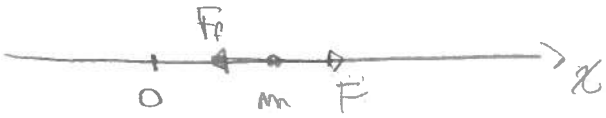
LESSON 0: RECAP ON LINEAR SYSTEMS

1) Example



$$m\ddot{x} = F - F_f = T - \bar{K}\dot{x} \quad \Rightarrow \quad m\ddot{x} = F - \bar{K}\dot{x} \qquad (1)$$

We are interested in the position (its evolution with time) of the mass m.

<u>Question</u> Can we write the second-order system as a first-order one?

Yes. Define

$$x_1 = x \qquad ; \qquad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{F}{m} - \frac{\bar{K}}{m} x_2 \end{cases} \qquad u := \frac{F}{m}$$
$$x_2 = \dot{x} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad K := -\frac{\bar{K}}{m}$$

$$\boxed{\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (2)} \qquad A := \begin{bmatrix} 0 & 1 \\ 0 & K \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1, 0]$$

In general, given a nth-order linear differential equation, we can always transform it into the form (2). More precisely:

$$x^{(n)} - a_{n-1} x^{(n-1)} - \dots - a_0 x = u$$

Define
$$x := \begin{bmatrix} x_1 = x \\ x_2 = x^{(1)} \\ x_3 = x^{(2)} \\ \vdots \\ x_n = x^{(n-1)} \end{bmatrix} \in \mathbb{R}^n \Rightarrow \dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

C depends on the variable we are interested in

Then,

$$x(t) = \exp(At)x_0 \Rightarrow \frac{d}{dt}x(t) = \dot{x} = A\exp(At)x_0 = Ax \Rightarrow$$

CASE $C=0, D=0$

$\dot{x} = Ax + Bu.$

It is possible to verify that the solution to the above differential equation is given by:

$$\boxed{x(t) = \exp(At)x_0 + \int_0^t \exp\big(A(t-\tau)\big)Bu(\tau)\,d\tau} \quad (4)$$

Let us see that.

Fact iii) The integral on the right hand side of equation (4 is a convolution integral. It can be written as

$$\int_0^t \exp\big(A(t-\tau)\big)Bu(\tau)\,d\tau = (f*g)(t) \quad , \text{with} \begin{cases} f(t) = \exp(At) \\ g(t) = Bu(t) \end{cases}$$

It is possible to verify that

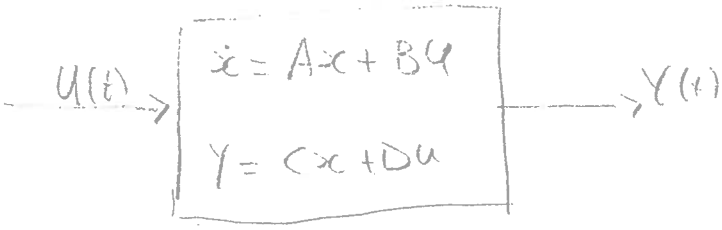
$$(5) \qquad \frac{d}{dt}(f*g) = \underline{\left(\frac{d}{dt}f\right)*g} \qquad \left(= f(t) * \frac{d}{dt}g(t)\right)$$

Fact iv) Recall that the fundamental property of integrals is

$$\begin{cases} \int f(x)\,dx = G(x) \\ \frac{d}{dx}G(x) = f(x) \end{cases} \Rightarrow \begin{cases} \int_0^t f(x)\,dx = G(t) - G(0) \\ \frac{d}{dt}G(t) = f(t) \end{cases}$$

General case

$$7)\quad \begin{cases} Y = Cx(t) + Du(t) \\ x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))\,Bu(\tau)\,d\tau \\ u: \text{control input} \end{cases}$$



iii) Evaluation of the unforced solution. With it we mean no input, $B=0$. Then from Eq. 7), it is clear that the fundamental point is to evaluate the exponential of a matrix A, i.e. $\exp(A)$, since $x(t) = \exp(At)x_0$

$\boxed{\text{Prove that } \exp(A) \text{ is always conv.}}$

Property: if $A = TDT^{-1}$, then $\exp(A) = T\exp(D)T^{-1}$

Proof $\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \sum_{k=0}^{+\infty} \frac{(TDT^{-1})^k}{k!} = \left\{ (TDT^{-1})^2 = TD\cancel{T^{-1}}\cancel{T}DT^{-1} = TD^2T^{-1} \right\}$

$$= \sum_{k=0}^{+\infty} T\frac{D^k}{k!}T^{-1} = T\exp(D)T^{-1}$$

The case of n distinct real eigenvalues

In this case D is diagonal, and on its diagonal there are the eigenvalues. Hence,

$$\exp(D) = \exp\left(\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_n \end{pmatrix}\right) = \sum_{k=0}^{+\infty} \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \lambda_n^k \end{pmatrix}\frac{1}{k!} = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & & e^{\lambda_n} \end{pmatrix}$$

Also, $A = TDT^{-1} \Leftrightarrow AT = TD \Leftrightarrow A[u_1, \ldots, u_n] = [\lambda_1 u_1, \ldots, \lambda_n u_n] \Rightarrow$

T is composed of the right eigenvectors of A! T^{-1} left eigenvectors!

the complex eigenvector

$$u_j = u_{a_j} + J u_{b_j}.$$

Clearly, the conjugate of μ_j, $\mu_j^* = \alpha_j - J\omega_j$, is associate with $u_j^* = u_{a_j} - J u_{b_j}$
Then, the matrix T is given by

$$T := \left[u_1, \dots, u_{m_1} \;\vdots\; u_{a_1}, u_{b_1} \dots\dots u_{a_{\frac{m_2}{2}}}, u_{b_{\frac{m_2}{2}}} \right]$$

and the matrix T^{-1} is composed by the left eigenvectors

$$T := \begin{bmatrix} v_1 \\ \vdots \\ v_{m_1} \\ \hdashline v_{a_1} \\ v_{b_1} \\ \vdots \\ v_{a_{\frac{m_2}{2}}} \\ v_{b_{\frac{m_2}{2}}} \end{bmatrix}$$

Also, it is possible to verify that

$$x(t) = \exp(At)x_0 \iff$$

8a) $x(t) = \sum_{j=1}^{m_1} e^{\lambda_j t} u_j v_j x_0 + \sum_{j=1}^{\frac{m_2}{2}} \beta_j e^{\alpha_j t} \left[u_{a_j} \sin(\omega_j t + \varphi_j) + u_{b_j} \cos(\omega_j t + \varphi_j) \right]$

8b) $\beta_j := \sqrt{(v_{a_j} x_0)^2 + (v_{b_j} x_0)^2}$

8c) $\varphi_j = \operatorname{atan}\left(\dfrac{v_{a_j} x_0}{v_{b_j} x_0} \right)$

0.5

$$(A - I)u_1 = 0 \iff \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & -3 \end{pmatrix} u_1 = 0 \implies u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\big(A + (1-j)I\big) u_j = 0 \quad \begin{pmatrix} 2-j & 0 & 0 \\ 0 & 1-j & 1 \\ 0 & -2 & -1-j \end{pmatrix} u_j = 0$$

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ -1+j \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{u_{a_1}} + j \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{u_{b_1}}$$

$$T = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ \begin{array}{ccc} u_1 & u_{a_1} & u_{b_1} \end{array} \end{array} \longrightarrow T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{c} v_1 \\ v_{a_1} \\ v_{b_1} \end{array}$$

$$\Downarrow$$

$$x(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + 2e^{-t} \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(t) \right]$$