LESSON 3

i) Recap

Lyptshiz condition $|f(t,x_1) - f(t,x_2)| < L|x_1 - x_2|$

Locally L $\Rightarrow$ Local uniqueness

Globally L. $\Rightarrow$ Global uniqueness

$|D_x f| < c$

- Stability of an equilibrium point

$$\forall \varepsilon > 0 \; \exists \delta(\varepsilon) : |x(0)| < \delta_\varepsilon \Rightarrow |x(t)| < \varepsilon$$

- Asymptotic stability: S + $\lim_{t \to +\infty} x(t) = 0$

- Boundedness $|x(t)| < c$

- Implications of those properties

- Boundedness criteria $V = \frac{1}{2}|x|^2 \Rightarrow \dot{V} < 0 \quad |x| > \delta \Rightarrow |x(t)| < \delta$

ii) Stability criteria for equilibria of autonomous systems

Consider the autonomous system

$$(1) \quad \dot{x} = f(x)$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose that $\bar{x} \in D$ is an equilibrium point of (1), i.e. $f(\bar{x}) = 0$ and, without loss of generality, assume $\bar{x} = 0$.

What follows presents theorems to ensure the stability of the equilibrium point $\bar{x} = 0$.

Theorem Let $x=0$ be an equilibrium point for (1) and $D \subset \mathbb{R}^n$ be a domain containing $x=0$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that

i) $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$

ii) $\dot{V}(x) \leq 0$ in D.

Then, $x=0$ is (locally) stable. Moreover, if

iii) $\dot{V}(x) < 0$ in $D - \{0\}$,

then $x=0$ is asymptotically stable.

Proof Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \left\{ x \in \mathbb{R}^n : |x| \leq r \right\} \subset D,$$

that is a sphere of radius r contained in D. Let

$$\alpha = \min_{|x| = r} V(x) > 0,$$

and consider

$$\Omega_\beta = \left\{ x \in B_r : V(x) \leq \beta \right\}, \quad \beta \in (0, \alpha).$$

Then $\Omega_\beta \subset B_r$. The set Ω_β has the property that any trajectory starting in Ω_β at $t=0$ stays in Ω_β $\forall t \geq 0$. In fact,

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta \quad \forall t \geq 0.$$

Because Ω_β is a compact set (Ω_β is closed and bounded),

we conclude that (1) has a unique solution defined $\forall t \geq 0$ [illegible] (see Khalil, third edition, pag 84, th 3.3). As $V(x)$ is continuous and $V(0)=0$, $\exists\ \delta>0$ such that (3.2)

$$|x|<\delta \Rightarrow V(x)<\beta$$

Then, $B_\delta \subset \Omega_\beta \subset B_r$ and $x(0)\in B_\delta \Rightarrow x(0)\in \Omega_\beta \Rightarrow x(t)\in \Omega_\beta \Rightarrow$

$$x(t)\in B_r$$

Therefore,

$$|x(0)|<\delta \Rightarrow \|x(t)\|<r\le \varepsilon \quad \forall t\ge 0$$

Now, assume that iii) holds. To show asymptotic stability, we need to show that $x(t)\to 0$ as $t\to+\infty$, that is,

$$\forall a>0\ \exists\ T>0:\ \|x(t)\|<a \text{ when } t>T.$$

By repeating the previous arguments, we can choose $b>0$ such that $\Omega_b\subset B_a$. Therefore, it is sufficient to show that

$$V(x(t))\to 0 \quad \text{as } t\to+\infty.$$

Since $V(x(t))$ is monotonically decreasing and bounded from below by zero, $V(x(t))$ necessarily converges to a constant $c\ge 0$, i.e.

$$\lim_{t\to+\infty} V(x(t)) = c\ge 0.$$

It is left to show that $c=0$. So, proceed by contradiction. Assume $c>0$. By continuity of $V(x)$, $\exists\ d>0$ such that $B_d\subset\Omega_c$. Then

$$\lim_{t\to+\infty} V(x) = c \Rightarrow x(t)\notin B_d.$$

Let $-\gamma = \max \dot{V}(x)$, which necessarily exists because the continuous function $\dot{V}$ has a maximum over the compact set $\{d \le |x| \le r\}$. Since iii) holds, then $\gamma < 0$. It follows that

$$V(x(t)) = V(0) + \int_0^t \dot{V}(x(s))\,ds \le V(0) - \gamma t.$$

Since $\exists T : V(x(T)) < 0$, then this contradicts $c > 0$.

__Global stability__: Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $\begin{cases} V(0) = 0,\ V(x) > 0 \ \forall x \neq 0 \\ \dot{V}(x) < 0 \ \forall x \neq 0,\ |x| \to \infty \Rightarrow V(x) \to +\infty. \end{cases}$ Then, $x = 0$ is G.A.S.

The power of the level surfaces $V(x) = c$ is that we can deduce

$c_1 \le c_2 \le c_3$

The condition $\dot{V} \le 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}$$

and can never come out. When $\dot{V} < 0$, the trajectory moves from one level surface to an inner one with a smaller c.

__ii) Quadratic forms__ Define meaning of positive definite $V(x)$

A class of scalar functions for which the sign definiteness can be easily checked, and consequently easily used to verify $\dot{V} < 0$, is the class of functions of quadratic forms: