Theorem A matrix A is Hurwitz, i.e. $Re(\lambda_i) < 0$ $\forall \lambda_i$ of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation (2). Moreover, if A is Hurwitz, then P is the unique solution to (2). (5.3)

Proof The sufficiency follows from the Lyapunov theorem with $V(x) = \frac{1}{2} x^T P x$. To prove necessity, assume that all eigenvalue of A satisfy $Re(\lambda_i) < 0$ and consider the matrix P defined by

$$(3) \quad P = \int_0^\infty \exp(A^T t) Q \exp(At)\, dt$$

The integrand is a sum of terms $t^{k-1} \exp(\lambda_i t)$, where $Re(\lambda_i) < 0$, and therefore the integral exists. The matrix P is symmetric and positive definite. In fact

i) $$P^T = \int_0^\infty \left(\exp(A^T t) Q \exp(At)\right)^T dt = \int_0^\infty \exp(A^T t) Q^T \exp(At)\, dt = P, \quad Q = Q^T$$

ii) $$x^T P x = \int_0^\infty x^T \exp(A^T t) Q \exp(At) x\, dt = \int_0^\infty \underbrace{Y^T(t) Q Y(t)}_{\geq 0 \text{ since } Q > 0} dt \geq 0$$ Positive Semi Definite

$Y(t) := \exp(At) x$

To show that P is positive definite, we have to show that $x = 0$ is the only solution to $Y = 0$, which is implied by

$$\det(\exp(At)) \neq 0 \; ; \; \det(\exp(At)) = \det(T \exp(\Lambda t) T^{-1}) \neq 0$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

Now, by substituting (3) into (2) we obtain

$$\int_0^\infty \exp(A^T t)\, Q \exp(At) A \, dt + \int_0^\infty A^T \exp(A^T t)\, Q \exp(At)\, dt =$$

$$= \int_0^\infty \frac{d}{dt}\Big[\exp(A^T t)\, Q \exp(At)\Big] dt = \exp(A^T t)\, Q \exp(At)\Big|_0^\infty = -Q,$$

which shows that P is a solution to (2). To show that it is the unique solution, suppose there exists another solution $\bar{P} \neq P$. Then

$$(P-\bar{P})A + A^T(P-\bar{P}) = 0.$$

Premultiplying and postmultiplying by $\exp(At)$ yields

$$0 = \exp(A^T t)\Big[(P-\bar{P})A + A^T(P-\bar{P})\Big]\exp(At)$$

$$= \frac{d}{dt}\Big[\exp(A^T t)\,(P-\bar{P})\exp(At)\Big] \Rightarrow$$

$\exp(A^T t)(P-\bar{P})\exp(At) = a$, a constant matrix. In particular, since $\exp(A0) = I$, one has

$$P - \bar{P} = \exp(A^T t)(P-\bar{P})\exp(At) \to 0 \Rightarrow P = \bar{P}.$$

Finding the matrix P is no easier than evaluating the eigenvalues of A. However, the above arguments can be used to assess the stability of $x = 0$ when $\dot{x} = Ax$ is perturbed. This perturbation

Consider now

$$V(x) = \frac{1}{2} x^T P x \Rightarrow \dot{V}(x) = -|x|^2 + x^T P R(x) \Rightarrow$$

$\dot{V}(x) \le -|x|^2 + |x||P||R(x)|$. The property that $\lim_{|x| \to 0} \frac{|R(x)|}{|x|} = 0 \Leftrightarrow$

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon): |x| < \delta_\varepsilon \Rightarrow \frac{|R(x)|}{|x|} < \varepsilon$$

This means that if $x \in B_{\delta_\varepsilon}$, i.e. an hypersphere of radius $\delta(\varepsilon)$, then $\frac{|R(x)|}{|x|} < \varepsilon$. Choosing $\varepsilon = \frac{1}{|P|}$, there exists $\delta_{1/|P|}$ such that if

$$|x| < \delta_{1/|P|}, \text{ then } \frac{|R(x)|}{|x|} < \frac{1}{|P|} \Rightarrow |R(x)| < \frac{|x|}{|P|} \Rightarrow$$

$$\dot{V}(x) < -|x|^2 + |x||P| \frac{|x|}{\cancel{|P|}} = 0 \quad \text{if} \quad |x| < \delta_{1/|P|} \Rightarrow \text{L.A.S.}$$

can represent higher order terms of the linearization of a nonlinear system.

<u>Theorem</u> Let $x=0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x)$$

where $f: D \to \mathbb{R}^n$ is continuously differentiable and D a neighborhood of the origin. Let

$$A := \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

Then,

i) The origin is L.A.S. (locally asymptotically stable) if $Re(\lambda_i) < 0$ for all eigenvalues of A.

ii) The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A

<u>Dim of i)</u> The dynamics $\dot{x} = f(x)$ can be approximated as

$$\dot{x} = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + R(x)$$

$$= Ax + R(x) \qquad \text{where } R(x) \text{ satisfies } \lim_{|x| \to 0} \frac{|R(x)|}{|x|} = 0$$

Now, by assumption all eigenvalues of A satisfy $Re(\lambda_i) < 0$.
Hence, A is Hurwitz and there exists P such that $P = P^T > 0$

$$PA + A^T P = -I$$