GNC - Problem Set Questions & Solutions

<u>Consider an LTI system and answer the following questions.</u>

Explain what is meant by stability of the system when no control inputs are there?

- Stability in the case of no control inputs is a property which ensures the system output remains bolunded or does not grow over time - the stability of a system without control iunputs is determined solely by characteristics and dynamics (transfer function/state-space representation)

Explain what is controllability?

- Assessment of ability to control the behaviour of a system, by steering the systems state to a desired final state by applying suitable control inputs in finite time.
- Controllability Matrix: $C = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
- Controllable if rank(C) is equal to dimension of grand state

Explain what is observability?

- Ability to infer/estimate the internal state of a dynamic state based on output - if a system is observable we can determine the system state by observing outputs over time
- Observability Matrix: $O = [C \quad CA \quad CA^2 \quad \ldots \quad CA^{n-1}]^T$
- Observable if rank(O) equal to dimension of grand state - meaning no state is unable to be observed

What is a state feedback controller, what is the process followed to design one?

- Control strategy used in control systems to manipulate behaviour of dynamic system by applying feedback based on internal state variables - instead of only controlling based on output measurements, state feedback controllers use knowledge of the system state to generate inputs
- Steps to design state feedback controller:
    - System Modelling: state space representation
    - Desired Pole Placement: determine desired closed-loop pole locations
    - Control Law Design: relate input to state vector, typically as $u = -Kx$
    - Stability/Performance Analysis: analyze eigenvalues, rise time, settling time, overshoot
    - Implementation/Tuning: adjustment of gain matrix (K) as required

What is meant by a tracking error model, find the tracking error model for the LTI deterministic system

- Tracking error: $e(t) = r(t) - y(t)$
- For LTI, using x(t) the grand state and output matrix C: $e(t) = r(t) - Cx(t)$
- Tracking error identifies the difference between desired and reference trajectory
- Tracking error modelling ensures the system tracks the trajectory accurately

What is an error state model. Find the error state model for the LTI deterministic system

- Error state model determines the difference between atual and desired state at any moment t
- Error State is defined as: $\Delta x(t) = x(t) - x_d(t)$
- Error state model with respect to the grand state can be defined as: $\Delta \dot{x} = \dot{x} - \dot{x_d} = Ax + Bu - \dot{x_d}$

What is a Luenberger observer? how do you design one?

- Also known as state observer/estimator
- Luenberger observer provides an estimate ($\hat{x}$) of actual state x(t) at any time t
- General form for LTI of Luenberger is: $\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$
- To design a Luenberger observer:
    - Model the system: state space representation
    - Select desired poles: desired poles for speed/stability
    - Compute observer gain: gain matrix L using poles and system matrices
    - Implement observer: estimate state variables in realtime

What advantage does an LQR provide over pole placement controllers?

- LQR optimizes performance and provides stability where pole placement controllers focus on stability only
- LQR allows a balance between performance factors (response speed/control effect)
- LQR considers disturbances and uncertainties for better real-world conditional performance
- LQR combines control and estimation and can function for time-varying nonlinear systems

What advantage does an LQE provide over pole placement observer?

- Minimizes estimation error covariance for better estimation of state
- Adapts to noise to continue to provide accurate state estimates of state
- EKF can estimate states of nonlinear and time-varying systems
- LQE can be combines with LQR enabling simultaneous estimation and optimization of control

What is time varying LQR what advantage does it provide over LQR (constant gain)?

- Designing time-varying control gains can be more complex than constant gains
- TV-LQR can adapt gains to achieve better performance in time-varying dynamics and parameter variations

What advantage does a Kalman filter provide over an LQE (constant gain)?

- Kalman filter can estimate while minimizing expected estimation error covariance
- The Kalman filter can adapt gain based on realtime noise conditions, and optimizes the estimation matrix
- Kalman filtering can deal with dynamic systems with time varying dynamics and parameters

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Consider a nonlinear system and answer the following questions.

Whats the difference between a vector space and a group e.g. R3 vs SO(3)?

- Vector Space:
  - Vectors (vector addition and scalar multiplication)
- Group (SO3)
  - Elements and binary operation which combine two elements to produce a third element
  - Special orthogonal group in 3 dimensions (all 3x3 orthogonal matrices with det = +1)
- Differences:
  - Vector spaces concern linear combinations, vector operations and vector transformations, while groups focus on element combination
  - Vectors can be added and multiplied, while in groups, there is a single binary operation

Explain briefly how rotation matrices in the state vector can be handled in a Kalman filter or an LQR implementation.

- Kalman:
  - Quaternion:
    - compact/numerically stable way to represent rotations, quaternion algebra can be used in Kalman filtering to propagate quaternion state estimation and covariance - quaternion based Kalman filters are usually linear
  - Euler Angle:
    - Use three angles from coordinate axes
    - Suffers singularities and gimbal lock
    - If rotations are limited, can still be used with Kalman
- LQR:
  - Linearization:
    - Linearize system around operational point
    - Approximate nonlinear rotation matrices to apply standard LQR
  - Quaternion:
    - Express system dynamics and cost functions in quaternions to design LQR control operational on quaternion state representations

Explain the main limitations of an LQR controller for non linear systems.
- Linearity Assumption: nonlinear systems can deviate from approximation from LQR
- Validity of State Feedback: state feedback can be more complex to implement and may require complex nonlinear transformations to map state to suitable control input
- Limited Region of Stability: nonlinear systems will exhibit different behaviours as they move from local region surrounding the operating point
- Sensitivity to Initial Conditions: small changes to initial state can lead to very different trajectories
- Handling Constraints: constraints on state and control may not be satisfied by LQR
- Performance near Extremes: nonlinear system vary differently near extremities and boundaries
- Complexity of Control Law Design: nonlinear control, adaptive control or model predictive control can be more involved than straightforward linear LQR control law design

Explain the main limitations of a Kalman filter for non linear systems.
- Linearity Assumption: Kalman filter linearization process can introduce errors and cause unstable behaviour
- Gaussian Assumption: distribution can become non-Gaussian due to nonlinear transformations
- Linearization Error: linearization introduces errors when system operates far from linearization point
- Unmodeled Dynamics: Kalman assumes specific form for system dynamics
- Convergence/Divergence: errors grow due to nonlinearities causing filter to converge/diverge to wrong solutions
- Computational Complexity: more complex calculatiorns (Jacobian) matrices and sigma-point transformations
- Tuning Parameters: selectring tuning parameters in nonlinear systems can require empirical tuning
- Degeneracy/Nonuniqueness: filter finds poor conditioning of covariance matrix or multiple solutions

Explain how outliers can be handled in an observer.
- Outlier Detection: compare observed measurements/residuals with expected values
- Outlier Rejection: reject/downweight outliers during estimation process
- Dynamic Thresholding: adapt outliers based on evolving estimation error/covariance which prevents over-rejection of valid data which may appear as outliers due to sudden changes
- Outlier Accomodations: accommodate temporary anomalies by allowing state estimate to deviate slightly
- Model Adaptation: model re-evaluation depending on model uncertainties/errors
- Multiple Observers: fuse estimates from multiple observers to reduce potential outliers

Explain how one can perform data correspondence in a filter and why it is needed.
- Purpose of Data Correspondence
  - Update Step: fusion of incoming measurements with predictions/estimations of state targets
  - Ambiguity Resolution: differentiation of measurements from different targets
  - Noise/Outlier Handling: distinguish valid measurements from outliers/noise
- Methods for Data Correspondence
  - Nearest Neighbor: assign measurement to nearest predicted/estimated target
  - Global Association: solve global optimization problem considering all possible assignment combinations
  - Probabilitistic Data Association: each measurement associated with target with certain probability
  - Joint Probablistic Data Association: multiple targets and measurements to account for interactions
  - Multiple Hypothesis Tracking: maintains set of possible target trajectories
- Challenges/Considerations
  - Ambiguities: difficult to uniquely associate measurements with targets
  - Detections/False Alarms: false/missed detections add complexity
  - Computation: some association can be computationally expensive
  - State Dependencies: influenced by dynamics of targets

What is meant by region of attraction of a controller.

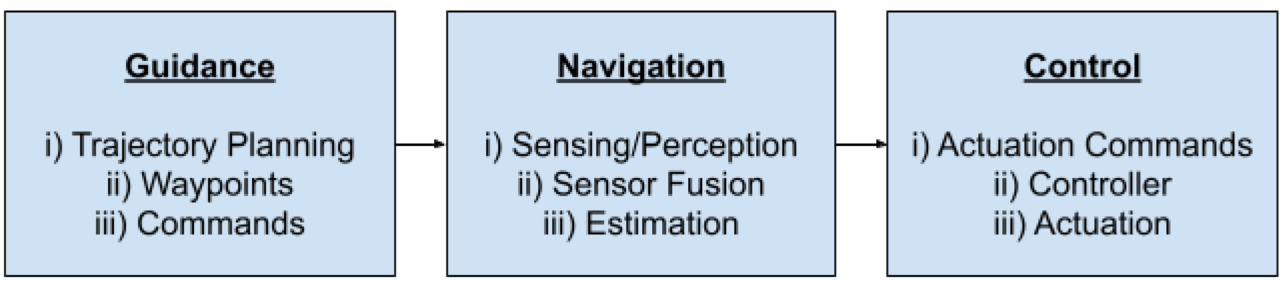
- Domain in which the controller can effectively stabilize the system
- A subset of state space where trajectories from points converge to desired equilibrium or stable behaviour

What is meant by Lyapunov stability and asymptotic stability?

- Lyapunov Stability:
    - Systems trajectories remain bounded and do not significantly deviate from the equilibrium point
    - Lyapunov stable iff state remains within certain vicinity of equilibrium point
- Asymptotic Stability:
    - Beyond Lyapunov, and requires trajectories to converge to equilibrium as time goes to infinity
    - This means the system must reach equilibrium at some point as time reaches infinity

Explain the role of a guidance system and show a block diagram of how guidance navigation and control modules work together on a given system (plant).

- Guidance systems make strategic decisions based on objectives, conditions and constraints, ensuring the system achieves the goal effectively and efficiently.
- Guidance systems can determine optimal paths/trajectories and set waypoints while considering factors such as efficiency, obstacle avoidance and mission requirements.



What is meant by non convexity of a cost function?

- Cost function is considered to be convex iff it curves upward in bowl like manner - meaning it has a single global minimum point - this guarantees that that the local minima are also the global minima
- Cost function is non-convex iff it has multiple local minima which leads to pitfalls while seeking an optimal solution
- Non-convexity can lead to the following problems/challenges in optimization:
    - Multiple local minima, making it difficult to determine the true optimal solution
    - Optimization algorithm sensitivity, meaning algorithms can be sensitivity to starting points/initial guess which leads to different convergence results depending on initial conditions
    - Convergence to sub-optimal solutions
    - Complex compytations
    - Uncertainty in quality of final solution

What is the difference between Djikstrar's and A star search?

- Dijkstrar's:
    - Finds shortest path from single source node to all other nodes in weighted graph
    - Guaranteed optimality in finding shortest path
    - Approach:
        - Iteratively expanding nodes and selects nodes with smallest distance from source
        - Requires non-negative edge weights
        - Priority queue to sort nodes to be explored, based on distance from source
        - Non heuristic to guide search, seeks purely based on distance
        - Guaranteed to find shortest path from source to all other nodes
- A* Search:
    - Shortest path from source node to goal node
    - Implements heuristics to guide the search
    - More efficient than Dijkstrar's, but does not guarantee optimality
    - Approach:
        - Combines actual cost and estimates cost to prioritize nodes
        - Requires non-negative edge weights
        - Priority of nodes determined by sum of cost to reach node from source and estimated cost to goal
        - Heuristic function provides estimate of remaining cost to reach goal
        - Optimality not guaranteed unless heuristic is admissible

What is the difference between Direct transcription and Direct collocation?

- Direct Transcription:
    - Optimization approach which discreizes continuous-time optimal control problem by dividing time interval into smaller segments/time steps
    - Each segment can be approximated by a set of control and state variables, parameterized using continuous functions
- Direct Collocation:
    - Focus on preserving continuity between time steps
    - Additional decision variables for dynamics at specific collocation points in each time segment - these variables allow continuity to be enforced within the system dynamics
- Comparison:
    - <u>Parameterization</u>: direct transcription directly parameterizes control/state variables, while direct collocation enforces continuity through collocation points
    - <u>Continuity</u>: transcription does not inherently ensure continuity of dynamics, while collocation explicitly encforces continuity constraints
    - <u>Numerical Integration</u>: transscription relies on numerical integration to compute state trajectories, while collocation provides accurate integrations
    - <u>Complexity</u>: collocation can be more complex due to additional constraints, but generally leads to more accurate solutions

Defintions & Additional Fun Information

<u>Heuristic</u>: estimate of distance/cost from particular node to goal node in search space, this helps guide search process by providing an informed *guess* about the effort/distance needed to reach the goal - very useful in optimization problems

<u>Probablistic</u>: uses probabilities/statistical techniques to make decisions about inclusion/exclusion of certain items/data points in filtered set - instead of making deterministic decisions, probabilistic decisions take into account probabilities that an item will meet certain criteria

<u>Quaternion</u>: defines complex numbers in four-dimensional space formed from one real part and three imaginary parts - quarternions allow for efficient computation of rotations and avoids gimbal lock which can occur while using Euler angles

<u>Stochastic Systems</u>: processes/systems which involve randomness/uncertainty - in a Stochastic system, the outcome of events/actions is not determinalistically predictable, but follows probabilistic distribution.

<u>LTI</u>: Linear-Time-Invariant
<u>LQR</u>: Linear Quadratic Regulator
<u>LQE</u>: Linear Quadratic Estimator
<u>TV-LQR</u>: Time-Variant LQR
<u>EKF</u>: Extended Kalman Filtering

Comparison of Dijkstrar and A*: [Link]

<u>Matrix Math</u>

Rank: maximum number of linearly independent columns or rows in a matrix
- Full rank iff number of columns or rows is equal to the rank
- Can be determined using Gaussian elimination or singular value decomposition

Determinant:
$$\det(M) = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

Multiplication:
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae + bf \\ ce + df \end{bmatrix}$$

Inverse:
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$