LESSON 2

i) Recap

$$\begin{cases} \dot{x} = f(t, x, u) \\ y = h(t, x, u) \end{cases}$$ STATE MODEL

$\dot{x} = f(t, x)$ UNFORCED STATE EQUATION: NONAUTONOMOUS SYSTEM

$\dot{x} = f(x)$ AUTONOMOUS SYSTEM

$f(\bar{x}) = 0$ $\bar{x}$: equilibrium points

$f(\bar{x}(t), t) = \dot{\bar{x}}(t)$ $\bar{x}(t)$: equilibrium trajectory

$\not\Longleftarrow$

1) $\dot{x} = f(t, x)$ Continuity of $f(\cdot)$ in (t, x) $\Rightarrow \exists$ of a solution

$\not\Rightarrow$ Uniqueness

The uniqueness of a solution starting with $x(0) = x_0$ is given by the existence of a constant L:

(2) $\boxed{\| f(t, x_1) - f(t, x_2) \| \leq L \| x_2 - x_1 \|}$ Lypshitz Condition

If this condition holds globally, i.e. $\forall x_1, x_2 \in \mathbb{R}^n$, $\forall t \in [t_0, t_1]$, then the solution is unique over $[t_0, t_1]$.

ii) Locally and globally Lypshitz functions and existence of solutions

i) A function is said to be locally lypshitz (L.L.) on a domain (open and connected) $D \subset \mathbb{R}^n$, if $\forall d \in D$ $\exists$ a neighborhood D_d : $f(\cdot, x)$ satisfies (2) with some L

f is said to be lypshitz if L is constant over D

ii) f is said to be globally lypshitz if it is lypshitz over $\mathbb{R}^n \Rightarrow \exists!$ global solution of 1

Lemma: If $f(x,t)$ and $\partial_x f$ are continuous on $D \times [a,b]$, where D is an open and connected set belonging to $\mathbb{R}^n$, then $f(\cdot)$ is L.L in x on $D \times [a,b]$

Proof

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} g(x)\,dx \Rightarrow$$

$$|f(x_2) - f(x_1)| \leq \int_{x_1}^{x_2} |g(x)||dx| \leq \int_{x_1}^{x_2} \max|g(x)|\,|dx| \leq L|x_2 - x_1|$$

Lemma: If $f(t,x)$ and $\partial_x f$ are continuous on $[a,b]\times\mathbb{R}^n$, then f is globally Lipschitz in x on $[a,b]\times\mathbb{R}^n$ if and only if $\partial_x f$ is uniformly bounded on $[a,b]\times\mathbb{R}^n$.

Fact i There exist systems having a unique solution, but whose dynamics is not globally lipschitz

Ex $\dot{x} = -x^2$ $\quad \partial_x f = -2x$ unbounded on $\mathbb{R}^n$

$$x(t) = \frac{1}{t-1} \qquad x(0) = 1$$

Fact ii If a function is L.L. in $\mathbb{R}^n$ $\cancel{\Rightarrow}$ the function is G.L.

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

$$\partial_x f = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$$

Lemma Linear systems have a unique solutions

Proof $|Ax_1 + g(t) - Ax_2 - g(t)| = |A(x_1 - x_2)| \leq L|x_1 - x_2|$ G.L.

ii) A consequence of the uniqueness of solutions

Lemma Consider the system $\dot{x} = f(x)$, with $f(\bar{x}) = 0$ and $f \in C^1$. If $\varphi(t)$ is a solution of the autonomous system defined on (t_0, t_1) and

$$\lim_{t \to t_1} \varphi(t) = \bar{x}, \quad \varphi(t) \neq \bar{x} \text{ for some } t,$$

then $t_1 = +\infty$.

Proof This is a proof by contradiction. Assume $t_1 < +\infty$. Then, define

$$\bar{\varphi}(t) = \begin{cases} \varphi(t) & t_0 < t < t_1 \\ \bar{x} & t_1 \leq t < \infty, \end{cases}$$

It is easy to verify that $\bar{\varphi}$ is a solution $C^{(1)}(t_0, +\infty)$ of the problem

$$\dot{x} = f(x), \quad x(t_1) = \bar{x}.$$

But the uniqueness of the solution forbids the existence of the above function.

Fact Equilibrium point can only be achieved at infinity

iii Stability definitions

Fact 0 Stability is intimately related to equilibrium points.

Consider the autonomous system

$$\dot{x} = \bar{f}(x),$$

where $\bar{f}(\bar{x}) = 0$, i.e. $\bar{x}$ is an equilibrium point. Then, define the error between the system's evolution x and $\bar{x}$ as follows

$$x := x - \bar{x}.$$

Then

$$(3) \quad \dot{x} = f(x) \quad , \quad f(0) = 0$$
$$f(x) := \bar{f}(x + \bar{x})$$

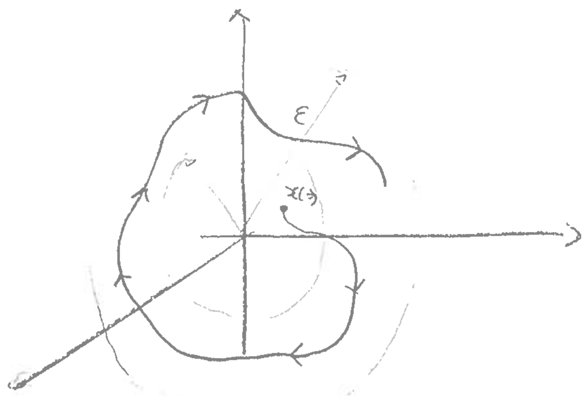
Fact i We can always consider $x=0$ as the equilibrium point and develop a general theory for this case.

Def's

* Stability The equilibrium point $x=0$ of system (3) is said to be stable (S) iff

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) : |x(0)| < \delta(\varepsilon) \Rightarrow |x(t)| \leq \varepsilon$$

Fact ii $\delta(\varepsilon) \leq \varepsilon$



Fact iii Stability is a local property by definitions, and it entails the property that

If you start close to the equilibrium point
$\Downarrow$
the system's evolution will stay close to it

* Local asymptotic stability

The equilibrium point $x=0$ of system (3) is said to be locally asymptotically stable iff it is stable and the trajectory converge to it, i.e.

$$\text{Stability} \ \oplus \ \lim_{t \to +\infty} |x(t)| = 0$$