LESSON 5

We know that if one has a Lyapunov function such that

$$V(x) > 0 \quad \forall x \text{ in } D - \{0\}$$

$$V(0) = 0$$

$$\dot{V}(x) \leq 0 \quad \forall x \text{ in } D,$$

then, $x=0$ is (only) locally stable. However, often the asymptotic stability of $x=0$ can be deduced by applying La Salle theorem.

Theorem. Let $x=0$ be an equilibrium point for $\dot{x}=f(x)$. Let $V: D \to \mathbb{R}$ be a continuously differentiable, positive definite function on a domain $D \subset \mathbb{R}^n$ containing the origin $x=0$ and such that $\dot{V}(x) \leq 0$ in D. Let

$$S = \left\{ x \in D : \dot{V}(x) = 0 \right\}$$

and suppose that no solution can stay identically in S other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable

Corollary: if $V(x)$ is radially unbounded ($|x| \to \infty \Rightarrow V(x) \to \infty$), then the origin is globally asymptotically stable. if the above theorem is satisfied

Examples

i) $\begin{cases} \dot{x}_1 = x_2^2 - x_1 \\ \dot{x}_2 = -x_1 x_2 \end{cases} = f(x) = \begin{pmatrix} x_2^2 - x_1 \\ -x_1 x_2 \end{pmatrix}$

Equilibrium points

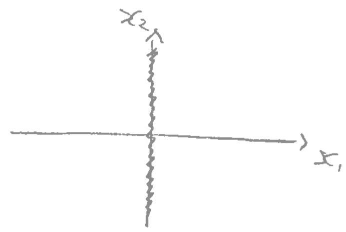
$\begin{cases} x_1 = x_2^2 \\ -x_2^3 = 0 \Leftrightarrow x_2 = 0 \Rightarrow x_1 = 0 \end{cases}$

$$V(x) := \frac{1}{2}|x|^2 \Rightarrow \dot{V} = x^T \dot{x} = (x_1, x_2)\begin{pmatrix} x_2^2 - x_1 \\ -x_1 x_2 \end{pmatrix} = x_1 x_2^2 - x_1^2 - x_1 x_2^2 = -x_1^2$$

$$\dot{V}(x) = -x_1^2 \leq 0 \Rightarrow x = 0 \text{ is globally stable.}$$

Fact i): $V(x)$ is radially unbounded

Fact ii) $S' = \{s \in \mathbb{R}^2 : s_1 = 0\}$



To see which are the solutions that can stay in S', we substitute into the dynamics $s = \begin{pmatrix} 0 \\ s_2(t) \end{pmatrix}$ and see if there exists $s_2(t)$ that satisfy the dynamics:

$$\dot{x}_2 = -x_1 x_2 \Rightarrow \dot{s}_2(t) \equiv 0 \Rightarrow s_2(t) = s_2 \text{, a constant}$$

$$\dot{x}_1 = x_2^2 - x_1 \Rightarrow 0 = s_2^2 - \cancel{0} \Rightarrow \boxed{s_2 = 0} \Rightarrow \text{Only the solution}$$

$s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ can stay in $S' \Rightarrow x = 0$ is G.A.S.

ii) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h_1(x_1) - h_2(x_2) \end{cases}$ Hp $h_i(0) = 0$

$y h_i(y) > 0 \quad \forall y \neq 0,\ y \in (-a, a)$

This is called "generalized pendulum equation" with $h_2(x_2)$ as the friction term. We then consider an energy-like function

$$V(x) = \int_0^{x_1} h_1(y)\,dy + \frac{1}{2} x_2^2$$

Let $D = \{x \in \mathbb{R}^2 : -a < x_i < a\}$. Then, $V(x)$ is P.D in D and

$$\dot{V}(x) = -x_2 h_2(x_2) \leq 0$$

(5.2)

To find $\dot{S} = \{ x \in D : \dot{V}(x) = \}$ note that

$$\dot{V}(x) = 0 \Rightarrow x_2 h(x_2) = 0 \Rightarrow x_2 = 0 , \text{ since } -a < x_2 < a \Rightarrow$$

$$\dot{S} = \{ x \in D : x_2 = 0 \}$$

Let $x(t)$ be a solution that belongs identically to $\dot{S}$:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0 \Rightarrow$$

The only solution that can stay in $\dot{S}$ is the trivial solution $\Rightarrow$ $x = 0$ is locally stable.

LINEAR SYSTEMS AND LINEARIZATION

Recall that given

$$\dot{x} = Ax \quad , \quad x \in \mathbb{R}^n , \; A \in \mathbb{R}^{n \times n} ,$$

the equilibrium point $x = 0$ is

- Globally asymptotically stable iff all eigenvalues of A satisfy

$$(1) \qquad Re(\lambda_i) < 0 \quad \forall i = \{1, \dots, n\}$$

- Unstable if at least one eigenvalue of A satisfies

$$\exists i \in \{1, \dots, n\} : Re(\lambda_i) > 0$$

Given a matrix A, when the condition (1) is satisfied, the matrix A is called Hurwitz matrix, or stability matrix. Hence,

<u>Def</u> The origin of $\dot{x} = Ax$ is GAS iff A is Hurwitz

The asymptotic stability of $x=0$ can also be investigated by using the Lyapunov's method. Consider

$$V(x) = x^T P x , \quad P = P^T > 0$$

Then

$$\dot{V}(x) = x^T [PA + A^T P] x = -x^T Q x$$

withe

$$-Q := PA + A^T P$$

If Q is positive definite, i.e. $Q > 0$, the Lyapunov's theorem ensures that the equilibrium point $x=0$ is G.A.S. This in turn implies that all eigenvalues of A satisfy $Re(\lambda_i) < 0$.

Hence, in the case of linear system, we can reverse[*] the order of the classical application of the Lyapunov theorem. In fact, we can choose a matrix Q, symmetric and positive definite, and then solve

$$(2) \qquad PA + A^T P = -Q$$

for the matrix P. If the solution is positive definite, then we can conclude that $x=0$ is G.A.S. The Eq (2) is called the Lyapunov equation. The next theorem characterizes the asymptotic stability of $x=0$ in terms of the solutions to the Lyapunov equation (2).

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[*] The normal order in establishing stability by means of the Lyapunov's theorem is to fix a candidate $V(x)$, and then verify that $\dot{V}(x) < 0$.