LESSON 4

i) Recap

Stability criteria for autonomous systems

Theorem (Lyapunov) Let $x=0$ an equilibrium point for (1), and $D \subset \mathbb{R}^n$ be a domain containing $x=0$. Let $V: D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$i) \; V(x) > 0 \quad \forall x \in D - \{0\}, \quad V(0) = 0$$

$$ii) \; \dot{V}(x) \leq 0 \quad \forall x \in D$$

Then, $x=0$ is (locally) stable. Moreover, if

$$iii) \; \dot{V} < 0 \quad \forall x \in D - \{0\},$$

then $x=0$ is (locally) asymptotically stable

Quadratic forms

$V(x) = x^T P x, \quad P = P^T$, Then

$V(x) > 0 \; \forall x - \{0\}$, i.e. positive definite, iff all leading principal minors of P have positive determinants.

ii) Applications

a) $\dot{x} = -ax, \quad a \in \mathbb{R}^+ \; (a > 0)$. Then we know that $x(t) = e^{-at} x_0 \Rightarrow$ $x(t) \rightarrow 0$, i.e. it is asymptotically stable. Consider

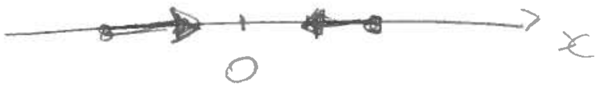
$$V = \frac{1}{2} x^2 \qquad i) \; V > 0 \quad \forall x - \{0\}, \; V(0) = 0$$

SO FAR, WE SAY THAT $V(x)$ IS A LYAPUNOV FUNCTION CANDIDATE

$$\dot V(x) = \frac{d}{dt}V(x) = x\dot x = -ax^2 < 0 \quad \forall x \in \mathbb{R}^n - \{0\} \Rightarrow$$

$x = 0$ is asymptotically stable.

<u>An interpretation of the attractivity of $x = 0$</u>

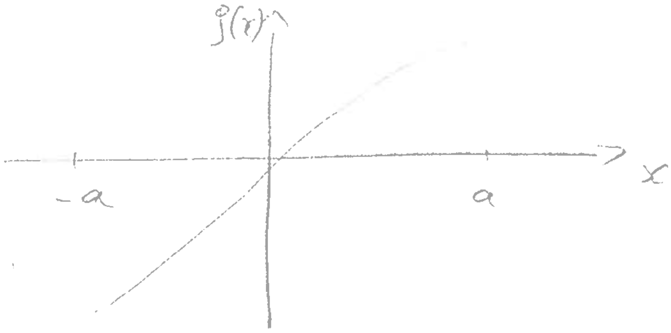


$\dot x = -ax$

This kind of diagrams can be very useful when analysing the system.

b) $\dot x = -g(x)$, with $g(x)$ l.l on $(-a, a)$ and

$$g(0) = 0 \,, \quad x g(x) > 0 \quad \forall x \neq 0 \in (-a, a)$$



$$V(x) = \int_0^x g(y)\,dy \qquad V(0) = 0 \,; \quad V(x) > 0 \quad \Rightarrow$$

$$\dot V(x) = \frac{\partial V}{\partial x}\dot x = -g^2(x) < 0 \quad \forall x \in D - \{0\} \quad \text{A.S. } x = 0$$

(a) $\dot x = -x^k$, $k = 2i+1$, $i \in \mathbb{N} \Rightarrow V(x) = \frac{1}{2}x^2 \Rightarrow \dot V = -x^{k+1} \leq 0 \ \forall x \Rightarrow x = 0$ is G.A.S.

c) Pendulum equation without wind

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -K_1 \sin(x_1) - K_2 x_2 \end{cases}$$

$$\boxed{V(x) = \frac{1}{2} x^T P x + K_1\left(1 - \cos(x_1)\right)} \quad (1) \qquad P := \begin{bmatrix} \frac{K_2^2}{2} & \frac{K_2}{2} \\ \frac{K_2}{2} & 1 \end{bmatrix}$$

P is positive definite, since $\frac{K_2^2}{2} > 0$ $\quad \frac{K_2^2}{2} - \frac{K_2^2}{4} > 0$

DIGRESSION: the gradient and derivative of quadratic forms

We have to now compute the derivative of the quadratic form $x^T P x$

Now, consider a $P = P^T$ and

P is constant

$$\bar{V}(x) := \frac{1}{2} x^T P x \Rightarrow \dot{\bar{V}}(x) = \frac{1}{2}\dot{x}^T P x + \cancel{\frac{1}{2} x^T \dot{P} x} + \frac{1}{2} x^T P \dot{x} \Rightarrow$$

$\left\{ \begin{array}{l} \text{since} \\ \dot{x}^T P x \text{ is a scalar,} \\ \text{it is equal to its transpose} \end{array} \right\} \Rightarrow \dot{\bar{V}}(x) = \frac{1}{2}\left(\dot{x}^T P x\right)^T + \frac{1}{2} x^T P \dot{x} = \frac{1}{2} x^T P^T \dot{x} + \frac{1}{2} x^T P \dot{x} \Rightarrow$

$\left\{ \begin{array}{l} \text{since P is symmetric} \\ P^T = P \end{array} \right\} \Rightarrow \boxed{\dot{\bar{V}}(x) = x^T P \dot{x}}$

Also, it is easy to verify that

$$\dot{\bar{V}}(x) = \frac{d}{dt}\bar{V}(x) = \frac{\partial \bar{V}(x)}{\partial x}\dot{x} \quad ; \quad \boxed{\frac{\partial \bar{V}(x)}{\partial x} = x^T P}$$

Then, $\dot{V}(x) = x^T P \dot{x} + K_1 \sin(x_1)\dot{x}_1$

$$\dot{V}(x) = x^T P \dot{x} + K_1 \sin(x_1)\dot{x}_1 = \left[\frac{K_2^2}{2}x_1 + \frac{K_2}{2}x_2, \frac{K_2}{2}x_1 + x_2\right] \begin{bmatrix} x_2 \\ -K_1 s(x_1) - K_2 x_2 \end{bmatrix} + K_1 s(x_1) x_2$$

$$= \cancel{\frac{K_2^2}{2} x_1 x_2} + \frac{K_2}{2}x_2^2 - \frac{K_1 K_2}{2} x_1 s(x_1) - \cancel{\frac{K_2^2}{2} x_1 x_2} - \cancel{K_1 x_2 s(x_1)} - K_2 x_2^2 + \cancel{K_1 s(x_1) x_2}$$

$$\boxed{\dot{V}(x) = -\frac{K_2}{2} x_2^2 - \frac{K_1 K_2}{2} x_1 \sin(x_1)} \quad (2) \quad ; \quad x_1 \sin(x_1) \geq 0 \quad ; \quad D = (-\pi, \pi)$$

$x = 0$ AS

Now, consider the same system with a different Lyapunov function:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 \sin(x_1) - k_2 x_2 \end{cases}$$

$$\boxed{V(x) = \frac{1}{2} x_2^2 + k_1 \left(1 - \cos(x_1)\right) \quad (3)} \Rightarrow$$

$$\dot{V}(x) = x_2 \dot{x}_2 + k_1 \sin(x_1) \dot{x}_1 = -k_1 \cancel{x_2 \sin(x_1)} - k_2 x_2^2 + k_1 \cancel{x_2 \sin(x_1)}$$

$$\boxed{\dot{V}(x) = -k_2 x_2^2 \quad (4)} \Rightarrow \dot{V}(x) \leq 0 \quad (\text{not } \dot{V}(x) < 0 \text{ as before})$$

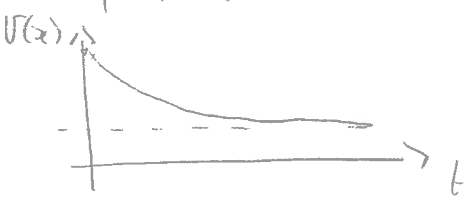
$\Rightarrow x = 0$ is stable. (globally)

- By using the Lyapunov function (1) we end up to (2) and conclude that $x = 0$ is globally stable and locally asymptotically stable
- By using the Lyapunov function (3) we end up to (4), so we can only conclude global stability.

Fact: the form (4) is hiding the property of asymptotic stability

Question: can we deduce the "hidden" property from (4)?

Important fact: the main idea for discovering this hidden property lies in the fact that (4) tells us that $V(x)$ will always decrease. Then, since $V(x)$ is bounded by zero from below, $V(x) \to c \geq 0$



$\Rightarrow \dot{V}(x) \to 0 \Rightarrow x \to$ To some specific set, called invariant!

In this case, the invariant set is

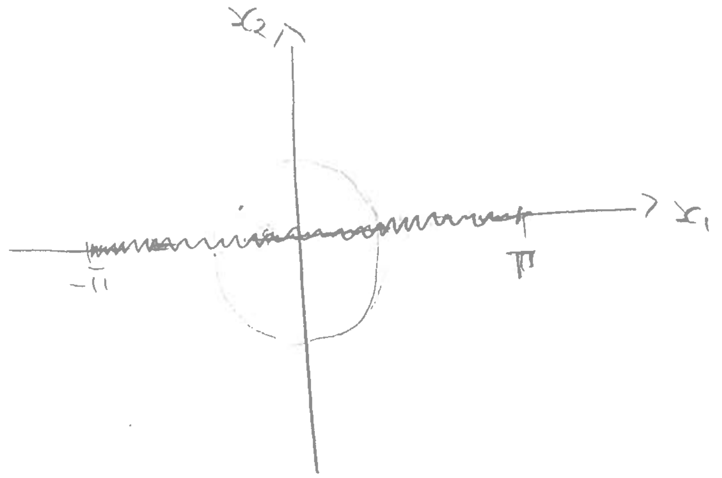
$$x_2 = 0$$

By substituting it into the dynamics

$$\dot{x}_1 = 0 \Rightarrow x_1 \text{ must be constant}$$

$$\dot{x}_2 = 0 = -k_1 \sin(x_1) = 0 \Rightarrow x_1 \to \{0, \pi\}$$

Hence, since $x = 0$ is stable, $x \to 0$ necessarily



The next time we'll see the general theory of invariant sets.