental solutions are $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

Then, $W(y_1, y_2) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}$.

Step-2: Find the particular solution y_p

$$y_p = -e^{-2x}\int \frac{e^{-x}(xe^{-2x})}{e^{-4x}}dx + xe^{-2x}\int \frac{e^{-x}(e^{-2x})}{e^{-4x}}dx$$

$$= -e^{-2x}\int xe^{x}dx + xe^{-2x}\int e^{x}dx = -e^{-2x}(xe^{x} - e^{x}) + xe^{-2x}(e^{x}) = e^{-x}$$

Therefore, the general solution is $y = y_c + y_p = c_1e^{-2x} + c_2xe^{-2x} + e^{-x}$.

h) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-4y'+4y=0$.

Here, $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ (we have repeated roots).

So. the fundamental solutions are $y_1 = e^{2x}$, $y_2 = xe^{2x}$ and their Wronskian

becomes $W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -y_1\int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1, y_2)}dx = -e^{2x}\int \frac{x^3e^{3x}}{e^{4x}}dx + xe^{2x}\int \frac{x^2e^{3x}}{e^{4x}}dx$$

$$= -e^{2x}\int x^3e^{-x}dx + xe^{2x}\int x^2e^{-x}dx = (x^2+4x+6)e^{2x}$$

Therefore, the general solution is $y = c_1e^{2x} + c_2xe^{2x} + (x^2+4x+6)e^{x}$.

3. Solve the following IVPs using **Variation of parameters.**

a) $y''-5y'+4y = e^{2x}, y(0) = 1, y'(0) = 0$

b) $y''-2y'+y = e^{x}\sin x, y(0) = 3, y'(0) = 0$

c) $y''+2y'+5y = e^{-x}\sin x, y(0) = 0, y'(0) = 1$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-5y'+4y = 0$

Here, the characteristics equation is $r^2 - 5r + 4 = 0$. Solving this gives us $r^2 - 5r + 4 = 0 \Rightarrow (r-4)(r-1) = 0 \Rightarrow r_1 = 4, r_2 = 1$.

Hence, $y_1 = e^{4x}$, $y_2 = e^{x}$ and $W(y_1, y_2) = \begin{vmatrix} e^{4x} & e^{x} \\ 4e^{4x} & e^{x} \end{vmatrix} = -3e^{5x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -e^{4x}\int \frac{e^{2x}(e^x)}{-3e^{5x}}dx + e^x \int \frac{e^{2x}(e^{4x})}{-3e^{5x}}dx$$

$$= \frac{e^{4x}}{3}\int e^{-2x}dx - \frac{e^x}{3}\int e^x dx = -\frac{e^{2x}}{6} - \frac{e^{2x}}{3} = -\frac{e^{2x}}{2}$$

Therefore, the general solution is $y = y_c + y_p = c_1e^{4x} + c_2e^x - \frac{e^{2x}}{2}$.

Now, let's determine the constants c_1 and c_2 using the initial conditions.

$$\begin{cases} y(0) = 1 \Rightarrow c_1 + c_2 - \frac{1}{2} = 1 \Rightarrow c_1 + c_2 = \frac{3}{2} \\ y'(0) = 0 \Rightarrow 4c_1 + c_2 - 1 = 0 \Rightarrow 4c_1 + c_2 = 1 \end{cases} \Rightarrow c_1 = -\frac{1}{6}, c_2 = \frac{5}{3}$$

Hence, the solution of the IVP becomes $y = \frac{5}{3}e^x - \frac{1}{6}e^{4x} - \frac{e^{2x}}{2}$.

b) **Step-1:** Here, $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$.

Hence, $y_1 = e^x$, $y_2 = xe^x$ and $W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -e^x\int \frac{e^x \sin x(xe^x)}{e^{2x}}dx + xe^x\int \frac{e^x \sin x(e^x)}{e^{2x}}dx$$

$$= -e^x\int x\sin x dx + xe^x\int \sin x dx$$

$$= -e^x(\sin x - x\cos x) - xe^x\cos x = -e^x\sin x$$

Therefore, the general solution is $y = y_c + y_p = c_1e^x + c_2xe^x - e^x\sin x$.

Now, let's determine the constants c_1 and c_2 using the initial conditions.,

$$\begin{cases} y(0) = 3 \Rightarrow c_1 = 3 \\ y'(0) = 0 \Rightarrow c_1 + c_2 - 1 = 0 \Rightarrow 3 + c_2 - 1 = 0 \end{cases} \Rightarrow c_1 = 3, c_2 = -2$$

Hence, the solution of the IVP becomes $y = 3e^x - 2xe^x - e^x\sin x$.

c) **Step-1:** Find the fundamental solutions y_1, y_2 of $y'' + 2y' + 5y = 0$

Here, $r^2 + 2r + 5 = 0 \Rightarrow r = -1 + 2i, -1 - 2i$.

Hence, the fundamental solutions are $y_1 = e^{-x}\cos 2x,\ y_2 = e^{-x}\sin 2x$ and

$$W(y_1, y_2) = \begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ -e^{-x}\cos 2x - 2e^{-x}\sin 2x & 2e^{-x}\cos 2x - e^{-x}\sin 2x \end{vmatrix} = 2e^{-2x}.$$

Step-2: Find the particular solution y_p

$$y_p = -e^{-x}\cos 2x \int \frac{e^{-x}\sin x(e^{-x}\sin 2x)}{2e^{-2x}}dx + e^{-x}\sin 2x \int \frac{e^{-x}\sin x(e^{-x}\cos 2x)}{2e^{-2x}}dx$$

$$= \frac{-e^{-x}\cos 2x}{2}\int \sin x \sin 2x dx + \frac{e^{-x}\sin 2x}{2}\int \sin x \cos 2x dx \quad \text{(Complete it!)}$$

4. Solve the following NHLDEs using Variation of parameters

a) $y''-4y'+4y = \dfrac{2e^{2x}}{x^2+1}$ *b)* $y''-y = \dfrac{e^x}{1+e^{2x}}$ *c)* $y''-3y'+2y = \dfrac{e^x}{1+e^x}$

Solution:

a) **Step-1:** Here, $r^2-4r+4=0 \Rightarrow (r-2)^2=0 \Rightarrow r=2$. Hence, the fundamental solutions are $y_1 = e^{2x},\ y_2 = xe^{2x}$ and $W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x}$

Step-2: Find the particular solution y_p

$$y_p = -e^{2x}\int \left(\frac{2e^{2x}}{x^2+1}\right)\frac{xe^{2x}}{e^{4x}}dx + xe^{2x}\int \left(\frac{2e^{2x}}{x^2+1}\right)\frac{e^{2x}}{e^{4x}}dx$$

$$= -e^{2x}\int \frac{2x}{x^2+1}dx + xe^{2x}\int \frac{2}{x^2+1}dx = -e^{2x}\ln(x^2+1) + 2xe^{2x}\tan^{-1}x$$

Therefore, the solution is $y = c_1e^{2x} + c_2xe^{2x} - e^{2x}\ln(x^2+1) + 2xe^{2x}\tan^{-1}x$.

b) **Step-1:** Here, $r^2-1=0 \Rightarrow r_1=1, r_2=-1$. Hence, the fundamental solutions are $y_1 = e^x,\ y_2 = e^{-x}$ and $W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

Step-2: Find the particular solution y_p

$$y_p(x) = -e^x\int \left(\frac{e^x}{1+e^{2x}}\right)\frac{e^{-x}}{-2}dx + e^{-x}\int \left(\frac{e^x}{1+e^{2x}}\right)\frac{e^x}{-2}dx$$

$$= \frac{e^x}{2}\int \frac{1}{1+e^{2x}}dx - \frac{e^{-x}}{2}\int \frac{e^{2x}}{1+e^{2x}}dx$$

Now consider the integrals $\int \frac{1}{1+e^{2x}}dx$ and $\int \frac{e^{2x}}{1+e^{2x}}dx$ separately.

1.9 System of First Order Linear Differential Equations

Revision on Matrix:

System of Linear Equations: Consider a 2×2 system: $\begin{cases} ax+by=k_1 \\ cx+dy=k_2 \end{cases}$

We know that such system of linear equations can be written or expressed in matrix form as $\mathrm{AX}=\mathrm{B}$ where $\mathrm{X}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathrm{A}=\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}, \mathrm{B}=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

Eigen-values and Eigen-vectors:

From the matrix representation, we have also seen how to determine the eigen-values and the corresponding eigen-vectors associated with the coefficient matrix A by forming its characteristics equation.

Characteristics Equation: $\det(\mathbf{A}-\lambda\mathbf{I})=0$ where **I** is identity matrix.

Eigen-values: The solution of the characteristics equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$.

Eigen-vectors: The vector with the property $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=0$.

These algebraic concepts are the basis for analysis of systems of DE.

1.12.1 Homogeneous Systems with Constant Coefficients

A system of linear differential equation is a system of equation involving the derivative of two or more variables about the same input parameter, usually denoted by t. Here under, we are going to see a system of two variables x and y about the same parameter t.

Notations: Derivatives wit respect to t: $\frac{dx}{dt}=x', \frac{dy}{dt}=y'$

Form of the system: $\begin{cases} \frac{dx}{dt}=ax+by \\ \frac{dy}{dt}=cx+dy \end{cases}$ **or** $\begin{cases} x'=ax+by \\ y'=cx+dy \end{cases}$

In matrix Form or in vector:

$$\mathrm{X}'=\mathrm{AX} \text{ or } \mathrm{X}'=\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{where } \mathrm{X}'=\begin{pmatrix} x' \\ y' \end{pmatrix}, \mathrm{A}=\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}, \mathrm{X}=\begin{pmatrix} x \\ y \end{pmatrix}.$$

Fundamental solutions: In general, like that of second order DE, such systems have two fundamental solutions X_1 and X_2. But the form of the fundamental solutions X_1 and X_2 depends on the nature of the roots of the characteristics equation $\det(A-\lambda I)=0$ of the system. So, our next task is how to find such fundamental solutions using eigenvector method.

Since the eigenvalues are the roots of $\det(A-\lambda I)=0$, they could be two distinct real roots, single root or complex roots. That is if the coefficient matrix of the system is $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\det(A-\lambda I)=0 \Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}=0 \Rightarrow \lambda^2-(a+d)\lambda+ad-bc=0$$

$$\Rightarrow \lambda^2-\text{trace}(A)\lambda+\det A=0 \quad (\text{Note: trace}(A)=a+d)$$

Now, consider the three cases for this quadratic equation.

Case-I: Two distinct real roots $\lambda_1 \neq \lambda_2$.

Then, the fundamental solutions are $X_1=v_1e^{\lambda_1 t}$, $X_2=v_2e^{\lambda_2 t}$.

Case-II: Single or repeated real root $\lambda_1=\lambda_2=\lambda$.

Then, the fundamental solutions are $X_1=v_1e^{\lambda t}$, $X_2=(v_2t+v_1)e^{\lambda t}$ where v_2 is a vector to be determined from the condition $(A-\lambda I)v_2=v_1$.

Case-III: Complex conjugate roots $\lambda=\alpha\pm\beta i$.

Now, for eigenvalue $\lambda=\alpha+\beta i$, calculate a complex eigenvector $v^*=\begin{pmatrix} a \\ b \end{pmatrix}$.

That is $(A-\lambda I)v^*=0 \Rightarrow (A-\lambda I)\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now, identify the real and imaginary parts of $v=v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

That is $\begin{cases} X_1=\text{Re}(v)=\text{Re}[v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)] \\ X_2=\text{Im}(v)=\text{Im}[v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)] \end{cases}$

Complementary Solution: Therefore, $X_c=c_1X_1+c_2X_2$.

Eugenvalue Approach to solve system of DE:

The process of solving system of linear differential equations using the characteristics equation is known as eigenvalue or matrix approach.

Procedures:

First: Find the eigenvalues of the coefficient matrix A using $|\mathbf{A}-\lambda\mathbf{I}|=0$.

Second: Find the eigenvectors corresponding to each eigenvalue. But be careful to analyze the three cases of the forms of the eigenvalues.

Third: Find the fundamental solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ and write the complementary solution $\mathbf{X}_c$. That is $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$.

Examples:

1. Find the general solution for the systems of differential equations.

a) $\begin{cases} \frac{dx}{dt}=3x+2y \\ \frac{dy}{dt}=x+4y \end{cases}$ b) $\begin{cases} x'=3x+y \\ y'=-x+y \end{cases}$ c) $\begin{cases} x'=2x+3y \\ y'=-3x+2y \end{cases}$ d) $\mathbf{X}'=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{X}$

Solution:

a) This is an example with two distinct real roots.

First, identify the coefficient matrix A and find its exigent-values.

The coefficient matrix is $A=\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix}=0 \Rightarrow (3-\lambda)(4-\lambda)-2=0$$

$$\Rightarrow (12-3\lambda-4\lambda+\lambda^2)-2=0$$

$$\Rightarrow \lambda^2-7\lambda+10=0 \Rightarrow (\lambda-2)(\lambda-5)=0$$

$$\Rightarrow \lambda_1=2,\ \lambda_2=5$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda=2$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a+2b=0 \\ a+2b=0 \end{cases} \Rightarrow a=-2b$

Hence, letting $b=1$, we get the basis vector to be $\mathbf{v}_1=\begin{pmatrix} -2b \\ b \end{pmatrix}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

ii) For $\lambda=5$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a+2b=0 \\ a-b=0 \end{cases} \Rightarrow a=b$

Hence, letting $a=1$, we get the basis vector to be $\mathbf{v}_2=\begin{pmatrix}a\\a\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+\mathbf{v}_2e^{\lambda_2 t}=cC_1\begin{pmatrix}2\\-1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{5t}=\begin{pmatrix}2C_1e^{2t}+C_2e^{5t}\\-C_1e^{2t}+C_2e^{5t}\end{pmatrix}$$

b) This is an example with single repeated real root.

The coefficient matrix is $A=\begin{pmatrix}3&1\\-1&1\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}3-\lambda&1\\-1&1-\lambda\end{vmatrix}=0\Rightarrow(3-\lambda)(1-\lambda)+1=0$$

$$\Rightarrow(3-3\lambda-\lambda+\lambda^2)+1=0$$

$$\Rightarrow\lambda^2-4\lambda+4=0\Rightarrow(\lambda-2)^2=0\Rightarrow\lambda=2$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\b\end{pmatrix}$

For $\lambda=2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}a+b=0\\-a-b=0\end{cases}\Rightarrow a=-b$

Hence, letting $b=1$, we get the basis vector to be $\mathbf{v}_1=\begin{pmatrix}-b\\b\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$.

Question: How do we get the second basis vector $\mathbf{v}_2$?

We need with the condition that $(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1$. Let $\mathbf{v}_2=\begin{pmatrix}c\\d\end{pmatrix}$.

$$(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1\Rightarrow\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}\Rightarrow c+d=-1\Rightarrow c=-1-d$$

Hence, letting $d=0$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix}-1-d\\d\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda t}+c_2(\mathbf{v}_2t+\mathbf{v}_1)e^{\lambda t}=c_1\begin{pmatrix}-1\\1\end{pmatrix}e^{2t}+c_2[\begin{pmatrix}-1\\0\end{pmatrix}t+\begin{pmatrix}-1\\1\end{pmatrix}]e^{2t}$$

c) This is an example with complex conjugate roots.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}2-\lambda & 3\\ -3 & 2-\lambda\end{vmatrix}=0\Rightarrow(2-\lambda)(2-\lambda)+9=0$$

$$\Rightarrow 4-2\lambda-2\lambda+\lambda^2+9=0$$

$$\Rightarrow \lambda^2-4\lambda+13=0\Rightarrow\lambda=2\pm 3i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $\mathbf{v}^{*}=\begin{pmatrix}a\\ b\end{pmatrix}$

For $\lambda=2+3i$, $(A-\lambda I)v=0$

$$\Rightarrow\begin{pmatrix}-3i & 3\\ -3 & -3i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-3ai+3b=0\\ -3a-3bi=0\end{cases}\Rightarrow b=ai$$

Hence, letting $a=-i$, we get $b=ai=-i^2=1$.

So, the complex basis vector becomes $\mathbf{v}^{*}=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}-i\\ 1\end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v}=\mathbf{v}^{*}e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

Here, $\lambda=2\pm 3i\Rightarrow\alpha=2,\ \beta=3$. Then, $\mathbf{v}=\mathbf{v}^{*}e^{2t}(\cos 3t+i\sin 3t)$.

$$\mathbf{v}=\mathbf{v}^{*}e^{2t}(\cos 3t+i\sin 3t)=\begin{pmatrix}-\mathbf{i}\\ 1\end{pmatrix}e^{2t}(\cos 3t+\mathbf{i}\sin 3t)$$

$$=\begin{pmatrix}-\mathbf{i}e^{2t}\cos 3t+e^{2t}\sin 3t)\\ e^{2t}\cos 3t+\mathbf{i}e^{2t}\sin 3t)\end{pmatrix}=\begin{pmatrix}e^{2t}\sin 3t)\\ e^{2t}\cos 3t\end{pmatrix}+\begin{pmatrix}-\mathbf{i}e^{2t}\cos 3t\\ \mathbf{i}e^{2t}\sin 3t)\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}e^{2t}\sin 3t)\\ e^{2t}\cos 3t\end{pmatrix}}_{=\mathrm{Re}(\mathbf{v})}+\mathbf{i}\underbrace{\begin{pmatrix}-e^{2t}\cos 3t\\ e^{2t}\sin 3t)\end{pmatrix}}_{=\mathrm{Im}(\mathbf{v}))}$$

Here, we got $\mathbf{v}_1=\mathbf{Re}(\mathbf{v})=\begin{pmatrix}e^{2t}\sin 3t)\\ e^{2t}\cos 3t\end{pmatrix}$, $\mathbf{v}_2=\mathbf{Im}(\mathbf{v})=\begin{pmatrix}-e^{2t}\cos 3t\\ e^{2t}\sin 3t)\end{pmatrix}$

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1+c_2\mathbf{v}_2=c_1\begin{pmatrix}e^{2t}\sin 3t)\\ e^{2t}\cos 3t\end{pmatrix}+c_2\begin{pmatrix}-e^{2t}\cos 3t\\ e^{2t}\sin 3t)\end{pmatrix}$$

d) This is an example with complex conjugate roots.

First, identify the coefficient matrix A and find its exigent-values.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)(1-\lambda)+1=0$$

$$\Rightarrow(1-\lambda-2\lambda+\lambda^2)+1=0$$

$$\Rightarrow\lambda^2-2\lambda+2=0\Rightarrow\lambda=1\pm i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\ b\end{pmatrix}$

i) For $\lambda=1+i,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-ai-b=0\\ a-bi=0\end{cases}\Rightarrow b=-ai$

Hence, letting $a=i$, we get $b=-ai=-i^2=1$.

So, the complex basis vector becomes $\mathbf{v}^*=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}i\\ 1\end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v}=\mathbf{v}^*e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

That is $\lambda=1\pm i\Rightarrow\alpha=1,\ \beta=1$. Then, $\mathbf{v}=\mathbf{v}^*e^t(\cos t+i\sin t)$.

$$\mathbf{v}=\mathbf{v}^*e^t(\cos t+i\sin t)=\begin{pmatrix}\mathbf{i}\\ 1\end{pmatrix}e^t(\cos t+\mathbf{i}\sin t)$$

$$=\begin{pmatrix}\mathbf{i}e^t\cos t-e^t\sin t\\ e^t\cos t+\mathbf{i}e^t\sin t\end{pmatrix}=\begin{pmatrix}-e^t\sin t\\ e^t\cos t\end{pmatrix}+\begin{pmatrix}\mathbf{i}e^t\cos t\\ \mathbf{i}e^t\sin t\end{pmatrix}=\underbrace{\begin{pmatrix}-e^t\sin t)\\ e^t\cos t\end{pmatrix}}_{=\mathrm{Re}(\mathbf{v})}+\mathbf{i}\underbrace{\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}}_{=\mathrm{Im}(\mathbf{v}))}$$

Here, we got $\mathbf{v}_1=\mathbf{Re}(\mathbf{v})=\begin{pmatrix}-e^t\sin t\\ e^t\cos t\end{pmatrix},\ \mathbf{v}_2=\mathbf{Im}(\mathbf{v})=\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$

Therefore, $\mathbf{X}_c=c_1\mathbf{v}_1+c_2\mathbf{v}_2=c_1\begin{pmatrix}-e^t\sin t\\ e^t\cos t\end{pmatrix}+c_2\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$

1.12.1 Non-homogeneous Systems with constant coefficients

A system of differential equation of the form $X' = AX + g(t)$ is said to be non-homogeneous when $g(t) \neq 0$. The function $g(t) \neq 0$ is said to be forcing or input function. In the system $X' = AX + g(t)$, the part $X' = AX$ is said to be the corresponding homogeneous part.

General form of non-homogeneous system: $\begin{cases} \frac{dx}{dt} = x' = ax + by + g_1(t) \\ \frac{dy}{dt} = y' = cx + dy + g_2(t) \end{cases}$

In matrix or in vector Form:

The above non-homogeneous system can be written in matrix or in vector form as follow: $X' = AX + g(t)$ where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, g(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

It can also be expressed in the form $X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

How to solve such non-homogeneous system?

Given: Suppose you a system is given in one of the notations or forms.

$$\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases} \text{ or } \begin{cases} \frac{dx}{dt} = ax + by + g_1(t) \\ \frac{dy}{dt} = cx + dy + g_2(t) \end{cases} \text{ or } X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$$

Objective: The main objective is to solve this system.

Fundamental solutions: The solutions of the corresponding homogeneous part which is $X' = AX$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then, the fundamental solutions are $X_1 = v_1 e^{\lambda_1 t}$, $X_2 = v_2 e^{\lambda_2 t}$.

Complementary (General) Solution: $X_c = c_1 X_1 + c_2 X_2$

General Solution of the non-homogeneous system:

The general solution of the system $X' = AX + g(t)$ is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$. Here, $\mathbf{X}_c$ is the complementary solution of the homogeneous part and $\mathbf{X}_p$ is any particular solution of the non-homogeneous.

What is the challenge to express the solution using $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$?

Since we have discussed how to determine $\mathbf{X}_c$, the challenge is how to get $\mathbf{X}_p$.

In general, to solve the system $X' = AX + g(t)$:

First: Find $\mathbf{X}_c$ of $X' = AX$. How? Use **Eigenvalue Approach**.

Second: Find $\mathbf{X}_p$ of $X' = AX + g(t)$.

Question: How to determine $\mathbf{X}_p$?

There are different approaches to find particular solution $\mathbf{X}_p$.

1. Method of Undetermined Coefficients
2. Variation of Parameters.
3. Laplace Transform method
4. Fourier Transform method
5. The Operator Method
6. Diagonalization method.
7. Power series Method

1. Method of Undetermined Coefficients

The method works in the same way as we used to solve non-homogeneous second order differential equation.

First: Assumption: Assume the form of the particular solution $\mathbf{X}_p$ based on the form of $\mathbf{g}(t)$. That means make educated guess for $\mathbf{X}_p$.

Second: Substitute $\mathbf{X}_p$ and $\mathbf{X}'_p$ in the system to determine constants.

Hints: To make educated guess, always notice the following points.

i) Since the form of $\mathbf{g}(t)$ is $\mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$, the form of $\mathbf{X}_p$ is also $\mathbf{X}_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$

That means guess x_p for $g_1(t)$ and y_p for $g_2(t)$.

Some trial forms: As example consider the following trial forms.

I) $\mathbf{g}(t) = \begin{pmatrix} ae^{kt} \\ be^{kt} \end{pmatrix}$, then the guess will be $\mathbf{X}_p = \begin{pmatrix} Ae^{kt} \\ Be^{kt} \end{pmatrix}$.

II) $\mathbf{g}(t) = \begin{pmatrix} at + b \\ ct + d \end{pmatrix}$, then the guess will be $\mathbf{X}_p = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix}$.

III) $\mathbf{g}(t)=\begin{pmatrix} a\sin kt+b\cos kt \\ c\sin kt+d\cos kt \end{pmatrix}$, then $\mathbf{X}_p=\begin{pmatrix} A\sin kx+B\cos kx \\ C\sin kx+D\cos kx \end{pmatrix}$.

IV) $\mathbf{g}(t)=\begin{pmatrix} (at+b)e^{kt} \\ (ct+d)e^{kt} \end{pmatrix}$, the guess will be $\mathbf{X}_p=\begin{pmatrix} (At+B)e^{kt} \\ (Ct+D)e^{kt} \end{pmatrix}$.

V) $\mathbf{g}(t)=\begin{pmatrix} ae^{k_1t} \\ be^{k_2t} \end{pmatrix}$, then the guess will be $\mathbf{X}_p=\begin{pmatrix} Ae^{k_1t}+Be^{k_2t} \\ Ce^{k_1t}+De^{k_2t} \end{pmatrix}$.

VI*) $\mathbf{g}(t)=\begin{pmatrix} ae^{kt} \\ bt+c \end{pmatrix}$, then the guess will be $\mathbf{X}_p=\begin{pmatrix} Ae^{kt}+Bt+C \\ De^{kt}+Et+F \end{pmatrix}$.

Notice: IN addition to these hints, modification rule also works here in a more generalized way. That means the trial form of $\mathbf{X}_p$ may not work directly. In such case, we use the rule depending on the situation.

Examples:

1. Find the general solution for the following systems of differential equations.

a) $\begin{cases} x'=x+8y+9t \\ y'=x-y+9t \end{cases}$ b) $\begin{cases} x'=2y+4t-5 \\ y'=2x-6t \end{cases}$ c) $\begin{cases} x'=x+y+10\cos t \\ y'=3x-y-10\sin t \end{cases}$

d) $\begin{cases} x'=x+y-2e^{-t} \\ y'=4x+y+3t \end{cases}$

Solution:

a) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$.

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix}=0\Rightarrow(1-\lambda)(-1-\lambda)-8=0$$

$$\Rightarrow(-1-\lambda+\lambda+\lambda^2)-8=0$$

$$\Rightarrow\lambda^2-9=0\Rightarrow\lambda^2=9\Rightarrow\lambda=-3,3$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = 3$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a+8b=0 \\ a-4b=0 \end{cases} \Rightarrow a = 4b$

Hence, letting $b=1$, we get $a = 4b = 4$.

So, the complex basis vector becomes $\mathbf{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

ii) For $\lambda = -3$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4a+8b=0 \\ a+2b=0 \end{cases} \Rightarrow a = -2b$

Hence, letting $b=-1$, we get $a = -2b = -2$.

So, the complex basis vector becomes $\mathbf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} = c_1\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-3t}$$

Second: Determine the particular solution X_p using MUCs.

Since $g(t) = \begin{pmatrix} 9t \\ 9t \end{pmatrix}$ is a linear polynomial in both components, X_p is also a linear polynomial. That is $X_p = \begin{pmatrix} at+b \\ ct+d \end{pmatrix}$. Then, $X'_p = \begin{pmatrix} a \\ c \end{pmatrix}$.

$$\begin{cases} x' = x+8y+9t \\ y' = x-y+9t \end{cases} \Rightarrow \begin{cases} x' = at+b+8(ct+d)+9t = a \\ y' = at+b-ct-d+9t = c \end{cases}$$

$$\Rightarrow \begin{cases} (a+8c+9)t+b+8d = a \\ (a-c+9)t+b-d = c \end{cases} \Rightarrow \begin{cases} a+8c+9=0, b+8d=a \\ a-c+9=0, b-d=c \end{cases}$$

$$\Rightarrow \begin{cases} a+8c+9=0 \\ a-c+9=0 \end{cases} \Rightarrow c=0, a=-9$$

$$\Rightarrow \begin{cases} b+8d=a \\ b-d=c \end{cases} \Rightarrow \begin{cases} b+8d=-9 \\ b-d=0 \end{cases} \Rightarrow \begin{cases} b+8d=-9 \\ b=d \end{cases} \Rightarrow b=d=-1$$

Hence, $X_p = \begin{pmatrix} -9t-1 \\ -1 \end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix}4\\1\end{pmatrix}e^{3t}+c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-3t}+\begin{pmatrix}-9t-1\\-1\end{pmatrix}$$

b) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix}0&2\\2&0\end{pmatrix}$.

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}-\lambda&2\\2&-\lambda\end{vmatrix}=0\Rightarrow\lambda^2-4=0\Rightarrow\lambda^2=4\Rightarrow\lambda=2,-2$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\b\end{pmatrix}$

i) For $\lambda=2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2&2\\2&-2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}-2a+2b=0\\2a-2b=0\end{cases}\Rightarrow a=b$

Hence, letting $a=1$, we get $b=a=1$.

So, the complex basis vector becomes $\mathbf{v}_1=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$.

ii) For $\lambda=-2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}2&2\\2&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}2a+2b=0\\2a+2b=0\end{cases}\Rightarrow b=-a$

Hence, letting $a=-1$, we get $b=-a=1$.

So, the complex basis vector becomes $\mathbf{v}_2=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-2t}$$

Second: Determine the particular solution $\mathbf{X}_p$ using MUCs.

Since $g(t)=\begin{pmatrix}4t-5\\-6t\end{pmatrix}$ is a linear polynomial in both components, $\mathbf{X}_p$ is also a linear polynomial. That is $\mathbf{X}_p=\begin{pmatrix}at+b\\ct+d\end{pmatrix}$. Then, $\mathbf{X}'_p=\begin{pmatrix}a\\c\end{pmatrix}$.

$$\begin{cases} x' = 2y + 4t - 5 \\ y' = 2x - 6t \end{cases} \Rightarrow \begin{cases} x' = 2ct + 2d + 4t - 5 = a \\ y' = 2at + 2b - 6t = c \end{cases} \Rightarrow \begin{cases} (2c+4)t + 2d - 5 = a \\ (2a-6)t + 2b = c \end{cases}$$

$$\Rightarrow \begin{cases} 2c + 4 = 0, 2d - 5 = a \\ 2a - 6 = 0, 2b = c \end{cases} \Rightarrow \begin{cases} c = -2, b = -1 \\ a = 3, d = 4 \end{cases}$$

Hence, the particular solution becomes $X_p = \begin{pmatrix} 3t - 1 \\ -2t + 4 \end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 3t - 1 \\ -2t + 4 \end{pmatrix}$$

c) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\Rightarrow (-1 - \lambda + \lambda + \lambda^2) - 3 = 0$$

$$\Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = 2, -2$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V = \begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = 2$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a + b = 0 \\ 3a - 3b = 0 \end{cases} \Rightarrow a = b$

Hence, letting $a = 1$, we get $b = a = 1$.

So, the complex basis vector becomes $\mathbf{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

ii) For $\lambda = -2$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a + b = 0 \\ 3a + b = 0 \end{cases} \Rightarrow b = -3a$

Hence, letting $a = -1$, we get $b = -3a = 3$.

So, the complex basis vector becomes $\mathbf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}-1\\3\end{pmatrix}e^{-2t}$$

Second: Determine the particular solution $\mathbf{X}_p$.

Since $g(t)=\begin{pmatrix}10\cos t\\-10\sin t\end{pmatrix}$ is a linear polynomial in both components, $\mathbf{X}_p$ is also a linear polynomial. That is $\mathbf{X}_p=\begin{pmatrix}a\sin t+b\cos t\\c\sin t+d\cos t\end{pmatrix}$. Then, $\mathbf{X}'_p=\begin{pmatrix}a\\c\end{pmatrix}$.

$$\begin{cases}x'=x+y+10\cos t\\ y'=3x-y-10\sin t\end{cases}$$

$$\Rightarrow\begin{cases}a\cos t-b\sin t=a\sin t+b\cos t+c\sin t+d\cos t+10\cos t\\ c\cos t-d\sin t=3(a\sin t+b\cos t)-c\sin t-d\cos t-10\sin t\end{cases}$$

$$\Rightarrow\begin{cases}a=b+d+10\\ -b=a+c\\ c=3b-d\\ -d=3a-c-10\end{cases}\Rightarrow\begin{cases}-b=b+d+10+3b-d\\ -d=3(b+d+10)-(3b-d)-10\end{cases}\Rightarrow\begin{cases}-5b=10\\ -d=4d+20\end{cases}$$

$$\Rightarrow\begin{cases}b=-2\\ d=-4\end{cases}\Rightarrow\begin{cases}c=3b-d\\ a=b+d+10\end{cases}\Rightarrow\begin{cases}c=-2\\ a=4\end{cases}$$

Hence, the particular solution becomes $\mathbf{X}_p=\begin{pmatrix}4\sin t-2\cos t\\-2\sin t-4\cos t\end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-1\\3\end{pmatrix}e^{-2t}+\begin{pmatrix}4\sin t-2\cos t\\-2\sin t-4\cos t\end{pmatrix}$$

d) First: Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix}1&1\\4&1\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)-4=0$$

$$\Rightarrow (1-2\lambda+\lambda^2)-4=0$$

$$\Rightarrow \lambda^2-2\lambda-3=0 \Rightarrow (\lambda+1)(\lambda-3)=0 \Rightarrow \lambda=-1,3$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda=-1$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a+b=0 \\ 4a+2b=0 \end{cases} \Rightarrow b=-2a$

Hence, letting $a=-1$, we get $a=-2b=2$.

So, the complex basis vector becomes $\mathbf{v}_1=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

ii) For $\lambda=3$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a+b=0 \\ 4a-2b=0 \end{cases} \Rightarrow b=2a$

Hence, letting $a=1$, we get $b=2a=2$.

So, the complex basis vector becomes $\mathbf{v}_2=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-t}+c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t}$$

Second: Determine the particular solution $\mathbf{X}_p$.

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}1-\lambda & 1\\ 4 & 1-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)(1-\lambda)-4=0$$

$$\Rightarrow(1-2\lambda+\lambda^2)-4=0$$

$$\Rightarrow\lambda^2-2\lambda-3=0\Rightarrow(\lambda+1)(\lambda-3)=0\Rightarrow\lambda=-1,3$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\ b\end{pmatrix}$

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Hence, letting $a=-1$, we get $a=-2b=2$.

So, the complex basis vector becomes $\mathbf{v}_1=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}-1\\ 2\end{pmatrix}$.

ii) For $\lambda=3$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-2a+b=0\\ 4a-2b=0\end{cases}\Rightarrow b=2a$

Hence, letting $a=1$, we get $b=2a=2$.

So, the complex basis vector becomes $\mathbf{v}_2=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix}-1\\ 2\end{pmatrix}e^{-t}+c_2\begin{pmatrix}1\\ 2\end{pmatrix}e^{3t}$$

Second: Determine the particular solution $\mathbf{X}_p$.

2. Variation of Parameters

Consider a non-homogeneous linear system: $\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases}$

In matrix Form:

The above system of equations can be written in matrix or in vector form as follow: $\mathrm{X}' = \mathrm{AX} + \mathbf{g}(t)$ where $\mathrm{X}' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \mathrm{A} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}, \mathrm{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

It can also be expressed in the form $\mathbf{X}' = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

What are we going to do?

Given: Suppose you a system is given in one of the notations or forms.

$$\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases} \text{ or } \begin{cases} \frac{dx}{dt} = ax + by + g_1(t) \\ \frac{dy}{dt} = cx + dy + g_2(t) \end{cases} \text{ or } \mathrm{X}' = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$$

Objective: The main objective is to solve this system.

Fundamental solutions: The solutions of the corresponding homogeneous part which is $\mathrm{X}' = \mathrm{AX}$ **or** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Then, the fundamental solutions are $\mathrm{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}$, $\mathrm{X}_2 = \mathbf{v}_2 e^{\lambda_2 t}$.

Complementary (General) Solution: $\mathrm{X}_c = c_1 \mathrm{X}_1 + c_2 \mathrm{X}_2$

Particular solution: The general solution of the non-homogeneous is of the form $\mathrm{X} = \mathrm{X}_c + \mathrm{X}_p$. Here, X_c is the complementary solution and X_p is any particular solution to the non-homogeneous system.

Since we have discussed how to get X_c, the challenge is how to get X_p.

Let's see how we can apply variation of Parameters.

Procedures to use Variation of Parameters effectively:

First: Determination of fundamental solutions

Using eigenvalue method, determine X_1 and X_2 of the homogeneous part.

That is solve the homogeneous part: $\mathrm{X}' = \mathrm{AX}$ **or** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

2. Variation of Parameters

Consider a non-homogeneous linear system: $\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases}$

In matrix Form:

The above system of equations can be written in matrix or in vector form as follow: $X' = AX + g(t)$ where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, g(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

It can also be expressed in the form $X' = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

What are we going to do?

Given: Suppose you a system is given in one of the notations or forms.

$$\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases} \text{ or } \begin{cases} \frac{dx}{dt} = ax + by + g_1(t) \\ \frac{dy}{dt} = cx + dy + g_2(t) \end{cases} \text{ or } X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$$

Objective: The main objective is to solve this system.

Fundamental solutions: The solutions of the corresponding homogeneous part which is $X' = AX$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then, the fundamental solutions are $X_1 = v_1 e^{\lambda_1 t}$, $X_2 = v_2 e^{\lambda_2 t}$.

Complementary (General) Solution: $X_c = c_1 X_1 + c_2 X_2$

Particular solution: The general solution of the non-homogeneous is of the form $X = X_c + X_p$. Here, X_c is the complementary solution and X_p is any particular solution to the non-homogeneous system.

Since we have discussed how to get X_c, the challenge is how to get X_p.

Let's see how we can apply variation of Parameters.

Procedures to use Variation of Parameters effectively:

First: Determination of fundamental solutions

Using eigenvalue method, determine X_1 and X_2 of the homogeneous part.

That is solve the homogeneous part: $X' = AX$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Second: Formation of solution matrix and computing its inverse.

Form the solution matrix **M** by using $\mathbf{X}_1$ as first column and $\mathbf{X}_2$ as second column of **M** and compute its inverse $\mathbf{M}^{-1}$. Use the formula for the inverse of 2×2 matrix. That is $\mathbf{M}=\begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{pmatrix}=\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\Rightarrow \mathbf{M}^{-1}=\frac{1}{\det \mathbf{M}}\begin{pmatrix}y_2 & -x_2\\ -y_1 & x_1\end{pmatrix}$

Third: Determination of the particular solution. .

Apply the formula $\mathbf{X}_p=\mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt$ where $\mathbf{g}(t)$ is the forcing function in the given system of differential equations.

Examples:

1. Find the general solution for the systems of differential equations.

a) $\begin{cases} x'=2y+e^t \\ y'=-x+3y-e^t\end{cases}$ b) $\mathbf{X}'=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\mathbf{X}+\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$

c) $\mathbf{X}'=\begin{pmatrix}1 & 8\\ 1 & -1\end{pmatrix}\mathbf{X}+\begin{pmatrix}12e^{-t}\\ 24te^t\end{pmatrix}$ d) $\begin{cases} x'=y+1\\ y'=-x+t\end{cases}$

Solution:

a) **First:** Determination of fundamental solutions using eigenvalue method.

The coefficient matrix is $A=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}-\lambda & 2\\ -1 & 3-\lambda\end{vmatrix}=0\Rightarrow -\lambda(3-\lambda)+2=0\Rightarrow \lambda^2-3\lambda+2=0$$
$$\Rightarrow(\lambda-1)(\lambda-2)=0\Rightarrow \lambda_1=1,\lambda_2=2$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\ b\end{pmatrix}$

i) For $\lambda_1=1$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-a+2b=0\\ -a+2b=0\end{cases}\Rightarrow a=2b$

Hence, letting $b=1$ in $a=2b=2$, we get the basis vector $\mathbf{v}_1=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}2\\ 1\end{pmatrix}$.

ii) For $\lambda_2=2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-2a+2b=0\\ -a+b=0\end{cases}\Rightarrow a=b$

Hence, letting $a=b=1$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$. So, the fundamental solutions are $\mathbf{X}_1=\mathbf{v}_1e^{\lambda_1 t}=\begin{pmatrix}2\\1\end{pmatrix}e^{t}, \mathbf{X}_2=\mathbf{v}_2e^{\lambda_2 t}=\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$ and the complementary solution is $\mathbf{X}_c=c_1\begin{pmatrix}2\\1\end{pmatrix}e^{t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$.

Second: Formation of solution matrix and computing its inverse.

Using $\mathbf{X}_1=\begin{pmatrix}2e^{t}\\e^{t}\end{pmatrix}$ as first column and $\mathbf{X}_2=\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix}$ as second column, the solution matrix is $\mathbf{M}=(\mathbf{X}_1\quad \mathbf{X}_2)=\begin{pmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{pmatrix}$ and $\det\mathbf{M}=\det\begin{pmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{pmatrix}=e^{3t}$.

Thus, $\mathbf{M}^{-1}=\dfrac{1}{\det\mathbf{M}}\begin{pmatrix}y_2 & -x_2\\ -y_1 & x_1\end{pmatrix}=\dfrac{1}{e^{3t}}\begin{pmatrix}e^{2t} & -e^{2t}\\ -e^{t} & 2e^{t}\end{pmatrix}=\begin{pmatrix}e^{-t} & -e^{-t}\\ -e^{-2t} & 2e^{-2t}\end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t)=\begin{pmatrix}e^{t}\\-e^{t}\end{pmatrix}$.

Hence, $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt=\begin{pmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{pmatrix}.\int\begin{pmatrix}e^{-t} & -e^{-t}\\ -e^{-2t} & 2e^{-2t}\end{pmatrix}\begin{pmatrix}e^{t}\\-e^{t}\end{pmatrix}dt$

$$=\begin{pmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{pmatrix}\int\begin{pmatrix}2\\-3e^{-t}\end{pmatrix}dt=\begin{pmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{pmatrix}\begin{pmatrix}2t\\3e^{-t}\end{pmatrix}=\begin{pmatrix}4te^{t}+3e^{t}\\2te^{t}+3e^{t}\end{pmatrix}$$

Thus, $\mathbf{X}_p=\begin{pmatrix}4te^{t}+3e^{t}\\2te^{t}+3e^{t}\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}te^{t}+\begin{pmatrix}3\\3\end{pmatrix}e^{t}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}2\\1\end{pmatrix}e^{t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\begin{pmatrix}4\\2\end{pmatrix}te^{t}+\begin{pmatrix}3\\3\end{pmatrix}e^{t}$$

b) **First: Determination of fundamental solutions using eigenvalue method.**

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix}=0\Rightarrow\lambda^2-2\lambda+2=0\Rightarrow\lambda=1\pm i$$

Here, we need how to form fundamental solutions for such complex roots.

i) For $\lambda = 1+i$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ai-b=0 \\ a-bi=0 \end{cases} \Rightarrow b=-ai$

Hence, letting $a = i$, we get $b = -ai = -i^2 = 1$.

So, the complex basis vector becomes $\mathbf{v}^* = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v} = \mathbf{v}^* e^{\alpha t}(\cos\beta t + i\sin\beta t)$.

That is $\lambda = 1 \pm i \Rightarrow \alpha = 1, \beta = 1$. Then, $\mathbf{v} = \mathbf{v}^* e^t(\cos t + i\sin t)$.

$$\mathbf{v} = \mathbf{v}^* e^t(\cos t + i\sin t) = \begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix} e^t(\cos t + \mathbf{i}\sin t) = \begin{pmatrix} \mathbf{i}e^t\cos t - e^t\sin t \\ e^t\cos t + \mathbf{i}e^t\sin t \end{pmatrix}$$

$$= \begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix} + \begin{pmatrix} \mathbf{i}e^t\cos t \\ \mathbf{i}e^t\sin t \end{pmatrix} = \underbrace{\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}}_{=\mathbf{Re}(\mathbf{v})} + \mathbf{i}\underbrace{\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}}_{=\mathbf{Im}(\mathbf{v})}$$

Hence, we have $\mathbf{X}_1 = \mathbf{Re}(\mathbf{v}) = \begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}, \mathbf{X}_2 = \mathbf{Im}(\mathbf{v}) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$.

Second: Formation of solution matrix and computing its inverse.

That is $\mathbf{M} = (\mathbf{X}_1 \quad \mathbf{X}_2) = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}$ and $\det\mathbf{M} = -e^{2t}$.

Thus, $\mathbf{M}^{-1} = -\frac{1}{e^{2t}}\begin{pmatrix} e^t\sin t & -e^t\cos t \\ -e^t\cos t & -e^t\sin t \end{pmatrix} = \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p = \mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$.

$$\mathbf{X}_p = \mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}\int\begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}dt$$

$$= \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}\int\begin{pmatrix} 0 \\ 1 \end{pmatrix}dt = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}\begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} te^t\cos t \\ te^t\sin t \end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix} + c_2\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}te^t$$

c) **First:** Determination of fundamental solutions using eigenvalue method.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}1-\lambda & 8\\ 1 & -1-\lambda\end{vmatrix}=0\Rightarrow\lambda^2-9=0\Rightarrow\lambda_1=3,\lambda_2=-3$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\ b\end{pmatrix}$

i) For $\lambda_1=3,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2 & 8\\ 1 & -4\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-2a+8b=0\\ a-4b=0\end{cases}\Rightarrow a=4b$

Hence, letting $b=1$ in $a=4b=4$, we get the basis vector $\mathbf{v}_1=\begin{pmatrix}4\\ 1\end{pmatrix}$.

ii) For $\lambda_2=-3,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}4 & 8\\ 1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}4a+8b=0\\ a+2b=0\end{cases}\Rightarrow a=-2b$

Hence, letting $b=1$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix}-2\\ 1\end{pmatrix}$.

So, the fundamental solutions are $\mathbf{X}_1=\mathbf{v}_1e^{\lambda_1 t}=\begin{pmatrix}4\\ 1\end{pmatrix}e^{3t},\mathbf{X}_2=\mathbf{v}_2e^{\lambda_2 t}=\begin{pmatrix}-2\\ 1\end{pmatrix}e^{-3t}$.

Second: Formation of solution matrix and computing its inverse.

Using $\mathbf{X}_1=\begin{pmatrix}4e^{3t}\\ e^{3t}\end{pmatrix}$ as first column and $\mathbf{X}_2=\begin{pmatrix}-2e^{-3t}\\ e^{-3t}\end{pmatrix}$ as second column, the solution matrix is $\mathbf{M}=\begin{pmatrix}4e^{3t} & -2e^{-3t}\\ e^{3t} & e^{-3t}\end{pmatrix}$ and $\det\mathbf{M}=\det\begin{pmatrix}4e^{3t} & -2e^{-3t}\\ e^{3t} & e^{-3t}\end{pmatrix}=6$.

Thus, $\mathbf{M}^{-1}=\dfrac{1}{\det\mathbf{M}}\begin{pmatrix}y_2 & -x_2\\ -y_1 & x_1\end{pmatrix}=\dfrac{1}{6}\begin{pmatrix}e^{-3t} & 2e^{-3t}\\ -e^{3t} & 4e^{3t}\end{pmatrix}=\begin{pmatrix}\frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t}\\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t}\end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt$.

$$\mathbf{X}_p = \mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \int \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix} \begin{pmatrix} 12e^{-t} \\ 24te^{t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \int \begin{pmatrix} 2e^{-4t} + 4te^{-2t} \\ -2e^{2t} + 16te^{4t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-4t} - 2te^{-2t} - e^{-2t} \\ -e^{2t} + 4te^{4t} - e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-t} - 8te^{t} - 4e^{t} + (2e^{-t} - 8te^{t} + 2e^{t}) \\ -\frac{1}{2}e^{-t} + te^{t} - \frac{1}{2}e^{t} + (-e^{-t} + 4te^{t} - e^{t}) \end{pmatrix} = \begin{pmatrix} -16te^{t} - 2e^{t} \\ -\frac{3}{2}e^{-t} + 5te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -16te^{t} - 2e^{t} \\ -\frac{3}{2}e^{-t} + 5te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

d) First: Determination of fundamental solutions using eigenvalue method.

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V = \begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = i$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ai + b = 0 \\ -a - bi = 0 \end{cases} \Rightarrow b = ai$

Hence, letting $a = -i$, we get $b = -ai = -i^2 = 1$.

So, the complex basis vector becomes $\mathbf{v}^* = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v} = \mathbf{v}^* e^{\alpha t}(\cos\beta t + i\sin\beta t)$.
That is $\lambda = i \Rightarrow \alpha = 0, \beta = 1$.

$$\mathbf{v} = \begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix}(\cos t + \mathbf{i}\sin t) = \begin{pmatrix} -\mathbf{i}\cos t + \sin t \\ \cos t + \mathbf{i}\sin t \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \mathbf{i}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Hence, we have $\mathbf{X}_1 = \mathbf{Re}(\mathbf{v}) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, $\mathbf{X}_2 = \mathbf{Im}(\mathbf{v}) = \begin{pmatrix} -\cos t \\ \sin t) \end{pmatrix}$ as the fundamental solutions.

Second: Formation of solution matrix and computing its inverse.

That is $\mathbf{M}=(\mathbf{X}_1 \quad \mathbf{X}_2)=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}$ and $\det \mathbf{M}=1$.

Thus, $\mathbf{M}^{-1}=\begin{pmatrix}\sin t & \cos t\\ -\cos t & \sin t\end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t)=\begin{pmatrix}1\\ t\end{pmatrix}$.

$$\mathbf{X}_p=\mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}.\int\begin{pmatrix}\sin t & \cos t\\ -\cos t & \sin t\end{pmatrix}\begin{pmatrix}1\\ t\end{pmatrix}dt$$

$$=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}\int\begin{pmatrix}\sin t+t\cos t\\ -\cos t+t\sin t\end{pmatrix}dt$$

$$=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}\begin{pmatrix}t\sin t\\ -t\cos t\end{pmatrix}=\begin{pmatrix}t\\ 0\end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}+c_2\begin{pmatrix}-\cos t\\ \sin t)\end{pmatrix}+\begin{pmatrix}t\\ 0\end{pmatrix}$$

2. Find the general solution for the systems of differential equations.

a) $\begin{cases} x'=-3x-4y+5e^t\\ y'=5x+6y-6e^t\end{cases}$ b) $\begin{cases} x'=x+4y-2\cos t\\ y'=x+y-\cos t+\sin t\end{cases}$

Solution:

a) First, find the complementary solution X_c of the homogeneous system.

That is find the solution of $\begin{cases} x'=-3x-4y\\ y'=5x+6y\end{cases}$

$$\begin{vmatrix}-3-\lambda & -4\\ 5 & 6-\lambda\end{vmatrix}=0\Rightarrow-(3+\lambda)(6-\lambda)+20=0$$

$$\Rightarrow-(18-3\lambda+6\lambda-\lambda^2)+20=0$$

$$\Rightarrow\lambda^2-3\lambda+2=0\Rightarrow(\lambda-1)(\lambda-2)=0$$

$$\Rightarrow\lambda_1=1,\ \lambda_2=2$$

Review Problems on Chapter-1

1. Determine the order n and degree d (if defined) of the following DEs.

a) $\frac{d^4y}{dx^4}-3x\left(\frac{d^2y}{dx^2}\right)^6=2x^7$ b) $y'''^2+y'=\ln y'$ c) $\left(\frac{dy}{dx}\right)^{\frac{7}{3}}=\sqrt{y'^6-x}$

d)* $y'''^4(y'''^7+y')^{\frac{1}{3}}=\sqrt{y''^9}$ e) $y''^2=(y''^4-2xy')^{\frac{2}{3}}$ f) $\frac{d^2y}{dx^2}-e^{y}=0$

g) $\left(\frac{d^3y}{dx^3}\right)^2+\cos(3y)=1$ h) $\left(\frac{d^3y}{dx^3}\right)^2+\cos(3y')=1$ i) $\frac{d^2y}{dx^2}=0$

Answer : a) $n=4, d=6$ b) $n=3$, No degree c) $n=1, d=18$
d) $n=3, d=38$ e) $n=2, d=8$ f) $n=2$, No degree
g) $n=3, d=2$ h) $n=2$, No degree i) $n=2, d=1$

2*. Show that $y=\tan^{-1}x$ is a solution of the DE $y''+2\sin y\cos^3 y=0$.

3. In each of the following, find the constants k and a.

a) If $y=e^{3x}$ is the solution of the DE $\frac{d^2y}{dx^2}-6y'+ky=0$, then find k.

b) If $y=x^2$ is the solution of the DE $8x\frac{dy}{dx}-ay=x^2$, then find a.

c) If $y=x^3+kx+1$ is the solution of the DE $y'''+xy''-2y'=0$, then find k.

d) If $y=x^k$ is the solution of the DE $16x^2y''+24xy'+y=0$, then find k.

Answer : a) $k=9$ b) $a=15$ c) $k=3$ d) $k=-1/4$

4*. Verify that the following DEs are homogeneous and solve them.

a) $xdy=(y+x\tan\frac{y}{x})dx$, b) $xdy=(y+x\cot\frac{y}{x})dx$

d) $x\sin\frac{y}{x}dy=(x+y\sin\frac{y}{x})dx$ e) $ydy=(\frac{y^2}{x}+x\tan\frac{y^2}{x^2})dx$

5. Verify that the following DEs are homogeneous and solve the IVPs.

a) $\dfrac{dy}{dx}=\dfrac{x+y}{x-y},\ y(1)=0$ b) $\dfrac{dy}{dx}=\dfrac{y^2+2xy}{x^2},\ y(1)=2$

c) $xydy=(x^2+2y^2)dx, y(1)=1$ d) $(x^3+y^3)dx-xy^2dy=0, y(-1)=0$

e) $(3x^2+4xy)dx+(2xy+x^2)dy=0, y(2)=-3$

f) $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=y\cos\left(\dfrac{y}{x}\right)+x, y(1)=\pi$

Answer : a) $2\tan^{-1}(y/x)=\ln(x^2+y^2)$ b) $y=\dfrac{2x^2}{3-2x}$ c) $y^2=2x^4-x^2$

d) $y^3=3x^3\ln|x|$ e) $x^3y^2+x^4y=24$ f) $\sin(y/x)=\ln|x|$

6. Verify that the following DES are exact and solve them.

a) $(6xy+2y^2)dx+(3x^2+4xy)dy=0$ b) $xdx+ydy=(x^2+y^2)dy$

c) $ye^{xy}dx+xe^{xy}dy=0$ d) $2xydx+(1+x^2)dy=0$

e) $(e^x\sin y+3x^2)dx+e^x\cos ydy=0$ f) $(ye^x+2x)dx+e^xdy=0$

g) $(2x+y^2)dx+2xydy=0$

Answer : a) $3x^2y+2xy^2=c$ b) $\ln(x^2+y^2)-2y=c$ c) $e^{xy}=c$

d) $x^2y+y=c$ e) $e^x\sin y+x^3=c.$ f) $ye^x+x^2=c$ g) $x^2+xy^2=c$

7. Verify that the DEs are not exact and solve them.

a) $(x^3-2y^2)dx+2xydy=0$ b) $(x^2y+4xy+2y)dx+(x^2+x)dy=0$

c) $(x^2+xy)y'+3xy+y^2=0$ d) $(x^4y^3+y)dx+(x^5y^2-x)dy=0$

e) $(y^2+2x^2)dx+xydy=0$ f) $(xy^3+y^2)dx+(y-xy)dy=0$

g) $(y^2+2x^2)dx-x(1+xy)dy=0$ h) $2xydx+(y^2-x^2)dy=0$

i) $y(1-xy)dx-xydy=0$ j) $y^3dx+(xy^3+3xy^2+1)dy=0$

Answer : a) $x+\dfrac{y^2}{x^2}=c$ b) $(x+1)e^xx^2y=c$ c) $x^3y+\dfrac{x^2y^2}{2}=c$

d) $x^4y^3-3y=cx$ e) $\dfrac{x^2y^2}{2}+\dfrac{x^4}{2}=c$ f) $\dfrac{x^2}{2}+\dfrac{x}{y}-\dfrac{1}{y}=c$

g) $2\ln x-\dfrac{y^2}{2x^2}=c$ h) $x^2+y^2=cy$ i) $xy^3=c$ j) $e^y(xy^3+1)=c$

8. Find the constants a, m, n so that the following DEs are exact.

a) $x^3+3xy+(ax^2+4y)y'=0$ b) $(x^3y^m+x^3)dx+(x^4y^n+y^3)dy=0$

c) $(y^3+x^2y)dx+(ax+1)dy=0$ d) $x^my^3\frac{dy}{dx}+7ax^4y^n=0$

Answer : a) $a=\frac{3}{2}$ b) $m=4, n=3$ c) $a=-2$ d) $m=5, n=4, a=\frac{5}{28}$

9. Solve the following *Bernoulli's Differential Equations*

a) $y'+3y=e^{3x}y^2, y(0)=1/4$ b) $y'+y=xy^4, y(0)=3$

c) $xy'+y=xy^3$ d) $xy'+y=x^2y^2\ln x, y(1)=1$

e) $y'=xy^2+y, y(0)=1$ f) $y'+2xy=-xy^4$

g) $x\frac{dy}{dx}+y=x^3y^6$ h) $\frac{dy}{dx}+y=xy^2$ i) $y'+y=y^2, y(0)=-1$

Answer : a) $y=\frac{e^{-3x}}{4-x}$ b) $y=\frac{3}{\sqrt[3]{27x-8e^{3x}+9}}$ c) $y=\frac{1}{\sqrt{ce^{2x}-x-1}}$

d) $y=\frac{1}{x^2-x^2\ln x}$ e) $y=\frac{1}{1-x}$ f) $\frac{1}{y^3}=ce^{3x^2}-\frac{1}{2}$

g) $x^3y^5(cx^2+5/2)=1$ h) $y=\frac{1}{ce^x+x+1}$ i) $y=\frac{1}{1-2e^x}$

10. Solve the following DEs using *Methods of Undetermined coefficients*

a) $y''-7y'+10y=24e^x$ b) $y''-4y'+4y=2e^{2x}+4x-12$

c) $y''+4y'+4y=e^{-2x}\sin 2x$ d) $y''-2y'+y=e^x\sin x$

e) $y''-3y'-4y=5e^x$ f) $y''-4y'+4y=e^{2x}\sin x$

g) $y''-y=1+2e^x$ h) $y''-2y'+y=2e^x$

i) $y''+2y'+y=2x\sin x$ j) $y''-3y'+2y=e^x$

k) $y''-2y'+y=e^x+x$ m) $y''+y'-2y=2xe^x$

Answer : a) $y=c_1e^{2x}+c_2e^{5x}+6e^x$ b) $y=c_1e^{2x}+c_2xe^{2x}+x^2e^{2x}+x-2$

c) $y=(c_1+c_2x)e^{-2x}-\frac{1}{4}e^{-2x}\sin 2x$ d) $y=(c_1+c_2x)e^x-e^x\sin x$

e) $y=c_1e^{4x}+c_2e^{-x}-xe^{-x}$ f) $y=(c_1+c_2x)e^{2x}-e^{2x}\sin x$

g) $y=c_1e^x+c_2e^{-x}+xe^{-x}-1$ h) $y=c_1e^x+c_2xe^x+x^2e^x$

11. Solve the following DEs using *Variation of Parameters*

a) $y''-2y'+y=\frac{e^x}{x^2}$ b) $y''-4y'+4y=\frac{12e^{2x}}{x^4}$ c) $4y''+36y=\csc 3x$

d) $y''+y=\sin x$ e) $y''-4y'+4y=(x+1)e^{2x}$ f) $y''+y=\sec x\tan x$

g) $y''-y=\cosh x$ h) $y''-4y=\frac{e^{2x}}{x}$

Answer: a) $y(x)=c_1e^x+c_2xe^x-(\ln|x|+1)e^x$ b) $y(x)=(c_1+c_2x)e^{2x}+2x^{-2}e^{2x}$

c) $y=c_1\cos 3x+c_2\sin 3x-\frac{x}{12}\cos 3x+\frac{1}{36}\sin 3x\ln|\sin 3x|$

d) $y=c_1\cos x+c_2\sin x-\frac{x}{2}\cos x$ e) $y=c_1e^{2x}+c_2xe^{2x}+\frac{x^3e^{2x}}{6}+\frac{x^2e^{2x}}{2}$

f) $y=c_1\cos x+c_2\sin x+x\cos x-\sin x\ln|\cos x|$

g) $y=c_1e^x+c_2e^{-x}-(\frac{e^{-x}}{8}+\frac{x}{4})e^{-x}+(\frac{x}{4}-\frac{1}{8})e^x$

12. Find the value(s) of the constant k for which the DE $y''+6y'+ky=0$ has a general solution of the form $y=(c_1+c_2x)e^{-3x}$. **Answer**: $k=9$

13*. Find a and k for which the DE $y''+ay'+ky=0$ has a general solution of the form $y=(c_1\cos 2x+c_2\sin 2x)e^{-3x}$. **Answer**: $a=6,\ k=13$

14. Using the appropriate methods, find the particular solution

a) $y''+y'+y=e^x+4$ b) $y''-y=\sinh x$ c) $y''+2y'+y=e^{-x}\ln x$

d) $y''+3y'+2y=\frac{1}{1+e^{2x}}$ e) $y''-2y'+2y=e^x\tan x$ f) $y''-y=\frac{1}{1+e^x}$

g) $y''-2y+y=4x^2-3$ h) $y''+3y+2y=\sin e^x$

Answer: a) $y_p=\frac{1}{3}e^x$ b) $y_p=\frac{x}{2}\cosh x$ c) $y_p=\frac{1}{2}x^2e^{-x}\ln x-\frac{3}{4}x^2e^{-x}$

d) $y_p=e^{-x}\tan^{-1}e^{-x}-\frac{1}{2}[e^{-2x}\ln(1+e^x)-1]$ e) $y_p=-e^x\cos x\ln(\sec x+\tan x)$

f) $y_p=\frac{1}{2}[e^x\ln(1+e^{-x})-e^{-x}\ln(1+e^x)-1]$

g) $y_p=4x^2+16x+21$ h) $y_p=-e^{-2x}\sin e^x$

CHAPTER-2

Laplace Transform and its Applications

2.1 Definition and Examples of Laplace Transform

Definition: Suppose f is a function defined for all $t \geq 0$. Then, tie Laplace Transform of f denoted by $L\{f(t)\}$ is a function $F(s)$ defined by

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$ where s is a parameter (real or conplex).

Note: The interval of integration is infinite which is an improper intgral and thus it is evaluated by the rule $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{n\to\infty} \int_0^n e^{-} f(t)dt$.

Examples: Find the Laplace Transform $L\{f(t)\}$ of the following functons.

a) $f(t) = 1, t \geq 0$ *b)* $f(t) = e^{at}$ *c)* $f(t) = t, t \geq 0$

d) $f(t) = \begin{cases} 1, 0 \leq t < 2 \\ t-2, 2 \leq t \end{cases}$ *e)* $f(t) = \begin{cases} -2, 0 < t < 4 \\ 2, t > 4 \end{cases}$ *f)* $f(t) = \begin{cases} 0, 0 < t < 1 \\ 2t-2, t > 1 \end{cases}$

Solution: Let's apply the definition.

a) $$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{n\to\infty} \int_0^n e^{-st} dt = \lim_{n\to\infty} \left[\frac{-e^{-st}}{s}\right]_{t=0}^{t=n} = \lim_{n\to\infty} \left[\frac{-e^{-sn}+1}{s}\right] = \frac{1}{s}$$

b) $$F(s) = L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{t(a-s)} dt = \lim_{u\to\infty} \int_0^u e^{t(a-s)} dt$$

$$= \lim_{u\to\infty} \left(\frac{-e^{t(a-s)}}{s-a}\right)\Bigg|_0^u = \lim_{u\to\infty} \left(\frac{-e^{u(a-s)}}{s-a}\right) + \frac{1}{s-a} = \begin{cases} \infty, \text{if } s \leq a \\ \dfrac{1}{s-a}, \text{if } s > a \end{cases}$$

Therefore, $L\{e^{at}\} = \dfrac{1}{s-a}; s > a$. For instance, $L\{e^{2t}\} = \dfrac{1}{s-2}, L\{e^{-t}\} = \dfrac{1}{s+1}$.

c) $F(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^\infty e^{-st}tdt=\lim_{n\to\infty}\int_0^n e^{-st}tdtx$

$$=\lim_{n\to\infty}\left[\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_{t=0}^{t=n}=\lim_{n\to\infty}\left[\frac{-ne^{-sn}}{s}-\frac{e^{-sn}}{s^2}+\frac{1}{s^2}\right]=\frac{1}{s^2}$$

d) $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^2 e^{-st}f(t)dt+\int_2^\infty e^{-st}f(t)dt=\int_2^\infty e^{-st}dt+\int_2^\infty (t-2)e^{-st}dt$

$$=\frac{e^{-st}}{-s}\Big|_0^2+\left(\frac{-(t-2)e^{-st}}{s}-\frac{e^{-st}}{s^2}\right)\Big|_{t=2}^{\infty}=\frac{1}{s}(1-e^{-2s})+\frac{e^{-2s}}{s^2}$$

e) $L\{f(t)=\int_0^4 -2e^{-st}dt+\lim_{u\to\infty}\int_4^u 2e^{-st}dt=(\frac{2e^{-st}}{s})\Big|_0^4+\lim_{u\to\infty}(-\frac{2e^{-st}}{s})\Big|_4^u$

$$=(\frac{2e^{-4s}}{s}-\frac{2}{s})+\lim_{u\to\infty}(-\frac{2e^{-us}}{s}+\frac{2e^{-4s}}{s})=\frac{4e^{-4s}}{s}-\frac{2}{s}$$

f) $L\{ft)\}=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt=\int_1^\infty(2t-2)e^{-st}dt=\frac{2e^{-s}}{s^2},s>0$

2.2 P'operties of Laplace Transform

Propety-I: Linearity Property:

If f nd g are functions whose Laplace Transforms exist, then for $a\in R$

i) $L\{af(t)\}=aL\{f(t)\}$

ii) $L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$

Exanple: Find the Laplace transform of $f(t)=2e^{3t}-3t+4$

Soltion: In the above example, we got $L\{1\}=\frac{1}{s},L\{e^{3t}\}=\frac{1}{s-3},L\{t\}=\frac{1}{s^2}$.

Thn, using linearity properties, we have

$L f(t)\}=L\{2e^{3t}-3t+4\}=L\{2e^{3t}\}+L\{-3t\}+L\{4\}$

$$=2L\{e^{3t}\}-3L\{t\}+4L\{1\}=\frac{2}{s-3}-\frac{3}{s^2}+\frac{4}{s}$$

The Laplace Transform of Trigonometric Functions:

The Laplace Transform of $f(t)=\sin at$ and $f(t)=\cos at$ can be derived directly by the definition using Integration by Parts which is too demanding. But it can be done easily using the **Euler's formula** with linearityproperties.

Recall: Euler's formula: $e^{iat}=\cos at+i\sin at$.

Using linearity properties, we have

$$e^{iat}=\cos at+i\sin at \Rightarrow L\{e^{iat}\}=L\{\cos at+i\sin at\}$$

$$\Rightarrow L\{e^{iat}\}=L\{\cos at\}+iL\{\sin at\}\text{......................(i)}$$

But as we discussed above, $L\{e^{at}\}=\dfrac{1}{s-a}\Rightarrow L\{e^{iat}\}=\dfrac{1}{s-ia}$.

Now rationalizing, $L\{e^{iat}\}=\dfrac{1}{s-ia}$ using the conjugate of $s-ia$, wehave

$$L\{e^{iat}\}=\frac{1}{(s-ia)(s+ia)}(s+ia)=\frac{s+ia}{s^2+a^2}=\frac{s}{s^2+a^2}+\frac{a}{s^2+a^2}i\text{.........(ii)}$$

Equating (i) and (ii), we have $L\{\cos at\}+iL\{\sin at\}=\dfrac{s}{s^2+a^2}+i\dfrac{a}{s^2+a^2}$.

But we know that two complex numbers are equal if their correspoding real parts at the same time imaginary parts are equal. Using this property,

$$\underbrace{L\{\cos at\}}_{\text{Realpart}}+\underbrace{L\{\sin at\}}_{\text{Imaginary part}}i=\underbrace{\frac{s}{s^2+a^2}}_{\text{Realpart}}+\underbrace{\frac{a}{s^2+a^2}}_{\text{Imaginary part}}i$$

$$\Rightarrow L\{\cos at\}=\frac{s}{s^2+a^2},\ L\{\sin at\}=\frac{a}{s^2+a^2}$$

Therefore, $L\{\cos at\}=\dfrac{s}{s^2+a^2}, L\{\sin at\}=\dfrac{a}{s^2+a^2}; s>0$

Examples:

$$L\{\cos t\}=\frac{s}{s^2+1}, L\{\sin t\}=\frac{1}{s^2+1}, L\{\cos 2t\}=\frac{s}{s^2+4}, L\{\sin 3t\}=\frac{3}{s^2+9}; s>0$$

The Laplace Transform of Hyperbolic Functions:

Consider the functions $f(t) = \sinh at$, $f(t) = \cosh at$.

Recall: By definition, we know that $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$, $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$.

Then, by linearity property, we have

i) $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\} = \dfrac{1}{2}L\{e^{at}\} - \dfrac{1}{2}L\{e^{-at}\} = \dfrac{1}{2(s-a)} - \dfrac{1}{2(s+a)} = \dfrac{a}{s^2 - a^2}$

ii) $L\{\cosh at\} = L\left\{\dfrac{e^{at} + e^{-at}}{2}\right\} = \dfrac{1}{2}L\{e^{at}\} + \dfrac{1}{2}L\{e^{-at}\} = \dfrac{1}{2(s-a)} + \dfrac{1}{2(s+a)} = \dfrac{s}{s^2 - a^2}$

Examples:

i) $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$, $L\{\sinh 3t\} = \dfrac{3}{s^2 - 9}$, $L\{\sinh \sqrt{3}t\} = \dfrac{\sqrt{3}}{s^2 - 3}$

ii) $L\{\cosh 2t\} = \dfrac{s}{s^2 - 4}$, $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$, $L\{\cosh \sqrt{2}t\} = \dfrac{s}{s^2 - 2}$

Property-II: Power property: $L\{t^n\} = \dfrac{n!}{s^{n+1}}, n = 0,1,2,\ldots$

Examples: $L\{t^2\} = \dfrac{2!}{s^{2+1}} = \dfrac{2}{s^3}, L\{t^3\} = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}, L\{t^4\} = \dfrac{4!}{s^{4+1}} = \dfrac{12}{s^5}; s > 0$

Property-III: S-Shifting Properties (First Shifting Rule):

Suppose $L\{f(t)\} = F(s)$. Then,

i) $L\{e^{at} f(t)\} = F(s-a)$ ii) $L\{e^{-at} f(t)\} = F(s+a)$

Examples:

1. Find the following Laplace transforms

a) $L\{e^{3t}\cos 2t\}$ b) $L\{e^{-2t}\sin 5t\}$ c) $L\{e^{-t}\cosh 6t\}$ d) $L\{e^{4t} t^5\}$

Solution: Identify f and compute $L\{f(t)\} = F(s)$. Then apply property-III.

a) Here, let $f(t) = \cos 2t$. Then $F(s) = L\{f(t)\} = L\{\cos 2t\} = \dfrac{s}{s^2 + 4}$.

Therefore, by S-shifting property, $L\{e^{3t}\cos 2t\} = F(s-3) = \dfrac{s-3}{(s-3)^2 + 4}$.

The Laplace Transform of Hyperbolic Functions:

Consider the functions $f(t)=\sinh at$, $f(t)=\cosh at$.

Recall: By definition, we know that $\sinh at=\dfrac{e^{at}-e^{-at}}{2}, \cosh at=\dfrac{e^{at}+e^{-at}}{2}$.

Then, by iinearity property, we have

i) $L\{\sinh at\}=L\left\{\dfrac{e^{at}-e^{-at}}{2}\right\}=\dfrac{1}{2}L\{e^{at}\}-\dfrac{1}{2}L\{e^{-at}\}=\dfrac{1}{2(s-a)}-\dfrac{1}{2(s+a)}=\dfrac{a}{s^2-a^2}$

ii) $L\{\cosh at\}=L\left\{\dfrac{e^{at}+e^{-at}}{2}\right\}=\dfrac{1}{2}L\{e^{at}\}+\dfrac{1}{2}L\{e^{-at}\}=\dfrac{1}{2(s-a)}+\dfrac{1}{2(s+a)}=\dfrac{s}{s^2-a^2}$

Examples:

i) $L\{\sinh 2t\}=\dfrac{2}{s^2-4}$, $L\{\sinh 3t\}=\dfrac{3}{s^2-9}, L\{\sinh\sqrt{3}t\}=\dfrac{\sqrt{3}}{s^2-3}$

ii) $L\{\cosh 2t\}=\dfrac{s}{s^2-4}$, $L\{\cosh 3t\}=\dfrac{s}{s^2-9}, L\{\cosh\sqrt{2}t\}=\dfrac{s}{s^2-2}$

Property-II: Power property: $L\{t^n\}=\dfrac{n!}{s^{n+1}}, n=0,1,2,...$

Examples: $L\{t^2\}=\dfrac{2!}{s^{2+1}}=\dfrac{2}{s^3}, L\{t^3\}=\dfrac{3!}{s^{3+1}}=\dfrac{6}{s^4}, L\{t^4\}=\dfrac{4!}{s^{4+1}}=\dfrac{12}{s^5}; s>0$

Property-III: S-Shifting Properties (First Shifting Rule):

Suppose $L\{f(t)\}=F(s)$. Then,

i) $L\{e^{at}f(t)\}=F(s-a)$ ii) $L\{e^{-at}f(t)\}=F(s+a)$

Examples:

1. Find the following Laplace transforms

a) $L\{e^{3t}\cos 2t\}$ b) $L\{e^{-2t}\sin 5t\}$ c) $L\{e^{-t}\cosh 6t\}$ d) $L\{e^{4t}t^5\}$

Solution: Identify f and compute $L\{f(t)\}=F(s)$. Then apply property-III.

a) Here, let $f(t)=\cos 2\text{t}$. Then $F(s)=\text{L}\{\text{f(t)}\}=L\{\cos 2\text{t}\}=\dfrac{s}{s^2+4}$.

Therefore, by S-shifting property, $L\{e^{3t}\cos 2t\}=F(s-3)=\dfrac{s-3}{(s-3)^2+4}$.

b) Here, let $f(t)=\sin 5t$. Then, $F(s)=L\{f(t)\}=L\{\sin 5t\}=\frac{5}{s^2+25}$.

Therefore, by S-shifting property, $L\{e^{-2t}\sin 5t\}=F(s+2)=\frac{5}{(s+2)^2+25}$.

c) Let $f(t)=\cosh 6t, F(s)=L\{\cosh 6t\}=\frac{s}{s^2-36}$.

Therefore, $L\{e^{-t}\cosh 6t\}=F(s+1)=\frac{s+1}{(s+1)^2-36}$.

d) Let $f(t)=t^5, F(s)=L\{t^5\}=\frac{5!}{s^{5+1}}=\frac{120}{s^6}\Rightarrow L\{e^{4t}t^5\}=F(s-4)=\frac{120}{(s-4)^6}$

2. Find the Laplace Transform of

a) $f(t)=\sin 3t\cos 2t$ b) $f(t)=\cos 4t\cos 2t$ c) $f(t)=\cos^2 t$

d) $f(t)=\sin 2t\cosh 3t$ e) $f(t)=\sin 2t\cos 2t$ f) $f(t)=4e^{-6t}\sin^2 t$

Solution: Recall: P*roduct to sum* formula of trigonometric functions.

$$\sin x\cos y=\frac{1}{2}[\sin(x-y)+\sin(x+y)], \cos x\cos y=\frac{1}{2}[\cos(x-y)+\cos(x+y)$$

a) $L\{\sin 3t\cos 2t\}=\frac{1}{2}L\{\sin t\}+\frac{1}{2}L\{\sin 5t\}=\frac{1}{2}(\frac{1}{s^2+1}+\frac{5}{s^2+25})$

b) $L\{\cos 4t\cos 2t\}=\frac{1}{2}L\{\cos 2t\}+\frac{1}{2}L\{\cos 6t\}=\frac{1}{2}(\frac{s}{s^2+4}+\frac{s}{s^2+36})$

c) $\cos^2 t=\frac{1}{2}+\frac{\cos 2t}{2}\Rightarrow L\{\cos^2 t\}=\frac{1}{2}L\{1\}+\frac{1}{2}L\{\cos 2t\}=\frac{1}{2s}+\frac{s}{2(s^2+4)}$

d) $f(t)=\sin 2t\cosh 3t=\sin 2t\left(\frac{e^{3t}+e^{-3t}}{2}\right)=\frac{1}{2}e^{3t}\sin 2t+\frac{1}{2}e^{-3t}\sin 2t.$

Then, $L\{f(t)\}=L\left\{\frac{1}{2}e^{3t}\sin 2t\right\}+L\left\{\frac{1}{2}e^{-3t}\sin 2t\right\}=\frac{1}{(s-3)^2+4}+\frac{1}{(s+3)^2+4}$.

e) $\sin 2t\cos 2t=\frac{\sin 4t}{2}\Rightarrow L\{\sin 2t\cos 2t\}=L\left\{\frac{\sin 4t}{2}\right\}=\frac{1}{2}L\{\sin 4t\}=\frac{2}{s^2+16}$

Property-IV: t-Shifting Property:

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Examples: Find the following Laplace Transforms using t-shifting property.

a) $L\{t\cos 2t\}$ b) $L\{te^{2t}\sin 3t\}$ c) $L\{t^4 e^{-t}\}$ d) $L\{t\cosh 3t\}$

Solution:

a) Let $f(t) = \cos 2t$. Then, $L\{f(t)\} = L\{\cos 2t\} = \frac{s}{s^2+4}$.

Therefore, by t-shifting property, $L\{t\cos 2t\} = -\frac{d}{ds}(\frac{s}{s^2+4}) = \frac{s^2-4}{(s^2+4)^2}$

b) Using $L\{\sin 3t\} = \frac{3}{s^2+9}$ and s-shifting, $F(s) = L\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$

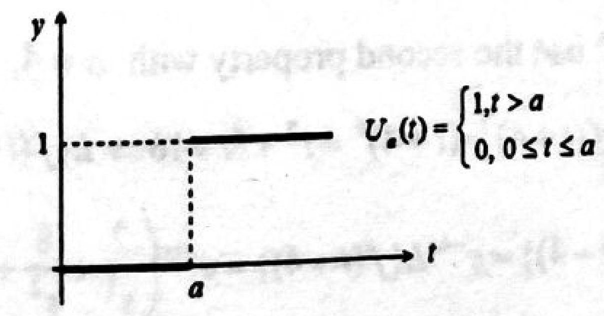
Therefore, by t-shifting, $L\{te^{2t}\sin 3t\} = -\frac{d}{ds}(\frac{3}{(s-2)^2+9}) = \frac{6(s-2)}{[(s-2)^2+9)^2}$

c) Here, $f(t) = e^{-t}, F(s) = L\{e^{-t}\} = \frac{1}{s+1} \Rightarrow L\{t^4 e^{-t}\} = (-1)^4 \frac{d^4}{ds^4}\left(\frac{1}{s+1}\right) = \frac{24}{(s+1)^5}$

d) Here, $L\{\cosh 3t\} = \frac{s}{s^2-9} \Rightarrow L\{t\cosh 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2-9}\right) = \frac{s^2+9}{(s^2-9)^2}$

2.3 Unit-Step (Heaviside) Function and its Laplace Transform

Definition: The unit-step function (also known as Heaviside function), denoted by $U_a(t) = U(t-a)$ or $H_a(t) = H(t-a)$ is defined as $U_a(t) = \begin{cases} 1, t > a \\ 0, 0 \le t \le a \end{cases}$



Graph of $U_a(t) = U(t-a)$

Laplace Transform of unit-step function $U_a(t)$

For any unit step function $U_a(t) = H(t-a)$, $L\{U_a(t)\} = L\{H(t-a)\} = \dfrac{e^{-as}}{s}$

This is obtained using the definition of transform as follow.

$$L\{U_a(t)\} = \int_0^\infty e^{-st}U_a(t)dt = \int_0^a e^{-st}U_a(t)dt + \int_a^\infty e^{-st}U_a(t)dt = \int_a^\infty e^{-st}dt$$

$$= \lim_{n\to\infty}\int_a^n e^{-st}dt = \lim_{n\to\infty}\left[\frac{-e^{-ts}}{s}\right]_{t=a}^{t=n} = \lim_{n\to\infty}\left[\frac{-e^{-ns}}{s} + \frac{e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

Examples: $L\{U_2(t)\} = \dfrac{e^{-2s}}{s}, L\{H(t-3)\} = \dfrac{e^{-3s}}{s}, L\{H(t-\pi)\} = \dfrac{e^{-\pi s}}{s}$

Theorem (Second-Shifting Rule): Let $L\{f(t)\} = F(s)$. Then,

i) $L\{f(t-a)U_a(t)\} = L\{f(t-a)H(t-a)\} = e^{-as}L\{f(t)\} = e^{-as}F(s)$

ii) $L\{f(t)U_a(t)\} = L\{f(t)H(t-a)\} = e^{-as}L\{f(t+a)\}$

Examples: Find the following Laplace transforms

a) $L\{4(t-2)H(t-2)\}$ *b)* $L\{2(t-3)^2H(t-3)\}$ *c)* $L\{t^2H(t-4)\}$

d) $L\{(5t+3)H(t-1)\}$ *e)* $L\{\cos 2(t-\pi)H(t-\pi)\}$ *f)* $L\{e^{2t}H(t-4)\}$

Solution: To use the above shifting rule, first identify $f(t)$ from the problem.

a) Here, $f(t-2) = 4(t-2) \Rightarrow f(t) = 4t \Rightarrow L\{f(t)\} = L\{4t\} = \dfrac{4}{s^2}$

Therefore, $L\{4(t-2)H(t-2)\} = e^{-2s}L\{f(t)\} = \dfrac{4e^{-2s}}{s^2}$

b) Here, $f(t-3) = 2(t-3)^2 \Rightarrow f(t) = 2t^2 \Rightarrow L\{f(t)\} = L\{2t^2\} = \dfrac{4}{s^3}$

Therefore, $L\{2(t-3)^2H(t-3)\} = e^{-3s}L\{f(t)\} = e^{-3s}L\{2t^2\} = \dfrac{4e^{-3s}}{s^3}$

c) In this case, let' use the second property with $a = 4$, $f(t) = t^2$.

Here, $f(t) = t^2 \Rightarrow f(t+4) = (t+4)^2 = t^2 + 8t + 16 \Rightarrow L\{f(t+4)\} = \dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}$

Therefore, $L\{t^2H(t-4)\} = e^{-4s}L\{f(t+4)\} = e^{-4s}\left(\dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}\right)$

$d)$ Here, $a=1, f(t)=5t+3 \Rightarrow f(t+1)=5t+8 \Rightarrow L\{f(t+1)\}=\frac{5}{s^2}+\frac{8}{s}$

Therefore, $L\{(5t+3)H(t-1)\}=e^{-s}L\{f(t+1)\}=e^{-s}L\{5t+8\}=e^{-s}\left(\frac{5}{s^2}+\frac{8}{s}\right)$

$e)$ Here, $f(t-\pi)=\cos 2(t-\pi) \Rightarrow f(t)=\cos 2t \Rightarrow L\{f(t)\}=L\{\cos 2t\}=\frac{s}{s^2+4}$

Therefore, $L\{\cos 2(t-\pi)H(t-\pi)\}=e^{-\pi s}L\{f(t)\}=\frac{se^{-\pi s}}{s^2+4}$

$f)$ Here, $a=4, f(t)=e^{2t} \Rightarrow f(t+4)=e^{2(t+4)}=e^{2t}e^{4} \Rightarrow L\{f(t+4)\}=\frac{e^4}{s-2}$

Therefore, $L\{e^{2t}H(t-4)\}=e^{-4s}L\{f(t+4)\}=e^{-4s}L\{e^{2t}e^4\}=\frac{e^4e^{-4s}}{s-2}=\frac{e^{4-4s}}{s-2}$

Laplace Transform of Piecewise Functions

First, let's see how any piecewise function is expressible in terms of step functions. Suppose $f(t)=\begin{cases} f_1(t), 0\le t\le a \\ f_2(t), \quad t>a \end{cases}$ is any piecewise continuous function.

Then, $f(t)$ can be expressed in terms of step function as

$$f(t)=f_1(t)+[f_2(t)-f_1(t)]U_a(t)=f_1(t)[1-U_a(t)]+f_2(t)U_a(t)$$

Laplace Transform:

$$L\{f(t)\}=L\{f_1(t)+[f_2(t)-f_1(t)]U_a(t)\}=L\{f_1(t)\}+L\{[f_2(t)-f_1(t)]U_a(t)\}$$

Examples: Express using step functions and find their Laplace Transforms.

$a)\ f(t)=\begin{cases} 3, 0\le t<1 \\ -1, t\ge 1 \end{cases}$ $\quad b)\ f(t)=\begin{cases} t, 0\le t<2 \\ t+3, t\ge 2 \end{cases}$ $\quad c)\ f(t)=\begin{cases} 0, 0\le t<1 \\ (t-1)^2, t\ge 1 \end{cases}$

$d)\ f(t)=\begin{cases} \sin t, 0\le t<\pi/2 \\ \sin t+\cos(t-\pi/2), t\ge \pi/2 \end{cases}$ $\quad e)\ f(t)=\begin{cases} 0, 0\le t<1 \\ t-1, 1\le t<8 \\ 7, t\ge 8 \end{cases}$

Solution:

$a)\ f(t)=f_1(t)+[f_2(t)-f_1(t)]U_1(t)=3+[-1-3]u_1(t)=3-4u_1(t)$

Therefore, $L\{f(t)\}=L\{3-4u_1(t)\}=L\{3\}-4L\{u_1(t)\}=\frac{3}{s}-\frac{4e^{-s}}{s}$

$b)\ f(t)=f_1(t)+[f_2(t)-f_1(t)]U_2(t)=t+[t+3-t]u_2(t)=t+3u_2(t)$

Therefore, $L\{f(t)\}=L\{t+3u_2(t)\}=L\{t\}+3L\{u_2(t)\}=\frac{1}{s^2}+\frac{3e^{-2s}}{s}$

$c)\ f(t)=(t-1)^2u_1(t)\Rightarrow L\{f(t)\}=L\{(t-1)^2u_1(t)\}=e^{-s}L\{t^2\}=\frac{2e^{-s}}{s^3}$

$d)\ f(t)=f_1(t)+[f_2(t)-f_1(t)]U_a(t)=\sin t+\cos(t-\pi/2)u_{\pi/2}(t)$

Therefore, $L\{f(t)\}=L\{\sin t\}+L\{\cos(t-\pi/2)u_{\pi/2}(t)\}=\frac{1+se^{-\pi s/2}}{s^2+1}$

$e)\ f(t)=f_1(t)[U_0(t)-U_1(t)]+f_2(t)[U_1(t)-U_8(t)]+f_3(t)U_8(t)$
$=(t-1)[U_1(t)-U_8(t)]+7U_8(t)=(t-1).U_1(t)-(t-8)U_8(t)+7U_8(t)$

Therefore, $L\{f(t)\}=L\{(t-1).U_1(t)\}-L\{(t-8)U_8(t)\}=(e^{-s}-e^{-8s})/s^2$

Laplace Transform of functions of the form $\frac{f(t)}{t}$

If $L\{f(t)\}=F(s)$, then, $L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(r)dr$.

Examples: Find the Laplace Transform of

$a)\ \frac{\sin 2t}{t}$ $b)\ \frac{e^t-1}{t}$ $c)\ \frac{2\sin^2 2t}{t}$ $d)\ \frac{e^{-2t}}{t}$

Solution:

$a)$ Here, $f(t)=\sin 2t\Rightarrow F(s)=L\{f(t)\}=L\{\sin 2t\}=\frac{2}{s^2+4}\Rightarrow F(r)=\frac{2}{r^2+4}$

Then, $L\left\{\frac{\sin 2t}{t}\right\}=\int_s^\infty F(r)dr=\int_s^\infty\frac{2}{r^2+4}dr=\lim_{n\to\infty}\int_s^n\frac{2}{r^2+4}dr=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}$

b) $f(t)=e^{t}-1 \Rightarrow F(s)=L\{f(t)\}=L\{e^{t}-1\}=\frac{1}{s-1}-\frac{1}{s} \Rightarrow F(r)=\frac{1}{r-1}-\frac{1}{r}$

Then, $L\left\{\frac{e^{t}-1}{t}\right\}=\int_{s}^{\infty}F(r)dr=\int_{s}^{\infty}(\frac{1}{r-1}-\frac{1}{r})dr=\lim_{n\to\infty}\ln\left(\frac{r-1}{r}\right)\Bigg|_{r=s}^{r=n}$.

$$=\lim_{n\to\infty}\ln\left(\frac{r-1}{r}\right)-\ln\left(\frac{s-1}{s}\right)-\ln\left(\frac{s-1}{s}\right)=\ln\left(\frac{s}{s-1}\right)$$

c) $f(t)=2\sin^{2}2t=2(\frac{1-\cos 4t}{2})=2(1-\cos 4t) \Rightarrow F(s)=\frac{1}{s}-\frac{s}{s^{2}+16}$

Then, $L\left\{\frac{2\sin^{2}2t}{t}\right\}=\int_{s}^{\infty}(\frac{1}{r}-\frac{r}{r^{2}+16})dr=\lim_{n\to\infty}[\ln r-\frac{1}{2}\ln(r^{2}+16)]\Bigg|_{r=s}^{r=n}$

$$=\lim_{n\to\infty}[\ln\frac{n}{\sqrt{n^{2}+16}}-\ln\frac{s}{\sqrt{s^{2}+16}}]=\ln\frac{\sqrt{s^{2}+16}}{s}$$

d) $f(t)=e^{-2t} \Rightarrow F(s)=L\{e^{-2t}\}=\frac{1}{s+2} \Rightarrow F(r)=\frac{1}{r+2}$

$L\left\{\frac{e^{-2t}}{t}\right\}=\int_{s}^{\infty}\frac{1}{r+2}dr=\infty$. This means that $L\left\{\frac{e^{-2t}}{t}\right\}$ does not exist.

2.4 Inverse Laplace Transform and Their Properties

If $L\{f(t)\}=F(s)$, then the function f is said to be the inverse Laplace transform of $F(s)$ and is given by $f(t)=L^{-1}\{F(s)\}$.

Examples: $a)\ L\{1\}=\frac{1}{s}\Rightarrow L^{-1}\{\frac{1}{s}\}=1 \qquad b)\ L\{e^{2t}\}=\frac{1}{s-2}\Rightarrow L^{-1}\{\frac{1}{s-2}\}=e^{2t}$

Properties of Inverse Laplace Transforms

Let $L\{f(t)\}=F(s)$, $L\{g(t)\}=G(s)$ and let a be any constant. Then,

i) $L^{-1}\{aF(s)\}=aL^{-1}\{F(s)\}=af(t)$

ii) $L^{-1}\{F(s)+G(s)\}=L^{-1}\{F(s)\}+L^{-1}\{G(s)\}=f(t)+g(t)$

iii) $L^{-1}\{F(s-a)\}=e^{at}L^{-1}\{F(s)\}=e^{at}f(t)$, $L^{-1}\{F(s+a)\}=e^{-at}f(t)$

iv) $L^{-1}\{\frac{e^{-as}}{s}\}=U(t-a)=H(t-a)$

v) $L^{-1}\{e^{-as}F(s)\}=U(t-a)f(t-a)$

How to find Inverse Laplace Transforms?

Simple Cases: Using manipulations or rearrangements and then properties of Laplace Transforms and basic tables of transforms.

Examples-1: Observe the following inverse computations.

$$a)\ L^{-1}\{\frac{4+s}{s^2+1}\}=L^{-1}\{\frac{4}{s^2+1}+\frac{s}{s^2+1}\}=L^{-1}\{\frac{4}{s^2+1}\}+L^{-1}\{\frac{s}{s^2+1}\}=4\sin t+\cos t$$

$$b)\ L^{-1}\{\frac{6}{s^2-2s+5}\}=L^{-1}\{\frac{6}{(s-1)^2+4}\}=3L^{-1}\{\frac{2}{(s-1)^2+4}\}=3e^{t}\sin 2t$$

$$c)\ L^{-1}\{\frac{s+3}{s^2+6s+13}\}=L^{-1}\{\frac{s+3}{(s+3)^2+4}\}=e^{-3t}\cos 2t$$

$$d)\ L^{-1}\{\frac{1+e^{-2s}}{s^4}\}=L^{-1}\{\frac{1}{s^4}+\frac{e^{-2s}}{s^4}\}=L^{-1}\{\frac{1}{s^4}\}+L^{-1}\{\frac{e^{-2s}}{s^4}\}=\frac{t^3}{6}+\frac{(t-2)^3}{6}H(t-2)$$

$$e)\ L^{-1}\left\{\frac{4s}{4s^2+1}\right\}=L^{-1}\left\{\frac{s}{s^2+(1/2)^2}\right\}=\cos\frac{1}{2}t$$

$$f)\ L^{-1}\left(\frac{s+7}{s^2+2s+5}\right)=L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+L^{-1}\left(\frac{6}{(s+1)^2+4}\right)=e^{-t}\cos 2t+3e^{-t}\sin 2t$$

General Cases: Inverse Laplace Transform by Partial Fractions

The tricky procedures used in the above examples to find inverse transforms are not useful for many cases. So, let's see the procedures to find inverse transform.

The procedure to find the inverse Laplace Transform of $F(s) = \dfrac{P(s)}{Q(s)}$

Step-1: Factorize the denominator $Q(s)$ completely.

Step-2: Decompose $F(s) = \dfrac{P(s)}{Q(s)}$ as sum of rational functions.

Step-3: Determine the constants in the decomposition.

Step-4: Take the Laplace Transform of each term separately.

In doing so, we may get different cases based on the factors of $Q(s)$

i) For each ***non-repeated linear factors*** like $s-a$ of $Q(s)$ assign a rational function of the form $\dfrac{A}{s-a}$ where A is a constant to be determined.

ii) For each ***repeated linear factors*** like $(s-a)^n$ assign a rational function of the form $\dfrac{A_0}{(s-a)^n} + \dfrac{A_1}{(s-a)^{n-1}} + \dfrac{A_2}{(s-a)^{n-2}} + \ldots + \dfrac{A_n}{s-a}$.

iii) For each ***non-repeated quadratic factors*** like as^2+bs+c assign a rational function of the form $\dfrac{As+B}{as^2+bs+c}$

Examples: Find the inverse Laplace Transforms.

$a)\ \dfrac{2s-7}{s^2-5s+6}$ $\quad b)\ \dfrac{18}{s^3-6s^2+9s}$ $\quad c)\ \dfrac{5s^2-6s+2}{(s-2)(s^2+1)}$ $\quad d)\ \dfrac{8s+10}{(s+1)(s+2)^3}$

Solution:

a) Here, $Q(s) = s^2-5s+6 = (s-2)(s-3)$. Then,

$$\frac{2s-7}{s^2-5s+6} = \frac{2s-7}{(s-2)(s-3)} = \frac{A_1}{s-2} + \frac{A_2}{s-3} = \frac{A_1(s-3)+A_2(s-2)}{(s-2)(s-3)}$$

$$\Rightarrow A_1(s-3)+A_2(s-2) = 2s-6 \Rightarrow (A_1+A_2)s - 3A_1 - 2A_2 = 2s-7$$

$$\Rightarrow A_1 + A_2 = 2, -3A_1 - 2A_2 = -7 \Rightarrow A_1 = 3,\ A_2 = -1$$

Hence, $L^{-1}\left\{\dfrac{2s-7}{s^2-5s+6}\right\}=L^{-1}\left\{\dfrac{3}{s-2}\right\}-L^{-1}\left\{\dfrac{1}{s-3}\right\}=3e^{2t}-e^{3t}$

b) Here, $Q(s)=s^3-6s^2+9s=s(s-3)^2$. Then,

$$\frac{18}{s(s-3)(s-3)}=\frac{A_1}{s}+\frac{B_1}{s-3}+\frac{B_2}{(s-3)^2}=\frac{A_1(s-3)(s-3)+B_1s(s-3)+B_2s}{s(s-3)(s-3)}$$

$$\Rightarrow A_1(s-3)(s-3)+B_1s(s-3)+B_2s=18$$

$$\Rightarrow A_1=2,\ B_1=-2,\ B_2=6$$

$$\therefore L^{-1}\left\{\frac{18}{s^3-6s^2+9s}\right\}=L^{-1}\left\{\frac{2}{s}\right\}-L^{-1}\left\{\frac{2}{s-3}\right\}+L^{-1}\left\{\frac{6}{(s-3)^2}\right\}=2-2e^{3t}+6te^{3t}$$

c) Here, we have mixed factors. So, the partial fraction decomposition is of the form

$$\frac{5s^2-6s+2}{(s-2)(s^2+1)}=\frac{A}{s-2}+\frac{Bs+C}{s^2+1}\Rightarrow A=2, B=3, C=0$$

Hence, $L^{-1}\left\{\dfrac{5s^2-6s+2}{(s-2)(s^2+1)}\right\}=L^{-1}\left\{\dfrac{2}{s-2}\right\}+L^{-1}\left\{\dfrac{3s}{s^2+1}\right\}=2e^{2t}+3\cos t$

d) Here, $\dfrac{8s+10}{(s+1)(s+2)^3}=\dfrac{A_1}{s+1}+\dfrac{B_1}{s+2}+\dfrac{B_2}{(s+2)^2}+\dfrac{B_3}{(s+2)^3}$

$$\Rightarrow A_1=2, B_1=-2, B_2=-2, B_3=6$$

$$\therefore L^{-1}\left\{\frac{8s+10}{(s+1)(s+2)^3}\right\}=L^{-1}\left\{\frac{2}{s+1}\right\}-L^{-1}\left\{\frac{2}{s+2}\right\}-L^{-1}\left\{\frac{2}{(s+2)^2}\right\}+L^{-1}\left\{\frac{6}{(s+2)^3}\right\}$$

$$=2e^{-t}-2e^{-2t}-2te^{-2t}+3t^2e^{-2t}$$

Inverse using Second –Shfting Rule: Suppose $L\{f(t)\}=F(s)$.

Then, $L\{f(t-a)u_a(t)\}=e^{-as}F(s)\Leftrightarrow L^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)$. We use this property to compute inerse Laplace Transforms of the form $L^{-1}\{e^{-as}F(s)\}$. To use this rule, first identify $F(s)$ and e^{-as} from the given problem, second find $f(t)$ using $f(t)=L^{-1}\{F(s)\}$, finally use the rule.

Examples: Find the inverse Laplace Transform of

a) $\dfrac{e^{-2s}}{s(s^2+3s+2)}$ b) $\dfrac{e^{-3s}-e^{-7s}}{s^2-4s+3}$ c) $\dfrac{e^{-3s}}{(s+1)^2+1}$ d) $\dfrac{4s+3e^{-20s}}{s^2-9}$

Hence, $L^{-1}\left\{\frac{2s-7}{s^2-5s+6}\right\}=L^{-1}\left\{\frac{3}{s-2}\right\}-L^{-1}\left\{\frac{1}{s-3}\right\}=3e^{2t}-e^{3t}$

b) Here, $Q(s)=s^3-6s^2+9s=s(s-3)^2$. Then,

$$\frac{18}{s(s-3)(s-3)}=\frac{A_1}{s}+\frac{B_1}{s-3}+\frac{B_2}{(s-3)^2}=\frac{A_1(s-3)(s-3)+B_1s(s-3)+B_2s}{s(s-3)(s-3)}$$

$$\Rightarrow A_1(s-3)(s-3)+B_1s(s-3)+B_2s=18$$

$$\Rightarrow A_1=2,\ B_1=-2,\ B_2=6$$

$$\therefore L^{-1}\left\{\frac{18}{s^3-6s^2+9s}\right\}=L^{-1}\left\{\frac{2}{s}\right\}-L^{-1}\left\{\frac{2}{s-3}\right\}+L^{-1}\left\{\frac{6}{(s-3)^2}\right\}=2-2e^{3t}+6te^{3t}$$

c) Here, we have mixed factors. So, the partial fraction decomposition is of the form

$$\frac{5s^2-6s+2}{(s-2)(s^2+1)}=\frac{A}{s-2}+\frac{Bs+C}{s^2+1}\Rightarrow A=2, B=3,\ C=0$$

Hence, $L^{-1}\left\{\frac{5s^2-6s+2}{(s-2)(s^2+1)}\right\}=L^{-1}\left\{\frac{2}{s-2}\right\}+L^{-1}\left\{\frac{3s}{s^2+1}\right\}=2e^{2t}+3\cos t$

d) Here, $\frac{8s+10}{(s+1)(s+2)^3}=\frac{A_1}{s+1}+\frac{B_1}{s+2}+\frac{B_2}{(s+2)^2}+\frac{B_3}{(s+2)^3}$

$$\Rightarrow A_1=2, B_1=-2, B_2=-2, B_3=6$$

$$\therefore L^{-1}\left\{\frac{8s+10}{(s+1)(s+2)^3}\right\}=L^{-1}\left\{\frac{2}{s+1}\right\}-L^{-1}\left\{\frac{2}{s+2}\right\}-L^{-1}\left\{\frac{2}{(s+2)^2}\right\}+L^{-1}\left\{\frac{6}{(s+2)^3}\right\}$$

$$=2e^{-t}-2e^{-2t}-2te^{-2t}+3t^2e^{-2t}$$

Inverse using Second –Shfting Rule: Suppose $L\{f(t)\}=F(s)$.

Then, $L\{f(t-a)u_a(t)\}=e^{-as}F(s)\Leftrightarrow L^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)$. We use this property to compute inerse Laplace Transforms of the form $L^{-1}\{e^{-as}F(s)\}$. To use this rule, first identify $F(s)$ and e^{-as} from the given problem, second find $f(t)$ using $f(t)=L^{-1}\{F(s)\}$, finally use the rule.

Examples: Find the inverse Laplace Transform of

a) $\frac{e^{-2s}}{s(s^2+3s+2)}$ *b)* $\frac{e^{-3s}-e^{-7s}}{s^2-4s+3}$ *c)* $\frac{e^{-3s}}{(s+1)^2+1}$ *d)* $\frac{4s+3e^{-20s}}{s^2-9}$

Solution:

a) Here, $\dfrac{e^{-2s}}{s(s^2+3s+2)}=e^{-2s}.\dfrac{1}{s(s^2+3s+2)}=e^{-2s}F(s)$ where

$$F(s)=\frac{1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\Rightarrow A=\frac{1}{2},B=-1,C=\frac{1}{2}$$

Thus, $f(t)=L^{-1}\left\{\dfrac{1/2}{s}\right\}-L^{-1}\left\{\dfrac{1}{s+1}\right\}+L^{-1}\left\{\dfrac{1/2}{s+2}\right\}=\dfrac{1}{2}-e^{-t}+\dfrac{1}{2}e^{-2t}$

Thus, $L^{-1}\{e^{-2s}F(s)\}=f(t-2)u(t-2)=(\dfrac{1}{2}-e^{-(t-2)}+\dfrac{1}{2}e^{-2(t-2)})u(t-2)$

b) Here, $\dfrac{e^{-3s}-e^{-7s}}{s^2-4s+3}=e^{-3s}.\dfrac{1}{s^2-4s+3}-e^{-7s}.\dfrac{1}{s^2-4s+3}=e^{-3s}F(s)-e^{-7s}F(s)$

$$F(s)=\frac{1}{s^2-4s+3}=\frac{1}{(s-3)(s-1)}=\frac{A}{s-3}+\frac{B}{s-1}\Rightarrow A=\frac{1}{2},B=-\frac{1}{2}$$

$$\Rightarrow f(t)=L^{-1}\left\{\frac{1}{s^2-4s+3}\right\}=L^{-1}\left\{\frac{1/2}{s-3}\right\}+L^{-1}\left\{\frac{-1/2}{s-1}\right\}=\frac{1}{2}e^{3t}-\frac{1}{2}e^{t}$$

Thus, $L^{-1}\{e^{-3s}F(s)-e^{-7s}F(s)\}=L^{-1}\{e^{-3s}F(s)\}-L^{-1}\{e^{-7s}F(s)\}$

$$=f(t-3)U(t-3)-f(t-7)u(t-7)$$

c) Here, $\dfrac{e^{-3s}}{(s+1)^2+1}=e^{-3s}.\dfrac{1}{(s+1)^2+1}=e^{-3s}F(s)$ where

$$F(s)=L\{f(t)\}=\frac{1}{(s+1)^2+1}\Rightarrow f(t)=L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$$

Thus, $L^{-1}\{e^{-3s}F(s)\}=f(t-3)u(t-3)=e^{3-t}\sin(t-3)u(t-3)$

d) Here, $\dfrac{4s+3e^{-20s}}{s^2-9}=\dfrac{4s}{s^2-9}+e^{-20s}.\dfrac{3}{s^2-9}=4G(s)+e^{-20s}F(s)$ where

$$g(t)=L^{-1}\left\{\frac{s}{s^2-9}\right\}=\cosh 3t,\ f(t)=L^{-1}\left\{\frac{3}{s^2-9}\right\}=\sinh 3t$$

Thus, $L^{-1}\{4G(s)+e^{-20s}F(s)\}=4L^{-1}\{G(s)\}+L^{-1}\{e^{-20s}F(s)\}$

$$=4\cosh 3t+\sinh 3(t-20)u(t-20)$$

2.5 Convolution Product and Laplace Transforms

Definition: Suppose f and g are continuous functions on $[a,b]$.
Then for $t\in[a,b]$, the convolution product of f and g, denoted by $f*g$, is defined as $(f*g)(t)=\int_0^t f(t-x)g(x)dx$.

Example:

Let $f(t)=3, g(t)=e^t$. Then, $(f*g)(t)=\int_0^t f(t-x)g(x)dx=\int_0^t 3e^x dx=3e^t-3$

Properties of convolution product

i) $f*g=g*f$ *ii)* $f*(g+h)=f*g+f*h$ *iii)* $f*(g*h)=(f*g)*h$

Convolution Theorem: If $L\{f(t)\}=F(s)$ and $L\{g(t)\}=G(s)$, then

i) $L\{(f*g)(t)\}=L\{f(t)\}L\{g(t)\}=F(s)G(s)$ ii) $L^{-1}\{F(s)G(s)\}=(f*g)(t)$

Examples:

Find $L\{(f*g)(t)\}$ using Convolution Theorem where

a) $f(t)=1, g(t)=t^4$ *b)* $f(t)=t^2, g(t)=te^t$ c) $f(t)=e^{3t}, g(t)=\sin 2t$

Solution: First find $L\{f(t)\}$ and $L\{g(t)\}$ so as to apply Convolution Theorem.

$$a)\ L\{f(t)\}=L\{1\}=\frac{1}{s}, L\{g(t)\}=L\{t^4\}=\frac{24}{s^5}$$

$$\Rightarrow L\{(f*g)(t)\}=L\{f(t)\}L\{g(t)\}=L\{1*t^4\}=L\{1\}L\{t^4\}=\frac{24}{s^6}$$

$$b)\ L\{f(t)\}=L\{t^2\}=\frac{2}{s^3}, L\{g(t)\}=L\{te^t\}=\frac{1}{(s-1)^2}$$

$$\Rightarrow L\{(f*g)(t)\}=L\{t^2*te^t\}=L\{t^2\}L\{te^t\}=\frac{2}{s^3}\cdot\frac{1}{(s-1)^2}=\frac{2}{s^3(s-1)^2}$$

$$c)\ L\{f(t)\}=L\{e^{3t}\}=\frac{1}{s-3}, L\{g(t)\}=L\{\sin 2t\}=\frac{2}{s^2+4}$$

$$\Rightarrow L\{(f*g)(t)\}=L\{e^{3t}*\sin 2t\}=L\{e^{3t}\}L\{\sin 2t\}=\frac{2}{(s-3)(s^2+4)}$$

2.6 Laplace Transform of Derivatives and Integrals

2.6.1 Laplace Transform of Derivatives

If $F(s)$ is the Laplace transform of f and if f has a value $f(0)$, when $t=0$, then $L\{f'(t)\}=sF(s)-f(0)$, $L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$.

In general, $L[f^{(n)}(t)]=s^nL[f(t)]-s^{n-1}f(0)-s^{n-2}f'(0)-...-f^{(n-1)}(0)$ where $f^{(n)}(0)$ is the value of the n^{th} derivative of $f(t)$ at $t=0$.

Example: Given f with $f''(t)-6f'(t)+5f(t)=0, f(0)=3, f'(0)=7$. **Find** the Laplace Transform of f and deduce $f(t)$ itself.

Solution: Take transform both sides.

$$L\{f''(t)-6f'(t)+5f(t)\}=0$$

$$\Rightarrow L\{f''(t)\}-6L\{f'(t)\}+5L\{f(t)\}=0$$

$$\Rightarrow s^2L\{f(t)\}-sf(0)-f'(0)-6[sL\{f(t)\}-f(0)]+5L\{f(t)\}=0$$

$$\Rightarrow s^2L\{f(t)\}-3s-7-6sL\{f(t)\}+18+5L\{f(t)\}=0$$

$$\Rightarrow (s^2-6s+5)L\{f(t)\}=3s-11$$

$$\Rightarrow L\{f(t)\}=\frac{3s-11}{s^2-6s+5}$$

Now let's deduce $f(t)$ itself by using inverse transform. That is

$$L\{f(t)\}=\frac{3s-11}{s^2-6s+5}$$

$$\Rightarrow f(t)=L^{-1}\left\{\frac{3s-11}{s^2-6s+5}\right\}=L^{-1}\left\{\frac{1}{s-5}+\frac{2}{s-1}\right\}$$

$$=L^{-1}\left\{\frac{1}{s-5}\right\}+L^{-1}\left\{\frac{2}{s-1}\right\}=e^{5t}+2e^{t}$$

2.6.2 Laplace Transform of Integrals

Suppose $f(t)$ is piece wise continuous for all $t \geq 0$ and $L\{f(t)\} = F(s)$.

Then, $L\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$ and thus $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(x)dx$.

Examples:

1. Find the following Laplace Transforms

a) $L\left\{\int_0^t \cos 3x dx\right\}$ b) $L\left\{\int_0^t x^2e^{2x}dx\right\}$ c) $L\left\{\int_0^t x\sin x dx\right\}$ d) $L\left\{\int_0^t e^x \cos x dx\right\}$

Solution:

a) $L\left\{\int_0^t \cos 3x dx\right\} = \frac{1}{s}L\{\cos 3t\} = \frac{1}{s}\cdot\frac{s}{s^2+9} = \frac{1}{s^2+9}$

b) $L\left\{\int_0^t x^2e^{2x}dx\right\} = \frac{1}{s}L\{t^2e^{2t}\} = \frac{1}{s}\frac{d^2}{ds^2}(L\{e^{2t}\}) = \frac{1}{s}\frac{d^2}{ds^2}(\frac{1}{s-2}) = \frac{2}{s(s-2)^3}$

c) $L\left\{\int_0^t x\sin x dx\right\} = \frac{1}{s}L\{t\sin t\} = -\frac{1}{s}\cdot\frac{-2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$

d) $L\left\{\int_0^t e^x \cos x dx\right\} = \frac{1}{s}L\{e^t \cos t\} = \frac{1}{s}\cdot\frac{s-1}{(s-1)^2+1} = \frac{s-1}{s[(s-1)^2+1]}$

2*. Find the Laplace Transform of the following functions.

a) $f(t) = e^{-6t}\int_0^t x^2e^{2x}dx$ b) $f(t) = 3t\int_0^t x\sin 2x dx$ c) $f(t) = \int_0^t \frac{\sin ax}{x}dx$

Solution: Transform step by step.

a) First using *t-shifting*, $L\{t^2e^{2t}\} = \frac{d^2}{ds^2}\left(L\{e^{2t}\}\right) = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$

Second, using Laplace Transform of integrals, find $L\left\{\int_0^t x^2e^{2x}dx\right\}$.

That is $F(s) = L\left\{\int_0^t x^2e^{2x}dx\right\} = \frac{1}{s}L\{t^2e^{2t}\} = \frac{1}{s}\cdot\frac{2}{(s-2)^3} = \frac{2}{s(s-2)^3}$

Then, by *S-shifting*, $L\left\{e^{-6t}\int_0^t x^2e^{2x}dx\right\} = F(s+6) = \dfrac{2}{(s+6)(s+4)^3}$

b) Using *t-shifting*, $L\{t\sin 2t\} = -\dfrac{d}{ds}(L\{\sin 2t\}) = -\dfrac{d}{ds}(\dfrac{2}{s^2+4}) = \dfrac{4s}{(s^2+4)^2}$

Second, using Laplace Transform of integrals, find $L\{\int_0^t x\sin 2xdx\}$.

That is $F(s) = L\{\int_0^t x\sin 2xdx\} = \dfrac{1}{s}L\{t\sin 2t\} = \dfrac{1}{s}.\dfrac{4s}{(s^2+4)^2} = \dfrac{4}{(s^2+4)^2}$

Finally, by *t-shifting*,

$$L\{f(t)\} = L\{3t\int_0^t x\sin 2xdx\} = -3\frac{d}{ds}\left(\frac{4}{(s^2+4)^2}\right) = \frac{48s}{(s^2+4)^3}$$

c) First find the transform of $\dfrac{\sin ax}{x}$. That is $L\{\dfrac{\sin ax}{x}\} = \int_s^\infty F(r)dr$ where

$F(r) = L\{\sin ax\} = \dfrac{a}{r^2+a^2}$.

So, $L\{\dfrac{\sin ax}{x}\} = \int_s^\infty \dfrac{a}{r^2+a^2}dr = \lim\limits_{n\to\infty}\tan^{-1}\dfrac{r}{a}\Big|_{r=s}^{r=n} = \dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{a}$

Therefore, $L\{f(t)\} = L\{\int_0^t \dfrac{\sin ax}{x}dx\} = \dfrac{1}{s}L\{\dfrac{\sin ax}{x}\} = \dfrac{\pi}{2s} - \dfrac{1}{s}\tan^{-1}\dfrac{s}{a}$

3. Given $L\{f(t)\} = \dfrac{se^{-6s}}{s^4+2s^2+1}, \int_0^2 f(x)dx = 8,\ g(t) = \int_2^t f(x)dx$. Find $L\{g(t)\}$.

Solution:

$$L\{g(t)\} = L\{\int_2^t f(x)dx\} = L\{\int_0^t f(x)dx - \int_0^2 f(x)dx\}$$

$$= L\{\int_0^t f(x)dx\} - L\{\int_0^2 f(x)dx\}$$

$$= \frac{1}{s}L\{f(t)\} - L\{8\} = \frac{1}{s}L\{f(t)\} - \frac{8}{s} = \frac{e^{-6s}}{s^4+2s^2+1} - \frac{8}{s}$$

2.6 Applications of Laplace Transform

Laplace Transform is widely applicable to solve many real life problems. Especially it is useful to solve Initial Value Problems, Integral-Equations, System of Differential Equations. Here, let's see some applications.

I) Initial value problems: Here, let's see how Laplace transform can be used to solve initial value problems for linear differential equations.

Procedures to solve IVPs using Laplace Transform:

Given the IVP: $ay''+by'+cy = f(x), y(0) = m, y'(0) = n$.

Then, to find $y(t)$ satisfying the IVP, we use the following procedures.

Step-1: Take Laplace Transform of the DE both sides.

That is $L\{ay''+by'+cy\} = L\{f(x)\} \Rightarrow aL\{y''\} + bL\{y'\} + cL\{y\} = y(s)$

At this step use Laplace transform of derivatives and solve for $y(s)$ by substituting the given initial conditions $y(0) = m,\ y'(0) = n$.

Step-2: Solve for $y(t)$ by taking inverse Laplace Transform of $Y(s)$

Take inverse Laplace Transform of $Y(s)$. That is $y(t) = L^{-1}\{Y(s)\}$

At this step, apply inverse finding techniques to solve $L^{-1}\{Y(s)\}$ for $y(t)$

Examples:

1. Solve the following initial value problem using Laplace Transforms.

a) $y'+3y = e^{t}$, $y(0) = 1$

b) $y''-2y'-3y = 3e^{2t}$, $y(0) = 0, y'(0) = 5$

c) $y''+4y = 24e^{2t}$, $y(0) = 0, y'(0) = 0$

d) $y''+y'-2y = 9te^{2t}$, $y(0) = y'(0) = 0$

e) $y''-3y'+2y = 6e^{-t}$, $y(0) = 0, y'(0) = 0$

f) $y''+2y'+5y = e^{-t}\sin t$, $y(0) = 0, y'(0) = 1$

g) $y''+4y'+5y = 25t^{2}$, $y(0) = 0, y'(0) = 1$

h) $y''-2y'+y = t^{3}e^{t}, y(0) = 0,\ y'(0) = 0$

i) $y''+2y'+y = te^{-2t}, y(0) = 1,\ y'(0) = 1$

Solution:

a) **Step-1:** Determine the Laplace Transform $Y(s)$ of $y(t)$. That is take Laplace Transform both sides and solve for $Y(s)$ by substituting initial conditions.

$$L\{y'+3y\} = L\{e^t\} \Rightarrow L\{y'\} + L\{3y\} = sy(s) - y(0) + 3y(s) = \frac{1}{s-1}$$

$$\Rightarrow sy(s) - 1 + 3y(s) = \frac{1}{s-1} \Rightarrow sy(s) + 3y(s) = \frac{1}{s-1} + 1$$

$$\Rightarrow y(s)(s+3) = \frac{s}{s-1} \Rightarrow y(s) = \frac{s}{(s+3)(s-1)}$$

Step-2: Solve for $y(t)$ from inverse Laplace Transform of $Y(s)$.

That is $y(t) = L^{-1}\{y(s)\} = L^{-1}\left(\dfrac{s}{(s+3)(s-1)}\right)$

Apply inverse finding techniques to solve $L^{-1}\{Y(s)\}$ for $y(t)$

That is find the inverse transform $L^{-1}\left(\dfrac{s}{(s+3)(s-1)}\right)$.

To determine the inverse, let's use Partial Fraction Decomposition.

Here, $y(s) = \dfrac{s}{(s+3)(s-1)} = \dfrac{A}{s+3} + \dfrac{B}{s-1}$ and determine A, B and C.

Now, determine the constants A, B and C.

By using Cover-Up Method, at $s=-3$, we have $A = \dfrac{3}{4}$ and at $s=1$, $B = \dfrac{1}{4}$.

So, the decomposition is $y(s) = \dfrac{s}{(s+3)(s-1)} = \dfrac{\frac{3}{4}}{s+3} + \dfrac{\frac{1}{4}}{s-1}$.

Therefore, the function $y(t)$ is determined as

$$y(t)=L^{-1}\left(\frac{s}{(s+3)(s-1)}\right)=L^{-1}\left(\frac{\frac{3}{4}}{s+3}+\frac{\frac{1}{4}}{s-1}\right)$$

$$=\frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)+\frac{1}{4}L^{-1}\left(\frac{1}{s-1}\right)=\frac{3}{4}e^{-3t}+\frac{1}{4}e^{t}$$

b) **Step-1:** Determine the Laplace Transform $Y(s)$ of $y(t)$. That is take Laplace Transform both sides and solve for $Y(s)$ by substituting initial conditions.

$$L\{y''-2y'-3y\}=L\{3e^{2t}\}\Rightarrow L\{y''\}-L\{2y'\}+L\{-3y\}=\frac{3}{s-2}$$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-2[sy(s)-y(0)]-3y(s)=\frac{3}{s-2}$$

$$\Rightarrow s^2y(s)-5-2sy(s)-3y(s)=\frac{3}{s-2}\Rightarrow (s^2-2s-3)y(s)=\frac{3}{s-2}+5$$

$$\Rightarrow y(s)=\frac{5s-7}{(s^2-2s-3)(s-2)}=\frac{5s-7}{(s+1)(s-3)(s-2)}.$$

Hence, the transform of $y(t)$ is found to be $y(s)=\dfrac{5s-7}{(s+1)(s-3)(s-2)}$.

Step-2: Solve for $y(t)$ by inverse Laplace Transform of $Y(s)$.

That is $y(t)=L^{-1}\{y(s)\}=L^{-1}\left(\dfrac{5s-7}{(s-3)(s+1)(s-2)}\right)$

To evaluate the inverse let's use Partial Fraction Decomposition.

Here, $y(s)=\dfrac{5s-7}{(s-3)(s+1)(s-2)}=\dfrac{A}{s-3}+\dfrac{B}{s+1}+\dfrac{C}{s-2}$.

By using cover-up method, at $s=3$, **we have** $A=\dfrac{15-7}{(4)(1)}=\dfrac{8}{4}=2$.

At $s=-1$, $B=\dfrac{-5-7}{-4(-3)}=-\dfrac{12}{12}=-1$ **and at** $s=2$, **we have** $C=\dfrac{10-7}{-3}=-1$.

So, the decomposition is $y(s)=\dfrac{5s-7}{(s-3)(s+1)(s-2)}=\dfrac{2}{s-3}-\dfrac{1}{s+1}-\dfrac{1}{s-2}$

Therefore, $y(t) = L^{-1}\left(\dfrac{5s-7}{(s-3)(s+1)(s-2)}\right) = L^{-1}\left(\dfrac{2}{s-3} - \dfrac{1}{s+1} - \dfrac{1}{s-2}\right)$

$$= 2L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 2e^{3t} - e^{-t} - e^{2t}$$

c) $L\{y''+4y\} = L\{24e^{2t}\} \Rightarrow L\{y''\} + L\{4y\} = 24L\{e^{2t}\}$

$$\Rightarrow s^2y(s) - sy(0) - y'(0) + 4y(s) = \frac{24}{s-2} \Rightarrow s^2y(s) + 4y(s) = \frac{24}{s-2}$$

$$\Rightarrow (s^2+4)y(s) = \frac{24}{s-2} \Rightarrow y(s) = \frac{24}{(s^2+4)(s-2)}$$

But $\dfrac{24}{(s^2+4)(s-2)} = \dfrac{As+B}{s^2+4} + \dfrac{C}{s-2} \Rightarrow A = -3, B = -6, C = 3$

So, we get $y(s) = \dfrac{-3s-6}{s^2+4} + \dfrac{3}{s-2} = -\dfrac{3s}{s^2+4} - \dfrac{6}{s^2+4} + \dfrac{3}{s-2}$

Therefore, $y(t) = L^{-1}\left\{-\dfrac{3s}{s^2+4}\right\} + L^{-1}\left\{-\dfrac{6}{s^2+4}\right\} + L^{-1}\left\{\dfrac{3}{s-2}\right\}$

$$= -3L^{-1}\left\{\frac{s}{s^2+4}\right\} - 3L^{-1}\left\{\frac{2}{s^2+4}\right\} + 3L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -3\cos 2t - 3\sin 2t + 3e^{2t}$$

d) $L\{y''+y'-2y\} = L\{8te^{2t}\}$

$$\Rightarrow L\{y''\} + L\{y'\} - 2L\{y\} = 8L\{te^{2t}\}$$

$$\Rightarrow s^2y(s) - sy(0) - y'(0) + sy(s) - y(0) - 2y(s) = \frac{8}{(s-2)^2}$$

$$\Rightarrow (s^2+s-2)y(s) = \frac{8}{(s-2)^2} \Rightarrow y(s) = \frac{8}{(s-2)^2(s^2+s-2)}$$

Now, using partial fraction decomposition,

$$y(s)=\frac{8}{(s-2)^2(s-1)(s+2)}=\frac{a}{s-2}+\frac{b}{(s-2)^2}+\frac{c}{s-1}+\frac{d}{s+2}$$

$$\Rightarrow a=-\frac{5}{2},b=2,c=\frac{8}{3},d=-\frac{1}{6}$$

Therefore, $y(t)=-\frac{5}{2}L^{-1}\left(\frac{1}{s-2}\right)+2L^{-1}\left(\frac{1}{(s-2)^2}\right)+\frac{8}{3}L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{6}L^{-1}\left(\frac{s}{s+2}\right)$

$$=-\frac{5}{2}e^{2t}+2te^{2t}+\frac{8}{3}e^{t}-\frac{1}{6}e^{-2t}$$

e) Step-1: Determine the Laplace Transform $y(s)$.

$$L\{y''-3y'+2y\}=L\{6e^{-t}\}\Rightarrow L\{y''\}-3L\{y'\}+2L\{y\}=6L\{e^{-t}\}$$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-3[sy(s)-y(0)]+2y(s)=\frac{6}{s+1}$$

$$\Rightarrow s^2y(s)-3sy(s)+2y(s)=\frac{6}{s+1}\Rightarrow(s^2-3s+2)y(s)=\frac{6}{s+1}$$

$$\Rightarrow y(s)=\frac{6}{(s^2-3s+2)(s+1)}=\frac{6}{(s-2)(s-1)(s+1)}$$

Step-2: Solve for $y(t)$ from $y(t)=L^{-1}\{y(s)\}$

To evaluate the inverse let's use Partial Fraction Decomposition.

Here, $\frac{6}{(s-2)(s-1)(s+1)}=\frac{A}{s-2}+\frac{B}{s-1}+\frac{C}{s+1}$.

Then, by using cover-up method, find the constants A, B and C.

At $s=2$, we have $A=2$, at $s=1, B=-3$ and at $s=-1, C=1$.

So, the decomposition is $y(s)=\frac{6}{(s-2)(s-1)(s+1)}=\frac{2}{s-2}-\frac{3}{s-1}+\frac{1}{s+1}$.

Therefore, $y(t)=L^{-1}\left\{\frac{6}{(s-2)(s-1)(s+1)}\right\}=L^{-1}\left\{\frac{2}{s-2}-\frac{3}{s-1}+\frac{1}{s+1}\right\}$

$$=2L^{-1}\left\{\frac{1}{s-2}\right\}-3L^{-1}\left\{\frac{1}{s-1}\right\}+L^{-1}\left\{\frac{1}{s+1}\right\}=2e^{2t}-3e^{t}+e^{-t}$$

f) $L\{y''+2y'+5y\}=L\{e^{-t}\sin t\}\Rightarrow L\{y''\}+L\{2y'\}+L\{5y\}=L\{e^{-t}\sin t\}$

$$\Rightarrow(s^2+2s+5)y(s)=\frac{s^2+2s+3}{(s+1)^2+1}\Rightarrow y(s)=\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

By partial fraction,

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}=\frac{As+B}{s^2+2s+5}+\frac{Cs+D}{s^2+2s+2}\Rightarrow A=C=0,B=\frac{2}{3},D=\frac{1}{3}$$

So, $y(t)=y(t)=\frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=\frac{1}{3}e^{-t}(\sin 2t+\sin t)$

g) $L\{y''+4y'+5y\}=L\{25t^2\}\Rightarrow L\{y''\}+4L\{y'\}+5L\{y\}=25L\{t^2\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+4[sy(s)-y(0)]+5y(s)=\frac{50}{s^3}$$

$$\Rightarrow(s^2+4s+5)y(s)=\frac{50}{s^3}+1\Rightarrow y(s)=\frac{50}{s^3(s^2+4s+5)}+\frac{1}{s^2+4s+5}$$

But by partial fraction decomposition, we have

$$\frac{50}{s^3(s^2+4s+5)}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{Ds+E}{s^2+4s+5}$$

$$\Rightarrow A=10,B=-8,C=\frac{22}{5},D=-\frac{22}{5},E=-\frac{48}{5}$$

So, $y(t)=L^{-1}\left(\frac{50}{s^3(s^2+4s+5)}\right)+L^{-1}\left(\frac{1}{s^2+4s+5}\right)$

$$=10L^{-1}\left(\frac{1}{s^3}\right)-8L^{-1}\left(\frac{1}{s^2}\right)+\frac{22}{5}L^{-1}\left(\frac{1}{s}\right)-\frac{22}{5}L^{-1}\left(\frac{s}{s^2+4s+5}\right)-\frac{43}{5}L^{-1}\left(\frac{1}{s^2+4s+5}\right)$$

$$=10L^{-1}\left(\frac{1}{s^3}\right)-8L^{-1}\left(\frac{1}{s^2}\right)+\frac{22}{5}L^{-1}\left(\frac{1}{s}\right)-\frac{22}{5}L^{-1}\left(\frac{s+2-2}{(s+2)^2+1}\right)-\frac{43}{5}L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$=5t^2-8t+\frac{22}{5}-\frac{22}{5}(e^{-2t}\cos t-2e^{-2t}\sin t)-\frac{43}{5}e^{-2t}\sin t$ (How?)

h) $L\{y''-2y'+y\}=L\{t^3e^t\} \Rightarrow L\{y''\}-2L\{y'\}+L\{y\}=L\{t^3e^t\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-2[sy(s)-y(0)]+y(s)=\frac{6}{(s-1)^4}$$

$$\Rightarrow (s^2-2s+1)y(s)=\frac{6}{(s-1)^4} \Rightarrow y(s)=\frac{6}{(s^2-2s+1)(s-1)^4}$$

$$\Rightarrow y(s)=\frac{6}{(s-1)^2(s-1)^4}=\frac{6}{(s-1)^6}$$

Therefore, using s-shifting and power properties of Laplace transform, we have $y(t)=L^{-1}\left(\frac{6}{(s-1)^6}\right)=6L^{-1}\left(\frac{1}{(s-1)^6}\right)=\frac{6}{120}t^5e^t=\frac{1}{20}t^5e^t$

i) $L\{y''+2y'+y\}=L\{te^{-2t}\} \Rightarrow L\{y''\}+2L\{y'\}+L\{y\}=L\{te^{-2t}\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+2[sy(s)-y(0)]+y(s)=\frac{1}{(s+2)^2}$$

$$\Rightarrow (s^2+2s+1)y(s)=\frac{1}{(s+2)^2}+s+3 \Rightarrow y(s)=\frac{s^3+7s^2+16s+12}{(s+1)^2(s+2)^2}$$

Continue with partial fraction and complete it.

2. Using ***Method of Laplace Transform***, solve

a) $y''-y=te^t$, $y(0)=y'(0)=1$

b) $y''-3y'-4y=t^2, y(0)=2, y'(0)=1$

c) $y''+4y'+3y=e^{-t}, y(0)=y'(0)=1$

d) $y''+9y=\sin 2t, y(0)=7, y'(0)=-3$

e) $y''+y'-2y=te^{2t}, y(0)=0, y'(0)=0$

f) $y''-2y'-3y=e^{-3t}, y(0)=0, y'(0)=0$

Solution:

a) Taking Laplace transform on both sides, we get

$$L\{y''-y\}=L\{te^t\} \Rightarrow s^2y(s)-sy(0)-y'(0)-y(s)=\frac{1}{(s-1)^2}$$

$$\Rightarrow y(s)=\frac{s+1}{s^2-1}+\frac{1}{(s-1)^2(s^2-1)}=\frac{1}{s-1}+\frac{1}{(s+1)(s-1)^3}$$

But by Partial Fraction decomposition,

$$\frac{1}{(s+1)(s-1)^3}=\frac{A}{s+1}+\frac{B}{(s-1)^3}+\frac{C}{(s-1)^2}+\frac{D}{s-1} \Rightarrow A=-\frac{1}{8}, B=\frac{1}{2}, C=-\frac{1}{4}, D=\frac{1}{8}$$

So, $y(t)=\frac{9}{8}e^{t}-\frac{1}{8}e^{-t}+\frac{1}{4}t^{2}e^{t}-\frac{1}{4}te^{t}$

b) $L\{y''-3y'-4y\}=L\{t^{2}\}\Rightarrow s^{2}y(s)-sy(0)-y'(0)-3[sy(s)-y(0)]-4y(s)=\frac{2}{s^{3}}$

$$\Rightarrow(s^{2}-3s-4)y(s)=2s-5+\frac{2}{s^{3}}\Rightarrow y(s)=\frac{2s-5}{(s-4)(s+1)}+\frac{2}{s^{3}(s-4)(s+1)}$$

But by Partial Fraction decomposition,

$$\frac{2s-5}{(s-4)(s+1)}=\frac{A}{s-4}+\frac{B}{s+1}\Rightarrow A=\frac{3}{5},B=\frac{7}{5}$$

$$\frac{2}{s^{3}(s-4)(s+1)}=\frac{C}{s-4}+\frac{D}{s+1}+\frac{E}{s^{3}}+\frac{F}{s^{2}}+\frac{G}{s}$$

$$\Rightarrow C=\frac{1}{160},D=\frac{2}{5},E=-\frac{1}{2},F=\frac{3}{8},G=-\frac{13}{32}$$

Thus, $y(t)=\frac{97}{160}e^{4t}+\frac{9}{5}e^{-t}-\frac{1}{4}t^{2}+\frac{3}{8}t-\frac{13}{32}$

c) Taking Laplace transform both sides, we get

$L\{y''+4y'+3y\}=L\{e^{-t}\}\Rightarrow L\{y''\}+L\{4y'\}+L\{3y\}=L\{e^{-t}\}$

$$\Rightarrow(s^{2}+4s+3)y(s)=\frac{s^{2}+6s+6}{s+1}\Rightarrow y(s)=\frac{s^{2}+6s+6}{(s+1)^{2}(s+3)}$$

But $\frac{s^{2}+6s+6}{(s+1)^{2}(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^{2}}+\frac{C}{s+3}$, $A=\frac{7}{4}$, $B=\frac{1}{2}$, $C=-\frac{3}{4}$

So, $y=\frac{7}{4}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^{2}}\right)-\frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)=\frac{7}{4}e^{-t}-\frac{3}{4}e^{-3t}+\frac{1}{2}te^{-t}$

d) $L\{y''+9y\}=L\{\sin 2t\}\Rightarrow y(s)=\frac{2}{(s^{2}+4)(s^{2}+9)}+\frac{7s-3}{s^{2}+9}$

$$\Rightarrow y(s)=\frac{2/5}{s^{2}+4}-\frac{2/5}{s^{2}+9}+\frac{7s-3}{s^{2}+9}=\frac{2/5}{s^{2}+4}-\frac{17/5}{s^{2}+9}+\frac{7s}{s^{2}+9}$$

$$\Rightarrow y(t)=\frac{1}{5}\sin 2t-\frac{17}{15}\sin 3t+7\cos 3t$$

$$\frac{1}{(s-2)^2(s^2+s-2)}=\frac{a}{s-2}+\frac{b}{(s-2)^2}+\frac{c}{s-1}+\frac{d}{s+2}$$

$$\Rightarrow d=-\frac{1}{48},c=\frac{1}{3},b=\frac{1}{4},a=-\frac{5}{16}$$

Thus, $y=L^{-1}\left[\frac{-5/16}{s-2}+\frac{1/4}{(s-2)^2}+\frac{1/3}{s-1}-\frac{1/48}{s+2}\right]=-\frac{5}{16}e^{2t}+\frac{1}{4}te^{2t}+\frac{1}{3}e^{t}-\frac{1}{48}e^{-2t}$

3. Using ***Method of Laplace Transform***, solve the following IVPs.

a) $y''-y=f(t),y(0)=1,y'(0)=0$ where $f(t)=\begin{cases}1, 0\le t<1\\ 0, t>1\end{cases}$

b) $y''+9y=f(t),y(0)=0,y'(0)=0$ where $f(t)=\begin{cases}1, 0\le t<2\\ 0, t\ge 2\end{cases}$

c) $y''+y=f(t),y(0)=0,y'(0)=0$ where $f(t)=\begin{cases}t, 0\le t<\pi\\ 0, t>\pi\end{cases}$

d) $y'-2y=\frac{\sin t}{t},y(0)=1$

Solution:

a) First observe that using unti step function $f(t)=1-u(t-1)$.

$$L\{f(t)\}=L\{1-u(t-1)\}=L\{1\}-L\{u(t-1)\}=\frac{1}{s}-\frac{e^{-s}}{s}$$

Then, $L\{y''-y\}=L\{f(t)\}\Rightarrow L\{y''\}-L\{y\}=L\{f(t)\}=\frac{1}{s}-\frac{e^{-s}}{s}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-L(y)=\frac{1}{s}-\frac{e^{-s}}{s}$$

$$\Rightarrow (s^2-1)y(s)-s=\frac{1}{s}-\frac{e^{-s}}{s}\Rightarrow y(s)=\frac{s}{s^2-1}+\frac{1}{s(s^2-1)}-\frac{e^{-s}}{s(s^2-1)}$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{s}{s^2-1}\right)+L^{-1}\left(\frac{1}{s(s^2-1)}\right)-L^{-1}\left(\frac{e^{-s}}{s(s^2-1)}\right)$$

Hence, $y(t)=L^{-1}\left(\dfrac{s}{s^2-1}\right)+L^{-1}\left(\dfrac{1}{s(s^2-1)}\right)-L^{-1}\left(\dfrac{e^{-s}}{s(s^2-1)}\right)$

$$=e^t+e^{-t}-1-(\frac{1}{2}e^{t-1}+\frac{1}{2}e^{1-t}-1)H(t-1)=\begin{cases}e^t+e^{-t}-1, 0\le t<1\\ e^t+e^{-t}-\frac{1}{2}(e^{t-1}+e^{1-t}), t>1\end{cases}$$

b) First observe that

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^2 e^{-st}f(t)dt+\int_2^\infty e^{-st}f(t)dt=\int_0^2 e^{-st}dt=\frac{1}{s}-\frac{e^{-2s}}{s}$$

Then, $L\{y''+9y\}=L\{f(t)\}\Rightarrow L\{y''\}+L\{9y\}=L\{f(t)\}=\dfrac{1}{s}-\dfrac{e^{-2s}}{s}=\dfrac{1-e^{-2s}}{s}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+9L(y)=\frac{1}{s}-\frac{e^{-2s}}{s}\Rightarrow s^2y(s)+9y(s)=\frac{1-e^{-2s}}{s}$$

$$\Rightarrow (s^2+9)y(s)=\frac{1}{s}-\frac{e^{-2s}}{s}\Rightarrow y(s)=\frac{1-e^{-2s}}{s(s^2+9)}=\frac{1}{s(s^2+9)}-\frac{e^{-2s}}{s(s^2+9)}$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{s(s^2+9)}-\frac{e^{-2s}}{s(s^2+9)}\right)=L^{-1}\left(\frac{1}{s(s^2+9)}\right)-L^{-1}\left(\frac{e^{-2s}}{s(s^2+9)}\right)$$

Now, by partial fraction, $\dfrac{1}{s(s^2+9)}=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+9}\Rightarrow A=\dfrac{1}{9}, B=-\dfrac{1}{9}, C=0$

Thus, $L^{-1}\left(\dfrac{1}{s(s^2+9)}\right)=L^{-1}\left(\dfrac{1/9}{s}\right)-\left(\dfrac{s/9}{s^2+9}\right)=\dfrac{1}{9}-\dfrac{1}{9}\cos 3t$

Therefore, $y(t)=\dfrac{1}{9}-\dfrac{1}{9}\cos 3t-\left(\dfrac{1}{9}-\dfrac{1}{9}\cos 3(t-2)\right)u(t-2)$

c) First observe that using unti step function $f(t)=t-tu(t-\pi)$ and using the t-shifting propety that $L\{tf(t)\}=-\dfrac{d}{ds}[L\{f(t)\}]$ we have

$$L\{f(t)\}=L\{t-tu(t-\pi)\}=L\{t)\}-L\{tu(t-\pi)\}$$

$$=\frac{1}{s^2}+\frac{d}{ds}[L\{u(t-\pi)\}]=\frac{1}{s^2}+\frac{d}{ds}\left(\frac{e^{-\pi s}}{s}\right)=\frac{1}{s^2}-e^{-\pi s}\left(\frac{\pi}{s}+\frac{1}{s^2}\right)$$

Then, $L\{y''+y\} = L\{f(t)\} \Rightarrow L\{y''\} + L\{y\} = L\{f(t)\} = \frac{1}{s^2} - e^{-\pi s}\left(\frac{\pi}{s} + \frac{1}{s^2}\right)$

$$\Rightarrow (s^2+1)y(s) = \frac{1}{s^2} - e^{-\pi s}\left(\frac{\pi}{s} + \frac{1}{s^2}\right)$$

$$\Rightarrow y(s) = \frac{1}{s^2(s^2+1)} - e^{-\pi s}\left(\frac{\pi}{s(s^2+1)} + \frac{1}{s^2(s^2+1)}\right)$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) - L^{-1}\left[e^{-\pi s}\left(\frac{\pi}{s(s^2+1)} + \frac{1}{s^2(s^2+1)}\right)\right]$$

Now, by partial fraction,

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \Rightarrow A = 0, B = 1, C = 0, D = -1$$

$$\frac{\pi}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow A = \pi, B = -\pi, C = 0$$

$$y(t) = L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{1}{s^2+1}\right) - L^{-1}\left[e^{-\pi s}\left(\frac{\pi}{s} - \frac{\pi s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}\right)\right]$$

$$= t - \sin t - (t - \pi\cos(t-\pi)u(t-\pi) - \sin(t-\pi))$$

$$= t - \sin t - (t + \pi\cos t + \sin t)u(t-\pi) = \begin{cases} t - \sin t, 0 < t < \pi \\ -2\sin t - \pi\cos t, t > \pi \end{cases}$$

d) $L\{y'-2y\} = L\left\{\frac{\sin t}{t}\right\} \Rightarrow sy(s) - y(0) - 2y(s) = \frac{\pi}{2} - \tan^{-1} s$

$$\Rightarrow y(s) = \left(\frac{1}{s-2} + \frac{\pi}{2(s-2)} - \frac{\tan^{-1} s}{s-2}\right)$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{s-2}\right) + \frac{\pi}{2}L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{\tan^{-1} s}{s-2}\right)$$

Solving [illegible] DE

An equation that involves an integral and derivative of a function is known as integral (integro) DE. Such equations are also solved easily by using Laplace Transforms of derivatives and integrals at the same time.

Examples: Solve the integral-DEs of IVPs using Laplace Transforms.

$$a)\ y'-\int_0^t y(x)dx = t^2,\ y(0)=1 \qquad b)\ y(t) = t+\int_0^t \sin(t-x)y(x)dx$$

$$c)\ y(t) = \cos t+\int_0^t (t-x)y(x)dx \qquad d)\ \int_0^t e^{t-x}y(x)dx = \sin t$$

Solution: a) Using Laplace Transform of derivatives and integrals,

$$L\{y'-\int_0^t y(x)dx\} = L\{t^2\} \Rightarrow sY(s)-y(0)-\frac{Y(s)}{s}=\frac{2}{s^3}$$

$$\Rightarrow Y(s)[s-\frac{1}{s}]=\frac{2}{s^3}+1 \Rightarrow Y(s)=\frac{s^3+2}{s^2(s^2-1)}$$

Then, $y(t)=L^{-1}\{y(s)\}=L^{-1}\left(\dfrac{s^3+2}{s^2(s^2-1)}\right)$.

But by Partial Fraction $\dfrac{s^3+2}{s^2(s^2-1)}=\dfrac{3/2}{s-1}-\dfrac{1/2}{s+1}-\dfrac{2}{s^2}$. Therefore,

$$y(t)=L^{-1}\left(\frac{s^3+2}{s^2(s^2-1)}\right)=L^{-1}\left(\frac{3/2}{s-1}\right)-L^{-1}\left(\frac{1/2}{s+1}\right)-2L^{-1}\left(\frac{2}{s^2}\right)$$

$$=\frac{3}{2}e^t-\frac{1}{2}e^{-t}-2t$$

b) Using the definition of convolution the given equation is transformed into $y(t)=t+\sin t * y(t)$. Now, by taking the Laplace transform both sides, we get

$$y(s)=\frac{1}{s^2}+\frac{y(s)}{1+s^2} \Rightarrow y(s)(1-\frac{1}{1+s^2})=\frac{1}{s^2} \Rightarrow y(s)=\frac{1+s^2}{s^4}=\frac{1}{s^4}+\frac{1}{s^2}.$$

Finally, taking inverse Laplace transform gives us

$$y(t)=L^{-1}\{y(s)\}=L^{-1}\left\{\frac{1}{s^4}+\frac{1}{s^2}\right\}=L^{-1}\left\{\frac{1}{s^4}\right\}+L^{-1}\left\{\frac{1}{s^2}\right\}=\frac{t^3}{6}+t$$

CHAPTER-3

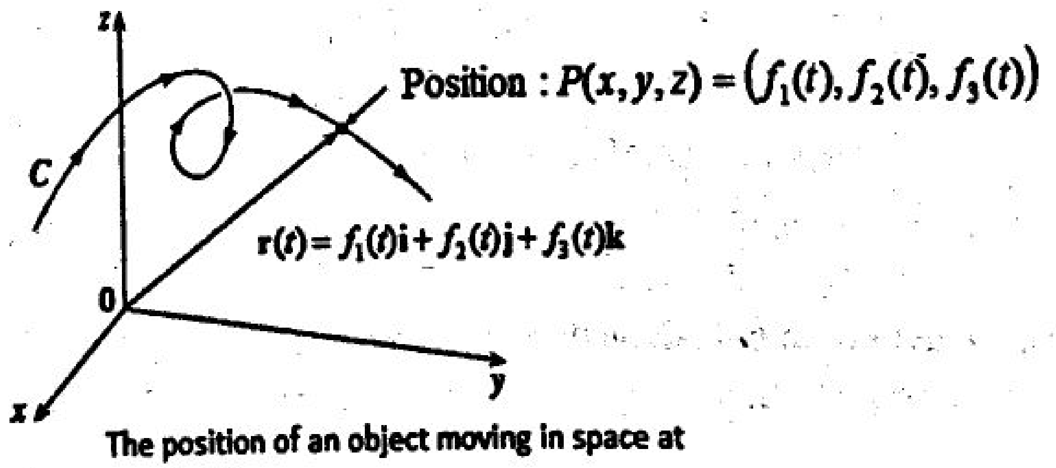
CALCULUS OF VECTOR VALUED FUNCTIONS

3.1 Definition and Examples of VVF

A function $F: R \to R^3$ of the form $F(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}, t \in R$ is called Vector Valued Function. Here each f_1, f_2, f_3 are called component functions.

Why we need Vector Valued Functions (VVF)?

Think about an object traveling in space along a curve C, particularly, in R^3. Then, its x, y, z coordinates are changing through time as it changes position. So, if we denote its coordinates as a function of time to be $x = f_1(t), y = f_2(t), z = f_3(t)$, then its position will be given by $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ at any time t as shown in the diagram below.



The position of an object moving in space at any time t.

In vector notation, the position vector of the object from the origin to its position is written as $\mathbf{r}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$. But this is really what we mean by a vector valued function. This means that to describe the motion of an object in space, to know its velocity, speed, acceleration, distance moved and other motion related concepts we need the concepts of vector valued functions.

3.2 Limit and Continuity of Vector Valued Functions

3.2.1 Limit of Vector-Valued Functions

Let $F: R \to R^3$ defined by $F(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}, t \in R$.

Then, F has a limit given by $\lim_{t \to t_0} F(t) = L = ai + bj + ck$ at t_0 if and only if the coordinate functions f_1, f_2, f_3 have limits at t_0.

That means

$$\lim_{t \to t_0} F(t) = ai + bj + ck \Leftrightarrow \lim_{t \to t_0} f_1(t) = a, \lim_{t \to t_0} f_2(t) = b, \lim_{t \to t_0} f_3(t) = c$$

$$i.e \lim_{t \to t_0} F(t) = [\lim_{t \to t_0} f_1(t)]\mathbf{i} + [\lim_{t \to t_0} f_2(t)]\mathbf{j} + [\lim_{t \to t_0} f_3(t)]\mathbf{k}$$

Examples: Evaluate $\lim_{t \to t_0} F(t)$ where

$$a)\ F(t) = \frac{3t}{\sqrt{t-1}} i + \frac{3}{t-1} j - \sqrt{5-t^2} k \text{ at } t_0 = 2$$

$$b)\ F(t) = \ln|t-1| i + \frac{\sin 6t}{3t} j + e^t k \text{ at } t_0 = 1$$

$$c)\ F(t) = \begin{cases} (2t+1)i + (2t-1)j + 8k, t < 3 \\ (10-t)i + (8-t)j + (t^2-1)k, \ t \geq 3 \end{cases} \text{ at } t_0 = 3$$

Solution:

$$a)\ \lim_{t \to 2} \frac{3t}{\sqrt{t-1}} = 6,\ \lim_{t \to 2} \frac{3}{t-1} = 3,\ \lim_{t \to 2} \sqrt{5-t^2} = 1 \Rightarrow \lim_{t \to 2} F(t) = 6i + 3j - k$$

b) $\lim_{t \to 1} \ln|t-1|$ does not exist. Hence, $\lim_{t \to 1} \mathrm{F(t)}$ does not exist

$$c)\ \lim_{t \to 3^-}(2t+1) = 7,\ \lim_{t \to 3^-}(2t-1) = 5,\ \lim_{t \to 3^-} 8 = 8 \Rightarrow \lim_{t \to 3^-} F(t) = 7i + 5j + 8k$$

$$\lim_{t \to 3^+}(10-t) = 7,\ \lim_{t \to 3^+}(8-t) = 5,\ \lim_{t \to 3^+}(t^2-1) = 8 \Rightarrow \lim_{t \to 3^+} F(t) = 7i + 5j + 8k$$

Here, $\lim_{t \to 3^-} F(t) = \lim_{t \to 3^+} F(t)$. Hence, $\lim_{t \to 3} \mathrm{F(t)} = 7i + 5j + 8k$.

CHAPTER-3

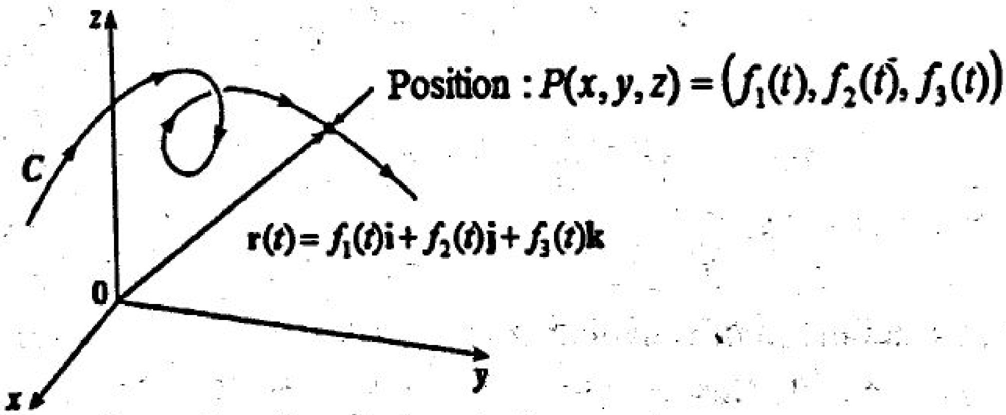
CALCULUS OF VECTOR VALUED FUNCTIONS

3.1 Definition and Examples of VVF

A function $F: R \to R^3$ of the form $F(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}, t \in R$ is called Vector Valued Function. Here each f_1, f_2, f_3 are called component functions.

Why we need Vector Valued Functions (VVF)?

Think about an object traveling in space along a curve C , particularly, in R^3. Then, its x, y, z coordinates are changing through time as it changes position. So, if we denote its coordinates as a function of time to be $x = f_1(t), y = f_2(t), z = f_3(t)$, then its position will be given by $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ at any time t as shown in the diagram below.



The position of an object moving in space at any time t.

In vector notation, the position vector of the object from the origin to its position is written as $\mathbf{r}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$. But this is really what we mean by a vector valued function. This means that to describe the motion of an object in space, to know its velocity, speed, acceleration, distance moved and other motion related concepts we need the concepts of vector valued functions.

Basic Properties of Limits of VVF:

Suppose F and G are vector valued functions whose limit exits. Then,

$$a)\ \lim_{t\to a}[F(t)+G(t)]=\lim_{t\to a}F(t)+\lim_{t\to a}G(t)$$

$$c)\ \lim_{t\to a}[F(t).G(t)]=\lim_{t\to a}F(t).\lim_{t\to a}G(t)$$

$$c)\ \lim_{t\to a}[F(t)\times G(t)]=\lim_{t\to a}F(t)\times\lim_{t\to a}G(t)$$

Notice that since the functions are vector valued, the operation in $F(t).G(t)$ is the usual dot product of vectors and the operation in $F(t)\times G(t)$ is the usual cross product of vectors.

Examples:

Let $F(t)=t^2i+(2t-1)j+(\frac{\ln t}{t-1})k, G(t)=t^4i+(\frac{t^2-1}{t-1})j+(3t^4-t^2+1)k$.

Evaluate

i) $\lim_{t\to 1}[F(t)+G(t)]$ ii) $\lim_{t\to 1}[F(t).G(t)]$

iii) $\lim_{t\to 1}[F(t)\times G(t)]$ iv) $\lim_{t\to 1}[3G(t)]$

Solution: Here, $\lim_{t\to 1}F(t)=i+j+k, \lim_{t\to 1}G(t)=i+2j+3k$.

Then, by the above properties,

$i)\ \lim_{t\to 1}[F(t)+G(t)]=\lim_{t\to 1}F(t)+\lim_{t\to 1}G(t)=2i+3j+4k$

$ii)\ \lim_{t\to 1}[F(t).G(t)]=\lim_{t\to 1}F(t).\lim_{t\to 1}G(t)=6$

$$iii)\ \lim_{t\to 1}[F(t)\times G(t)]=\lim_{t\to 1}F(t)\times\lim_{t\to 1}G(t)=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}=i-2j+k$$

$iv)\ \lim_{t\to 1}[3G(t)]=3i+6j+9k$

3.2.2 Continuity of Vector-Valued Functions

A vector valued function F is said to be continuous at a if and only if $\lim_{t\to a} F(t) = F(a) = f_1(a)\mathbf{i} + f_2(a)\mathbf{j} + f_3(a)\mathbf{k}$.

This means, a vector valued function is continuous if and only if all of its component functions are continuous.

Examples:

1. Check the continuity of the following VVFs at the indicted point.

a) $F(t) = (\cos t)i + (\sin t)j + (\tan t)k$ at $t = \dfrac{\pi}{3}$ and $t = \dfrac{\pi}{2}$

b) $F(t) = \begin{cases} 2t\mathbf{i} + \sqrt{t+7}\,\mathbf{j}, & t < 2 \\ (5t-6)\mathbf{i} + (t^2+1)\mathbf{j}, & t \geq 2 \end{cases}$ at $t = 2$

Solution:

a) $\lim_{t\to\frac{\pi}{3}} F(t) = \dfrac{1}{2}i + \dfrac{\sqrt{3}}{2}j + \sqrt{3}k = F(\frac{\pi}{3})$. Hence, F is continuous at $t = \dfrac{\pi}{3}$.

But at $t = \dfrac{\pi}{2}$, $f_3(t) = \tan t$ is not continuous and thus F is not continuous.

b) $\lim_{t\to 2^-} 2t = 4$, $\lim_{t\to 2^-} \sqrt{t+7} = 3 \Rightarrow \lim_{t\to 2^-} F(t) = 4i + 3j$

$\lim_{t\to 2^+} (5t-6) = 4$, $\lim_{t\to 2^+} (t^2+1) = 5 \Rightarrow \lim_{t\to 2^+} F(t) = 4i + 5j$

Here, $\lim_{t\to 2^-} F(t) \neq \lim_{t\to 2^+} F(t)$. Hence, $\lim_{t\to 2}$ F(t) does not exist

Therefore, the function is not continuous at $t = 2$.

2. If $F(t) = \begin{cases} \sqrt{t+1}\,i + (2a^2+t)j + 2tk, & t < 3 \\ bt^2 i + (4t-1)j + (t^2-t)k, & t \geq 3 \end{cases}$ is continuous at $t = 3$, find a and b.

Solution: To be continuous at $t = 3$, we have $\lim_{t\to 3^-} F(t) = \lim_{t\to 3^+} F(t) = F(3)$.

Therefore, $\lim_{t\to 3^-}(\sqrt{t+1}) = \lim_{t\to 3^+} bt^2 \Rightarrow 9b = 2 \Rightarrow b = \dfrac{2}{9}$,

$\lim_{t\to 3^-}(2a^2+t) = \lim_{t\to 3^+}(4t-1) \Rightarrow 2a^2 + 3 = 11 \Rightarrow a = \pm 2$

3.3 Derivatives and Integrals of Vector Valued Functions

3.3.1 Derivatives of Vector Valued Functions

Let $F(t)=f_1(t)\mathbf{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}, t\in R$ be any vector valued function.

Then, the derivative of F denoted by $F'(t)$ or $D_t[F(t)]$, is defined by

$$D_t[F(t)]=F'(t)=\lim_{h\to 0}\frac{F(t+h)-F(t)}{h}=f_1'(t)\mathbf{i}+f_2'(t)\mathbf{j}+f_3'(t)\mathbf{k}.$$

Examples: Find $F'(t)$ and compute $F'(2)$ where

a) $F(t)=(3t^3+5)\mathbf{i}-5t\mathbf{j}+(t^2+t)\mathbf{k}$ b) $F(t)=3t\mathbf{i}+|t-2|\mathbf{j}+t^2\mathbf{k}$

Solution:

a) $F'(t)=9t^2\mathbf{i}-5\mathbf{j}+(2t+1)\mathbf{k}$, $F'(2)=36\mathbf{i}-5\mathbf{j}+5\mathbf{k}$.

b) Since $f_2(t)=|t-2|$ is not differentiable at $t=2$, F itself is not differentiable.

Properties: Suppose F and G are vector valued functions whose derivative exits and let f be differentiable scalar function. Then,

i) Dot - Product Rule : $D_t[F(t).G(t)]=D_t[F(t)].G(t)+F(t).D_t[G(t)]$

ii) Cross - Product Rule : $D_t[F(t)\times G(t)]=D_t[F(t)]\times G(t)+F(t)\times D_t[G(t)]$

iii) Chain - Rule : $D_t[F(f(t))]=f'(t)F'(f(t))$

Examples: If $f(t)=F(t).G(t), H(t)=F(t)\times G(t)$, find $f'(1)$, $H'(1)$. Given that

$F(t)=(2t+1)i+j+(t^5-4)k, F'(1)=3i+5k$, $G(t)=ti+t^2j-t^3k$

Solution:

i) Here, $F(t)=(2t+1)i+j+(t^5-4)k\Rightarrow F(1)=3i+j-3k$, $F'(1)=2i+5k$ and

$G(t)=ti+t^2j-t^3k\Rightarrow G(1)=i+j-k$, $G'(1)=i+2j-3k$

$f'(t)=F'(t).G(t)+F(t).G'(t)\Rightarrow f'(1)=F'(1).G(1)+F(1).G'(1)$

$$\Rightarrow f'(1)=(2i+5k).(i+j-k)+(3i+j-3k).(i+2j-3k)=11$$

ii) $H'(1)=F'(1)\times G(1)+F(1)\times G'(1)=\begin{vmatrix} i & j & k \\ 3 & 0 & 5 \\ 1 & 1 & -1 \end{vmatrix}+\begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 2 & -3 \end{vmatrix}=-2i+11j+6k$

3.3.2 Integration of Vector Valued Functions

Let $F(t)=f_1(t)\mathbf{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}$ where f_1, f_2, f_3 are continuous on $[a,b]$.

Indefinite integral: $\int F(t)dt=\int f_1(t)dt.\mathbf{i}+\int f_2(t)dt.\mathbf{j}+\int f_3(t)dt.\mathbf{k}$.

Definite integral: $\int\limits_a^b F(t)dt=\left(\int f_1(t)dt.\mathbf{i}+\int f_2(t)dt.\mathbf{j}+\int f_3(t)dt\,\mathbf{k}\right)\Big|_{t=a}^{t=b}$

Examples:

1. Let $F(t)=10t^4\mathbf{i}+2t\mathbf{j}+(3t^2-4t)\mathbf{k}$. Find $\int F(t)dt$ and compute $\int\limits_1^2 F(t)dt$.

Solution:

i) $\int F(t)dt=\int f_1(t)dt.\mathbf{i}+\int f_2(t)dt.\mathbf{j}+\int f_3(t)dt\,\mathbf{k}$

$$=\int 10t^4dt\,\mathbf{i}+\int 2tdt.\mathbf{j}+\int(3t^2-4t)dt.\mathbf{k}$$

$$=2t^5\mathbf{i}+t^2\mathbf{j}+(t^3-2t^2)\mathbf{k}$$

ii) $\int\limits_1^2 F(t)dt=\left(\int f_1(t)dt.\mathbf{i}+\int f_2(t)dt.\mathbf{j}+\int f_3(t)dt\,\mathbf{k}\right)\Big|_{t=1}^{t=2}$.

$$=\left(\int 10t^4dt\,\mathbf{i}+\int 2tdt.\mathbf{j}+\int(3t^2-4t)dt.\mathbf{k}\right)\Big|_{t=1}^{t=2}$$

$$=(2t^5\mathbf{i}+t^2\mathbf{j}+(t^3-2t^2)\mathbf{k})\Big|_{t=1}^{t=2}=62\mathbf{i}+3\mathbf{j}+\mathbf{k}$$

2. Let $F'(t)=2t\mathbf{i}+5\mathbf{j}+3t^2\mathbf{k}$, $F(1)=2\mathbf{i}+5\mathbf{j}+3\mathbf{k}$. Find $F(t)$.

Solution:

$F(t)=\int F'(t)dt=(\int 2tdt)\mathbf{i}+(\int 5dt)\mathbf{j}+(\int 3t^2dt)\mathbf{k}$

$$=t^2\mathbf{i}+5t\mathbf{j}+t^3\mathbf{k}+\mathbf{C},\ (\mathbf{C}=c_1i+c_2j+c_3k)$$

But $F(1)=2\mathbf{i}+5\mathbf{j}+3\mathbf{k}\Rightarrow\mathbf{i}+5\mathbf{j}+\mathbf{k}+\mathbf{C}=2\mathbf{i}+5\mathbf{j}+3\mathbf{k}\Rightarrow\mathbf{C}=\mathbf{i}+2\mathbf{k}$.

Thus, $F(t)=t^2\mathbf{i}+5t\mathbf{j}+t^3\mathbf{k}+\mathbf{C}=(t^2+1)\mathbf{i}+5t\mathbf{j}+(t^3+2)\mathbf{k}$

3.4 Space Curves and Length of Space Curves

Curves: The graph of any continuous vector valued function $\mathbf{r}(t)$ on a given interval $[a,b]$ is said to be a curve.

For instance, line segments, circles, rectangles, ellipses are curves.

From now onwards, we use the notation C to denote a curve and $\mathbf{r}(t)$ to denote a vector valued function whose graph is the curve C. In such cases, we say that $\mathbf{r}(t)$ is the parameterization of C on $[a,b]$. We write as $C:\mathbf{r}(t), a \le t \le b$.

i) **Closed curve:** A curve C is said to be closed curve if it has a parameterization $\mathbf{r}(t)$ on the closed interval $[a,b]$ such that $\mathbf{r}(a)=\mathbf{r}(b)$. In other words, a curve C is said to be closed if its initial and terminal points concides.

Examples:

a) The curve parameterized by $C:\mathbf{r}(t)=\cos(2\pi t)i+\sin(2\pi t)j,\ 0\le t\le 1$ is closed because $\mathbf{r}(0)=\mathbf{r}(1)$.

b) The curve $C:\mathbf{r}(t)=ti+t^2j, -1\le t\le 1$ is not closed because $\mathbf{r}(-1)\ne \mathbf{r}(1)$.

ii) **Smooth curve**: A vector valued function $\mathbf{r}(t)$ is said to be smooth on a given interval I if $\mathbf{r}'$ is continuous on I and $\mathbf{r}'(t)\ne 0$ for all points in I (with the possible exception at the end points of I).

Furthermore, $\mathbf{r}(t)$ is said to be **piecewise smooth** on the interval I if I is expressible as the union of finite number of sub intervals such that $\mathbf{r}(t)$ is smooth on each of the sub intervals and the one sided derivatives $\mathbf{r}_+'(t), \mathbf{r}_-'(t)$ exists at each interior points of I.

A curve C is said to be *smooth* if its parameterization $\mathbf{r}(t)$ is smooth and it is said to be piecewise smooth if its parameterization $\mathbf{r}(t)$ is piecewise smooth.

Examples:

a) Consider the curve parameterized by $C:\mathbf{r}(t)=t^2i+2tj+t^3k$.

Here, $\mathbf{r}'(t)=2ti+2j+3t^2k$ is continuous. Besides, $\mathbf{r}'(t)\ne 0$ for any t because the $\mathbf{j}$−component of $\mathbf{r}'(t)$ can never be zero. Hence, the curve C is smooth.

b) Consider the curve parameterized by $C:\mathbf{r}(t)=t^2i+(e^{-t}+t)j+t^3k$.
Here, $\mathbf{r}'(t)=2ti+(1-e^{-t})j+3t^2k$ is continuous but, $\mathbf{r}'(t)=0$ at $t=0$ and thus $\mathbf{r}(t)$ is not smooth. Thus, the curve C itself is not smooth.

c) Verify that the curve parameterized by
$C:\mathbf{r}(t)=3t^2i+(e^{2t}-2t)j+[(t+1)\ln(t+1)-t]k$ on $0\le t\le 1$ is not smooth.
Here, $\mathbf{r}'(t)=6ti+(2e^{2t}-2)j+\ln(t+1)k$ but, $\mathbf{r}'(t)=0$ at $t=0$ and thus $\mathbf{r}(t)$ is not smooth. Hence, the curve C itself is not smooth.

d) Consider the curve $C:\mathbf{r}(t)=2(1-\cos t)i+2(t-\sin t)j, I=[-2\pi,2\pi]$. Here, $\mathbf{r}'(t)=2\sin ti+2(1-\cos t)j$ is continuous yet $\mathbf{r}(t)$ is not smooth because $\mathbf{r}'(-2\pi)=\mathbf{r}'(0)=\mathbf{r}'(2\pi)=0$ and thus C itself is not smooth. But if we break the interval as $I=[-2\pi,2\pi]=[-2\pi,0]\cup[0,2\pi]$, we get $\mathbf{r}'(t)\neq 0$ for any $-2\pi<t<0,\ 0<t<2\pi$. This means $\mathbf{r}(t)$ is smooth on $[-2\pi,0]$ and $[0,2\pi]$. So, $\mathbf{r}(t)$ is piecewise smooth on $[-2\pi,2\pi]$. Therefore, C is piecewise smooth.

e) Consider the curve parameterized by $C:\mathbf{r}(t)=t^{\frac{2}{3}}i+tj+t^2k$.

Here, $\mathbf{r}'(t)=\dfrac{2}{3t^{\frac{1}{3}}}i+j+2tk$ is not continuous at $t=0$ and thus $\mathbf{r}(t)$ is not smooth. Thus, the curve C itself is not smooth.

Common Parameterizations: If you remember, we have seen in *Applied I*, how to give the parametric equations of a line in space. Likewise, it is also possible to express different curves using parameter. Here, under let's see parametric equations of important figures which we will use frequently.

i) Parametric equations of a line segment:

Suppose L is a line segment connecting the points $A(x_0,y_0,z_0)$ and $B(x_1,y_1,z_1)$. Then, its parametric equation is given by

$$L:X=A+t\overrightarrow{AB}\Rightarrow L:\begin{cases}x=x_0+at\\ y=y_0+bt\\ z=z_0+ct\end{cases}\text{ for } 0\le t\le 1 \text{ where } ai+bj+ck=\overrightarrow{AB}$$

In short, the parametrization is $\mathbf{r}(t)=A+t\overrightarrow{AB}=(1-t)A+tB$.

Example: Find the parametric equations of a line segment connecting the points $A(2,-1,1)$ and $B(1,3,-2)$.

Solution: Here, $ai+bj+ck=\overrightarrow{AB}=B-A=-i+4j-3k$.

Hence, using $(x_0,y_0,z_0)=(2,-1,1)$, we have $L:\begin{cases} x=2-t \\ y=-1+4t \\ z=1-3t \end{cases}$ for $0\le t\le 1$

ii) Parametric equations of a circle:

Suppose C is a circle given by $C: x^2+y^2=r^2$. Then, its parametric equation is given by $C: x=r\cos t,\ y=r\sin t,$ for $0\le t\le 2\pi$.

Similarly, if the center is not the origin like $C:(x-a)^2+(y-b)^2=r^2$,

$$C:\begin{cases} x=a+r\cos t, \\ y=b+r\sin t, \end{cases} \text{ for } 0\le t\le 2\pi$$

Example: Find the parametric equations of a circle given by

a) $C: x^2+y^2=4$ b) $C:(x-2)^2+(y+3)^2=16$.

Solution:

a) Here, $r=2$. Hence, $C: x=2\cos t,\ y=2\sin t,$ for $0\le t\le 2\pi$

b) Here, $r=4$ and the center is $(2,-3)$.

Hence, $C: x=2+4\cos t,\ y=-3+4\sin t,$ for $0\le t\le 2\pi$

iii) Parametric equations of an Ellipse:

Suppose E is an ellipse given by $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, its parametric equation is given by $E:\begin{cases} x=a\cos t, \\ y=b\sin t, \end{cases}$ for $0\le t\le 2\pi$.

Example: Find the parametrization of the ellipse $E: 9x^2+4y^2=36$.

Solution: Here, first write in standard form as $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and identify a and b. That is $E: 9x^2+4y^2=36 \Rightarrow E:\frac{x^2}{4}+\frac{y^2}{9}=1 \Rightarrow a=2, b=3$

Hence, $E: x=2\cos t,\ y=3\sin t,$ for $0\le t\le 2\pi$.

iv) Parametric equations of functions:

Suppose $C: y = g(x),\ a \le x \le b$. Then, its parametric equation is given by $x = t, y = g(t),\ a \le t \le b \Rightarrow C: r(t) = ti + g(t)j,\ \ a \le t \le b$.

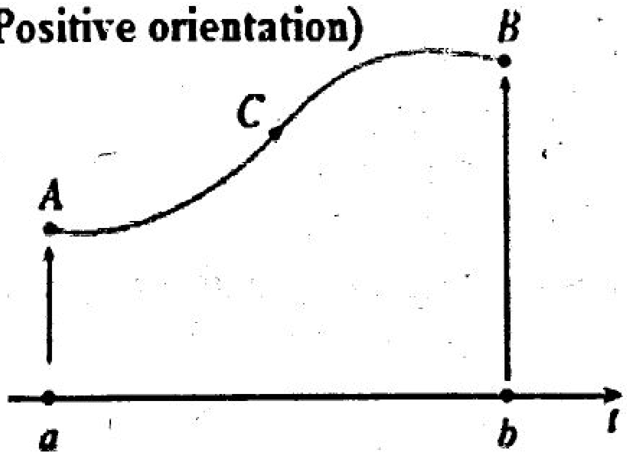
Example: Find the parametrization of $C: y = x^2 - 2x$ from $(0,0)$ to $(2,8)$.

Solution: Here, $x = t, y = t^2 - 2t, 0 \le t \le 2 \Rightarrow C: r(t) = ti + (t^2 - 2t)j, 0 \le t \le 2$

Oriented curves and types of orientations:

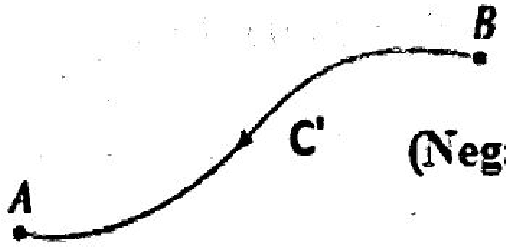
Suppose C is a curve parameterized by $C: x = f(t), y = g(t),\ a \le t \le b$ as shown in the diagram below.

(Positive orientation)



Types of orientations:

- **Positive oprientation**
- **Negative orientation**



(Negative orientation)

If we move from point A to point B along the curve C (in the direction of the arrow), the value of the parameter t is increasing from a to b. Such type of direction in which we move or trace out the curve in the increasing order of the parameter values is said to be *positive orientation.* On the other hand, if we move from point B to point A, the value of the parameter t is decreasing from b to a as shown in the lower diagram. Such type of direction in which we move or trace out the curve in the decreasing order of the parameter values is said to be *negative orientation.* A curve C with either types of orientations (positive or negative) is said to be *oriented curve.*

iv) Parametric equations of functions:

Suppose $C: y = g(x),\ a \le x \le b$. Then, its parametric equation is given by $x = t, y = g(t),\ a \le t \le b \Rightarrow C: r(t) = ti + g(t)j,\ \ a \le t \le b$.

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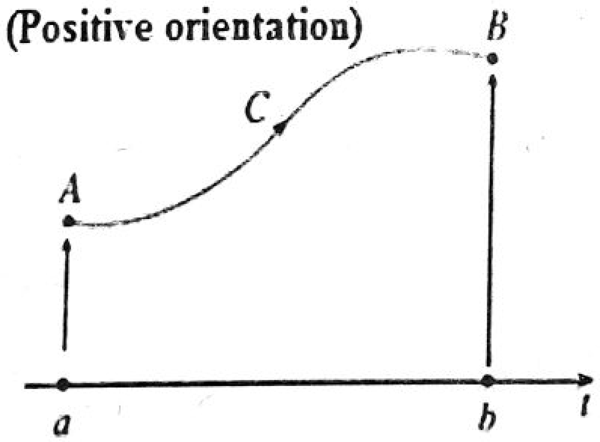

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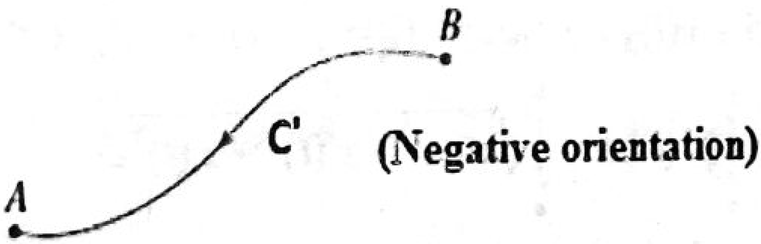
- **Negative orientation**



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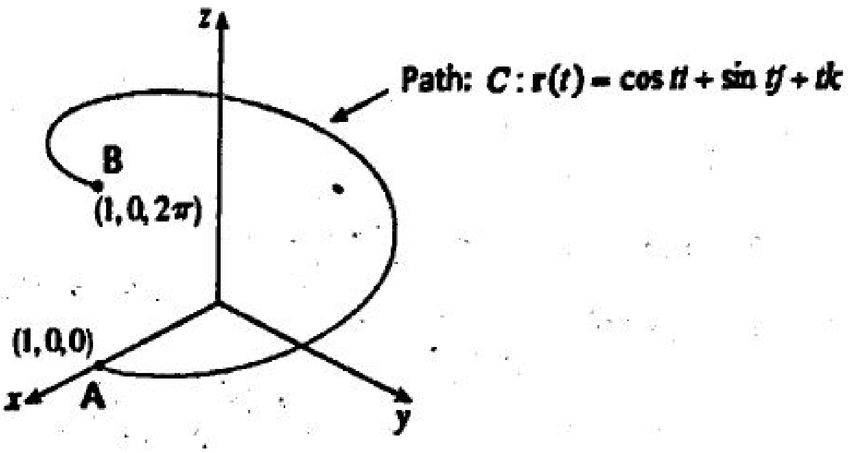
Lengths of Space curves:

Just think How to find the total length of the curve below?

Suppose a particle moves from A to B along a circular helix $C:\mathbf{r}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t\mathbf{k}$ for $0\le t\le 2\pi$ as shown in the diagram below.



What idea do you have to compute this length?

Arc Length: Let C be a curve having a piecewise smooth parameterization $\mathbf{r}(t)$ on $[a,b]$. Then the length L of C is defined by $L=\int_a^b\|\mathbf{r}'(t)\|dt$.

Particularly, if $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$, $\|\mathbf{r}'(t)\|=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}$, the arc length is given by $L=\int_a^b\|\mathbf{r}'(t)\|dt=\int_a^b\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$.

Examples: Find the arc length L of a curve C on the given interval.

a) $\mathbf{r}(t)=t\mathbf{i}+\frac{\sqrt{6}}{2}t^2\mathbf{j}+t^3\mathbf{k};-1\le t\le 1$

b) $\mathbf{r}(t)=\cos 4t\mathbf{i}+\sin 4t\mathbf{j}-3t\mathbf{k};0\le t\le \pi$

c) $\mathbf{r}(t)=\frac{1}{3}(1+t)^{3/2}\mathbf{i}+\frac{1}{3}(1-t)^{3/2}\mathbf{j}+\frac{1}{2}t\mathbf{k};-1\le t\le 1$

d) $\mathbf{r}(t)=e^t\mathbf{i}+e^{-t}\mathbf{j}+\sqrt{2}t\mathbf{k};0\le t\le 1$

e) $C:\mathbf{r}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t^{\frac{3}{2}}\mathbf{k};0\le t\le \frac{20}{3}$

f) $\mathbf{r}(t)=4t\mathbf{i}+\sqrt{2}t^2\mathbf{j}+\frac{t^3}{3}\mathbf{k};0\le t\le 3$

Solution:

a) $\mathbf{r}'(t)=\mathbf{i}+\sqrt{6}t\mathbf{j}+3t^2\mathbf{k}$

Then, $L=\int\limits_{-1}^{1}\|\mathbf{r}'(t)\|dt=\int\limits_{-1}^{1}\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$

$$=\int\limits_{-1}^{1}\sqrt{1+6t^2+9t^4}\,dt=\int\limits_{-1}^{1}\sqrt{(1+3t^2)^2}\,dt=\int\limits_{-1}^{1}(1+3t^2)dt=t+t^3\Big|_{t=-1}^{t=1}=4$$

b) $\mathbf{r}'(t)=-4\sin 4t\mathbf{i}+4\cos 4t\mathbf{j}-3\mathbf{k}$

Then, $L=\int\limits_{0}^{\pi}\|\mathbf{r}'(t)\|dt=\int\limits_{0}^{\pi}\sqrt{16\cos^2 4t+16\sin^2 4t+9}\,dt$

$$=\int\limits_{0}^{\pi}\sqrt{16\underbrace{(\cos^2 4t+\sin^2 4t)}_{=1}+9}\,dt=\int\limits_{0}^{\pi}\sqrt{25}\,dt=\int\limits_{0}^{\pi}5dt=5\pi$$

c) $\mathbf{r}'(t)=\dfrac{1}{2}\sqrt{1+t}\mathbf{i}-\dfrac{1}{2}\sqrt{1-t}\mathbf{j}+\dfrac{1}{2}\mathbf{k}$

Then, $L=\int\limits_{-1}^{1}\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt=\int\limits_{-1}^{1}\sqrt{\dfrac{1}{4}+\dfrac{1}{4}+\dfrac{1}{4}}\,dt=\int\limits_{-1}^{1}\sqrt{\dfrac{3}{4}}\,dt=\sqrt{3}$

d) $L=\int\limits_{0}^{1}\sqrt{e^{2t}+e^{-2t}+2}\,dt=\int\limits_{0}^{1}\sqrt{(e^t+e^{-t})^2}\,dt=\int\limits_{0}^{1}(e^t+e^{-t})dt=e-\dfrac{1}{e}$

e) $L=\int\limits_{0}^{\frac{20}{3}}\sqrt{\underbrace{\cos^2 t+\sin^2 t}_{=1}+\dfrac{9}{4}t}\,dt=\int\limits_{0}^{\frac{20}{3}}\sqrt{\dfrac{9}{4}t+1}\,dt=\dfrac{8}{27}\left(\dfrac{9}{4}t+1\right)^{\frac{3}{2}}\Bigg|_{t=0}^{t=\frac{20}{3}}=\dfrac{8}{9}$

f) $\mathbf{r}'(t)=4\mathbf{i}+2\sqrt{2}t\mathbf{j}+t^2\mathbf{k}$

Then, $L=\int\limits_{0}^{3}\|\mathbf{r}'(t)\|dt=\int\limits_{0}^{3}\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt=\int\limits_{0}^{3}\sqrt{16+8t^2+t^4}\,dt$

$$=\int\limits_{0}^{3}\sqrt{(4+t^2)^2}\,dt=\int\limits_{0}^{3}(4+t^2)dt=4t+\frac{t^3}{3}\Bigg|_{t=0}^{t=3}=21$$

3.5 Tangent and Normal Vectors and Curvatures

Let C be a smooth curve with parameterization $\mathbf{r}(t)$ on an open interval I.

Then, we have

i) Unit Tangent Vector: For any point t in I where $\mathbf{r}'(t) \neq 0$, the *unit tangent vector* $T(t)$ to the curve C is defined by $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$.

ii) Unit Normal Vector: If $\mathbf{r}'(t)$ is smooth and $T'(t) \neq 0$, the unit *normal vector* $N(t)$ to the curve C is defined by $N(t) = \dfrac{T'(t)}{\|T'(t)\|}$.

iii) Curvature and radius of Curvature:

The curvature κ of C is defined by $\kappa(t) = \dfrac{\|T'(t)\|}{\|\mathbf{r}'(t)\|}$. Furthermore, the radius of curvature, ρ at a given point t is given by $\rho(t) = \dfrac{1}{\kappa(t)}$.

Examples: Find the unit tangent, unit normal, curvature and give the radius of curvature at the point $t = t_0$ for the curves whose parameterization is given.

a) $\mathbf{r}(t) = \cos 3t\mathbf{i} - 4t\mathbf{j} + \sin 3t\mathbf{k}; t_0 = \pi$ *b)* $\mathbf{r}(t) = \sin e^t\mathbf{i} + \cos e^t\mathbf{j} - 5\mathbf{k}; t_0 = 0$

c) $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}; t_0 = 0$ *d)* $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}; t_0 = 1$

e) $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}; t_0 = 0$ *f)* $\mathbf{r}(t) = (t^2 + 4)\mathbf{i} + 2t\mathbf{j}; t_0 = 1$

Solution:

a) $\mathbf{r}'(t) = -3\sin 3t\mathbf{i} - 4\mathbf{j} + 3\cos 3t\mathbf{k}; \|r'(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16} = 5$

Hence, $T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{-3\sin 3t}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j} + \dfrac{3\cos 3t}{5}\mathbf{k}$.

Again, $T'(t) = \dfrac{-9\cos 3t}{5}\mathbf{i} - \dfrac{9\sin 3t}{5}\mathbf{k}; \|T'(t)\| = \dfrac{9}{5}$

So, $N(t) = \dfrac{T'(t)}{\|T'(t)\|} = -\cos 3t\mathbf{i} - \sin 3t\mathbf{k}; \kappa(t) = \dfrac{\|T'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{9}{25}; \rho(\pi) = \dfrac{25}{9}$

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c) $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}; t_0 = 0$ *d)* $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}; t_0 = 1$

e) $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}; t_0 = 0$ *f)* $\mathbf{r}(t) = (t^2 + 4)\mathbf{i} + 2t\mathbf{j}; t_0 = 1$

Solution:

a) $\mathbf{r}'(t) = -3\sin 3t\mathbf{i} - 4\mathbf{j} + 3\cos 3t\mathbf{k}; \|r'(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16} = 5$

Hence, $T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{-3\sin 3t}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j} + \dfrac{3\cos 3t}{5}\mathbf{k}.$

Again, $T'(t) = \dfrac{-9\cos 3t}{5}\mathbf{i} - \dfrac{9\sin 3t}{5}\mathbf{k}; \|T'(t)\| = \dfrac{9}{5}$

So, $N(t) = \dfrac{T'(t)}{\|T'(t)\|} = -\cos 3t\mathbf{i} - \sin 3t\mathbf{k}; \kappa(t) = \dfrac{\|T'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{9}{25}; \rho(\pi) = \dfrac{25}{9}$

$b)$ $\mathbf{r}'(t)=\mathbf{r}(t)=e^t\cos e^t\mathbf{i}-e^t\sin e^t\mathbf{j};\|r'(t)\|=\sqrt{e^{2t}(\cos^2 e^t+\sin^2 e^t)}=\sqrt{e^{2t}}=e^t$

Hence, $T(t)=\dfrac{r'(t)}{\|r'(t)\|}=\dfrac{e^t\cos e^t\mathbf{i}-e^t\sin e^t\mathbf{j}}{e^t}=\cos e^t\mathbf{i}-\sin e^t\mathbf{j}$

Here, $T'(t)=-e^t\sin e^t\mathbf{i}-e^t\sin e^t\mathbf{j};\|T'(t)\|=\sqrt{e^{2t}(\cos^2 e^t+\sin^2 e^t)}=e^t$

So, $N(t)=\dfrac{T'(t)}{\|T'(t)\|}=\dfrac{-e^t\sin e^t\mathbf{i}-e^t\sin e^t\mathbf{j}}{e^t}=-\sin e^t\mathbf{i}-\sin e^t\mathbf{j}$

$\kappa(t)=\dfrac{\|T'(t)\|}{\|\mathbf{r}'(t)\|}=\dfrac{e^t}{e^t}=1;\rho(0)=\dfrac{1}{\kappa(1)}=1.$

$c)$ $\mathbf{r}'(t)=2\mathbf{i}+2t\mathbf{j}+t^2\mathbf{k}\,;\|r'(t)\|=\sqrt{4+2t^2+t^4}=\sqrt{(2+t^2)^2}=2+t^2$

Hence, $T(t)=\dfrac{r'(t)}{\|r'(t)\|}=\dfrac{2}{2+t^2}\mathbf{i}+\dfrac{2t}{2+t^2}\mathbf{j}+\dfrac{t^2}{2+t^2}\mathbf{k}$

Here, $T'(t)=\dfrac{-4t}{(2+t^2)^2}\mathbf{i}+\dfrac{4-2t^2}{(2+t^2)^2}\mathbf{j}+\dfrac{4t}{(2+t^2)^2}\mathbf{k};\|T'(t)\|=\dfrac{2}{2+t^2}$

So, $N(t)=\dfrac{T'(t)}{\|T'(t)\|}=\dfrac{-2t}{2+t^2}\mathbf{i}+\dfrac{2-t^2}{2+t^2}\mathbf{j}+\dfrac{2t}{2+t^2}\mathbf{k};\kappa(t)=\dfrac{2}{(2+t^2)^2};\rho(0)=2.$

$d)$ $\mathbf{r}'(t)=2\mathbf{i}+2t\mathbf{j}+\dfrac{1}{t}\mathbf{k};\|r'(t)\|=\dfrac{2t^2+1}{t}\Rightarrow T(t)=\dfrac{2t}{2t^2+1}\mathbf{i}+\dfrac{2t^2}{2t^2+1}\mathbf{j}+\dfrac{1}{2t^2+1}\mathbf{k}$

Here, $T'(t)=\dfrac{2-4t^2}{(2t^2+1)^2}\mathbf{i}+\dfrac{4t}{(2t^2+1)^2}\mathbf{j}-\dfrac{4t}{(2t^2+1)^2}\mathbf{k};\|T'(t)\|=\dfrac{2}{2t^2+1}$

So, $N(t)=\dfrac{T'(t)}{\|T'(t)\|}=\dfrac{1-2t^2}{2t^2+1}\mathbf{i}+\dfrac{2t}{2t^2+1}\mathbf{j}-\dfrac{2t}{2t^2+1}\mathbf{k};\ \rho(1)=\dfrac{1}{\kappa(1)}=\dfrac{9}{2}.$

$e)$ $\|r'(t)\|=e^t+e^{-t}$. Hence, $T(t)=\dfrac{r'(t)}{\|r'(t)\|}=\dfrac{e^{2t}}{e^{2t}+1}\mathbf{i}-\dfrac{1}{e^{2t}+1}\mathbf{j}+\dfrac{\sqrt{2}e^t}{e^{2t}+1}\mathbf{k}$

$N(t)=\dfrac{T'(t)}{\|T'(t)\|}=\dfrac{\sqrt{2}e^t}{e^{2t}+1}\mathbf{i}-\dfrac{\sqrt{2}e^t}{e^{2t}+1}\mathbf{j}-\dfrac{e^{2t}-1}{e^{2t}+1}\mathbf{k}.\ \kappa(t)=\dfrac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2};\rho(0)=2\sqrt{2}.$

$f)$ $T(t)=\dfrac{2t\mathbf{i}+2\mathbf{j}}{2\sqrt{t^2+1}}=\dfrac{t}{\sqrt{t^2+1}}\mathbf{i}+\dfrac{1}{\sqrt{t^2+1}}\mathbf{j},N(t)=\dfrac{1}{\sqrt{t^2+1}}\mathbf{i}-\dfrac{t}{\sqrt{t^2+1}}\mathbf{j}$

3.6 The concepts of Scalar and Vector Fields

3.6.1 Divergence and Curl of Vector Fields

A function that assigns a vector to each point P in a region R (in a plane or in space) is called a *vector field* and usually denoted by $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

a) $F(x,y) = 2x\mathbf{i} + y^2\mathbf{j}$ is a vector field in a plane

b) $F(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ is a vector field in space.

Note that a vector field F formed by the gradient of a scalar function f is said to be *gradient vector field* and denoted by $F = \nabla.f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.

Definition: Let $F = \mathbf{M.i} + \mathbf{N.j} + \mathbf{P.k}$ be a vector field such that all the partial derivatives $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}$ and $\frac{\partial P}{\partial z}$ exist. Then,

i) Divergence of Vector Fields:

The divergence of F denoted by $divF$ or $\nabla.F$ is the scalar function defined by

$$\text{div}F = \nabla.F = \frac{\partial \mathbf{M}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y} + \frac{\partial \mathbf{P}}{\partial z} = \mathbf{M}_x + \mathbf{N}_y + \mathbf{P}_z.$$

In a plane, where $F = \mathbf{M.i} + \mathbf{N.j}$, $\text{div}F = \nabla.F = \mathbf{M}_x + \mathbf{N}_y$.

ii) The Curl of Vector Fields:

The curl of F which is denoted by $curlF$ or $\nabla \times F$ is a vector function defined by $\text{curl}F = \nabla \times F = (\frac{\partial \mathbf{P}}{\partial y} - \frac{\partial \mathbf{N}}{\partial z})\mathbf{i} + (\frac{\partial \mathbf{M}}{\partial z} - \frac{\partial \mathbf{P}}{\partial x})\mathbf{j} + (\frac{\partial \mathbf{N}}{\partial x} - \frac{\partial \mathbf{M}}{\partial y})\mathbf{k}$

We can express, $curlF$ in compact form as $curlF = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{vmatrix}$

In a plane, where $F = \mathbf{M}\mathbf{i} + \mathbf{N}\mathbf{j}$ $\text{curl}F = (\mathbf{N}_y - \mathbf{M}_x)\mathbf{k}$

Note:

i) If $divF = 0$, then the field F is said to be divergence free or *solenoidal.*

ii) If curl $\mathbf{F} = \nabla \times F = 0$, then the field F is said to be *irrotational.*

Examples: Find the divergence and curl of the following vector fields.

$a)\ F(x,y)=xy^3\mathbf{i}+5x^2y^2\mathbf{j}$ $\quad b)\ F(x,y,z)=x^2\mathbf{i}+(3x+yz^2)\mathbf{j}-2z^3\mathbf{k}$

$c)\ F(x,y,z)=z^3\mathbf{i}-3xy\mathbf{j}+y^2z\mathbf{k}$ $\quad d)\ F(x,y,z)=e^x\cos y\mathbf{i}-e^x\sin y\mathbf{j}+xy^2\mathbf{k}$

$e)\ F(x,y,z)=3x^2\mathbf{i}+xy\mathbf{j}-y^2z\mathbf{k}$ $\quad f)\ F(x,y,z)=xy\mathbf{i}+y^2\mathbf{j}+xyz\mathbf{k}$

Solution:

$a)$ Here, $M=xy^3, N=5x^2y^2$. Then,

$$\text{i) div}F=\nabla.F=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}=\frac{\partial}{\partial x}(xy^3)+\frac{\partial}{\partial y}(5x^2y^2)=y^3+10x^2$$

$$\text{ii) curl}F=(N_y-M_x)\mathbf{k}=(10xy^2-3xy^2)\mathbf{k}=7xy^2\mathbf{k}$$

$$b)\ \text{i) div}F=M_x+N_y+P_z=2x+z^2-6z^2=2x-5z^2$$

$$\text{ii) curl}F=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & 3x+yz^2 & -2z^3\end{vmatrix}=-2yz\mathbf{i}+3\mathbf{k}$$

$$c)\ \text{i) div}F=M_x+N_y+P_z=-3x+y^2$$

$$\text{ii) curl}F=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z^3 & -3xy & y^2\end{vmatrix}=2y\mathbf{i}+3z^2\mathbf{j}-3y\mathbf{k}$$

$$d)\ \text{i) div}F=\nabla.F=M_x+N_y+P_z=e^x\cos y-e^x\cos y=0$$

$$\text{ii) curl}F=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^x\cos y & -e^x\sin y & xy^2\end{vmatrix}=2xy\mathbf{i}-y^2\mathbf{j}$$

$e)$ i) $\text{div}F=M_x+N_y+P_z=7x-y^2$ $\quad$ ii) $\text{curl}F=-2yz\mathbf{i}+y\mathbf{k}$

2. Find a if $F(x,y,z)=e^x\sin y\mathbf{i}+(2ay+e^x\cos y)\mathbf{j}-6z\mathbf{k}$ is divergence free.

Solution:

$$divF=0\Rightarrow\frac{\partial}{\partial x}(e^x\sin y)+\frac{\partial}{\partial y}(2ay+e^x\cos y)+\frac{\partial}{\partial z}(-6z)=2a-6=0\Rightarrow a=3$$

3.6.2 Conservative Vector Fields and Test of Conservativeness

Definition: A vector field F is said to be *conservative* if there exists a differentiable function f such that $F = \nabla f$. The function f with this property is called potential function for F.

Example: The function $F(x,y,z) = 3x^2\mathbf{i} + 3y^2\mathbf{j} + 3z^2\mathbf{k}$ is conservative because there exists a potential function $f(x,y,z) = x^3 + y^3 + z^3$ for F such that $F = \nabla f$.

Test of Conservativeness:

A vectror field $F(x,y,z) = \mathbf{M.i} + \mathbf{N.j} + \mathbf{P.k}$ is conservative vector field if and only if curl $\mathbf{F(x,y,z)} = \nabla \times F = 0 \Leftrightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{vmatrix} = 0.$

Test of Conservative in the plane:

In a plane, $F(x,y) = \mathbf{M.i} + \mathbf{N.j}$ is conservative only if $\text{curl}\mathbf{F} = 0 \Leftrightarrow \frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial x}$.

Procedures for Finding potential Function:

If a vector field $F(x,y,z) = \mathbf{M\,i} + \mathbf{N.j} + \mathbf{P.k}$ is conservative, we can find its potential function f such that $F = \nabla f = f_x\mathbf{.i} + f_y\mathbf{.j} + f_z\mathbf{.k} = \mathbf{M\,i} + \mathbf{N.j} + \mathbf{P.k}$.

That means $f_x\mathbf{.i} + f_y\mathbf{.j} + f_z\mathbf{.k} = \mathbf{M\,i} + \mathbf{N.j} + \mathbf{P.k} \Rightarrow f_x = M, f_y = N, f_z = P.$

Step-1: Variable separation: That means express the function f as a sum of three functions in the from $f(x,y,z) = k(x,y,z) + g(y,z) + h(z)$.

Then, $f_x = k_x, f_y = k_y + g_y, f_z = k_z + g_z + h'(z)$.

Notice about the functions k, g, h in the decomposition!

The functions are separated in such a way that k is the function of all the three variables x,y,z ,the function g is the function of two of variables y,z and thus $g_x = 0$. Again, h is a function of only variable z and thus $h_x = 0, h_y = 0$.

Step-2: Integrate $f_x = k_x = \mathbf{M}$ with respect to x.

That is $k_x = \mathbf{M} \Rightarrow \int k_x dx = \int \mathbf{M} dx \Rightarrow k(x,y,z) = \int \mathbf{M} dx$

Step-3: Integrate $f_y = k_y + g_y = \mathbf{N}$ with respect to y. That is

$$f_y = k_y + g_y = \mathbf{N} \Rightarrow g_y = \mathbf{N} - k_y \Rightarrow \int g_y dy = \int (\mathbf{N} - k_y) dy$$

$$\Rightarrow g(y,z) = \int (\mathbf{N} - k_y) dy$$

Step-4: Integrate $f_z = k_z + g_z + h'(z) = \mathbf{P}$ with respect to z. That is

$$k_z + g_z + h'(z) = \mathbf{P} \Rightarrow h'(z) = \mathbf{P} - k_z - g_z \Rightarrow \int h'(z) dz = \int (\mathbf{P} - k_z - g_z) dz$$

$$\Rightarrow h(z) = \int (\mathbf{P} - k_z - g_z) dz$$

Finally: Write $f(x,y,z) = k(x,y,z) + g(y,z) + h(z)$ using the above results.

Examples:

1. Determine whether the following vector fields are conservative or not for these which are conservative, find the potential function.

a) $F(x,y,z) = 4xe^z\mathbf{i} - 6y\mathbf{j} + (2x^2e^z - 9)\mathbf{k}$

b) $F(x,y,z) = 2x\cos y\mathbf{i} + (6yz - x^2\sin y)\mathbf{j} + (3y^2 - 4z)\mathbf{k}$

c) $F(x,y,z) = (2xy + z^2)\mathbf{i} + x^2\mathbf{j} + (2xz - 6\cos 3z)\mathbf{k}$

d) $F(x,y,z) = (2xy + e^z)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + xe^z)\mathbf{k}$

e) $F(x,y) = e^y\mathbf{i} + (xe^y + y)\mathbf{j}$

f) $F(x,y,z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

g) $F(x,y,z) = \frac{z}{y}\mathbf{i} - \frac{xz}{y^2}\mathbf{j} + \frac{x}{y}\mathbf{k}$

Solution:

a) Here, $M = 4xe^z, N = -6y, P = 2x^2e^z - 9$

Test Conservativeness: curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xe^z & -6y & 2x^2e^z - 9 \end{vmatrix} = 0.$

Therefore, the vector field is conservative.

Determine Potential Function:

Variable separation: Suppose $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$.

First: Evaluate $k(x,y,z)=\int \mathrm{M}dx$.

That is $k(x,y,z)=\int \mathrm{M}dx=\int 4xe^{z}dx=2x^2e^{z}$.

Second: Evaluate $g(y,z)=\int(\mathrm{N}-k_y)dy$

That is $g(y,z)=\int(\mathrm{N}-k_y)dy=\int(-6y-0)dy=\int-6ydy=-3y^2$.

Third: Evaluate $h(z)=\int(\mathrm{P}-k_z-g_z)dz$.

That is $h(z)=\int(\mathrm{P}-k_z-g_z)dz=\int(2x^2e^{z}-9-2x^2e^{z})dz=\int-9dz=-9z$.

Finally: Express $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$ using the results.

Therefore, $f(x,y,z)=2x^2e^{z}-3y^2-9z$.

b) **Test Conservativeness:** Here, $M=2x\cos y, N=6yz-x^2\sin y, P=3y^2-4z$.

$$\text{Then curl F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{M} & \mathrm{N} & \mathrm{P} \end{vmatrix}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\cos y & 6yz-x^2\sin y & 3y^2-4z \end{vmatrix}=0.$$

Therefore, the vector field is conservative.

Determine Potential function:

Variable separation: Suppose $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$.

First: Evaluate $k(x,y,z)=\int \mathrm{M}dx$.

That is $k(x,y,z)=\int \mathrm{M}dx=\int 2x\cos y dx=x^2\cos y$.

Second: Evaluate $g(y,z)=\int(\mathrm{N}-k_y)dy$

That is $g(y,z)=\int(\mathrm{N}-k_y)dy=\int(6yz-x^2\sin y+x^2\cos y)dy=\int 6yzdy=3y^2z$.

Third: Evaluate $h(z)=\int(\mathrm{P}-k_z-g_z)dz$.

That is $h(z)=\int(\mathrm{P}-k_z-g_z)dz=\int(3y^2-4z-3y^2)dz=\int-4zdz=-2z^2$.

Finally: Express $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$ using the results.

Therefore, $f(x,y,z)=x^2\cos y+3y^2z-2z^2$.

c) Test Conservativeness: $\text{curl F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+z^2 & x^2 & 2xz-6\cos 3z\end{vmatrix}=0$

Therefore, the vector field is conservative.

Determine Potential Function:

Variable separation: Suppose $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$.

First: Evaluate $k(x,y,z)=\int \mathbf{M}dx$.

That is $k(x,y,z)=\int \mathbf{M}dx=\int(2xy+z^2)dx=x^2y+xz^2$.

Second: Evaluate $g(y,z)=\int(\mathbf{N}-k_y)dy$

That is $g(y,z)=\int(\mathbf{N}-k_y)dy=\int(x^2-x^2)dy=\int 0dy=c$.

Third: Evaluate $h(z)=\int(\mathbf{P}-k_z-g_z)dz$.

That is $h(z)=\int(\mathbf{P}-k_z-g_z)dz=\int(2xz-6\cos 3z-2xz)dz$

$$=\int -6\cos 3z dz=-2\sin 3z$$

Finally: Express $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$ using the results.

Therefore, $f(x,y,z)=x^2y+xz^2-2\sin 3z$.

d) Test Conservativeness: Here, $M=2xy+e^z, N=x^2+2yz, P=y^2+xe^z$

Then, $\dfrac{\partial P}{\partial y}=\dfrac{\partial N}{\partial z},\dfrac{\partial P}{\partial x}=\dfrac{\partial M}{\partial z},\dfrac{\partial N}{\partial x}=\dfrac{\partial M}{\partial y}$. Hence, F is conservative.

Potential function:

Variable separation: Suppose $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$.

First: Evaluate $k(x,y,z)=\int \mathbf{M}dx$.

That is $k(x,y,z)=\int \mathbf{M}dx=\int(2xy+e^z)dx=x^2y+xe^z$.

Second: Evaluate $g(y,z)=\int(\mathrm{N}-k_y)dy$

That is $g(y,z)=\int(\mathrm{N}-k_y)dy=\int(x^2+2yz-x^2)dy=\int 2yz\,dy=y^2z$.

Third: Evaluate $h(z)=\int(\mathrm{P}-k_z-g_z)dz$.

That is $h(z)=\int(\mathrm{P}-k_z-g_z)dz=\int(y^2+xe^z-xe^z-y^2)dz=\int 0dz=0$.

Finally: Express $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$ using the results.

Therefore, $f(x,y,z)=x^2+xe^z+y^2z$.

e) Here, $\mathrm{M}=e^y, N=xe^y+y\Rightarrow\frac{\partial N}{\partial x}=e^y=\frac{\partial M}{\partial y}$. Hence, its is conservative.

$$\frac{\partial f}{\partial x}=M=e^y\Rightarrow f(x,y)=xe^y+g(y),\frac{\partial f}{\partial y}=N\Rightarrow g'(y)=y\Rightarrow g(y)=\frac{y^2}{2}+c$$

Hence, $f(x,y)=xe^y+\frac{y^2}{2}+\mathrm{c}$.

f) **First:** Evaluate $k(x,y,z)=\int\mathrm{M}dx$.

That is $k(x,y,z)=\int\mathrm{M}dx=\int xy^2z^2dx=\frac{x^2y^2z^2}{2}$.

Second: $g(y,z)=\int(\mathrm{N}-k_y)dy=\int(x^2yz^2-x^2yz^2)dy=\int 0dy=c$.

Third: $h(z)=\int(\mathrm{P}-k_z-g_z)dz=\int(x^2y^2z-x^2y^2z)dz=\int 0dz=0$.

Therefore, $f(x,y,z)=\frac{1}{2}x^2y^2z^2+c$.

g) Similarly, we get F is conservative and $\mathrm{f(x,y,z)}=\frac{\mathrm{xz}}{\mathrm{y}}+c$.

2. Find a so that $F(x,y,z)=2xy\mathbf{i}+(x^2-3ayz)\mathbf{j}+(y^2-3a)\mathbf{k}$ is conservative.

Solution: To be conservative, we have $\mathrm{curl}F=0$.

$$\text{Hence, } \mathrm{curl}F=0\Rightarrow\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy & x^2-3ayz & y^2-3a\end{vmatrix}=(2y+3ay)\mathbf{i}=0$$

$$\Rightarrow y(3a+2)=0\Rightarrow a=-\tfrac{3}{2}$$

3.7 Line Integrals of Scalar and Vector Fields

3.7.1 Line Integrals of Scalar Fields

Definition: If f is a continuous scalar field on a smooth curve C parameterized by $\mathbf{r}(t)$ on $[a,b]$, then the integral $\int_C f(x,y,z)ds$ is said to be the line integral of f on C. It is defined as $\int_C f(x,y,z).ds = \int_a^b f(x(t),y(t),z(t)).\|\mathbf{r}'(t)\|dt$.

As we see from this formula, to evaluate line integrals;

First: Compute $ds = \|\mathbf{r}'(t)\|dt$ where $\|\mathbf{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

Second: Use the formula: $\int_C f(x,y,z).ds = \int_a^b f(x(t),y(t),z(t)).\|\mathbf{r}'(t)\|dt$ by replacing x,y,z in the function f with their parametric expressions.

Examples:

1. Evaluate the following line integrals

a) $\int_C (8xy^2 - 3yz - x^2z)ds$ where $C: r(t) = 2t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 1 \le t \le 2$.

b) $\int_C \sqrt{1+y}ds$ where $C: r(t) = 2t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 1$.

c) $\int_C 3x^2yzds$ where $C: r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}, 0 \le t \le 1$.

d) $\int_C \frac{e^{-z}}{x^2+y^2}ds$ where $C: r(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le 2\pi$.

e) $\int_C (5+xy)ds$ where $C: r(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}, 0 \le t \le \pi$.

f) $\int_C (x+y)ds$ where $C: r(t) = (e^t+1)i + (e^t-1)j, 0 \le t \le \ln 2$

Solution:

a) **First:** Compute $ds = \|\mathbf{r}'(t)\|dt$: Here, $r(t) = 2t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$.

Then, $\mathbf{r}'(t) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{4+1+4} = 3 \Rightarrow ds = \|\mathbf{r}'(t)\|dt = 3dt$.

Second: Express the integrand f as a function of t and apply the formula.

From the parametrization $r(t)=2t\mathbf{i}+t\mathbf{j}+2t\mathbf{k}$,we have $x=2t, y=t, z=2t$.

Then, $f(x,y,z)=8xy^2-3yz-x^2z \Rightarrow f(t)=16t^3-6t^2-8t^3=8t^3-6t^2$

Therefore, $\int_C f(x,y,z)ds=\int_1^2 f(t)\|\mathbf{r}'(t)\|dt=\int_1^2 3(8t^3-6t^2)dt=3(2t^4-2t^3)\Big|_1^2=48.$

b) Here, $\mathbf{r}'(t)=2\mathbf{i}+2t\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\|=\sqrt{4+4t^2}=2\sqrt{1+t^2}$

Hence, $\int_C f(x,y,z)ds=\int_0^1 f(x(t),y(t),z(t))\|\mathbf{r}'(t)\|dt=\int_0^1 \sqrt{1+t^2}(2\sqrt{1+t^2})dt$

$$=2\int_0^1(1+t^2)dt=2(t+\frac{1}{3}t^3)\Big|_0^1=2(1+\frac{1}{3})=\frac{8}{3}$$

c) $\mathbf{r}'(t)=\mathbf{i}+2t\mathbf{j}+2t^2\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\|=\sqrt{1+4t^2+4t^4}=\sqrt{(1+2t^2)^2}=1+2t^2$

Hence, $\int_C f(x,y,z)ds=\int_0^1 f(x(t),y(t),z(t))\|\mathbf{r}'(t)\|dt=\int_0^1 2t^7(1+2t^2)dt$

$$=\int_0^1(2t^7+4t^9)dt=(\frac{t^8}{4}+\frac{2}{5}t^{10})\Big|_0^1=\frac{1}{4}+\frac{2}{5}=\frac{13}{20}$$

d) $\mathbf{r}'(t)=-2\sin t\mathbf{i}+2\cos t\mathbf{j}+\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\|=\sqrt{4\sin^2 t+4\cos^2 t+1}=\sqrt{5}$

Hence, $\int_C f(x,y,z)ds=\sqrt{5}\int_0^{2\pi}\frac{e^{-t}dt}{4\cos^2 t+4\sin^2 t}=\frac{\sqrt{5}}{4}\int_0^{2\pi}e^{-t}dt=\frac{\sqrt{5}}{4}(1-e^{-2\pi})$

e) $x(t)=4\cos t, y(t)=4\sin t, z(t)=3t, \mathbf{r}'(t)=-4\sin t\mathbf{i}+4\cos t\mathbf{j}+3\mathbf{k},$

$\|\mathbf{r}'(t)\|=\sqrt{16\sin^2 t+16\cos^2 t+9}=\sqrt{16(\sin^2 t+\cos^2 t)+9}=\sqrt{25}=5$

Hence, $\int_C f(x,y,z)ds=\int_0^{\pi}(5+16\cos t\sin t)5dt=25\pi$

f) $\mathbf{r}'(t)=e^t\mathbf{i}+e^t\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\|=\sqrt{e^{2t}+e^{2t}}=\sqrt{2e^{2t}}=\sqrt{2}e^t$

$\int_C(x+y)ds=\int_0^{\ln 2}(e^t+1+e^t-1)\sqrt{2}e^t dt=\sqrt{2}\int_0^{\ln 2}2e^{2t}dt=\sqrt{2}e^{2t}\Big|_{t=0}^{t=\ln 2}=3\sqrt{2}$

2. Evaluate the following line integrals .

a) $\int_C \frac{1}{1+x} ds$ where $C: x=t, y=\frac{2}{3}t^{3/2}, 0 \le t \le 3.$

b) $\int_C (xy+z)ds$ where $C: \sin t\mathbf{i}+\sin t\mathbf{j}+\sin t\mathbf{k}, 0 \le t \le \frac{\pi}{6}.$

c) $\int_C (7xy-2yz+xz)ds$ where $C: r(t)=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}, 1 \le t \le 2.$

d) $\int_C (x^3+y^3)ds$ where $C: \cos^3 t\mathbf{i}+\sin^3 t\mathbf{j}, 0 \le t \le \frac{\pi}{2}.$

Solution:

a) $\mathbf{r}'(t)=\mathbf{i}+\sqrt{t}\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{1+t}$

Hence, $\int_C f(x,y)ds = \int_0^3 f(x(t),y(t))\|\mathbf{r}'(t)\|dt = \int_0^3 \frac{\sqrt{1+t}dt}{1+t} = \int_0^3 \frac{1}{\sqrt{1+t}}dt = 2$

b) $\mathbf{r}'(t)=\cos t\mathbf{i}+\cos t\mathbf{j}+\cos t\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{3\cos^2 t} = \sqrt{3}|\cos t|$

Since $\cos t \ge 0$ on $0 \le t \le \frac{\pi}{6}, \|\mathbf{r}'(t)\| = \sqrt{3}|\cos t| = \sqrt{3}\cos t.$

So, $\int_C f(x,y,z)ds = \sqrt{3}\int_0^{\pi/6}(\sin^2 t+\sin t)\cos t dt = \sqrt{3}\left(\frac{\sin^3 t}{3}+\frac{\sin^2 t}{2}\right)\Big|_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{6}$

c) $\mathbf{r}'(t)=\mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{1+1+1} = \sqrt{3}$ and $f(t)=7t^2-2t^2+t^2=6t^2$.

Therefore, $\int_C f(x,y,z)ds = \int_1^2 f(t)\|\mathbf{r}'(t)\|dt = \sqrt{3}\int_1^2 6t^2 dt = 2\sqrt{3}t^3\Big|_1^2 = 14\sqrt{3}$

d) $\|\mathbf{r}'(t)\| = 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)} = 3\sqrt{\cos^2 t\sin^2 t} = 3|\cos t\sin t|$

Since $\cos t \ge 0, \sin t \ge 0$ on $0 \le t \le \frac{\pi}{2}, \|\mathbf{r}'(t)\| = 3|\cos t\sin t| = 3\cos t\sin t$

Hence, $\int_C f(x,y,z)ds = 3\int_0^{\pi/2}\cos^{10} t\sin t dt + 3\int_0^{\pi/2}\sin^{10} t\cos t dt$

$$= 3\left(\frac{-\cos^{11} t}{11}\right)\Big|_0^{\pi/2} + 3\left(\frac{\sin^{11} t}{11}\right)\Big|_0^{\pi/2} = \frac{3}{11}+\frac{3}{11} = \frac{6}{11}$$

3. Think About: Suppose $\int_C (5+y)ds = 30\pi$ where C is a circle centered at the origin. Find the radius of this circle.

Solution: Since C is a circle centered at the origin, it is parameterized by

$\mathbf{r}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j},\ 0 \le t \le 2\pi$. Hence, $\int_C (5+y)ds = \int_0^{2\pi} (5 + r\sin t)dt = 10\pi r$.

Then, using the given value we have $10\pi r = 30\pi \Rightarrow r = 3$ units.

Remark: In each of the above examples, we are given the curve C with its parametric equations. But in many cases, we may need to evaluate a line integral in which the curve C is not given in parametric form. In such cases, we use the basic parameterization techniques that we discussed in *Section 3.4.*

Examples: Evaluate the following line integrals.

a) $\int_C (x+y+z)ds$ where C is the line segment from $(0,1,1)$ to $(2,2,3)$.

b) $\int_C xyds$ where C is the circle $x^2 + y^2 = 4$

c) $\int_C (x+y)ds$ where C is the upper half of the circle $x^2 + y^2 = 25$ in CCW direction.

d) $\int_C \sqrt{1+9xy}ds$ where C is the graph of $y = x^3$ for $0 \le x \le 1$

e) $\int_C (2xy - 5yz)ds$ where C is the line segment from $(1,0,1)$ to $(0,3,2)$.

Solution: In each of the problems, the curve C is not given in parametric form. So, our first task is to find the parametric representation of the curves.

a) Using the parametric equations for line segment, the line segment from $(0,1,1)$ to $(2,2,3)$ is parameterized by

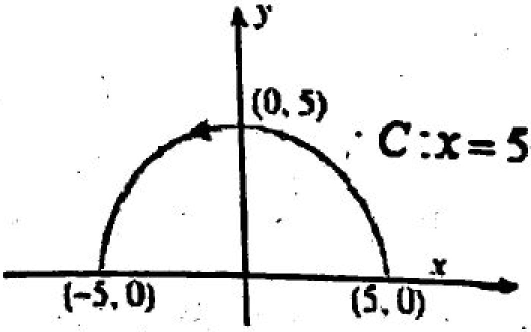
$x = 2t, y = 1+t, z = 1+2t, 0 \le t \le 1$, $\mathbf{r}'(t) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{9} = 3$

Hence, $\int_C f(x,y,z)ds = \int_0^1 f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|dt = 3\int_0^1 (2+5t)dt = \frac{27}{2}$

b) The circle $x^2 + y^2 = 4$ is parameterized by $x = 2\cos t, y = 2\sin t, 0 \le t \le 2\pi$ and then $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = 2$

Hence, $\int_C xyds = 2\int_0^{2\pi} 4\cos t\sin t dt = 4\int_0^{2\pi} \sin 2t dt = -2\cos 2t\Big|_0^{2\pi} = 0$

c) The upper half of the circle $x^2 + y^2 = 25$ in CCW direction means the semi-circle from the positive x-axis to the negative x-axis as shown below.



$C: x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$

The circle is parameterized by $C: x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$ (observe that we used the interval to be $0 \le t \le \pi$ but not $0 \le t \le 2\pi$ because C is the semicircle not the full circle) and then $\mathbf{r}'(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = 5$

Hence, $\int_C (x+y)ds = 5\int_0^{\pi} (5\cos t + 5\sin t)dt = 25(\sin t - \cos t)\Big|_0^{\pi} = 50$

d) Here, $y = x^3$ suggests a parameterization of $x = t, y = t^3$ for $0 \le t \le 1$.

Thus $f(x,y) = \sqrt{1+9xy} \Rightarrow f(t) = \sqrt{1+9t^4}, \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{1+9t^4}$

Hence, $\int_0^1 (\sqrt{1+9t^4})(\sqrt{1+9t^4})dt = \int_0^1 (1+9t^4)dt = t + \frac{9t^5}{5}\Big|_{t=0}^{t=1} = \frac{14}{5}$

e) The line segment from $(1,0,1)$ to $(0,3,2)$ is parameterized by

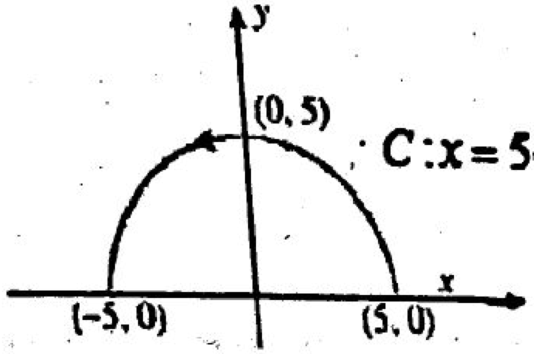
$x = 1-t, y = 3t, z = 1+t, 0 \le t \le 1,\ \mathbf{r}'(t) = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{11}$

Hence, $\int_C f(x,y,z)ds = \sqrt{11}\int_0^1 [6t(1-t) - 15t(1+t)]dt = \sqrt{11}\int_0^1 (-21t^2 - 9t)dt$

$$= \sqrt{11}(-7t^3 - \frac{9}{2}t^2)\Big|_0^1 = \frac{-23\sqrt{11}}{2}$$

Hence, $\int_C xyds = 2\int_0^{2\pi} 4\cos t\sin tdt = 4\int_0^{2\pi}\sin 2tdt = -2\cos 2t\big|_0^{2\pi} = 0$

c) The upper half of the circle $x^2+y^2=25$ in CCW direction means the semi-circle from the positive x-axis to the negative x-axis as shown below.



$C: x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$

The circle is parameterized by $C: x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$ (observe that we used the interval to be $0\le t\le\pi$ but not $0\le t\le 2\pi$ because C is the semicircle not the full circle) and then $\mathbf{r}'(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = 5$

Hence, $\int_C (x+y)ds = 5\int_0^{\pi}(5\cos t + 5\sin t)dt = 25(\sin t - \cos t)\big|_0^{\pi} = 50$

d) Here, $y = x^3$ suggests a parameterization of $x = t, y = t^3$ for $0 \le t \le 1$.

Thus $f(x,y) = \sqrt{1+9xy} \Rightarrow f(t) = \sqrt{1+9t^4}, \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{1+9t^4}$

Hence, $\int_0^1 (\sqrt{1+9t^4})(\sqrt{1+9t^4})dt = \int_0^1 (1+9t^4)dt = t + \frac{9t^5}{5}\bigg|_{t=0}^{t=1} = \frac{14}{5}$

e) The line segment from $(1,0,1)$ to $(0,3,2)$ is parameterized by

$x = 1-t, y = 3t, z = 1+t, 0 \le t \le 1$, $\mathbf{r}'(t) = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \|\mathbf{r}'(t)\| = \sqrt{11}$

Hence, $\int_C f(x,y,z)ds = \sqrt{11}\int_0^1 [6t(1-t) - 15t(1+t)]dt = \sqrt{11}\int_0^1 (-21t^2 - 9t)dt$

$$= \sqrt{11}\left(-7t^3 - \frac{9}{2}t^2\right)\bigg|_0^1 = \frac{-23\sqrt{11}}{2}$$

3.7.2 Line Integrals of Vector Fields

Definition: Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be continuous vector field defined on a smooth curve C parameterized by $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \le t \le b$.

Then, the line integral of F on C, denoted by $\int_C F.d\mathbf{r}$, is defined as

$$\int_C F.d\mathbf{r} = \int_a^b F(x(t), y(t), z(t)).\mathbf{r}'(t)dt.$$

More precisely, using $r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ and the dot product operator in $F.\mathbf{r}'(t)dt$, we get $\int_C F.d\mathbf{r} = \int_a^b \left(M.x'(t) + N.y'(t) + P.z'(t)\right)dt$.

Using the notation, $x'(t) = \frac{dx}{dt}, y'(t) = \frac{dy}{dt}, z'(t) = \frac{dz}{dt}$, we have

$$\int_C F.d\mathbf{r} = \int_a^b F(x(t), y(t), z(t)).\mathbf{r}'(t)dt = \int_a^b \left(M.\frac{dx}{dt} + N.\frac{dy}{dt} + P.\frac{dz}{dt}\right)dt$$

Evaluation Procedures: To evaluate $\int_C F.d\mathbf{r}$;

First: Express the given field F as $F = F(t)$ in terms of the parameter t and evaluate the dot product $F.\mathbf{r}'dt$ using $r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Second: Evaluate the integral using, $\int_C F.d\mathbf{r} = \int_a^b F(t).\mathbf{r}'(t)dt$.

Examples: Evaluate the line integral $\int_C F.d\mathbf{r}$ where

a) $F(x,y,z) = 8x^2yz\mathbf{i} + 5z\mathbf{j} - 4xy\mathbf{k}$ where $C: r(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \le t \le 1$.

b) $F(x,y,z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$ and C is boundary of $x^2 + y^2 = 4$ in the xy plane

c) $F(x,y,z) = zi - yj - xk$ and $C: r(t) = 5\mathbf{i} - \sin t\mathbf{j} - \cos t\mathbf{k}, 0 \le t \le \pi/4$

d) $F(x,y,z) = 3y\mathbf{i} + 3x\mathbf{j}$ and C is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in the xy plane.

e) $F(x,y,z) = ye^{x^2}\mathbf{i} + x\cos y^3 e^{y^2}\mathbf{k}$ and $C: r(t) = e^t\mathbf{i} + e^t\mathbf{j} + 3\mathbf{k}, 0 \le t \le 1$.

f) $F(x,y,z) = ye^{x^2}\mathbf{i} + xe^{y^2}\mathbf{j} + xy^3\mathbf{k}$ and $C: r(t) = e^t\mathbf{i} + e^t\mathbf{j} + 3\mathbf{k}, 0 \le t \le 1$

Solution:

Here, $x(t)=t, y(t)=t^2, z(t)=t^3, \dfrac{dx}{dt}=1, \dfrac{dy}{dt}=2t, \dfrac{dz}{dt}=3t^2$

From the field F, $M(x,y,z)=8x^2yz, N(x,y,z)=5z, P(x,y,z)=-4xy.$

So, $M(t)=8t^7, N(t)=5t^3, P(t)=-4t^3 \Rightarrow F(t)=8t^7i+5t^3j-4t^3k$

Hence, $\displaystyle\int_C F.dr=\int_0^1\left(M.\frac{dx}{dt}+N.\frac{dy}{dt}+P.\frac{dz}{dt}\right)dt=\int_0^1\left(8t^7+10t^4-12t^5\right)dt=1$

b) In the xy plane the circle C is parameterized by, $r(t)=2\cos ti+2\sin tj, z=0$ for $0\le t\le 2\pi$ so that $r'(t)=-\sin ti+2\cos tj.$

Besides, $M(x,y,z)=2z, N(x,y,z)=x, P(x,y,z)=y^2,$

Then, $M(t)=0, N(t)=2\cos t, P(t)=4\sin^2 t$ and $F(t)=2\cos tj+4\sin^2 tk$

Hence, $\displaystyle\int_C F.dr=\int_0^{2\pi}F(t).r'(t)dt=\int_0^{2\pi}4\cos^2 tdt=2\int_0^{2\pi}(1+\cos 2t)dt=4\pi$

c) Here, $\displaystyle\int_C F.dr=\int_0^{\pi/4}(-\sin t\cos t-5\sin t)dt=\frac{\cos 2t}{4}+5\cos t\Big|_0^{\pi/4}=\frac{10\sqrt{2}-21}{4}$

d) The ellipse is parameterized by $r(t)=3\cos ti+2\sin tj, 0\le t\le 2\pi$

Hence, $\displaystyle\int_C F.dr=\int_0^{2\pi}F(t).r'(t)dt=\int_0^{2\pi}(-18\sin^2 t+18\cos^2 t)dt$

$$=18\int_0^{2\pi}(\cos^2 t-\sin^2 t)dt=18\int_0^{2\pi}\cos 2tdt=9\sin 2t\Big|_0^{2\pi}=0$$

e) $\displaystyle\int_C F.dr=\int_0^1[e^te^{e^{2t}}.e^t+e^t\cos(e^{3t}e^{e^{2t}}).0]dt=\int_0^1 e^{2t}e^{e^{2t}}dt=\frac{1}{2}e^{e^{2t}}\Big|_{t=0}^{t=1}=\frac{1}{2}(e^{e^2}-e)$

f) Here, $x(t)=e^t, y(t)=e^t, z(t)=3, \dfrac{dx}{dt}=e^t, \dfrac{dy}{dt}=e^t, \dfrac{dz}{dt}=0.$

Using the substitution, $u=e^{e^{2t}} \Rightarrow du=2e^{2t}e^{e^{2t}}dt$, we have

$$\int_C F.dr=\int_a^b F(t).r'(t)dt=\int_0^1\left(e^{2t}e^{e^{2t}}+e^{2t}e^{e^{2t}}\right)dt=\int_0^1 2e^{2t}e^{e^{2t}}dt=e^{e^{2t}}\Big|_0^1=e^{e^2}-e.$$

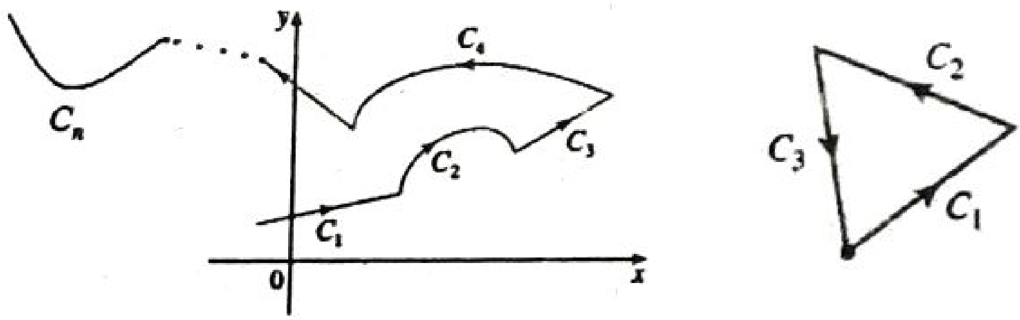
Additivity of Line Integrals:

Suppose an oriented curve C is not smooth but piece wise smooth composed of smooth curves $C_1, C_2, ..., C_n$ such that $C = C_1 \cup C_2 \cup ... \cup C_n$ as shown.

Then the line integral over C is given by $\int_C F.dr = \sum_{k=1}^{n} \int_{C_k} F.dr$.

Particularly, for $C = C_1 \cup C_2 \cup C_3$, $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + \int_{C_3} F.dr$

C: Piecewise smooth curves



i) $C = C_1 \cup C_2 \cup ... \cup C_n$ ii) $C = C_1 \cup C_2 \cup C_3$

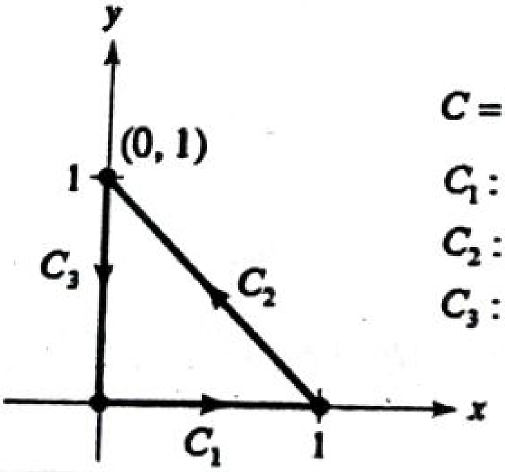
Examples: Evaluate $\int_C F \cdot dr$ where

a) $F(x, y) = x\mathbf{i} + y\mathbf{j}$ and C is the triangle in a plane with vertices $A(0,0), B(1,0)$ and $C(0,1)$ oriented counter clockwise (CCW).

b) $F(x, y, z) = x\mathbf{i} - y\mathbf{j} + (x+z)\mathbf{k}$ and C is the triangle in space with vertices $A(1,0,0), B(0,1,0)$ and $C(0,0,1)$ oriented counter clockwise (CCW).

Solution:

a) Consider the diagram to determine the parametric equations of the curve.



$C = C_1 \cup C_2 \cup C_3$ where

$C_1 : x = t, y = 0, 0 \le t \le 1$

$C_2 : x = 1 - t, y = t, 0 \le t \le 1$

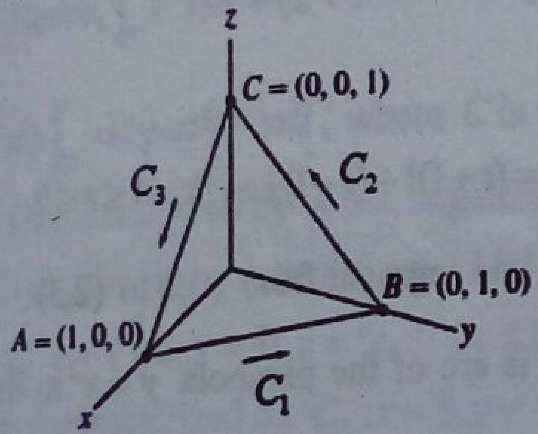
$C_3 : x = 0, y = 1 - t, 0 \le t \le 1$

On $C_1 : \int_{C_1} F.d\mathbf{r} = \int_0^1 t\,dt = \frac{1}{2}$; On $C_2 : \int_{C_2} F.d\mathbf{r} = \int_0^1 [(1-t)(-1)+t]dt = \int_0^1 (2t-1)dt = 0;$

On $C_3 : \int_{C_3} F.d\mathbf{r} = \int_0^1 (1-t)(-1).dt = \int_0^1 (t-1)dt = -\frac{1}{2}$

Therefore, $\int_C F.d\mathbf{r} = \int_{C_1} F.d\mathbf{r} + \int_{C_2} F.d\mathbf{r} + \int_{C_3} F.d\mathbf{r} = \frac{1}{2} - \frac{1}{2} = 0$

b) Consider the diagram to determine the parametric equations of the curve.



$C = C_1 \cup C_2 \cup C_3$ where

$C_1 : x = 1-t, y = t, z = 0, 0 \le t \le 1$

$C_2 : x = 0, y = 1-t, z = t, 0 \le t \le 1$

$C_3 : x = t, y = 0, z = 1-t, 0 \le t \le 1$

On $C_1 : \int_{C_1} F.d\mathbf{r} = \int_0^1 [(1-t)(-1)-t]dt = \int_0^1 -1dt = -1$; On $C_2 : \int_{C_2} F.d\mathbf{r} = \int_0^1 1dt = 1;$

On $C_3 : \int_{C_3} F.d\mathbf{r} = \int_0^1 [t+(t+1-t)(-1)]dt = \int_0^1 (t-1)dt = -\frac{1}{2}$

Therefore, $\int_C F.d\mathbf{r} = \int_{C_1} F.d\mathbf{r} + \int_{C_2} F.d\mathbf{r} + \int_{C_3} F.d\mathbf{r} = -1+1-\frac{1}{2} = -\frac{1}{2}$

3.7.3 Line Integral in Differential Form

A line integral of the form $\int_C Mdx + Ndy + Pdz$ is kown as differential form.

The procedure of evaluation of this differential form is the same as the procedure of a line integral for the vector field $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

That is $\int_C Mdx + Ndy + Pdz = \int_C F.dr = \int_a^b \left(M.\frac{dx}{dt} + N.\frac{dy}{dt} + P.\frac{dz}{dt} \right) dt.$

Examples:

1. Evaluate the following line integrals

a) $\int_C xydx + (x+z)dy + z^2 dz$,where $C: r(t) = (t+1)\mathbf{i} + (t-1)\mathbf{j} + t^2\mathbf{k}, -1 \le t \le 2$

b) $\int_C (e^x + y)dx + (x+2y)dy$, where C is a line segment from $(0,1)$ to $(2,3)$.

c) $\int_C x^2 ydx + (x+z)dy - xy^2z^2 dz$, where C is arc of the parabola $y = x^2$ in the plane $z = 2$ from $A(0,0,2)$ to $B(1,1,2)$.

d) $\int_C ydx + zdy + xdz$, where C consists of the line segment from $(2,0,0)$ to $(3,4,5)$ followed by a line segment from $(3,4,5)$ to $(3,4,0)$.

e) $\int_C (y + 2xe^y)dx + (x + x^2e^y)dy$ where $C: r(t) = \sqrt{t}\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k}, 1 \le t \le 4$

f) $\int_C -zydx + xzdy + 3zdz$ where $C: x = \cos t, y = \sin t, z = 3t,\ 0 \le t \le 2\pi$.

Solution:

a) Here, $x(t) = t+1, y(t) = t-1, z(t) = t^2, \frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 2t$

Besides $M(t) = t^2 - 1, N(t) = t^2 + t + 1, P(t) = t^4$

So, $\int_c xydx + (x+z)dy + z^2dz = \int_{-1}^{2} \left(t^2 - 1 + t + 1 + t^2 + 2t^5\right)dt$

$$= \int_{-1}^{2} \left(2t^5 + 2t^2 + t\right)dt = \left(\frac{2}{6}t^6 + \frac{2}{3}t^3 + \frac{t^2}{2}\right)\Bigg|_{-1}^{2} = \frac{57}{2}$$

b) Here, a line segment in the xy-plane from $(0,1)$ to $(2,3)$ is parameterized by $x = 2t, y = 1 + 2t,\ 0 \le t \le 1$ and $dx = 2dt, dy = 2dt$.

Hence, $\int_C (e^x + y)dx + (x + 2y)dy = 2\int_0^1 (e^{2t} + 8t + 3)dt = e^2 + 13$

c) Here, the curve from $A(0,0,2)$ to $B(1,1,2)$ is parameterized by

$x = t, y = t^2, z = 2,\ 0 \le t \le 1$ and $dx = dt, dy = 2tdt,\ dz = 0$.

Hence, $\int_C x^2 ydx + (x + z)dy - xy^2z^2dz = \int_0^1 (t^4 + 2t^2 + 4t)dt = \dfrac{43}{15}$

d) $\int_C ydx + zdy - xdz$, where C is consisting of the line segment from $(2,0,0)$ to $(3,4,5)$ followed by a line segment from $(3,4,5)$ to $(3,4,0)$.

Here, the line segment from $(2,0,0)$ to $(3,4,5)$ is parameterized by

$C_1 : x = t + 2, y = 4t, z = 5t,\ 0 \le t \le 1$ and the line segment from $(3,4,5)$ to $(3,4,0)$ is parameterized by $C_2 : x = 3, y = 4, z = -5t,\ 0 \le t \le 1$.

Evaluate the line integral on each segement separately:

On $C_1 : \int_C ydx + zdy + xdz = \int_{C_1} F.dr = \int_0^1 [4t + 20t + 5(t+2)]dt = \int_0^1 (29t + 10)dt = \dfrac{49}{2}$

On $C_2 : \int_C ydx + zdy + xdz = \int_{C_2} F.dr = \int_0^1 -15dt = -15$

Thus, by additivity of line integral, $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr = \dfrac{49}{2} - 15 = \dfrac{19}{2}$

e) $\int_C (y + 2xe^y)dx + (x + x^2e^y)dy = \int_1^4 \left(\dfrac{\ln t}{2\sqrt{t}} + t + \dfrac{1}{\sqrt{t}} + t\right)dt = 2\ln 4 + 15$

f) Here, $dx = -\sin tdt, dy = \cos tdt, dz = 3dt$.

Hence, $\int_C -zydx + xzdy + 3zdz = \int_0^{2\pi} \left(3t\sin^2 t + 3t\cos^2 t + 3t\right)dt = \int_0^{2\pi} 6tdt = 12\pi^2$.

2. Evaluate the following line integrals

a) $\int_C ydx - xdy + xyz^2dz$ and $C: r(t) = e^{-t}\mathbf{i} + e^t\mathbf{j} + t\mathbf{k}, 0 \le t \le 1$.

b) $\int_C e^x dx + xydy + xyzdz$ and $C: r(t) = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, -1 \le t \le 1$

c) $\int_C x^2dx + y^2dy + 3z^2dz$ where $C: x = \sin t, y = \cos t, z = t^2, 0 \le t \le \frac{\pi}{2}$

d) $\int_C \frac{xdy - ydx}{x^2 + y^2}$ where $C: x^2 + y^2 = 9$

e) $\int_C ydx - xdy$ where C is the parabola $y = x^2$ for $0 \le x \le 3$

Solution:

a) Here, $\int_C ydx - xdy + xyz^2dz = \int_0^1 (-1 - 1 + t^2)dt = \int_0^1 (t^2 - 2)dt = \frac{t^3}{3} - 2t\Big|_0^1 = -\frac{5}{3}$

b) Here, $\int_C e^x dx + xydy + xyzdz = \int_{-1}^1 (e^t + t^2 + 4t^3)dt = e^t + \frac{t^3}{3} + t^4\Big|_{-1}^1 = e - \frac{1}{e} + \frac{2}{3}$

c) $\int_C x^2dx + y^2dy + 3z^2dz = \int_0^{\pi/2} 6t^5 dt = t^6\Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{\pi^6}{64}$

d) $\int_C \frac{xdy - ydx}{x^2 + y^2} = \int_0^{2\pi} \frac{9\cos^2 tdt + 9\sin^2 tdt}{9\cos^2 t + 9\sin^2 t} = \int_0^{2\pi} dt = 2\pi$.

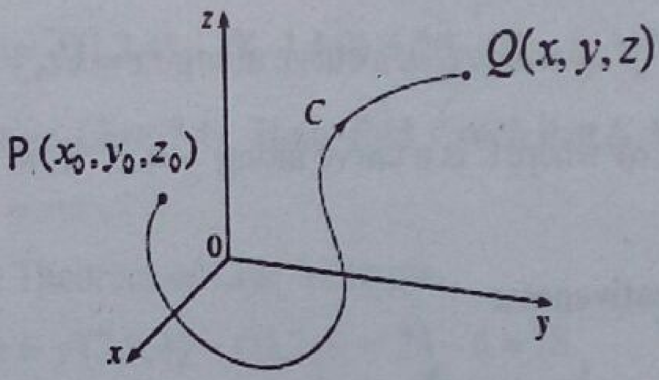
e) Here, the curve is parameterized by $x = t, y = t^2, 0 \le t \le 3$.

Thus, $\int_C ydx - xdy = \int_0^3 (t^2 - 2t^2)dt = -\int_0^3 t^2 dt = -\frac{t^3}{3}\Big|_{t=0}^{t=3} = -9$

3.7.4 Fundamental Theorem of Line Integral

(For Conservative Vector Fields)

Let C be an oriented curve with initial point $P(x_0, y_0, z_0)$ and terminal point $Q(x, y, z)$ as shown in the diagram below.



Fundamental Theorem of Line integral:

Let F be a conservative vector field with arbitrary potential function f (that is $F = \nabla f$). Then, a line integral from P to Q along a curve C is given by

$$\int_C F \cdot dr = \int_P^Q F . dr = \int_C \nabla f \cdot dr = f(Q) - f(P) = f(x, y, z) - f(x_0, y_0, z_0).$$

This theorem tells us that if a vector field is conservative, then the line integral depends only at the end points of the curve but not the parameterizations.

The basic advantage of this Theorem is that it helps us to evaluate the line integral without finding the parametric equations of the curve. Even when it is impossible to find the parametric equations of a curve, we can evaluate the line integral $\int_C F \cdot dr$ provided that F is conservative.

Remarks:

i) If F is conservative, then there is a potential function f such that $F = \nabla f$.

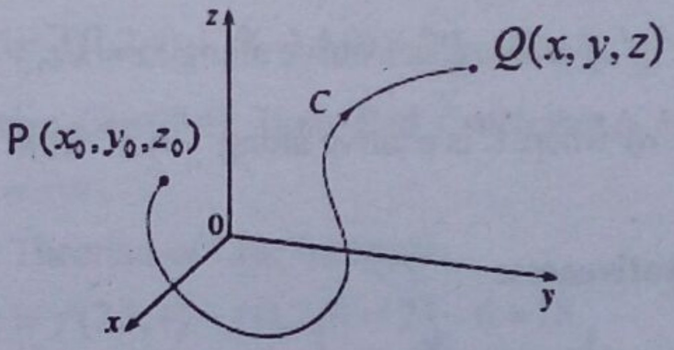
ii) If F is conservative, then the line integral $\int_C F \cdot dr$ is independent of path.

iii) If a vector field F is conservative, then $\int_C F \cdot dr = 0$ for every closed path C.

3.7.4 Fundamental Theorem of Line Integral

(For Conservative Vector Fields)

Let C be an oriented curve with initial point $P(x_0, y_0, z_0)$ and terminal point $Q(x, y, z)$ as shown in the diagram below.



Fundamental Theorem of Line integral:

Let F be a conservative vector field with arbitrary potential function f (that is $F = \nabla f$). Then, a line integral from P to Q along a curve C is given by

$$\int_C F \cdot dr = \int_P^Q F.dr = \int_C \nabla f \cdot dr = f(Q) - f(P) = f(x, y, z) - f(x_0, y_0, z_0).$$

This theorem tells us that if a vector field is conservative, then the line integral depends only at the end points of the curve but not the parameterizations.

The basic advantage of this Theorem is that it helps us to evaluate the line integral without finding the parametric equations of the curve. Even when it is impossible to find the parametric equations of a curve, we can evaluate the line integral $\int_C F \cdot dr$ provided that F is conservative.

Remarks:

i) If F is conservative, then there is a potential function f such that $F = \nabla f$.

ii) If F is conservative, then the line integral $\int_C F \cdot dr$ is independent of path.

iii) If a vector field F is conservative, then $\int_C F \cdot dr = 0$ for every closed path C .

Examples:

1. Verify that F is conservative and evaluate $\int_C F \cdot dr$ where

a) $F(x,y,z)=(2xy+z^2)\mathbf{i}+x^2\mathbf{j}+(2xz-\pi\sin\pi z)\mathbf{k}$ and C is a curve from $(3,-1,1)$ to $(2,3,1)$ along a certain path.

b) $F(x,y,z)=2xy\mathbf{i}+(x^2+y)\mathbf{j}+2z\mathbf{k}$ and C is the curve from $(-1,4,0)$ to $(1,2,4)$

c) $F(x,y,z)=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$ and C is a curve from $(1,2,3)$ to $(2,3,4)$

d) $\int_C x^3y^4dx+x^4y^3dy$ where C is a curve along $x=\sqrt[5]{t}, y=3+t^5, 0\le t\le 1$

e) $\int_C y^3dx+3xy^2dy$ where C is a curve along $y=x^2$ from $(1,1)$ to $(2,4)$

Solution:

a) Test Conservativeness:

$$\text{curl F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^2 & x^2 & 2xz-\pi\sin\pi z \end{vmatrix}=(2z-2z)j+(2x-2x)k=0$$

Therefore, the vector field is conservative.

Determine Potential Function:

Variable separation: Suppose $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$.

First: Evaluate $k(x,y,z)=\int \mathrm{M}dx$.

That is $k(x,y,z)=\int \mathrm{M}dx=\int(2xy+z^2)dx=x^2y+xz^2$.

Second: Evaluate $g(y,z)=\int(\mathrm{N}-k_y)dy$

That is $g(y,z)=\int(\mathrm{N}-k_y)dy=\int(x^2-x^2)dy=\int 0dy=c$.

Third: Evaluate $h(z)=\int(\mathrm{P}-k_z-g_z)dz$.

That is $h(z)=\int(2xz-\pi\sin\pi z-2xz)dz=\int-\pi\sin\pi zdz=\cos\pi z$

Finally: Express $f(x,y,z)=k(x,y,z)+g(y,z)+h(z)$ using the results.

Therefore, $f(x,y,z)=x^2y+xz^2+\cos\pi z$.

Hence, by Fundamental Theorem of Line integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,3,1) - f(3,-1,1) = 13 + 7 = 20.$$

b) Clearly F is conservative. (Justify!). Then, find f such that $F = \nabla f$.

After some ups and downs we get $f(x,y,z) = x^2 y + \frac{y^2}{2} + z^2$.

Hence, by Fundamental Theorem of Line integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,2,4) - f(-1,4,0) = 20 - 12 = 8.$$

c) Clearly F is conservative. (Justify!). Then, find f such that $F = \nabla f$.

Here, we get $f(x,y,z) = xyz$.

Hence, by Fundamental Theorem of Line Integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,3,4) - f(1,2,3) = 24 - 6 = 18.$$

d) In this case the line integral is given in differential form $\int_C Mdx + Ndy$.

For such situation, conservative test is applied by assuming as the force is of the form $F = M\mathbf{i} + N\mathbf{j}$. In our case, $F(x,y) = x^3 y^4 \mathbf{i} + x^4 y^3 \mathbf{j}$.

Hence it is conservative. Here, we get $f(x,y) = \frac{1}{4} x^4 y^4$. Besides, find the end points of the curve. Observe that the end point of the curve corresponds to the end points of the parameter. That is $P = (0,3)$ at $t = 0, Q = (1,4)$ at $t = 1$

Hence, $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,4) - f(0,3) = 64 - 0 = 64.$

e) Here, $M_y = 3y^2, N_x = 3y^2 \Rightarrow M_y = N_x$.So, F is conservative.

Besides, its potential function is $f(x,y) = xy^3$.

Therefore, $\int_C F \cdot dr = f(2,4) - f(1,1) = 128 - 1 = 127$.

2. Given $F(x,y,z) = (2xy + 3z)\mathbf{i} + (x^2 - y^2)\mathbf{j} + (3x - 6e^{2z})\mathbf{k}$.

a) Show that F is conservative

b) Find the potential function of F

c) Evaluate $\int_C F.dr$ using FTLI where C is the curve from $(0,0,0)$ to $(1,3,1)$

Hence, by Fundamental Theorem of Line integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,3,1) - f(3,-1,1) = 13 + 7 = 20.$$

b) Clearly F is conservative. (Justify!). Then, find f such that $F = \nabla f$.

After some ups and downs we get $f(x,y,z) = x^2 y + \frac{y^2}{2} + z^2$.

Hence, by Fundamental Theorem of Line integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,2,4) - f(-1,4,0) = 20 - 12 = 8.$$

c) Clearly F is conservative. (Justify!). Then, find f such that $F = \nabla f$.

Here, we get $f(x,y,z) = xyz$.

Hence, by Fundamental Theorem of Line Integrals,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,3,4) - f(1,2,3) = 24 - 6 = 18.$$

d) In this case the line integral is given in differential form $\int_C Mdx + Ndy$.

For such situation, conservative test is applied by assuming as the force is of the form $F = M\mathbf{i} + N\mathbf{j}$. In our case, $F(x,y) = x^3 y^4 \mathbf{i} + x^4 y^3 \mathbf{j}$.

Hence it is conservative. Here, we get $f(x,y) = \frac{1}{4} x^4 y^4$. Besides, find the end points of the curve. Observe that the end point of the curve corresponds to the end points of the parameter. That is $P = (0,3)$ at $t = 0, Q = (1,4)$ at $t = 1$

Hence, $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,4) - f(0,3) = 64 - 0 = 64.$

e) Here, $M_y = 3y^2, N_x = 3y^2 \Rightarrow M_y = N_x$. So, F is conservative.

Besides, its potential function is $f(x,y) = xy^3$.

Therefore, $\int_C F \cdot dr = f(2,4) - f(1,1) = 128 - 1 = 127$.

2. Given $F(x,y,z) = (2xy + 3z)\mathbf{i} + (x^2 - y^2)\mathbf{j} + (3x - 6e^{2z})\mathbf{k}$.

a) Show that F is conservative

b) Find the potential function of F

c) Evaluate $\int_C F.dr$ using FTLI where C is the curve from $(0,0,0)$ to $(1,3,1)$

Solution:

a) **Test Conservativeness:** $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+3z & x^2-y^2 & 3x-6e^{2z} \end{vmatrix} = 0$

Therefore, the vector field F is conservative.

b) **Determine Potential Function:**

Variable separation: Suppose $f(x,y,z) = k(x,y,z) + g(y,z) + h(z)$.

First: Evaluate $k(x,y,z) = \int \mathrm{M}dx$.

That is $k(x,y,z) = \int \mathrm{M}dx = \int (2xy+3z)dx = x^2y + 3xz$.

Second: Evaluate $g(y,z) = \int (\mathrm{N} - k_y)dy$

That is $g(y,z) = \int (\mathrm{N} - k_y)dy = \int (x^2 - y^2 - x^2)dy = \int -y^2 dy = -\dfrac{y^3}{3}$.

Third: Evaluate $h(z) = \int (\mathrm{P} - k_z - g_z)dz$.

That is $h(z) = \int (\mathrm{P} - k_z - g_z)dz = \int (3x - 6e^{2z} - 3x)dz = \int -6e^{2z}dz = -3e^{2z}$

Finally: Express $f(x,y,z) = k(x,y,z) + g(y,z) + h(z)$ using the results.

Therefore, $f(x,y,z) = x^2y + 3xz - \dfrac{y^3}{3} - 3e^{2z}$.

c) By the Fundamental Theorem of Line integral (FTLI), we have

$$\int_C F.d\mathbf{r} = \int_{(0,0,0)}^{(1,3,1)} F.d\mathbf{r} = f(1,3,1) - f(0,0,0) = -3 - 3e^2 + 3 = -3e^2.$$

3. a) Verify $F = e^x \ln y\mathbf{i} + \dfrac{e^x}{y}\mathbf{j}$ is independent of path and evaluate $\displaystyle\int_{(0,1)}^{(0,e^2)} F.dr$

b) Evaluate $\displaystyle\int_{(1,0,2)}^{(0,2,3)} F.dr$ where $F(x,y,z) = 6xye^{x^2}\mathbf{i} + 3e^{x^2}\mathbf{j} + 3z^2\mathbf{k}$

c) Let $F(x,y,z) = \dfrac{x}{1+x^2+y^2+z^2}\mathbf{i} + \dfrac{y}{1+x^2+y^2+z^2}\mathbf{j} + \dfrac{z}{1+x^2+y^2+z^2}\mathbf{k}$.

Evaluate $\int_C F \cdot dr$ where the curve is $C: \mathbf{r}(t) = ti + t^2 j + t^4 k; 0 \le t \le 1$.

Solution:

a) Here, $M = e^x \ln y, N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = \frac{e^x}{y} = \frac{\partial N}{\partial x}$. So F is conservative.

Thus, by part (ii) of the above remark, the line integral is independent of path. Besides, the potential function is $f(x,y) = e^x \ln y$. Hence, by Fundamental Theorem of Line integral, $\int_{(0,1)}^{(0,e^2)} F.dr = f(0,e^2) - f(0,1) = 2 - 0 = 2.$

b) Clearly F is conservative. (Justify!). Here, we get $f(x,y,z) = 3ye^{x^2} + z^3$.

Hence, $\int_{(1,0,2)}^{(0,2,3)} F.dr = \int_{(1,0,2)}^{(0,2,3)} \nabla f.dr = f(0,2,3) - f(1,0,2) = 33 - 8 = 25.$

c) Clearly F is conservative with potential function

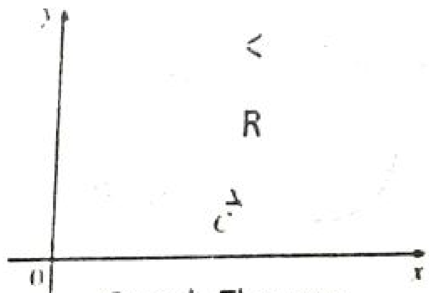
$f(x,y,z) = \frac{1}{2}\ln(1 + x^2 + y^2 + z^2)$ (Verify this!). Besides, from the parametric equation $\mathbf{r}(t) = ti + t^2 j + t^4 k; 0 \le t \le 1$, we have

$x(t) = t, y(t) = t^2, z(t) = t^4; 0 \le t \le 1 \Rightarrow P = (0,0,0)$, at $t = 0, Q = (1,1,1)$, at $t = 1$.

Therefore, $\int_C F \cdot dr = f(Q) - f(P) = f(1,1,1) - f(0,0,0) = \frac{1}{2}\ln 4 = \ln 2.$

3.7.4 Green's Theorem for Line Integrals

Let C be a simple closed piecewise smooth curve oriented counterclockwise and let R be the region enclosed by C as shown in the diagram below.



Green's Theorem

Let M and N be functions of two variables having continuous partial derivatives on R. Then, $\int_C M(x,y)dx + N(x,y)dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA$.

Alternatively, if $F(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ where M and N have continuous partial derivatives on R, then $\int_C F.dr = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA$.

Remarks:

i) **The conditions:** The curve C is *simple* means it does not intersect itself, *closed* means its initial and terminal points are the same (like a circle, a triangle), *piecewise smooth* means the parametric equations of the curve are differentiable except a finite number of points, *oriented counter clockwise* (CCW) means the region R is always lying to the left of the path while moving around C. Unless and otherwise stated assume the curve is oriented CCW.

ii) This Theorem states that the value of a line integral along a simple closed and oriented curve C is evaluated by a double integral over a simply connected plane region R enclosed by the boundary of C.

iii) One of the advantage of Green's Theorem is that we can evaluate the line integral $\int_C F \cdot dr$ without using the parameterization of the curve C. It is also useful to evaluate difficult line integrals that cannot be evaluated using the parameterization of the curve.

Examples:

1. Using Green's Theorem, evaluate

a) $\int_C (y^3 + e^{x^3} \sin x)dx + (x^3 + 3xy^2)dy$ where C is the path from $(0,0)$ to $(1,1)$ along $y = x^3$ and from $(1,1)$ to $(0,0)$ along $y = x$ oriented counter clockwise.

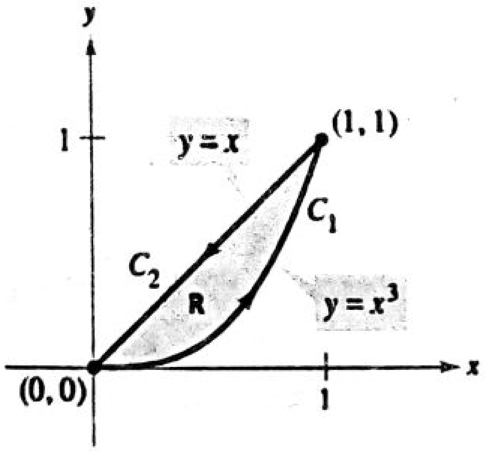
b) $\int_C (e^{x^2} + y^2)dx + (e^{y^2} + x^2)dy$ where C is the boundary of the region between the graphs of $y = x$ and $y = x^2$.

c) $\int_C \sin(x^2 + x)dx + 3x^2y^3dy$ where C is the triangle with vertices $(0,0), (2,0)$ and $(2,2)$ oriented counter clockwise.

d) $\int_C (\sin^{-1} x^2 - 4y^3)dx + (4x^3 + 27y^2)dy$ where C is the closed path along the semicircles of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ above the x-axis oriented CCW.

Solution: Always, to use Green's Theorem effectively, first identify the region R enclosed by the boundary of the curve C.

a) The region enclosed by the boundary of the curves in CCW is as follow.



$C = C_1 + C_2$

$$R: \begin{cases} x^3 \le y \le x \\ 0 \le x \le 1 \end{cases}$$

Hence, by Green's Theorem,

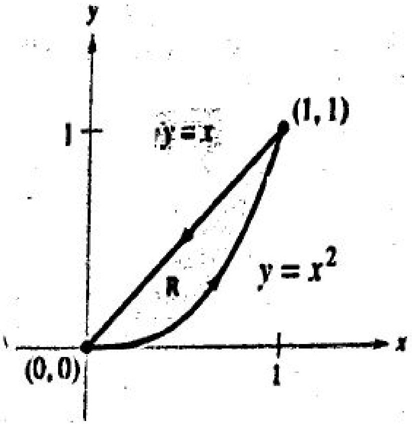
$$\int_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

$$\Rightarrow \int_C (y^3 + e^{x^3} \sin x)dx + (x^3 + 3xy^2)dy = \iint_R 3x^2 dA = 3\int_0^1 \int_{x^3}^{x} x^2 dy dx$$

$$= 3\int_0^1 (x^2 y)\Big|_{y=x^3}^{y=x} dx = 3\int_0^1 (x^3 - x^5)dx = 3\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{4}$$

b) First find the points where $y = x$ and $y = x^2$ intersect.

That is $x^2 = x \Rightarrow x = 0, x = 1$ and thus the region is as shown below.



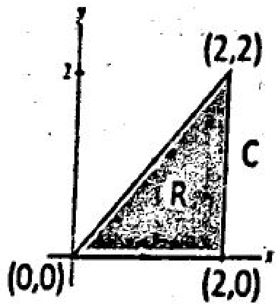
$$\int_C (e^{x^2} + y^2)dx + (e^{y^2} + x^2)dy = \iint_R (2x - 2y)dA$$

$$= \int_0^1 \int_{x^2}^{x} (2x - 2y)dy\,dx = \int_0^1 (2xy - y^2)\Big|_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - 2x^3 + x^4)dx = \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\Big|_{x=0}^{x=1}$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$$

c) The equation of the line joining the origin to $(2,2)$ is given by $y = x$.



The region Is described as:

$R: 0 \le x \le 2$

$\quad 0 \le y \le x$

The triangular path with the enclosed region .

Hence, by Green's Theorem,

$$\int_C \sin(x^2 + x)dx + 3x^2y^3dy = \iint_R 6xy^3 dA = 6\int_0^2 \int_0^x xy^3 dy\,dx = \frac{3}{2}\int_0^2 x^5 dx = 16$$

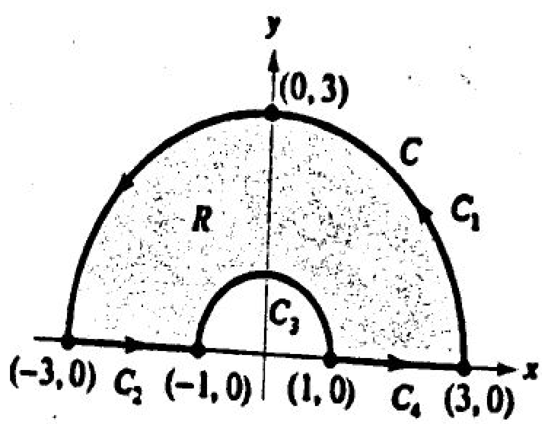
d) The region enclosed by the boundary of the two semicircles above the x-axis in CCW and its description in polar coordinates is as shown below.

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$R: \begin{cases} 0 \le \theta \le \pi \\ 1 \le r \le 3 \end{cases}$$

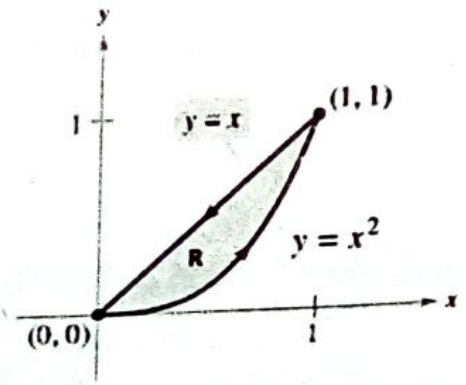
$x^2 + y^2 = r^2$

$dA = r\,dr\,d\theta$



b) First find the points where $y = x$ and $y = x^2$ intersect.

That is $x^2 = x \Rightarrow x = 0, x = 1$ and thus the region is as shown below.



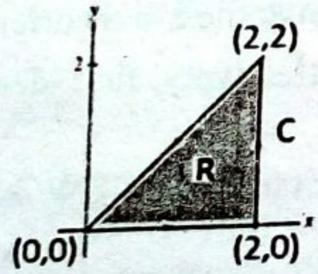
$$\int_C (e^{x^2} + y^2)dx + (e^{y^2} + x^2)dy = \iint_R (2x - 2y)dA$$

$$= \int_0^1 \int_{x^2}^{x} (2x - 2y)dydx = \int_0^1 (2xy - y^2)\Big|_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - 2x^3 + x^4)dx = \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\Big|_{x=0}^{x=1}$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$$

c) The equation of the line joining the origin to $(2,2)$ is given by $y = x$.



The region is described as:

$R: 0 \le x \le 2$

$0 \le y \le x$

The triangular path with the enclosed region .

Hence, by Green's Theorem,

$$\int_C \sin(x^2 + x)dx + 3x^2y^3dy = \iint_R 6xy^3 dA = 6\int_0^2\int_0^x xy^3 dydx = \frac{3}{2}\int_0^2 x^5 dx = 16$$

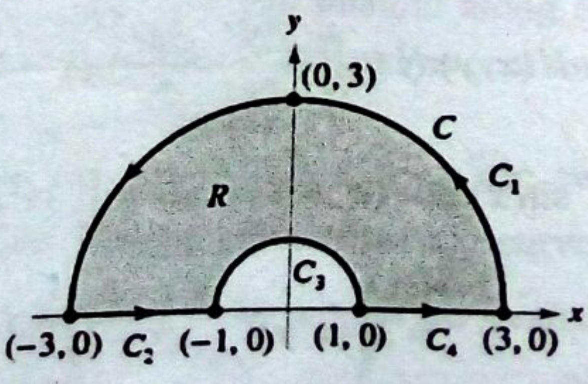
d) The region enclosed by the boundary of the two semicircles above the x-axis in CCW and its description in polar coordinates is as shown below.

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$R: \begin{cases} 0 \le \theta \le \pi \\ 1 \le r \le 3 \end{cases}$$

$x^2 + y^2 = r^2$

$dA = rdrd\theta$



Hence, by Green's Theorem,

$$\int_C (\sin^{-1} x^2 - 4y^3)dx + (4x^3 + 27y^2)dy = \iint_R (12x^2 + 12y^2)dA = 12\int_0^{\pi}\int_1^{3} r^3 dr d\theta$$

$$= 3\int_0^{\pi} r^4\Big|_{r=1}^{r=3} d\theta = 3\int_0^{\pi} 80 d\theta = 240\pi$$

2. Using Green's Theorem, evaluate

a) $\int_C (e^{e^x} - 2y)dx + \tan^{-1}(ye^{-y} + 1)dy$ where C is the ellipse $9x^2 + 4y^2 = 36$.

b) $\int_C \sqrt{1 + x^2 + x^4}\,dx + xe^{y^3} dy$ where C is the path for $0 \le x \le 1, \sqrt{x} \le y \le 1$

c) $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$ where C is the boundary of the region bounded by $y = x^2$ and $x = y^2$ oriented counterclockwise.

d) $\int_C y\tan^2 x dx + \tan x dy$ where C is the circle $x^2 + (y+1)^2 = 1$.

e) $\int_C -3x^2 y dx + 3xy^2 dy$ where C is the boundary of the region in the first quadrant bounded between the coordinate axes and the circle $x^2 + y^2 = 4$.

f) $\int_C 2y dx + 6x dy$ where C is the boundary of the region between the graphs of $y = x^2 - 1$ and $y = 1 - x$.

g) $\int_C y^2 dx + (x^2 + 2xy)dy$ where C is the triangle with vertices $(-1,0),(1,0),(0,1)$ oriented counter clockwise.

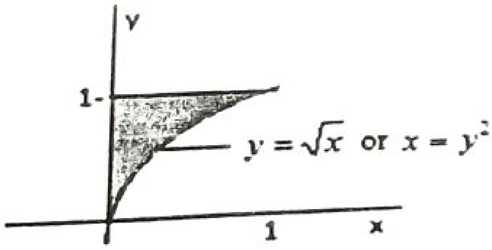
Solution:

a) $\int_C (e^{e^x} - 2y)dx + \tan^{-1}(ye^{-y} + 1)dy = \iint_R 2dA = 2(\text{Area}) = 12\pi$

b) $\int_C \sqrt{1 + x^2 + x^4}\,dx + xe^{y^3} dy = \iint_R e^{y^3} dA = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy dx$.

Here, if we need to evaluate the integral $\int_{\sqrt{x}}^1 e^{y^3} dy$ directly it is difficult.

So, changing the order of integration is a must.



From the diagram, to change the order of integration x has to run from $x=0$ to $x=y^2$ and y from $y=0$ to $y=1$.

Hence, $\int_C \sqrt{1+x^2+x^4}\,dx + xe^{y^3}dy = \iint_R e^{y^3}dA = \int_0^1\int_{\sqrt{x}}^1 e^{y^3}dydx = \int_0^1\int_0^{y^2} e^{y^3}dxdy$

$$= \int_0^1 xe^{y^3}\Big|_{x=0}^{x=y^2} dy = \int_0^1 y^2e^{y^3}dy = \frac{1}{3}(e-1)$$

c) Here, $M = y+e^{\sqrt{x}}, N = 2x+\cos y^2, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 2$.

Besides the intersection of the boundary curves $y=x^2, x=y^2$ is found to be $x=x^4 \Rightarrow x^4-x=0 \Rightarrow x(x^3-1)=0 \Rightarrow x=0, x=1$

Hence, the region of integration is described as $0\le x\le 1, x^2\le y\le \sqrt{x}$

Thus, $\int_C (y+e^{\sqrt{x}})dx + (2x+\cos y^2)dy = \int_0^1\int_{x^2}^{\sqrt{x}} dxdy = \int_0^1 (\sqrt{x}-x^2) = -\frac{1}{3}$

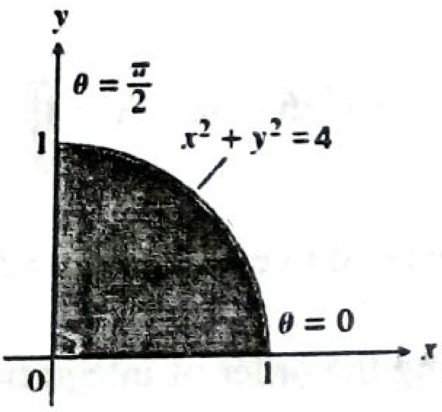
d) $\int_C y\tan^2 x dx + \tan x dy = \iint_R (\sec^2 x - \tan^2 x)dA = \iint_R 1dA = \pi$

e) The region between the coordinate axes (the x-and y-axes) and the given circle in the first quadrant is drawn as follow.

So. $\int_C -3x^2ydx + 3xy^2dy = \iint_R (3y^2+3x^2)dA$

$$= \int_0^{\frac{\pi}{2}}\int_0^2 3r^3drd\theta$$

$$= \int_0^{\frac{\pi}{2}} 12d\theta = 6\pi$$



f) First find the points where $y = x^2 - 1$ and $y = 1 - x$ intersect.

That is $x^2 - 1 = 1 - x \Rightarrow x^2 + x - 2 = 0, x = -2, x = 1$. Besides, by drawing the region, we get that $x^2 - 1 \le y \le 1 - x,\ -2 \le x \le 1$.

Hence, by Green's Theorem,

$$\int_C 2ydx + 6xdy = \iint_R 4dA = \int_{-2}^{1}\int_{x^2-1}^{1-x} 4dydx = 4\int_{-2}^{1}(2 - x - x^2)dx = 4(2x - \frac{x^2}{2} - \frac{x^3}{3})\Big|_{-2}^{1} = 18$$

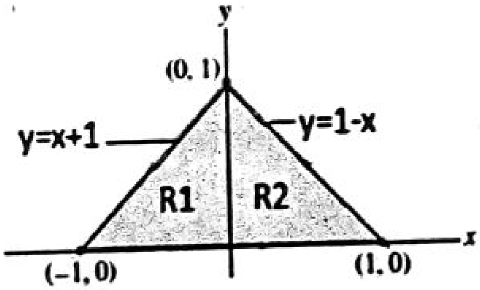
g) First draw the triangular region enclosed by the boundary of the triangle with the given vertices in CCW. The equation of the line joining $(-1,0)$ to $(0,1)$ is given by $y = x + 1$ and $(0,1)$ to $(1,0)$ is given by $y = 1 - x$ as shown below.

The region is described as a union of two simple regions as follow:

$R = R_1 \cup R_2$

$R_1 : -1 \le x \le 0, 0 \le y \le x + 1$

$R_2 : 0 \le x \le 1, 0 \le y \le 1 - x$



$$\int_C y^2dx + (x^2 + 2xy)dy = \iint_{R_1} 2xdA + \iint_{R_2} 2xdA = \int_{-1}^{0}\int_{0}^{x+1} 2xdydx + \int_{0}^{1}\int_{0}^{1-x} 2xdydx$$

$$= \int_{-1}^{0}(2x^2 + 2x)dx + \int_{0}^{1}(2x - 2x^2)dx = 0$$

3. Using Green's Theorem, evaluate

a) $\int_C (e^x \ln y - 4xy)dx + \frac{e^x}{y}dy$ where C is the boundary of the region bounded above by $y = 3 - x^2$ and below by $y = x^4 + 1$.

b) $\int_C \tan^{-1}\left(\frac{y}{x}\right)dx + \ln(x^2 + y^2)dy$ where C is the circular region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ in the first quadrant.

c) $\int_C (2y - e^{\sin x})dx + (6x + \sqrt{y^4 + 1})dy$ where C is the circle $x^2 + y^2 = 9$.

d) $\int_C 2y^3dx + (x^4 + 6xy^2)dy$ where C is the closed path bounded by $x^4 + y^4 = 1$ and the coordinate axes in the first quadrant oriented CCW.

Solution:

a) First find the points where $y = 3 - x^2$ and $y = x^4 + 1$ intersect. That is $3 - x^2 = x^4 + 1 \Rightarrow x^4 + x^2 - 2 = 0$. Using the substitution, $t = x^2$, we have $x^4 + x^2 - 2 = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = 1, t = -2$. Then, $t = x^2, t = 1 \Rightarrow x = \pm 1$.

Hence, by Green's Theorem,

$$\int_C (e^x \ln y - 4xy)dx + \frac{e^x}{y}dy = \iint_R 4x\,dA = \int_{-1}^{1}\int_{x^4+1}^{3-x^2} 4x\,dy\,dx = \int_{-1}^{1} 4(2x - x^3 - x^5)dx = 0$$

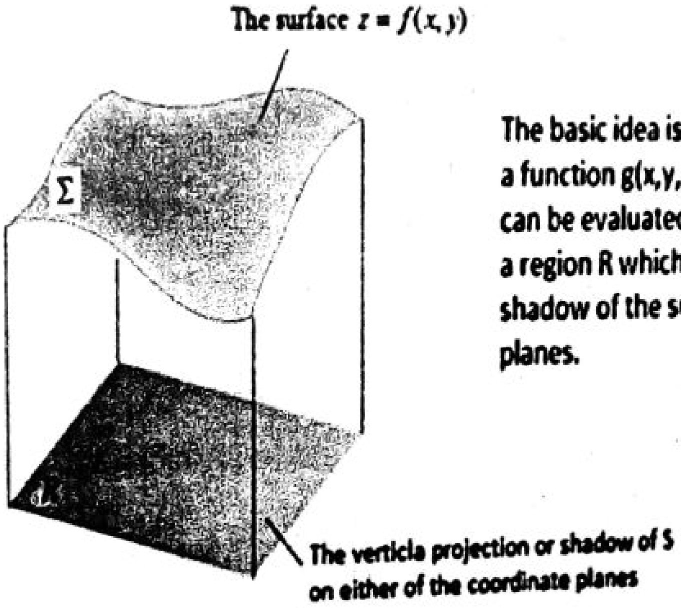
b) In the first quadrant the circular region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ is given in polar coordinates as $0 \le \theta \le \frac{\pi}{2}, 1 \le r \le 3$. Hence, by Green's Theorem; $\int_C \tan^{-1}\left(\frac{y}{x}\right)dx + \ln(x^2 + y^2)dy = \iint_R \frac{x}{x^2 + y^2}dA = \int_0^{\pi/2}\int_1^3 \cos\theta\,dr\,d\theta = 2$

c) $\int_C (2y - e^{\sin x})dx + (6x + \sqrt{y^4 + 1})dy = \iint_R 4dA = \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = 36\pi$.

d) $\int_C 2y^3dx + (x^4 + 6xy^2)dy = \iint_R 4x^3dA = \int_0^1\int_0^{\sqrt[4]{1-x^4}} 4x^3\,dy\,dx = \int_0^1 (1 - x^4)dx = \frac{4}{5}$.

3.8 Surface Integrals

Let Σ be the graph of a function having continuous partial derivative and defined on a region R in the xy plane that is composed of a finite number of vertically or horizontally simple regions as shown below.



The basic idea is that the surface integral of a function g(x,y,z) over a surface S in space can be evaluated using double integral over a region R which is the projection or shadow of the surface on the coordinate planes.

Let R be a simple region and suppose f has continuous partial derivative on R. If Σ is the graph of f on R, then the surface area of small differential element is given by $dS=\sqrt{f_x^{\,2}+f_y^{\,2}+1}\,dA$ .By taking integration, the total surface area S of Σ is obtained to be $S=\iint\limits_R \sqrt{f_x^{\,2}+f_y^{\,2}+1}\,dA$.

Case-1: If Σ is the surface with equation, $z=f(x,y)$ and R is its projection on the xy −plane, then $\iint\limits_{\Sigma} g(x,y,z)ds=\iint\limits_R g(x,y,z)\sqrt{f_x^{\,2}+f_y^{\,2}+1}\,dA$.

Case-2: If Σ is the surface with equation, $y=f(x,z)$ and R is its projection on the xz −plane, then $\iint\limits_{\Sigma} g(x,y,z)ds=\iint\limits_R g(x,y,z)\sqrt{f_x^{\,2}+f_z^{\,2}+1}\,dA$.

Case-3: If Σ is the surface with equation, $x=f(y,z)$ and R is its projection on the yz −plane, then $\iint\limits_{\Sigma} g(x,y,z)ds=\iint\limits_R g(x,y,z)\sqrt{f_y^{\,2}+f_z^{\,2}+1}\,dA$.

3.8 Surface Integrals

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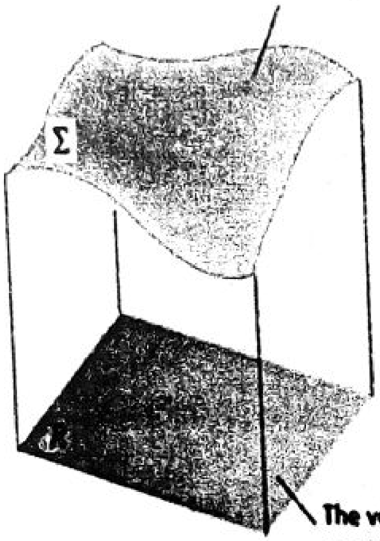

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Case-1: If Σ is the surface with equation, $z = f(x,y)$ and R is its projection on the xy-plane, then $\iint_\Sigma g(x,y,z)ds = \iint_R g(x,y,z)\sqrt{f_x^2 + f_y^2 + 1}\,dA$.

Case-2: If Σ is the surface with equation, $y = f(x,z)$ and R is its projection on the xz-plane, then $\iint_\Sigma g(x,y,z)ds = \iint_R g(x,y,z)\sqrt{f_x^2 + f_z^2 + 1}\,dA$.

Case-3: If Σ is the surface with equation, $x = f(y,z)$ and R is its projection on the yz-plane, then $\iint_\Sigma g(x,y,z)ds = \iint_R g(x,y,z)\sqrt{f_y^2 + f_z^2 + 1}\,dA$.

Examples:

1. Evaluate

a) $\iint_S 3xdS$ where S is part of $z = x^2 + y$ for $0 \le x \le 2, 0 \le y \le 1$.

b) $\iint_\Sigma \sqrt{4x^2 + 4y^2 + 1}dS$ where Σ is the paraboloid $z = x^2 + y^2$ below $z = 4$.

c) $\iint_\Sigma z^2 ds$ where Σ is the cone $z = \sqrt{x^2 + y^2}$ for which $1 \le \sqrt{x^2 + y^2} \le 3$.

d) $\iint_S (x^2 + y^2)z^{\frac{3}{2}}dS$ where S is portion of $x^2 + y^2 + z^2 = 4$ above $z = 1$.

e) $\iint_S ydS$ where S is the parabolic sheet $z = 4 - y^2$ for $0 \le x \le 12, 0 \le y \le 2$.

f) $\iint_S (1 + z)dS$ where S is portion of the cylinder $y^2 + z^2 = 4$ in the first octant between the planes $x = 0$ and $x = 3$.

g) $\iint_\Sigma zds$ where Σ is the part of the plane $x + y + z = 1$ in the first octant.

h) $\iint_S xdS$ where S is tetrahedron or triangular region with vertices $A(1,0,0), B(0,-2,0), C(0,0,4)$.

Solution:

a) Here, $f_x(x,y) = 2x, f_y(x,y) = 1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 2}$.

Hence,

$$\iint_S 3xds = 3\iint_R x\sqrt{4x^2 + 2}dA = 3\int_0^2 \int_0^1 x\sqrt{4x^2 + 2}dydx$$

$$= 3\int_0^2 (x\sqrt{4x^2 + 2})dx = 13\sqrt{2}$$

b) Let $z = f(x,y) = x^2 + y^2$ such that S is the graph of f below $z = 4$. The surface and the region of integration R (in the xy-plane) is drawn as shown.

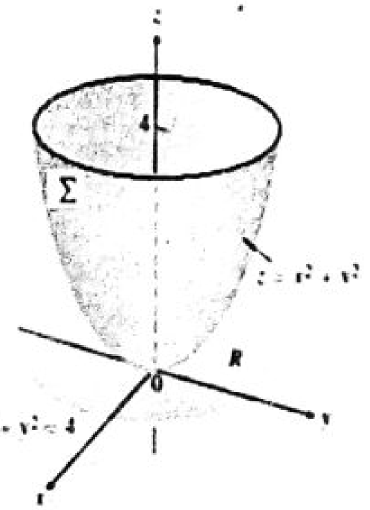
Here, $f_x(x,y)=2x, f_y(x,y)=2y \Rightarrow \sqrt{f_x^2+f_y^2+1}=\sqrt{4x^2+4y^2+1}$.

Hence, using polar coordinates

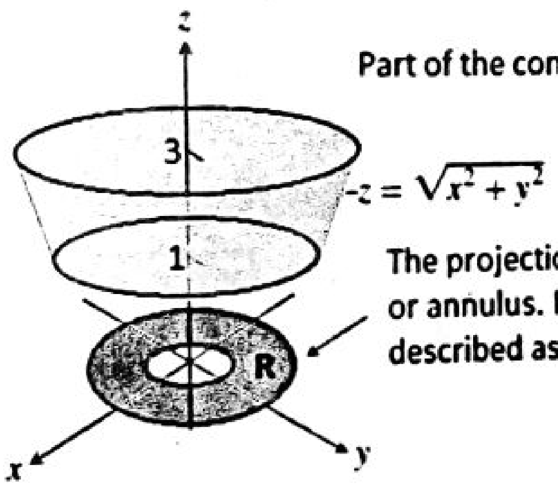
$$\iint_{\Sigma}\sqrt{4x^2+4y^2+1}\,ds=\iint_R \sqrt{4x^2+4y^2+1}\sqrt{f_x^2(x,y)+f_y^2(x,y)+1}\,dA$$

$$=\iint_R (4x^2+4y^2+1)\,dA=\int_0^{2\pi}\int_0^2 (4r^2+1)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(r^4+\frac{r^2}{2}\right)\Big|_0^2 d\theta=\int_0^{2\pi}18\,d\theta=18\theta\Big|_0^{2\pi}=36\pi$$



c) Let $z=f(x,y)=\sqrt{x^2+y^2}$ for which $1\le\sqrt{x^2+y^2}\le 3$ is as shown.



Part of the cone between the planes z=1 and z=3

The projection **R** in the xy-plane is a ring or annulus. In polar coordinates, it is described as:

$$\mathbf{R}:\begin{cases}0\le\theta\le 2\pi\\ 1\le r\le 3\end{cases}$$

$dA = r\,dr\,d\theta$

Then, $f_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}, f_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{f_x^2+f_y^2+1}=\sqrt{2}$.

Hence, using polar coordinates

$$\iint_{\Sigma} z^2\,ds=\iint_R\left(\sqrt{x^2+y^2}\right)^2\sqrt{2}\,dA=\sqrt{2}\iint_R (x^2+y^2)\,dA=\sqrt{2}\int_0^{2\pi}\int_1^3 (r^2)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\frac{r^4}{4}\Big|_1^3 d\theta=\sqrt{2}\left(\frac{81}{4}-\frac{1}{4}\right)\int_0^{2\pi}d\theta=20\sqrt{2}\,\theta\Big|_0^{2\pi}=40\sqrt{2}\pi$$

d) Portion of the sphere above the plane $z=1$ is $z=\sqrt{4-x^2-y^2}$.

Now, find the projection of $z=\sqrt{4-x^2-y^2}$ on the plane $z=1$.

That is $\sqrt{4-x^2-y^2}=1 \Rightarrow x^2+y^2=3$. It is a circle of radius $r=\sqrt{3}$.

$$\iint_S (x^2+y^2)z^{\frac{3}{2}}dS=\iint_R (x^2+y^2)(4-x^2-y^2)^{\frac{3}{2}}\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}dA$$

$$=2\iint_R (x^2+y^2)(4-x^2-y^2)dA=2\int_0^{2\pi}\int_0^{\sqrt{3}} r^2(4-r^2)rdrd\theta$$

$$=2\int_0^{2\pi}\int_0^{\sqrt{3}}(4r^3-r^5)drd\theta=2\int_0^{2\pi}\frac{9}{2}d\theta=18\pi$$

e) $f(x,y)=4-y^2 \Rightarrow f_x(x,y)=0, f_y(x,y)=-2y \Rightarrow \sqrt{f_x^{\,2}+f_y^{\,2}+1}=\sqrt{4y^2+1}$.

Hence, $\iint\limits_S yds=\iint\limits_R y\sqrt{4y^2+1}dA=\int_0^{12}\int_0^2 y\sqrt{4y^2+1}dydx$

$$=\frac{1}{12}\int_0^{12}(4y^2+1)^{3/2}\Big|_{y=0}^{y=2})dx=\frac{1}{12}\int_0^{12}(17\sqrt{17}-1)dx=17\sqrt{17}-1$$

f) Since $z\geq 0$ in the first octant, $y^2+z^2=4 \Rightarrow z=f(x,y)=\sqrt{4-y^2}$ for $0\leq y\leq 2$. Then, $f_x(x,y)=0, f_y(x,y)=\frac{y}{\sqrt{4-y^2}} \Rightarrow \sqrt{f_x^{\,2}+f_y^{\,2}+1}=\frac{2}{\sqrt{4-y^2}}$

But $f_y(x,y)=\frac{y}{\sqrt{4-y^2}}$ is not continuous at $y=2$ and thus we cannot evaluate the integral formula directly. Rather we use the concept of improper integrals.

Hence, $\iint\limits_S (1+z)ds=\iint\limits_R (1+\sqrt{4-y^2})\frac{2}{\sqrt{4-y^2}}dA=2\lim\limits_{u\to 2}\int_0^3\int_0^u(\frac{1}{\sqrt{4-y^2}}+1)dydx$

$$=2\lim_{u\to 2}\int_0^3[\sin^{-1}\left(\frac{u}{2}\right)+u]dx=6\lim_{u\to 2}[\sin^{-1}\left(\frac{u}{2}\right)+u]=3\pi+12$$

g) $\iint\limits_{\Sigma} zds=\iint\limits_R (1-x-y)\sqrt{f_x^{\,2}+f_y^{\,2}+1}dA=\sqrt{3}\int\limits_0^1\int\limits_0^{1-x}(1-x-y)dydx$

$$=\sqrt{3}\int\limits_0^1(1-2x+x^2-\frac{(1-x)^2}{2})dx=\frac{\sqrt{3}}{6}$$

h) First find the equation of the plane (tetrahedron) formed by the given points $(1,0,0),(0,-2,0),(0,0,4)$. Using vectors (the idea from *Applied I*), the normal vector to the plane is $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k\\ -1 & -2 & 0\\ -1 & 0 & 4\end{vmatrix}=-8i+4j-2k$. Then, using one of the points say $A(1,0,0)$, the equation of the plane is determined to be $-8(x-1)+4y-2z=0\Rightarrow 4x-2y+z=4$. Thus, using

$z=f(x,y)=4-4x+2y$, $f_x(x,y)=-4, f_y(x,y)=2\Rightarrow\sqrt{f_x^{\,2}+f_y^{\,2}+1}=\sqrt{21}$.

Besides, in the xy-plane the region is $R: 0\le x\le 1, 2x-2\le y\le 0$. Hence,

$$\iint_S x\,ds=\sqrt{21}\iint_R x\,dA=\sqrt{21}\int_0^1\int_{2x-2}^0 x\,dy\,dx=\sqrt{21}\int_0^1(2x-2x^2)dx=\frac{\sqrt{21}}{3}.$$

2. Evaluate $\iint_\Sigma g(x,y,z)ds$ where

a) $g(x,y,z)=\sqrt{4x^2+4y^2+1}$ and Σ is part of $z=x^2+y^2$ below $z=y$.

b) $g(x,y,z)=z(x^2+y^2)$ and S is the hemisphere given by $\sqrt{x^2+y^2+z^2}=2$.

c) $g(x,y,z)=4z^2$ where Σ is the portion $z=\sqrt{x^2+y^2}$ for $1\le x^2+y^2\le 4$

d) $g(x,y,z)=x$ and S is the cone $z=\sqrt{x^2+y^2}$ inside the cylinder $x^2+y^2=4$

e) $g(x,y,z)=3x^2+3y^2+3z^2$ and S is the portion of $x^2+y^2+z^2=16$ inside the cone $z=\sqrt{x^2+y^2}$

Solution:

a) Let $z=f(x,y)=x^2+y^2$ such that Σ is the graph of f below $z=y$. First let's find the intersection of $z=x^2+y^2$ with the plane $z=y$. That is

$x^2+y^2=y\Rightarrow x^2+y^2-y=0\Rightarrow x^2+(y-\frac{1}{2})^2=\frac{1}{4}$ with radius $r=\frac{1}{2}$.

Then, $f_x(x,y)=2x, f_y(x,y)=2y\Rightarrow\sqrt{f_x^{\,2}+f_y^{\,2}+1}=\sqrt{4x^2+4y^2+1}$.

Hence, using polar coordinates

$$\iint_{\Sigma} \sqrt{4x^2+4y^2+1}\,ds = \iint_R (4x^2+4y^2+1)\,dA = \int_0^{2\pi}\int_0^{\frac{1}{2}} (4r^2+1)\,r\,dr\,d\theta = \frac{3\pi}{8}$$

b) Here, $\sqrt{x^2+y^2+z^2} = 2 \Rightarrow z = f(x,y) = \sqrt{4-x^2-y^2}$

In cylindrical, $x = r\cos\theta, y = r\sin\theta, z = \sqrt{4-x^2-y^2} \Rightarrow z = \sqrt{4-r^2}$

$$\iint_S z(x^2+y^2)\,dS = \iint_R \sqrt{4-x^2-y^2}\,(x^2+y^2)\left(\frac{2}{\sqrt{4-x^2-y^2}}\right)dA$$

$$= 2\iint_R (x^2+y^2)\,dA = 2\int_0^{2\pi}\int_0^{2} r^3\,dr\,d\theta = 2\int_0^{2\pi} 4\,d\theta = 16\pi$$

c) Let R be the ring $1 \le x^2+y^2 \le 4$ and let $f(x,y) = \sqrt{x^2+y^2}$ for (x, y) in R

such that Σ is the graph of f on R.

Using polar coordinates, we have

$$\iint_{\Sigma} 4z^2\,ds = 4\sqrt{2}\iint_R (x^2+y^2)\,dA = 4\sqrt{2}\int_0^{2\pi}\int_1^{2} (r^2)\,r\,dr\,d\theta = 4\sqrt{2}\int_0^{2\pi} \frac{r^4}{4}\bigg|_1^2\,d\theta$$

$$= 4\sqrt{2}\left(4-\frac{1}{4}\right)\int_0^{2\pi} d\theta = 15\sqrt{2}\,\theta\Big|_0^{2\pi} = 30\sqrt{2}\pi$$

d) Here, $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}, f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{f_x^{\,2}+f_y^{\,2}+1} = \sqrt{2}$

$$\iint_S x\,dS = \iint_R x\sqrt{f_x^2+f_y^2+1}\,dA = \iint_R x\sqrt{2}\,dA = \sqrt{2}\iint_R x\,dA$$

In the xy-plane, the projection of $x^2+y^2=4$ is a circle of radius 2. So, using polar coordinates, the region is described as $R: 0 \le \theta \le 2\pi, 0 \le r \le 2$. Hence,

$$\iint_S x\,dS = \sqrt{2}\iint_R x\,dA = \sqrt{2}\int_0^{2\pi}\int_0^{2} r^2\cos\theta\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \cos\theta\,\frac{r^3}{3}\bigg|_0^2\,d\theta = \frac{8\sqrt{2}}{3}\int_0^{2\pi}\cos\theta\,d\theta = \frac{8\sqrt{2}}{3}\sin\theta\Big|_0^{2\pi} = 0$$

3. Let S be part of the paraboloid $z = x^2 + y^2$ within the cylinder $x^2 + y^2 = 1$ Then, evaluate $\iint_S dS$ and $\iint_S \sqrt{1+4z}\,dS$.

Solution: Here, $\iint_S dS$ simply represents the surface area of S.

$$\text{Thus, } \iint_S dS = \iint_R \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\,dA = \int_0^{2\pi}\int_0^1 (\sqrt{4r^2 + 1})r\,dr\,d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} (4r^2+1)^{3/2}\Big|_{r=0}^{r=1} d\theta = \frac{1}{12}(5\sqrt{5}-1)\int_0^{2\pi} d\theta = \frac{\pi}{6}(5\sqrt{5}-1)$$

In cylindrical coordinates, $z = x^2 + y^2 \Rightarrow z = r^2$

$$\iint_S \sqrt{1+4z}\,dS = \iint_R \sqrt{1+4z}\sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dA = \iint_R \sqrt{1+4r^2}\sqrt{4x^2 + 4y^2 + 1}\,dA$$

$$= \int_0^{2\pi}\int_0^1 (\sqrt{4r^2+1})(\sqrt{4r^2+1})r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (4r^3 + r)\,dr\,d\theta$$

$$= \int_0^{2\pi} (r^4 + \frac{r^2}{2})\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi$$

3.9 Surface Integrals over Oriented Surfaces

Oriented Surfaces and Types of Orientations:

In *Applied-II*, (*Hand Book-II, Section 3.8*), we have determined the equation of a tangent plane at a given point (x_0, y_0, z_0) to the surface $f(x, y, z) = 0$ using the normal vector $\mathbf{N} = f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}$.

Particularly, to the surface $z = f(x, y)$, the normal vector is given by $\mathbf{N} = -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$ or $f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$.

But either of these normal vectors may not be differentiable at all points.

If it is possible to find a normal vector $\mathbf{n}$ that varies continuously at any point (x, y, z) over the surface S (possibly except at the boundary of S), then S is said to be *oriented* or *two-sided* surface. As we explained above, there are two choices for the normal vector at any point (x, y, z),

$\mathbf{N} = -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$ or $f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$. So, either of these normals specifies the orientation of the surface S .Surfaces in which it is possible to give unit normal vectors at any points of the surface are said to be *orientable* surfaces. The unit normal vectors of the surface are then said to be *orientation* of the surface.

In general, surfaces can be oriented in different directions:

i) Surfaces oriented *up-ward* or *out-ward* will have positive **k** components.
In such cases the normal vectors are $\mathbf{N} = -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$.

ii) Surfaces oriented *down- ward* or *in-ward* will have negative **k** components.
In such cases the normal vectors are $\mathbf{N} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$.

iii) Surfaces oriented to the *right* will have positive **j** components.
In such cases the normal vectors are $\mathbf{N} = -f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$.

iv) Surfaces oriented by forward (backward) unit vectors will have positive (negative) **i** components respectively.

Examples: Spheres, cones, paraboloids, cylinders are orientable surfaces.

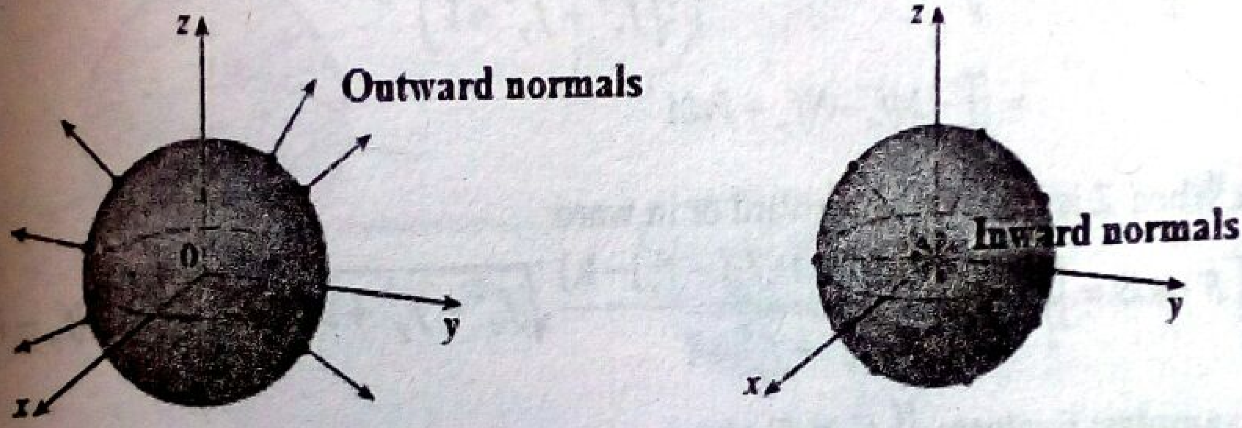
By convention, (unless otherwise specified), closed surfaces are assumed to have two types of orientation.

Types of Orientation:

Positive orientation: An orientation specified by outward normals.

Negative orientation: An orientation specified by inward normals. Particularly, for spheres the two types of orientations are as shown in the diagram below.

Possible oreintations of spheres



i) Positive orientation **ii) Negative orientaion**

Here, notice that for half of a sphere (a hemisphere), the orientation is upward for the upper hemispheres and downward for the lower hemispheres.

Surface Integrals over oriented surfaces:

Now, let's see what happens to the surface integral for oriented surfaces.

From the normal vectors, $\mathbf{N}=-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}$ and $\mathbf{N}=f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k}$ of the surface Σ, the unit normal $\mathbf{n}$ is $\mathbf{n}=\dfrac{-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}}{\sqrt{f_x^{\,2}+f_y^{\,2}+1}}$ or $\mathbf{n}=\dfrac{f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k}}{\sqrt{f_x^{\,2}+f_y^{\,2}+1}}$ depending on the orientation of Σ. When $\mathbf{n}$ is directed upward/outward (that is its k component is positive) , then $\mathbf{n}$ has the first form and when $\mathbf{n}$ is directed downward/inward $\mathbf{n}$ has the second form.

Besides, from the above discussion we have that $dS=\sqrt{f_x^{\,2}+f_y^{\,2}+1}\,dA$.

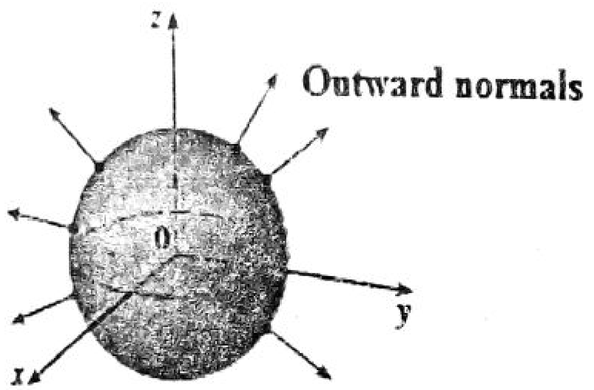
By convention, (unless otherwise specified), closed surfaces are assumed to have two types of orientation.

Types of Orientation:

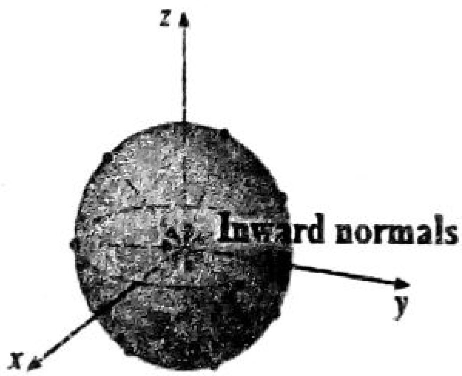
Positive orientation: An orientation specified by outward normals.

Negative orientation: An orientation specified by inward normals. Particularly, for spheres the two types of orientations are as shown in the diagram below.

Possible oreintations of spheres



i) Positive orientation



ii) Negative orientaion

Here, notice that for half of a sphere (a hemisphere), the orientation is upward for the upper hemispheres and downward for the lower hemispheres.

Surface Integrals over oriented surfaces:

Now, let's see what happens to the surface integral for oriented surfaces.

From the normal vectors, $\mathbf{N}=-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}$ and $\mathbf{N}=f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k}$ of the surface Σ, the unit normal $\mathbf{n}$ is $\mathbf{n}=\dfrac{-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}}{\sqrt{f_x^{\,2}+f_y^{\,2}+1}}$ or $\mathbf{n}=\dfrac{f_x\mathbf{i}+f_y\mathbf{j}-\mathbf{k}}{\sqrt{f_x^{\,2}+f_y^{\,2}+1}}$ depending on the orientation of Σ. When $\mathbf{n}$ is directed upward/outward (that is its k component is positive) , then $\mathbf{n}$ has the first form and when $\mathbf{n}$ is directed downward/inward $\mathbf{n}$ has the second form.

Besides, from the above discussion we have that $dS=\sqrt{f_x^{\,2}+f_y^{\,2}+1}\,dA$.

Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and let Σ be oriented surface given by $z = f(x,y)$. Suppose R is the projection of Σ on the $xy-$plane. Then, the integral given by $\iint_\Sigma F \cdot dS = \iint_\Sigma F \cdot \mathbf{n} dS$ is said to be *surface integral* (*flux integral*) of F over Σ.

Depending on orientation of Σ, the surface integral can be evaluated as follow:

i) When Σ is oriented upward or out ward:

$$\iint_\Sigma F \cdot \mathbf{n} dS = \iint_R (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot \left(\frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{f_x^{\,2} + f_y^{\,2} + 1}} \right) \sqrt{f_x^{\,2} + f_y^{\,2} + 1} dA$$

$$= \iint_R -Mf_x - Nf_y + PdA$$

ii) When Σ is oriented downward or in ward:

$$\iint_\Sigma F \cdot \mathbf{n} dS = \iint_R \frac{(M\mathbf{i} + N\mathbf{j} + P\mathbf{k}).(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})}{\sqrt{f_x^{\,2} + f_y^{\,2} + 1}} \sqrt{f_x^{\,2} + f_y^{\,2} + 1} dA = \iint_R Mf_x + Nf_y - PdA$$

Examples: Evaluate $\iint_\Sigma F \cdot \mathbf{n} dS$ where

a) $F(x,y,z) = 6z\mathbf{k}$, Σ is the sphere $x^2 + y^2 + z^2 = 9$ with outward normal.

b) $F(x,y,z) = y\mathbf{j} + \mathbf{k}$ and Σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, oriented by down ward normals.

c) $F(x,y,z) = y\mathbf{i} - x\mathbf{j} + 3\mathbf{k}$, Σ is the part of the paraboloid $z = 6 - x^2 - y^2$ above $z = 2 + x^2 + y^2$ oriented by upward normal.

d) $F(x,y,z) = y\mathbf{i} - x\mathbf{j} + 8\mathbf{k}$, Σ is part of the paraboloid $z = 9 - x^2 - y^2$ above the xy plane, $\mathbf{n}$ is directed upward.

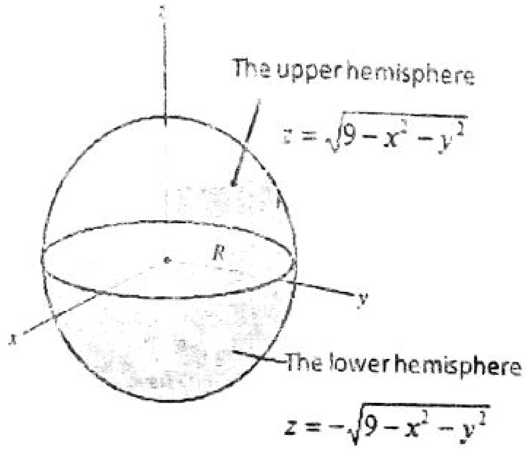
e) $F(x,y,z) = x\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$, Σ is the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = -2$ and $z = 1$ oriented by upward normal.

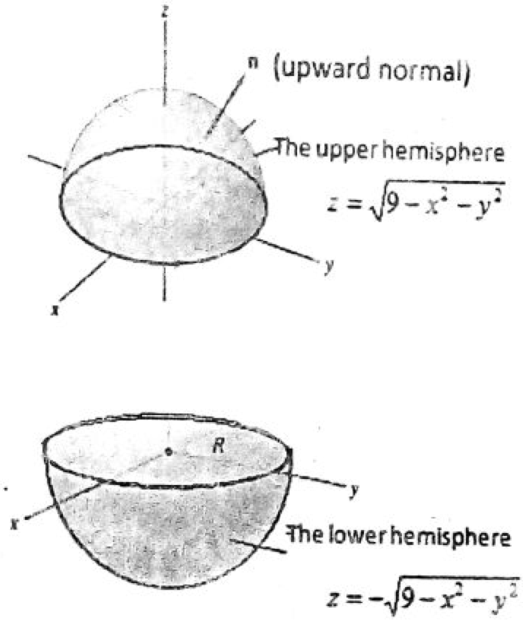
f) $F(x,y,z) = 2x\mathbf{i} - y\mathbf{j} + 6y^2\mathbf{k}$, Σ is the part of $z = 6 - x^2 - y^2$ above the plane $z = 2$ oriented by upward normal.

g) $F(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, Σ is part of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and $\mathbf{n}$ is directed upward.

Solution:

a) Refer the following diagrams. As we see from the diagrams, the surface of the sphere can be broken into two surfaces, one with the lower hemisphere and the other with the upper hemisphere.





Commonly, the normal vector on the lower hemisphere points downward and that of the upper hemisphere points upward. By symmetry, the two parts of the sphere gives the same surface integrals. So, let's evaluate over the upper hemisphere and multiply the result by 2 to get the total surface integral.

Here, $f(x,y) = \sqrt{9-x^2-y^2}, f_x(x,y) = \dfrac{-x}{\sqrt{9-x^2-y^2}}, f_y(x,y) = \dfrac{-y}{\sqrt{9-x^2-y^2}}$

Besides, $F(x,y,z) = 6z\mathbf{k} \Rightarrow M = 0, N = 0, P = 6z$ and the projection of the hemisphere on the xy plane is the circle $x^2 + y^2 = 9$.

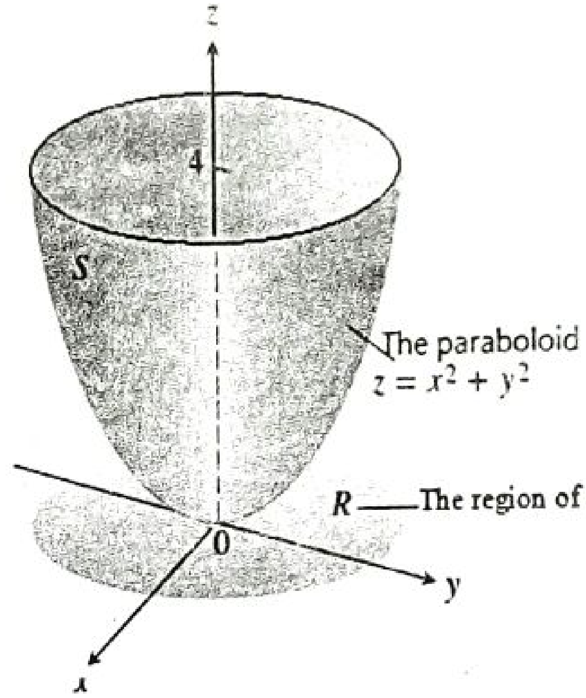
Therefore, $\iint\limits_{\Sigma} F \cdot n dS = \iint\limits_{R} \left(-Mf_x - Nf_y + P\right) dA = \iint\limits_{R} 6z dA$

$$= \iint\limits_{R} 6\sqrt{9-x^2-y^2}\, dA = 6\int_0^{2\pi}\int_0^3 \sqrt{9-r^2}\, r dr d\theta$$

$$= 6\int_0^{2\pi} -\frac{(9-r^2)^{3/2}}{3}\Bigg|_0^3 d\theta = 54\int_0^{2\pi} d\theta = 108\pi$$

Hence, the total surface integral is $I = 2(108\pi) = 216\pi$

b) Since the orientation is down ward, the normal vector is $n = f_x i + f_y j - k$ and the region in the plane $z = 4$ is the circle $x^2 + y^2 = 4$ as shown below.



In polr coordinates, the region is given by

$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$$

Besides, $f_x(x, y) = 2x, f_y(x, y) = 2y, M = 0, N = y, P = 1$.

$$\iint_{\Sigma} F \cdot n dS = \iint_R \left(Mf_x + Nf_y - P\right) dA = \iint_R (2y^2 - 1) dA$$

$$= \int_0^{2\pi} \int_0^2 (2r^2 \sin^2 \theta - 1) r dr d\theta = \int_0^{2\pi} (\sin^2 \theta \frac{r^4}{2} - \frac{r^2}{2}) \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} (8\sin^2 \theta - 2) d\theta = \int_0^{2\pi} (4 - 4\cos 2\theta - 2) d\theta = 4\pi$$

c) Here, $z = f(x, y) = 6 - x^2 - y^2 \Rightarrow f_x(x, y) = -2x, f_y(x, y) = -2y,$

$F(x, y, z) = y\mathbf{i} - x\mathbf{j} + 3\mathbf{k} \Rightarrow M(x, y, z) = y, N(x, y, z) = -x, P(x, y, z) = 3$

Besides, the intersection of $z = 6 - x^2 - y^2$ and $z = 2 + x^2 + y^2$ is obtained as $6 - x^2 - y^2 = 2 + x^2 + y^2 \Rightarrow x^2 + y^2 = 2$ which is a circle of radius $r = \sqrt{2}$.

So, $\iint_{\Sigma} F \cdot n dS = \iint_R \left(-Mf_x - Nf_y + P\right) dA = \iint_R \left(2xy - 2xy + 3\right) dA = 3\iint_R dA$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r dr d\theta = 3\int_0^{2\pi} \frac{r^2}{2} \Big|_0^{\sqrt{2}} d\theta = 3\int_0^{2\pi} d\theta = 3(2\pi) = 6\pi$$

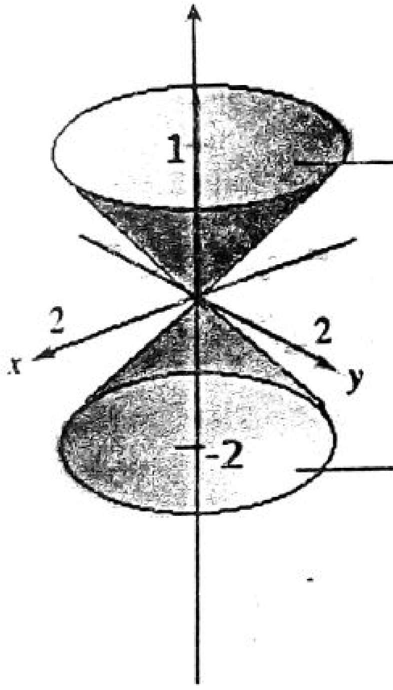
d) Here, $z = f(x,y) = 9 - x^2 - y^2 \Rightarrow f_x(x,y) = -2x, f_y(x,y) = -2y,$

$F(x,y,z) = y\mathbf{i} - x\mathbf{j} + 8\mathbf{k} \Rightarrow M(x,y,z) = y, N(x,y,z) = -x, P(x,y,z) = 8$

Besides, the projection of $z = 9 - x^2 - y^2$ on the xy- plane is $x^2 + y^2 = 9$.

So, $\iint_{\Sigma} F \cdot n dS = \iint_R (2xy - 2xy + 8) dA = 8\iint_R dA = \int_0^{2\pi}\int_0^3 r dr d\theta = 72\pi$

e) $z^2 = x^2 + y^2 \Rightarrow z = \pm\sqrt{x^2 + y^2}$. But the portion of the cone $z^2 = x^2 + y^2$ between $z = -2$ and $z = 1$ is $z = -\sqrt{x^2 + y^2}$ on $-2 \le z \le 0$ and $z = \sqrt{x^2 + y^2}$ on $0 \le z \le 1$ as shown in the diagram below.



The upper cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$

Region of integration $R_1 : \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$

The lower cone $z = -\sqrt{x^2 + y^2}$ above the plane $z = -2$

Region of integration $R_2 : \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$

Thus, $f_x(x,y) = \dfrac{x}{\sqrt{x^2 + y^2}}, f_y(x,y) = \dfrac{y}{\sqrt{x^2 + y^2}}$ on $0 \le z \le 1$ and

$f_x(x,y) = \dfrac{-x}{\sqrt{x^2 + y^2}}, f_y(x,y) = \dfrac{-y}{\sqrt{x^2 + y^2}}$ on $-2 \le z \le 0$.

Besides, $z = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$, and

$z = -2 \Rightarrow -\sqrt{x^2 + y^2} = -2 \Rightarrow x^2 + y^2 = 4$.

Hence, we have two regions described by $R_1 : 0 \le \theta \le 2\pi, 0 \le r \le 1$ and

$R_2 : 0 \le \theta \le 2\pi, 0 \le r \le 2$ such that the surface integral is the sum of the integrals over these two regions. Therefore,

$$\iint_{\Sigma} F \cdot n dS = \iint_{R_1} (\frac{-x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 4z)dA + \iint_{R_2} (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + 4z)dA$$

$$= \iint_{R_1} [\frac{-(x^2+y^2)}{\sqrt{x^2+y^2}} + 4\sqrt{x^2+y^2}]dA + \iint_{R_2} (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + 4(-\sqrt{x^2+y^2})dA$$

$$= \iint_{R_1} 3\sqrt{x^2+y^2} dA - \iint_{R_2} 3\sqrt{x^2+y^2})dA$$

$$= \int_0^{2\pi}\int_0^1 3r^2 dr d\theta - \int_0^{2\pi}\int_0^2 3r^2 dr d\theta = \int_0^{2\pi} d\theta - \int_0^{2\pi} 8d\theta = -14\pi$$

f) Here, $z = f(x,y) = 6 - x^2 - y^2 \Rightarrow f_x(x,y) = -2x, f_y(x,y) = -2y.$

and $F(x,y,z) = 2x\mathbf{i} - y\mathbf{j} + 6y^2\mathbf{k} \Rightarrow M = 2x, N = -y, P = 6y^2$

Besides, the intersection of $z = 6 - x^2 - y^2$ and $z = 2$ is obtained as

$6 - x^2 - y^2 = 2 \Rightarrow x^2 + y^2 = 4$ which is a circle of radius $r = 2$.

So, $$\iint_{\Sigma} F \cdot n dS = \iint_R \left(-Mf_x - Nf_y + P\right)dA = \iint_R \left(4x^2 - 2y^2 + 6y^2\right)dA$$

$$= \iint_R (4x^2 + 4y^2 dA = \int_0^{2\pi}\int_0^2 4r^3 dr d\theta = \int_0^{2\pi} 16 d\theta = 32\pi$$

g) $z = f(x,y) = \sqrt{9 - x^2 - y^2} \Rightarrow f_x(x,y) = \dfrac{-x}{\sqrt{9-x^2-y^2}}, f_y(x,y) = \dfrac{-y}{\sqrt{9-x^2-y^2}}$

But the partial derivatives are not continuous at $r = 3$ and thus we use the concept of improper integrals. Beside, the projection of $z = \sqrt{9-x^2-y^2}$ on the xy plane becomes $x^2 + y^2 = 9$. So, using polar coordinates, we get

$$\iint_{\Sigma} F \cdot n dS = \iint_R \left(\frac{x^2+y^2}{\sqrt{9-x^2-y^2}} + \sqrt{9-x^2-y^2}\right)dA = \iint_R \frac{9}{\sqrt{9-x^2-y^2}} dA$$

$$= \lim_{u \to 3^-} \int_0^{2\pi}\int_0^u \frac{9}{\sqrt{9-r^2}} r dr d\theta = 9 \lim_{u \to 3^-} \int_0^{2\pi} (-\sqrt{9-r^2})\Big|_0^u d\theta = 54\pi$$

3.10 Stokes's and Gauss's Divergence Theorems

3.10.1 Stokes's Theorem (Relates Line and Surface integrals)

Let Σ be an oriented surface with unit normal **n** and finite surface area. Assume Σ is bounded by closed piecewise smooth curve C whose orientation is induced by Σ. Let F be a continuous vector field defined on Σ and assume that the component functions of F have continuous partial derivatives at each non boundary points of Σ. Then, $\int_C F.dr = \iint_\Sigma (curlF).\mathbf{n}dS = \iint_\Sigma (\nabla \times F).\mathbf{n}dS$.

Examples:

1. Using Stokes's Theorem, evaluate $\int_C F.dr$,where

a) $F(x,y,z) = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$ and C is boundary of the upper hemisphere of the sphere $x^2+y^2+z^2=9$ oriented by upward normal.

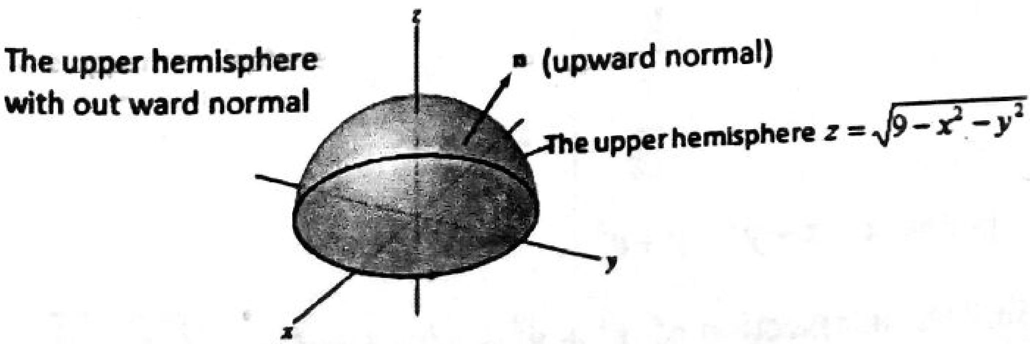
b) $F(x,y,z) = 3y^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y+z=3$ and the cylinder $x^2+y^2=1$ oriented counterclockwise.

c) $F(x,y,z) = (z+\sin x)\mathbf{i} + (x+y^2)\mathbf{j} + (y+e^z)\mathbf{k}$ and C is the intersection of the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$ oriented CCW.

d) $F(x,y,z) = (x+y^2)\mathbf{i} + (y+z^2)\mathbf{j} + (z+x^2)\mathbf{k}$ and C is the triangle with vertices $(1,0,0),(0,1,0),(0,01)$ oriented counterclockwise.

Solution:

a) The boundary of the upper hemisphere (the upper half) means part of the sphere above the xy-plane (when $z \geq 0$) as shown in the diagram below.



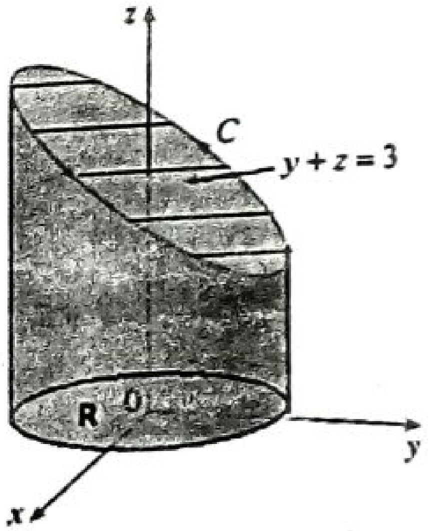
$$\text{Then, } curlF = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k} \text{ and upper half of the sphere}$$

$x^2+y^2+z^2=9$ is $z=f(x,y)=\sqrt{9-x^2-y^2}$

$$\int_c F.dr = \iint_{\Sigma}(curlF).\mathbf{n}dS = \iint_R (\mathbf{k}).\left(-f_x(x,y)\mathbf{i}-f_y(x,y)\mathbf{j}+\mathbf{k}\right)dA$$

$$= \iint_R \mathbf{k}.\mathbf{k}dA = \iint_R dA = \int_0^{2\pi}\int_0^{3} r\,dr\,d\theta = \int_0^{2\pi}\frac{9}{2}d\theta = 9\pi$$

b) The intersection of the cylinder and the palne is as shown.



$$curlF = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2 & 2x & z^2 \end{vmatrix} = (2-6y)\mathbf{k}$$

Here, $z=f(x,y)=3-y$

$\Rightarrow f_x(x,y)=0,\ f_y(x,y)=-1$

$$\int_c F.dr = \iint_{\Sigma}(curlF).\mathbf{n}dS = \iint_R (2-6y)\mathbf{k}.\left(-f_x(x,y)\mathbf{i}-f_y(x,y)\mathbf{j}+\mathbf{k}\right)dA$$

$$= \iint_R (2-6y)dA = \int_0^{2\pi}\int_0^{1}(2r-6r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}(1-2\sin\theta)d\theta = 2\pi$$

$$\text{c) } \mathrm{curl}F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z+\sin x & x+y^2 & y+e^z \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k}$$

Besides, find the intersection of $x^2+y^2+z^2=1$ and $z=\sqrt{x^2+y^2}$.

That is $x^2+y^2+z^2=1 \Rightarrow x^2+y^2=\frac{1}{2}$ which is a with radius $r=\frac{1}{\sqrt{2}}$.

So, by taking the surface to be $z=f(x,y)=\frac{1}{\sqrt{2}}$, $f_x(x,y)=0, f_y(x,y)=0$.

Hence, $\int_c F.dr=\iint_\Sigma (curlF).\mathbf{n}ds$

$$=\iint_R (\mathbf{i}+\mathbf{j}+\mathbf{k}).\left(-f_x(x,y)\mathbf{i}-f_y(x,y)\mathbf{j}+\mathbf{k}\right)dA$$

$$=\iint_R dA=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} rdrd\theta=\int_0^{2\pi}\frac{1}{4}d\theta=\frac{\pi}{2}$$

d) Very important Tip! Always if vertices of a rianle or some plane are given , first find the equation of the plane formed by the given vertices.

In our case, we are given $(1,0,0),(0,1,0),(0,0,1)$.

Using vectors (the idea from *Applied I*), the normal vector to the plane is

$\overline{AB}\times\overline{AC}=\begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}=i+j+k$. Then, using A(1,0,0), the equation of the plane is determined to be $(x-1)+y+z=0 \Rightarrow x+y+z=1$.

Thus, $z=f(x,y)=1-x-y \Rightarrow f_x(x,y)=-1, f_y(x,y)=-1$

Here, $curlF=-2z\mathbf{i}-2x\mathbf{j}-2y\mathbf{k}$

$$\int_c F.dr=\iint_R (-2z\mathbf{i}-2x\mathbf{j}-2y\mathbf{k}).\left(-f_x(x,y)\mathbf{i}-f_y(x,y)\mathbf{j}+\mathbf{k}\right)dA$$

$$=\iint_R (-2z-2x-2y)dA=\iint_R -2dA$$

$$=-2\int_0^1\int_0^{1-x} dydx=-2\int_0^1 (1-x)dx=-1$$

2. Using Stokes's Theorem, evaluate $\int_C F.dr$,where

a) $F(x,y,z)=-y^2\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ and C is the oriented triangle lying in the plane $2x+2y+z=6$ with vertices on the coordinate axes, with upward normal.

b) $F(x,y,z)=-3y^2\mathbf{i}+4z\mathbf{j}+6x\mathbf{k}$ and C is the triangle in the plane $z=\frac{1}{2}y$ with vertices $(2,0,0),(0,2,1),(0,0,0)$ oriented counterclockwise.

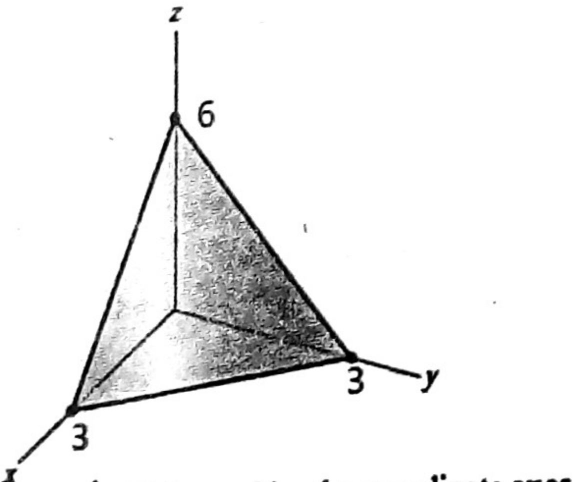
c) $F(x,y,z)=(x^2+z)\mathbf{i}+(y^2+x)\mathbf{j}+(z^2+y)\mathbf{k}$ and C is the intersection of the cone $z=3\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=40$ oriented with outward.

d) $F(x,y,z)=2z\mathbf{i}+x\mathbf{j}+y^2\mathbf{k}$ and Σ is part of the paraboloid $z=4-x^2-y^2$ above the xy plane oriented by upward normal.

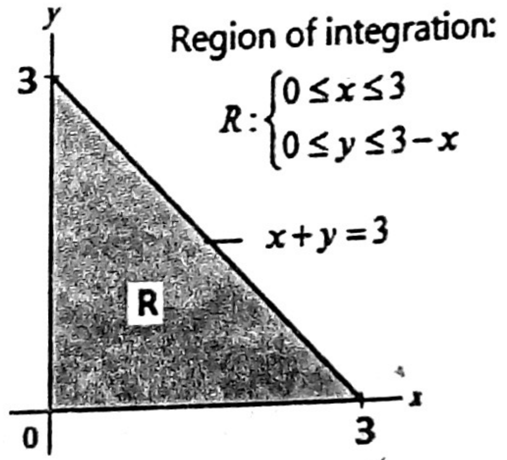
Solution:

a) $curlF=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y^2 & z & x\end{vmatrix}=-\mathbf{i}-\mathbf{j}+2y\mathbf{k}$ and let $z=f(x,y)=6-2x-2y$.

Then, the solid and the region of integration are as shown.



The solid formed by the coordinate axes and the plane $2x+2y+z=6$



The region of integration in the xy-plane

Therefore, $\int_c F.dr=\iint_\Sigma (curlF).\mathbf{n}dS=\int_0^3\int_0^{3-x}(2y-4)dydx=\int_0^3(x^2-2x-3)dx=-9$

b) Here, $curlF = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = 4\mathbf{i} - 6\mathbf{j} + 6y\mathbf{k}$

Thus, using the given plane $z = f(x,y) = \frac{1}{2}y$, $f_x(x,y) = 0, f_y(x,y) = \frac{1}{2}$.

Besides, in the xy-plane the region is $R: 0 \le x \le 2, 0 \le y \le 2-x$. (How?)

Then, by Stokes's Theorem, we have

$$\int_c F.dr = \iint_\Sigma (curlF).\mathbf{n}ds = \iint_R (4\mathbf{i} - 6\mathbf{j} + 6y\mathbf{k}).\left(-f_x(x,y)\mathbf{i} - f_y(x,y)\mathbf{j} + \mathbf{k}\right)dA$$

$$= \iint_R (3+6y)dA = \int_0^2 \int_0^{2-x} (3+6y)dydx = \int_0^2 (18 - 15x + 3x^2)dx = 14$$

c) $curlF = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and let $z = 3\sqrt{x^2+y^2} = f(x,y)$.

Now, to find the limits of integration from their intersection.

Then, $x^2 + y^2 + z^2 = 40 \Rightarrow x^2 + y^2 + 9x^2 + 9y^2 = 40 \Rightarrow x^2 + y^2 = 4$

This means the intersection is a circle of radius 2. Thus,

$$\int_c F.dr = \iint_R (curlF).\mathbf{n}ds = \iint_R (\mathbf{i} + \mathbf{j} + \mathbf{k}).\left(\frac{-3x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{-3y}{\sqrt{x^2+y^2}}\mathbf{j} + \mathbf{k}\right)dA$$

$$= \iint_R \left(\frac{-3x}{\sqrt{x^2+y^2}} + \frac{-3y}{\sqrt{x^2+y^2}} + 1\right)dA = \lim_{u \to 0} \int_0^{2\pi} \int_u^2 (1 - 3\cos\theta - 3\sin\theta)rdrd\theta$$

$$= 2\int_0^{2\pi} (1 - 3\cos\theta - 3\sin\theta)d\theta = 2(\theta - 3\sin\theta + 3\cos\theta)\Big|_0^{2\pi} = 4\pi$$

d) Here, $z = f(x,y) = 4 - x^2 - y^2$ and $curlF = 2y\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$\int_c F.dr = \iint_R (4xy + 4y + 1)dA = \int_0^{2\pi} \int_0^2 (4r^2\cos\theta\sin\theta + 4r\sin\theta + 1)rdrd\theta$$

$$= \int_0^{2\pi} (16\cos\theta\sin\theta + \frac{32}{3}\sin\theta + 2)d\theta = 4\pi$$

Remark: Stokes's Theorem relates the line integral with the double integral of the normal component of the curl of a vector field. But sometimes, it is simpler to evaluate the line integral as compared to the double integral. That is one of the advantage of this Theorem is to evaluate complex surface integrals by using the corresponding line integrals.

Examples: Using Stokes's Theorem, evaluate

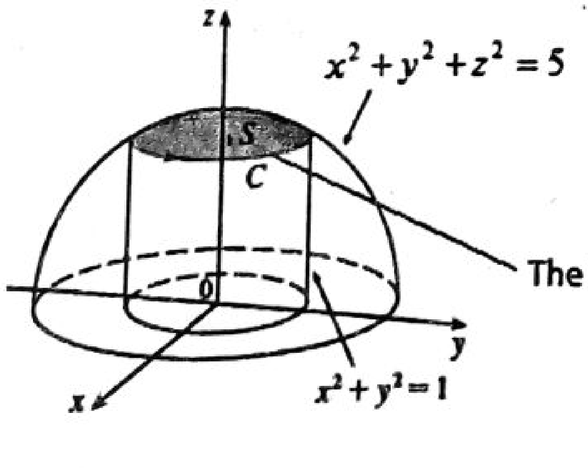
a) $\iint_S curlF.ndS$,where $F(x,y,z)=xz\mathbf{i}+yz\mathbf{j}+xy\mathbf{k}$ and S is the part of the sphere $x^2+y^2+z^2=5$ inside the cylinder $x^2+y^2=1$ above the xy-plane.

b) $\iint_S (\nabla\times F).\mathbf{n}dS$ where $F(x,y,z)=zx^2\,\mathbf{i}+(ze^{xy^2}-x)\mathbf{j}+x\ln y^2\mathbf{k}$ and S is the portion of $z=1-x^2-y^2$ above the xy-plane.

c) $\iint_S curlF.ndS$,where $F(x,y,z)=x^2z^2\mathbf{i}+y^2z^2\mathbf{j}+xyz\mathbf{k}$ and S is the part of $z=x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$ oriented upward.

Solution:

a) To find the boundary curve first solve the intersections of $x^2+y^2+z^2=5$ and $x^2+y^2=1$. Subtracting $x^2+y^2=1$ from $x^2+y^2+z^2=5$ gives $z^2=4\Rightarrow z=\pm 2$. But above the xy-plane is $z=2$.



In parametric form, $C: x=\cos t, y=\sin t, z=2, 0\le t\le 2\pi$ .Hence,

$$\iint_S (curlF).\mathbf{n}ds=\int_C F.dr=\int_0^{2\pi}(-2\cos t\sin t+2\sin t\cos t)dt=\int_0^{2\pi}0\,dt=0$$

b) By Stokes's Theorem, $\iint_S (\nabla \times F).\mathbf{n}dS = \int_C F.dr$ where C is the boundary of the surface above the xy-plane.

But the boundary is the circle $x^2 + y^2 = 1$ which can be parametrized as $x = \cos t, y = \sin t, z = 0,\ 0 \le t \le 2\pi \Rightarrow dx = -\sin t dt, dy = \cos t dt, dz = 0$.

Hence,

$$\iint_S (\nabla \times F).\mathbf{n}dS = \int_C F.dr = \int_C zx^2\,dx + (ze^{xy^2} - x)dy + x\ln y^2 dz$$

$$= \int_0^{2\pi} (0.\cos^2 t(-\sin t)dt + (0.e^{\cos t\sin^2 t} - \cos t)\cos t dt + (\cos t \ln \sin^2 t).0$$

$$= \int_0^{2\pi} -\cos^2 t dt = -\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)dt = -\frac{1}{2}\int_0^{2\pi}(1+\cos 2t)dt = -\pi$$

c) By Stokes's Theorem, $\iint_S curlF.\mathbf{n}dS = \int_C F.dr$ where C is the boundary of the intersection of $z = x^2 + y^2$ and $x^2 + y^2 = 4$.

The boundary is the circle $x^2 + y^2 = 4$ on a plane $z = 4$. It can be parametrized as $x = 2\cos t, y = 2\sin t, z = 4, 0 \le t \le 2\pi \Rightarrow dx = -2\sin t dt, dy = 2\cos t dt, dz = 0$

Hence, $\iint_S curlF.\mathbf{n}dS = \int_C F.dr = \int_C x^2z^2dx + y^2z^2dy + xyzdz$

$$= \int_0^{2\pi}(-128\cos^2 t\sin t + 128\sin^2 t\cos t)dt$$

$$= 128(\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3})\Big|_0^{2\pi} = 0$$

b) By Stokes's Theorem, $\iint_S (\nabla \times F).\mathbf{n}dS = \int_C F.dr$ where C is the boundary of the surface above the xy-plane.

But the boundary is the circle $x^2 + y^2 = 1$ which can be parametrized as

$x = \cos t, y = \sin t, z = 0,\ 0 \le t \le 2\pi \Rightarrow dx = -\sin t dt, dy = \cos t dt, dz = 0$.

Hence,

$$\iint_S (\nabla \times F).\mathbf{n}dS = \int_C F.dr = \int_C zx^2\, dx + (ze^{xy^2} - x)dy + x\ln y^2 dz$$

$$= \int_0^{2\pi} (0.\cos^2 t(-\sin t)dt + (0.e^{\cos t \sin^2 t} - \cos t)\cos t dt + (\cos t \ln \sin^2 t).0$$

$$= \int_0^{2\pi} -\cos^2 t dt = -\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt = -\frac{1}{2}\int_0^{2\pi}(1+\cos 2t)dt = -\pi$$

c) By Stokes's Theorem, $\iint_S curlF.\mathbf{n}dS = \int_C F.dr$ where C is the boundary of the intersection of $z = x^2 + y^2$ and $x^2 + y^2 = 4$.

The boundary is the circle $x^2 + y^2 = 4$ on a plane $z = 4$. It can be parametrized as $x = 2\cos t, y = 2\sin t, z = 4, 0 \le t \le 2\pi \Rightarrow dx = -2\sin t dt, dy = 2\cos t dt, dz = 0$

Hence, $\iint_S curlF.\mathbf{n}dS = \int_C F.dr = \int_C x^2z^2dx + y^2z^2dy + xyzdz$

$$= \int_0^{2\pi}(-128\cos^2 t\sin t + 128\sin^2 t\cos t)dt$$

$$= 128\left(\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3}\right)\Big|_0^{2\pi} = 0$$

3.10.2 Gauss's Divergence Theorem

Let D be a simple solid region whose boundary surface Σ is oriented by the normal directed outward from D and let F be a vector field whose component functions have continuous partial derivatives on D.

Then $\iint_{\Sigma} F \cdot n dS = \iiint_{D} divF(x,y,z)dV$.

Examples:

1. Use the Divergence Theorem to evaluate $\iint_{\Sigma} F \cdot n dS$ where

a) $F(x,y,z) = xy^2 i + x^2 yj - 3e^{x^2+y^2}k$ and Σ is the surface of the solid bounded by the parabolids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$.

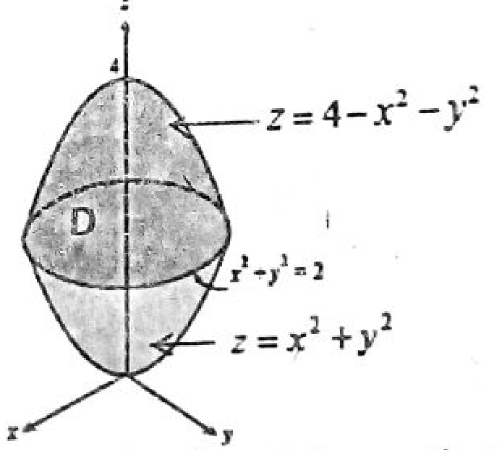
b) $F(x,y,z) = 20xy\mathbf{i} - 10y^2\mathbf{j} + 6z\mathbf{k}$ and Σ is the solid bounded by the parabolid $z = 6 - x^2 - y^2$ and the cone $z = \sqrt{x^2 + y^2}$.

c) $F(x,y,z) = x^3 i + y^3 j + z^3 k$ and Σ is the surface of the sphere $x^2 + y^2 + z^2 = 1$ with outward directed normal.

d) $F(x,y,z) = x^2 y.\mathbf{i} - xy^2.\mathbf{j} + (4z+2).\mathbf{k}$ and Σ is the surface of the solid bounded above by $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.

Solution:

a) For more understanding refer the diagram below.



The solid above $z = x^2 + y^2$ and below $z = 4 - x^2 - y^2$

Firs, find the intersection of the surfaces $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$.

That is $x^2+y^2=4-x^2-y^2 \Rightarrow x^2+y^2=2$. Thus the solid using cylindrical coordinates is given by $D: 0\le\theta\le 2\pi, 0\le r\le\sqrt{2}, r^2\le z\le 4-r^2$.

$$\iint_\Sigma F.ndS=\iiint_D divF(x,y,z)dV=\iiint_D(x^2+y^2)dV=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2}r^3dzdrd\theta$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}(4r^3-2r^5)drd\theta=\int_0^{2\pi}(r^4-\frac{r^6}{3})\Big|_{r=0}^{r=\sqrt{2}}d\theta=\int_0^{2\pi}\frac{4}{3}d\theta=\frac{8\pi}{3}$$

b) $divF(x,y,z)=\nabla.F(x,y,z)=20y-20y+6=6$. Since the surface is bounded by $z=6-x^2-y^2$ and $z=\sqrt{x^2+y^2}$, first find their intersection.

Here, to simplify the calculation we use polar coordinates.

That is $z=6-x^2-y^2, z=\sqrt{x^2+y^2}\Rightarrow 6-r^2=r\Rightarrow r^2+r-6=0\Rightarrow r=2$

Thus the solid using cylindrical coordinates is given by

$D: 0\le\theta\le 2\pi, 0\le r\le 2, r\le z\le 6-r^2$.

$$\iint_\Sigma F.ndS=\iiint_D divF(x,y,z)dV=\iiint_D 6dV=\int_0^{2\pi}\int_0^{2}\int_{r}^{6-r^2}6rdzdrd\theta$$

$$=\int_0^{2\pi}\int_0^{2}(36r-6r^2-6r^3)drd\theta=\int_0^{2\pi}(18r^2-2r^3-\frac{3}{2}r^4)\Big|_{r=0}^{r=2}d\theta=64\pi$$

c) $divF(x,y,z)=\nabla.F(x,y,z)=3x^2+3y^2+3z^2$. Since the surface upon which the triple integral to be evaluated is a sphere, we use spherical coordinates. So, in spherical coordinates, the solid is given by

$D: 0\le\theta\le 2\pi, 0\le\phi\le\pi, 0\le\rho\le 1, dV=\rho^2\sin\phi d\rho d\phi d\theta$. Therefore,

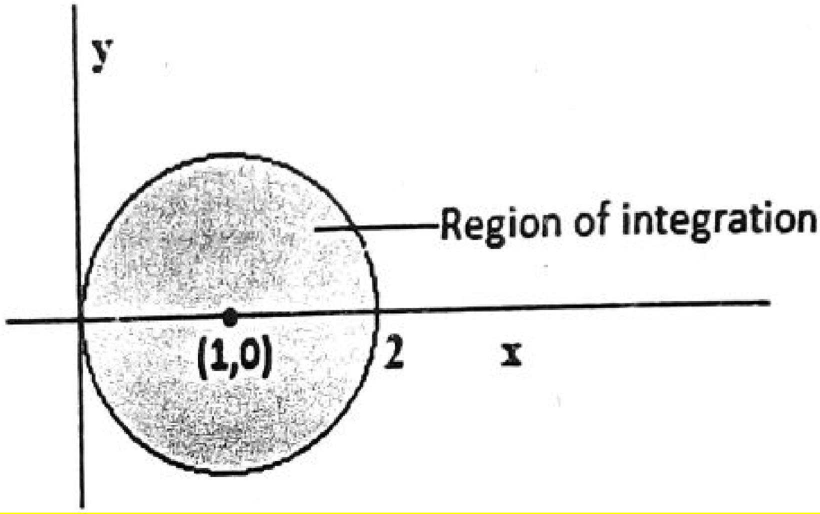
$$\iint_\Sigma F.ndS=\iiint_D divF(x,y,z)dV=\iiint_D(3x^2+3y^2+3z^2)dV$$

$$=3\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(\rho^4\sin\phi)d\rho d\phi d\theta=3\int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^5}{5}\right)\Big|_0^1\sin\phi d\phi d\theta$$

$$=\frac{3}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi d\phi d\theta=\frac{3}{5}\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi}d\theta=\frac{6}{5}\int_0^{2\pi}d\theta=\frac{12\pi}{5}$$

d) Here, $\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D}(2xy-2xy+4)dV = \iiint_{D} 4dV$

Now find the limits of integration by finding the intersections of $z=2x$ and $z=x^2+y^2$. That is $x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1$ is a circle as shown.



Besides, using $r^2=x^2+y^2, x=r\cos\theta$, we have

$x^2+y^2=2x \Rightarrow r^2=2r\cos\theta \Rightarrow r=0, r=2\cos\theta$.

Hence, the region is described in cylindrical coordinates as

$0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, x^2+y^2 \le z \le 2x \Rightarrow r^2 \le z \le 2r\cos\theta$.

$$\iint_{S} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} 4dV = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\cos\theta}\int_{r^2}^{2r\cos\theta} rdzdrd\theta$$

$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\cos\theta}(2r^2\cos\theta-r^3)drd\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{16}{3}\cos^4\theta-4\cos^4\theta)d\theta$$

$$=\frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos^4\theta d\theta = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)^2 d\theta = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2\theta+\cos^2 2\theta)d\theta$$

$$=\frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta)d\theta = \frac{4}{3}(\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = 2\pi$$

2. Using the divergence theorem, evaluate $\iint\limits_{\Sigma} F.ndS$ where

a) $F(x,y,z)=(xy^2+\cos z)\mathbf{i}+(x^2y+\sin x)\mathbf{j}+z(x^2+y^2)\mathbf{k}$ and Σ is the solid bounded by the parabolid $z=2-x^2-y^2$ and the cone $z=\sqrt{x^2+y^2}$.

b) $F(x,y,z)=z^3\mathbf{i}+x^2y\mathbf{j}+y^2z\mathbf{k}$ and S is the portion of $z=4-x^2-y^2, z=1, z=0$.

c) $F(x,y,z)=-2x\mathbf{i}+4y\mathbf{j}-7z\mathbf{k}$ and Σ is the boundary of the solid region inside the sphere $x^2+y^2+z^2=4$ and outside the cylinder $x^2+y^2=1$

d) $F(x,y,z)=x^4i-x^3z^2j+4xy^2zk$ and Σ is the surface bounded by the cylinder $x^2+y^2=1$ and the planes $z=x+2, z=0$

e) $F(x,y,z)=2xi-yzj+3z^2k$ and Σ is the surface of the paraboloid $z=x^2+y^2$ capped by the disk $x^2+y^2\le 1$ in the plane $z=1$.

f) $F(x,y,z)=x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ and Σ is the teterahedron with vertices $(4,0,0),(0,0,0),(0,0,4)$ and $(0,4,0)$.

Solution:

a) $divF(x,y,z)=\nabla.F(x,y,z)=2(x^2+y^2)$. Since the surface is bounded by $z=2-x^2-y^2$ and $z=\sqrt{x^2+y^2}$, first find their intersection to determine the limits of integration.

That is $z=2-x^2-y^2, z=\sqrt{x^2+y^2}\Rightarrow r^2+r-2=0\Rightarrow r=1$. Thus the solid using cylindrical coordinates is given by $D: 0\le\theta\le 2\pi, 0\le r\le 1, r\le z\le 2-r^2$.

$$\iint\limits_{\Sigma} F.nds=\iiint\limits_{D} divF(x,y,z)dV=\iiint\limits_{D} 2(x^2+y^2)dV=\int_0^{2\pi}\int_0^1\int_r^{2-r^2} 2r^3dzdrd\theta$$

$$=2\int_0^{2\pi}\int_0^1 (2r^3-r^4-r^5)drd\theta=\frac{4}{15}\int_0^{2\pi}d\theta=\frac{8\pi}{15}$$

$$b) \iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} (x^2+y^2)dV = \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{4-r^2} r^3 dzdrd\theta$$

$$= \int_0^{2\pi}\int_{\sqrt{3}}^{2} r^3(4-r^2)drd\theta = \int_0^{2\pi}(16-\frac{32}{3}-9+\frac{9}{2})d\theta = \frac{5\pi}{6}$$

c) Here, the projection of the sphere on the xy plane is the circle $x^2+y^2=4$ and that of the cylinder is also $x^2+y^2=1$.

Besides, $x^2+y^2+z^2=4 \Rightarrow z=\pm\sqrt{4-x^2-y^2}$

So, in cylindrical coordinates the solid D is described as

$D: 0\le\theta\le 2\pi, 1\le r\le 2, -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}$. Hence,

$$\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} -5dV = -5\int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dzdrd\theta$$

$$= -10\int_0^{2\pi}\int_1^2 \sqrt{4-r^2}\,rdrd\theta = -10\int_0^{2\pi}\sqrt{3}d\theta = -20\sqrt{3}\pi$$

d) Using cylindrical coordinates, we have

$x = r\cos\theta, y = r\sin\theta, z = x+2 = r\cos\theta+2, 0\le r\le 1, 0\le\theta\le 2\pi$

Hence, $\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} 4x(x^2+y^2)dV$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2} r^4\cos\theta dzdrd\theta = 4\int_0^{2\pi}\int_0^1 r^4\cos\theta(r\cos\theta+2)drd\theta$$

$$= 4\int_0^{2\pi}\left(\frac{1}{6}\cos^2\theta+\frac{2}{5}\cos\theta\right)d\theta = \frac{2\pi}{3}$$

e) Since the surface is $z=x^2+y^2$ for $x^2+y^2\le 1$, using cylindrical coordinates the solid is given by $D: 0\le\theta\le 2\pi, 0\le r\le 1, r^2\le z\le 1$.

$$\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} (2+5z)dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 (2+5z)rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1 (\frac{9}{2} - 2r^2 - \frac{5}{2}r^4)rdrd\theta = \int_0^{2\pi}\int_0^1 (\frac{9}{2}r - 2r^3 - \frac{5}{2}r^5)drd\theta = \frac{8\pi}{3}$$

f) The solid is the tetrahedron with the given vertices and the plane containing the teterahedron is obtained using the vertices to be $x+y+z=4$. (Explain how we get this?)It is described by $D: 0 \le x \le 4, 0 \le y \le 4-x, 0 \le z \le 4-x-y$. Besides, verify that $\iiint_D (2x+2y+2z)dV = 3\iiint_D 2zdV$.Therefore,

$$\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D} (2x+2y+2z)dV = 3\int_0^4\int_0^{4-x}\int_0^{4-x-y} 2zdzdydx$$

$$= 3\int_0^4\int_0^{4-x} (4-x-y)^2 dydx = 64$$

3. Use the Divergence Theorem to evaluate $\iint_S F \cdot ndS$ where

a) $F(x,y,z) = 2xi + 3yj + zk$ and S is the surface of a solid inside the sphere $x^2+y^2+z^2=2$ and the double cone $z^2 = x^2+y^2$ with outward normal.

b) $F(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and Σ is the surface of the cylindrical solid bounded above by $x^2+y^2=4, z=0, z=3$.

c) $F(x,y,z) = (x^3 - e^{2z})\mathbf{i} + (y^3 + \tan z^2)\mathbf{j} + (z^3 + xy)\mathbf{k}$ and Σ is the surface of the solid bounded above by $z = \sqrt{4-x^2-y^2}$ and the xy-plane.

d) $F(x,y,z) = x^2ye^{2z}i + xy^2z^3j - xye^{2z}k$ and Σ is the surface of the box bounded by the coordinate planes and the planes $x=3, y=2, z=1$

e) $F(x,y,z) = x^2\mathbf{i} + y\mathbf{j} - 2z^2\mathbf{k}$ and Σ is the boundary of the solid region bounded below by the xy plane, above by the plane $z=x$ and on the sides by the parabolic sheet $y^2 = 2-x$

Solution:

a) $divF(x,y,z) = \nabla.F(x,y,z) = 6$. Since the surface is bounded by $x^2+y^2+z^2=2$ and $z^2=x^2+y^2$, first find their intersection to determine the limits of integration. Here, to simplify the calculation we use polar coordinates with $x^2+y^2=r^2$. That is $x^2+y^2+z^2=2 \Rightarrow 2r^2=2 \Rightarrow r^2=1 \Rightarrow r=1$.

Hence, the projection of the solid on the xy plane is the circle $x^2+y^2=1$. Since S is the boundary of a spherical solid, we use spherical coordinates. Besides, $x^2+y^2+z^2=2 \Rightarrow 2z^2=2 \Rightarrow z^2=2 \Rightarrow z=\pm1$.

Since we are interested in the double cone both $z=1$ and $z=-1$ are valid solutions. Let's use these values to determine the spherical coordinates.

First of all $x^2+y^2+z^2=2 \Rightarrow \rho^2=2 \Rightarrow \rho=\sqrt{2}$.

$$z=1, z=\rho\cos\phi \Rightarrow \sqrt{2}\cos\phi=1 \Rightarrow \cos\phi=\frac{1}{\sqrt{2}} \Rightarrow \phi=\frac{\pi}{4},$$

$$z=-1, z=\rho\cos\phi \Rightarrow \sqrt{2}\cos\phi=-1 \Rightarrow \cos\phi=-\frac{1}{\sqrt{2}} \Rightarrow \phi=\frac{3\pi}{4}$$

Thus the solid using spherical coordinates is given by

$$D: 0\le\theta\le2\pi, 0\le\rho\le\sqrt{2}, 0\le\phi\le\frac{\pi}{4}, \frac{3\pi}{4}\le\phi\le\pi.$$

$$\iint_S F.ndS = \iiint_D divF(x,y,z)dV = \iiint_D 6dV = 6\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\rho^2\sin\varphi d\rho d\varphi d\theta + 6\int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^{\sqrt{2}}\rho^2\sin\varphi d\rho d\varphi d\theta$$

$$=12\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\rho^3}{3}\Big|_0^{\sqrt{2}}\sin\varphi d\varphi d\theta = 8\sqrt{2}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin\varphi d\varphi d\theta = 8\sqrt{2}\int_0^{2\pi}-\cos\varphi\Big|_0^{\frac{\pi}{4}}d\theta$$

$$=4(2-\sqrt{2})\int_0^{2\pi}d\theta = 8(2-\sqrt{2})\pi$$

(Since the double cone is symmetric,

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\rho^2\sin\varphi d\rho d\varphi d\theta = \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^{\sqrt{2}}\rho^2\sin\varphi d\rho d\varphi d\theta)$$

b) $\iint\limits_S F.nds = \iiint\limits_D (3x^2+3y^2+3z^2)dV = 3\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)rdzdrd\theta = 180\pi$

c) Here, $\iint\limits_S F.nds = \iiint\limits_D divF(x,y,z)dV = \iiint\limits_D (3x^2+3y^2+3z^2)dV$.

Since the solid is spherical, we use spherical coordinates, to evaluate it.

$$\iint\limits_S F.nds = \iiint\limits_D (3x^2+3y^2+3z^2)dV = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\phi d\rho d\phi d\theta = \frac{192\pi}{5}$$

d) Hence, $\iint\limits_\Sigma F.nds = \iiint\limits_D 2xyz^3 dV = \int_0^3\int_0^2\int_0^1 2xyz^3 dzdydx = \frac{1}{2}\int_0^3\int_0^2 xydydx = \frac{9}{2}$

e) Here, $y^2 = 2-x \Rightarrow y = \pm\sqrt{2-x}$. So, the solid is described by

$D: 0 \le x \le 2, -\sqrt{2-x} \le y \le \sqrt{2-x}, 0 \le z \le x$. Therefore,

$$\iint\limits_\Sigma F.nds = \iiint\limits_D (2x-4z+1)dV = \int_0^2\int_{-\sqrt{2-x}}^{\sqrt{2-x}}\int_0^x (2x-4z+1)dzdydx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{2-y^2}\int_0^x (2x-4z+1)dzdxdy = \int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{2-y^2} xdzdxdy = \frac{32\sqrt{2}}{15}$$

4. Use the divergence theorem to evaluate $\iint\limits_\Sigma F\cdot nds$ where

a) $F(x,y,z) = 2xy\mathbf{i} + 10y^2\mathbf{j} + 6z\mathbf{k}$ and Σ is the solid bounded by the paraboloid $z = 6 - x^2 - y^2$ and the cone $z = \sqrt{x^2+y^2}$.

b) $F(x,y,z) = y(x^2+y^2)^{\frac{3}{2}}\mathbf{i} - x(x^2+y^2)^{\frac{3}{2}}\mathbf{j} + (z+1)\mathbf{k}$ and Σ is the boundary of the solid bounded above by $z = 2x$ and below by the paraboloid $z = x^2+y^2$.

c) $F(x,y,z) = (x^3+\sin y)\mathbf{i} + x^2y\mathbf{j} + \ln(x^2+y^2)\mathbf{k}$ and Σ is the surface of the solid bounded by the cylinder $z = 1-x^2$, the planes $y+z=5, z=0, y=0$.

d) $F(x,y,z) = (x^2+y^2+z^2)(xi+yj)$ and Σ is the sphere $x^2+y^2+z^2=9$ with outward normal.

e)* $F(x,y,z) = (x^2+y^2+z^2)(xi+yj+zk)$ and Σ is an outward surface of a solid bounded by the planes $z=0, z=3$ and by the cylinder $x^2+y^2=4$.

Solution:

a) First find the intersection of $z = 6 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2}$. Here, to simplify the calculation we use polar coordinates.

That is $z = 6 - x^2 - y^2, z = \sqrt{x^2 + y^2} \Rightarrow 6 - r^2 = r \Rightarrow r^2 + r - 6 = 0 \Rightarrow r = 2$

Thus the solid using cylindrical coordinates is given by

$$D: 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 6 - r^2.$$

$$\iint_\Sigma F.ndS = \iiint_D divF(x,y,z)dV = \iiint_D (22y + 6)dV$$

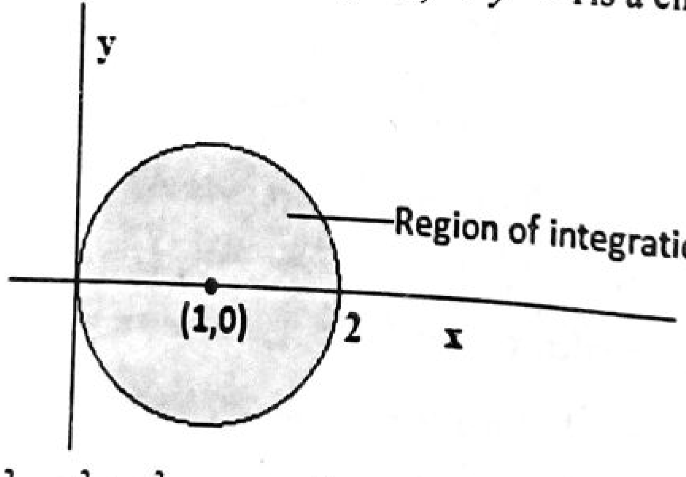
$$= \int_0^{2\pi}\int_0^2\int_r^{6-r^2} (22r\sin\theta + 6)rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2\int_r^{6-r^2} (22r\sin\theta)rdzdrd\theta + \int_0^{2\pi}\int_0^2\int_r^{6-r^2} 6rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2 (36r - 6r^2 - 6r^3)drd\theta = \int_0^{2\pi} (18r^2 - 2r^3 - \frac{3}{2}r^4)\Big|_{r=0}^{r=2} d\theta = 64\pi$$

b) Here, $\iint_\Sigma F.nds = \iiint_D divF(x,y,z)dV = \iiint_D dV$

Now find the limits of integration by finding the intersections of $z = 2x$ and $z = x^2 + y^2$. That is $x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ is a circle as shown.



Besides, using $r^2 = x^2 + y^2, x = r\cos\theta$, we have

$x^2 + y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$.

Hence, the region is described in cylindrical coordinates as

$0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, x^2 + y^2 \le z \le 2x \Rightarrow r^2 \le z \le 2r\cos\theta$.

$$\iint_S F.nds = \iiint_D dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r dz dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2r^2\cos\theta - r^3) dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{16}{3}\cos^4\theta - 4\cos^4\theta) d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^4\theta d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)^2 d\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta + \cos^2 2\theta) d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta) d\theta$$

$$= \frac{1}{3}(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = \frac{\pi}{2}$$

c) $z = 1 - x^2, z = 0 \Rightarrow -1 \le x \le 1, 0 \le z \le 1 - x^2, y + z = 5, y = 0 \Rightarrow 0 \le y \le 5 - z$.

Hence, $\iint_\Sigma F.nds = \iiint_D 4x^2 dV = 4\int_{-1}^{1}\int_0^{1-x^2}\int_0^{5-z} x^2 dy dz dx = 4\int_{-1}^{1}\int_0^{1-x^2} x^2(5-z) dz dx$

$$= 4\int_{-1}^{1} x^2\left((5-5x^2) - \frac{1}{2}(1-x^2)^2\right) dx = 4\int_{-1}^{1} (\frac{9}{2}x^2 - 4x^4 - \frac{x^6}{2}) dx = \frac{176}{35}$$

d) $F(x,y,z) = (x^2+y^2+z^2)(xi+yj) = (x^3+xy^2+xz^2)\mathbf{i} + (x^2y+y^3+yz^2)\mathbf{j}$

Thus, $divF(x,y,z) = \nabla.F(x,y,z) = 4x^2 + 4y^2 + 2z^2$

Since the surface Σ is the sphere $x^2+y^2+z^2 = 9$, spherical coordinate is appropriate to evaluate the triple integrals. So, in spherical coordinates the solid D enclosed by the sphere $x^2+y^2+z^2 = 9$ is described as

$D: 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le 3$, $z = \rho\cos\phi$, $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.

Hence,

$$\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D}(4x^2+4y^2+2z^2)dV = \iiint_{D}(4r^2+2z^2)dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^3 (4\rho^2\sin^2\phi + 2\rho^2\cos^2\phi)\rho^2\sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^3 (2\rho^2\sin^2\phi + 2\rho^2)\rho^2\sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^3 [2(1-\cos^2\phi)\sin\phi + 2\sin\phi]\rho^4 d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} [2(1-\cos^2\phi)\sin\phi + 2\sin\phi]\left(\frac{\rho^5}{5}\right)\Bigg|_0^3 d\phi d\theta$$

$$= \frac{243}{5}\int_0^{2\pi}\int_0^{\pi} [2(1-\cos^2\phi)\sin\phi + 2\sin\phi]d\phi d\theta$$

$$= \frac{243}{5}\int_0^{2\pi}[\frac{2}{3}\cos^3\phi - 4\cos\phi]\Bigg|_{\phi=0}^{\phi=\pi} d\theta = \frac{243}{5}\int_0^{2\pi}\frac{20}{3}d\theta = 648\pi$$

e) $F(x,y,z) = (x^2+y^2+z^2)(xi+yj+zk)$

$$= (x^3+xy^2+xz^2)\mathbf{i} + (x^2y+y^3+yz^2)\mathbf{j} + (x^2z+y^2z+z^3)\mathbf{j}$$

Thus, $divF(x,y,z) = 5x^2+5y^2+5z^2$

Besides, the solid D enclosed by the planes $z=0, z=3$ and the cylinder $x^2+y^2=4$ is described as $D: 0\le\theta\le 2\pi, 0\le r\le 2, 0\le z\le 2\pi$. Hence,

$$\iint_{\Sigma} F.nds = \iiint_{D} divF(x,y,z)dV = \iiint_{D}(5x^2+5y^2+5z^2)dV = \iiint_{D}(5r^2+5z^2)dV$$

$$= 5\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)rdzdrd\theta = 5\int_0^{2\pi}\int_0^2 (3r^3+9r)drd\theta = 5\int_0^{2\pi} 30\,d\theta = 300\pi$$

CHAPTER-4

Fourier-Analysis

4.1 Fourier-Series

4.1.1 Periodic Functions

A function f is said to be periodic if there exists anon zero constant p such that $f(x+p)=f(x)$ for all x in the domain of f.

The smallest positive constant p such that $f(x+p)=f(x)$ for all x in the domain of f is said to be primitive or fundamental period.

Examples: Verify that the functions are periodic with period as indicated.

a) $f(x)=\sin x, p=2\pi$ *b)* $f(x)=\tan x, p=\pi$ *c)* $f(x)=\sin(4x), p=\frac{\pi}{2}$

d) $f(x)=\cos x, p=2\pi$ *e)* $f(x)=\sec x, p=2\pi$ *f)* $f(x)=\csc x, p=2\pi$

Solution:

a) Here, for f to be periodic, $f(x+p)=f(x)$ for x in the domain of f.

But $f(x+2\pi)=\sin(x+2\pi)=\sin x\cos 2\pi+\cos x\sin 2\pi=\sin x=f(x)$.

Therefore, $f(x)=\sin x$ is periodic with period $p=2\pi$.

b) Here, $f(x+\pi)=\tan(x+\pi)=\dfrac{\tan x-\tan\pi}{1-\tan x\tan\pi}=\dfrac{\tan x-0}{1-0}=\tan x=f(x)$.

Therefore, $f(x)=\tan x$ is periodic with period $p=\pi$.

c) Here $f(x+\frac{\pi}{2})=\sin(4(x+\frac{\pi}{2}))=\sin(4x+2\pi)$

$$=\sin 4x\cos 2\pi+\cos 4x\sin 2\pi=\sin 4x=f(x)$$

Therefore, $f(x)=\sin(4x)$ is periodic with period $p=\frac{\pi}{2}$.

In general, we can summarize the periodic trigonometric functions and their fundamental periods using table as follow.

Functions	Periods (p)	Fundamental period
$f(x)=\sin x,\ g(x)=\cos x$	$p=2\pi,4\pi,6\pi,8\pi...$	$p=2\pi$
$f(x)=\sin ax,\ g(x)=\cos ax,$ $f(x)=\sec ax,\ g(x)=\csc ax,\ a\neq 0$	$p=\left(\dfrac{2\pi}{\lvert a\rvert}\right)k,\ k\in N$	$p=\dfrac{2\pi}{\lvert a\rvert}$
$f(x)=\tan x, g(x)=\cot x$	$p=k\pi,\ k\in N$	$p=\pi$
$f(x)=\tan ax, g(x)=\cot ax,\ a\neq 0$	$p=(\dfrac{\pi}{\lvert a\rvert})k,\ k\in N$	$p=\dfrac{\pi}{\lvert a\rvert}$

Periods for Sum and Difference of Functions

The sum and difference of a number of periodic functions with *commensurable period* is a periodic function with period the least common multiple of their periods. That is if the period of f is p_1, the period of g is p_2 and the period of h is p_3, such that the periods are commensurable, then the fundamental period of their combination $F=af+bg+ch$ is $p=\text{LCM}(p_1,p_2,p_3)$.

(Here, functions with *commensurable period* means functions that have common periods).

Examples: Identify the fundamental period of the following functions.

a) $f(x)=3\sin\dfrac{x}{2}+6\cos\dfrac{x}{3}$ b) $f(x)=3\cos(\dfrac{x}{3})-4\sin(\dfrac{x}{2})+9\tan(\dfrac{x}{5})$

c) $f(x)=\tan(\dfrac{\pi}{3}x)+\sin(\dfrac{\pi}{2}x)$ d) $f(x)=8\tan(\dfrac{x}{3})-\cos(\dfrac{3x}{2})$

e) $f(x)=\sin(\pi x)+\cos(2x)$ f) $f(x)=4\sin(\dfrac{2x-7}{3})-5\csc(\dfrac{x}{5})+\cot\dfrac{x}{4}$

Solution: First observe that coefficients have no impact on periods.

For instance in $f(x)=3\sin\dfrac{x}{2}+6\cos\dfrac{x}{3}$, the coefficients 3 and 6 have no impact.

Now, let's find the fundamental period of the function.

a) From the table, the primitive period of $\sin ax$ is $p=\dfrac{2\pi}{|a|}$. Thus, the primitive period of $\sin\dfrac{x}{2}$ is $p_1=\dfrac{2\pi}{1/2}=4\pi$. Again, since the primitive period of $\cos ax$ is $p=\dfrac{2\pi}{|a|}$, the primitive period of $\cos\dfrac{x}{3}$ is $p_2=6\pi$. Hence, the primitive period of f is the least common integral multiple given by $p=\text{LCM}(4\pi,6\pi)=12\pi$.

b) Since the primitive period of $\cos x$ and $\sin x$ is 2π, the primitive period of $\cos(\dfrac{x}{3})$ is $p_1=6\pi$ and $\sin(\dfrac{x}{2})$ is $p_2=4\pi$. Again, since the primitive period of $\tan x$ is π, the primitive period of $\tan(\dfrac{x}{5})$ is $p_3=\dfrac{\pi}{1/5}=5\pi$.

Hence, the primitive period of f is the least common integral multiple of $p_1=6\pi, p_2=4\pi$ and $p_3=5\pi$. That is $p=\text{LCM}(6\pi,4\pi,5\pi)=60\pi$.

c) As the primitive period of $\tan x$ is π, the primitive period of $\tan(\dfrac{\pi}{3}x)$ is $p_1=\dfrac{\pi}{\pi/3}=3$. Again, since the primitive period of $\sin x$ is 2π, the primitive period of $\sin(\dfrac{\pi}{2}x)$ is $p_2=\dfrac{2\pi}{\pi/2}=4$. Hence, the primitive period of f is the least common integral multiple of 3 and 4. That is $p=\text{LCM}(3,4)=12$.

d) The primitive period of $\sin x$ is 2π and that of $\tan\dfrac{x}{3}$ is as $p_1=\dfrac{\pi}{1/3}=3\pi$. Again, since the primitive period of $\cos x$ is 2π, the primitive period of $\cos(\dfrac{3x}{2})$ is $p_2=\dfrac{2\pi}{3/2}=\dfrac{4\pi}{3}$. Hence, the primitive period of f is the least common integral multiple given by $p=\text{LCM}\left(2\pi,3\pi,\dfrac{4\pi}{3}\right)=12\pi$.

e) The function has no primitive period. (Do you see why?)

f) The primitive period is $p=\text{LCM}(p_1,p_2,p_3)=\text{LCM}(3\pi,10\pi,4\pi)=60\pi$.

4.1.2 Fourier Series and Euler's Formula

Definition: The representation or expansion of a function f of period 2π in an open interval $(a, a+2\pi)$ in the form $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ is known as *Fourier Series representation*. Simply, it is called Fourier series.

The coefficients a_0, a_n, b_n are called *Fourier coefficients* and they are given by

$$\begin{cases} a_0 = \dfrac{1}{\pi}\displaystyle\int_a^{a+2\pi} f(x)dx \\ a_n = \dfrac{1}{\pi}\displaystyle\int_a^{a+2\pi} f(x)\cos nx dx \\ b_n = \dfrac{1}{\pi}\displaystyle\int_a^{a+2\pi} f(x)\sin nx dx \end{cases} \qquad \text{(Euler's formula)}$$

The most commonly used interval is determine by letting $a=-\pi$.
In this case, the Euler's formula becomes as follow:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

Remark: Commonly encountered important integrals. (Please bear in mind!)

1) $\int x\sin nxdx=\frac{1}{n^2}(\sin nx-nx\cos nx)$

2) $\int x\cos nxdx=\frac{1}{n^2}(nx\sin nx+\cos nx)$

3) $\int x^2\sin nxdx=\frac{-x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}$

4) $\int x^2\cos nxdx=\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}$

5) $\int e^{ax}\sin bxdx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$

6) $\int e^{ax}\cos bxdx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

4.1.2 Fourier Series and Euler's Formula

Definition: The representation or expansion of a function f of period 2π in an open interval $(a, a+2\pi)$ in the form $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ is known as *Fourier Series representation*. Simply, it is called Fourier series.

The coefficients a_0, a_n, b_n are called *Fourier coefficients* and they are given by

$$\begin{cases} a_0=\frac{1}{\pi}\int_a^{a+2\pi}f(x)dx \\ a_n=\frac{1}{\pi}\int_a^{a+2\pi}f(x)\cos nxdx \qquad \text{(Euler's formula)} \\ b_n=\frac{1}{\pi}\int_a^{a+2\pi}f(x)\sin nxdx \end{cases}$$

The most commonly used interval is determine by letting $a=-\pi$.
In this case, the Euler's formula becomes as follow:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

Remark: Commonly encountered important integrals. (Please bear in mind!)

1) $\int x\sin nxdx=\frac{1}{n^2}(\sin nx-nx\cos nx)$

2) $\int x\cos nxdx=\frac{1}{n^2}(nx\sin nx+\cos nx)$

3) $\int x^2\sin nxdx=\frac{-x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}$

4) $\int x^2\cos nxdx=\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}$

5) $\int e^{ax}\sin bxdx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$

6) $\int e^{ax}\cos bxdx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

Examples:

1. Find the Fourier series expansion on the given intervals.

a) $f(x)=4x, -\pi<x<\pi$ b) $f(x)=3-2x, -\pi<x<\pi$

c) $f(x)=e^{-x}, 0<x<2\pi$ d) $f(x)=x\sin x, 0\le x\le 2\pi$

e) $f(x)=\dfrac{\pi-x}{2}, 0\le x\le 2\pi$ f) $f(x)=x+\dfrac{x^2}{4}, -\pi\le x\le\pi$

Solution:

a) $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}4xdx=\frac{2x^2}{\pi}\Big|_{x=-\pi}^{x=\pi}=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}4x\cos nxdx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}4x\sin nxdx=\frac{-8\cos n\pi}{n}=\frac{8(-1)^{n+1}}{n}$$

Therefore, $f(x)=8\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx=8\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\right)$

b) $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)dx=\frac{1}{\pi}\left(3x-x^2\right]_{-\pi}^{\pi}=6$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\cos nxdx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\sin nxdx=\frac{4(-1)^n}{n}$$

Therefore, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=3+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

c) $$a_0=\frac{1}{\pi}\int_{0}^{2\pi}f(x)dx=\int_{0}^{2\pi}e^{-x}dx=e^{-x}\Big|_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{0}^{2\pi}e^{-x}\cos(nx)dx=\frac{e^{-x}}{\pi(n^2+1)}(n\sin nx-\cos nx)\Big|_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)dx$$

$$= \frac{e^{-x}}{\pi(n^2+1)}(-n\cos nx - \sin nx)\Big|_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}$$

Therefore, $f(x) = \dfrac{1-e^{-2\pi}}{\pi}\left(\dfrac{1}{2} + \sum_{n=1}^{\infty}\dfrac{1}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\dfrac{n}{n^2+1}\sin nx\right)$

$$d)\ a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x dx = \frac{1}{\pi}(\sin x - x\cos x)\Big|_0^{2\pi} = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx dx = \frac{1}{2\pi}\left(\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]dx\right)$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{2}{n^2-1}, \text{for } n \neq 1$$

Here, we calculated above $a_n = \dfrac{2}{n^2-1}$ for $n \neq 1$. So, we have to compute a_1 separately. But for $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}(2\sin x\cos x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x dx = \frac{1}{2\pi}\left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{2\pi} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx dx = \frac{1}{2\pi}\left(\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]dx\right)$$

$$= \frac{1}{2\pi}\left[\left(\frac{x\sin(n-1)x}{n-1} - \frac{x\sin(n+1)x}{n+1}\right) + \left(\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= 0, \text{for } n \neq 1$$

Here, $b_n = 0$ for $n \neq 1$. So, we have to compute b_1 separately. But for $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx = \frac{1}{2\pi}(2\pi^2) = \pi$$

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

e) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} (\frac{\pi - x}{2})dx = \frac{1}{\pi}\left(\frac{\pi x}{2} - \frac{x^2}{4}\right)\Bigg|_0^{2\pi} = 0$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\frac{\pi - x}{2})\cos nx dx = \frac{1}{2\pi}\int_0^{2\pi} (\pi \cos nx - x\cos nx)dx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} (\frac{\pi - x}{2})\sin nx dx = \frac{1}{2\pi}\int_0^{2\pi} (\pi \sin nx - x\sin x)dx$$

$$= \frac{1}{2\pi}\left(\frac{-\pi\cos nx}{n} - \frac{1}{n^2}(\sin nx - nx\cos nx)\right)\Bigg|_0^{2\pi} = \frac{1}{n}$$

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$

f) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \frac{x^2}{4})dx = \frac{1}{\pi}\left(\frac{x^2}{2} + \frac{x^3}{12}\right)\Bigg|_{-\pi}^{\pi} = \frac{\pi^2}{6}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \frac{x^2}{4})\cos nx dx = \frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \frac{x^2}{4})\sin nx dx = \frac{-2(-1)^n}{n}$$

Therefore, $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$

2. Find the Fourier series of $f(x)=x^2, -\pi \le x \le \pi$ and deduce the sums

$$i)\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6} \qquad ii)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$$

Solution:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{1}{\pi}\left(\frac{x^3}{3}\right)\Bigg|_{-\pi}^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}x^2\cos nx dx\right)=\frac{1}{\pi}\left(\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right)\Bigg|_{-\pi}^{\pi}=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}x^2\sin nx dx\right)=\frac{1}{\pi}\left(\frac{-x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}\right)\Bigg|_{-\pi}^{\pi}=0$$

Therefore, $f(x)=x^2=\dfrac{\pi^2}{3}+4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos(nx)$.

i) The series $\sum_{n=1}^{\infty}\dfrac{1}{n^2}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\ldots$ is obtained when $x=-\pi$.

So, equating $f(-\pi)$ and its Fourier series representation at $x=-\pi$ gives us

$$f(-\pi)=\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi)$$

$$\Rightarrow\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n=\pi^2, \quad \left(\text{Note}:\cos(n\pi)=(-1)^n \ \& \ (-1)^n(-1)^n=1\right)$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{2\pi^2}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

ii) The series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}=1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots$ is obtained when $x=0$.

$$f(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(0)\Rightarrow\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=0\Rightarrow 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{-\pi^2}{3}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$$

4.1.3 Fourier Series of Discontinuous Functions

Suppose f is discontinuous at $x = c$ where $a < c < a + 2\pi$. In this case, f is defined piece wisely as follow: $f(x) = \begin{cases} g(x), a < x < c \\ h(x), c < x < a + 2\pi \end{cases}$ where g and h are continuous on the indicated intervals. Then, using properties of definite integral, the Fourier coefficients are obtained as follow:

$$a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx = \frac{1}{\pi}\left(\int_a^c f(x)dx + \int_c^{a+2\pi} f(x)dx\right) = \frac{1}{\pi}\left(\int_a^c g(x)dx + \int_c^{a+2\pi} h(x)dx\right)$$

$$a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nxdx = \frac{1}{\pi}\left(\int_a^c g(x)\cos nxdx + \int_c^{a+2\pi} h(x)\cos nxdx\right)$$

$$b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nxdx = \frac{1}{\pi}\left(\int_a^c g(x)\sin nxdx + \int_c^{a+2\pi} h(x)\sin nxdx\right)$$

Remark: Dirichlet's Conditions

Since f is discontinuous at the point $x = c$, it is impossible to guess the value of the function at $x = c$ directly from the expansion. In such case, its value at $x = c$ is given by the average of the left and right limits of f at $x = c$.

Remember that as f has a finite jump at $x = c$, the left and right limits of f exists at $x = c$ but they may not be the same. Hence, the value of the function at $x = c$ is given by $f(c) = \dfrac{\lim\limits_{x \to c^-} f(x) + \lim\limits_{x \to c^+} f(x)}{2} = \dfrac{\lim\limits_{x \to c^-} g(x) + \lim\limits_{x \to c^+} h(x)}{2}$.

Examples:

1. Find the Fourier series expansion of the following piecewise functions.

a) $f(x)=\begin{cases}-2, -\pi<x<0\\ 2, 0<x<\pi\end{cases}$ and deduce $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$

b) $f(x)=\begin{cases}-\pi, -\pi<x<0\\ 3x, 0<x<\pi\end{cases}$ and deduce that $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$

Solution:

a) Observe that f is discontinuous at $x=0$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2dx+\int_{0}^{\pi}kdx\right)=\frac{1}{\pi}\left(-2x\Big|_{-\pi}^{0}+2x\Big|_{0}^{\pi}\right)=2\pi-2\pi=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2\cos nxdx+\int_{0}^{\pi}2\cos nxdx\right)$$

$$=\frac{1}{\pi}\left(\frac{-2}{n}\sin nx\Big|_{-\pi}^{0}+\frac{2}{n}\sin nx\Big|_{0}^{\pi}\right)=0$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2\sin nxdx+\int_{0}^{\pi}2\sin nxdx\right)=\frac{4}{n\pi}\left(1-(-1)^n\right)$$

Therefore, $f(x)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\left(1-(-1)^n\right)\sin nx=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)$

Besides, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ is obtained when $x=\frac{\pi}{2}$. But when $x=\frac{\pi}{2}$, we have $f(\pi/2)=2$ (Because $f(x)=2$ for $0<x<\pi$).

So, $\frac{8}{\pi}\left(\sin(\pi/2)+\frac{\sin(3\pi/2)}{3}+\frac{\sin(5\pi/2)}{5}+\ldots\right)=f(\pi/2)=2$

$$\Rightarrow\frac{8}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)=2\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right)=\frac{1}{\pi}\left(\int_{-\pi}^{0}-\pi dx+\int_{0}^{\pi}3xdx\right)$$

$$=\frac{1}{\pi}(-\pi x)\Big|_{-\pi}^{0}+\frac{1}{\pi}\frac{3x^2}{2}\Big|_{0}^{\pi}=-\pi+\frac{3\pi}{2}=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right)$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}-\pi\cos nxdx+\int_{0}^{\pi}3x\cos nxdx\right)$$

$$=\frac{1}{\pi}\left(-\frac{\pi\sin nx}{n}\right)\Big|_{-\pi}^{0}+\frac{1}{\pi}\left(\frac{3}{n^2}(nx\sin nx+\cos nx)\right)\Big|_{0}^{\pi}=\frac{3}{\pi n^2}[(-1)^n-1]$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right)=\frac{1}{\pi}\left(\int_{-\pi}^{0}-\pi\sin nxdx+\int_{0}^{\pi}3x\sin nxdx\right)$$

$$=\frac{1}{\pi}\left(\frac{\pi\cos nx}{n}\right)\Big|_{-\pi}^{0}+\frac{1}{\pi}\left(\frac{3}{n^2}(\sin nx-nx\cos nx)\right)\Big|_{0}^{\pi}=\frac{1}{n}[1-4(-1)^n]$$

Therefore, $f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{3}{\pi n^2}[(-1)^n-1]\cos nx+\sum_{n=1}^{\infty}\frac{1}{n}[1-4(-1)^n]\sin nx$

Now let's deduce the sum $1+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8}$. Since f is discontinuous at $x=0$, it converges at $x=0$ to

$$\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{3}{\pi n^2}[(-1)^n-1]\cos nx+\sum_{n=1}^{\infty}\frac{1}{n}[1-4(-1)^n]\sin nx=f(x)$$

$$\Rightarrow\frac{\pi}{4}-\frac{6}{\pi}\left(\cos 0+\frac{\cos 0}{3^2}+\frac{\cos 0}{5^2}+...\right)=\frac{\lim_{x\to 0^-}f(x)+\lim_{x\to 0^+}f(x)}{2}=-\frac{\pi}{2}$$

$$\Rightarrow-\frac{6}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right)=-\frac{\pi}{2}-\frac{\pi}{4}=-\frac{3\pi}{4}\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{3\pi^2}{24}=\frac{\pi^2}{8}$$

2. Find the Fourier series expansion of the following piecewise functions.

a) $f(x)=\begin{cases}0,-\pi<x<0\\ x,0<x<\pi\end{cases}$ b) $f(x)=\begin{cases}0,-\pi<x<0\\ x^2,0<x<\pi\end{cases}$

c) $f(x)=\begin{cases}x+x^2,-\pi<x<\pi\\ \pi^2,x=\pm\pi\end{cases}$

Solution:

a) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx=\frac{1}{\pi}\int_{0}^{\pi}xdx=\frac{\pi}{2}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right)=\frac{1}{\pi}\int_{0}^{\pi}x\cos nxdx$$

$$=\frac{1}{\pi}\left(\frac{x\sin(nx)}{n}+\frac{\cos(nx)}{n^2}\right)\Bigg|_0^{\pi}=\frac{\cos(n\pi)-1}{n^2\pi}=\frac{(-1)^n-1}{n^2\pi}$$

$$b_n=\frac{1}{\pi}\int_{0}^{\pi}x\sin nxdx=\frac{1}{\pi}\left(\frac{-x\cos(nx)}{n}+\frac{\sin(nx)}{n^2}\right)\Bigg|_0^{\pi}=\frac{-\cos(n\pi)}{n}=\frac{(-1)^{n+1}}{n}$$

Therefore, $f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi}\cos nx+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

b) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right)=\frac{1}{\pi}\int_{0}^{\pi}x^2dx=\frac{x^3}{3\pi}\Bigg|_0^{\pi}=\frac{\pi^2}{3}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{0}^{\pi}x^2\cos nxdx=\frac{2(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{\pi}x^2\sin nxdx=\frac{\pi(-1)^{n+1}}{n}+\frac{2[(-1)^n-1]}{n^3\pi}$$

Therefore, $f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left(\frac{\pi(-1)^{n+1}}{n}+\frac{2[(-1)^n-1]}{n^3\pi}\right)\sin nx$

c) $f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n^2}\cos(nx)-\frac{2}{n}\sin(nx)\right)$

4.1.4 Fourier Series of Functions with Arbitrary Periods

The Fourier series of the periodic function f of period $2L$ defined on $(a, a+2L)$ is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$

where $a_0=\frac{1}{L}\int_a^{a+2L}f(x)dx$, $a_n=\frac{1}{L}\int_a^{a+2L}f(x)\cos\frac{n\pi x}{L}dx$, $b_n=\frac{1}{L}\int_a^{a+2L}f(x)\sin\frac{n\pi x}{L}dx$

Remarks: Recall the following important integrals to save your time.

i) $a_n=\int_a^b x\cos\left(\frac{n\pi x}{L}\right)dx=\frac{L^2}{(n\pi)^2}\left(\frac{n\pi}{L}x.\sin(\frac{n\pi x}{L})+\cos(\frac{n\pi x}{L})\right)\Bigg|_a^b$

ii) $b_n=\int_a^b x\sin\left(\frac{n\pi x}{L}\right)dx=\frac{L^2}{(n\pi)^2}\left(\sin(\frac{n\pi x}{L})-\frac{n\pi}{L}x\cos(\frac{n\pi x}{L})\right)\Bigg|_a^b$

Examples:

1. Find the Fourier series of

a) $f(x)=x,\ 0<x<2$ and $f(x+2)=f(x)$

b) $f(x)=\begin{cases}2, & -2<x<0\\ x, & 0<x<2\end{cases}$ and $f(x+4)=f(x)$

c) $f(x)=\begin{cases}0, -2<x<0\\ 1, 0<x<2\end{cases}$ $f(x+2)=f(x)$

d) $f(x)=\begin{cases}1, & 0<x<2\\ x, & 2<x<4\end{cases}$, $f(x+4)=f(x)$

e) $f(x)=\begin{cases}1, & -5<x<0\\ x+1, & 0<x<5\end{cases}$, $f(x+10)=f(x)$

f) $f(x)=\begin{cases}x, -2<x<0\\ 4, 0<x<2\end{cases}$, $f(x+2)=f(x)$

g) $f(x)=\begin{cases}0, -5<x<0\\ 3, 0<x<5\end{cases}$, $f(x+10)=f(x)$

Solution:

a) Here, $a = 0, a + 2L = 2 \Rightarrow L = 1. a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx = \int_0^2 xdx = \frac{x^2}{2}\Big|_0^2 = 2$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos(\frac{n\pi x}{L})dx = \int_0^2 x\cos(n\pi x)dx = \frac{x\sin(n\pi x)}{n\pi}\Big|_0^2 - \frac{1}{n\pi}\int_0^2 \sin(n\pi x)dx$$

$$= \frac{2}{n\pi}\sin(2n\pi) - \frac{1}{n\pi}\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Big|_0^2 = 0$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin(\frac{n\pi x}{L})dx = \int_0^2 x\sin(n\pi x)dx = \frac{-x\cos(n\pi x)}{n\pi}\Big|_0^2 + \frac{1}{n\pi}\int_0^2 \cos(n\pi x)dx$$

$$= \frac{-1}{n\pi}[2\cos(2n\pi) - 0] + \frac{1}{(n\pi)^2}[\sin(2n\pi) - 0] = \frac{-2}{n\pi}$$

Therefore, $f(x) = 1 - \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)$

b) Here, $a = -2, a + 2L = 2 \Rightarrow L = 2.$

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} f(x)dx + \frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_{-2}^{0} 2dx + \frac{1}{2}\int_0^2 xdx = 3$$

$$a_n = \frac{1}{2}\int_{-2}^{0} f(x)\cos(\frac{n\pi x}{2})dx + \frac{1}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2\cos(\frac{n\pi x}{2})dx + \frac{1}{2}\int_0^2 x\cos(\frac{n\pi x}{2})dx$$

$$= \frac{2}{n\pi}\sin(\frac{n\pi x}{2})\Big|_{-2}^{0} + \frac{4}{(n\pi)^2}\left(\frac{n\pi}{2}x.\sin(\frac{n\pi x}{2}) + \cos(\frac{n\pi x}{2})\right)\Big|_0^2 = \frac{4}{(n\pi)^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{0} 2\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \frac{4}{(n\pi)^2}\left(\sin\frac{n\pi x}{2} - \frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\Big|_0^2 = \frac{-2[1 + (-1)^n]}{n\pi}$$

Therefore, $f(x)=\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}[(-1)^n-1]\cos(\frac{n\pi x}{2})+\sum_{n=1}^{\infty}\frac{-2[1+(-1)^n]}{n\pi}\sin(\frac{n\pi x}{2})$

c) Here, a = -2, a + 2L = 2 ⇒ L = 2.

$$a_0=\frac{1}{L}\int_a^{a+2L}f(x)dx=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}\int_{-2}^{0}f(x)dx\frac{1}{2}\int_0^2 f(x)dx=\frac{1}{2}\int_0^2 1dx=1$$

$$a_n=\frac{1}{L}\int_{-L}^{a+2L}f(x)\cos\left(\frac{n\pi}{L}\right)dx=\frac{1}{2}\int_0^2\cos\left(\frac{n\pi}{2}x\right)dx=\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\Big|_0^2=0$$

$$b_n=\frac{1}{L}\int_{-L}^{a+2L}f(x)\sin\left(\frac{n\pi}{L}\right)dx=\frac{1}{2}\int_0^2\sin\left(\frac{n\pi}{2}x\right)dx=\frac{-1}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2=\frac{1}{n\pi}[1-(-1)^n]$$

Therefore, $f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi}{2}x\right)$

d) Here, a = 0, a + 2L = 4 ⇒ 2L = 4 ⇒ L = 2.

$$a_0=\frac{1}{L}\int_a^{a+2L}f(x)dx=\frac{1}{2}\int_0^4 f(x)dx=\frac{1}{2}\int_0^2 f(x)dx+\frac{1}{2}\int_2^4 f(x)dx$$

$$=\frac{1}{2}\int_0^2 1dx+\frac{1}{2}\int_2^4 xdx=\frac{x}{2}\Big|_0^2+\frac{x^2}{4}\Big|_2^4=1+3=4$$

$$a_n=\frac{1}{L}\int_a^{a+2L}f(x)\cos(\frac{n\pi x}{L})dx=\frac{1}{2}\int_0^4 f(x)\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})dx+\frac{1}{2}\int_2^4 f(x)\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{2}\int_0^2\cos(\frac{n\pi x}{2})dx+\frac{1}{2}\int_2^4 x\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{n\pi}\sin(\frac{n\pi x}{2})\Big|_0^2+\frac{2}{(n\pi)^2}\left(\frac{n\pi}{2}x.\sin(\frac{n\pi x}{2})+\cos(\frac{n\pi x}{2})\right)\Big|_2^4$$

$$=\frac{2}{(n\pi)^2}(1-\cos n\pi)=\frac{2}{(n\pi)^2}[1-(-1)^n]$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^4 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_2^4 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_2^4 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 + \frac{2}{(n\pi)^2}\left(\sin\frac{n\pi x}{2} - \frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\Bigg|_2^4$$

$$= -\frac{1}{n\pi}[\cos n\pi - 1] + \frac{2}{(n\pi)^2}[-2n\pi + n\pi\cos n\pi]$$

$$= -\frac{1}{n\pi}[(-1)^n - 1] + \frac{2}{n\pi}[(-1)^n - 2] = \frac{1}{n\pi}[(-1)^n - 3]$$

Therefore,

$$f(x) = 2 + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}[1-(-1)^n]\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty}\frac{1}{n\pi}[(-1)^n - 3]\sin\left(\frac{n\pi x}{2}\right)$$

e) Here, $a = -5, a + 2L = 5 \Rightarrow L = 5$.

$$a_0 = \frac{1}{L}\int_{-L}^{a+2L} f(x)dx = \frac{1}{5}\int_{-5}^{5} f(x)dx = \frac{1}{5}\int_{-5}^{0} 1dx + \frac{1}{5}\int_0^5 (x+1)dx = \frac{9}{2}$$

$$a_n = \frac{1}{5}\int_{-5}^{0}\cos\left(\frac{n\pi}{5}x\right)dx + \frac{1}{5}\int_0^5 (x+1)\cos\left(\frac{n\pi}{5}x\right)dx = \frac{5}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{5}\int_{-5}^{0}\sin\left(\frac{n\pi x}{5}\right)dx + \frac{1}{5}\int_0^5 x\sin\left(\frac{n\pi x}{5}\right)dx + \frac{1}{5}\int_0^5 \sin\left(\frac{n\pi x}{5}\right)dx = \frac{5}{n\pi}(-1)^{n+1}$$

Therefore, $f(x) = \frac{9}{4} + \sum_{n=1}^{\infty}\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi}{5}x\right) + \sum_{n=1}^{\infty}\frac{5}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{5}x\right)$

2. Find the Fourier series expansion of $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 2x, & 0 < x < 1 \end{cases}$, $f(x+2) = f(x)$ and show that $f(0) = -\frac{1}{2}$.

Solution: Here, $p = 2L = 2 \Rightarrow L = 1$. Thus,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = \int_{-1}^{0} -1dx + \int_{0}^{1} 2xdx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x)dx = \int_{-1}^{1} f(x)\cos(n\pi x)dx = -\int_{-1}^{0}\cos(n\pi x)dx + \int_{0}^{1} 2x\cos(n\pi x)dx$$

$$= -\frac{\sin(n\pi x)}{n\pi}\bigg|_{0}^{-1} + \left[\frac{2}{n\pi}x\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]_{0}^{1} = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x)dx = \int_{-1}^{1} f(x)\sin(n\pi x)dx$$

$$= \int_{-1}^{0} -\sin(n\pi x)dx + \int_{0}^{1} 2x\sin(n\pi x)dx$$

$$\int_{0}^{2} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L^2}{(n\pi)^2}\left(\sin(\frac{n\pi x}{L}) - \frac{n\pi}{L}x\cos(\frac{n\pi x}{L})\right)\Bigg|$$

$$= \frac{\cos(n\pi x)}{n\pi}\bigg|_{-1}^{0} - \frac{2}{n\pi}\left[x\cos(n\pi x) - \frac{\sin(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$= \frac{1}{n\pi}[1 - (-1)^n] - \frac{2}{n\pi}(-1)^n = \frac{1 - (1)^n - 2(-1)^n}{n\pi} = \frac{1 - 3(-1)^n}{n\pi}$$

Therefore, $f(x) = \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x) + \sum_{n=1}^{\infty}\frac{1 - 3(-1)^n}{n\pi}\sin(n\pi x)$

Besides, since f is discontinuous at $x = 0$, by Dirichilet's Condition, we have

$$f(0) = \frac{\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)}{2} = \frac{0 - 1}{2} = -\frac{1}{2}$$

4.1.5 Fourier Series of Even and Odd Functions

Revision about Even and Odd Functions:

Definition of even and odd functions:

i) A function f is *even* if and only if $f(-x) = f(x)$ for all x .

ii) A function f is odd if and only if $f(-x) = -f(x)$ for all x .

Examples:

a) $f(x) = x^2$ is even function because $f(-x) = (-x)^2 = x^2 = f(x)$.

b) $f(x) = e^{x^2}$ is an even function because $f(-x) = e^{(-x)^2} = e^{x^2} = f(x)$.

c) $f(x) = e^{2x} - e^{-2x}$ is odd because $f(-x) = e^{-2x} - e^{2x} = -(e^{2x} - e^{-2x}) = -f(x)$.

d) $f(x) = 2x^3$ is an odd function because $f(-x) = 2(-x)^3 = -2x^3 = -f(x)$.

Properties of even and odd functions:

i) The product of any two or more even functions is even.

ii) The product of an even and an odd function is odd.

iii) The product of any two odd functions is even.

Frequently use even and odd functions:

i) The **cosine** function: $g(x) = \cos ax$ is an even function.

ii) The **sine** function: $h(x) = \sin ax$ is an odd function.

iii) The product function $f(x) = x\cos ax$ is an odd function.

iv) The product function $f(x) = x\sin ax$ is an even function.

Integral Properties of even and odd functions:

i) $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$ if f is even

ii) $\int_{-L}^{L} f(x)dx = 0$ if f is odd

Now let's discuss the Fourier Series of even and odd periodic functions. Suppose f is a periodic function on $(-L, L)$ with period $p = 2L$.

Case-1: Suppose f is even. That is $f(-x)=f(x)$. The Fourier series of f is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L} \text{ where}$$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx.$$

Since f is even, the product $f(x)\cos\frac{n\pi x}{L}$ is even. Then, by the above integral properties of even and odd functions, we have

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx=\frac{2}{L}\int_{0}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

Besides, since the product of an odd and an even function is odd, we have that $f(x)\sin\frac{n\pi x}{L}$ is odd. So, we have $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=0$.

Therefore, the Fourier series of an even function f of period $2L$ is reduced to

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L} \text{ where } a_0=\frac{2}{L}\int_{0}^{L}f(x)dx,\ a_n=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

This series involves only the cosine terms and thus it is said to be cosine Fourier series expansion of f.

Case-2: When f is odd. That is $f(-x)=-f(x)$. Then, since f is odd

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx=0,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}=0,\ b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

Therefore the Fourier series of an odd function f of period $2L$ is reduced to

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L} \text{ where } b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi x}{L}dx.$$

This series involves only sine terms and thus it is said to be **sine** Fourier series expansion of f.

Examples

1. Find the Fourier series of the following functions:

a) $f(x)=x, -2<x<2$ and deduce $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...=\frac{\pi}{4}$

b) $f(x)=|x|, -\pi\le x\le\pi$ and deduce $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...=\frac{\pi^2}{8}$

Solution:

a) Since f is odd, $a_0=0, a_n=0$.

$$b_n=\frac{2}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{2}{2}\int_{0}^{2}f(x)\sin nxdx=\int_{0}^{2}x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$=\frac{4}{(n\pi)^2}\left(\sin\frac{n\pi x}{2}-\frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\Bigg|_0^2=\frac{-4n\pi\cos n\pi}{(n\pi)^2}=\frac{4(-1)^{n+1}}{n\pi}$$

Therefore, $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}=\frac{4}{\pi}(\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+...)$

Besides, since the function is continuous at $x=1$, we have

$$f(1)=\frac{4}{\pi}(\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+...)$$

$$\Rightarrow 1=\frac{4}{\pi}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...)\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{\pi}{4}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$$

b) Since $f(-x)=|-x|=|x|=f(x), f$ *is even* $\Rightarrow b_n=0$

$$a_0=\frac{2}{\pi}\int_{0}^{\pi}f(x)dx=\frac{2}{\pi}\int_{0}^{\pi}xdx=\pi$$

$$a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_{0}^{\pi}x\cos nxdx=\frac{2}{n^2\pi}[(-1)^n-1]=\frac{-4}{(2n-1)^2\pi}$$

Therefore, $f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n-1]\cos nx=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}$

The series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...$ is obtained when $x=0$.

Thus, putting $x=0$ in $|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}$ gives

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(0)}{(2n-1)^2} = 0 \Rightarrow \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi}{2} \Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

2. Find the Fourier series expansions of

a) $f(x) = x,\ -\pi \le x \le \pi$ b) $f(x) = x\cos, -\pi \le x \le \pi$

c) $f(x) = \pi^2 - x^2, -\pi < x < \pi$ d) $f(x) = x|x|, -1 \le x \le 1$

Solution:

a) Since $f(x) = x$ is an odd function, we have $a_0 = 0, a_n = 0$. Besides,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = \frac{1}{\pi}\left(\frac{1}{n^2}(\sin nx - nx\cos nx)\right)\Bigg|_{-\pi}^{\pi} = \frac{2(-1)^{n+1}}{n}$$

Therefore, $x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = 2(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots)$

b) Since $f(x) = x\cos x$ is the product of an odd and an even function ($h(x) = x$ is odd and $g(x) = \cos x$ is even), f is an odd function and thus $a_n = 0$. Besides,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx dx = \frac{1}{\pi}\left(\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]dx\right)$$

$$= \left(\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right) = (-1)^n\frac{2n}{n^2-1}, \text{for } n \ne 1$$

Here, we calculated above $(-1)^n\frac{2n}{n^2-1}$, for $n \ne 1$. So, we have to compute b_1 separately. But for $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x dx = \frac{1}{4\pi}(\sin 2x - 2x\cos 2x)\Bigg|_0^{\pi} = -\frac{1}{2}$$

Therefore, $f(x) = \cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

$$= b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}(-1)^n\frac{n}{n^2-1}\sin nx$$

4.1.6 Half-Range Expansions

A half range Fourier sine series is a series in which only sine terms are present and a half range Fourier cosine series is a series in which only cosine terms are present. When we want to determine a half range series to a given function, the function is generally defined in the interval $(0, L)$. This is half of the interval $(-L, L)$. That is why the expansion is referred as half range expansion.

i) Half-range sine Expansion:

It is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}$ where $a_n = 0, b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

ii) Half-range cosine Expansion:

It is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}$ where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, b_n = 0$$

Examples: Find the half range sine and cosine series expansion of

a) $f(x) = x, 0 < x < 2$ b) $f(x) = 2x, 0 < x < 1$

c) $f(x) = \sin x, 0 < x < \pi$ d) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$

e) $f(x) = x(\pi - x), 0 \le x \le \pi$ f) $f(x) = \sin\left(\frac{\pi x}{3}\right), f(x+3) = f(x), 0 < x < 3$

Solution:

a) To find the half range sine expansion, $0 < x < L \Rightarrow 0 < x < 2 \Rightarrow L = 2$.
Since f is odd, $a_0 = a_n = 0$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx = \int_0^2 x\sin(\frac{n\pi x}{L})dx$$

$$= \left(\frac{-2x\cos(\frac{n\pi x}{2})}{n\pi} + \frac{4\sin(\frac{n\pi x}{2})}{n^2\pi^2}\right)\Bigg|_0^2 = \frac{-4\cos n\pi}{n\pi} = \frac{4(-1)^{n+1}}{n\pi}$$

Therefore, the half range sine series expansion of f(x) is given by

$$f(x)=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(\tfrac{n\pi x}{2})=\frac{4}{\pi}\left(\sin(\tfrac{\pi x}{2})-\frac{\sin(\pi x)}{2}+\frac{1}{3}\sin(\tfrac{3\pi x}{2})-...\right)$$

ii) Half range cosine series expansion:

Since f is even, $b_n=0$, $a_0=\frac{2}{L}\int_0^L f(x)dx=\int_0^2 xdx=2$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos(\tfrac{n\pi x}{L})dx=\int_0^2 x\cos(\tfrac{n\pi x}{2})dx$$

$$=\left(\frac{2x\sin(\frac{n\pi x}{2})}{n\pi}+\frac{4\cos(\frac{n\pi x}{2}))}{n^2\pi^2}\right)\Bigg|_0^2=\frac{4[\cos(n\pi)-1]}{n^2\pi^2}=\frac{4[(-1)^n-1]}{n^2\pi^2}$$

Therefore, the half range cosine series is $f(x)=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}\cos(\frac{n\pi x}{2})$

b) $b_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx=2\int_0^1 2x\sin(n\pi x)dx$

$$=4\left(\frac{1}{(n\pi)^2}(\sin(n\pi x)-n\pi x\cos(n\pi x))\right)\Bigg|_0^1=\frac{-4\cos n\pi}{n\pi}=\frac{4(-1)^{n+1}}{n\pi}$$

Therefore, the half range sine series expansion of $f(x)$ is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{L}x=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(n\pi x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$$

For the half range cosine expansion, $b_n=0$.

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\int_0^1 2xdx=1$$

$$a_n=2\int_0^1 2x\cos(n\pi x)dx=4\left(\frac{1}{(n\pi)^2}(n\pi x.\sin(n\pi x)+\cos(n\pi x))\right)\Bigg|_0^1=\frac{4[(-1)^n-1]}{n^2\pi^2}$$

Therefore, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}\cos(n\pi x)$

c) $a_0 = \frac{2}{\pi}\int_0^L f(x)dx = \frac{2}{\pi}\int_0^\pi \sin x dx = \frac{2}{\pi}(-\cos x)\Big|_0^\pi = \frac{4}{\pi}$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx dx$$

$$= \frac{2}{\pi}\int_0^\pi [\tfrac{1}{2}\sin[(1-n)x + \tfrac{1}{2}\sin[(1+n)x]dx = \frac{1}{\pi}\int_0^\pi [\sin[(1-n)x + \sin[(1+n)x]dx$$

$$= \frac{-\cos[(1-n)x]}{(1-n)\pi}\Big|_0^\pi - \frac{\cos[(1+n)x]}{(1+n)\pi}\Big|_0^\pi = \frac{1-\cos[(1-n)\pi]}{(1-n)\pi} + \frac{1-\cos[(1+n)\pi]}{(1+n)\pi}$$

$$= \frac{1+\cos n\pi}{(1+n)\pi} + \frac{1+\cos(n\pi)}{(1-n)\pi} = \frac{-2(1+\cos n\pi}{(n^2-1)\pi}, \text{if } n \neq 1$$

If $n=1, a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Big|_0^\pi = 0.$

Therefore, $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right).$

d) Since we are interested in the half range cosine series expansion, $b_n = 0$.

$$a_0 = \int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 2xdx + \int_1^2 2(2-x)dx = 2$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 2(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 4\cos\left(\frac{n\pi x}{2}\right)dx - \int_1^2 2x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{16}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{8}{(n\pi)^2}(1+\cos n\pi)$$

Therefore, $f(x) = 1 - \frac{32}{\pi^2}\left(\frac{\cos \pi x}{2^2} + \frac{\cos 3\pi x}{6^2} + \frac{\cos 5\pi x}{10^2} + \ldots\right)$

e) Here, $a_0 = \frac{\pi^2}{3}$, $a_n = -\frac{1}{n^2}$ and thus $f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2}\cos(2nx)$

4.1.7 Parseval's Identity (PI)

Let f be a periodic function with period $p = 2L$ such that its Fourier series expansion on $(a, a+2L)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

Then, $\int_{-L}^{L} [f(x)]^2 dx = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right)$ where $a_0, a_n,$ and b_n are the Fourier coefficients. This formula is known as Parseval's Identity.

Note: Parserval's Identity also works for half range expansions.

i) If the half-range cosine series of f in $(0, L)$, is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ then } \int_0^L [f(x)]^2 dx = \frac{L}{2}\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right).$$

ii) If the half-range sine series of f in $(0, L)$, is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ then } \int_0^L [f(x)]^2 dx = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2.$$

Examples:

1. Find the Fourier series expansion of

a) $f(x) = x^2$ in $-\pi < x < \pi$ and use Parseval's identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{90}.$$

b) $f(x) = |\sin x|, -\pi < x < \pi$ and deduce using Parserval's identity that

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

Solution:

We have found the expansion earlier $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$.

From this expansion, we get $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}(-1)^n, b_n = 0.$

Then, by Parseval's formula,

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \pi\left[\frac{a_0^{\ 2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\ 2}+b_n^{\ 2}\right)\right]$$

$$\Rightarrow \int_{-\pi}^{\pi}[f(x)]^2dx = \pi\left[\frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\right)^2\right]$$

$$\Rightarrow \pi\left[\frac{4\pi^4}{18}+\sum_{n=1}^{\infty}\frac{16}{n^4}\right]=\frac{2\pi^5}{5}\Rightarrow \frac{4\pi^4}{18}+16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}-\frac{4\pi^4}{18}\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{16\pi^4}{90}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

$$\text{i.e } 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{90}$$

b) Since $f(-x)=|\sin(-x)|=|-\sin x|=|\sin x|=f(x)$, f is even. So, $b_n=0$.

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi}\sin x dx=\left.\frac{-2\cos x}{\pi}\right|_0^{\pi}=\frac{4}{\pi}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx dx$$

$$=\frac{2}{\pi}\int_0^{\pi}[\tfrac{1}{2}\sin[(1-n)x+\tfrac{1}{2}\sin[(1+n)x]dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin[(1-n)x+\sin[(1+n)x]dx$$

$$=\left.\frac{-\cos[(1-n)x]}{(1-n)\pi}\right|_0^{\pi}-\left.\frac{\cos[(1+n)x]}{(1+n)\pi}\right|_0^{\pi}=\frac{1-\cos[(1-n)\pi]}{(1-n)\pi}+\frac{1-\cos[(1+n)\pi]}{(1+n)\pi}$$

$$=\frac{2}{(1-2n)\pi}+\frac{2}{(1+2n)\pi}=\frac{-4}{(4n^2-1)\pi}$$

Therefore, $f(x)=\dfrac{2}{\pi}+\displaystyle\sum_{n=1}^{\infty}\frac{-4}{(4n^2-1)\pi}\cos(2nx)=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\ldots\right)$

Now from Parserval's Identity, sung, we have

$$\int_{-\pi}^{\pi}[f(x)]^2\,dx=\pi\left[\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\,2}+b_n^{\,2}\right)\right]$$

$$\Rightarrow \int_{-\pi}^{\pi}\sin^2 x\,dx=\pi\left[\frac{1}{2}\left(\frac{4}{\pi}\right)^2+\sum_{n=1}^{\infty}\left(\frac{-4}{(4n^2-1)\pi}\right)^2\right]$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)dx=\pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]$$

$$\Rightarrow \pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]=\left(\frac{x}{2}-\frac{1}{4}\sin 2x\right)\Big|_{-\pi}^{\pi}$$

$$\Rightarrow \pi=\pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]\Rightarrow 1=\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}$$

$$\Rightarrow \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=1-\frac{8}{\pi^2}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\frac{\pi^2}{16}-\frac{1}{2}$$

2. Find the half range cosine expansion of $f(x)=x,\ 0<x<4$. Using the result, show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$.

Solution: As we did earlier , the expansion is From the expansion, observe that is given by $x=2-\frac{16}{\pi^2}\left(\cos(\pi x/4)+\frac{\cos(3\pi x/4)}{3^2}+\frac{\cos(5\pi x/4)}{5^2}+\ldots\right)$.

Then, $a_0=4, a_n=\frac{16}{(2n-1)^2\pi^2}, b_n=0$. So, by Parseval's formula,

$$\int_0^L[f(x)]^2dx=\frac{L}{2}\left(\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}a_n^{\,2}\right)\Rightarrow \int_0^4 x^2dx=2\left[\frac{16}{2}+\sum_{n=1}^{\infty}\left(\frac{16}{(2n-1)^2\pi^2}\right)^2\right]$$

$$\Rightarrow 2\left[8+\sum_{n=1}^{\infty}\frac{256}{(2n-1)^4\pi^4}\right]=\frac{64}{3}\Rightarrow \sum_{n=1}^{\infty}\frac{256}{(2n-1)^4\pi^4}=\frac{8}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}\Rightarrow \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$$

3*. Using Parserval's Identity, show that $\int_{-\pi}^{\pi}\cos^4 x dx = \frac{3\pi}{4}$

Solution: Here, we infer. But $f(x)=\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x.$

So, we have $a_0 = 1,\ a_1 = \frac{1}{2}, a_n = 0, b_n = 0, \forall n \geq 2$

Therefore, $\int_{-\pi}^{\pi}[f(x)]^2 dx = \pi\left[\frac{a_0^{\ 2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\ 2}+b_n^{\ 2}\right)\right] \Rightarrow \int_{-\pi}^{\pi}\cos^4 x dx = \pi[\frac{1}{2}+\frac{1}{4}] = \frac{3\pi}{4}$

4*. Using Parserval's Identity, evaluate $\int_{-\pi}^{\pi}[4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x]^2 dx$

Solution: Here, infer $f(x) = 4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x$. But using half angle formula, $f(x) = 4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x = 2 + \frac{1}{2}\sin x$.

So, we have $a_0 = 2,\ a_n = 0,\ b_1 = \frac{1}{2}, b_n = 0, \forall n \geq 2$

Therefore, $\int_{-\pi}^{\pi}[f(x)]^2 dx = \pi\left[\frac{a_0^{\ 2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\ 2}+b_n^{\ 2}\right)\right] = \pi[\frac{4}{2}+\frac{1}{4}] = \frac{9\pi}{4}$

4.2 Fourier Integrals and Transforms

4.2.1 Fourier Integrals

The Fourier Integral representation (simply the Fourier Integral) of a function f is an integral given by $f(x)=\int_0^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$

In this integral, $A(\omega)$ and $B(\omega)$ are called Fourier Integral coefficients.

They are given by $\begin{cases} A(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx \\ B(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx \end{cases}$

Examples: Given the function $f(x)=\begin{cases}1, \text{ if } |x|<1\\ 0, \text{ if } |x|>1\end{cases}$

a) Find the Fourier Integral of f

b) Show that $\displaystyle\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\begin{cases}\frac{\pi}{2}, |x|<1\\ \frac{\pi}{4}, |x|=1\\ 0, |x|>1\end{cases}$ and deduce $\displaystyle\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$.

Solution: By definition, $f(x)=\int_0^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$.

a) So, the basic task is to determine the coefficients $A(\omega)$ and $B(\omega)$.

First, redefine f as $f(x)=\begin{cases}1, \text{ if } |x|<1\\ 0, \text{ if } |x|>1\end{cases} \Rightarrow f(x)=\begin{cases}1, \text{ if } -1<x<1\\ 0, \text{ if } x<-1 \text{ or } x>1\end{cases}$

Then, using the properties if definite integrals, we have

i) $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} f(\omega)\cos\omega x dx+\int_{-1}^{1} f(\omega)\cos\omega x dx+\int_{1}^{\infty} f(\omega)\cos\omega x dx\right)$$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} 0.\cos\omega x dx+\int_{-1}^{1} 1.\cos\omega x dx+\int_{1}^{\infty} 0.\cos\omega x dx\right)$$

$$=\frac{1}{\pi}\int_{-1}^{1}\cos\omega x dx=\frac{\sin\omega x}{\pi\omega}\Big|_{x=-1}^{x=1}=\frac{\sin\omega-\sin(-\omega)}{\pi\omega}=\frac{2\sin\omega}{\pi\omega}$$

ii) $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} f(\omega)\sin\omega x dx+\int_{-1}^{1} f(\omega)\sin\omega x dx+\int_{1}^{\infty} f(\omega)\sin\omega x dx\right)$$

$$=\frac{1}{\pi}\int_{-1}^{1}\sin\omega x dx=-\frac{\cos\omega x}{\pi\omega}\Big|_{x=-1}^{x=1}=-\frac{\cos\omega-\cos\omega}{\pi\omega}=0$$

$$[\text{Notice}: \sin(-\omega)=-\sin\omega,\ \cos(-\omega)=\cos\omega]$$

Therefore, the Fourier Integral of f becomes;

$$f(x)=\int_{0}^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

b) Equate the Fourier Integral and the function on the given interval.

For $|x|<1$, we have $f(x)=1$. So, using the result of part (a), we have

$$f(x)=1\Rightarrow\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=1\Rightarrow\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\frac{\pi}{2}.$$

For $|x|>1$, $f(x)=0 \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = 0 \Rightarrow \int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = 0$.

For $|x|=1$, the function is discontinuous. So, the Fourier Integral is equal to the average of the limits as stated in the Dirichlet's conditions. That is

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = \frac{\lim_{x\to 1^-} f(x)+\lim_{x\to 1^+} f(x)}{2} = \frac{1}{2} \Rightarrow \int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = \frac{\pi}{4}.$$

Finally, at $x=0$, $f(0)=1 \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos(0)\sin\omega}{\omega}d\omega = 1 \Rightarrow \int_0^\infty \frac{\sin\omega}{\omega}d\omega = \frac{\pi}{2}$.

4.2.2 Sine and cosine Fourier Integrals

Revision about Even and Odd Functions:

i) A function f is *even* if and only if $f(-x)=f(x)$ for all x .

ii) A function f is odd if and only if $f(-x)=-f(x)$ for all x .

Properties of even and odd functions:

i) The of any two or more even functions is even.

ii) The product of an even and an odd function is odd.

iii) The product of any two odd functions is even.

Integral Properties of even and odd functions:

i) $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$ if f is even

ii) $\int_{-L}^{L} f(x)dx = 0$ if f is odd

Fourier sine and Fourier cosine Integrals:

As we have discussed above, the Fourier Integral representation of a function f is given by $f(x)=\int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$.

Case-1: Suppose f is an even function. Since $\cos(\omega x)$ is even and $\sin(\omega x)$ is odd, the product $f(\omega)\cos(\omega x)$ is even while the product $f(\omega)\sin(\omega x)$ is odd. So, using the integral property of even and odd functions,

i) $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\cos\omega x dx$

ii) $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx=0$

Then, the Fourier Integral representation of a function f becomes

$$f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega \text{ where } A(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\cos\omega x dx.$$

The representation $f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega$ is known as Fourier cosine Integral.

Case-2: Suppose f is an odd function.
In this case, the product $f(\omega)\cos(\omega x)$ is odd and $f(\omega)\sin(\omega x)$ is even.
So, using the integral property of even and odd functions,

i) $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx=0$

ii) $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx=\frac{2}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx$

Then, the Fourier Integral representation of a function f becomes

$$f(x)=\int_{0}^{\infty} B(\omega)\sin\omega x d\omega \text{ where } B(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\sin\omega x dx.$$

The representation $f(x)=\int_{0}^{\infty} B(\omega)\sin\omega x d\omega$ is known as Fourier sine Integral.

Examples:

1. Find the Fourier sine and cosine Integrals of $f(x)=e^{-ax}, x>0, a>0$.

Solution:

i) Fourier cosine integral: Using case-1; $f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega$.

Here, using the important integral given in (4), we have

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(\omega)\cos\omega x dx=\frac{2}{\pi}\int_0^\infty e^{-ax}\cos\omega x dx=(\frac{2}{\pi})\frac{a}{a^2+\omega^2}=\frac{2a}{\pi(a^2+\omega^2)}$$

Therefore, the Fourier cosine integral becomes

$$f(x)=\int_0^\infty A(\omega)\cos\omega x d\omega=\int_0^\infty\frac{2a}{\pi(a^2+\omega^2)}\cos\omega x d\omega=\frac{2a}{\pi}\int_0^\infty\frac{\cos\omega x}{a^2+\omega^2}d\omega$$

ii) Fourier sine integral: Using case-2; $f(x)=\int_0^\infty B(\omega)\sin\omega x d\omega$.

Here, using the important integral given in (3), we have

$$B(\omega)=\frac{2}{\pi}\int_0^\infty f(\omega)\sin\omega x dx=\frac{2}{\pi}\int_0^\infty e^{-ax}\cos\omega x dx$$

$$=(\frac{2}{\pi})\frac{e^{-ax}}{a^2+\omega^2}\left[a\sin\omega x-\omega\cos\omega x\right]_{x=0}^{\infty}=(\frac{2}{\pi})\frac{\omega}{a^2+\omega^2}=\frac{2\omega}{\pi(a^2+\omega^2)}$$

Therefore, the Fourier sine integral becomes

$$f(x)=\int_0^\infty B(\omega)\sin\omega x d\omega=\int_0^\infty\frac{2\omega}{\pi(a^2+\omega^2)}\sin\omega x d\omega=\frac{2}{\pi}\int_0^\infty\frac{\omega\sin\omega x}{a^2+\omega^2}d\omega$$

2. Given $\int_0^\infty f(t)\cos\omega t dt=\begin{cases}1-\omega, \text{ if } 0\le\omega\le 1\\ 0, \text{ if } \omega>1\end{cases}$. Find f if it is even.

Solution: From the Fourier cosine integral, $f(t)=\int_0^\infty A(\omega)\cos\omega t d\omega$.

Now, by using the given value of $\int_0^\infty f(t)\cos\omega t dt$ in the problem, we have

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(\omega)\cos\omega t dt=\frac{2}{\pi}\left(\int_0^1 f(\omega)\cos\omega t dt+\int_1^\infty f(\omega)\cos\omega t dt\right)$$

$$=\frac{2}{\pi}\int_0^1 f(\omega)\cos\omega t dt=\frac{2}{\pi}(1-\omega)$$

Then, $f(t)=\int_0^\infty A(\omega)\cos\omega x d\omega=\int_0^\infty\frac{2}{\pi}(1-\omega)\cos\omega t d\omega$.

Now, by using the important integral (2) revised above (integration by parts).

$$f(t)=\int_0^{\infty}\frac{2}{\pi}(1-\omega)\cos\omega t d\omega \frac{2\sin\omega t}{\pi t}-\frac{2}{\pi t^2}(\omega t\sin\omega t+\cos\omega t)\Big|_{\omega=0}^{\omega=1}$$

$$=(\frac{2\sin t}{\pi t}-\frac{2\sin t}{\pi t}-\frac{2\cos t}{\pi t^2})-(-\frac{2}{\pi t^2})=\frac{2}{\pi t^2}(1-\cos t)$$

3. Solve the equation $\int_0^{\infty} f(t)\sin\omega t dt = g(x)$ where $g(x)=\begin{cases}1, \text{if } 0\le x<\pi\\ 0, \text{ if } x>\pi\end{cases}$

Solution: Similarly as in problem 2, we get

$$f(t)=\int_0^{\infty}B(\omega)\sin\omega t d\omega=\int_0^{\pi}B(\omega)\sin\omega t d\omega+\int_{\pi}^{\infty}B(\omega)\sin\omega t d\omega$$

$$=\int_0^{\pi}\frac{2}{\pi}\sin\omega t d\omega=\frac{2}{\pi}(-\frac{\cos\omega t}{t})\Big|_{\omega=0}^{\omega=\pi}=\frac{2}{\pi t}(1-\cos\pi t)$$

4.2.3 Fourier Transforms

Suppose f is a function defined on $(-\infty,\infty)$. Then, the Fourier transform of f is given by $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$. Alternatively, using Euler's formula $e^{-iwx}=\cos wx-i\sin wx$, the Fourier transform of the function can be written as

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[\cos wx-i\sin wx]dx.$$

Note:

i) The form $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$ is said to be *exponential form.*

ii) The form $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[\cos wx-i\sin wx]dx$ is *trigonometric form.*

iii) Which form is better? Since both forms give the same answer, we can use any of the forms. But we select one of the forms depending on the nature of the function whose Fourier transform is needed.

Parserval's Identity: $\int_{-\infty}^{\infty}[F(w)]^2\,dw = \int_{-\infty}^{\infty}[f(x)]^2\,dx$

Examples: 1. Find the Fourier transform of the function $f(x) = \begin{cases} 1, \text{ if } |x| < 1 \\ 0, \text{ if } |x| > 1 \end{cases}$ and deduce that $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi$ and thus $\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

Solution:

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-1} f(x)e^{-iwx}dx + \int_{-1}^{1} f(x)e^{-iwx}dx + \int_{1}^{\infty} f(x)e^{-iwx}dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}dx = -\frac{1}{\sqrt{2\pi}}\frac{e^{-iwx}}{iw}\Bigg|_{x=-1}^{x=1} = \frac{1}{\sqrt{2\pi}}.\frac{e^{iw}-e^{-iw}}{iw} = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$$

Now, apply Parserval's Identity.

$$\int_{-\infty}^{\infty}[F(w)]^2dw = \int_{-\infty}^{\infty}[f(x)]^2dx \Rightarrow \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}\right)^2 dw = \int_{-1}^{1}[f(x)]^2dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 w}{w^2}dw = \int_{-1}^{1}1dx \Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}dw = 2 \Rightarrow \int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}dw = \pi$$

Hence, replacing w by x in this result, we get $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi$.

Furthermore, since both $\sin^2 x$ and x^2 are even, the question $\frac{\sin^2 x}{x^2}$ is even.

Thus, by integral property of even function, $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = 2\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx$.

So, $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi \Rightarrow 2\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi \Rightarrow \int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

2. Find the Fourier transform of

$$a)\ f(x)=\begin{cases} e^{-ax}, & \text{if } x>0 \\ 0, & \text{if } x<0 \end{cases} \qquad b)\ f(x)=\begin{cases} e^{2x}, & \text{if } x>0 \\ e^{-2x}, & \text{if } x<0 \end{cases}.$$

Solution:

$$\text{a) } F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax}.e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+iw)x}dx=\frac{1}{\sqrt{2\pi}}.\frac{1}{(a+iw)}$$

$$\text{b) } F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-(2+iw)x}dx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(2-iw)x}dx$$

$$=\frac{1}{\sqrt{2\pi}}.\frac{1}{(2+iw)}+\frac{1}{\sqrt{2\pi}}.\frac{1}{(2-iw)}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2+iw}+\frac{1}{2-iw}\right)=\frac{4}{\sqrt{2\pi}(4+w^2)}$$

3. Find the Fourier transform of $f(x)=\begin{cases} x^2e^{-x}, & \text{if } x>0 \\ 0, & \text{if } x<0 \end{cases}$.

Solution: Using integration by parts twice on $\int_{0}^{\infty} x^2e^{-(1+iw)x}dx$, we have

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-x}.e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-(1+iw)x}dx$$

$$=\frac{1}{\sqrt{2\pi}}.\frac{2}{(1+iw)^3}=\sqrt{\frac{2}{\pi}}.\frac{1}{(1+iw)^3}$$

4.2.4 Fourier Sine and Cosine Transforms

Fourier Transform of odd and even functions:

Consider the trigonometric form of the Fourier transform representation formula and expand it. That is

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[\cos wx-i\sin wx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\cos wxdx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\sin wxdx$$

Recall that is cosine function even and sine function is odd.

When f is **EVEN**, the product $f(x)\cos wx$ is even but $f(x)\sin wx$ is odd.

This gives $\int_{-\infty}^{\infty}f(x)\cos wxdx=2\int_{0}^{\infty}f(x)\cos wxdx$ and $\int_{-\infty}^{\infty}f(x)\sin wxdx=0$.

Therefore, the above transform formula is reduced as

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\cos wxdx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\sin wxdx\pi$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}f(x)\cos wxdx=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos wxdx$$

When f is **ODD**, the product $f(x)\cos wx$ is odd but $f(x)\sin wx$ is even.

This gives $\int_{-\infty}^{\infty}f(x)\cos wxdx=0$ and $\int_{-\infty}^{\infty}f(x)\sin wxdx=2\int_{0}^{\infty}f(x)\sin wxdx$.

Again, the above transform formula is reduced to

$$F(w)=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}f(x)\sin wxdx=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin wxdx$$

Fourier Sine and Cosine Transforms:

From the discussion of Fourier Transforms of even and odd functions, we can summarize the results for Fourier sine and cosine transforms. But to talk about sine and cosine transforms from the transforms of even and odd functions, we have to restrict the domain of definition to be $(0,\infty)$ rather than $(-\infty,\infty)$. So, for a function on $(0,\infty)$, its Fourier sine and cosine transforms are as follow:

Fourier cosine transform: $F_c(w)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\cos wx dx$.

Fourier sine transform: $F_s(w)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin wx dx$.

Important relation between Fourier Integrals and Transforms:

It is crucially advisable to readers to understand the following important relations between Fourier sine and cosine Integrals with Fourier sine and cosine transforms. Because in many problematic situations, students are challenged to determine Fourier Integrals from the given Fourier transforms or Fourier transforms from the given Fourier integrals.

Fourier cosine transform: $F_c(w)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\cos wx dx$.

Fourier cosine integral: $f(x)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} F_c(w)\cos \omega x d\omega$.

The two formula are said to be cosine transform of each other. If we are given $F_c(w)$, we can determine $f(x)$ from the cosine integral formula or vice-versa.

Fourier sine transform: $F_s(w)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin wx dx$.

Fourier sine integral: $f(x)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} F_s(w)\sin \omega x d\omega$.

The two formula are said to be sine transform of each other. If we are given $F_s(w)$, we can determine $f(x)$ from the sine integral formula or vice-versa.

Examples:

1. Find the Fourier cosine and sine transforms of $f(x)=\begin{cases}1, \text{ if } 0<x<1\\ 0, \text{ if } x>1\end{cases}$

Using the result, show that $\int_0^{\infty}\frac{\sin^4 x}{x^2}dx=\frac{\pi}{2}$ & $\int_0^{\infty}\frac{(1-\cos x)^2}{x^2}dx=\frac{\pi}{2}$.

Solution:

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos wx dx=\sqrt{\frac{2}{\pi}}\int_0^{1}f(x)\cos wx dx+\sqrt{\frac{2}{\pi}}\int_1^{\infty}f(x)\cos wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{1}\cos wx dx=\sqrt{\frac{2}{\pi}}\frac{\sin \omega x}{\omega}\Big|_{x=0}^{x=1}=\sqrt{\frac{2}{\pi}}.\frac{\sin\omega}{\omega}$$

Fourier sine transform:

$$F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin wx dx=\sqrt{\frac{2}{\pi}}\int_0^{1}f(x)\sin wx dx+\sqrt{\frac{2}{\pi}}\int_1^{\infty}f(x)\sin wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{1}\sin wx dx=\sqrt{\frac{2}{\pi}}(-\frac{\cos \omega x}{\omega})\Big|_{x=0}^{x=1}=\sqrt{\frac{2}{\pi}}.\frac{(1-\cos\omega)}{\omega}$$

2. Find the Fourier sine and cosine transforms of $f(x)=e^{-x}, x\geq 0$.

i) Using the resulting transform, deduce that $\int_0^{\infty}\frac{x\sin ax}{x^2+1}dx=\frac{\pi}{2}e^{-a}, a>0$.

ii) At $x=0$, the representation $e^{-x}=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin wx}{1+\omega^2}dw \Rightarrow 1=0$. Why it is wrong?

iii) Using the transform, deduce that $\int_0^{\infty}\frac{\cos \omega x}{\omega^2+1}d\omega=\frac{\pi}{2}e^{-x}, x\geq 0$.

Solution:

Fourier sine transform:

$$F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin wx dx=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\sin wx dx$$

$$=(\sqrt{\frac{2}{\pi}})\frac{e^{-x}}{1+\omega^2}[\sin\omega x-\omega\cos\omega x]\Big|_{x=0}^{\infty}=\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{1+\omega^2}\right)$$

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos wx\,dx$$

$$=\left(\sqrt{\frac{2}{\pi}}\right)\frac{e^{-x}}{1+\omega^2}\left[-\cos\omega x+\sin\omega x\right]_{x=0}^{\infty}=\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\omega^2}\right)$$

Now, apply the dual formula that relates integrals and transforms.

i) Use Fourier sine integral and sine transform.

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(w)\sin wx\,dw=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{1+\omega^2}\right)\sin wx\,dw=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin wx}{1+\omega^2}dw$$

Hence, we get the representation $f(x)=e^{-x}=\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{\omega\sin wx}{1+\omega^2}dw.$

Then, using $x=a$ in $f(x)=e^{-x}$ and its Fourier integral representation, we have

$$f(x)=e^{-x}\Rightarrow\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin a\omega}{1+\omega^2}dw=e^{-a}\Rightarrow\int_0^{\infty}\frac{\omega\sin a\omega}{1+\omega^2}dw=\frac{\pi}{2}e^{-a}.$$

Fourier cosine integral: $f(x)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f_c(w)\cos\omega x\,d\omega$.

iii) Use Fourier sine integral and sine transform.

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(w)\cos\omega x\,d\omega=\frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{1+\omega^2}\right)\cos\omega x\,d\omega$$

Then, equating $f(x)=e^{-x}$ with its Fourier integral representation, we have

$$f(x)=e^{-x}\Rightarrow\frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{1+\omega^2}\right)\cos\omega x\,d\omega=e^{-x}\Rightarrow\int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}dw=\frac{\pi}{2}e^{-x},x\geq 0.$$

3. Find the Fourier transform of $f(x)=e^{-|x|}$.

Solution: From the trigonometric form of the definition, we have

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos wx\,dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin wx\,dx.$$

Since $f(x)=e^{-|x|}$ is even function, its Fourier transform is the same as its cosine transform. That is $F(w)=F_c(w)$.

So, using the Fourier cosine transform formula, we have

$$F(w)=F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\cos wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos wx dx \qquad (e^{-|x|}=e^{-x} \text{ for } x>0)$$

$$=\sqrt{\frac{2}{\pi}}\frac{e^{-x}}{1+\omega^2}\left[-\cos\omega x+\omega\sin\omega x\right]_{x=0}^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$$

4. Find the Fourier cosine and sine transforms of $f(x)=\begin{cases} x, & \text{if } 0<x<1 \\ 0, & \text{otherwise}\end{cases}$

Solution:

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx=\sqrt{\frac{2}{\pi}}\int_0^1 f(x)\cos wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 x\cos wx dx=\sqrt{\frac{2}{\pi}}(\omega x\sin\omega x+\cos\omega x)\Big|_{x=0}^{x=1}$$

$$=\sqrt{\frac{2}{\pi}}.\frac{\omega\sin\omega+\cos\omega-1}{\omega^2}$$

CHAPTER-5

CALCULUS OF COMPLEX FUNCTIONS

5.1 Functions of Complex Variables

Definition: A function of a complex variable z is a rule that assigns to each complex number z in a set D a unique complex number w. That is $f:D\to C$ such that $\forall z\in D, \exists w\in C, \ni f(z)=w$. Here, the set D is called the domain of definition of f and the set of all images $R=\{w=f(z):z\in D\}$ is called the *range* of f. Since $z=x+iy$, the image is also written in the form $w=u+iv$. Thus, $f(z)=f(x+iy)=u+iv$. Since, all z,u,v are functions of x and y, from now on wards we use the notation $f(z)=f(x,y)=u(x,y)+iv(x,y)$.

Examples:

Express the complex functions in the form $f(x,y)=u(x,y)+iv(x,y)$.

a) $f(z)=z^2+3z$ *b)* $f(z)=z.\bar{z}$ *c)* $f(z)=e^{3z}$

Solution: Let $z=x+iy$. Then,

a) $f(z)=z^2+3z=(x+iy)^2+3(x+iy)$

$$=x^2+2ixy+i^2y^2+3x+3iy=x^2-y^2+3x+i(2xy+3y)$$

Hence, it is of the form $f(x,y)=u(x,y)+iv(x,y)$ where

$u(x,y)=x^2-y^2+3x, v(x,y)=2xy+3y$.

b) $f(z)=z.\bar{z}=(x+iy)(x-iy)=x^2-ixy+ixy-i^2y^2=x^2+y^2$.

Hence, $f(x,y)=u(x,y)+iv(x,y)$ where $u(x,y)=x^2+y^2, v(x,y)=0$.

c) $f(z)=e^{3z}=e^{3(x+iy)}=e^{3x}.e^{i(3y)}=e^{3x}(\cos 3y+i\sin 3y)$.

Hence, $f(x,y)=u(x,y)+iv(x,y)$ where $u(x,y)=e^{3x}\cos 3y, v(x,y)=e^{3x}\sin 3y$.

5.2 Limits and Continuity of Complex Functions

Limits of Complex Functions: Intuitively, a complex number L is said to be the limit of the complex function f at a point $z_0 = a + bi$ if $f(z)$ gets closer and closer to L as the point $z = x + yi$ gets closer and closer to $z_0 = a + bi$. This is denoted by $\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(a,b)} f(x,y) = L$ or $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$.

Limit in terms of real and imaginary parts:

Suppose $f(z) = f(x,y) = u(x,y) + iv(x,y), z_0 = a + bi, L = u_0 + iv_0$.

Then, $\lim_{z\to z_0} f(z) = L = u_0 + iv_0 \Leftrightarrow \lim_{(x,y)\to(a,b)} u(x,y) = u_0$ and $\lim_{(x,y)\to(a,b)} v(x,y) = v_0$

This approach has dual implications. On one hand it tells us that the limit of the complex function exists if and only if the limits of its real and imaginary parts exist at the same time.

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} u(x,y) + \mathbf{i}\left(\lim_{(x,y)\to(a,b)} v(x,y)\right).$$

i) By Factorization: Factorize the numerator and the denominator and cancel the factor that poses trouble to evaluate the limit.

Examples: Evaluate

$a)\lim_{z\to 2i}\frac{z^4 - 2iz^3}{z^2+4}$ $\quad b)\lim_{z\to 3i}\frac{z^2+3}{z-\sqrt{3}i}$ $\quad c)\lim_{z\to 3+3i}\frac{x^2-xy}{x-y}+i\frac{x^4-y^4}{x^2-y^2}$

Solution: Here, we cannot evaluate the limit directly.

$$a)\ \lim_{z\to 2i}\frac{z^4-2iz^3}{z^2+4} = \lim_{z\to 2i}\frac{z^3(z-2i)}{(z-2i)(z+2i)} = \lim_{z\to 2i}\frac{z^3}{(z+2i)} = \frac{-8i}{4i} = -2$$

$$b)\ \lim_{z\to 3i}\frac{z^2+3}{z-\sqrt{3}i} = \lim_{z\to 3i}\frac{(z-\sqrt{3}i)(z+\sqrt{3}i)}{z-\sqrt{3}i} = \lim_{z\to 3i}(z+\sqrt{3}i) = 2\sqrt{3}i$$

$$c)\ \lim_{z\to 3+3i}\frac{x^2-xy}{x-y}+i\frac{x^4-y^4}{x^2-y^2} = \lim_{(x,y)\to(3,3)}\frac{x(x-y)}{x-y}+i\left(\lim_{(x,y)\to(3,3)}\frac{(x^2-y^2)(x^2+y^2)}{x^2-y^2}\right)$$

$$= \lim_{(x,y)\to(3,3)} x + i\left(\lim_{(x,y)\to(3,3)}(x^2+y^2)\right) = 3+18i$$

ii) By Rationalization: Evaluate $\lim\limits_{z\to 3-i} \dfrac{z+i-3}{\sqrt{z+2+i}-\sqrt{5}}$.

Solution: By rationalizing the denominator,

$$\lim_{z\to 3-i} \frac{z+i-3}{\sqrt{z+2+i}-\sqrt{5}} = \lim_{z\to 3-i} \frac{z+i-3}{\sqrt{z+2+i}-\sqrt{5}}\left(\frac{\sqrt{z+2+i}+\sqrt{5}}{\sqrt{z+2+i}+\sqrt{5}}\right)$$

$$= \lim_{z\to 3-i} \frac{(z+i-3)(\sqrt{z+2+i}+\sqrt{5})}{z+i-3} = 2\sqrt{5}$$

Continuity of Complex Functions: A complex function f defined at z_0 is said to be continuous at a point $z = z_0$ if and only if $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

Examples:

1. Show that

a) $f(z) = \dfrac{z+5i}{z^2+4i+6}$ is continuous at $z_0 = 1-2i$.

b) $f(z) = \begin{cases} \dfrac{z^2-2iz}{z^2+4}, & z_0 \neq 2i \\ 3i & , z_0 = 2i \end{cases}$ is not continuous at $z_0 = 2i$.

Solution:

a) $\lim\limits_{z\to 1-2i} \dfrac{z+5i}{z^2+4i+6} = \lim\limits_{z\to 1-2i} \dfrac{1-2i+5i}{(1-2i)^2+4i+6} = \lim\limits_{z\to 1-2i} \dfrac{1-2i+5i}{1-4i-4+4i+6} = \dfrac{1}{3}+i$.

Besides, $f(z) = \dfrac{z+5i}{z^2+4i+6} \Rightarrow f(z_0) = f(1-2i) = \dfrac{1-2i+5i}{(1-2i)^2+4i+6} = \dfrac{1}{3}+i$.

That means $\lim\limits_{z\to 1-2i} f(z) = f(1-2i) = \dfrac{1}{3}+i$. Therefore, f is continuous.

b) Here, $\lim\limits_{z\to 2i} f(z) = \lim\limits_{z\to 2i} \dfrac{z^2-2iz}{z^2+4} = \lim\limits_{z\to 2i} \dfrac{z(z-2i)}{(z-2i)(z+2i)} = \lim\limits_{z\to 2i} \dfrac{z}{z+2i} = \dfrac{1}{2}$.

However, $f(z_0) = f(2i) = 3i \neq \lim\limits_{z\to 2i} f(z)$. So, f is not continuous at $z_0 = 2i$

5.3 Differentiability and Cauchy-Riemann Equations

Conditions of Differentiability and Cauchy-Riemann Equations:

A function $f(x,y)=u(x,y)+iv(x,y)$ is said to be differentiable in a region R (the derivative exists in R) if and only if the two conditions are satisfied:

i) The partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ are continuous in R

ii) The equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$ are satisfied

Note:

a) The equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \&\ \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$ are called *Cauchy-Riemann-Equations*.

b) Here, by checking these two conditions (continuity and CRE), we can verify differentiability of f. But the basic question is how to find $f'(z)$ if it exists because it is not an easy task to find derivative of complex functions.

Derivative Formulas:

The formula or techniques of finding derivative $f'(z)$ of complex function are easily derived using Cauchy-Riemann-Equations (CRE).

If a function $f(z)=f(x,y)=u(x,y)+iv(x,y)$ is differentiable, its derivative is obtained using one of the following formulas.

i) $f'(z)=u_x(x,y)+iv_x(x,y)$ (In terms of u and v)

ii) $f'(z)=u_x(x,y)-iu_y(x,y)$ (In terms of u only)

iii) $f'(z)=v_y(x,y)+iv_x(x,y)$ (In terms of v only)

iv) $f'(z)=v_y(x,y)-iu_y(x,y)$ (In terms of u and v)

Important Notice! Each of these formulas gives the same result for $f'(z)$.

But which of these to use depends on the situation of the problem. In the problem, only u, or only v or both may be given. If only u is given, we prefer the second formula, if only v is given, we prefer the third formula, if both u and v are given, we may prefer either of these formula.

Examples:

1. Show that the following functions satisfy Cauchy-Riemann equations and conclude that they are differentiable and find their derivatives.

a) $f(x,y)=x^3-3xy^2+\mathrm{i}(3x^2y-y^3)$ *b)* $f(x,y)=e^{-4y}\cos 4x+\mathrm{i}e^{-4y}\sin 4x$

c) $f(x,y)=x^2-y^2+3y+\mathrm{i}(2xy-3x)$ *d)* $f(x,y)=\ln\sqrt{x^2+y^2}+\mathrm{i}\tan^{-1}\left(\frac{y}{x}\right)$

e) $f(x,y)=e^{2x}\cos y+\mathrm{i}(e^{2x}\sin 2y)$

Solution:

$$a)\begin{cases} u(x,y)=x^3-3xy^2 \Rightarrow \dfrac{\partial u}{\partial x}=3x^2-3y^2, \dfrac{\partial u}{\partial y}=-6xy \\ v(x,y)=3x^2y-y^3 \Rightarrow \dfrac{\partial v}{\partial x}=6xy, \dfrac{\partial v}{\partial y}=3x^2-3y^2 \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Here, each of the partial derivatives are continuous and satisfies CREs. Hence, by the above condition of differentiability, f is differentiable. Therefore, using either of the derivative formula, we have

i) $f'(z)=u_x(x,y)+iv_x(x,y)=3x^2-3y^2+i(6xy)$ **or**

ii) $f'(z)=u_x(x,y)-iu_y(x,y)=3x^2-3y^2+i(6xy)$ **or**

iii) $f'(z)=v_y(x,y)+iv_x(x,y)=3x^2-3y^2+i(6xy)$ **or**

iv) $f'(z)=v_y(x,y)-iu_y(x,y)=3x^2-3y^2+i(6xy)$ **or**

(Do you see? We get the same result using either of the derivative formulas)

$$b)\begin{cases} u(x,y)=e^{-4y}\cos 4x \Rightarrow \dfrac{\partial u}{\partial x}=-4e^{-4y}\sin 4x, \dfrac{\partial u}{\partial y}=-4e^{-4y}\cos 4x \\ v(x,y)=e^{-4y}\sin 4x \Rightarrow \dfrac{\partial v}{\partial x}=4e^{-4y}\cos 4x, \dfrac{\partial v}{\partial y}=-4e^{-4y}\sin 4x \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Hence, by the above condition of differentiability, f is differentiable.

Therefore, $f'(z)=u_x(x,y)+iv_x(x,y)=-4e^{-4y}\sin 4x+i(4e^{-4y}\cos 4x)$

$$c)\begin{cases} u(x,y)=x^2-y^2+3y \Rightarrow \dfrac{\partial u}{\partial x}=2x, \dfrac{\partial u}{\partial y}=-2y+3 \\ v(x,y)=2xy-3x \Rightarrow \dfrac{\partial v}{\partial x}=2y-3, \dfrac{\partial v}{\partial y}=2x \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Hence, by the above condition of differentiability, f is differentiable.

Therefore, $f'(z)=u_x(x,y)+iv_x(x,y)=2x+\mathbf{i}(2y-3)$.

$$d)\begin{cases} u(x,y)=\ln\sqrt{x^2+y^2} \Rightarrow \dfrac{\partial u}{\partial x}=\dfrac{x}{x^2+y^2}, \dfrac{\partial u}{\partial y}=\dfrac{y}{x^2+y^2} \\ v(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right) \Rightarrow \dfrac{\partial v}{\partial x}=\dfrac{-y}{x^2+y^2}, \dfrac{\partial v}{\partial y}=\dfrac{x}{x^2+y^2} \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Hence, by the above condition of differentiability, f is differentiable.

Therefore, $f'(z)=u_x(x,y)+iv_x(x,y)=\dfrac{x}{x^2+y^2}-i(\dfrac{y}{x^2+y^2})$.

$$e)\begin{cases} u(x,y)=e^x\cos y \Rightarrow \dfrac{\partial u}{\partial x}=e^x\cos y, \dfrac{\partial u}{\partial y}=-e^x\sin y \\ v(x,y)=e^x\sin y \Rightarrow \dfrac{\partial v}{\partial x}=e^x\sin y, \dfrac{\partial v}{\partial y}=e^x\cos y \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Hence, by the above condition of differentiability, f is differentiable.

Therefore, $f'(z)=u_x(x,y)+iv_x(x,y)=e^x\cos y+i(e^x\sin y)$

2. Show that the following functions are not differentiable.

a) $f(z)=x^3-3xy^2+2x+i(3x^2y-y^3)$ b) $f(x,y)=x^4+iy^4$

c) $f(x,y)=x^2-y^2-5y+\mathbf{i}(2xy+5y)$

Solution:

$$a)\begin{cases} u(x,y)=x^3-3xy^2+2x \Rightarrow \dfrac{\partial u}{\partial x}=3x^2-3y^2+2, \dfrac{\partial u}{\partial y}=-6xy \\ v(x,y)=3x^2y-y^3 \Rightarrow \dfrac{\partial v}{\partial x}=6xy, \dfrac{\partial v}{\partial y}=3x^2-3y^2 \end{cases} \Rightarrow \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$$

Therefore, f is not differentiable.

5.5 Analytic and Harmonic Functions

Analytic Functions: A function f is said to be *analytic at* a if its derivative exists at each point z in some open set containing a.

If f is analytic at each point in a region R, then we say f is *analytic* on R.

If f is analytic on the whole complex plane, f is said to be *entire function.*

Examples:

a) Consider $f(z)=z^3-3z^2+5e^{2z}$. Here, $f'(z)=3z^2-6z+10e^{2z}$ is defined for any complex number $z=a$. So, the function is analytic and it is also entire.

b) Consider $f(z)=4x^2+5x-4y^2+9+i(8xy+5y-1)$.

$$\frac{\partial u}{\partial x}=8x+5, \frac{\partial u}{\partial y}=-8y, \frac{\partial v}{\partial x}=8y, \frac{\partial v}{\partial y}=8x+5 \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}, \forall z=x+iy$$

Thus, it is analytic and it is also an entire function.

c) Consider $f(z)=\ln(x^2+y^2)+9+i\tan^{-1}(\frac{y}{x})$. Here, f is analytic at all points in its domain except the origin. But it is not an entire function.

Note: Differentiability and analyticity are different concepts. A function may be differentiable at a point but may not be analytic.

However, every analytic function is differentiable.

Harmonic Functions: A real valued function $f(x,y)$ in a region R that has continuous second partial derivatives is said to be *Harmonic function* if it satisfies the Laplace's equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ in the region R.

Examples: Determine whether the following functions are harmonic or not.

a) $f(x,y)=x^3y-xy^3+4$ *b)* $f(x,y)=2x^2-2y^2+4xy$

c) $f(x,y)=e^{3x}\cos 3y$ *d)* $f(x,y)=x^2+y^2-xy$

Solution:

a) Here, $\frac{\partial f}{\partial x}=3x^2y-y^3, \frac{\partial^2 f}{\partial x^2}=6xy, \frac{\partial f}{\partial y}=x^3-3xy^2, \frac{\partial^2 f}{\partial y^2}=-6xy$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=6xy-6xy=0$$

Thus, the function is harmonic.

b) $\frac{\partial f}{\partial x}=2x+4y, \frac{\partial^2 f}{\partial x^2}=2, \frac{\partial f}{\partial y}=-2y+4x, \frac{\partial^2 f}{\partial y^2}=-2 \Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=2-2=0$

c) Here, $\frac{\partial^2 f}{\partial x^2}=9e^{3x}\cos 3y, \frac{\partial^2 f}{\partial y^2}=-9e^{3x}\cos 3y \Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

d) Here, $\frac{\partial^2 f}{\partial x^2}=2, \frac{\partial^2 f}{\partial y^2}=2 \Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4\neq 0$

Thus, the function is not harmonic.

Relation between Harmonic and Analytic Functions:

Suppose $f(x,y)=u(x,y)+iv(x,y)$ is analytic function in a region R. Then, it satisfies the Cauchy-Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$.

Differentiating $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ with respect to x and $\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$ with respect to y, we get $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}, \frac{\partial^2 u}{\partial y^2}=\frac{-\partial^2 v}{\partial y\partial x}$.

Now adding these gives us $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}$ (Since the function is analytic, $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous and thus $\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$.

Hence, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. This means the real part $u(x,y)$ of the analytic function $f(x,y)=u(x,y)+iv(x,y)$ is Harmonic. Similarly, by differentiating $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ with respect to y and $\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$ with respect to x, we get $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

This means the imaginary part of $f(x,y)=u(x,y)+iv(x,y)$ is also Harmonic.

Therefore, if $f(x,y)=u(x,y)+iv(x,y)$ is analytic, then both $u(x,y)$ and $v(x,y)$ are harmonic.

In such cases, $v(x,y)$ is said to be the *harmonic conjugate* of $u(x,y)$.

Remark:

If a harmonic function $u(x,y)$ is given, then it is possible to find the harmonic conjugate $v(x,y)$ by applying Cauchy-Riemann equations so that the function $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

Examples:

1. Verify that the following functions are harmonic and find the corresponding harmonic conjugate $v(x,y)$ such that $f(z)=u(x,y)+iv(x,y)$ is analytic.

$a)\ u(x,y)=x^3-3xy^2$ $\quad b)\ u(x,y)=e^{-y}\sin x$ $\quad c)\ u(x,y)=xy$

$d)\ u(x,y)=x^3-3xy^2+4xy$ $\quad e)\ u(x,y)=e^{2x}\cos 2y$

Solution:

a) $\dfrac{\partial u}{\partial x}=3x^2-3y^2,\ \dfrac{\partial u}{\partial y}=-6xy \Rightarrow \dfrac{\partial^2 u}{\partial x^2}=6x, \dfrac{\partial^2 u}{\partial y^2}=-6x \Rightarrow \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

Hence, $u(x,y)=x^3-3xy^2$ is a harmonic function. So, there exists a harmonic function $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\Rightarrow\dfrac{\partial v}{\partial y}=3x^2-3y^2$. Integrating both sides with respect to y gives us $v(x,y)=3x^2y-y^3+g(x)$. Again from the second Riemann equation, $\dfrac{\partial u}{\partial y}=\dfrac{-\partial v}{\partial x}\Rightarrow -6xy+g'(x)=-6xy\Rightarrow g'(x)=0\Rightarrow g(x)=c$

Hence, $v(x,y)=3x^2y-y^3+c$ and the required analytic function is

$f(x,y)=x^3-3xy^2+i(3x^2y-y^3+c)$.

b) $u_x=e^{-y}\cos x, u_{xx}=-e^{-y}\sin x, u_y=-e^{-y}\sin x, u_{yy}=e^{-y}\sin x\Rightarrow u_{xx}+u_{yy}=0.$

Hence, $u(x,y)=e^{-y}\sin x$ is a harmonic function. So, there exists a harmonic function $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

Now, from the Cauchy-Riemann equations, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\Rightarrow\dfrac{\partial v}{\partial y}=e^{-y}\cos x$.

Integrating both sides with respect to y gives us $v(x,y)=-e^{-y}\cos x+g(x)$.

A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow -e^{-y}\sin x=-e^{-y}\sin x-g'(x)\Rightarrow g'(x)=0\Rightarrow g(x)=c \cdot$$

Hence, $v(x,y)=-e^{-y}\cos x+c$ and the required analytic function.

c) $\frac{\partial u}{\partial x}=y,\ \frac{\partial u}{\partial y}=x\Rightarrow\frac{\partial^2 u}{\partial x^2}=0,\frac{\partial^2 u}{\partial y^2}=0\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Hence, $u(x,y)$ is a harmonic function. So, there exists a harmonic function $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=y$. Integrating both sides with respect to y gives us $v(x,y)=\frac{y^2}{2}+g(x)$. A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow g'(x)=-x\Rightarrow\int g'(x)dx=\int -xdx\Rightarrow g(x)=-\frac{x^2}{2}.$$

Hence, $v(x,y)=\frac{y^2}{2}-\frac{x^2}{2}$.

d) $\frac{\partial u}{\partial x}=3x^2-3y^2+4y,\frac{\partial u}{\partial y}=-6xy+4x\Rightarrow\frac{\partial^2 u}{\partial x^2}=6x,\frac{\partial^2 u}{\partial y^2}=-6x\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Hence, $u(x,y)$ is a harmonic function. So, there exists a harmonic conjugate $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=3x^2-3y^2+4y$. Integrating both sides with respect to y gives us $v(x,y)=3x^2y-y^3+2y^2+g(x)$.

A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow -6xy-g'(x)=-6xy+4x\Rightarrow g'(x)=-4x\Rightarrow g(x)=-2x^2$$

Hence, $v(x,y)=3x^2y-y^3+2y^2-2x^2+c$.

e) $\frac{\partial^2 u}{\partial x^2}=e^x\cos y,\frac{\partial^2 u}{\partial y^2}=-e^x\cos y\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

Hence, $u(x,y)=e^{x}\cos y$ is a harmonic function. So, there exists a harmonic function $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=e^{x}\cos y$.

Integrating both sides with respect to y gives us

$v(x,y)=e^{x}\sin y+g(x)$. A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow -e^{x}\sin y+g'(x)=-e^{x}\sin y\Rightarrow g'(x)=0\Rightarrow g(x)=c$$

Hence, $v(x,y)=e^{x}\sin y+c$ and the required analytic function is

$f(x,y)=e^{x}\cos y+ie^{x}\sin y$. (You can add a constant!)

2. Show that $u(x,y)$ is harmonic and find its harmonic conjugate $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

$a)$ $u(x,y)=x^{3}-3xy^{2}+3x^{2}-3y^{2}+1$ $\quad$ $b)$ $u(x,y)=e^{x}\cos y+e^{y}\cos x+xy$

Solution:

$$a)\ \frac{\partial u}{\partial x}=3x^{2}-3y^{2}+6x,\ \frac{\partial u}{\partial y}=-6xy-6y\Rightarrow\frac{\partial^{2}u}{\partial x^{2}}=6x+6,\frac{\partial^{2}u}{\partial y^{2}}=-6x-6$$

$$\Rightarrow\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{2}u}{\partial y^{2}}=0$$

Hence, $u(x,y)$ is a harmonic function. So, there exists a harmonic conjugate $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=3x^{2}-3y^{2}+6x$.

Integrating both sides with respect to y gives us

$v(x,y)=3x^{2}y-y^{3}+6xy+g(x)$

A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow -6xy-6y-g'(x)=-6xy-6y\Rightarrow g'(x)=0\Rightarrow g(x)=c$$

Hence, $v(x,y)=3x^{2}y-y^{3}+6xy+c$.

b) $\frac{\partial u}{\partial x}=e^x\cos y-e^y\sin x+y,\ \frac{\partial u}{\partial y}=-e^x\sin y+e^y\cos x+x$

$$\Rightarrow\frac{\partial^2 u}{\partial x^2}=e^x\cos y-e^y\cos x,\frac{\partial^2 u}{\partial y^2}=-e^x\cos y+e^y\cos x\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Hence, $u(x,y)$ is a harmonic function. So, there exists a harmonic function $v(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=e^x\cos y-e^y\sin x+y$.

Integrating both sides gives us $v(x,y)=e^x\sin y-e^y\sin x+\frac{y^2}{2}+g(x)$.

A gain from the second Riemann equation,

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow-e^x\sin y+e^y\cos x-g'(x)=-e^x\sin y+e^y\cos x+x$$

$$\Rightarrow g'(x)=-x\Rightarrow\int g'(x)dx=\int-xdx\Rightarrow g(x)=-\frac{x^2}{2}$$

Hence, $v(x,y)=e^x\sin y-e^y\sin x+\frac{y^2}{2}-\frac{x^2}{2}$.

3. Show that $v(x,y)=2xy+x$ is harmonic and find another harmonic function $u(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

Solution: $\frac{\partial v}{\partial x}=2y+1,\ \frac{\partial v}{\partial y}=2x\Rightarrow\frac{\partial^2 v}{\partial x^2}=0,\frac{\partial^2 v}{\partial y^2}=0\Rightarrow\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

Hence, $v(x,y)=2xy+x$ is a harmonic function. So, there exists a harmonic function $u(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic. Now, from the Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial u}{\partial x}=2x$. Integrating both sides with respect to x gives us $u(x,y)=x^2+g(y)$. A gain from the second Riemann equation, $\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}\Rightarrow g'(y)=-2y-1\Rightarrow g(y)=-y^2-y$

Hence, $u(x,y)=x^2-y^2-y$ and $f(x,y)=x^2-y^2-y+i(2xy+x)$.

5.6 Complex Integrations

5.6.1 Contour Integral

If f is continuous on smooth curve C given by the parameterization $z(t)=x(t)+iy(t), a\le t\le b$, then the integral along the curve C denoted by $\int_C f(z)dz$ is known as *contour* or *line integral of* f and it is given by

$\int_C f(z)dz=\int_a^b f(z(t))z'(t)dt$. Here, C is known as *path of integration.*

Examples:

1. Evaluate the following contour integrals.

a) $\int_C z^2dz$ where C is given by $z(t)=3t+2it, -1\le t\le 1$

b) $\int_C \bar{z}dz$ where C is given by $x(t)=3,\ y(t)=t^2,\ 1\le t\le 2$

c) $\int_C \frac{1}{z}dz$ where $C: z(t)=e^{it},\ -\frac{\pi}{2}\le t\le \frac{\pi}{2}$.

d) $\int_C f(z)dz$, where $f(z)=x^2+y+ixy$ and C is part of curve $y=x^2$, from $a=0$ to $b=1+i$.

e) $\int_C (x^2+y^2)dz$ where C is the line segment from $z=2+4i$ to $z=5+6i$.

Solution:

a) Here, $f(z)=z^2, z=3t+2it, z'=(3+2i)dt \Rightarrow f(z)=(3t+2it)^2=5t^2+12it^2$.

$$\int_C z^2dz=\int_{-1}^{1} f(z)z'(t)dt=\int_{-1}^{1}(3t+2it)^2(3+2i)dt=\int_{-1}^{1}(-9t^2+46it^2)dt=-6+\frac{92}{3}i$$

b) Here, $f(z)=\bar{z}, z(t)=3+it^2, z'(t)=2itdt \Rightarrow f(z(t))=\bar{z}(t)=3-it^2$.

$$\int_C \bar{z}\,dz = \int_1^2 (3-it^2)2it\,dt = \int_1^2 (6ti+2t^3)\,dt = 3t^2 i + \left.\frac{t^4}{2}\right|_1^2 = 9i + \frac{15}{2}$$

c) $z(t)=e^{it} \Rightarrow z'(t)=ie^{it}dt \Rightarrow \int_C \frac{1}{z}dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{z'(t)}{z(t)}dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ie^{it}}{e^{it}}dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\,dt = \pi i$

d) Recall from *Chapter 3* that if $y=f(x), a\le x\le b$, then its parameterization is of the form $x=t, y=f(t) \Rightarrow z(t)=t+if(t), a\le t\le b$. As a result, part of the parabola $y=x^2$ connecting the two points is $x(t)=t,\ y(t)=t^2,\ 0\le t\le 1$.

So, $\int_C f(z)dz = \int_0^1 (2t^2+it^3)(1+2ti)dt = \int_0^1 (2t^2-2t^4+5t^3 i)dt = \frac{4}{15}+\frac{5}{4}i$

2. Evaluate the following integrals using appropriate parameterizations.

a) $\int_C (x^3-iy^3)dz$ **where** C **is the lower half of** $|z|=1$ **from** $z=-1$ **to** $z=1$

b) $\int_C \left(\frac{1}{(z+i)^3}-\frac{5}{z+i}+8\right)dz$ **where** C **is the circle** $|z+i|=1$

c) $\int_C f(z)dz$ **where** $f(z)=\begin{cases}2, x<0\\ 6x, x\ge 0\end{cases}$ **and** C **is the parabola** $y=x^2$ **from** $z=-1+i$ **to** $z=1+i$

d) $\int_C dz$ **where** C **is the left half of the ellipse** $\frac{x^2}{36}+\frac{y^2}{4}=1$ **from** $z=2i$ **to** $z=-2i$

e) $\int_C (3z^2-4z)dz$ **where** C **is the curve along** $y=x^3$ **from** $z=0$ **to** $z=2+8i$

Solution:

a) The lower half of the circle $|z|=1$ **from** $z=-1$ **to** $z=1$ **is parameterized by** $z=e^{it}, \pi\le t\le 2\pi$ **. Besides,**

$z=e^{it}=\cos t+i\sin t \Rightarrow dz=ie^{it}dt, x=\cos t=\frac{e^{it}+e^{-it}}{2}, y=\sin t=\frac{e^{it}-e^{-it}}{2i}$.

$$\text{Hence, } \int_C (x^3 - iy^3)dz = \int_{\pi}^{2\pi}\left(\left(\frac{e^{it}+e^{-it}}{2}\right)^3 - i\left(\frac{e^{it}-e^{-it}}{2i}\right)^3 \right) ie^{it}\,dt$$

$$= \int_{\pi}^{2\pi}\left(\frac{1}{8}(e^{3it}+3e^{it}+3e^{-it}+e^{-3it}) + \frac{1}{8}(e^{3it}-3e^{it}+3e^{-it}-e^{-3it}) \right) ie^{it}\,dt \text{ b)}$$

$$= \int_{\pi}^{2\pi}\left(\frac{1}{8}(e^{3it}+3e^{-it}) + \frac{1}{8}(e^{3it}+3e^{-it}) \right) ie^{it}\,dt = \frac{i}{8}\int_{\pi}^{2\pi}(2e^{4it}+6)dt = \frac{3\pi i}{4}$$

The circle $|z+i| = 1$ is parameterized by $z+i = e^{it}, 0 \le t \le 2\pi$.

Besides, $z+i = e^{it} \Rightarrow dz = ie^{it}dt$. Hence,

$$\int_C \left(\frac{1}{(z+i)^3} - \frac{5}{z+i} + 8 \right) dz = \int_0^{2\pi}\left(\frac{1}{e^{3it}} - \frac{5}{e^{it}} + 8 \right) ie^{it}\,dt = \int_0^{2\pi}\left(ie^{-2it} - 5i + 8ie^{it} \right) dt$$

$$= \left(\frac{-e^{-2it}}{2} - 5ti + 4e^{it} \right)\Bigg|_{t=0}^{t=2\pi} = -10\pi i$$

c) Here, $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ where

$C_1 : x = t, y = t^2, -1 \le t \le 0, z = t + it^2, dz = (1+2it)dt$ and

$C_2 : x = t, y = t^2, 0 \le t \le 1, z = t + it^2, dz = (1+2it)dt$

$$\text{Hence, } \int_C f(z)dz = \int_{-1}^{0} 2(1+2it)dt + \int_0^1 6t(1+2it)dt = 5+2i$$

d) The left half of the ellipse is $x = 6\cos t, y = 2\sin t, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

$$\text{Hence, } \int_C dz = \int_{\pi/2}^{3\pi/2} dz = \int_{\pi/2}^{3\pi/2}(-6\sin t + 2i\cos t)dt = \left(6\cos t + 2i\sin t\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -4i$$

5.6.2 Cauchy's Theorem

Suppose a function f is analytic in a simply connected domain D and f' is continuous in D. Then, for every simple closed curve C in D, $\int_C f(z)dz = 0$.

Examples: Evaluate the following integrals.

a) $\int\limits_C (z^4 + z^2 + 4)dz$ where $C: |z| = 1$ b) $\int\limits_C \frac{z^6 e^z}{(z-1)^3}dz$ where $C: |z-3| = 1$

c) $\int\limits_C \frac{e^z \sin z}{z^2+4}dz$ where $C: |z| = 1$ d) $\int\limits_C \frac{e^z}{z^2+1}dz$ where $C: |z+i| = 3$

Solution:

a) Here, $f(z) = z^4 + z^2 + 4$ is analytic everywhere and so is on the circle $C: |z| = 1$. Therefore, by Cauchy's Theorem, $\int\limits_C (z^4 + z^2 + 4)dz = 0$.

b) $f(z) = \frac{z^6 e^z}{(z-1)^3}$ is analytic everywhere except $z = 1$. But the point $z = 1$ is not inside or on the circle $C: |z-3| = 1$ because at $z = 1$, $|z-3| = |1-3| = 2 > 1$.

Therefore, by Cauchy's Theorem, $\int\limits_C \frac{z^6 e^z}{(z-1)^3}dz = 0$.

c) $f(z) = \frac{e^z \sin z}{z^2+4}$ is analytic everywhere except at $z = \pm 2i$ but each of these points are outside $C: |z| = 1$ and thus it is analytic on the given circle.

Therefore, by Cauchy's Theorem, $\int\limits_C \frac{e^z \sin z}{z^2+4}dz = 0$.

d) Here, $f(z) = \frac{e^z}{z^2+1}$ is not analytic at $z = \pm i$ but each of these points are inside the circle $C: |z+i| = 1$. (So, what? Wait for a while for such cases!)

5.6.3 Cauchy-Integral-Formula

If f is analytic inside and on a closed curve C and a is any point within C, then

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \text{ or } \int_C \frac{f(z)}{z-a}dz = 2\pi i f(a).$$

In general, Cauchy-Integral-Formula can be generalized as follow:

$$\int_C \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(a), \int_C \frac{f(z)}{(z-a)^3}dz = \frac{2\pi i}{2!}f''(a),\ldots,\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)$$

What is confusing part of the formula when to use?

To use this formula, the difficulty is to identify the function $f(z)$. To avoid such difficulty, please understand the following procedures.

First: Identify the point $z=a$ within C that makes the integrand undefined.

Second: Cover $z-a$ or $(z-a)^k$ in the integrand and take the remaining as $f(z)$

Third: Apply Cauchy-Integral-Formula.

Notice: If you get more than one point $z=a$ within C that make the integrand undefined, first decompose the integrand as sum of partial fractions.

Examples: Evaluate the integrals using Cauchy-Integral-Formula

a) $\int_C \frac{z^2}{z-3}dz$ where $C:|z|=4$

b) $\int_C \frac{6z^2}{z^2+9}dz$ where $C:|z+2i|=4$

c) $\int_C \frac{\cos(\pi z)}{(z-2)(z-3)}dz$ where $C:|z|=5$

d) $\int_C \frac{6}{2z^3+3z^2}dz$ where $C:|z|=1$

e) $\int_C \frac{e^{2z}}{(z-2)^2(z-5)}dz$ where $C:|z|=3$

f) $\int_C \frac{z^2+1}{z^2-1}dz$ where $C:|z-1|=1$

g) $\int_C \frac{z^3}{(z+i)^3}dz$ where $C:|z|=2$

h) $\int_C \frac{\sin^6 z}{z-\frac{\pi}{6}}dz$ where $C:|z|=1$

i) $\int_C \frac{\sin z}{4z+\pi}dz$ where $C:|z|=1$

j) $\int_C \frac{\sin(\pi z^2)}{(z-1)^2(z+3)}dz$ where $C:|z|=2$

k) $\int_C \frac{\cos 3z}{z^5}dz$ where $C:|z|=1$

l) $\int_C \frac{e^{-z}\sin z}{z^3}dz$ where $C:|z-1|=3$

Solution: Apply the above procedures to identify the function.

a) In this problem, the integrand is discontinuous at $z=a=3$ which is inside the circle $C:|z|=4$. So, if we cover the factor $z-3$, we get $f(z)=z^2$. But it is analytic on $C:|z|=4$. So, we have $\frac{z^2}{z-3}=\frac{f(z)}{z-3}$. Therefore, by Cauchy Integral Formula, $\int_C \frac{z^2}{z-3}dz=\int_C \frac{f(z)}{z-3}dz=2\pi i f(3)=2\pi i(3^2)=18\pi i$.

b) Here, the integrand $\frac{6z^2}{z^2+9}=\frac{6z^2}{(z-3i)(z+3i)}$. So, the discontinuity points are $z=3i, z=-3i$ but $z=3i$ is outside $C:|z+2i|=4$. So, $f(z)=\frac{6z^2}{z-3i}$ is analytic within $C:|z+2i|=4$. Hence, for this choice of $f(z)$, by Cauchy' integral formula, we have $\int_C \frac{6z^2}{z^2+9}dz=\int_C \frac{f(z)}{z-3i}dz=2\pi i f(-3i)=2\pi i\left(\frac{-54}{-6i}\right)=18\pi$.

(What happens if both were inside?)

c) Here, the integrand $\frac{\cos(\pi z)}{(z-2)(z-3)}$ undefined at the points $z=2, z=3$. But both of these points are inside the given circle. (So, what shall we do?). As noticed in the above procedures, in such case first decompose the integrand as sum of partial fractions. Using partial fractions, $\frac{1}{(z-2)(z-3)}=\frac{1}{z-3}-\frac{1}{z-2}$.

Thus, $\int_C \frac{\cos(\pi z)}{(z-2)(z-3)}dz=\int_C \frac{\cos(\pi z)}{z-3}dz-\int_C \frac{\cos(\pi z)}{z-2}dz$.

For $\int_C \frac{\cos(\pi z)}{z-3}dz$, using $f(z)=\cos(\pi z)$, $\int_C \frac{\cos(\pi z)}{z-3}dz=2\pi i\cos(3\pi)=-2\pi i$.

For $\int_C \frac{\cos(\pi z)}{z-2}dz$, using $f(z)=\cos(\pi z)$, $\int_C \frac{\cos(\pi z)}{z-2}dz=2\pi i\cos(2\pi)=2\pi i$.

Therefore, $\int_C \frac{\cos(\pi z)}{(z-2)(z-3)}dz=\int_C \frac{\cos(\pi z)}{z-3}dz-\int_C \frac{\cos(\pi z)}{z-2}dz=-2\pi i-2\pi i=-4\pi i$.

d) Here, $2z^3+3z^2=z^2(2z+3)$. So, the discontinuity points are $z=0, z=-\frac{3}{2}$ but $z=-\frac{3}{2}$ is outside $C:|z|=1$. So, by covering z^2, we get $f(z)=\dfrac{6}{2z+3}$ which is analytic on $C:|z|=1$. Besides, its derivative is $f'(z)=-\dfrac{12}{(2z+3)^2}$.

Hence, for this choice of $f(z)$, by Cauchy' integral formula, we have

$$\int_C \frac{6}{2z^3+3z^2}dzz=\int_C \frac{f(z)}{z^2}dz=2\pi i f'(0)=2\pi i\left(-\frac{12}{9}\right)=-\frac{4\pi i}{3}.$$

e) In this case, we use Cauchy's integral formula for the case $n=1$. In our problem, the discontinuity points are $z=2, z=5$ but $z=5$ is outside the $C:|z|=3$. So, $f(z)=\dfrac{e^{2z}}{z-5}$ is analytic on $C:|z|=3$. So, for this choice of $f(z)$, we have $\dfrac{e^{2z}}{(z-2)^2(z-5)}=\dfrac{f(z)}{(z-2)^2}$ and $f'(z)=\dfrac{2e^{2z}(z-5)-e^{2z}}{(z-5)^2}$. Therefore, by

$$\int_C \frac{e^{2z}}{(z-2)^2(z-5)}dz=\int_C \frac{f(z)}{(z-2)^2}dz=2\pi i f'(2)=2\pi i\left(\frac{-7e^4}{9}\right)=\frac{-14\pi i e^4}{9}.$$

f) The discontinuity points are $z=1, z=-1$ but $z=-1$ is outside the path of integration $C:|z-1|=1$. So, $f(z)=\dfrac{z^2+1}{z+1}$ is analytic on $C:|z-1|=1$.

Hence, for this choice of $f(z)$, $\displaystyle\int_C \frac{z^2+1}{z^2-1}dz=\int_C \frac{f(z)}{z-1}dz=2\pi i f(1)=2\pi i$.

g) The discontinuity point is $z=-i$. So, $f(z)=z^3$ is analytic on $C:|z|=2$.

Hence, by using the generalization $\displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2\pi i}{n!}f^{(n)}(a)$ of Cauchy's formula, we have $\displaystyle\int_C \frac{z^3}{(z+i)^3}dz=\int_C \frac{f(z)}{(z+i)^3}dz=\frac{2\pi i}{2!}f^{(2)}(-i)=\pi i(-6i)=6\pi$.

h) The discontinuity point is $z=\frac{\pi}{6}$. So, $f(z)=\sin^6 z$ is analytic on $C:|z|=1$. Hence, for this choice of $f(z)$, by Cauchy' integral formula, we have

$$\int_C \frac{\sin^6 z}{z-\frac{\pi}{6}}dz=\int_C \frac{f(z)}{z-\frac{\pi}{6}}dz=2\pi i f\left(\tfrac{\pi}{6}\right)=2\pi i\sin^6\left(\tfrac{\pi}{6}\right)=2\pi i\left(\tfrac{1}{2}\right)^6=\frac{\pi i}{32}.$$

i) Here, the discontinuity point is $z=-\frac{\pi}{4}$. So, by covering the factor $z-\frac{\pi}{4}$, we get $f(z)=\frac{\sin z}{4}$. Hence, $\int_C \frac{\sin z}{4z+\pi}dz=\int_C \frac{f(z)}{z+\frac{\pi}{4}}dz=2\pi i f(-\frac{\pi}{4})=\frac{-\sqrt{2}\pi i}{4}$.

j) The discontinuity points are $z=1, z=-3$ but $z=-3$ is outside $C:|z|=2$.

So, $f(z)=\frac{\sin(\pi z^2)}{z+3}$ is analytic on $C:|z|=2$. Hence, for this choice of $f(z)$, we have $\int_C \frac{\sin(\pi z^2)}{(z-1)^2(z+3)}dz=\int_C \frac{f(z)}{(z-1)^2}dz=2\pi i f'(1)=\frac{-\pi^2 i}{4}$.

k) The discontinuity point is $z=0$ with $n=5$. So, $f(z)=\cos 3z$ is analytic on $C:|z|=1$ and $f'(z)=-3\sin 3z, f''(z)=-9\cos 3z, f^{(4)}(z)=81\cos 3z$

Hence, $\int_C \frac{\cos 3z}{z^5}dz=\int_C \frac{f(z)}{z^{4+1}}dz=\frac{2\pi i}{4!}f^{(4)}(0)=\frac{2\pi i}{24}(81)=\frac{27\pi}{4}i$.

l) Here, $\int_C \frac{e^{-z}\sin z}{z^3}dz=\int_C \frac{f(z)}{z^{2+1}}dz=\frac{2\pi i}{2!}f''(0)=\pi i(-2)=-2\pi i$.

5.7 Power Series Representations of Complex Functions

Definition: For a given complex variable z, an infinite series of the form

$\sum_{n=0}^{\infty} a_n(z-a)^n = a_0+a_1(z-a)+a_2(z-a)^2+\dots$ is called *power series.*

5.7.1 Taylor Series

Taylor's Theorem: Suppose f is analytic within a domain D. Then, for any point z in D and for a given point a in D, f has a power series representation of the form

$$f(z)=f(a)+\frac{(z-a)}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\dots+\frac{(z-a)^n}{n!}f^{(n)}(a)+\dots$$

This representation is known as *Taylor Series*. In general, the Taylor series is given by $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$ where $c_n=\frac{f^{(n)}(a)}{n!}$.

In particular, if $a=0$, the series is known as Maclaurine series.

i) Here, the discontinuity point is $z=-\frac{\pi}{4}$. So, by covering the factor $z-\frac{\pi}{4}$, we get $f(z)=\frac{\sin z}{4}$. Hence, $\int_C \frac{\sin z}{4z+\pi}dz=\int_C \frac{f(z)}{z+\frac{\pi}{4}}dz=2\pi i f(-\frac{\pi}{4})=\frac{-\sqrt{2}\pi i}{4}$.

j) The discontinuity points are $z=1, z=-3$ but $z=-3$ is outside $C:|z|=2$.

So, $f(z)=\frac{\sin(\pi z^2)}{z+3}$ is analytic on $C:|z|=2$. Hence, for this choice of $f(z)$, we have $\int_C \frac{\sin(\pi z^2)}{(z-1)^2(z+3)}dz=\int_C \frac{f(z)}{(z-1)^2}dz=2\pi i f'(1)=\frac{-\pi^2 i}{4}$.

k) The discontinuity point is $z=0$ with $n=5$. So, $f(z)=\cos 3z$ is analytic on $C:|z|=1$ and $f'(z)=-3\sin 3z, f''(z)=-9\cos 3z, f^{(4)}(z)=81\cos 3z$

Hence, $\int_C \frac{\cos 3z}{z^5}dz=\int_C \frac{f(z)}{z^{4+1}}dz=\frac{2\pi i}{4!}f^{(4)}(0)=\frac{2\pi i}{24}(81)=\frac{27\pi}{4}i$.

l) Here, $\int_C \frac{e^{-z}\sin z}{z^3}dz=\int_C \frac{f(z)}{z^{2+1}}dz=\frac{2\pi i}{2!}f''(0)=\pi i(-2)=-2\pi i$.

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In particular, if $a=0$, the series is known as Maclaurine series .

Examples:

1. Find the Taylor series of the following functions at the indicated point a.

$a)\ f(z)=\ln z;\ a=1 \qquad b)\ f(z)=\dfrac{1}{1-z};\ a=2i \qquad c)\ f(z)=2\sin z,\ a=\dfrac{\pi}{6}$

Solution:

$$a)\ f(z)=\ln z \Rightarrow f(1)=0,\ f'(z)=\frac{1}{z}\Rightarrow f'(1)=1,\ f^{(2)}(1)=-1,\ f^{(3)}(1)=2$$

$$\therefore f(z)=f(1)+f'(1)(z-1)+\frac{f^{(2)}(1)(z-1)^2}{2!}+\frac{f^{(3)}(1)(z-1)^3}{3!}+\dots$$

$$\Rightarrow \ln z = z-1-\frac{(z-1)^2}{2!}+\frac{2(z-1)^3}{3!}-\frac{6(z-1)^4}{4!}+\frac{24(z-1)^5}{5!}+\dots$$

5.7.2 Laurent Series

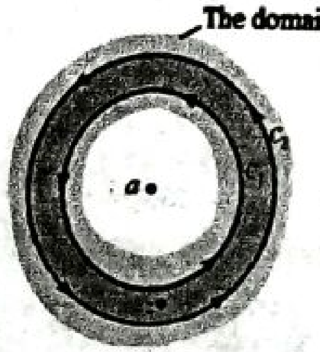
Laurent's Theorem: Suppose f is analytic inside the annulus formed by the concentric circles C_1 and C_2 having center at a with radii r and R respectively where $r<R$ (as shown in diagram (a) below). Then, f has a series representation within the annulus between C_1 and C_2 given by

$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ valid on $r<|z-a|<R$ where

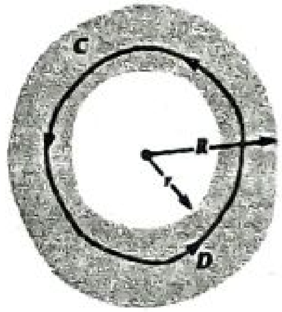
$a_n=\dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{(z-a)^{n+1}}dz=\dfrac{1}{n!}f^{(n)}(a)$ for which C is any circle of the form

$C: a+\mathrm{Re}^{i\theta},\ r_1<R<r_2$ (as shown in diagram (b) below).

This representation is known as *Laurent's series expansion.*



a) Concentric circles and the annulus formed



b) The contour C

Now, by separating the negative and positive powers of $z-a$, we can express the series as $f(z)=\sum_{-\infty}^{\infty}a_n(z-a)^n=\underbrace{\sum_{n=0}^{\infty}a_n(z-a)^n}_{R(z)}+\underbrace{\sum_{-\infty}^{n=-1}b_n(z-a)^n}_{P(z)}$.

Here, the first part $R(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$ is called *regular part*. It is good at a, that is it is analytic. The second part $P(z)=\sum_{-\infty}^{n=-1}b_n(z-a)^n$ is called *principal part*. It is not good at a, it is singular at a (not analytic at a). If we write in expanded form $P(z)=\sum_{-\infty}^{n=-1}b_n(z-a)^n$, we obtain

$$P(z)=\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\ldots+\frac{b_n}{(z-a)^n}+\ldots=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

However, as we have seen above the coefficients a_n and b_n are obtained by differentiating or by using contour integrals which is tedious and time taking. Thus, let's see simpler techniques to obtain Laurent's series of a function.

Using basic Maclaurine series: The following lists of power series are frequently used in many situations.

i) $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ valid on $|z|<\infty$

ii) $\frac{1}{1-z}=1+z+z^2+z^3+z^4+\ldots=\sum_{n=0}^{\infty}z^n$ valid on $|z|<1$

iii) $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ valid on $|z|<\infty$

iv) $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$ valid on $|z|<\infty$

v) $\sinh z=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\ldots=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$ valid on $|z|<\infty$

vi) $\cosh z=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\ldots=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$ valid on $|z|<\infty$

Examples: Find the Laurent's series about their singularity points.

$a)\ f(z)=\dfrac{\cos z}{z^4}\quad b)\ f(z)=z^3e^{\frac{1}{z}}\quad c)\ f(z)=\dfrac{z^2}{1-z}$

$d)\ f(z)=z^4\sin\left(\dfrac{1}{z}\right)\quad e)\ f(z)=\dfrac{e^z-1}{z^3}\quad f)\ f(z)=\dfrac{\sinh 3z-3z}{z^6}$

Solution:

$a)$ Here, $\cos z=1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dfrac{z^6}{6!}+\dots$

$$\Rightarrow\frac{\cos z}{z^4}=\frac{1}{z^4}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right)=\frac{1}{z^4}-\frac{1}{2!z^2}+\frac{1}{4!}-\frac{z^2}{6!}+\dots$$

$b)\ e^z=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^4}{4!}+\dots\Rightarrow e^{\frac{1}{z}}=1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+\dfrac{1}{4!z^4}+\dots$

Therefore, $z^3e^{\frac{1}{z}}=z^3\left(1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+\dfrac{1}{4!z^4}+\dots\right)$

$$=z^3+z^2+\frac{1}{2!z}+\frac{1}{3!}+\frac{1}{4!z^4}+\frac{1}{5!z^2}+\dots$$

$c)\dfrac{1}{1-z}=1+z+z^2+z^3+\dots\Rightarrow\dfrac{z^2}{1-z}=z^2(1+z+z^2+z^3+\dots)=z^2+z^3\dots$

$d)\ \sin z=z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+\dots\Rightarrow\sin\left(\dfrac{1}{z}\right)=\dfrac{1}{z}-\dfrac{1}{3!z^3}+\dfrac{1}{5!z^5}-\dfrac{1}{7!z^7}+\dots$

Therefore, $z^4\sin\left(\dfrac{1}{z}\right)=z^4\left(\dfrac{1}{z}-\dfrac{1}{3!z^3}+\dfrac{1}{5!z^5}-\dots\right)=z^3-\dfrac{z}{3!}+\dfrac{1}{5!z}-\dfrac{1}{7!z^3}+\dots$

$e)$ Here, $e^z=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^4}{4!}+\dfrac{z^5}{5!}+\dots$

$$\Rightarrow\frac{e^z-1}{z^3}=\frac{1}{z^3}\left[\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\dots\right)-1\right]=\frac{1}{z^3}\left(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\dots\right)$$

$$=\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{z}{4!}+\frac{z^2}{5!}+\dots$$

5.8 Calculation of Residues and Residue Theorem

5.8.1 Singularities, Types of Singularities and Poles

Singular points: A point $z = a$ at which a function f is not analytic is called a *singularity point* or *singular point* of f. At such point the function f is said to be *singular*.

Isolated Singularity points: A singularity point $z = a$ is said to be *isolated singularity* if there exists an open set containing a where f is analytic but no other singularity points of f.

Examples:

a) Let $f(z)=\dfrac{5}{z-3}$. Then, f has a singular point at $z=3$. But for any open set of the form $I = \{z : 0 < |z-3| < r,\ r > 0\}$ f is continuous and analytic.

So, $z=3$ is an isolated singularity point.

b) Let $f(z)=\dfrac{1}{(z^2+1)(z^2+9)}$. Then, f has singular points at $z=\pm i,\ z=\pm 3i$.

But for each open set of the form

$$I = \{z : 0 < |z-i| < r,\ r > 0\}, I = \{z : 0 < |z+i| < r,\ r > 0\},$$
$$I = \{z : 0 < |z-3i| < r,\ r > 0\}, I = \{z : 0 < |z+3i| < r,\ r > 0\}'$$

the function f is analytic. So, all the singularity points are isolated singularities.

c) Let $f(z)=\ln(z-2)$. Then, f has a singular point at $z=2$. But for any open set $I = \{z : 0 < |z-2| < r,\ r > 0\}$, f is not analytic. Because, for $r>0$, any open set $I = \{z : 0 < |z-2| < r, r > 0\}$ contains numbers less than 2 and greater than 2 which means any open set containing 2 always contains singularity points of f. So, $z=2$ is a singularity point but not an isolated singularity point.

Singularities:

Suppose $z=a$ is an isolated singularity point of f such that the Laurent's Series expansion of f valid for $0<|z-a|<R$ is given by

$$f(z)=\sum_{-\infty}^{\infty}a_n(z-a)^n=\underbrace{\sum_{n=0}^{\infty}a_n(z-a)^n}_{R(z)}+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{P(z)}.$$

Here, consider the principal part (the terms with negative powers),

$$P(z)=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}=\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\ldots+\frac{b_n}{(z-a)^n}+\ldots$$

Now, the singularity point $z=a$ is classified depending on the number of terms present in $P(z)$ (number of terms with negative powers).

i) Removable Singularity:

If all the coefficients b_n's in $P(z)$ are zero, we say that $z=a$ is a *removable singularity*.

In this case, the Laurent's Series expansion of f valid for $0<|z-a|<R$ is reduced to $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n=a_0+a_1(z-a)+a_2(z-a)^2+a_3(z-a)^3+\ldots$

ii) Pole of order n:

If $P(z)$ contains only a finite number of non-zero coefficients b_n, then $z=a$ is called a *pole of f*. If n is the highest integer such that $b_n\neq 0$, then $z=a$ is called a pole of order n.

In particular, if $n=1$ (only $b_1\neq 0$), then $z=a$ is called a *simple pole*.

In this case, the Laurent's Series expansion of f valid for $0<|z-a|<R$ is reduced to

$$f(z)=\frac{b_n}{(z-a)^n}+\ldots+\frac{b_2}{(z-a)^2}+\frac{b_1}{z-a}+a_0+a_1(z-a)+a_2(z-a)^2+\ldots$$

iii) Essential Singularity:

If an infinite number of $b_n^{'}s$ are non-zero, then $z=a$ is called an *essential singularity*.

In this case, the Laurent's Series expansion of f valid for $0<|z-a|<R$ is

$$f(z)=...+\frac{b_3}{(z-a)^3}+\frac{b_2}{(z-a)^2}+\frac{b_1}{z-a}+a_0+a_1(z-a)+a_2(z-a)^2+...$$

Examples:

1. Determine the singularities and classify the types of singularity.

a) $f(z)=\dfrac{e^{z^2}}{z^3}$ b) $f(z)=z^3\sin\left(\dfrac{1}{z}\right)$ c) $f(z)=(z-3)^2e^{\frac{1}{z-3}}$

d) $f(z)=\dfrac{\sin z}{z(z-1)}$ e) $f(z)=\dfrac{1+3i}{(z+1)(z-i)^4}$ f) $f(z)=\dfrac{\cot\pi z}{z^2}$

Solution:

a) Here, the singularity is at $z=0$ and using the expansion of e^z, we have

$e^{z^2}=1+\dfrac{z^2}{1!}+\dfrac{z^4}{2!}+\dfrac{z^6}{3!}+...$. Now dividing both sides by z^3 gives

$$\frac{e^{z^2}}{z^3}=\frac{1}{z^3}\left(1+\frac{z^2}{1!}+\frac{z^4}{2!}+\frac{z^6}{3!}+...\right)=\frac{1}{z^3}+\frac{1}{1!z}+\frac{z}{2!}+\frac{z^3}{3!}+...$$

Here, there is only two (finite) terms with negative power, that is only the terms $\dfrac{1}{z^3},\dfrac{1}{1!z}$ in the principal part. Hence, the type of singularity is a *pole* of order 3.

b) Here, the singularity is at $z=0$ and using the expansion of $\sin z$, we have

$\sin\left(\dfrac{1}{z}\right)=\dfrac{1}{z}-\dfrac{1}{3!z^3}+\dfrac{1}{5!z^5}-...$ Now multiplying both sides by z^3 gives

$z^3\sin\left(\dfrac{1}{z}\right)=z^3\left(\dfrac{1}{z}-\dfrac{1}{3!z^3}+\dfrac{1}{5!z^5}-...\right)=z^2-\dfrac{1}{3!}+\dfrac{1}{5!z^2}-...$ Here, the principal part contains infinite number of negative powers of z, that is $\dfrac{1}{3!},\dfrac{1}{5!z^2},...$ and thus the type of singularity at $z=0$ is essential singularity.

5.8.2 Calculations of Residues

Residues: In the Laurent's series expansion,

$$f(z)=\ldots+\frac{b_3}{(z-a)^3}+\frac{b_2}{(z-a)^2}+\frac{b_1}{z-a}+a_0+a_1(z-a)+a_2(z-a)^2+\ldots,$$

the coefficient b_1, simply the coefficient of $\frac{1}{z-a}$, is called the *residue* of f at $z=a$ and denoted by $\operatorname{Res}(f;a)=b_1$.

Method I (Without using Laurent's Series Expansion):

Our next goal is to develop techniques of finding the residues of a function without actually obtaining its Laurent series expansion.

Case-1: When $z=a$ is a simple pole (A pole of order 1):

Then, $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\frac{b_1}{z-a}$. Thus,

$$(z-a)f(z)=\sum_{n=0}^{\infty}a_n(z-a)^{n+1}+b_1 \Rightarrow \lim_{z\to a}(z-a)f(z)=\lim_{z\to a}\left(\sum_{n=0}^{\infty}a_n(z-a)^{n+1}+b_1\right)$$

$$\Rightarrow \lim_{z\to a}(z-a)f(z)=\underbrace{\lim_{z\to a}\sum_{n=0}^{\infty}a_n(z-a)^{n+1}}_{=0}+\underbrace{\lim_{z\to a}b_1}_{=b_1}$$

Therefore, $\operatorname{Res}(f;a)=b_1=\lim_{z\to a}(z-a)f(z)$ whenever $z=a$ is simple pole.

Examples (Residue at simple poles): Determine the singularities or poles and compute the residues at each poles.

a) $f(z)=\frac{z+1}{z^2-3z+2}$ b) $f(z)=\frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}$

Solution:

a) Here, $f(z)=\frac{z+1}{z^2-3z+2}=\frac{z+1}{(z-1)(z-2)}$.

Hence, its singularity points are $z=1$ and $z=2$ which are both simple poles.

Therefore, $\operatorname{Res}(f;1)=\lim_{z\to 1}(z-1)\dfrac{z+1}{(z-1)(z-2)}=\lim_{z\to 1}\dfrac{z+1}{z-2}=-2$

$$\operatorname{Res}(f;2)=\lim_{z\to 2}(z-2)\frac{z+1}{(z-1)(z-2)}=\lim_{z\to 1}\frac{z+1}{z-1}=3$$

b) The singularity points of f are $z=-1,-2,-3$

$$\operatorname{Res}(f;-1)=\lim_{z\to -1}(z+1)\frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}=\lim_{z\to -1}\frac{5z^2-4z+3}{(z+2)(z+3)}=6$$

$$\operatorname{Res}(f;-2)=\lim_{z\to -2}(z+2)\frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}=\lim_{z\to -2}\frac{5z^2-4z+3}{(z+1)(z+3)}=-31$$

$$\operatorname{Res}(f;-3)=\lim_{z\to -3}(z+3)\frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}=\lim_{z\to -3}\frac{5z^2-4z+3}{(z+1)(z+2)}=30$$

Case -2: When $z=a$ is a pole of order k (Multiple pole):

Then $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\dfrac{b_1}{z-a}+\dfrac{b_2}{(z-a)^2}+\ldots+\dfrac{b_k}{(z-a)^k}$

Thus, $f(z)(z-a)^k=\sum_{n=0}^{\infty}a_n(z-a)^{n+k}+b_1(z-a)^{k-1}+b_2(z-a)^{k-2}+\ldots+b_k$

Then, differentiating $(k-1)$ times of both sides, we obtain that

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]=\sum_{n=0}^{\infty}a_n\frac{(n+k)!}{(n+1)!}(z-a)^{n+1}+b_1(k-1)!$$

$$\Rightarrow \lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]=\lim_{z\to a}\sum_{n=0}^{\infty}a_n\frac{(n+k)!}{(n+1)!}(z-a)^{n+1}+\lim_{z\to a}b_1(k-1)!$$

$$\Rightarrow \lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]=b_1(k-1)!\Rightarrow b_1=\frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]$$

Therefore, $\operatorname{Res}(f;a)=b_1=\dfrac{1}{(k-1)!}\lim_{z\to a}\dfrac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]$ whenever $z=a$ is a pole of order k.

Examples: Determine the singularities or the poles with their orders and compute the residues of each of the following functions.

$$a)\ f(z)=\left(\frac{z-1}{z+1}\right)^3 \quad b)\ f(z)=\frac{1}{(z^2-2z+2)^2} \quad c)\ f(z)=\frac{e^{2z}}{(z+1)^4}$$

Solution:

a) The singularity point of $f(z)$ is $z=-1$ which is a pole of order 3.

Therefore, $\operatorname{Res}(f;-1)=\dfrac{1}{(3-1)!}\lim\limits_{z\to-1}\dfrac{d^{3-1}}{dz^{3-1}}\left[(z+1)^3 f(z)\right]=\dfrac{1}{2}\lim\limits_{z\to-1}\dfrac{d^2}{dz^2}(z-1)^3$

$$=\frac{1}{2}\lim_{z\to-1}6(z-1)=-6$$

b) $f(z)=\dfrac{1}{(z^2-2z+2)^2}$ has singularities at $z^2-2z+2=0$. Using quadratic formula, we get $z=1+i,1-i$ which are poles of order 2.

Thus, $R_1=\operatorname{Res}(f;1-i)=\lim\limits_{z\to1-i}\dfrac{d}{dz}(z-1+i)^2 f(z)$

$$=\lim_{z\to1-i}\frac{d}{dz}(z-1+i)^2\frac{1}{(z-1+i)^2(z-1-i)^2}$$

$$=\lim_{z\to1-i}\frac{d}{dz}\left(\frac{1}{(z-1-i)^2}\right)=\lim_{z\to1-i}\frac{-2}{(z-1-i)^3}=\frac{-2}{8i}=\frac{i}{4}$$

$$R_2=\operatorname{Res}(f;1+i)=\lim_{z\to1+i}\frac{d}{dz}(z-1-i)^2\frac{1}{(z-1+i)^2(z-1-i)^2}$$

$$=\lim_{z\to1+i}\frac{d}{dz}\left(\frac{1}{(z-1+i)^2}\right)=\lim_{z\to1+i}\frac{-2}{(z-1+i)^3}=\frac{-2}{-8i}=-\frac{i}{4}$$

c) $f(z)=\dfrac{e^{2z}}{(z+1)^4}$ has a pole at $(z+1)^4=0 \Rightarrow z=-1$ of order 4.

Thus, $\operatorname{Res}(f;-1)=\dfrac{1}{3!}\lim\limits_{z\to-1}\dfrac{d^3}{dz^3}(z+1)^4 f(z)=\dfrac{1}{6}\lim\limits_{z\to-1}\dfrac{d^3}{dz^3}(e^{2z})=\dfrac{4}{3}e^{-2}$

Case-3: When f is of the form $f(z)=\dfrac{p(z)}{q(z)}$ where p and q are analytic at a such that $p(a)\neq 0, q(a)=0, q'(a)\neq 0$. Then, $f(z)=\dfrac{p(z)}{q(z)}$ has a simple pole at a. Then, by **Case 1**, using some rearrangement, we have

$$\operatorname{Res}(f;a)=\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}=\frac{\lim\limits_{z\to a}p(z)}{\lim\limits_{z\to a}\dfrac{q(z)-q(a)}{z-a}}=\frac{p(a)}{q'(a)}$$

Examples: Determine the singularities or the poles and compute the residues.

a) $f(z)=\dfrac{\cos \pi z}{1-z^{100}}$ b) $f(z)=\cot(\pi z)$ c) $f(z)=\dfrac{8z^4}{z^4+1}$

Solution:

a) Here, $f(z)=\dfrac{\cos \pi z}{1-z^{100}}$ has a simple pole at $z=1$. Besides,

$p(z)=\cos \pi z \Rightarrow p(1)=-1\neq 0,\ q(z)=1-z^{100}\Rightarrow q(1)=0, q'(1)\neq 0$

Therefore, by *case 3*, $\operatorname{Res}(f;1)=\dfrac{p(1)}{q'(1)}=\dfrac{1}{100}$.

b) Here, $f(z)=\cot(\pi z)=\dfrac{\cos(\pi z)}{\sin(\pi z)}$ has singularity points when

$\sin(\pi z)=0\Rightarrow \pi z=n\pi, n\in Z\Rightarrow z=n, n\in Z$ and each of them are simple poles because $q'(z)=\pi\cos(\pi z)\Rightarrow q'(n)\neq 0, \forall n\in Z$.

Besides, $p(z)=\cos(\pi z)\Rightarrow p(n)\neq 0,\ q(z)=\sin(\pi z)\Rightarrow q(n)=0, q'(n)\neq 0$

Therefore, by *case 3*, $\operatorname{Res}(f;n)=\dfrac{p(n)}{q'(n)}=\dfrac{1}{\pi}$.

c) Here, the poles of $f(z)$ occur when $z^4+1=0$.That is,

$$z^4=-1\Rightarrow z=\sqrt[4]{-1}\Rightarrow z_k=2\left(\cos\left(\frac{2k+1}{4}\right)\pi+i\sin\left(\frac{2k+1}{4}\right)\pi\right), k=0,1,2,3$$

$$\Rightarrow z_0=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, z_1=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, z_2=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i, z_3=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

Method II (Using Laurent's Series Expansion):

Remarks: For the cases, that never fit to either of the above cases, we may need to manipulate $f(z)$ and get its Laurent's expansion to get residues.

Examples: Determine the singularities and compute the residues

$a)\ f(z)=e^{\frac{1}{z}}$ $\quad b)\ f(z)=z^2e^{\frac{1}{z}}$ $\quad c)\ f(z)=\sin\left(\frac{1}{z}\right)$ $\quad d)\ f(z)=\frac{1}{z^2\sin z}$

$e)\ f(z)=\frac{\sinh 3z-3z}{z^6}$ $\quad f)\ f(z)=\frac{e^z-1}{z^3}$ $\quad g)\ f(z)=(z-3)^2\sin\left(\frac{1}{z-3}\right)$

Solution: First write the Laurent's expansion at the singularity point $z=a$

a) Here, the singularity is at $z=0$ and lets find the expansion.

$$f(z)=e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\Rightarrow \mathrm{Res}(f,0)=1$$

b) Here, the singularity is at $z=0$ and lets find the expansion.

$$e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\Rightarrow z^2e^{\frac{1}{z}}=z^2+z+\frac{1}{2!}+\frac{1}{3!z}+\ldots\Rightarrow \mathrm{Res}(f,0)=\frac{1}{3!}=\frac{1}{6}$$

c) Here, the singularity is at $z=0$ and lets find the expansion.

$$\sin\left(\frac{1}{z}\right)=\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-\ldots\Rightarrow \mathrm{Res}(f,0)=1$$

d) Here, the singularity is at $z=0, z=k\pi, k\in Z$.

$$f(z)=\frac{1}{z^2\sin z}=\frac{1}{z^3}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)^{-1}=\frac{1}{z^3}+\frac{1}{3!z}+\frac{z}{5!}+\ldots\Rightarrow \mathrm{Res}(f,0)=\frac{1}{3!}=\frac{1}{6}$$

e) $$\frac{\sinh 3z-3z}{z^6}=\frac{(3z+\frac{3^3z^3}{3!}+\frac{3^5z^5}{5!}+\frac{3^7z^7}{7!}+\ldots)-3z}{z^6}=\frac{\frac{3^3z^3}{3!}+\frac{3^5z^5}{5!}+\frac{3^7z^7}{7!}+\ldots}{z^6}$$

$$=\frac{3^3}{3!z^3}+\frac{3^5}{5!z}+\frac{3^7z^2}{7!}+\ldots\Rightarrow \mathrm{Res}(f,0)=\frac{3^5}{5!}=\frac{243}{120}$$

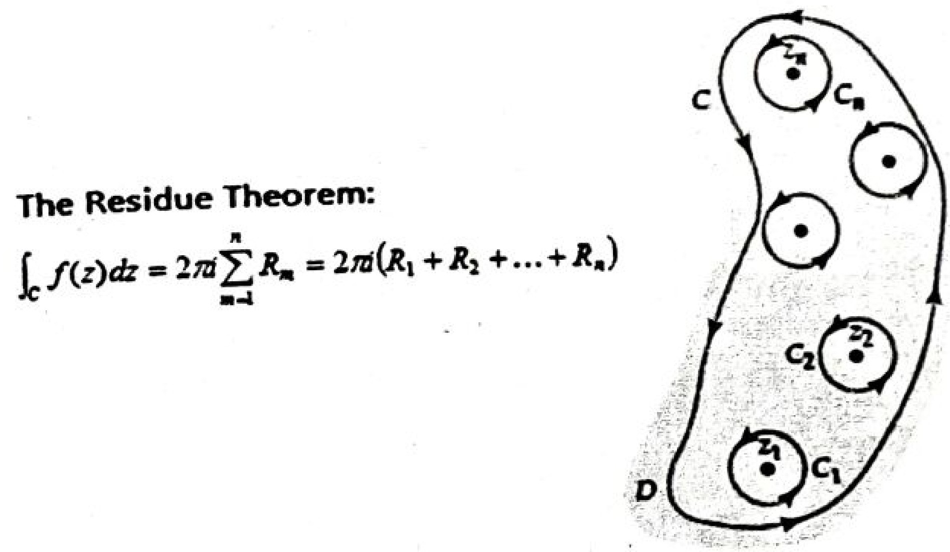
f) $$\frac{e^z-1}{z^3}=\frac{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots}{z^3}=\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{z}{4!}+\ldots\Rightarrow \mathrm{Res}(f,0)=\frac{1}{2!}=\frac{1}{2}$$

g) $$z-3)^2\sin\left(\frac{1}{z-3}\right)=z-3-\frac{1}{3!(z-3)}+\frac{1}{5!(z-3)^3}-\frac{1}{7!(z-3)^5}+\ldots\Rightarrow \mathrm{Res}(f,3)=-\frac{1}{6}$$

5.8.3 The Residue Theorem

If C is a simple closed curve such that $f(z)$ is analytic inside and on C except at a finite number of poles or isolated singularities in the interior of C, then

$\int_C f(z)dz = 2\pi i\sum_{m=1}^{n} R_m = 2\pi i(R_1 + R_2 + \ldots + R_n)$ where $R_1, R_2, \ldots, R_n$ are the residues of f at the poles or isolated singularities in the interior of C.



Procedures to apply Residue Theorem:

Suppose we want to evaluate $\int_C f(z)dz; C: |z-a| = R$ using Residue Theorem.

Step-1: Find all the singularity points of f. Say $a_1, a_2, \ldots, a_n$.

Step-2: Identify the singularities inside C and discard theses outside C.

Step-3: Compute the residues at all the points identified in *step 2*.

Step-4: Apply the Residue Theorem.

Question: How to identify singularity points that lie in $C: |z-a| = R$?

Singularity points that lie inside C are these which satisfy the inequality $|z - a_i| < R$. For instance for the circle $C: |z-1| = 3$, the point $a_1 = 2i + 1$ is inside C because $|a_1 - 1| = |2i + 1 - 1| = |2i| = 2 < 3$. But the point $a_2 = 4i + 4$ is outside C because $|a_2 - 1| = |4i + 4 - 1| = |4i + 3| = 5 > 3$. The point $a_3 = -2$ is also not in the interior of C because $|a_3 - 1| = |-2-1| = |-3| = 3$ is on the boundary.

Examples:

1. Evaluate the following integrals using Residue Theorem

a) $\int_C \frac{3z+2}{z-2}dz$; $C:|z|=3$

b) $\int_C \frac{z-3i}{z+i}dz$; $C:|z|=2$

c) $\int_C \frac{z+3}{z(z+1)(z-1)}dz$; $C:|z|=3$

d) $\int_C \frac{z-10i}{z^2+4}dz$; $C:|z|=4$

e) $\int_C \frac{3z-4}{z(z-1)(z-3)}dz$; $C:|z|=\frac{3}{2}$

f) $\int_C \frac{4z^2-4z+1}{(z-1)(4+z^2)}dz$; $C:|z|=\frac{3}{2}$

g) $\int_C \frac{z^2}{(z^2+1)(z^2+4)}dz$; $C:|z|=3$

h) $\int_C \frac{z+4}{z^2+2z+5}dz$; $C:|z+1-i|=2$

Solution:

a) $f(z)=\frac{3z+2}{z-2}$ has a simple pole at $z-2=0 \Rightarrow z=2$.

Thus, $R_1 = \mathrm{Res}(f;2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(3z+2) = 8$

Therefore, $\int_C \frac{3z+2}{z-2}dz = 2\pi i R_1 = 2\pi i(8) = 16\pi i$

b) $f(z)=\frac{z-3i}{z+i}$ has a simple pole at $z+i=0 \Rightarrow z=-i$.

Thus, $R_1 = \mathrm{Res}(f;-i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}(z-3i) = -4i$

Therefore, $\int_C \frac{z-3i}{z+i}dz = 2\pi i R_1 = 2\pi i(-4i) = 8\pi$

c) $f(z)=\frac{z+3}{z(z+1)(z-1)}$ has simple poles at $z=0, z=-1, z=1$ and all inside C

Thus, $R_1 = \mathrm{Res}(f;0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\left(\frac{z+3}{(z+1)(z-1)}\right) = -3$

$R_2 = \mathrm{Res}(f;-1) = \lim_{z\to -1}(z+1)f(z) = 1$

$R_3 = \mathrm{Res}(f;1) = \lim_{z\to 1}(z-1)f(z) = 2$

Therefore, $\int_C \frac{z+3}{z(z+1)(z-1)}dz = 2\pi i(R_1+R_2+R_3) = 2\pi i(-3+1+2) = 0$

d) $f(z)=\dfrac{z-10i}{z^2+4}$ has simple poles at $z^2+4=0 \Rightarrow (z-2i)(z+2i)=0$.

That is at $z=2i$, $z=-2i$ each of which are inside C.

Thus, $R_1=\operatorname{Res}(f;2i)=\lim\limits_{z\to 2i}(z-2i)f(z)=\lim\limits_{z\to 2i}\left(\dfrac{z-10i}{z+2i}\right)=-2$

$$R_2=\operatorname{Res}(f;-2i)=\lim_{z\to -2i}(z+2i)f(z)=\lim_{z\to -2i}\left(\frac{z-10i}{z-2i}\right)=3$$

Therefore, $\displaystyle\int_C \frac{z-10i}{z^2+4}dz=2\pi i(R_1+R_2)=2\pi i(-2+3)=2\pi i$

e) $f(z)=\dfrac{3z-4}{z(z-1)(z-3)}$ has simple poles at $z=0, z=1, z=3$. But the pole $z=3$ is outside the given contour C.So, only the poles $z=0, z=1$ are relevant.

Thus, $R_1=\operatorname{Res}(f;0)=\lim\limits_{z\to 0} zf(z)=\lim\limits_{z\to 0}\left(\dfrac{3z-4}{(z-1)(z-3)}\right)=\dfrac{-4}{3}$

$$R_2=\operatorname{Res}(f;1)=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}\left(\frac{3z-4}{z(z-3)}\right)=\frac{1}{3}$$

Therefore, $\displaystyle\int_C \frac{3z-4}{z(z-1)(z-3)}dz=2\pi i(R_1+R_2)=2\pi i\left(\frac{-4}{3}+\frac{1}{3}\right)=-2\pi i$

f) The poles occur at $(z-1)(4+z^2)=0 \Rightarrow z=1,2i,-2i$ but $z=1$ is inside.

Thus, $R_1=\operatorname{Res}(f;1)=\lim\limits_{z\to 1}(z-1)f(z)=\lim\limits_{z\to 1}\dfrac{4z^2-4z+1}{4+z^2}=\dfrac{1}{5}$

Therefore, $\displaystyle\int_C \frac{4z^2-4z+1}{(z-1)(4+z^2)}dz=2\pi i R_1=\frac{2\pi i}{5}$

2. Evaluate the following integrals using Residue Theorem.

a) $\displaystyle\int_C \frac{dz}{(z^2+4)^2}$; $C:|z-i|=2$ b) $\displaystyle\int_C \frac{2z+1}{(2z-1)^2}dz$; $C:|z|=1$

c) $\displaystyle\int_C \frac{1}{(z^2-2z+2)^2}dz$; $C:|z+i|=2$ d) $\displaystyle\int_C \frac{e^{4z}}{(z+1)^3}dz$; $C:|z|=2$

Solution:

$a)$ $f(z)=\dfrac{1}{(z^2+4)^2}$ has poles at $(z^2+4)^2=0 \Rightarrow z=-2i,\ z=2i$ of order 2.

Here, only $z=2i$ is in the given circle because $|z-i|=|i|=1<2$ but

$|z-i|=|-2i-i|=|-3i|=3>2$ which is outside C.

Thus, $R_1=\lim\limits_{z\to 2i}\dfrac{d}{dz}(z-2i)^2 f(z)=\lim\limits_{z\to 2i}\dfrac{d}{dz}\left(\dfrac{1}{(z+2i)^2}\right)=\lim\limits_{z\to 2i}\left(\dfrac{-2}{(z+2i)^3}\right)=\dfrac{1}{32i}$

Therefore, $\displaystyle\int_C \frac{dz}{(z^2+4)^2}=2\pi iR_1=2\pi i(\frac{1}{32i})=\frac{\pi}{16}$

$b)$ f has a pole at $(2z-1)^2=0 \Rightarrow z=1/2$ of order 2.

Thus, $R_1=\mathrm{Res}(f;1/2)=\lim\limits_{z\to 1/2}\dfrac{d}{dz}(2z-1)^2 f(z)=\lim\limits_{z\to 1/2}\dfrac{d}{dz}(2z+1)=2$

Therefore, $\displaystyle\int_C \frac{2z+1}{(2z-1)^2}dz=2\pi iR_1=2\pi i(2)=4\pi i$

$c)$ $f(z)=\dfrac{1}{(z^2-2z+2)^2}$ has singularities at $z^2-2z+2=0$. Using quadratic formula, we get $z=1+i,1-i$ as poles of order 2 but only $z=1-i$ lies in C.

Thus, $R_1=\mathrm{Res}(f;1-i)=\lim\limits_{z\to 1-i}\dfrac{d}{dz}(z-1+i)^2 f(z)$

$$=\lim_{z\to 1-i}\frac{d}{dz}\left(\frac{1}{(z-1-i)^2}\right)=\lim_{z\to 1-i}\frac{-2}{(z-1-i)^3}=\frac{-2}{8i}=\frac{i}{4}$$

Therefore, $\displaystyle\int_C \frac{1}{(z^2-2z+2)^2}dz=2\pi iR_1=2\pi i\left(\frac{i}{4}\right)=-\frac{\pi}{2}$

$d)$ $f(z)=\dfrac{e^{4z}}{(z+1)^3}$ has a pole at $(z+1)^3=0 \Rightarrow z=-1$ of order 3.

Thus, $R_1=\mathrm{Res}(f;-1)=\dfrac{1}{2!}\lim\limits_{z\to -1}\dfrac{d^2}{dz^2}(z+1)^3 f(z)=\dfrac{1}{2}\lim\limits_{z\to -1}\dfrac{d^2}{dz^2}(e^{4z})=8e^{-4}$

Therefore, $\displaystyle\int_C \frac{e^{4z}}{(z+1)^3}dz=2\pi iR_1=2\pi i(8e^{-4})=16\pi ie^{-4}$

3. Evaluate the following integrals using Residue Theorem.

a) $\int_C \frac{z+1}{z^2(z-1)}$; $C:|z-i|=\sqrt{2}$ b) $\int_C \frac{e^z}{z(z+1)^3}dz$; $C:|z|=2$

c) $\int_C \frac{z^2-2z}{(z+1)^2(z^2+1)}dz$; $C:|z+i|=3$ d) $\int_C \frac{z^2}{z^4-1}dz$; $C:|z-1|=3$

Solution:

a) The singularity points occur at $z^2(z-1)=0 \Rightarrow z=0,\ z=1$. Hence, $z=0$ is a pole of order 2 and $z=1$ a simple pole. But only $z=0$ is inside.

$$\operatorname{Res}(f;0)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}z^2 f(z)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{z+1}{z-1}\right)=\lim_{z\to 0}\frac{-2}{(z-1)^2}=-2$$

Therefore, $\int_C \frac{z+1}{z^2(z-1)}=2\pi i\operatorname{Res}(f;0)=2\pi i(-2)=-4\pi i$

b) The singularity points of $f(z)$ occur at $z(z+1)^3=0 \Rightarrow z=0, z=-1$. Hence, $z=-1$ is a pole of order 3, $z=0$ a simple pole and each lies in $C:|z|=2$.

Therefore, $\operatorname{Res}(f;-1)=\frac{1}{2!}\lim_{z\to -1}\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)=\frac{1}{2}\lim_{z\to -1}\left(\frac{e^z}{z}-\frac{2e^z}{z^2}+\frac{2e^z}{z^3}\right)=\frac{-5}{2e}$

$$\operatorname{Res}(f;0)=\lim_{z\to 0}zf(z)=\lim_{z\to 0}\frac{e^z}{(z+1)^3}=1$$

Therefore, $\int_C \frac{e^z}{z(z+1)^3}dz=2\pi i(R_1+R_2)=2\pi i(1-\frac{5}{2e})$

c) $f(z)$ has simple poles at $z=i,-i$ and a pole of order 2 at $z=-1$.

Thus, $R_1=\operatorname{Res}(f;-1)=\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z)=\lim_{z\to -1}\frac{d}{dz}(\frac{z^2-2z}{(z^2+1)})=-\frac{1}{2}$

$$R_2=\operatorname{Res}(f;i)=\lim_{z\to i}(z-i)f(z)=\frac{1}{4}(2i+1)$$

$$R_3=\operatorname{Res}(f;-i)=\lim_{z\to -i}(z+i)f(z)=\frac{1}{4}(1-2i)$$

Therefore, $\int_C \frac{z^2-2z}{(z+1)^2(z^2+1)}dz=2\pi i(R_1+R_2+R_3)=2\pi i(0)=0$

d) The poles occur at $z^4 - 1 = 0 \Rightarrow (z^2-1)(z^2+1) = 0 \Rightarrow z = 1, -1, i, -i$ each of which are simple poles inside the given circle. Thus,

$$R_1 = \operatorname{Res}(f;1) = \frac{1}{4}, R_2 = \operatorname{Res}(f;-1) = \frac{-1}{4},$$

$$R_3 = \operatorname{Res}(f;i) = \frac{-i}{4}, R_4 = \operatorname{Res}(f;-i) = \frac{i}{4}$$

Therefore, $\int_C \frac{z^2}{z^4-1}dz = 2\pi i(R_1 + R_2 + R_3 + R_4) = 2\pi i(0) = 0$

e) The poles occurs at $z^3 + 2z^2 = 0 \Rightarrow z^2(z+2) = 0 \Rightarrow z = 0, -2$ but only $z = 0$ is inside the given circle. Besides it is a pole of order 2.

Thus, $\operatorname{Res}(f;0) = \lim_{z\to 0}\frac{d}{dz}[z^2 f(z)] = \lim_{z\to 0}\frac{d}{dz}\left(\frac{e^z}{z+2}\right) = \lim_{z\to 0}\frac{(z+1)e^z}{(z+2)^2} = \frac{1}{4}$

Therefore, $\int_C \frac{e^z}{z^3+2z^2}dz = 2\pi i \operatorname{Res}(f;0) = \frac{\pi i}{2}$

4. Evaluate the following integrals using Residue Theorem.

$a) \int_C \cot(\pi z)dz; C: |z| = 2 \quad b) \int_C \frac{\cot(\pi z)}{z^2}dz; C: |z| = \frac{1}{4} \quad c) \int_C \frac{dz}{\sinh(4z)}; C: |z| = 2$

Solution:

$a)$ $f(z) = \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ has singularities at $\sin(\pi z) = 0 \Rightarrow z = n, \forall n \in Z$ and only, $z = -1, z = 0, z = 1$ lies in $C: |z| = 2$.

That is $R_1 = \frac{p(-1)}{q'(-1)} = \frac{\cos(-\pi)}{\pi\cos(-\pi)} = \frac{1}{\pi}, R_2 = \frac{p(0)}{q'(0)} = \frac{1}{\pi}, R_3 = \operatorname{Res}(f;1) = \frac{1}{\pi}$

Therefore, $\int_C \cot(\pi z)dz = 2\pi i(R_1 + R_2 + R_3) = 6i$

b) Here, $q(z) = z^2\sin(\pi z) = 0 \Rightarrow z^2 = 0, \sin(\pi z) = 0 \Rightarrow z = 0, z = n, n \in Z$. Using the expansion of $\cot \pi z$, $z = 0$ is a pole of order 3 but all the other poles $z = n, n \in Z/\{0\}$ are simple poles and only $z = 0$ lies inside C.

Therefore, $R_1 = \operatorname{Res}(f;0) = -\frac{\pi}{3} \Rightarrow \int_C \frac{\cot(\pi z)}{z^2}dz = 2\pi i(-\frac{\pi}{3}) = -\frac{2\pi^2}{3}i$

c) f has simple pole at $z = 0$. Therefore, $\int_{C:|z|=2} \frac{dz}{\sinh(4z)} = 2\pi i\left(\frac{1}{4}\right) = \frac{\pi i}{2}$.

4.9 Applications of Residue Theorem

So far, we have seen how to use Residue Theorem in evaluating integrals around a contour by using some points found inside the contour. Here, under let's see how to use that theorem to evaluate difficult real integrals.

5.9.1 Evaluation of Real Trigonometric Integrals

Here, under we will see how to evaluate integrals involving rational functions of $\cos at$ and $\sin bt$. That is integrals of the form $\int_0^{2\pi} F(\cos at, \sin bt)dt$.

To evaluate such forms of integrals, we usually choose the unit circle $C: |z| = 1$.

But we know that a unit circle is parameterized by $z = e^{it}, dz = ie^{it}dt, 0 \le t \le 2\pi$.

Besides, using $z = e^{it}$, in Euler's Formula,

$$\begin{cases} e^{iat} = \cos at + i\sin at \\ e^{-iat} = \cos at - i\sin at \end{cases} \Rightarrow \cos at = \frac{e^{iat} + e^{-iat}}{2}, \sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\Rightarrow \frac{(e^{it})^a + (e^{it})^{-a}}{2} = \frac{z^a + z^{-a}}{2}, \sin at = \frac{e^{iat} - e^{-iat}}{2i} = \frac{z^a - z^{-a}}{2i}$$

Therefore, the required integral is transformed into complex integral as follow:

$$\int_0^{2\pi} F(\cos at, \sin bt)dt = \int_C F\left(\frac{z^a + z^{-a}}{2}, \frac{z^a - z^{-a}}{2i}\right)\frac{dz}{iz} \text{ where } C \text{ is the unit circle.}$$

Examples: Evaluate the following integrals using the above procedures.

a) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ b) $\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta} d\theta$ c) $\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta} d\theta$

Solution: To apply the above substitutions, first consider a unit circle with parameterization of $C: z = e^{i\theta}, 0 \le \theta \le 2\pi, dz = ie^{i\theta}d\theta = izd\theta$.

a) Here, $\cos 3\theta = \frac{z^3 + z^{-3}}{2} = \frac{z^6 + 1}{2z^3}$ and $\cos\theta = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$.

So, $I=\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta=\int_C\left(\frac{(z^6+1)/2z^3}{5-4(z^2+1)/2z}\right)\frac{dz}{iz}=\frac{1}{i}\int_C\frac{z^6+1}{z^3(10z-4z^2-4)}dz$

$$=-\frac{1}{2i}\int_C\frac{z^6+1}{z^3(2z^2-5z+2)}dz=-\frac{1}{2i}\int_C\frac{z^6+1}{z^3(z-2)(2z-1)}dz$$

Here, the singularity points are $z=0, z=2, z=1/2$ but $z=2$ is outside the unit circle. Hence, using $f(z)=\frac{z^6+1}{z^3(z-2)(2z-1)}$ by Residue Theorem,

$$\int_C\frac{z^6+1}{z^3(z-2)(2z-1)}dz=2\pi i[\mathrm{Res}(f,0)+\mathrm{Res}(f,1/2)]=2\pi i\left(\frac{21}{8}-\frac{65}{24}\right)=-\frac{\pi i}{6}.$$

Therefore, $I=\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta=-\frac{1}{2i}\int_C\frac{z^6+1}{z^3(z-2)(2z-1)}dz=-\frac{1}{2i}\left(-\frac{\pi i}{6}\right)=\frac{\pi}{12}.$

b) Here, $\cos\theta=\frac{z+z^{-1}}{2}=\frac{z^2+1}{2z}, \sin\theta=\frac{z-z^{-1}}{2i}=\frac{z^2-1}{2iz}.$

So, $\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta=\int_C\left(\frac{[(z^2-1)/2iz]^2}{5+4(z^2+1)/2z}\right)\frac{dz}{iz}=-\frac{1}{4i}\int_C\frac{(z^2-1)^2}{z^2(2z^2+5z+2)}dz$

$$=-\frac{1}{4i}\int_C\frac{(z^2-1)^2}{z^2(z+2)(2z+1)}dz$$

Here, the singularity points are $z=0, z=-2, z=-1/2$ but $z=-2$ is outside the unit circle. Hence, using $f(z)=\frac{(z^2-1)^2}{z^2(z+2)(2z+1)}$ by Residue Theorem,

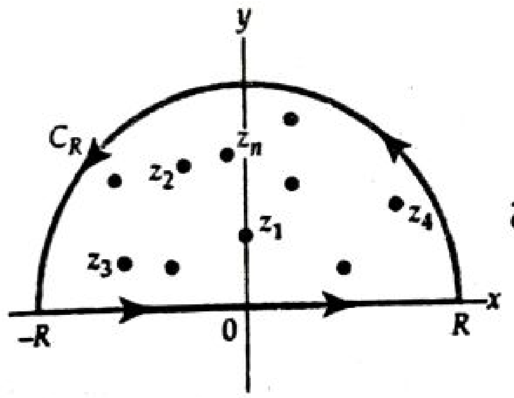
$$\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta=-\frac{1}{4i}\int_C\frac{(z^2-1)^2}{z^2(z+2)(2z+1)}dz$$

$$=-\frac{1}{4i}\left(2\pi i[\mathrm{Res}(f,0)+\mathrm{Res}(f,-1/2)]\right)=\frac{\pi}{4}$$

c) $\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}d\theta=\int_C\frac{z^2+1}{z(z^2+6iz-1)}dz=2\pi i[\mathrm{Res}(f,0)+\mathrm{Res}(f,-3i+2\sqrt{2}i)]=0$

5.9.2 Evaluation of Real Improper Integrals

Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$: Before discussing how this improper integral is related with Residue Theorem, let's see one important result that could be useful in the discussion. To evaluate an improper integrals $\int_{-\infty}^{\infty} f(x)dx$, first consider a contour C consisting of a semicircle C_R together with its diameter $(-R,R)$ lying along the x-axis. Make the radius as large as possible so as to enclose all the poles of f in the upper half plane (where $\operatorname{Im} z > 0$) as shown in the diagram below. Then the contour $C = C_R \cup (-R,R)$.



$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

The contour $C = C_R \cup (-R,R)$

Let's consider the case when there are no poles along the real axis. Now the idea is to evaluate $\int_{-\infty}^{\infty} f(x)dx$ using Residue Theorem, first replace all the real variable x by a complex variable z. Besides, notice that along the real axis, the function f depends only on the real variable x. This means the integral along the diameter is real improper integral given by $\int_{-R}^{R} f(x)dx$. Then, integrate the resulting complex function $f(z)$ over a closed contour $C = C_R \cup (-R,R)$. That is $\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$ where z_k denotes all the poles of f in the upper-half plane (as in the diagram above).

But as $R \to \infty$), we have $\int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$ .Hence,

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

$$\Rightarrow \lim_{R\to\infty} \int_C f(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)dz + \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k) - \lim_{R\to\infty} \int_{C_R} f(z)dz,$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k) \quad (\text{By JL}, \lim_{R\to\infty} \int_{C_R} f(z)dz = 0)$$

Examples: Evaluate the definite integrals using complex integrations:

$a)$ $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)} dx$ $\qquad$ $b)$ $\int_{0}^{\infty} \frac{dx}{x^4+16}$

Solution:

a) Consider a semi-circle of radius R with center at the origin in the upper half plane. Consider the complex integral $\int_C f(z)dz = \int_C \frac{z^2}{(z^2+1)(z^2+9)} dz$ where

$f(z) = \frac{z^2}{(z^2+1)(z^2+9)}$ and C is a closed contour from –R to +R and an arc of the semi-circle $C_R : |z| = R$. Now, by Residue Theorem, $\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} R_i$

where R_i's are the poles of f in the upper half plane. Thus, as $R \to \infty$,

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \lim_{R\to\infty} \int_{C_R} f(z)dz = \int_{-\infty}^{\infty} f(x)dx \Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{i=1}^{n} R_i$$

Here, the poles of $f(z)$ are at $z = \pm i$ and $z = \pm 3i$. Of these poles, only poles $z = i$ and $z = 3i$ lies in the upper half plane and they are simple poles.

So, $\operatorname{Res}(f;i)=\lim_{z\to i}(z-i)\left(\frac{z^2}{(z-i)(z+i)(z^2+9)}\right)=\frac{-1}{(2i)(-1+9)}=\frac{-1}{16i}$

$$\operatorname{Res}(f;3i)=\lim_{z\to 3i}(z-3i)\left(\frac{z^2}{(z^2+1)(z-3i)(z+3i)}\right)=\frac{-9}{(-9+1)(3i+3i)}=\frac{3}{16i}$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+9)}dx=2\pi i\sum_{i=1}^{2}R_i=2\pi i(R_1+R_2)=2\pi i\left(\frac{-1}{16i}+\frac{3}{16i}\right)=\frac{4\pi i}{16i}=\frac{\pi}{4}.$$

b) Since $f(x)=\frac{1}{x^4+16}$ is even , we have $\int_0^{\infty}\frac{dx}{x^4+16}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+16}$.

Now, the poles of $f(z)=\frac{1}{z^4+16}$ occur when $z^4+16=0$. That is,

$$z^4=-16\Rightarrow z=\sqrt[4]{-16}\Rightarrow z_k=2\left(\cos\left(\frac{2n+1}{4}\right)\pi+i\sin\left(\frac{2n+1}{4}\right)\pi\right),k=0,1,2,3$$

$$\Rightarrow z_0=\sqrt{2}+\sqrt{2}i,z_1=-\sqrt{2}+\sqrt{2}i,z_2=-\sqrt{2}-\sqrt{2}i,z_3=\sqrt{2}-\sqrt{2}i$$

Of these, only $z_0=\sqrt{2}+\sqrt{2}i,\ z_1=-\sqrt{2}+\sqrt{2}i$ are inside the upper half plane.

So, $R_1=\operatorname{Res}(f;z_0)=\lim_{z\to z_0}(z-z_0)\left(\frac{1}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)}\right)$

$$=\frac{1}{(2\sqrt{2})(2\sqrt{2}+2\sqrt{2}i)(2\sqrt{2}i)}=\frac{\sqrt{2}-\sqrt{2}i}{64}$$

$$R_2=\operatorname{Res}(f;z_1)=\lim_{z\to z_1}(z-z_1)\left(\frac{1}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)}\right)$$

$$=\frac{1}{(-2\sqrt{2})(2\sqrt{2}i)(-2\sqrt{2}+2\sqrt{2}i)}=\frac{-\sqrt{2}-\sqrt{2}i}{64}$$

Hence, $\int_{-\infty}^{\infty}\frac{dx}{x^4+16}=2\pi i(R_1+R_2)=2\pi i\left(\frac{\sqrt{2}-\sqrt{2}i}{64}+\frac{-\sqrt{2}-\sqrt{2}i}{64}\right)=\frac{4\sqrt{2}\pi}{64}=\frac{\pi}{8\sqrt{2}}$.

Therefore, $\int_0^{\infty}\frac{dx}{x^4+16}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+16}=\frac{\pi}{16\sqrt{2}}$.

Review Problems on Chapter-5

1. Check whether the functions are harmonic or not and find the corresponding harmonic conjugate and analytic function $f(x,y)=u(x,y)+iv(x,y)$.

$a)\ u(x,y)=x^2-y^2$ $\quad c)\ u(x,y)=x^2-y^2+7x+3y$ $\quad c)\ u(x,y)=2x(1-y)$

$d)\ u(x,y)=e^{-x/2}\cos\left(\frac{y}{2}\right)$ $\quad e)\ u(x,y)=2x-x^3+3xy^2$ $\quad f)\ u(x,y)=\frac{y}{x^2+y^2}$

Answer :$a)\ v(x,y)=2xy$ $\quad b)\ v(x,y)=2xy+7y-3x$ $\quad c)\ v(x,y)=x^2-y^2+2y$

$d)\ v(x,y)=e^{-x/2}\sin(y/2)$ $\quad e)\ v(x,y)=2y-3x^2y+y^3$ $\quad f)\ v(x,y)=\frac{x}{x^2+y^2}$

2. Show that $v(x,y)$ is harmonic and find a harmonic conjugate $u(x,y)$ such that $f(x,y)=u(x,y)+iv(x,y)$ is analytic.

$a)v(x,y)=2xy$ $\quad b)\ v(x,y)=2xy+7y-3x$ $\quad c)\ v(x,y)=e^{3x}\sin 3y$

Answer :$a)\ u=x^2-y^2$ $\quad b)\ u=x^2-y^2+7x+3y$ $\quad c)\ u=e^{3x}\cos 3y$

2. Evaluate the following integrals using Residue Theorem

$a)\ \int_C \frac{z^6+1}{z^3(2z^2-5z+2)}dz; C:|z|=1$ $\quad b)\ \int_C \frac{dz}{z^3(z+4)}; C:|z|=2$

$c)\ \int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz; C:|z-2|=2$ $\quad d)\ \int_C \frac{z^3+1}{(z^2+1)(z-2)}dz\ ;\ C:|z|=\sqrt{3}$

Answer :$a)\ -\frac{\pi i}{6}$ $\quad b)\ \frac{\pi i}{32}$ $\quad c)\ \pi i$

3. Evaluate the following definite integrals using Residue Theorem:

$a)\ \int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}d\theta$ $\quad b)\ \int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta}d\theta$ $\quad$ **Answer** :$a)\ \frac{3\pi}{8}$ $\quad b)\ \sqrt{2}\pi$

4. Evaluate the following definite integrals using Residue Theorem:

$a)\ \int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$ $\quad b)\ \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}dx$ $\quad$ **Answer** :$a)\ \frac{\pi}{3}$ $\quad b)\ \sqrt{2}\pi$