+ Local exponential stability (L.E.S.)

The equilibrium point $x=0$ of system (3) is said to be locally exponentially stable iff

$$|x(0)| < \delta \Rightarrow |x(t)| \leq \alpha |x(0)| e^{-\beta t},$$

with $\alpha, \beta \in \mathbb{R}^+$.

Instability: The equilibrium point $x=0$ of system (3) is said to be unstable if it is not stable

As we have already exposed, the stability concept is related to the system's equilibria and not to the system itself.
A property that may be considered directly connected to the system is the boundedness of its solutions.

Boundedness: The solutions of (3) are said to be bounded if $\forall x(0) \in D \subset \mathbb{R}^n$, there exists a constant $c \in \mathbb{R}^+$ such that

$$|x(t)| \leq c.$$

The rest of the course will present some conditions under which stability, and its derivatives, can be assessed.

Fact iv)

The above stability properties are considered to be "global" when they hold $\forall x(0) \in \mathbb{R}^n$. (G.S., G.A.S., G.E.S., G.B.)

iv) Various implications of the above properties

a) (L.G.) Stability $\Rightarrow$ (L.G.) Boundedness

b) Boundedness $\not\Rightarrow$ Stability (The concept of "start close $\Rightarrow$ stay close" is not in general guaranteed by the boundedness)

c) Instability $\not\Rightarrow$ Trajectory explosions

d) Instability $\not\Rightarrow$ Non-convergence of trajectories

a): Remember that the definition of stability is

$$\forall \varepsilon > 0 \;\; \exists \delta(\varepsilon): \; |x(0)| < \delta(\varepsilon) \Rightarrow |x(t)| < \varepsilon,$$

then the constant ε defines the bound of the system trajectories

b): The main problem for this implication is that we are mixing a property of the system (boundedness) with a property of an equilibrium point. For a counterexample, consider

$$(4) \qquad \dot{x} = -x^3 + x, \quad x \in \mathbb{R}.$$

Now, this system has three equilibrium points: $\bar{x} = \{0, 1, -1\}$. Focus on the equilibrium point $\bar{x} = 0$. Without bothering the theory of linearization, it is clear that close to $\bar{x} = 0$ the system will behave as

$$\dot{x} = x,$$

the solution of which are $x(t) = x(0)\, e^t$, and this shows that $\bar{x} = 0$ is unstable. It is simple to show, however, that the trajectories of system (4) are bounded. More precisely, consider the function

$$V = \frac{1}{2} x^2.$$

The time derivative of V along the system's trajectories results

$$\dot{V} = x\dot{x} = x(x - x^3) = x^2(1 - x^2)$$

Then, $\dot{V} < 0$ when $|x| > 1$. Now, assume that $|x(0)| < 1$. If $\exists T$: $|x(T)| = 1$, then $\dot{V}(T) \leq 0 \Rightarrow V$ must decrease $\Rightarrow (V = \frac{1}{2}x^2)\ |x(t)| \leq 1$.

If $|x(0)| > 1$, then $|x(t)| < |x(0)|\ \forall t$. This proves that independently of $x(0)$, the system has bounded trajectories

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Fact v) The criteria we have just applied holds for general autonomous/non autonomous systems. To show boundedness, it is sufficient to show that

$$V = \frac{1}{2}|x|^2$$

decreases, i.e. $\dot{V} \leq 0$, when the norm of x exceeds a certain value, i.e. $|x| \geq c$. More generally, the next Lemma holds.

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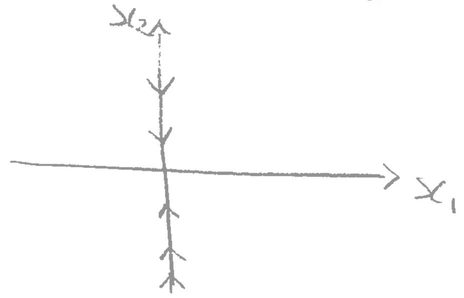
c-d): To show this, let's consider the following example.

$$(5) \quad \begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases} \Leftrightarrow \dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x$$

The (unique) equilibrium point $x = 0$ is unstable because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has a positive-real-part eigenvalue. Note, however, that the solutions to system (5) are given by

$$\begin{cases} x_1(t) = x_1(0)e^{t} \\ x_2(t) = x_2(0)e^{-t} \end{cases}$$



So, if we choose $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \alpha \neq 0,$ we have $x(t) \to 0$

v) A lemma on ultimate boundedness

Let $V(x)$ be a continuously differentiable function such that

$$\alpha_1(|x|) \le V(t,x) \le \alpha_2(|x|) \qquad W(x) > 0 \ \forall x - \{0\}$$
$$W(0) = 0$$
$$\dot V \le -W(x), \quad \forall |x| \ge \mu > 0$$

where the functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are strictly increasing functions and $\alpha_1(|0|) = \alpha_2(|0|) = 0$. Then, for any initial state $x(t_0)$ there exists $T(x_0, \mu) \ge 0$ such that the solutions of $\dot x = f(t,x)$ satisfy

$$|x(t)| \le \beta(|x(t_0)|, t - t_0) \qquad t_0 \le t \le t_0 + T$$
$$|x(t)| \le \alpha_1^{-1}(\alpha_2(\mu)) \qquad t \ge t_0 + T,$$

where $\beta(x,t)$ has the same properties of $\alpha(\cdot)$ at each fixed t, but is decreasing w.r.t the variable t and $\beta(r,t) \to 0$.

Example Consider the mechanical system

$$f_m|x|^2 \le x^T F x \le f_M |x|^2$$

$$M(q)\ddot q + C(q,\dot q)q + g(q) + F\dot q = 0$$

Where i) $M = M^T$, $\underline{\lambda}|x|^2 \le x^T M x \le \bar\lambda |x|^2$

ii) $x^T(\dot M - 2C)x = 0$; $x^T F x > 0 \ \forall x - \{0\}$

iii) $|g(q)| < c \quad \forall q$

$$V = \frac{1}{2}\dot q^T M(q)\dot q \Rightarrow \dot V = \dot q^T g(q) - \dot q^T F \dot q \le -f_m|\dot q|^2 + |\dot q|c + \theta f_m|\dot q|^2 - \theta f_m |\dot q|^2$$

$$= -f_m(1-\theta)|\dot q|^2 + \underbrace{|\dot q|c - \theta f_m |\dot q|^2}_{\le 0 ?} \qquad \theta \in (0,1)$$

$$|\dot q| \ge \frac{c}{\theta f_m} \Rightarrow \dot V \le -f_m(1-\theta)|\dot q|^2 \Rightarrow |\dot q| \text{ is bounded}$$