# LESSON 1

## i) Introduction

We will deal with dynamical systems that are modeled by a finite number of coupled first-order ordinary differential equations

$$(1) \qquad \frac{d}{dt}x := \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, \dots, x_n, u_1, \dots, u_m) \\ \vdots \\ f_n(t, x_1, \dots, x_n, u_1, \dots, u_m) \end{bmatrix} := f(t, x, u)$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $t \in \mathbb{R}$

$x$: state (variables)

$u$ input (variables)

$t$: time such that $\dot{t} = 1$

Often, we also refer to

$$(2) \qquad y = h(t, x, u) \; ; \quad y \in \mathbb{R}^q,$$ where $y$ is the output of the considered system.

**Def** STATE MODEL: Eq. (1)-(2)

Often, we will work with the so-called unforced state equation, i.e.

$$(3) \qquad \dot{x} = f(t, x),$$

which does not necessarily means that $u \equiv 0$, e.g., $u = g(t)$ or $u = \varphi(x)$.

A special case of (3) arises when the right hand side does not depend on time, i.e. $\partial_t f = 0$, which implies

$$(4) \qquad \dot{x} = f(x),$$

also called autonomous system. The importance of (4) resides in the fact that changing the initial time $t_0$ at which an experiments associated with (4) starts will produce the same time evolution for $x(t)$, i.e., the same results. In other words, changing the time variable from $t$ to $\tau = t - t_0$ does not affect the right hand side of (4).

ii) System's equilibria

Consider the autonomous system (4). Then, the equilibria of the system are all points $\bar{x}$ such that

$$(5) \qquad f(\bar{x}) = 0$$

Note that in the case of a non-autonomous system, we have equilibrium trajectories $\bar{x}(t)$ such that

$$(6) \qquad f(t, \bar{x}(t)) \equiv \dot{\bar{x}}(t) \;\forall t$$

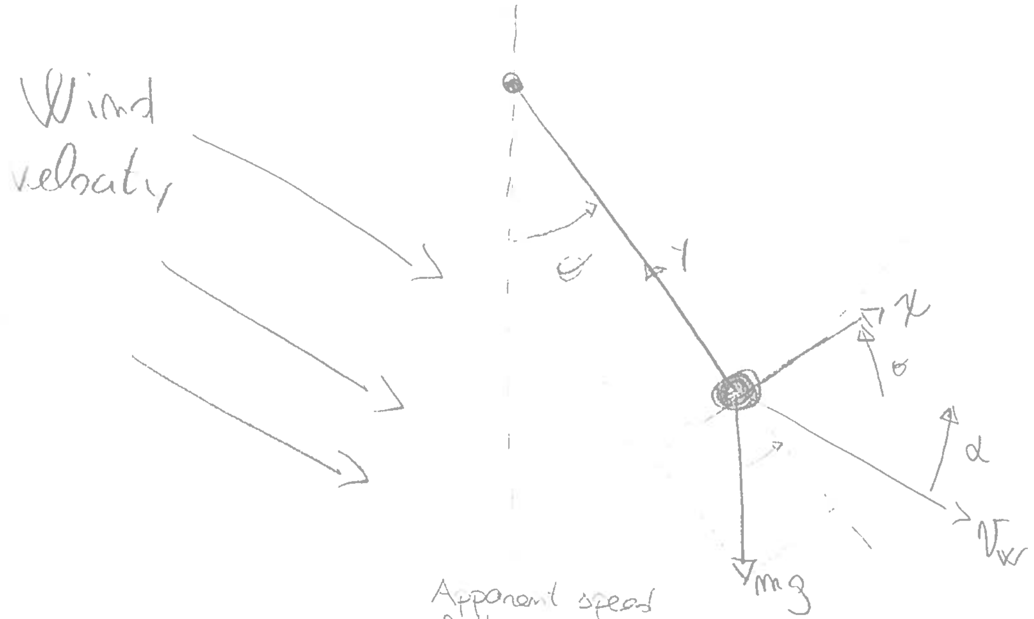
Furthermore, if control variables are available, we say that a trajectory $\bar{x}(t)$ is an equilibrium trajectory if there exists an equilibrium control input $\bar{u}(t) \in \mathbb{R}^m$ such that

$$(7) \qquad f\big(t, \bar{x}(t), \bar{u}(t)\big) \equiv \dot{\bar{x}}(t) \;\forall t$$

The main property of an equilibrium point, or trajectory, is that whenever the state of the system starts at the equilibrium point, it will remain at this point (or follow the trajectory) for all future time. In the case of equilibrium trajectories, they must be differentiable

## iii) Example

a) Pendulum with wind perturbation




$$m\ddot{x} = -mg\sin(\theta) - K\overbrace{(\dot{x} - \dot{x}_w)}^{\text{Apparent speed of the mass}} \qquad x = l\theta \Rightarrow \begin{cases} \dot{x} = l\dot{\theta} \\ \ddot{x} = l\ddot{\theta} \end{cases}$$

$$ml\ddot{\theta} = -mg\sin(\theta) - Kl\dot{\theta} + K\dot{x}_w(t) \Rightarrow$$

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{Kl}{m}\dot{\theta} + \frac{K}{ml}x_w(t)$$

$$:= -K_1\sin(\theta) - K_2\dot{\theta} + K_3 x_w(t)$$

$$\begin{matrix} x_1 = \theta \\ x_2 = \dot{\theta} \end{matrix}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} x_2 \\ -K_1\sin(x_1) - K_2 x_2 + K_3 x_w(t) \end{bmatrix} := f(t, x)$$

Equilibrium points: $\boxed{f(t, \bar{x}) = 0 \quad \forall t}$

$\bar{x}_2 = 0 \Rightarrow$ (Zero velocity, constant position $\bar{x}_1$)

$K_3 x_w(t) - K_1\sin(\bar{x}_1) = 0 \Rightarrow x_w$ must be constant $\Rightarrow$

$$\sin(\bar{x}_1) = \frac{K_3}{K_1}x_w(t) = \frac{K}{mg}|V_w|\cos(\alpha + \bar{x}_1) \Rightarrow \boxed{\begin{matrix} \sin(\bar{x}_1) = \beta\cos(\alpha + \bar{x}_1) \\ \beta := \frac{K|V_w|}{mg} \end{matrix}}$$

## Case of no wind, $|V_w| = 0$

$$\sin(\bar{x}_1) = 0 \iff \bar{x}_1 = \{0, \pi\}$$

$$\boxed{\text{What is the different nature of these two points?}}$$

## Case of constant wind gusts,

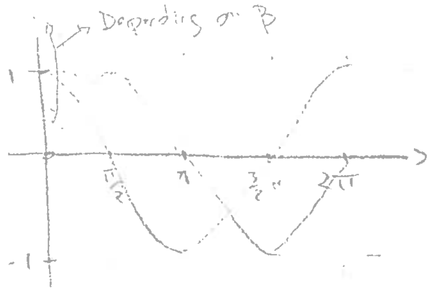
$$\frac{1}{\beta}\sin(\bar{x}_1) = \cos(\alpha)\cos(\bar{x}_1) - \sin(\alpha)\sin(\bar{x}_1).$$

Horizontal wind $\quad \alpha = 0 \Rightarrow \operatorname{tg}(\bar{x}_1) = \beta = \frac{K}{mg}|V_w|$

Vertical wind $\quad \alpha = \pm\frac{\pi}{2} \Rightarrow \sin(\bar{x}_1) = \mp\beta\sin(\bar{x}_1) \Rightarrow \bar{x}_1 = \{0, \pi\}$ if $-\beta$

$\forall \bar{x}_1$ if $+\beta$ and $1 - \beta = 0 \iff mg = K|V_w|$

General solution left for willing students. Note that there always exists solutions since $\sin(\bar{x}_1) = \beta\cos(\alpha + \bar{x}_1)$




$$g(\alpha, x_1) = \sin(x_1) - \beta\cos(\alpha + x_1)$$

$$g(\alpha, x_1) = -g(\alpha, x_1 + \pi) \quad \forall \alpha \Rightarrow$$

$$\exists\, \bar{x}_1 \in (x_1, x_1 + \pi) : g(\alpha, \bar{x}_1) = 0 \;!$$

**IMPORTANT FACT** The existence of an equilibrium configuration cannot be given for grant a priori.