$$V(x) = \sum_{i=1}^{m} \sum_{j=1}^{m} P_{ij} x_i x_j = x^T P x .$$

Without loss of generality, we can always assume that P is symmetric. In fact, in the case it is not, one has:

$$V(x) = x^T P x = x^T \left[\frac{P - P^T}{2} + \frac{P + P^T}{2} \right] x$$

$$= x^T \left[\frac{P - P^T}{2} \right] x + x^T \left[\frac{P + P^T}{2} \right] x = x^T \left[\frac{P + P^T}{2} \right] x ,$$

since $\frac{P - P^T}{2}$ is skew symmetric. Note that $\frac{P + P^T}{2}$ is symmetric, so if P is not symmetric we can always refer to the "simmetrization" of P, i.e. $\frac{P + P^T}{2}$. When $V(x)$ is, e.g. positive define, we say that P is positive definite, and we'll write $P > 0$. Analogously, when $V(x)$ is positive semidefinite, we'll say that P is so, and we'll write $P \geq 0$.

There exist many criteria to asses the positive definiteness of a symmetric matrix P.

(*) Given $V(x): D \to \mathbb{R}$ we say that V is

Positive definite : if $V(x) > 0 \ \forall x - \{0\}$, $V(0) = 0$

Positive semidefinite : if $V(x) \geq 0 \ \forall x - \{0\}$, $V(0) = 0$

Negative definite : if $V(x) < 0 \ \forall x - \{0\}$, $V(0) = 0$

Negative semidefinite : if $V(x) \leq 0 \ \forall x - \{0\}$, $V(0) = 0$

Eigenvalues: when P is symmetric with real entries

$$\Downarrow$$

It is transformable in Jordan's form such that

$$\Lambda := \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & & & \lambda_m \end{bmatrix} = TPT^{T}, \quad T^{-1} = T^{T}$$

where λ_i is the i-th eigen value of P. In fact, one has in general

$$P = T\Lambda T^{-1}. \text{ Let } T := [t_1, \dots, t_m] \Rightarrow Pt_i = \lambda_i t_i$$

$$T^{-1} := \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad u_i P = \lambda_i u_i \Rightarrow P^T u_i^T = \lambda_i u_i^T = P u_i^T$$

$$\Rightarrow u_i = t_i^T$$

Now, observe that

$$V(x) = x^T P x = x^T T^T \Lambda T x = y^T \Lambda y \quad ; \; y := Tx$$

$$= \sum_{i=1}^{m} y_i^2 \lambda_i \quad . \text{ As a consequence}$$

if $\lambda_j < 0$, then $\bar{y} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ jth coordinate renders $V(T^T\bar{y}) < 0$. Hence

$P > 0 \Leftrightarrow$ all eigen values are positive, $\lambda_i > 0$

$P \geq 0 \Leftrightarrow$ all eigenvalues are nonnegative, $\lambda_i \geq 0$

Also, assume that

$$0 < \lambda_{min} \leq \lambda_i \leq \lambda_{MAX},$$

then

$$\lambda_{min}|y|^2 = \sum_{i=1}^{m} \lambda_{min}\, y_i^2 \leq x^T P x \leq \sum_{i=1}^{m} \lambda_{MAX}\, y_i^2 = \lambda_{MAX}|y|^2$$

Since $T^T T = I$, one has that $|y|^2 = y^T y = x^T T^T T x = x^T x = |x|^2 \Rightarrow$

$$\boxed{\lambda_{min}|x|^2 \leq x^T P x \leq \lambda_{MAX}|x|^2}$$

Sylvester: A real symmetric matrix P is positive definite if and only if all leading principal minors have positive determinants.

Fact i) If some determinant of the leading principal minors is zero, we cannot conclude that the matrix is positive semidefinite.

Fact ii) All elements on the diagonal of the matrix must be >0 for the matrix to be positive definite $\Rightarrow e_i^T P e_i > 0$

Fact iii) If one element on the diagonal is equal to zero, then the matrix is not positive definite

Fact iv) If one element on the diagonal is equal to zero and the associated row and column is zero, the matrix is positive semidefinite if the Sylvester criteria holds.

Ex

$$P := \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

a) $1 > 0$

b) $5-4=1>0$

c) $35-28>0$

P is positive definite. Another way to asses that was to compute the eigen values

$$\begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{pmatrix}$$

$$(1-\lambda)(5-\lambda)(7-\lambda)-4(7-\lambda)=0$$

$$(7-\lambda)\left[(1-\lambda)(5-\lambda)-4\right]=0$$

$$(7-\lambda)\left[\lambda^2-6\lambda+1\right]=0 \qquad \lambda_{1,2}=\frac{6\pm\sqrt{32}}{2}>0$$

DERIVATIVE OF QUADRATIC FORMS