iv) Existence and uniqueness of solutions

Consider the automous system

$$\dot{x} = f(x).$$

Clearly, an equilibrium point $\bar{x}$ such that $f(\bar{x}) = 0$ is a solution to this differential equation, i.e. $x(t) = \bar{x}$ if $x(t_0) = \bar{x}$.

This already raises the question: is this constant solution the only solution to the above differential equation?

in general, no. As a matter of fact, consider

$$\dot{x} = x^{\frac{1}{3}}$$

which has a unique equilibrium point at $x = 0$. Now,

$$\dot{x} = \frac{dx}{dt} = x^{1/3} \Rightarrow x^{-\frac{1}{3}} dx = dt \Leftrightarrow \frac{3}{2} x^{\frac{2}{3}} \Big|_{t_0}^{t} = t - t_0 \Rightarrow$$

$$x(t)^{\frac{2}{3}} - x(t_0)^{\frac{2}{3}} = \frac{2}{3}(t - t_0) \Rightarrow x(t) = \left[x(t_0)^{\frac{2}{3}} + \frac{2}{3}(t - t_0) \right]^{\frac{3}{2}}$$

If $x(t_0) = 0$, then $\boxed{x(t) = \sqrt{\frac{4}{27}} (t - t_0)^{\frac{3}{2}}}$

Question: if I start the system at this initial condition, what is the evolution of the system?

Now, consider the non autonomous system

$$(8) \qquad \dot{x} = f(t, x).$$

A solution $\varphi(t)$ to this system is usually defined over a time interval $[t_0, t_1]$, and on this interval one has

$$\dot{\varphi}(t) = f(t, \varphi(t)), \text{ where}$$

$$\varphi(t) \text{ is continuous}$$

Fact i If $f(t, x)$ is continuous in (t, x)

$$\Downarrow$$

The solution $x(t)$ exists and is continuously differentiable

Hp $f(t, x)$ is continuous in x, but only piece-wise continuous in t

As a consequence, the solution $x(t)$ can only be piece-wise continuously differentiable

Fact ii The example $\dot{x} = x^{1/3} = f(x)$ shows that

Continuity of $f(x)$ $\not\Rightarrow$ Uniqueness of $x(t)$

Theorem Let $f(t, x)$ be piecewise continuous in t and satisfy

$$|f(t, x) - f(t, y)| < \mathcal{L}|x - y|, \quad \mathcal{L} \in \mathbb{R}^+$$

$\forall x, y \in B_r = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$, $\forall t \in [t_0, t_1]$. Then, $\exists \, \delta > 0 : \dot{x} = f(t, x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

We wont detail the proof. However, this proof is based on the <u>contraction mapping theorem</u>.

<u>Theorem</u> (Contraction mapping). Consider the equation

$$x = T(x),$$

where $x \in S \subset \mathcal{X}$, an euclidian space (more generally, a Banach space). Assume that:

hi) $T : S \longrightarrow S$

hii) $\|T(x) - T(y)\| \leq \rho \|x - y\| \quad \forall x, y \in S, \ 0 \leq \rho < 1,$

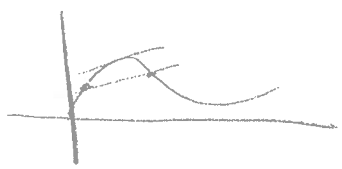
then

i) there exists a <u>unique</u> vector $x^* \in S$ satisfying

$$x^* = T(x^*)$$

ii) x^* can be obtained by the method of successive approximation, starting from any point in S.

<u>Examples</u> $\quad x = \cos(x) \Rightarrow \exists\, \xi : \cos(x_2) - \cos(x_1) = D_\xi(\cdot)(x_{1} \ldots$



(Mean value theorem)

$$|\cos(x_2) - \cos(x_1)| \leq |\sin(\xi)|\,|x_2 - x_1| \leq$$

Very powerful tool!

Show that $f(x) = x^{1/3}$ is not Lypschitz

Do it later

Then, given

$$\dot{x} = f(x,t),$$

the uniqueness of the solution $x(t)$ is proven by observing that

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds = T(x(t))$$

~~and~~ by showing that $\exists\, \rho < 1$ such that $|T(x_2) - T(x_1)| < \rho |x_2 - x_1|$.